ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

GUY MÉTIVIER, BENJAMIN TEXIER, KEVIN ZUMBRUN

Existence of quasilinear relaxation shock profiles in systems with characteristic velocities

Tome XXI, nº 1 (2012), p. 1-23.

<http://afst.cedram.org/item?id=AFST_2012_6_21_1_1_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXI, n° 1, 2012
pp. 1–24

Existence of quasilinear relaxation shock profiles in systems with characteristic velocities

Guy Métivier[(1)], Benjamin Texier[(2)], Kevin Zumbrun[(3)]

Abstract. — We revisit the existence problem for shock profiles in quasilinear relaxation systems in the case that the velocity is a characteristic mode, implying that the profile ODE is degenerate. Our result states existence, with sharp rates of decay and distance from the Chapman–Enskog approximation, of small-amplitude quasilinear relaxation shocks. Our method of analysis follows the general approach used by Métivier and Zumbrun in the semilinear case, based on Chapman–Enskog expansion and the macro–micro decomposition of Liu and Yu. In the quasilinear case, however, in order to close the analysis, we find it necessary to apply a parameter-dependent Nash-Moser iteration due to Texier and Zumbrun, whereas, in the semilinear case, a simple contraction-mapping argument sufficed.

Résumé. — Pour des systèmes de relaxation quasi-linéaires, dans le cas dégénéré où la vitesse est un mode caractéristique, nous donnons un résultat d'existence de profils de relaxation de petite amplitude, avec des taux de décroissance. Comme dans le cas semi-linéaire traité dans un

(*) Reçu le 12/03/2010, accepté le 17/04/2011

(1) IMB, Université de Bordeaux, CNRS, IMB, 33405 Talence Cedex, France
metivier@math.u-bordeaux.fr.
G.M. thanks Indiana University for its hospitality during a visit in which this work was partly carried out
(2) Université Paris Diderot (Paris 7), Institut de Mathématiques de Jussieu, UMR CNRS 7586
texier@math.jussieu.fr
Research of B.T. was partially supported under NSF grant number DMS-0505780.
(3) Indiana University, Bloomington, IN 47405
kzumbrun@indiana.edu
K.Z. thanks the University of Bordeaux I and the Institut Henri Poincaré for their hospitality during the visits in which this work was carried out. Research of K.Z. was partially supported under NSF grants number DMS-0070765 and DMS-0300487.

Article proposé par Thierry Gallay.

travail antérieur de Métivier et Zumbrun, nous construisons un profil approché par un développement de Chapman-Enskog et nous utilisons la décomposition “micro-macro” de Liu et Yu. L’ingrédient nouveau dans le cas quasi-linéaire est le recours à un théorème de Nash-Moser à paramètre, du à Texier et Zumbrun, par opposition au cas semi-linéaire dans lequel un simple argument de point fixe permet de conclure la preuve.

1. Introduction

We consider the problem of existence of relaxation profiles

$$U(x,t) = \bar U(x-st), \qquad \lim_{z\to\pm\infty} \bar U(z) = U_\pm \tag{1.1}$$

of a general relaxation system

$$U_t + A(U)U_x = Q(U), \tag{1.2}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q \end{pmatrix}, \tag{1.3}$$

in one spatial dimension, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^r$, where, for some smooth v_* and f,

$$q(u, v_*(u)) \equiv 0, \quad \Re\sigma(\partial_v q(u, v_*(u))) \leqslant -\theta,\ \theta > 0, \tag{1.4}$$

$\sigma(\cdot)$ denoting spectrum, and

$$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \begin{pmatrix} \partial_u f & \partial_v f \end{pmatrix}. \tag{1.5}$$

Here, we are thinking particularly of the case n bounded and $r \gg 1$ arising through discretization or moment closure approximation of the Boltzmann equation or other kinetic models; that is, we seek estimates and proof independent of the dimension of v.

For fixed n, r, the existence problem has been treated in [26, 11] under the additional assumption

$$\det(A - sI) \neq 0 \tag{1.6}$$

corresponding to nondegeneracy of the traveling-wave ODE. However, as pointed out in [12, 13], this assumption is unrealistic for large models, and in particular is not satisfied for the Boltzmann equations, for which the eigenvalues of A are constant particle speeds of all values, hence cannot be uniformly satisfied for discrete velocity or moment closure approximations. Our goal here, therefore, is to revisit the existence problem without the assumption (1.6).

The latter problem was treated in [17] for the semilinear case, which includes discrete velocity approximations of Boltzmann's equations, and for Boltzmann's equation (semilinear but infinite-dimensional) in [18]. We mention also the proof, by similar methods, of positivity of Boltzmann shock profiles in [9] and the original proof, by different methods, of existence of Boltzmann profiles in [2]. The new application here is to moment closure approximations of Boltzmann's and other kinetic equations, which are in general quasilinear.

Our main result is to show existence with sharp rates of decay and distance from the Chapman–Enskog approximation of small-amplitude quasilinear relaxation shocks in the general case that the profile ODE may become degenerate. See Sections 2 and 3 for model assumptions and description of the Chapman–Enskog approximation, and Section 4 for a statement of the main theorem. Our method of analysis, as in [17, 18] is based on Chapman–Enskog expansion and the macro-micro decomposition of [9]. The main difference in this analysis from those of the previous works is that, due to a subtle loss of derivatives, *in the quasilinear case, we find it necessary to apply Nash-Moser iteration to close the analysis*, whereas in the semilinear case a simple contraction-mapping argument sufficed. (See Remark 5.9 for further discussion of this point.) Indeed, we require a nonstandard, parameter-dependent, Nash–Moser iteration scheme, indexed by amplitude $\varepsilon \to 0$, for which the linear solution operator loses not only derivatives but powers of ε. In this, we make convenient use of a general scheme developed in [23] for the treatment of such problems, which also arise in certain hyperbolic problems involving oscillatory solutions with large amplitudes or times of existence (see [23], Section 4).

We note that spectral stability has been shown for general small-amplitude quasilinear relaxation profiles in [13], without the assumption (1.6), under the assumption that the profile exist and satisfy exponential bounds like those of the viscous case. The results obtained here verify that assumption, completing the analysis of [13]. Existence results in the absence of condition (1.6) have been obtained in special cases in [14, 4] by quite different methods. (For example, center-manifold expansion near an assumed single degenerate point [4]. However, the decay bounds as stated, though exponential, are not sufficiently sharp with respect to ε for the needs of [13].)

2. Model, assumptions, and the reduced system

Taking without loss of generality $s = 0$, we study the traveling-wave ODE

$$A(U)U' = Q(U), \tag{2.1}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} \partial_u f(u,v) & \partial_v f(u,v) \\ A_{21}(u,v) & A_{22}(u,v) \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix} \tag{2.2}$$

governing solutions (1.1), where

$$q(u, v_*(u)) \equiv 0, \quad \Re\sigma(\partial_v q(u, v_*(u))) \leqslant -\theta, \ \theta > 0. \tag{2.3}$$

We make the standard assumption of *symmetric–dissipativity* [25]:

Assumption 2.1 (SD). — *There exists a smooth, symmetric and uniformly positive definite matrix $S(U)$ such that*

(i) *for all U, $S(U)A$ is symmetric,*

(ii) *for all equilibria $U_* = (u, v_*(u))$, $\Re S\, dQ(U_*)$ is nonpositive with*

$$\dim\ker \Re S dQ = \dim\ker dQ \equiv n. \tag{2.4}$$

In (2.4) and below, $\Re M$ denotes symmetric part of the matrix M, i.e.

$$\Re M := \frac{1}{2}(M + M^*). \tag{2.5}$$

By the change of coordinates $v \to v - v_*(u,v)$, we may take without loss of generality

$$v_*(u,v) \equiv 0, \quad dQ = \begin{pmatrix} 0 & 0 \\ 0 & \partial_v q \end{pmatrix} \tag{2.6}$$

changing neither the assumed structure (2.1) nor (since it is coordinate-independent) the property of symmetrizability. Note that symmetry of SdQ, together with (2.4), then implies both block-diagonal structure

$$S = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \tag{2.7}$$

and definiteness and proper rank of $\Re S_{22}\partial_v q$. Likewise, symmetry of SA together with (2.7) yields symmetry of $S_{11}A_{11}$ and $S_{22}A_{22}$ as well as

$$(S_{11}A_{12})^T = S_{22}A_{21}. \tag{2.8}$$

We make the simplifying assumption (2.6) throughout the paper. We make also the Kawashima assumption of *genuine coupling* [8]:

Assumption 2.2 (GC). — *For all equilibria $U_* = (u, v_*(u))$, there exists no eigenvector of A in the kernel of $dQ(U_*)$. Equivalently* [8], *given Assumption* 2.1, *there exists in a neighborhood $\mathcal{N}$ of the equilibrium manifold a skew symmetric $K = K(U)$ such that*

$$\Re(KA - SdQ)(U) \geqslant c > 0, \qquad \textit{for all } U \in \mathcal{N}. \tag{2.9}$$

Recall [25] that the reduced, Navier–Stokes type equations obtained by Chapman–Enskog expansions are

$$f_*(u)' = (b_*(u)u')' \tag{2.10}$$

where, under the simplifying assumption (2.6),

$$\begin{aligned} f_*(u) &:= f(u,0) \\ b_*(u)u' &:= -A_{12}\partial_v q^{-1} A_{21}(u,0). \end{aligned} \tag{2.11}$$

For the reduced system (2.10), symmetric–dissipativity becomes:

(sd) There exists $s(u)$ symmetric positive definite such that $s\,df_*$ is symmetric and sb_* is symmetric positive semidefinite, with $\dim\ker \Re sb_* = \dim\ker b_*$.

