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Captures, matings and regluings

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ABSTRACT. — In parameter slices of quadratic rational functions, we identify arcs represented by matings of quadratic polynomials. These arcs are on the boundaries of hyperbolic components.

RÉSUMÉ. — Dans des tranches de l'espace des paramètres de fractions rationnelles de degré 2, nous identifions des arcs représentés par des accouplements de polynômes quadratiques. Ces arcs sont contenus dans le bord des composantes hyperboliques.

1. Introduction

The operation of *mating* has been introduced by Douady and Hubbard. Mating can be applied to a pair of polynomials of the same degree, and gives a continuous self-map (the *mating map*) of some topological space (the *mating space*). In many cases, the mating space is homeomorphic to the 2-sphere, and the mating map is a branched covering topologically conjugate

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to a rational function. In this paper, we only consider matings of degree 2 polynomials.

Recall that any quadratic polynomial in one complex variable is affinely conjugate to $p_c(z) = z^2 + c$ for some c . The *filled Julia set* K_c of p_c is defined as the set of all points $z \in \mathbb{C}$ such that the p_c -orbit of z is bounded, and the *Julia set* J_c as the boundary of K_c . The *Fatou set* of p_c is defined as the complement of J_c in the Riemann sphere; its connected components are called *Fatou components*. The *Mandelbrot set* M is the locus of all c such that K_c is connected, equivalently, $c \in K_c$. Consider two quadratic polynomials p_c and $p_{c'}$ such that K_c and $K_{c'}$ are locally connected, equivalently, J_c and $J_{c'}$ are locally connected. In this case, it is well known that there are *Caratheodory loops* $\gamma_c : \mathbb{R}/\mathbb{Z} \rightarrow J_c$ and $\gamma_{c'} : \mathbb{R}/\mathbb{Z} \rightarrow J_{c'}$ that semi-conjugate the map $\theta \mapsto 2\theta$ on \mathbb{R}/\mathbb{Z} with the maps p_c and $p_{c'}$ on J_c and $J_{c'}$, respectively. Define the mating space $X = X_{c,c'}$ as the quotient of $K_c \sqcup K_{c'}$ by the minimal equivalence relation $\sim = \sim_{c,c'}$ such that $z \in K_c$ is equivalent to $z' \in K_{c'}$ if $z = \gamma_c(\theta)$ and $z' = \gamma_{c'}(-\theta)$ for some $\theta \in \mathbb{R}/\mathbb{Z}$. Since the self-map of $K_c \sqcup K_{c'}$ acting as p_c on K_c and as $p_{c'}$ on $K_{c'}$ takes \sim -classes to \sim -classes, it descends to a self-map $p_c \sqcup p_{c'}$ of the mating space X . This map is called the *topological mating* of p_c and $p_{c'}$. If a rational function $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is topologically conjugate to the topological mating $p_c \sqcup p_{c'}$, then this rational function is called a *conformal mating* of p_c and $p_{c'}$.

In this paper, we identify certain arcs in parameter slices of quadratic rational functions that consist of conformal matings. More precisely, we consider the slices $Per_k(0)$ consisting of conformal conjugacy classes of degree 2 rational functions f with marked critical points c_1, c_2 such that $f^{\circ k}(c_1) = c_1$. These slices were first defined and studied by M. Rees [13] and J. Milnor [8]. In his thesis, B. Wittner [19] described an operation that provides topological models for many hyperbolic components of $Per_k(0)$. These are called *capture components*. In this paper, we prove that the boundaries of capture components of $Per_k(0)$ contain arcs of matings. The main theorems are Theorem 4.5, 4.7 and 4.8.

Organization of the paper. A significant part of this paper is expository. In Section 2, we recall the terminology of quadratic invariant laminations [17]. We also use this terminology to give several equivalent definitions of matings and to describe topological models for captures [13]. In Section 3, we recall the topological surgery called *regluing* [18]. Regluing will be used to redescribe topological models for captures. Finally, in Section 4, we consider parameter slices $Per_k(0)$ of quadratic rational functions. Topological models for rational functions representing boundary points of some hyperbolic components in $Per_k(0)$ were described in [18] in terms of regluing. Compar-

ing this description with the description of captures as regluings of matings, we obtain arcs of matings on the boundaries of capture components.

