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An invariant for difference field extensions

ZOÉ CHATZIDAKIS⁽¹⁾, EHUD HRUSHOVSKI⁽²⁾

ABSTRACT. — In this paper we introduce a new invariant for extensions of difference fields, the *distant degree*, and discuss its properties.

RÉSUMÉ. — Dans cet article nous introduisons un nouvel invariant pour les extensions de corps aux différences, le *degré distant*, et discutons ses propriétés.

Introduction

A *difference field* is a field with a distinguished endomorphism σ . In this short note, we introduce a new invariant for finitely generated difference field extensions of finite transcendence degree, the *distant degree*. If (K, σ) is a difference field, and a a finite tuple in some difference field extending K , and which satisfies $\sigma(a) \in K(a)^{alg}$ (the field-theoretic algebraic closure of $K(a)$), we define

$$\text{dd}(a/K) = \lim_{k \rightarrow +\infty} [K(a, \sigma^k(a)) : K(a)]^{1/k}.$$

One shows easily that $\text{dd}(a/K)$ is bounded by a classical invariant of difference field extensions, the *limit degree* of a over K , and which is defined

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⁽¹⁾ Université Paris Diderot Paris 7 – IMJ UFR de Mathématiques case 7012, site Chevaleret – 75205 Paris Cedex 13, France.

⁽²⁾ Institute of Mathematics – Hebrew University (Giv'at Ram) – Jerusalem 91904, Israel.

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by

$$\text{ld}(a/K) = \lim_{k \rightarrow +\infty} [K(a, \sigma(a), \dots, \sigma^{k+1}(a)) : K(a, \sigma(a), \dots, \sigma^k(a))].$$

Our main result, Theorem 1.9, is that there is some $b \in K(a)_\sigma$ (the difference field generated by a over K) such that $a \in K(b)^{\text{alg}}$, and $\text{dd}(b/K) = \text{ld}(a/K)$. In characteristic 0, this result is a consequence of a result of George Willis on scale functions of automorphisms of totally disconnected locally compact groups, see [5], [6].

Theorem 1.9 follows immediately from Theorem 1.8, which asserts that there is $b \in K(a)_\sigma$ such that $a \in K(b)^{\text{alg}}$ and $\sigma(b) \in K(b, \sigma^\ell(b))$ for every $\ell > 0$. This latter result is particularly useful for difference fields - it is quite convenient to find a tuple satisfying $[K(a, \sigma^\ell(a)) : K(a)] = \text{ld}(a/K)^\ell$ for all $\ell > 0$. We then proceed to derive other properties of these tuples b satisfying “ $\text{ld}=\text{dd}$ ”, see Proposition 1.10. We conclude the study of dd with Proposition 1.11, which among other things shows that $\text{dd}(a, b/K) \geq \text{dd}(a/K(b)_\sigma)\text{dd}(b/K)$. Unfortunately, the distant degree is not multiplicative in towers (see 1.12).

The above results continue to hold for the class of perfect fields, in place of the class of fields. More generally, the statements and proofs go through verbatim for *strongly minimal sets*, cf. e.g. [4] for a definition. Fields should be replaced by definably closed substructures K of a model M of the given strongly minimal theory. We then obtain an invariant of automorphisms of such substructures.

The results for strongly minimal sets admit a purely group theoretic presentation. Namely let G be a group, σ an automorphism of G , and H a subgroup of G such that $H^\sigma \cap H$ has finite index in H and in H^σ . Then one can define the distant degree in terms of (G, H, σ) alone. When \mathcal{U} is a strongly minimal structure with an automorphism σ , K a substructure, $a \in \mathcal{U} \setminus K$, setting $G = \text{Aut}(\mathcal{U}/K)$, $H = \text{Aut}(\mathcal{U}/K(a))$, and $H^\sigma = \text{Aut}(\mathcal{U}/K(\sigma(a)))$, we recover the previous definitions. See the earlier ArXiv version of the paper for details, <http://arxiv.org/abs/0902.0844v2>.

After formulating the results group-theoretically, we found earlier results of Willis extending most of ours in this context ¹. Willis starts out from a totally disconnected locally compact group, rather than an abstract group G with a subgroup H as above; one can however complete the abstract group G above with respect to the topology generated by the finite index subgroups of H ; so again the two settings are equivalent. It follows that our

⁽¹⁾ thanks to Dugald Macpherson for drawing Willis’s results to our attention

invariant $\text{dd}(a/K)$ coincides with the *scale* of σ in the sense of Willis. This yields two new ways of computing the scale function: the definition of dd , and Lemma 1.6(3).

Willis' results allowed us to strengthen our original results. A key observation towards Theorem 1.8 comes from a result hidden in Lemma 3(a) of [5]. Further help comes from the definition of Willis' group \mathcal{L} , but the other ingredients in our proof are different.

We conclude the paper with a discussion of the three settings. In 2.1 – 2.3 we compare our results in the field setting with Willis' in the group setting; naturally they bring in intuitions from different directions. We then show the equivalence of the setting of strongly minimal structures with the one of totally disconnected locally compact groups, see 2.4.

At the end of chapter 1, we also refine the main results for definable groups. By a *difference subgroup* we mean here a subgroup of an algebraic group defined by difference equations; by a *morphism*, we mean a group homomorphism given locally in the σ -topology by difference-rational functions. We show in Proposition 1.15 that if H is a difference subgroup, has finite order and is connected for the σ -topology, then there is a morphism $f : H \rightarrow H'$ with finite central kernel, such that if b is a generic of the difference subgroup H' , then $\text{ld}(b/K) = \text{dd}(b/K)$.

1. The results

1.1. Setting, notation and convention

A *difference field* is a field with a distinguished endomorphism σ . If σ is onto, it is called an *inversive* difference field. Every difference field (K, σ) has an *inversive closure*, denoted K^{inv} , which is characterised by admitting a unique K -embedding into any inversive difference field containing K ([1], 2.5.II). We will work in some large inversive difference field (\mathcal{U}, σ) .

If a is a tuple in \mathcal{U} , then $K(a)_\sigma$ denotes the difference field generated by a over K , i.e., $K(a)_\sigma = K(\sigma^i(a) \mid i \in \mathbb{N})$. If E is a field, then E^{alg} denotes the (field-theoretic) algebraic closure of E , E^s its separable closure, and E^{perf} its perfect hull. If a is a tuple in E^{alg} , then $\mu(a/E)$ denotes $[E(a) : E]$.

We will say that a sequence $(a_n)_{n \in \mathbb{N}}$ is *increasing* if $a_n \leq a_{n+1}$ for any $n \in \mathbb{N}$. Similarly for decreasing.

1.2. Definitions

Let K be a difference subfield of \mathcal{U} , a be a finite tuple in \mathcal{U} , and assume that $\sigma(a) \in K(a)^{alg}$.