We have likewise a notion of genuine coupling [8]:

(gc) There is no eigenvector of df_* in $\ker b_*$.

We note first the following important observation of [25].

Proposition 2.3 [25]. — *Let (2.1) as described above be a symmetric–dissipative system satisfying the genuine coupling condition* (GC). *Then, the reduced system (2.10) is a symmetric–dissipative system satisfying genuine coupling condition* (gc).

Proof. — Assuming without loss of generality (2.6), we find that $s = S_{11}$ is a symmetrizer, since $sdf_* = S_{11}A_{11}$ is symmetric as already observed, and $sb_* = -S_{11}A_{12}(S_{22}\partial_v q)^{-1}S_{22}A_{21}$ is definite with proper rank by the corresponding properties of $S_{22}\partial_v q$ together with (2.8). Computing that (gc) is the condition that no eigenvector of A_{11} lie in $\ker A_{21}$, we see that (GC) and (gc) are equivalent. □

Besides the basic properties guaranteed by Lemma 2.3, we assume that the reduced system satisfies the following important additional conditions.

Assumption 2.4. — (i) The matrix $b_*(u)$ has constant left kernel, with associated eigenprojector π_* onto $\ker b_*$, and (ii) The matrix $a_* := \pi_* df_* \pi_*(u)|_{\ker b_*}$ is uniformly invertible.

Assumption 2.4 ensures that the zero-speed profile problem for the reduced system,

$$f_*(u)' = (b_*(u)u')', \qquad \lim_{z\to\pm\infty} u(z) = u_\pm \tag{2.12}$$

or, after integration from $-\infty$ to x,

$$b_*(u)u' = f_*(u) - f_*(u_\pm), \tag{2.13}$$

may be expressed as a nondegenerate ODE in u_2, coordinatizing $u = (u_1, u_2)$ with $u_1 = \pi_* u$ and $u_2 = (I - \pi_*)u$ [13, 27, 5]. Next, we assume that the classical theory of weak shocks can be applied to (2.12), assuming that the flux f_* has a genuinely nonlinear eigenvalue near 0:

Assumption 2.5. — *In a neighborhood $\mathcal{U}_*$ of a given base state u^0, df_* has a simple eigenvalue α near zero, with $\alpha(u^0) = 0$, and such that the associated hyperbolic characteristic field is genuinely nonlinear, i.e., after a choice of orientation, $\nabla\alpha \cdot r(u^0) < 0$, where r denotes the eigendirection associated with α.*

Remark 2.6. — Assumption 2.5 is standard, and is satisfied in particular for the compressible Navier–Stokes equations resulting from Chapman–Enskog approximation of the Boltzmann equation. Assumptions 2.1 and 2.2 are verified in [25] for a wide variety of discrete kinetic models.[1] Assumptions 2.4 and 2.5 on the reduced equations must be checked in individual cases.

3. Chapman–Enskog approximation

We construct in this Section an approximate solution $U_{CE} = (u_{CE}, v_{CE})$ to the traveling-wave ODE (2.1) that satisfies $U_{CE} \to U_\pm = (u_\pm, 0)$ at $\pm\infty$, under a smallness assumption for the amplitude

$$|u_+ - u_-| =: \varepsilon. \tag{3.1}$$

We work in an $O(\varepsilon)$ neighborhood of the base state u^0 given in Assumption 2.5, in the sense that, for some $C > 0$,

$$|u_\pm - u^0| \leqslant C\varepsilon. \tag{3.2}$$

(1) For example, both discrete kinetic models [21] used to approximate the Boltzmann equation [21] and BGK models [7, 19] used to approximate general hyperbolic conservation laws; see pp. 289–294 [25]. Note for each of these examples that the symmetrizer S is not constant, but depends nontrivially on U.

Integrating the first equation of (2.1), we obtain

$$\begin{cases} f(u,v) = f_*(u_-), \\ q(u,v) = A_{21}(u,v)u' + A_{22}(u,v)v'. \end{cases} \tag{3.3}$$

Our ansatz is

$$U(x) = \dot{U}(\varepsilon x) = (\dot{u}(\varepsilon x), \dot{v}(\varepsilon x)), \qquad (\dot{u}, \dot{v}) := \sum_{k=0}^{N} \varepsilon^k (u_k, v_k), \tag{3.4}$$

where the profiles $U_k := (u_k, v_k)$ satisfy

$$\sup_{\varepsilon} \|U_k\|_{W^{k+1,\infty}} < \infty, \tag{3.5}$$

and the boundary conditions

$$\lim_{\pm\infty} u_0 = u_\pm, \quad \lim_{\pm\infty} (u_{k+1}, v_k) = (0,0), \quad k \geqslant 0. \tag{3.6}$$

3.1. Leading term

By (2.6), we necessarily have $v_0 = 0$. Taylor expanding (3.3) and neglecting $O(\varepsilon^2)$ terms, we then obtain

$$\begin{cases} f(u_0, 0) + \varepsilon \partial_u f(u_0, 0) u_1 + \varepsilon \partial_v f(u_0, 0) v_1 &= f_*(u_-), \\ \varepsilon \partial_v q(u_0, 0) v_1 &= \varepsilon A_{21}(u_0, 0) u_0', \end{cases} \tag{3.7}$$

Equation (3.7) can be solved for u_0, u_1 satisfying (3.5) only under the polarization condition

$$f_*(u_0) - f_*(u_-) = O(\varepsilon), \tag{3.8}$$

uniformly in x. If ε is small enough, then the condition (3.2), together with simplicity (hence regularity) of the eigenvalue α given in Assumption 2.5, implies $\alpha(u_-) = O(\varepsilon)$. Then, under $v_0 = 0$, condition (3.8) is equivalent to

$$\Pi_-(u_0 - u_-) = O(\sqrt{\varepsilon}), \qquad (1 - \Pi_-)(u_0 - u_-) = O(\varepsilon), \tag{3.9}$$

uniformly in x, where Π_- is the projection onto the eigendirection $r(u_-)$ associated with $\alpha(u_-)$. Under (3.8), the system (3.7) becomes

$$\begin{cases} \varepsilon^{-1}(f_*(u_-) - f_*(u_0)) + f_*'(u_0) u_1 &= -(A_{12} \partial_v q^{-1} A_{21})(u_0, 0) u_0', \\ v_1 &= (\partial_v q^{-1} A_{21})(u_0, 0) u_0', \end{cases} \tag{3.10}$$

Then, under the uniform polarization condition for u_1 :

$$(1-\Pi_0)u_1 = O(\varepsilon), \tag{3.11}$$

where Π_0 is the projection onto $r(u_0)$, we obtain the approximate viscous profile ODE

$$b_*(u_0)u_0' = \frac{1}{\varepsilon}\big(f_*(u_0) - f_*(u_-)\big), \tag{3.12}$$

with b_* defined in (2.11).