2. Laminations, matings and captures

In this section, we discuss topological models for quadratic polynomials and matings of quadratic polynomials. We also define captures.

2.1. Invariant laminations

Topological models for quadratic polynomials can be described in terms of Thurston's invariant quadratic laminations in the disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. We first consider a slightly more general notion.

Let Ω be an open simply connected domain in $\mathbb{C}\mathbb{P}^1$, whose complement consists of more than one point (hence of infinitely many points). Then Ω is conformally isomorphic to \mathbb{D} . We will also assume that the boundary of Ω is locally connected. Then the closure in $\mathbb{C}\mathbb{P}^1$ of any geodesic in Ω with respect to the Poincaré metric on Ω consists of the geodesic itself and at most two limit points of it that belong to $\partial\Omega$. The closure of a geodesic in Ω is called a *geodesic chord* of Ω . A *geodesic lamination* in $\Omega \subset \mathbb{C}\mathbb{P}^1$ is a set of geodesic chords in Ω , whose union is closed. Elements of a geodesic lamination \mathcal{L} are called *leaves of \mathcal{L}* . Let Z and W be prime ends of Ω . We will write $(ZW)_\Omega$ for the geodesic chord of Ω connecting the prime ends Z and W . We allow for $Z = W$, in which case $(ZW)_\Omega$ is defined as the single point, which is the prime end impression of $Z = W$, and called a *degenerate leaf* of \mathcal{L} . For convenience, we will assume that every geodesic lamination contains all degenerate leaves $(ZZ)_\Omega$. Note that there may be different prime ends $Z \neq W$ with the same prime end impression; they still define a non-degenerate leaf $(ZW)_\Omega$, which is a closed curve.

Let $f : \partial\Omega \rightarrow \partial\Omega$ be any continuous map that extends to the set Ω as a proper orientation-preserving branched covering of degree two. Then f acts on prime ends of Ω . We say that a lamination \mathcal{L} in Ω is *forward invariant* with respect to f if, for every leaf $\ell = (ZW)_\Omega \in \mathcal{L}$, the curve $f[\ell] = (f(Z)f(W))_\Omega$ is also a leaf of \mathcal{L} , possibly degenerate. We use the square brackets in the notation $f[\ell]$ to emphasize that this curve is, in general, different from the image of the curve ℓ under the map f .

We now define the notion of an f -invariant lamination in Ω . This is a forward f -invariant lamination \mathcal{L} that satisfies the following properties: for every leaf $\ell \in \mathcal{L}$, there is another leaf $\tilde{\ell}$ of \mathcal{L} such that $f[\ell] = f[\tilde{\ell}]$, and there are two leaves ℓ_1 and ℓ_2 such that $f[\ell_1] = f[\ell_2] = \ell$.

People usually consider laminations in the unit disk \mathbb{D} that are invariant under the map $\sigma_2 : z \mapsto z^2$. These laminations are called (Thurston) *quadratic invariant laminations*. The boundary of the unit disk is the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The unit circle \mathbb{S}^1 is identified with \mathbb{R}/\mathbb{Z} by means of the map

$$\theta \in \mathbb{R}/\mathbb{Z} \mapsto \bar{\theta} = e^{2\pi i\theta} \in \mathbb{S}^1.$$

If $z = \bar{a}$ and $w = \bar{b}$, then we write \overline{ab} or zw for the leaf connecting z with w . In the case $\Omega = \mathbb{D}$, we will identify prime ends with points on the unit circle that are the corresponding prime end impressions.