1. The *limit degree of a over K* (or of $K(a)_\sigma$ over K) is

$$\text{ld}(a/K) = \lim_{k \rightarrow \infty} \mu(\sigma^{k+1}(a)/K(a, \sigma(a), \dots, \sigma^k(a))),$$

and the *inverse limit degree of a over K* is

$$\text{ild}(a/K) = \lim_{k \rightarrow \infty} \mu(\sigma^{-(k+1)}(a)/K^{inv}(a, \sigma^{-1}(a), \dots, \sigma^{-k}(a))).$$

2. We define the *distant degree* and *inverse distant degree of a over K* by

$$\text{dd}(a/K) = \lim_{k \rightarrow \infty} \mu(\sigma^k(a)/K(a))^{1/k},$$

$$\text{idd}(a/K) = \lim_{k \rightarrow \infty} \mu(\sigma^{-k}(a)/K^{inv}(a))^{1/k}.$$

1.3. Properties of the limit degree

The limit and inverse limit degrees are invariants of the extension $K(a)_\sigma/K$, they are multiplicative in towers, and $\text{ld}(a/K) = \text{ld}(a/K^{inv})$, see [1], section 5.16. If $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$, then for every $i \in \mathbb{N}$, the fields $K(\sigma^j(a) \mid j \geq i)$ and $K^{inv}(\sigma^j(a) \mid j \leq i)$ are linearly disjoint over $K(\sigma^i(a))$. Indeed, the numbers $\mu(\sigma^k(a)/K(a, \dots, \sigma^{k-1}(a)))$ form a decreasing sequence, and $\text{ld}(a/K)$ is the value at which it stabilises. Thus, when $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$, $\text{ld}(\sigma^i(a)/K) = \mu(\sigma^{i+1}(a)/K(\sigma^i(a)))$ for every $i \geq 0$. From

$$\begin{aligned} \mu(\sigma^{i+1}(a)/K(\sigma^i(a))) &= \text{ld}(\sigma^i(a)/K) = \text{ld}(\sigma^i(a)/K^{inv}) \\ &= \mu(\sigma^{i+1}(a)/K^{inv}(\sigma^j(a) \mid j \leq i)), \end{aligned}$$

we obtain that $K^{inv}(\sigma^j(a) \mid j \leq i)$ and $K(\sigma^i(a), \sigma^{i+1}(a))$ are linearly disjoint over $K(\sigma^i(a))$. An easy induction argument gives the result. In this case one also has $\text{ild}(a/K) = \mu(a/K^{inv}(\sigma(a)))$. Furthermore, if $i < j < k$, then

$$\mu(\sigma^j(a)/K^{inv}(\sigma^i(a), \sigma^k(a))) = \mu(\sigma^j(a)/K^{inv}(\sigma^\ell(a), \ell \in (-\infty, i] \cup [k, +\infty))). \quad (\#)$$

1.4. Lemma

Let a and b be tuples in \mathcal{U} such that $b, \sigma(a) \in K(a)^{alg}$, $\sigma(b) \in K(b)^{alg}$.

- (1) There is a constant D such that for all $k \in \mathbb{N}$, $\mu(\sigma^k(a), \sigma^k(b)/K(a, b)) \leq D\mu(\sigma^k(a)/K(a))$. Hence $\text{dd}(b/K) \leq \text{dd}(a/K)$.
- (2) There is a constant D' such that for every $k > 0$, $\mu(\sigma^k(a)/K(a)) \leq D'\mu(\sigma^k(a)/K^{inv}(a))$.
- (3) $\text{ld}(a, b/K)\text{ild}(a/K) = \text{ild}(a, b/K)\text{ld}(a/K)$.

Proof. — (1) One verifies easily that

$$\begin{aligned} \mu(\sigma^k(a), \sigma^k(b)/K(a, b)) &\leq \mu(\sigma^k(b)/K(\sigma^k(a)))\mu(\sigma^k(a)/K(a)) \\ &\leq \mu(b/K(a))\mu(\sigma^k(a)/K(a)). \end{aligned}$$

Take $D = \mu(b/K(a))$.

(2) Let n be such that $\mu(\sigma^{n+1}(a)/K(a, \dots, \sigma^n(a))) = \text{ld}(a/K)$, and let $E = K(a, \dots, \sigma^n(a))$. Then, for $m \geq n$, we have

$$\mu(\sigma^m(a)/K(a)) \leq \mu(\sigma^m(a)/E)\mu(E/K(a)),$$

and

$$\mu(\sigma^m(a)/E) = \mu(\sigma^m(a)/K^{inv}(E)) \leq \mu(\sigma^m(a)/K^{inv}(a)).$$

Take $D' = \mu(E/K(a))$.

(3) Using the multiplicativity in towers of the limit degrees and inverse limit degrees the desired equality becomes

$$\text{ld}(a/K)\text{ld}(b/K(a)_\sigma)\text{ild}(a/K) = \text{ild}(a/K)\text{ild}(b/K(a)_\sigma)\text{ld}(a/K).$$

Therefore (and using (2)), it suffices to show that if $L = K(a)_\sigma^{inv}$ then $\text{ld}(b/L) = \text{ild}(b/L)$. We have $\mu(b/L) = \mu(\sigma(b)/L)$, so that

$$\mu(\sigma(b)/L(b)) = \frac{\mu(b, \sigma(b)/L)}{\mu(b/L)} = \mu(b/L(\sigma(b))).$$

If $\text{ld}(b/L) = \mu(\sigma(b)/L(b))$, this gives the result. Else, it suffices to replace b by $(b, \sigma(b), \dots, \sigma^n(b))$ for some n .

1.5. Setting

The previous lemma has three immediate consequences: if $K(a)_\sigma^{alg} = K(b)_\sigma^{alg}$, then $\text{dd}(a/K) = \text{dd}(b/K)$ (item 1); $\text{dd}(a/K) = \text{dd}(a/K^{inv})$ (item 2); and $\text{dd}(a/K)\text{ild}(a/K) = \text{idd}(a/K)\text{ld}(a/K)$ (item 3). This reduces the study of dd to the following setting: we work inside a large algebraically closed difference field \mathcal{U} , over a difference field $K = \sigma(K)$, and a is a tuple such that $\sigma(a) \in K(a)^{alg}$ and $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$.