3.2. First corrector

Further expanding (3.3) and neglecting $O(\varepsilon^3)$ terms, we obtain a triangular system in the second corrector U_2 :

$$\left\{\begin{array}{rcl} \partial_u f(u_0,0)u_2 + \partial_v f(u_0,0)v_2 &=& -\frac{1}{\varepsilon}f_*'(u_0)\cdot u_1 - \frac{1}{2}d^2 f(u_0,0)\cdot(U_1,U_1), \\ \partial_v q(u_0,0)\cdot v_2 &=& A_{21}(u_0,0)u_1' + (\partial_u A_{21}(u_0,0)\cdot u_1 \\ && \quad +\partial_v A_{21}(u_0,0)\cdot v_1)u_0' \\ && \quad +A_{22}(u_0,0)v_1' - \partial_v^2 q(u_0,0)\cdot(v_1,v_1). \end{array}\right. \tag{3.13}$$

We impose the uniform polarization condition

$$(1-\Pi_0)u_2 = O(\varepsilon). \tag{3.14}$$

By the triangular structure of system (3.13), equation (3.13)(ii) can be solved for v_2 as a linear function of u_1', with a source depending on u_0 :

$$v_2 = \partial_v q(u_0,0)^{-1}\left(A_{21}u_1' + (\partial_U A_{21}\cdot U_1)u_0' + A_{22}v_1' - \partial_v^2 q\cdot(v_1,v_1)\right). \tag{3.15}$$

Then, equation (3.13)(i) can be solved under a compatibility condition that states that the right-hand side belongs to the image of the matrix to the left-hand side; under (3.11) and (3.14), this condition takes the form of a differential equation in u_1 with quadratic non-linearity:

$$b_*(u_0,0)u_1' = \tilde f_*(u_0)u_1 + \frac{1}{2}f_*''(u_0)\cdot(u_1,u_1) + \mathbf{u}_1, \tag{3.16}$$

where

$$\tilde f_*(u_0)u_1 := -(\partial_u b^*(u_0,0)\cdot u_1)u_0' + \frac{1}{\varepsilon}f_*'(u_0)u_1 + \partial_{uv}^2 f(u_0,0)\cdot(u_1,v_1),$$

and the source $\mathbf{u}_1$ depends on derivatives of the lower-order terms:

$$\mathbf{u}_1 := \frac{1}{2}\partial_v^2 f(u_0,0)\cdot(v_1,v_1)$$
$$-(A_{12}\partial_v q^{-1})(u_0,0)\left(A_{22}(u_0,0)v_1' - \frac{1}{2}\partial_v^2 q(u_0,0)\cdot(v_1,v_1)\right).$$

3.3. Higher-order terms

By induction, we can continue this process of Chapman–Enskog expansion to all orders, and, for $k \geqslant 2$, under the polarization conditions

$$(1-\Pi_0)u_{k'} = O(\varepsilon), \qquad k' \leqslant k, \tag{3.17}$$

formally derive linear equations

$$\begin{cases} b_*(u_0,0)u_k' &= \tilde{f}_*(u_0)u_k + f_*''(u_0)\cdot(u_k,u_1) + \mathbf{u}_k, \\ \qquad v_{k+1} &:= (\partial_v q^{-1}A_{21})(u_0,0)u_k' + \mathbf{v}_k, \end{cases} \tag{3.18}$$

where $\mathbf{u}_k$ is linear in u_k, and $\mathbf{u}_k$ and $\mathbf{v}_k$ both depend on $\partial^{k''}u_{k'}$, for $0 \leqslant k'' \leqslant k' < k$, with $0 < k''$ if $k' = 0$.

Remark 3.1. — Equation (3.18)(i) for the higher-order corrector is the linearization at $(u_1, 0)$ of equation (3.16) for the first-order corrector, whereas in typical Chapman-Enskog expansions [3], the equation for the first corrector is linear, being the linearization of the equation for the leading term.

3.4. Existence and decay bounds

Small amplitude shock profiles solutions of (3.12) are constructed using the center manifold analysis of [20] under conditions (i)-(ii) of Assumption 2.4; see discussion in [14].

Proposition 3.2. — *Under Assumptions* 2.4 *and* 2.5, *in a neighborhood of* $(u^0, u^0) \in \mathbf{R}^n \times \mathbf{R}^n$, *there is a smooth manifold* $\mathcal{S}$ *of dimension* n *passing through* (u^0, u^0), *such that for* $(u_-, u_+) \in \mathcal{S}$ *with amplitude* $\varepsilon := |u_+ - u_-| > 0$ *sufficiently small, and direction* $(u_+ - u_-)/\varepsilon$ *sufficiently close to* $r(u^0)$, *the zero speed shock profile equation (3.12) has a unique (up to translation) solution* u_0 *in the neighborhood* $\mathcal{U}_*$ *of* u^0 *introduced in Assumption* 2.5, *with* u_0 *satisfying (3.9), and, for* $k \geqslant 1$, *the corrector equations (3.16), (3.18)*(i), *have unique (up to translation) solutions* u_k *in* $\mathcal{U}_*$ *satisfying (3.11) and (3.17).*

Moreover, there is $\theta > 0$ *and for all* k, k', *there is* $C_{k,k'} > 0$, *independent of* (u_-, u_+) *and* ε, *such that*

$$|\partial_x^{k'}(u_0 - u_\pm)| \leqslant \varepsilon C_{0,k'} e^{-\theta|x|}, \quad x \gtrless 0. \tag{3.19}$$

and, for $k \geqslant 1$,

$$|\partial_x^{k'} u_k| \leqslant \varepsilon C_{k,k'} e^{-\theta|x|}, \quad x \gtrless 0. \tag{3.20}$$

The shock profile u_0 is necessarily of Lax type: i.e., with dimensions of the unstable subspace of $df_*(u_-)$ and the stable subspace of $df_*(u_+)$ summing to one plus the dimension of u, that is $n+1$. We denote by $\mathcal{S}_+$ the set of $(u_-,u_+)\in\mathcal{S}$ with amplitude $\varepsilon := |u_+-u_-|>0$ sufficiently small and direction $(u_+-u_-)/\varepsilon$ sufficiently close to $r(u_0)$ such that the profile U_{CE} exists. Given $(u_-,u_+)\in\mathcal{S}_+$, with associated profiles $u_0,\dots u_N$, given in Proposition 3.2, we define $v_1,\dots v_N$ by (3.10)(ii), (3.15), (3.18)(ii), and

$$U_{CE} := (u_{CE}, v_{CE}) := \sum_{k=0}^{N} \varepsilon^k (u_k, v_k)(\varepsilon x). \tag{3.21}$$

It is an approximate solution of (3.3) in the following sense:

Corollary 3.3. — *For fixed u_- and amplitude $\varepsilon := |u_+-u_-|$ sufficiently small, the remainder $\mathcal{R} := (\mathcal{R}_1,\mathcal{R}_2)$, defined by*

$$\begin{aligned} \mathcal{R}_{1\pm} &:= f(u_{CE}, v_{CE}) - f_*(u_\pm), \qquad x \gtrless 0,\\ \mathcal{R}_2 &:= A_{21}(u_{CE}, v_{CE}) u'_{CE} + A_{22}(u_{CE}, v_{CE}) v'_{CE} - q(u_{CE}, v_{CE}). \end{aligned} \tag{3.22}$$

satisfies, for $k \geqslant 0$,

$$|\partial_x^k \mathcal{R}_{1\pm}(x)| \leqslant \tilde C_{k,N} \varepsilon^{N+k+2} e^{-\theta\varepsilon|x|}, \quad |\partial_x^k \mathcal{R}_2(x)| \leqslant \tilde C_{k,N} \varepsilon^{N+k+1} e^{-\theta\varepsilon|x|}, \tag{3.23}$$

uniformly in x, where the constants $\tilde C_{k,N}>0$ are independent of (u_-,u_+) and $\varepsilon = |u_+-u_-|$.

Proof. — A direct consequence of the formal Chapman-Enskog expansion of Sections 3.1 to 3.3 and the existence and decay bounds of Proposition 3.2. □

4. Statement of the main theorem

We are now ready to state the main result. Define a base state $U_0 = (u_0, 0)$ and a neighborhood $\mathcal{U} = \mathcal{U}_* \times \mathcal{V}$.

Theorem 4.1. — *Let Assumptions 2.1, 2.2 hold in $\mathcal{U}$, with $f, A, Q \in C^\infty$, and let Assumptions 2.4 and 2.5 hold in $\mathcal{U}_*$. Then, there are $\varepsilon_0>0$ and $\delta>0$ such that for $(u_-,u_+)\in\mathcal{S}_+$ with amplitude $\varepsilon := |u_+-u_-| \leqslant \varepsilon_0$, the standing-wave equation (2.1) has a solution $\bar U = (\bar u, \bar v)$ in $\mathcal{U}$, with associated Lax-type equilibrium shock (u_-,u_+), satisfying for all k, N:*

$$\begin{aligned} \left|\partial_x^k(\bar U - U_{CE})\right| &\leqslant \varepsilon^{k+N} C_{k,N} e^{-\delta\varepsilon|x|},\\ |\partial_x^k(\bar u - u_\pm)| &\leqslant \varepsilon^{k+1} C_k e^{-\delta\varepsilon|x|}, \quad x \gtrless 0,\\ \left|\partial_x^k(\bar v - v_*(\bar u))\right| &\leqslant \varepsilon^{k+2} C_k e^{-\delta\varepsilon|x|}, \end{aligned} \tag{4.1}$$

where U_{CE} is the approximating Chapman–Enskog profile defined in (3.21), and C_k, $C_{k,N}$ are independent of ε. Moreover, up to translation, this solution is unique within a ball of radius $c\varepsilon$ about U_{CE} in norm

$$\varepsilon^{-1/2}\|\cdot\|_{L^2}+\varepsilon^{-3/2}\|\partial_x\cdot\|_{L^2}+\ldots+\varepsilon^{-11/2}\|\partial_x^5\cdot\|_{L^2}, \tag{4.2}$$

for $c>0$ sufficiently small.