Let \mathcal{L} be a geodesic lamination in the unit disk, and Ω an arbitrary simply connected domain in $\mathbb{C}\mathbb{P}^1$, whose complement contains more than one point and whose boundary is locally connected. We can transform the lamination \mathcal{L} into a geodesic lamination in Ω as follows. Let $\phi : \mathbb{D} \rightarrow \Omega$ be a Riemann map. Suppose that we fixed the Riemann map, i.e. we specified the point $\phi(0)$ and the argument of the derivative $\phi'(0)$. By Caratheodory's theory, the map ϕ acts on prime ends, i.e. $\phi(z)$ is a well-defined prime end in Ω for every $z \in \mathbb{S}^1$. With every leaf $\ell = zw \in \mathcal{L}$, we associate the curve $\ell_\Omega = (\phi(z)\phi(w))_\Omega$. The set of all such curves is a geodesic lamination $\mathcal{L}(\Omega, \phi)$ in Ω . We call this lamination *the (ϕ) -image of the lamination \mathcal{L} in Ω* . Sometimes, we write simply $\mathcal{L}(\Omega)$ if the choice of the Riemann map is clear. Clearly, any geodesic lamination in Ω is the image of some geodesic lamination in the unit disk. Moreover, if $f : \bar{\Omega} \rightarrow \bar{\Omega}$ is a continuous map such that f is holomorphic on Ω and has degree two, and the continuous extension $\bar{\phi} : \bar{\mathbb{D}} \rightarrow \bar{\Omega}$ of a Riemann map $\phi : \mathbb{D} \rightarrow \Omega$ semi-conjugates the map σ_2 on \mathbb{S}^1 with the map $f|_{\partial\Omega}$, then any f -invariant geodesic lamination in Ω is the ϕ -image of some invariant quadratic lamination.

We now introduce some notions for laminations in the disk. By the construction just described, they automatically carry over to laminations in any simply connected domain $\Omega \subset \mathbb{C}\mathbb{P}^1$, whose complement consists of more than one point, and whose boundary is locally connected. Let \mathcal{L} be a geodesic lamination in \mathbb{D} . *Gaps* of \mathcal{L} are defined as closures (in $\mathbb{C}\mathbb{P}^1$) of connected components of $\mathbb{D} - \bigcup \mathcal{L}$. The *basis* of a gap G is defined as $G' = \mathbb{S}^1 \cap G$. Clearly, a gap is uniquely determined by its basis. Gaps may be *finite* or *infinite* according to whether their bases are finite or infinite. The lamination \mathcal{L} is called *clean* if, whenever two leaves of \mathcal{L} share an endpoint, they are on the boundary of a finite gap. If \mathcal{L} is clean, then it defines an equivalence relation \sim on $\mathbb{C}\mathbb{P}^1$: two different points z and w are equivalent if they belong to the same leaf or a finite gap of \mathcal{L} . One can prove using Moore's theorem [6] that the quotient space $\mathbb{C}\mathbb{P}^1 / \sim$ is homeomorphic to the sphere. Let $J_{\mathcal{L}}$ be the image of \mathbb{S}^1 under the quotient projection $\mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1 / \sim$. If \mathcal{L} is a quadratic invariant lamination, then the map σ_2 descends to a self-map $F_{\mathcal{L}}$ of $J_{\mathcal{L}}$. Actually, the map $F_{\mathcal{L}}$ can be extended to the entire sphere

\mathbb{CP}^1 / \sim as a branched covering (we keep the same notation $F_{\mathcal{L}}$ for the extended map). This branched covering is called a *topological polynomial*.

Let $p_c(z) = z^2 + c$ be a polynomial, whose Julia set J_c is locally connected. Then p_c is topologically conjugate to a topological polynomial $F_{\mathcal{L}}$ corresponding to an invariant lamination \mathcal{L} . In this sense, we will say that the lamination \mathcal{L} *models* the polynomial p_c .

Example 2.1. — Consider the quadratic polynomial $z^2 - 1$. Its critical point 0 is periodic of period 2: $0 \mapsto -1 \mapsto 0$. The lamination \mathcal{L} that models $z^2 - 1$ can be constructed as follows. Consider the chord $\ell_0 = \frac{1}{3}\frac{2}{3}$. Note that $\sigma_2[\ell_0] = \ell_0$. Set $\ell_1 = -\ell_0$, i.e. the leaf ℓ_1 is obtained from ℓ_0 by the half-turn around the origin. The chords ℓ_0 and ℓ_1 divide the unit circle into four arcs. Divide each of these four arcs into four equal parts and connect the 1/4 with the 3/4 of this subdivision by an arc. In this way, we obtain 6 chords that divide the unit circle into 12 arcs. Repeat the same operation with each of these 12 arcs, and so on. All chords that can be obtained by this countable procedure form a σ_2 -invariant lamination \mathcal{L} that models $z^2 - 1$.