1.6. Lemma

- (1) The sequence $\mu(\sigma(a)/K(a, \sigma^\ell(a)))$, $\ell \in \mathbb{N}^{>0}$, is an increasing sequence.
- (2) Let $m = \sup\{\mu(\sigma(a)/K(a, \sigma^\ell(a))), \ell \in \mathbb{N}^{>0}\}$, let ℓ_0 be the smallest ℓ at which this value is attained, and let $C = \mu(\sigma(a), \dots, \sigma^{\ell_0-1}(a)/a, \sigma^{\ell_0}(a))$. If $\ell, j \geq \ell_0$, then

$$\mu(a/K(\sigma^{-j}(a), \sigma^\ell(a))) = \frac{m^{\ell_0}}{C}.$$

- (3) With m as in (2),

$$\text{dd}(a/K) = \frac{\text{ld}(a/K)}{m}.$$

Proof. — We will omit K from the notation, i.e., $\mu(a/b)$ denotes $\mu(a/K(b))$. We will use equation (#) of 1.3 repeatedly.

- (1) One has

$$\mu(\sigma(a)/a, \sigma^\ell(a)) = \mu(\sigma(a)/a, \sigma^\ell(a), \sigma^{\ell+1}(a)) \leq \mu(\sigma(a)/a, \sigma^{\ell+1}(a)).$$

(The first equality is an example of the use of 1.3 (#)).

- (2) If $\ell \geq \ell_0$, then

$$\begin{aligned} \mu(\sigma(a), \dots, \sigma^{\ell-1}(a)/a, \sigma^\ell(a)) &= \prod_{j=1}^{\ell-1} \mu(\sigma^j(a)/K(\sigma^{j-1}(a), \sigma^\ell(a))) \\ &= \prod_{j=1}^{\ell-\ell_0} \mu(\sigma^j(a)/\sigma^{j-1}(a), \sigma^\ell(a)) \mu(\sigma^{\ell-\ell_0+1}(a), \dots, \sigma^{\ell-1}(a)/\sigma^{\ell-\ell_0}(a), \sigma^\ell(a)) \\ &= m^{\ell-\ell_0} C. \end{aligned}$$

If $j \geq \ell_0$, applying σ^{-j} to the above equation with $\ell = j$ gives $\mu(\sigma^{-j+1}(a), \dots, \sigma^{-1}(a)/\sigma^{-j}(a), a) = m^{j-\ell_0} C$.

On the other hand,

$$\begin{aligned} \mu(\sigma^{-j+1}(a), \dots, \sigma^{\ell-1}(a)/\sigma^{-j}(a), \sigma^\ell(a)) \\ &= \mu(a/\sigma^{-j}(a), \sigma^\ell(a)) \mu(\sigma^{-j+1}(a), \dots, \sigma^{-1}(a)/\sigma^{-j}(a), a) \\ &\quad \mu(\sigma(a), \dots, \sigma^{\ell-1}(a)/a, \sigma^\ell(a)) \\ &= \mu(a/\sigma^{-j}(a), \sigma^\ell(a)) C m^{j-\ell_0} C m^{\ell-\ell_0}, \end{aligned}$$

which implies that

$$\mu(a/\sigma^{-j}(a), \sigma^\ell(a)) = \frac{C m^{j+\ell-\ell_0}}{C^2 m^{j+\ell-2\ell_0}} = \frac{m^{\ell_0}}{C}.$$

(3) We computed in the proof of (2) that for $\ell \geq \ell_0$, $\mu(\sigma(a), \dots, \sigma^{\ell-1}(a)/a, \sigma^\ell(a)) = Cm^{\ell-\ell_0}$. Hence

$$\mu(\sigma^\ell(a)/a) = \frac{\mu(\sigma(a), \dots, \sigma^\ell(a)/a)}{\mu(\sigma(a), \dots, \sigma^{\ell-1}(a)/a, \sigma^\ell(a))} = \frac{\text{ld}(a/K)^\ell}{Cm^{\ell-\ell_0}} = \left(\frac{\text{ld}(a/K)}{m} \right)^\ell \frac{m^{\ell_0}}{C}.$$

1.7. Definition

Let $a = (a_1, \dots, a_n)$ be algebraic over the field L . We define the *tuple of minimal monic polynomials of a over L* as follows: $p = (p_1, \dots, p_n)$, with $p_i \in L[X_1, \dots, X_i]$, $i = 1, \dots, n$, are such that $p_1(X_1)$ is the minimal monic polynomial of a_1 over L , and for $1 < i \leq n$, $p_i(a_1, \dots, a_{i-1}, X_i)$ is the minimal monic polynomial of a_i over $L(a_1, \dots, a_{i-1}) = L[a_1, \dots, a_{i-1}]$. Then $\mu(a/L) = \prod_i \deg_{X_i} p_i$.

Let L_0 be a subfield of L , and assume that $\mu(a/L_0) = \mu(a/L)$. Then the tuple p has its coefficients in L_0 . This follows from the fact that for any subfield L_0 of L , one always has $\mu(a_i/L(a_1, \dots, a_{i-1})) \leq \mu(a_i/L_0(a_1, \dots, a_{i-1}))$ for $i = 1, \dots, n$, so that our assumption on the degree of the extension forces equality everywhere.

1.8. Theorem

Let $K = \sigma(K)$, and a a tuple such that $\sigma(a) \in K(a)^{alg}$. Then there is $c \in K(a)_\sigma$ such that $a \in K(c)^{alg}$, and for every $\ell > i > 0$, $\sigma^i(c) \in K(c, \sigma^\ell(c))$.

Proof. — We may assume that $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$. We let ℓ_0 , m and C be defined as in Lemma 1.6, and let c be the tuple of coefficients of the tuple of minimal monic polynomials of a over $K(\sigma^{-\ell_0}(a), \sigma^{\ell_0}(a))$.

Since $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$, we have $\mu(a/K(\sigma^i(a) \mid |i| \geq \ell_0)) = \mu(a/K(\sigma^{-\ell_0}(a), \sigma^{\ell_0}(a)))$. Hence, using Lemma 1.6, if $j, \ell \geq \ell_0$, then c belongs to $K(\sigma^{-j}(a), \sigma^\ell(a))$. Let

$$F = \bigcap_{\ell-n \geq 2\ell_0} K(\sigma^i(a) \mid i \in (-\infty, n] \cup [\ell, +\infty)).$$

Then $c \in F$ and $\sigma(F) = F$. We have $\mu(a/F) = \mu(a/K(c)) := N$. Let $\ell \geq \ell_0$. Then $\mu(\sigma^{-\ell}(a)/F(a, \sigma^\ell(a))) = N$ because $F(a, \sigma^\ell(a)) \subseteq K(\sigma^i(a) \mid i \in (-\infty, -\ell - \ell_0] \cup [-\ell + \ell_0, +\infty))$ and $\sigma^{-\ell}(c) \in F$; and $\mu(\sigma^\ell(a)/F(a)) = N$ because $F(a) \subseteq K(\sigma^i(a) \mid i \in (-\infty, \ell - \ell_0] \cup [\ell + \ell_0, +\infty))$ and $\sigma^\ell(c) \in F$. This implies that

$$[K(\sigma^{-\ell}(a), \sigma^\ell(a), \sigma^{-\ell}(c), c, \sigma^\ell(c)) : K(\sigma^{-\ell}(c), c, \sigma^\ell(c))] = N^2,$$

and therefore that

$$c \in K(\sigma^{-\ell}(c), \sigma^\ell(c)).$$

The first implication is clear; for the second, we know that c belongs to $K(\sigma^{-\ell}(a), \sigma^\ell(a))$, so if $c \notin K(\sigma^{-\ell}(c), \sigma^\ell(c))$, we would have $\mu(\sigma^{-\ell}(a), \sigma^\ell(a)/K(\sigma^{-\ell}(c), c, \sigma^\ell(c))) < N^2$.