By (2.6), the equilibrium v_* in (4.1) is $v_*\equiv 0$. Note that $U_{CE}-U_\pm$ is order $O(\varepsilon)$ in the norm (4.2), by (4.1)(ii)–(iii).

Theorem 4.1 certainly holds under an assumption of finite, although large, pointwise regularity for f, A and Q. The uniqueness result in space (4.2) follows from application of Theorem A.5 with $s_0=3$, $m=1$ and $r'=0$ (see Proposition 5.2).

Bounds (4.1) show that (i) the behavior of profiles is indeed well-described by the Navier–Stokes approximation, and (ii) profiles indeed satisfy the exponential decay rates required for the proof of spectral stability in [13]. From the second observation, we obtain immediately from the results of [13] the following stability result.

Corollary 4.2 [13]. — *Under the assumptions of Theorem 4.1, the resulting profiles $\bar U$ are spectrally stable for amplitude ε sufficiently small, in the sense that the linearized operator $L:=\partial_x A(\bar U)-dQ(\bar U)$ about $\bar U$ has no L^2 eigenvalues λ with $\Re\lambda\geqslant 0$ and $\lambda\neq 0$.*

Proof. — In [13], under the same structural conditions assumed here, it was shown that small-amplitude profiles of general quasilinear relaxation systems are spectrally stable, provided that $|\bar U'|_{L^\infty}\leqslant C|U_+-U_-|^2$, $|\bar U''(x)|\leqslant C|U_+-U_-|\,|\bar U'(x)|$, and

$$\left|\frac{\bar U'}{|\bar U'|}+\operatorname{sgn}(\eta)R_0\right|\leqslant C\,|U_+-U_-|,\qquad R_0:=\begin{pmatrix} r(u_0)\\ dv_*(U_0)r(u_0)\end{pmatrix}, \tag{4.3}$$

where $r(u_0)$ as defined in Theorem 4.1 is the eigenvector of df_* at base point U_0 in the principal direction of the shock. These conditions are readily verified using (4.1). $\square$

The remainder of the paper is devoted to the proof of Theorem 4.1.

5. Proof

5.1. Linear and nonlinear perturbation equations

Defining the perturbation variable $U := \bar U - U_{CE}$, where U_{CE} is defined in (3.21), we obtain from (3.3) the nonlinear perturbation equations $\Phi^\varepsilon(U) = 0$, where

$$\Phi^\varepsilon(U) := \begin{pmatrix} f(U_{CE}+U) - f_*(u_-) \\ A_{21}(U_{CE}+U)(u_{CE}+u)' + A_{22}(U_{CE}+U)(v_{CE}+v)' - q(U_{CE}+U) \end{pmatrix}. \tag{5.1}$$

Formally linearizing Φ^ε about a background profile $\underline{U}$, we obtain

$$(\Phi^\varepsilon)'(\underline{U})U = \begin{pmatrix} A_{11}u + A_{12}v \\ A_{21}u' + A_{22}v' + b_2 U - \partial_v q\, v \end{pmatrix}, \tag{5.2}$$

where

$$A = A(U_{CE} + \underline{U}), \quad \partial_v q = \partial_v q(U_{CE} + \underline{U}),$$

and

$$b_2 U = \big(\partial_u (A_{21} + A_{22})(U_{CE} + \underline{U}) \cdot u + \partial_v (A_{21} + A_{22})(U_{CE} + \underline{U}) \cdot v\big)(U_{CE} + \underline{U})'.$$

The associated linearized equation for a given forcing term $h = (h_1, h_2)$ is

$$(\Phi^\varepsilon)'(\underline{U})U = h. \tag{5.3}$$

5.2. Functional analytic setting

The coefficients and the error term $\mathcal{R}$ from Corollary 3.3 are smooth functions of U_{CE} and its derivatives, so behave like smooth functions of εx. Thus, it is natural to solve the equations in spaces which reflect this scaling. We observe that

$$\|f(\varepsilon\cdot)\|_{L^2} = \varepsilon^{-1/2}\|f\|_{L^2}, \quad \|f(\varepsilon\cdot)\|_{H^s} = \varepsilon^{-1/2}\sum_{k=0}^{s}\varepsilon^k\|\partial_x^k f\|_{L^2}, \tag{5.4}$$

in one space dimension, for $s \in \mathbb{N}$. We do not introduce explicitly the change of variables $\tilde x = \varepsilon x$, but introduce exponentially weighted norms which correspond to usual weighted H^s norms in the $\tilde x$ variable: for $s \in \mathbb{N}$ and $\delta \geqslant 0$, we let, in accordance with (5.4),

$$\|f\|_{\varepsilon,\delta,s} := \varepsilon^{1/2}\sum_{0\leqslant k\leqslant s}\varepsilon^{-k}\|e^{\delta\varepsilon(1+|\cdot|^2)^{1/2}}\partial_x^k f\|_{L^2}, \tag{5.5}$$

the exponential weight accounting for the exponential decay of the source and the solution. For fixed δ, we introduce the spaces $E_s := H^s(\mathbb{R})$, and $F_s := H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, with norms

$$|h|_{E_s} := \|h\|_{\varepsilon,\delta,s}, \qquad |(h_1, h_2)|_{F_s} := \|h_1\|_{\varepsilon,\delta,s+1} + \|h_2\|_{\varepsilon,\delta,s}.$$

In particular, the Chapman-Enskog approximate solution of Section 3 satisfies, by (3.19) and (3.20),

$$|\partial_x^j U_{CE}|_{L^\infty} \leqslant \varepsilon^{j+1} C_j, \quad |\partial_x^{j+1} U_{CE}|_{E_s} \leqslant \varepsilon^{j+2} C_{j,s}, \qquad \text{for } j \geqslant 0, \tag{5.6}$$

where the constants $C_j > 0$, $C_{j,s} > 0$ do not depend on ε, for all $s \in \mathbb{N}$.

5.3. Nash Moser iteration scheme

LEMMA 5.1. — *The application Φ^ε, defined in (5.1), maps smoothly E_s to F_{s-1}, for any s. It satisfies Assumption* A.1 *with $s_0 = 1$, $\gamma_0 = 1$, $\bar{s} = +\infty$, and Assumption* A.3, *with $k = N + 1$.*

Proof. — The bounds of Assumption A.1, describing the action of Φ^ε and its first two derivatives, follow directly from Moser's inequality and the definition of the weighted Sobolev norms. The bound on $\Phi^\varepsilon(0)$ is immediate from (3.23) and (5.5). □

PROPOSITION 5.2. — *Under the assumptions of Theorem* 4.1, *for ε and δ small enough, the map Φ^ε satisfies Assumption* A.2 *with $s_0 = 3$, $\gamma = 1$, $r = 1$, $r' = 0$.*

The proof of this proposition is carried out in Sections 5.4–5.6. Once it is established, existence and uniqueness follow by Theorems A.4 and A.5 from [23]:

Proof of Theorem 4.1 *(Existence).* — The profiles U_{CE} exist if ε is small enough, by Proposition 3.2. By Lemma 5.1 and Proposition 5.2, we can apply Theorem A.4, and thus obtain existence of a solution U^ε of (5.1) with $|U^\varepsilon|_{E_{s+1}} \leqslant C\varepsilon^N$. Defining $\bar{U}^\varepsilon := U_{CE} + U^\varepsilon$, and noting by Sobolev embedding that $|h|_{E_{s+1}}$ controls $|e^{\delta\varepsilon(1+|\cdot|)^{1/2}} h|_{L^\infty}$, we obtain the result. □

Proof of Theorem 4.1 *(Uniqueness).* — Applying Theorem A.5 for $s_0 = 3$, $\gamma_0 = 0$, $\gamma = 1$, $k = 3$, $r = 1$, $r' = 0$, we obtain uniqueness in a ball of radius $c_0\varepsilon$ in $\|\cdot\|_{\varepsilon,0,4}$, $c_0 > 0$ sufficiently small, under the additional phase condition (A.29). We obtain unconditional uniqueness from this weaker version by the observation that phase condition (A.29) may be achieved for any solution $\bar{U} = U_{CE} + U$ with

$$\|U'\|_{L^\infty} \leqslant C\varepsilon^2 << U'_{CE}(0) \sim \varepsilon^2$$

by translation in x, yielding $\bar U_a(x) := \bar U(x+a) = U_{CE}(x) + U_a(x)$ with

$$U_a(x) := U_{CE}(x+a) - U_{CE}(x) + U(x+a)$$

so that, defining $\phi := \bar U'/|\bar U'|$, we have $\partial_a\langle \phi, U_a\rangle \sim \langle \phi, U'_{CE} + U'\rangle = \langle \phi, (1+o(1))\bar U' + U'\rangle = (1+o(1))|\bar U'| \sim \varepsilon^2$ and so (by the Implicit Function Theorem applied to $h(a) := \varepsilon^{-2}\langle \phi, U_a\rangle$, together with the fact that $\langle \phi, U_0\rangle = o(\varepsilon)$ and that $\langle \phi, \bar U'_{NS}\rangle \sim |\bar U'_{NS}| \sim \varepsilon^2$) the inner product $\langle \phi, U_a\rangle$, hence also ΠU_a may be set to zero by appropriate choice of $a = o(\varepsilon^{-1})$ leaving U_a in the same $o(\varepsilon)$ neighborhood, by the computation $U_a - U_0 \sim \partial_a U \cdot a \sim o(\varepsilon^{-1})\varepsilon^2$. □

It remains to prove existence of the linearized solution operator and the linearized bounds of Assumption A.2, which tasks will be the work of the rest of the paper. We concentrate first on estimates, Sections 5.4 and 5.5, and mention next, in Section 5.6, how to prove existence using a viscosity method.