2.2. Critical leaf laminations

Consider a point $\bar{\vartheta}$ in the unit circle. It defines a diameter ℓ_0 of the unit circle, namely, the geodesic chord connecting the two σ_2 -preimages $\frac{\bar{\vartheta}}{2}$ and $\frac{\bar{\vartheta}+1}{2}$ of $\bar{\vartheta}$. We can now define an invariant quadratic lamination $\mathcal{L}_{\bar{\vartheta}}$ as follows. A *pullback* of ℓ_0 is defined as any geodesic chord zw such that, for some integer $m > 0$, we have $\sigma_2^{2m}(z)\sigma_2^{2m}(w) = \ell_0$, and for all $i < m$, the geodesic chord $\sigma_2^{2i}(z)\sigma_2^{2i}(w)$ does not cross the leaf ℓ_0 in \mathbb{D} (although it may have an endpoint in common with ℓ_0). Consider the set of all pullbacks of ℓ_0 , and also all geodesic chords obtained as limits of pullbacks with respect to the Hausdorff metric. We obtain a quadratic invariant lamination $\mathcal{L}_{\bar{\vartheta}}$, not necessarily clean. The lamination $\mathcal{L}_{\bar{\vartheta}}$ is called the *critical leaf lamination*.

There are two cases, in which the lamination $\mathcal{L}_{\bar{\vartheta}}$ is unclean.

Case 1: one endpoint of ℓ_0 is σ_2 -periodic. In this case, there is an infinite concatenation of leaves $\ell_0, \ell_1, \ell_2, \dots$ of $\mathcal{L}_{\bar{\vartheta}}$ such that ℓ_{i+1} shares an endpoint with ℓ_i for every $i = 0, 1, 2, \dots$. There is also a periodic leaf such that one of its endpoints is an endpoint of ℓ_0 , and the other endpoint is the limit of the leaves ℓ_i as $i \rightarrow \infty$. The infinite concatenation of leaves ℓ_i is called a *caterpillar*. To make the unclean lamination $\mathcal{L}_{\bar{\vartheta}}$ into a clean lamination, we can apply the *cleaning procedure*, i.e. remove the caterpillar and all its pullbacks. What remains is a clean lamination $\mathcal{L}_{\bar{\vartheta}}^c$ called the *cleaning* of $\mathcal{L}_{\bar{\vartheta}}$.

Case 2: no endpoint of ℓ_0 is periodic; however, there are two finite gaps of \mathcal{L}_ϑ such that ℓ_0 is their common edge. In this case, the cleaning of the lamination \mathcal{L}_ϑ is the removal of the critical leaf ℓ_0 and all its pullbacks. Thus all leaves that survive this cleaning procedure are limits of pullbacks of ℓ_0 ; they form a clean lamination \mathcal{L}_ϑ^c . The lamination \mathcal{L}_ϑ^c has a finite critical gap, i.e. a finite gap containing a pair of opposite (different by a half-turn) points on the boundary. It is proved in [17] that this gap is necessarily preperiodic.

Cleanings \mathcal{L}_ϑ^c of laminations \mathcal{L}_ϑ , whose critical leaves have no periodic endpoints, have the following meaning. Consider a polynomial p_c such that c is in the Julia set J_c , and the Julia set is locally connected. Then p_c is modeled by the lamination \mathcal{L}_ϑ^c , where the angle ϑ is chosen so that $\gamma_c(\vartheta) = c$ (there may be several angles satisfying this equality, they give rise to the same cleaning \mathcal{L}_ϑ^c). The cleanings \mathcal{L}_ϑ^c of critical leaf laminations \mathcal{L}_ϑ , whose critical leaves have a periodic endpoint, model polynomials, whose parameters are in the interior of the Mandelbrot set.