Assume that $\sigma(c) \notin K(c, \sigma^\ell(c))$ for some $\ell > 0$, and let n be the maximum value of $\mu(\sigma(c)/K(c, \sigma^\ell(c)))$, attained at ℓ_2 but not before. As we saw in Lemma 1.6, if $\ell \geq \ell_2$ and $C' := \mu(\sigma(c), \dots, \sigma^{\ell_2-1}(c)/K(c, \sigma^{\ell_2}(c)))$, then $\mu(c/K(\sigma^{-\ell}(c), \sigma^\ell(c))) = n^{\ell_2}/C'$, i.e., $n^{\ell_2} = C'$ (since for $\ell \gg 0$, $c \in K(\sigma^{-\ell}(c), \sigma^\ell(c))$). But by definition of ℓ_2 , if $j < \ell_2$, then $\mu(\sigma(c)/c, \sigma^j(c)) < n$. Hence

$$C' = \prod_{i=1}^{\ell_2-1} \mu(\sigma^i(c)/\sigma^{i-1}(c), \sigma^{\ell_2}(c)) = n^{\ell_2},$$

which implies $n = 1$, since the second term is $\leq n^{\ell_2-1}$. I.e., $\sigma(c) \in K(c, \sigma^\ell(c))$ for all $\ell > 0$. An easy induction then gives that $\sigma^i(c) \in K(c, \sigma^\ell(c))$ if $0 < i < \ell$. The proof gives that $c \in K(a)_{\sigma}^{inv}$; if m is such that $\sigma^m(c) \in K(a)_{\sigma}$, then $\sigma^m(c)$ is our desired element.

1.9. We will now derive some consequences of Theorem 1.8. First note a very easy corollary:

THEOREM. — *Let $K = \sigma(K)$, a such that $\sigma(a) \in K(a)^{alg}$, and let c be given by Theorem 1.8. Then $\text{dd}(a/K) = \text{ld}(c/K)$.*

Proof. — By Lemma 1.4, $\text{dd}(a/K) = \text{dd}(c/K)$. On the other hand, since $\sigma(c) \in K(c, \sigma^\ell(c))$ for every $\ell > 0$, we have $\mu(\sigma^\ell(c)/K(c)) = \text{ld}(c/K)^\ell$.

We now proceed to list properties of elements satisfying $\text{ld} = \text{dd}$.

1.10. Proposition

Let $K = \sigma(K)$, a a tuple such that $\sigma(a) \in K(a)^{alg}$, and $c \in K(a)_{\sigma}$ given by Theorem 1.8.

- (1) The following conditions are equivalent, for a tuple d which is equi-algebraic with a over $K = \sigma(K)$:
 - (i) $\text{ld}(d/K) = \text{dd}(a/K)$ ($= \text{dd}(d/K)$).
 - (ii) $\text{ld}(d/K) = \inf\{\text{ld}(e/K) \mid K(e)^{alg} = K(a)^{alg}\}$.

If in addition $\mu(\sigma(d)/K(d)) = \text{ld}(d/K)$, then each of the above conditions is equivalent to each of the following:

- (iii) For every $\ell > 0$, $\sigma(d) \in K(d, \sigma^\ell(d))$.
- (iv) For every $\ell > 0$, $d \in K(\sigma^{-\ell}(d), \sigma^\ell(d))$.

Furthermore, any of the above conditions is equivalent to the analogous one for σ^{-1} .

- (2) Let b be the set of conjugates of a over $K(c)_\sigma$, and let d be a code for the set b (i.e., $K(c)_\sigma(d)$ is the subfield of $K(c)_\sigma(b)$ fixed under $\text{Aut}(K(c)_\sigma(b)/K(c)_\sigma)$). Then for some n , $\text{ld}(d^{p^n}/K) = \text{dd}(d^{p^n}/K) = \text{dd}(a/K)$. If K is perfect, then $\text{ld}(d/K) = \text{dd}(a/K)$, and $a \in K(d)^s$.
- (3) The number $\text{dd}_{\sigma^n}(a/K)$ computed in the σ^n -difference field \mathcal{U} , equals the n -th power of $\text{dd}_\sigma(a/K)$.
- (4) $\text{dd}(a/K) = 1$ if and only if $\{\mu(\sigma^\ell(a)/K(a)) \mid \ell \in \mathbb{N}\}$ is bounded. In that case, $\sigma(c) \in K(c)$.
- (5) $\text{dd}(a/K)$ divides $\text{ld}(a/K)$.
- (6) Assume that $\text{ld}(d/K) = \text{dd}(d/K)$. Then also $\text{ld}(c, d/K) = \text{dd}(c, d/K)$.
- (7) Assume that d is equi-algebraic with a over K , and that for some ℓ_1 , $d \in \bigcap_{\ell \geq \ell_1} K(\sigma^{-\ell}(a), \sigma^\ell(a))$. Then $\text{dd}(d/K) = \text{ld}(d/K)$.

Proof. — (1) For every n , the limit degree satisfies $\text{ld}(a/K) = \text{ld}(a, \sigma(a), \dots, \sigma^n(a)/K)$ and we may therefore assume that $\text{ld}(a/K) = \mu(\sigma(a)/K(a))$ since this change will not affect the first two conditions. We will show the equivalence of (i) – (iv).

We know by Lemma 1.4 that $\text{dd}(a/K) = \text{dd}(d/K)$. Assume that (iii) does not hold. Then for some $\ell > 0$, we have $\sigma(d) \notin K(d, \sigma^\ell(d))$; by Lemma 1.6 (1) and (3), we have $\text{dd}(d/K) < \text{ld}(d/K)$, whence $\text{dd}(a/K) < \text{ld}(d/K)$. Thus (i) implies (iii). Clearly (iii) implies (i).

Similarly, $\text{dd}(e/K) \leq \text{ld}(e/K) < \text{dd}(a/K)$ is impossible unless $K(e)^{\text{alg}}$ is strictly contained in $K(a)^{\text{alg}}$, and this proves the equivalence of (i) and (ii).