5.4. Internal and high frequency estimates

We begin by establishing a priori estimates on solutions of the equation (5.3). This will be done in two stages. In the first stage, carried out in this section, we establish energy estimates showing that "microscopic", or "internal", variables consisting of v and derivatives of (u, v) are controlled by and small with respect to the "macroscopic", or "fluid" variable, u. In the second stage, carried out in Section 5.5, we estimate the macroscopic variable u by Chapman–Enskog approximation combined with finite-dimensional ODE techniques such as have been used in the study of fluid-dynamical shocks [16, 15, 22, 27].

5.4.1. The basic H^1 estimate

Let $s \in \mathbb{N}$, and some background profile $\underline{U} \in H^s$. We consider equation (5.3), and its differentiated form:

$$(AU' - dQ + b)U = (h_1', h_2), \tag{5.7}$$

in which $bU := (b_1 U, b_2 U)$, where b_2 is defined in Section 5.1, and b_1 is defined similarly, by differentiating the coefficients A_{11}, A_{12} in the first line of (5.3). The coefficients A, b, and dQ, defined in (2.6), are smooth functions of $U_{CE} + \underline{U}$. The bound for U_{CE}, (5.6), and the assumed bound for $\underline{U}$ imply the coefficient bounds

$$\begin{cases} |\partial_x^{j+1} C|_{L^\infty} + |\partial_x^j b|_{L^\infty} \leqslant c_j \varepsilon^{2+j}, & 0 \leqslant j \leqslant s-1, \\ \|\partial_x^{k+1} C\|_{L^2} + \|\partial_x^k b\|_{L^2} \leqslant C_k \varepsilon^{1/2+k}(\varepsilon + \underline{U}|_{\varepsilon,0,s+1}), & 0 \leqslant k \leqslant s, \end{cases} \tag{5.8}$$

where $C = A, Q, K$, the matrix K being the Kawashima multiplier (a smooth function of A). In (5.8), the constants c_j depend on $|\partial_x^{j'}(U_{CE} + \underline{U})|_{L^\infty}$, for $0 \leqslant j' \leqslant j$, while, by the classical Moser's inequality, the constants C_k depend on $|U_{CE} + \underline{U}|_{L^\infty}$.

We give in the following Proposition an estimate for the internal variables $U' = (u', v')$ and v.

Proposition 5.3. — *Under the assumptions of Theorem* 4.1, *for some* $C > 0$, *for* ε *and* δ *small enough, given* $(h_1, h_2) \in F_1$, *if* U *solves (5.3) with* $|\underline{U}|_{E_2} \leqslant \varepsilon$, *there holds*

$$|U'|_{E_0} + |v|_{E_0} \leqslant C\big(|H|_{E_0} + \varepsilon |u|_{E_0}\big), \tag{5.9}$$

where $H = (h_1, h_1', h_1'', h_2, h_2')$.

Proposition 5.3 follows from an L^2 estimate given in Lemma 5.4 for the symmetrized equations, defined as follows.

Multiplying (5.7) by symmetrizer S (block-diagonal, (2.7)), we obtain an ODE

$$\widetilde{A}U' - \widetilde{Q}\,U + \tilde{b}U = S(h_1', h_2), \tag{5.10}$$

where

$$\widetilde{A} = SA, \quad \widetilde{Q} = SdQ = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{Q}_{22} \end{pmatrix}, \quad \tilde{b} = Sb, \tag{5.11}$$

with $\tilde{A}$ symmetric, $\Re\widetilde{Q}_{22}$ negative definite, and $\tilde{b} = O(\varepsilon^2)$, by (5.8). The genuine coupling condition, valid by Assumption 2.2 for A and dQ, still holds for $\tilde{A}$ and $\tilde{Q}$. By the results of [8], this is equivalent to the *Kawashima condition*, and there is a smooth $\tilde{K} = \tilde{K}(U_{CE} + \underline{U}) = -\tilde{K}^*$, such that $\Re(\tilde{K}\tilde{A} - \tilde{Q})$ is definite positive: there is $c > 0$ such that for ε small enough, there holds, uniformly in $x \in \mathbb{R}$,

$$\tilde{Q} \leqslant -c\mathrm{Id}, \qquad \Re(\tilde{K}\tilde{A} - \tilde{Q}) \geqslant c\mathrm{Id}. \tag{5.12}$$

Lemma 5.4. — *For some* $C > 0$, *for* ε *sufficiently small, given* $(h_1, h_2) \in H^2 \times H^1$, *if* $U \in H^1$ *satisfies (5.10) with* $\|\underline{U}\|_{\varepsilon,0,2} \leqslant \varepsilon$, *there holds*

$$\|U'\|_{L^2} + \|v\|_{L^2} \leqslant C\big(\|h_1\|_{H^2} + \|h_2\|_{H^1}) + \varepsilon\|u\|_{L^2}\big). \tag{5.13}$$

Proof. — Introduce the symmetrizer

$$\mathfrak{S} = \partial_x^2 + \partial_x \circ \tilde{K} - \lambda,$$

where $\lambda \in \mathbb{R}$. We bound the (real) L^2 scalar product $(\mathfrak{S}h, U)_{L^2}$ from above and from below. If M is a differential operator, we note that $(Mu, u)_{L^2} =$

$(\Re Mu, u)_{L^2}$, where $\Re M$ is defined as in (2.5), M^* denoting here the adjoint operator of M. Using only the symmetry of $\tilde A$, we find

$$\begin{aligned}&\Re\partial_x^2\circ(\tilde A\partial_x-\tilde Q)=\frac12\partial_x\circ\tilde A'\circ\partial_x-\partial_x\circ\Re\tilde Q\circ\partial_x-\Re\partial_x\circ\tilde Q'\\&\Re\partial_x\circ\tilde K(\tilde A\partial_x-\tilde Q)=\partial_x\circ\Re K\tilde A\circ\partial_x-\Re\partial_x\circ K\tilde Q\\&\Re(\tilde A\partial_x-\tilde Q)=\frac12\tilde A'-\tilde Q.\end{aligned}$$

Thus

$$\begin{aligned}\Re\mathfrak{S}\circ(\tilde A\partial_x-\tilde Q)\quad&=\partial_x\circ\Re(\tilde K\tilde A-\tilde Q)\circ\partial_x+\frac12\Big(\partial_x\circ\tilde A'\circ\partial_x-\lambda\tilde A'\Big)\\&\qquad+\lambda\tilde Q-\Re\partial_x\circ(\tilde Q'+K\tilde Q).\end{aligned}$$

Therefore, if $U\in H^2(\mathbb{R})$ solves (5.10), then (5.12) implies that

$$\begin{aligned}-(\mathfrak{S}S(h_1',h_2),U)_{L^2}\geqslant c\|U'\|^2_{L^2}+\lambda c\|v\|^2_{L^2}-\lambda\Big(\tfrac12\|\tilde A'\|_{L^\infty}+|\tilde b|_{L^\infty}\Big)\|U\|^2_{L^2}\\-\Big(\tfrac12|\tilde A'|_{L^\infty}\|U'\|_{L^2}+\|\tilde Q'\|_{L^\infty}\|U\|_{L^2}+\|K\|_{L^\infty}\|\tilde Q_{22}v\|_{L^2}\Big)\|U'\|_{L^2}\\-\Big(|\tilde b|_{L^\infty}\|U'\|_{L^2}+\|\tilde b'\|_{L^2}|U|_{L^\infty}+|\tilde K\tilde b|_{L^\infty}\|U\|_{L^2}\Big)\|U'\|_{L^2}.\end{aligned}$$

Note that we used an L^2 bound, and not an L^∞ bound, for the term $\tilde b'$ which contains the largest number of derivatives of the background $U_{CE}+\underline{U}$. In the above lower bound, all the terms with a minus sign have small prefactors, by (5.8), except the term $\|\tilde Q_{22}v\|_{L^2}\|U'\|_{L^2}$. We handle this term by Young's product inequality:

$$\|K\|_{L^\infty}\|\tilde Q_{22}v\|_{L^2}\|U'\|_{L^2}\leqslant\frac12c\|U'\|^2_{L^2}+\frac1c\|K\|^2_{L^\infty}\|\tilde Q\|^2_{L^\infty}\|v\|^2_{L^2},$$

and this implies that for some λ, depending on c, $\|K\|_{L^\infty}$ and $\|\tilde Q_{22}\|_{L^\infty}$, the above upper bound can be absorbed in $c(\|U'\|^2_{L^2}+\lambda\|v\|_{L^2})$. Using (5.8) together with the assumed bound on $\underline{U}$, which implies $\|\tilde b'\|_{L^2}\leqslant C\varepsilon^{5/2}$, and using the bound

$$|U|_{L^\infty}\lesssim\varepsilon^{-1/2}\|U\|_{L^2}+\varepsilon^{1/2}\|U'\|_{L^2},$$

we obtain

$$\|U'\|^2_{L^2}+\|v\|^2_{L^2}\leqslant C\big|(\mathfrak{S}Sh,U)_{L^2}\big|+\varepsilon^2C_2\big(\|U\|^2_{L^2}+\|U'\|^2_{L^2}\big),$$

where

$$\varepsilon^2C_2:=|U'_{CE}+\underline{U}'|_{L^\infty}+\varepsilon^{-1/2}\|U''_{CE}+\underline{U}''\|_{L^2}.$$

In the opposite direction,

$$\begin{aligned}\big|(\mathfrak{S}S(h_1',h_2),U)_{L^2}\big|\leqslant C_1\big(\|h_1'\|_{H^1}+\|h_2\|_{H^1}\big)\|U'\|_{L^2}\\+\lambda\big(\|h_1\|_{L^2}\|(S_{11}u)'\|_{L^2}+\|h_2\|_{L^2}\|v\|_{L^2}\big),\end{aligned}$$

where C_1 depends on the L^∞ norm of $U'_{CE} + \underline{U}'$, and where we integrated by parts the term $(h_1', S_{11} u)_{L^2}$ in order to convert the "fluid" variable u into a "microscopic" variable u', up to an error that depends only on one derivative of the coefficients. The estimate (5.13) follows provided that ε is small enough. This proves the lemma under the additional assumption that $U \in H^2$. When $U \in H^1$, the estimates follows using Friedrichs mollifiers. □

Proof. — [Proof of Proposition 5.3] We use Lemma 5.4 for $\varepsilon^{1/2} e^{\delta\varepsilon\langle x\rangle} U$, which solves (5.10) with the source term

$$\varepsilon^{1/2} e^{\delta\varepsilon\langle x\rangle}\big((h_1', h_2) + \delta\varepsilon\langle x\rangle' \tilde{A} U\big),$$

from which (5.9) follows. □

5.4.2. Higher order estimates

Proposition 5.5. — *For $k \geqslant 1$, for come $C > 0$, for ε and δ small enough, given $h \in F_{k+1}$, if $U \in H^k$ satisfies (5.10) with $|\underline{U}|_{E_2} \leqslant \varepsilon$, there holds*

$$\begin{aligned} |\partial_x^k U'|_{E_0} + |\partial_x^k v|_{E_0} &\leqslant C\big(|\partial_x^k H|_{E_0} + \varepsilon^k\big(|U'|_{E_{k-1}} + \varepsilon |v|_{E_{k-1}} + \varepsilon |u|_{E_0}\big)\big) \\ &\quad + C\varepsilon^{k+1} |\underline{U}|_{E_{k+2}} (|v|_{E_1} + \varepsilon |U|_{E_2}), \end{aligned} \tag{5.14}$$

where $H = (h_1, h_1', h_1'', h_2, h_2')$.

Proof. — Differentiating (5.10) k times, we obtain

$$\tilde{A}\partial_x^{k+1} U - \tilde{Q}\partial_x^k U + \tilde{b}\partial_x^k U = (\partial_x^{k+1} h_1, \partial_x^k h_2) + r_k, \tag{5.15}$$

where

$$r_k = -\partial_x^{k-1}\big((\partial_x \tilde{A})\, \partial_x U\big) + \partial_x^{k-1}\big((\partial_x \tilde{Q}) U\big) - \partial_x^{k-1}\big((\partial_x \tilde{b})\, U\big).$$

Note that in the case $k = 1$, the source r_1 in (5.15) does not have the structure of the source term in (5.10). It is however straightforward to adapt the proof of Proposition 5.3 to (5.15) with $k = 1$, by the bound

$$((\partial_x \tilde{C}) U, \partial_x U)_{L^2} \leqslant |\partial_x \tilde{C}|_{L^\infty} \|U\|_{L^2} \|U'\|_{L^2}, \tag{5.16}$$

in which $\partial_x \tilde{C} = \partial_x(\tilde{A}, \tilde{Q}, \tilde{b}) = O(\varepsilon^2)$ in L^∞, by (5.8), hence the contribution of (5.16) is absorbed as in the proof of Proposition 5.3. Thus we apply Proposition 5.3 to (5.15), and obtain

$$|\partial_x^k U'|_{E_0} + |\partial_x^k v\|_{E_0} \leqslant C\big(|\partial_x^k (h_1, h_1', h_1'', h_2, h_2')|_{E_0} + \varepsilon |\partial_x^k u|_{E_0} + |r_k'|_{E_0} + |r_k|_{E_0}\big),$$

in which there is no r_k'' term by the reason indicated above. Thus we are led to estimate terms

$$\left\|\partial_x^\alpha\left((\partial_x^{1+k_1}\tilde{C})(\partial_x^{k-k_1-1+\beta}U)\right)\right\|_{E_0}, \qquad 0 \leqslant k_1 \leqslant k-1, \quad 0 \leqslant \alpha \leqslant 1, \tag{5.17}$$

in which $\tilde{C} = \tilde{A}, \tilde{Q}, \tilde{b}$, and $\beta = 1$ if $\tilde{C} = \tilde{A}$, $\beta = 0$ otherwise. We handle these terms as in the proof of Lemma 5.4, by bounding the coefficients in L^∞, save for the term with the largest numbers of derivatives of the coefficients, namely $(\partial_x^k\tilde{C})(\partial_x^\beta U)$, $(\partial_x^{k+1}\tilde{C})\partial_x^\beta U$, which we bound by taking the L^2 norm of the coefficients and the L^∞ norm of $\partial_x^\beta U$, and obtain (5.14). □

5.5. Linearized Chapman–Enskog estimate

It remains only to estimate the weighted L^2 norm $|u|_{E_0}$ in order to close the estimates and establish the bound claimed in Proposition (5.2). To this end, we work with the first equation in (5.3) and estimate it by comparison with the Chapman-Enskog approximation of Section 3.

5.5.1. The linearized profile equation

From the second equation in (5.3), in which, by (5.8), $b = O(\varepsilon^2)$, we find, for small ε,

$$v = (\partial_v q - b_{22})^{-1}\Big(A_{21}u' + A_{22}v' + b_{21}u - h_2\Big), \tag{5.18}$$

where $b_2 U =: b_{21}u + b_{22}v$. Introducing now (5.18) in the first equation of (5.3), we obtain the linearized profile equation

$$A_{12}(\partial_v q - b_{22})^{-1}A_{21}u' + \big(A_{11} + A_{12}(\partial_v q - b_{22})^{-1}b_{21}\big)u = h^\sharp, \tag{5.19}$$

where $h^\sharp$ depends on the source h and on v', but not on v nor on u :

$$h^\sharp := -A_{12}(\partial_v q - b_{22})^{-1}A_{22}v' + h_1 + A_{12}(\partial_v q - b_{22})^{-1}h_2.$$

5.5.2. L^2 estimates and proof of the main estimates

Introduce the notation

$$b^\sharp := \big(A_{12}(\partial_v q - b_{22})^{-1}A_{21}\big)\,(U_{CE} + \cdot),$$
$$f^\sharp := \big(A_{11} + A_{12}(\partial_v q - b_{22})^{-1}b_{21}\big)\,(U_{CE} + \cdot).$$

Then (5.19) takes the form

$$(b^\sharp\partial_x - f^\sharp)(\underline{U})u = -h^\sharp. \tag{5.20}$$

We estimate the solution of (5.20) by the following:

Proposition 5.6. — *Given* $\underline{U} \in H^4$, *with* $|\underline{U}|_{E_4} \leqslant \varepsilon$, *if* ε *is sufficiently small, then the operator* $(b^\sharp \partial_x - f^\sharp)(\underline{U})$ *has a right inverse* $(b^\sharp \partial_x - f^\sharp)(\underline{U})^\dagger$, *satisfying the bound*

$$\|(b^\sharp \partial_x - f^\sharp)(\underline{U})^\dagger h\|_{E_0} \leqslant C\varepsilon^{-1}\|h\|_{E_0}, \tag{5.21}$$

and uniquely specified by the property that the solution u *to (5.20) satisfies*

$$\ell_\varepsilon \cdot u(0) = 0. \tag{5.22}$$

for certain unit vector ℓ_ε.