(iii) implies (iv) is an easy induction, and (iv) implies (iii) is proved in the last part of the proof of Theorem 1.8.

Finally, for the last assertion it suffices to show that one of the above conditions is equivalent to its analogue for σ^{-1} . We know that the quotient $\frac{\text{ld}(a/K)}{\text{ld}(a/K)}$ is an invariant of the extension $K(a)^{\text{alg}}/K$, by Lemma 1.4(3). Hence, (ii) for σ implies (ii) for σ^{-1} .

(2) By definition of d the extension $K(b)/K(d)$ is separable, and the extension $K(c)_\sigma(d)/K(c)_\sigma$ is purely inseparable. This implies that the extension $K(c, d)_\sigma/K(c)_\sigma$ is purely inseparable, and $a \in K(d)^s$. If n is such that $d^{p^n} \in K(c)_\sigma$, then $\text{ld}(d^{p^n}/K)$ divides $\text{ld}(c/K)$, and by minimality of the latter, must be equal to it. Hence $\text{ld}(d^{p^n}/K) = \text{dd}(a/K)$.

If K is perfect, then $\text{ld}(d^{p^n}/K) = \text{ld}(d/K^{p^{-n}}) = \text{ld}(d/K) = \text{dd}(a/K)$.

(3) Clear from the definition of dd .

(4) Clear by Lemma 1.4(1) and Theorem 1.9.

(5) As $c \in K(a)_\sigma$, $\text{dd}(a/K) = \text{ld}(c/K)$ divides $\text{ld}(a/K)$.

(6) By (1), we have $\sigma(c) \in K(c, \sigma^\ell(c))$ and $\sigma(d) \in K(d, \sigma^\ell(d))$ for every $\ell > 0$. Hence, $\sigma(c, d) \in K(c, d, \sigma^\ell(c, d))$ for every $\ell > 0$, which by (1) implies that $\text{ld}(c, d/K) = \text{dd}(c, d/K)$.

(7) We use the notation of Theorem 1.8. Without loss of generality, we may assume that $\ell_1 \geq \ell_0$. Let e be a tuple such that $K(e) = \bigcap_{\ell \geq \ell_1} K(\sigma^{-\ell}(a), \sigma^\ell(a))$. Then $c \in K(e)$ (since $\ell_1 \geq \ell_0$), and therefore is equi-algebraic with e over K . As $d \in K(e)$, it suffices to show that $\text{ld}(e/K) = \text{dd}(e/K)$, since $\text{ld}(d/K) \leq \text{ld}(e/K)$, and by (1).

Let F_0 be the inversive difference field generated by $K(e)$. Then $F_0 \subseteq F$, and $c \in F_0$. These imply that $\mu(\sigma^{-\ell}(a)/F_0(a, \sigma^\ell(a))) = N = \mu(\sigma^\ell(a)/F_0(a))$. Reasoning as in the proof of Theorem 1.8 one gets $e \in K(\sigma^{-\ell}(e), \sigma^\ell(e))$. Now use (1) to conclude.

We now investigate the behaviour of dd in towers of extensions. Unfortunately, it is not multiplicative, as we will see in 1.12.

1.11. Proposition

Let $K \subset \mathcal{U}$ be a difference field, a and b two tuples in \mathcal{U} such that $\sigma(a) \in K(a)^{\text{alg}}$, $\sigma(b) \in K(b)^{\text{alg}}$.

(1) $\text{dd}(a, b/K) \geq \text{dd}(a/K(b)_\sigma)\text{dd}(b/K)$.

(2) If $b \in K(a)^{\text{alg}}$, then $\text{dd}(b/K) \leq \text{dd}(a/K)$.

Proof. — (1) By Lemma 1.4(2) we may assume that K is inversive. Let d be a finite tuple of $K(b)^{\text{alg}}$ such that $K(a, b, d)$ is a regular extension of

$K(b, d)$. If $C = [K(b, d) : K(b)]$, then for any $\ell > 0$,

$$\mu(\sigma^\ell(b)/K(b)) \leq C\mu(\sigma^\ell(b)/K(a, b)).$$

Thus

$$\begin{aligned} \mu(\sigma^\ell(a), \sigma^\ell(b)/K(a, b)) &= \mu(\sigma^\ell(a)/K(a, b, \sigma^\ell(b)))\mu(\sigma^\ell(b)/K(a, b)) \\ &\geq \mu(\sigma^\ell(a)/K(b)_\sigma(a))C^{-1}\mu(\sigma^\ell(b)/K(b)). \end{aligned}$$

This gives the result.

(2) Follows immediately from Lemma 1.4.

1.12. An example

Unfortunately, Proposition 1.11(1) is the best we can hope for, the invariant dd is not multiplicative in towers. Here is an example.

Let a be a generic solution of $\sigma(a^2) = a^2 + 1$ over an algebraically closed inversive difference field K of characteristic 0, and b a solution of $\sigma(b) = b + a$. Then $\text{dd}(a/K) = \text{dd}(a^2/K) = \text{ld}(a^2/K) = 1$, $\text{ld}(a/K) = 2$, and $\text{ld}(b/K(a)_\sigma) = 1 = \text{dd}(b/K(a)_\sigma)$, so that $\text{ld}(a, b/K) = 2$. If $\ell > 0$, then $\sigma^\ell(b) - b = a + \sqrt{a^2 + 1} + \cdots + \sqrt{a^2 + \ell - 1}$, $K(a^2, \sigma^\ell(b) - b) = K(a, \sqrt{a^2 + 1}, \dots, \sqrt{a^2 + \ell - 1})$ is an extension of degree 2^ℓ of $K(a^2)$.

Thus, if $\ell > 1$, then $\sigma(a), \sigma(b) \in K(a, b, \sigma^\ell(a), \sigma^\ell(b))$, so that $\text{dd}(a, b/K) = \text{ld}(a, b/K) = 2$, but $\text{dd}(a/K)\text{dd}(b/K(a)_\sigma) = 1$.

1.13. Remark

Note that the example shows that the failure of multiplicativity in towers is fundamental: taking $L = K(a)^{\text{alg}}$ and $M = K(a, b)^{\text{alg}}$, we obtain a tower $K \subset L \subset M$ of algebraically closed inversive difference fields with

$$\text{dd}(M/K) = 2 \neq \text{dd}(L/K)\text{dd}(M/L).$$

1.14. The case of difference subgroups of algebraic groups

In case our tuple a is the generic of some difference subgroup, we will show that the tuple c can be chosen to be the generic of a difference subgroup, with the map $a \mapsto c$ a morphism. We first need a lemma:

LEMMA. — *Let K be a field, G_1, G_2, U (connected) algebraic groups defined over K with $U \subset G_1 \times G_2$, and $\pi_i : G_1 \times G_2 \rightarrow G_i$ the natural*

projections. Assume that $\pi_i(U) = G_i$ for $i = 1, 2$. If $S_1 = \pi_1(U \cap (G_1 \times 1))$, then S_1 is a normal subgroup of G_1 . Moreover, if the restriction of π_2 to U is finite, if $g = (a, b)$ is a generic of U over K , then the field conjugates of a over $K(b)$ are the elements of $a + S_1$, and S_1 is central.