Proof. — Standard asymptotic ODE techniques, using the gap and reduction lemmas of [16, 13, 22], where the assumption $\|\underline{U}\|_{E_4} \leqslant C\varepsilon$ gives the needed control on coefficients; see the proof of Proposition 7.1, [17]. □

Proposition 5.7. — *For some* $C > 0$, *for* ε *and* δ *small enough, given* $h \in F_2$, *and* $\underline{U} \in H^4$ *satisfying* $|\underline{U}|_{E_4} \leqslant \varepsilon$, *if* $U = (u, v) \in H^2$ *satisfies (5.3), with* u *satisfying (5.22), there holds*

$$|U|_{E_2} \leqslant C\varepsilon^{-1}|h|_{F_2}. \tag{5.23}$$

Proof. — If U solves (5.3), then u solves (5.19), and if in addition u satisfies (5.22), then by Proposition (5.6), there holds

$$|u|_{E_0} \leqslant C\varepsilon^{-1}|h^\sharp|_{E_0} \leqslant C\varepsilon^{-1}(|h|_{E_0} + |v'|_{E_0}). \tag{5.24}$$

If we now use Proposition 5.3 to bound v', we are left with a term in $C|u|_{E_0}$ in the upper bound, which a priori cannot be absorbed by the left-hand side of (5.24). We use instead Proposition 5.5 with $k = 1$, which together with Proposition 5.3 gives a better estimate for v', namely

$$|v'|_{E_0} \lesssim |H'|_{E_0} + \varepsilon|H|_{E_0} + \varepsilon^2|u|_{E_0} + \varepsilon^2|\underline{U}|_{E_3}(|v|_{E_1} + \varepsilon|U|_{E_2}),$$

and with (5.24) we find, for small ε,

$$\varepsilon|u|_{E_0} \lesssim |h|_{E_0} + \varepsilon|H|_{E_0} + |H'|_{E_0} + \varepsilon|\underline{U}|_{E_3}|U''|_{E_0}. \tag{5.25}$$

Plugging this estimate in (5.9), we find

$$|U'|_{E_0} + |v|_{E_0} + \varepsilon|u|_{E_0} \lesssim |h|_{E_0} + \varepsilon|H|_{E_0} + |H'|_{E_0} + \varepsilon|\underline{U}|_{E_3}|U''|_{E_0}, \tag{5.26}$$

from which we deduce, using again Proposition 5.5 with $k = 1$,

$$|U''|_{E_0} + |v'|_{E_0} \lesssim |h|_{E_0} + \varepsilon|H|_{E_0} + |H'|_{E_0}. \tag{5.27}$$

By definition of the E_2 and F_2 norms, (5.23) follows from (5.26) and (5.27). □

Knowing a bound for $|u|_{E_0}$, Proposition 5.5 implies by induction the following final result.

Proposition 5.8. — *For $s \geqslant 3$, for some $C > 0$, for ε and δ small enough, given $h \in F_s$ and $\underline{U} \in H^{s+1}$ with $|\underline{U}|_{E_4} \leqslant \varepsilon$, if $U \in H^s$ satisfies (5.3) and (5.22), then*

$$|U|_{E_s} \leqslant \varepsilon^{-1} C\big(|\underline{U}|_{E_{s+1}} |h|_{F_2} + |h|_{F_s}\big) \tag{5.28}$$

Remark 5.9. — The loss of derivative on $\tilde{U}$ comes from the conservative form of the linearized equations, through the microscopic energy estimates on the solution. A similar loss in derivative may be seen in the resolvent equation for linear hyperbolic equations in conservative form, $\lambda U + (A(\tilde{U})u)' = f$; see [23] for further discussion. We could avoid this by writing the differentiated equations in quasilinear form, but this would prevent us from integrating back to carry out linearized Chapman–Enskog estimates. That is, the loss of derivatives is due to a subtle incompatibility between the integrated form needed for linearized Chapman–Enskog estimates and the nonconservative (quasilinear) form needed for optimal energy estimates with no loss of derivative.

5.6. Existence for the linearized problem

To complete the proof of Proposition 5.2, it remains to demonstrate existence for the linearized problem. This can be carried out as in [17] by the vanishing viscosity method, with viscosity coefficient $\eta > 0$, obtaining existence for each positive η by standard boundary-value theory, and noting that our previous A Priori bounds (5.28) persist under regularization for sufficiently small viscosity $\eta > 0$, so that we can obtain a weak solution in the limit by extracting a weakly convergent subsequence. We omit these details, referring the reader to Section 8, [17]. The asserted estimates then follow in the limit by continuity.

A. A Nash–Moser Theorem with losses

We give in this appendix the parameter-dependent Nash–Moser theory developed in [23]. The main novelty of this treatment is to allow losses of powers of the parameter $\varepsilon \to 0$ in the linearized solution operator. For a proof of this result, see [23]; for a more general discussion of Nash–Moser iteration methods, see [6, 1, 24], and references therein.

Consider two families of Banach spaces $\{E_s, |\cdot|_{E_s}\}_{s\in\mathbb{R}}$, $\{F_s, |\cdot|_{F_s}\}_{s\in\mathbb{R}}$, where the norms $|\cdot|_{E_s}$ and $|\cdot|_{F_s}$ may be ε-dependent, as in our application here, and a family of equations $\Phi^\varepsilon(u^\varepsilon) = 0$, $u^\varepsilon \in E_s$, indexed by $\varepsilon \in (0,1)$, where for all ε, $\Phi^\varepsilon \in C^2(E_s, F_{s-1})$, for all $s \leqslant \bar{s}$, and some $\bar{s} \in \bar{\mathbb{R}}$.

We assume (i) for $s \leqslant s'$, the embeddings $E_{s'} \hookrightarrow E_s$, $F_{s'} \hookrightarrow F_s$, hold, with $|\cdot|_{E_s} \leqslant |\cdot|_{E_{s'}}$, $|\cdot|_{F_s} \leqslant |\cdot|_{F_{s'}}$, (ii) the interpolation property $|\cdot|_{E_{s+\sigma}} \lesssim |\cdot|_{E_s}^{(\sigma'-\sigma)/\sigma'} |\cdot|_{E_{s+\sigma'}}^{\sigma/\sigma'}$, for $0 < \sigma < \sigma'$, and (iii) the existence of a family of regularizing operators $S_\theta : E_s \to E_s$, for $\theta > 0$, such that for all $s \leqslant s,'$, $|S_\theta u - u|_{E_s} \lesssim \theta^{s-s'} |u|_{E_{s'}}$, and $|S_\theta u|_{E_{s'}} \lesssim \theta^{s'-s} |u|_{E_s}$. (In Sobolev spaces, we can take S_θ to be high-frequency truncations.)

Assumption A.1. — *For some $s_0 \in \mathbb{R}$, some $\gamma_0 \geqslant 0$, for all s such that $s_0 + 1 \leqslant s + 1 \leqslant \bar{s}$, for all $u, v, w \in E_{s+1}$,*

$$\begin{aligned}
|\Phi^\varepsilon(u)|_{F_s} &\leqslant C_0(1 + |u|_{E_{s+1}} + |u|_{E_{s_0+1}} |u|_{E_s}), \\
|(\Phi^\varepsilon)'(u) \cdot v|_{F_s} &\leqslant C_0(|v|_{E_{s+1}} + |v|_{E_{s_0+1}} |u|_{E_{s+1}}), \\
|(\Phi^\varepsilon)''(u) \cdot (v, w)|_{F_s} &\leqslant C_0\big(|v|_{E_{s_0+1}} |w|_{E_{s+1}} + |v|_{E_{s+1}} |w|_{E_{s_0+1}} \\
&\qquad + |u|_{E_{s+1}} |v|_{E_{s_0+1}} |w|_{E_{s_0+1}}\big)
\end{aligned}$$

where $C_0 = C_0(\varepsilon, |u|_{E_{s_0+1}})$ satisfies $\sup_\varepsilon \sup_{|u|_{E_{s_0+1}} \lesssim \varepsilon^{\gamma_0}} C_0 < +\infty$.

Assumption A.2. — *For some $\gamma \geqslant 0, r \geqslant 0, r' \geqslant 0$, for all s such that $s_0 + 1 + \max(r, r') \leqslant s + \max(r, r') \leqslant \bar{s}$, for all $u \in E_{s+r}$ such that $|u|_{E_{s_0+1}} \lesssim \varepsilon^\gamma$, the map $(\Phi^\varepsilon)'(u) : E_{s+1} \to F_s$ has a right inverse $\Psi^\varepsilon(u)$:*

$$(\Phi^\varepsilon)'(u) \Psi^\varepsilon(u) = \mathrm{Id} : \quad F_s \to F_s,$$

satisfying, for all $\phi \in F_{s+r'}$,

$$|\Psi^\varepsilon(u)\phi|_{E_s} \leqslant \varepsilon^{-1} C(|\phi|_{F_{s_0+1+r'}} |u|_{E_{s+r}} + |\phi|_{F_{s+r'}}),$$

where C is a non-decreasing function of its arguments s and $|u|_{s_0+1+r}$.

Assumption A.3. — *There holds the bound*

$$\|\Phi^\varepsilon(0)\|_s \lesssim \varepsilon^k,$$

for some k and s satisfying $\max(2, 1 + \gamma_0, 1 + \gamma) < k$, $C(k) \leqslant \bar{s} - s_0 - 1$, *where $C(k)$ is a certain positive function (see* [23]*) and* $s \in [s_0 + 1, \bar{s} - C(k)]$.

Theorem A.4 (Existence). — *Under Assumptions* A.1, A.2 *and* A.3, *for ε small enough, there exists a real sequence θ_j^ε, satisfying $\theta_j^\varepsilon \to +\infty$ as $j \to +\infty$ and ε is held fixed, such that the sequence $u_0^\varepsilon := 0$, $u_{j+1}^\varepsilon := u_j^\varepsilon + S_{\theta_j^\varepsilon} v_j^\varepsilon$, $v_j^\varepsilon := -\Psi^\varepsilon(u_j^\varepsilon)\Phi^\varepsilon(u_j^\varepsilon)$, is well defined and converges, as $j \to \infty$ and ε is held fixed, to a solution u^ε of $\Phi^\varepsilon(u^\varepsilon) = 0$, in $s + 1$ norm, which satisfies the bound $|u^\varepsilon|_s \lesssim \varepsilon^{k-1}$.*

THEOREM A.5 (Uniqueness). — *Under Assumptions* A.1, A.2 *and* A.3, *for ε small enough, if $(\Phi^\varepsilon)'$ is invertible, i.e., Ψ^ε is also a left inverse, then the solution described in Thm A.4 is unique in a ball of radius $o(\varepsilon^{\max(1,\gamma_0,\gamma)})$ in s_0+2+r' norm. More generally, if $\hat{u}^\varepsilon$ is a second solution within this ball, then $(\hat{u}^\varepsilon - u^\varepsilon)$ is approximately tangent to* $\mathrm{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$, *in the sense that its distance in s_0 norm from* $\mathrm{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ *is $o(|\hat{u}^\varepsilon - u^\varepsilon|_{s_0})$. In particular, if* $\mathrm{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ *is finite-dimensional, then u is the unique solution in the ball satisfying the additional "phase condition"*

$$\Pi_{(\Phi^\varepsilon)'(u^\varepsilon)}(\hat{u}^\varepsilon - u^\varepsilon) = 0, \tag{A.29}$$

where $\Pi_{(\Phi^\varepsilon)'(u^\varepsilon)}$ is any uniformly bounded projection onto $\mathrm{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ *(in a Hilbert space, any orthogonal projection onto* $\mathrm{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$*).*

Bibliography

[1] ALINHAC (S.) and GÉRARD (P.). — Pseudo-differential operators and the Nash-Moser theorem. Graduate Studies in Mathematics, 82. American Mathematical Society, Providence, RI. viii+168 pp (2007).

[2] CAFLISCH (R.) and NICOLAENKO (B.). — Shock profile solutions of the Boltzmann equation, Comm. Math. Phys. 86, no. 2, p. 161-194 (1982).

[3] DEGOND (P.), LEMOU (M.). — On the viscosity and thermal conduction of fluids with multivalued internal energy. Eur. J. Mech. B Fluids 20, no. 2, p. 303-327 (2001).

[4] DRESSEL (A.) and YONG (W.-A.). — Existence of traveling-wave solutions for hyperbolic systems of balance laws, Arch. Ration. Mech. Anal. 182, no. 1, p. 49-75 (2006).

[5] GUÈS (O.), MÉTIVIER (G.), WILLIAMS (M.) and ZUMBRUN (K.). — Navier-Stokes regularization of multidimensional Euler shocks, Ann. Sci. École Norm. Sup. (4) 39, no. 1, p. 75-175 (2006).

[6] HAMILTON (R. S.). — The inverse function theorem of Nash and Moser. Bull. Amer. Math. Soc. (N.S.) 7, no. 1, p. 65-222 (1982).

[7] JIN (S.) and XIN (Z.). — The relaxation schemes for systems of conservation laws in arbitrary space dimensions, Comm. Pure Appl. Math. 48, no. 3, p. 235-276 (1995).

[8] KAWASHIMA (S.). — Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, thesis, Kyoto University (1983).

[9] LIU (T.-P.) and YU (S.-H.). — Boltzmann equation: micro-macro decompositions and positivity of shock profiles, Comm. Math. Phys. 246, no. 1, p. 133-179 (2004).

[10] MAJDA (A.) and PEGO (R.). — Stable viscosity matrices for systems of conservation laws, J. Diff. Eqs. 56 229-262, (1985).

[11] MASCIA (C.) and ZUMBRUN (K.). — Pointwise Green's function bounds and stability of relaxation shocks. Indiana Univ. Math. J. 51, no. 4, p. 773-904 (2002).

[12] Mascia (C.) and Zumbrun (K.). — Stability of large-amplitude shock profiles of general relaxation systems, SIAM J. Math. Anal. 37, no. 3, p. 889-913 (2005).

[13] Mascia (C.) and Zumbrun (K.). — Spectral stability of weak relaxation shock profiles, Comm. Partial Differential Equations 34, no. 1-3, p. 119-136 (2009).

[14] Mascia (C.) and Zumbrun (K.). — Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems, Comm. Pure Appl. Math. 57, no.7, p. 841-876 (2004).

[15] Mascia (C.) and Zumbrun (K.). — Pointwise Green function bounds for shock profiles of systems with real viscosity, Arch. Rational Mech. Anal. 169, no.3, p. 177-263 (2003).

[16] Métivier (G.) and Zumbrun (K.). — Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems, Memoir of the American Mathematical Society, p. 826 (2005).

[17] Métivier (G.) and Zumbrun (K.). — Existence of semilinear relaxation shocks, Journal de Mathématiques Pures et Appliquées, Volume 92, Issue 3, September, p. 209-231, (2009).

[18] Métivier (G.) and Zumbrun (K.). — Existence and sharp localization in velocity of small-amplitude Boltzmann shocks, Kinetic and Related Models, volume 2, number 4, December, p. 667-705 (2009).

[19] Natalini (R.). — Recent mathematical results on hyperbolic relaxation problems, TMR Lecture Notes (1998). Analysis of systems of conservation laws (Aachen, 1997), p. 128-198, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, Chapman & Hall/CRC, Boca Raton, FL (1999).

[20] Pego (R.L.). — Stable viscosities and shock profiles for systems of conservation laws, Trans. Amer. Math. Soc. 282, p. 749-763 (1984).

[21] Platkowski (T.) and Illner (R.). — Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory, SIAM Rev. 30, no. 2, p. 213-255 (1988).

[22] Plaza (R.) and Zumbrun (K.). — An Evans function approach to spectral stability of small-amplitude shock profiles, Discrete Contin. Dyn. Syst. 10, p. 885-924 (2004).

[23] Texier (B.) and Zumbrun (K.). — Nash-Moser iteration and singular perturbations, Ann. Inst. H. Poincaré Anal. Non Linéaire 28, no. 4, p. 499-527 (2011).

[24] Saint-Raymond (X.). — A simple Nash-Moser implicit function theorem. Enseign. Math. (2) 35, no. 3-4, p. 217-226 (1989).

[25] Yong (W.-A.). — Basic structures of hyperbolic relaxation systems, Proc. Roy. Soc. Edinburgh Sect. A 132, no. 5, p. 1259-1274 (2002).

[26] Yong (W.-A.) and Zumbrun (K.). — Existence of relaxation shock profiles for hyperbolic conservation laws, SIAM J. Appl. Math. 60, no.5, p. 1565-1575 (2000).

[27] Zumbrun (K.). — Multidimensional stability of planar viscous shock waves, “Advances in the theory of shock waves”, p. 307-516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA (2001).