Proof. — $S_1 \times 1 = \text{Ker}(\pi_2)$ is normal in U , and because $\pi_1(U) = G_1$, S_1 is normal in G_1 . The finiteness of $\pi_2|_U$ implies that S_1 is finite, and therefore central since G_1 is connected. If $a' \in a + S_1$, then (a', b) is also a generic of U over K^{alg} , and therefore the fields $K(a, b)$ and $K(a', b)$ are $K(b)$ -isomorphic.

1.15. Proposition

Assume that K is a difference field, let H be a difference subgroup of some algebraic group G , both defined over K , and assume that if a is a generic of H (for the σ -topology), then $K(a)_\sigma$ is a regular extension of K of finite transcendence degree over K . Then there are a difference subgroup H' , a morphism $f : H \rightarrow H'$ with finite central kernel, defined by a tuple of difference rational functions, and such that $\text{ld}(f(a)/K) = \text{dd}(a/K)$.

Proof. — Let a be a generic of H over K . Choose n such that $\sigma^{n+1}(a) \in K(a, \dots, \sigma^n(a))^{alg}$ and $\mu(\sigma^{n+1}(a)/K(a, \dots, \sigma^n(a))) = \text{ld}(a/K)$. Let $b = (a, \sigma(a), \dots, \sigma^n(a))$; then $\text{ld}(a/K) = \text{ld}(b/K) = \mu(\sigma(b)/K(b))$. Furthermore, b is a generic of the difference subgroup H_n (of $G \times \dots \times G^{\sigma^n}$) defined by

$$H_n = \{(g_0, \dots, g_n) \mid g_0 \in H, \bigwedge_{i=1}^n g_i = \sigma(g_{i-1})\}.$$

As $b = g(a)$ for some isomorphism $g : H \rightarrow H_n$ given by tuples of difference polynomials, it suffices to prove the result for b and H_n .

Hence, replacing a by b , H by H_n , we may assume that $\sigma(a) \in K(a)^{alg}$ and $\text{ld}(a/K) = \mu(\sigma(a)/K(a))$. Without loss of generality, H is Zariski dense in G , so that G is connected.

Let ℓ_0 be defined as in Lemma 1.6, take $\ell \geq \ell_0$, and consider the algebraic groups U_ℓ, V_ℓ , where U_ℓ is the algebraic locus of $(\sigma^{-\ell}(a), \sigma^\ell(a))$ over K^{inv} , and V_ℓ the algebraic locus of $(a, \sigma^{-\ell}(a), \sigma^\ell(a))$ over K^{inv} . Then V_ℓ is an algebraic subgroup of $G \times U_\ell$, and its images under the projections $\pi_1 : G \times U_\ell \rightarrow G$ and $\pi_2 : G \times U_\ell \rightarrow U_\ell$ equal G and U_ℓ respectively.

We now apply Lemma 1.14, and use its notation and the notation of Theorem 1.8. Note that S_1 is finite, so that in particular S_1 is central in G (since G is connected). Let f be the isogeny $G \rightarrow G/S_1$, and

$d = f(a)$. Then d encodes the set $a + S_1$ of field conjugates of a over $K(\sigma^{-\ell}(a), \sigma^\ell(a))$. Recall that this field contains the tuple c of Theorem 1.8, and that $\mu(a/K^{inv}(c)) = \mu(a/K^{inv}(\sigma^{-\ell}(a), \sigma^\ell(a)))$. By Proposition 1.10(2), for some r we obtain $\text{ld}(f(a)^{p^r}/K^{inv}) = \text{dd}(a/K^{inv})$.

As constructed, our element $f(a)^{p^r}$ is in $K(a)_\sigma^{inv}$, not necessarily in $K(a)_\sigma$. But for some m , $\sigma^m f(a)^{p^r} \in K(a)_\sigma$. We let H' be the σ -closure of $\sigma^m \circ \text{Frob}^r \circ f(H)$ in $(G/S_1)^{\sigma^m \circ \text{Frob}^r}$. Then $\sigma^m(d^{p^r})$ is a generic of H' , and $\sigma^m \circ \text{Frob}^r \circ f$ defines a group homomorphism $H \rightarrow H'$. (Here Frob denotes the Frobenius automorphism).

If K is perfect, then $\text{ld}(d/K) = \text{dd}(a/K)$, so we may take H' to be the σ -closure of $\sigma^m \circ f(H)$.

2. Comparison and/or equivalence of the various settings

In this section we first recall Willis' definitions and results on totally disconnected locally compact groups (see [5], [6]) and explain how they give our results for difference fields of characteristic 0. We then compare the two sets of results, in the group case and in the field case; and exhibit some interesting translations. We end the section with the proof that any totally disconnected locally compact group is the inverse limit of automorphism groups of strongly minimal structures.

2.1. The scale of a totally disconnected locally compact group

Let G be a totally disconnected locally compact group, with a continuous automorphism α . Let U be an open compact subgroup of G , and define

$$U_+ = \bigcap_{n \in \mathbb{N}} \alpha^n(U), \quad U_- = \bigcap_{n \in \mathbb{N}} \alpha^{-n}(U).$$

Say that U is *tidy* for α if it satisfies

T1 $U = U_+U_- = U_-U_+$, and

T2 $\bigcup_{n \in \mathbb{N}} \alpha^n(U_+)$ and $\bigcup_{n \in \mathbb{N}} \alpha^{-n}(U_-)$ are closed in G .

One then defines the *scale function* of α on G by

$$s_G(\alpha) = [\alpha(U) : \alpha(U) \cap U],$$

where U is a tidy subgroup. That tidy subgroups exist and that the scale function is well-defined is shown in [5], Theorems 1 and 2.

Let us now go to difference fields and see how the duality works. For simplicity of notation we will assume that the characteristic is 0; in positive characteristic, analogous results are obtained if one replaces everywhere the degree of a field extension by its separable degree. Let $K = \sigma(K)$ be a difference subfield of \mathcal{U} , a a tuple in \mathcal{U} such that $\sigma(a) \in K(a)^{alg}$, and $L = K(a)^{alg}$. Set

$$G = \text{Aut}(L/K), \quad V = \text{Aut}(L/K(a)).$$

Then G is locally compact, and V is a compact open subgroup which is profinite. The action of σ on L induces a continuous action α on G :

$$\tau \mapsto \sigma\tau\sigma^{-1},$$

which maps $V = \text{Aut}(L/K(a))$ onto $\text{Aut}(L/K(\sigma(a)))$. Then

$$V_+ = \text{Aut}(L/K(a)_\sigma),$$

and

$$V_- = \text{Aut}(L/K(a)_{\sigma^{-1}})$$

(where $K(a)_{\sigma^{-1}} = K(\sigma^{-n}(a), n \in \mathbb{N})$).

Condition T1 then corresponds to $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$. Condition T2 is not so clear, until one inspects Lemma 3(a) of [5]: $\bigcup_{n \in \mathbb{N}} \alpha^n(U_+)$ is closed if and only if $\bigcup_{n \in \mathbb{N}} \alpha^n(U_+) \cap U = U_+$. This implies that $\alpha^\ell(U_+) \cap U \subseteq U_+$ for $\ell > 0$ and, assuming T1, a moment's thought shows that it gives $\alpha(U) \supseteq U \cap \alpha^\ell(U)$. Thus, if V is tidy, this tells us that $\sigma(a) \in K(a, \sigma^\ell(a))$.

Thus, in characteristic 0, the existence of tidy subgroups of G together with this lemma give us (almost) Theorem 1.8. Indeed, Theorem 1 of [5] gives a tidy subgroup U which is compact open, and therefore commensurable with V . I.e., if $K(b)$ is the subfield of L fixed by V then $K(a, b)$ is a finite extension of $K(a)$ and of $K(b)$. However, inspection of the construction of this subgroup U (see e.g. Lemma 3.3 in [6]) shows that it contains (a finite intersection of transforms of) V . I.e., $b \in K(a)_\sigma$.

The fact that an element which satisfies $\text{ld} = \text{dd}$ must also satisfy the conclusion of Theorem 1.8 is fairly clear, so the existence of tidy subgroups led us to look closely at the proof of Theorem 1 of [5] and to discover the above mentioned implication of Lemma 3(a). It suggested that the result might be true in all characteristic, but for that we needed to find a proof slightly more precise. We got more help from Willis' definition of the group \mathcal{L} (see [5] page 347), which suggested that the field F of 1.8 might be large. However, the rest of our proof is somewhat different from Willis'.

2.2. Comparison of the results in the group and in the field context

Below we will give a dictionary of how the various results relate to each other. We first list the group-theoretic result (g), then immediately below its field analogue (f). Many results are very similar, some are unexpected.

(1)(g) The scale function does not depend on the chosen tidy subgroup (Theorem 2 and/or Lemma 10 of [5]).

(f) Lemma 1.4 tells us that $\text{dd}(a/K)$ is an invariant of the difference field extension $K(a)_\sigma^{\text{alg}}/K$. See also 1.6(6): if c, d satisfy $\text{ld} = \text{dd}$, then so does (c, d) .

(2)(g) The modular function $\Delta(\alpha)$ of α equals $s(\alpha)s(\alpha^{-1})^{-1}$ (Corollary 1 of [5])

(f) If a and b are equi-algebraic over K , then $\frac{\text{ld}(a/K)}{\text{id}(a/K)} = \frac{\text{ld}(b/K)}{\text{id}(b/K)}$ (Lemma 1.4(3)).

(3)(g) $s(\alpha^n) = s(\alpha)^n$ for $n > 0$ (Corollary 3 of [5]).

(f) $\text{dd}_{\sigma^n}(a/K) = \text{dd}(a/K)^n$ (Lemma 1.6(6)).

(4)(g) If U is tidy for α , and β is conjugation by some element $\tau \in U$, then U is tidy for $\alpha\beta$, and $s(\alpha\beta) = s(\alpha)$ (Theorem 3 of [5], p. 356).

(f) This one is totally unexpected on the field side. Translated, it becomes:

If $\text{ld}(a/K) = \text{dd}(a/K)$ and $\tau \in \text{Aut}(L/K(a))$, then $\text{ld}_{\sigma\tau}(a/K) = \text{dd}_{\sigma\tau}(a/K) = \text{dd}(a/K)$. This is a direct consequence of the following striking result, inspired by the proof given in [5]:

PROPOSITION. — *If a satisfies $\mu(\sigma(a)/K(a)) = \text{ld}(a/K)$, and $\tau \in \text{Aut}(K(a)^{\text{alg}}/K(a))$, then the difference fields $(K(a)_\sigma, \sigma)$ and $(K(a)_{\sigma\tau}, \sigma\tau)$ are isomorphic (by a K -isomorphism taking a to a).*

Proof. — Observe first that if $\rho_1, \rho_2 \in \text{Aut}(K(a)^{\text{alg}}/K(a))$, then the linear disjointness of $K(a)_{\sigma^{-1}}$ and $K(a)_\sigma$ over $K(a)$ implies the linear disjointness of $\rho_1(K(a)_{\sigma^{-1}})$ and $\rho_2(K(a)_\sigma)$ over $K(a)$. In particular, there is $\rho \in \text{Aut}(K(a)^{\text{alg}}/K(a))$ which agrees with ρ_1 on $K(a)_{\sigma^{-1}}$ and with ρ_2 on $K(a)_\sigma$.

One shows by induction on n , that $K(a, \sigma(a), \dots, \sigma^n(a)) \simeq K(a, \sigma\tau(a), \dots, (\sigma\tau)^n(a))$ by a K -isomorphism (of fields) f_n which sends $\sigma^i(a)$ to $(\sigma\tau)^i(a)$ for $0 \leq i \leq n$. For $n = 1$, $\tau\sigma^{-1}$ sends $(a, \sigma(a))$ to $(a, \tau(a))$.

Assume given f_n , and observe that the field $K((\sigma\tau)^{-1}(a), a)$ is precisely the image by τ^{-1} of the field $K(\sigma^{-1}(a), a)$; we let f_{-1} denote the restriction

of τ^{-1} to $K(\sigma^{-1}(a), a)$. By the remark above, and because f_{-1} and f_n are the identity on $K(a)$, there is an element $\rho \in \text{Aut}(K(a)^{\text{alg}}/K(a))$ which extends $f_{-1} \cup f_n$. Let f_{n+1} be the restriction of $(\sigma\tau)\rho\sigma^{-1}$ to $K(a, \dots, \sigma^{n+1}(a))$.

(5)(g) $s(\alpha) = \min\{[\alpha(U) : U \cap \alpha(U)] \mid U \text{ compact open}\}$; $[\alpha(U) : \alpha(U) \cap U] = s(\alpha) \iff [\alpha^{-1}(U) : \alpha^{-1}(U) \cap U] = s(\alpha^{-1})$ (Theorem 3.1 and Corollary 3.11 of [6]).

(f) The equivalence of items (i) and (ii) of Proposition 1.10(1), and their equivalence with the statement for σ^{-1} .

(6)(g) Let H be a closed subgroup of G such that $\alpha(H) = H$. Then there is a tidy subgroup U of G , such that $U \cap H$ is tidy for $\alpha|_H$; furthermore $s(\alpha|_H) \leq s(\alpha)$ (Corollary 4.2 and Proposition 4.3 of [6]).

(f) Let M be a difference subfield of L containing K . If $\text{ld}(a/K) = \text{dd}(a/K)$, then $\text{ld}(a/M) = \text{dd}(a/M)$: this is clear using 1.10(1); $\text{dd}(a/M) \leq \text{dd}(a/K)$ is obvious. However, Example 6.4 of [6] tells us that this is not the exact analogue of the group statement.

(7)(g) Let H be a closed normal subgroup of G satisfying $\alpha(H) = H$, and $\dot{\alpha}$ the automorphism of G/H induced by α . Then $s(\alpha|_H)s(\dot{\alpha})$ divides $s(\alpha)$ (Proposition 4.7 of [6]).

(f) $\text{dd}(a, b/K) \geq \text{dd}(a/K(b)_\sigma)\text{dd}(b/K)$ (Proposition 1.11(1)). Thus we get a weaker result, but also under weaker assumptions. On the other hand $\text{Aut}(L/K^{\text{alg}})$ has no proper closed normal subgroup.

2.3. Additional remark and results

We conclude with a remark on some ingredients of our proof. We constantly use equation 1.3(#), it is easy to derive the analogue in the group context. The other ingredient we are using is the tuple c which encodes the tuple of minimal polynomials of a over a given field, see 1.7; its existence and properties guarantee that certain infinite intersections are large. The analogue in the group context exists, and can be stated as follows:

Let U be a compact open subgroup, V a compact subgroup of G , such that $[V : V \cap U] = N < \infty$. There is a compact open subgroup W of G which contains V , satisfies $[W : W \cap U] = N$, and contains all subgroups with these properties.

This result is not difficult to prove, here is a sketch. Let \mathcal{W} be the family of compact subgroups of G which contain V and satisfy $[W : W \cap U] = N$. Note that this last condition is equivalent to $W \cdot U = V \cdot U$ (where $W \cdot U$ denotes $\{wu \mid w \in W, u \in U\}$). The family \mathcal{W} is non-empty ($V \in \mathcal{W}$);

observe that if $W_1, W_2 \in \mathcal{W}$, so does $W_1 \cap W_2$, and therefore also $\langle W_1 W_2 \rangle$: this follows easily from $W_1 \cdot W_2 \cdot U = W_1 \cdot (W_1 \cap W_2) \cdot U = W_1 \cdot U$. Also, the closure of an element of \mathcal{W} is in \mathcal{W} , and this implies that \mathcal{W} has a unique maximal element, say W_0 . As $\bigcap_{v \in V} v^{-1} U v$ is an open subgroup which is normalized by V , it is contained in W_0 , and therefore W_0 is open compact.

When translated, our proof gives a slightly different proof of the result in the group situation. Note the alternate definition of the scale function as

$$s(\alpha) = \lim_{k \rightarrow +\infty} [\alpha^k(U) : U \cap \alpha^k(U)]^{1/k},$$

where U is any compact open subgroup of G , and which comes from the analogue of Lemma 1.4(1). (This fact was already observed by R. G. Möller, [3].) One can also easily obtain the result corresponding to 1.11(7):

If U satisfies T1, and W is a compact open subgroup which contains $\alpha^{-\ell}(U) \cap \alpha^\ell(U)$ for all $\ell \gg 0$, then W is tidy.

These results does not seem to appear in either [5] or [6].

2.4. Totally disconnected locally compact groups and strongly minimal sets

If T is a disintegrated strongly minimal theory², and M is a model of T , then for any non-algebraic singleton $a \in M$, the group $\text{Aut}(\text{acl}(a)/\text{acl}(\emptyset))$ has the natural structure of a totally disconnected locally compact group (basic open sets are translates of stabilisers of finite sets; note that $\text{Aut}(\text{acl}(a)/a)$ is profinite and therefore compact). Conversely, we will now explain why any totally disconnected locally compact group G is a projective limit of ones that arise in this way.

Let O be an open compact subgroup of G , and let N_O be the intersection of all conjugates of O . If O' is an open subgroup of O , then we have a natural onto map $G/N_{O'} \rightarrow G/N_O$, and the intersection of all subgroups $N_{O'}$, O' open compact, is 1, so that

$$G = \varprojlim G/N_{O'}.$$

We will show that each $G/N_{O'}$ is the automorphism group of a strongly minimal disintegrated set. Without loss of generality, $N_O = 1$, i.e., O contains no proper normal subgroup of G .

⁽²⁾ Recall that a theory T is *strongly minimal* iff in any model M of T , every definable subset of M is finite or cofinite. It is *disintegrated* iff for any $A \subset M$, one has $\text{acl}(A) = \bigcup_{a \in A} \text{acl}(a)$.

Let $X = G/O$, with n -ary relations $R_a = Ga$ for any $a = (a_1, \dots, a_n) \in X^n$ and $n \in \mathbb{N}$, i.e. R_a is the G -orbit of a . So G acts on $M = (X, R_a)_a$ automorphically, transitively, and faithfully because O contains no proper normal subgroup. Let \bar{O} be the image of O in X . As G acts transitively on X , to show that the homomorphism $G \rightarrow \text{Aut}(M)$ is surjective, it suffices to show that $O \rightarrow \text{Aut}(M/\bar{O})$ is surjective.

To show that $O \rightarrow \text{Aut}(M/\bar{O})$ is surjective, since O is compact it suffices to see that the image is dense.

Indeed if $h \in \text{Aut}(M/\bar{O})$ and $h(a) = b$ for two k -tuples a, b of X , then (b, \bar{O}) must be in the orbit of (a, \bar{O}) since they have the same (quantifier-free) type; so $ga = b$ for some $g \in G$ with $g\bar{O} = \bar{O}$, i.e. $g \in O$.

Now M is strongly minimal and disintegrated since the automorphism group is transitive, and for any basic relation $R = R_a$, for some m , $R(\bar{O}, x_1, \dots, x_m)$ holds for only finitely many elements x_1, \dots, x_m ; see [2] and the references therein.

Each element g of G defines an automorphism α of M (via the natural action of G on X) and the corresponding action on G (viewed as $\text{Aut}(M)$) is conjugation by g . Thus the analogues of Theorems 1.8 and 1.9 for strongly minimal sets would give us Willis' Theorems 1 and 2 for inner automorphisms of G (since quotienting by N_O is irrelevant). On the other hand, if G is totally disconnected locally compact, so is $H = G \rtimes \langle \sigma \rangle$ for any automorphism σ of G , so that only considering inner automorphisms is not a restriction.

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