2.3. Matings

An equivalent definition of a topological mating is the following. Let p_{c_1} and p_{c_2} be quadratic polynomials with connected and locally connected Julia sets. Then p_{c_1} is modeled by some quadratic invariant lamination \mathcal{L}_1 , and p_{c_2} is modeled by some quadratic invariant lamination \mathcal{L}_2 . We can draw the leaves of \mathcal{L}_1 in the unit disk, and the leaves of \mathcal{L}_2 in the complement of \mathbb{D} in \mathbb{CP}^1 (which is also a disk), i.e. we consider laminations \mathcal{L}_1 and $\mathcal{L}_2^{-1} = \mathcal{L}_2(\mathbb{CP}^1 - \overline{\mathbb{D}})$. The lamination $\mathcal{L}_2(\mathbb{CP}^1 - \overline{\mathbb{D}})$ is formed using the map $z \mapsto 1/z$ as the Riemann map for $\mathbb{CP}^1 - \overline{\mathbb{D}}$. Thus the leaves of \mathcal{L}_2^{-1} are images of the leaves of \mathcal{L}_2 under the map $z \mapsto 1/z$. We write ℓ^{-1} instead of $\ell_{\mathbb{CP}^1 - \overline{\mathbb{D}}}$ for the image of the leaf $\ell \in \mathcal{L}_2$. Let \sim_1 be the equivalence relation associated with \mathcal{L}_1 , and \sim_2 the equivalence relation associated with \mathcal{L}_2^{-1} (i.e. two different points z and w are equivalent with respect to \sim_2 if they lie in the same leaf or finite gap of \mathcal{L}_2^{-1}). Finally, let \sim be the minimal equivalence relation on \mathbb{CP}^1 containing both \sim_1 and \sim_2 . The mating space X_{c_1, c_2} of p_{c_1} and p_{c_2} identifies with the space \mathbb{CP}^1 / \sim . Clearly, the image of \mathbb{S}^1 in X_{c_1, c_2} is a quotient of $J_{\mathcal{L}_1}$, and, simultaneously, a quotient of $J_{\mathcal{L}_2}$. The map σ_2 descends to a continuous self-map of the quotient space $\mathbb{S}^1 / \sim \subset X_{c_1, c_2}$. This map coincides with the restriction of the mating map to \mathbb{S}^1 / \sim . The set \mathbb{S}^1 / \sim , which can also be obtained by pasting the Julia sets J_{c_1} and J_{c_2} together as described in the introduction is called the *Julia set of the mating*. If the mating is topologically conjugate to a rational function, then the conjugacy takes the Julia set of the mating to the Julia set of this rational function. Note that this construction explains the minus sign in the

definition of matings: the image of a point $\bar{\theta}$ on the unit circle under the map $z \mapsto 1/z$ is the point $-\bar{\theta}$.

The two definitions of matings can be combined into the following non-symmetric construction. Consider the filled Julia set K_{c_1} of the first polynomial p_{c_1} , and the quadratic invariant lamination \mathcal{L}_2 that models the second polynomial p_{c_2} . By the Böttcher theorem, there exists a unique Riemann map $\phi_{c_1} : \mathbb{D} \rightarrow \mathbb{C}\mathbb{P}^1 - K_{c_1}$ that conjugates the map $z \mapsto z^2$ with the map p_{c_1} restricted to the basin of infinity. Clearly, ϕ_{c_1} must map 0 to ∞ . We can now take the image of the lamination \mathcal{L}_2 in the basin of infinity $\Omega_{c_1} = \mathbb{C}\mathbb{P}^1 - K_{c_1}$. Taking the quotient of the sphere by the equivalence relation generated by the lamination $\mathcal{L}_2(\Omega_{c_1})$, we also obtain a topological model for the mating $p_{c_1} \sqcup p_{c_2}$. We will use this non-symmetric construction below, when discussing topological models for captures.

The non-symmetric construction of matings can be generalized to the case, where \mathcal{L}_2 does not necessarily model a quadratic polynomial. Thus we can talk about a mating of a polynomial and an invariant lamination. Similarly, we can talk about a mating of two invariant laminations.

2.4. Internal and external angles

Let $p_c(z) = z^2 + c$ be a polynomial such that the critical point 0 of p_c is periodic of minimal period k . Let A be the Fatou component of p_c containing 0. The map p_c^{ok} takes A to itself. By the Böttcher theorem, there exists a conformal isomorphism $\psi : \mathbb{D} \rightarrow A$ that conjugates the map $z \mapsto z^2$ with the restriction of p_c^{ok} to A . It is well-known that the Julia set of p_c , as well as the boundary of any Fatou component of p_c , are locally connected. Therefore, the map ψ admits a continuous extension $\bar{\psi} : \bar{\mathbb{D}} \rightarrow \bar{A}$. A point of ∂A of *internal angle* \varkappa is defined as $\bar{\psi}(\bar{\varkappa})$. Since ∂A is homeomorphic to the circle, the map $\bar{\psi}$ must be a homeomorphism (as follows e.g. from Caratheodory's theory), and then internal angles are in one-to-one correspondence with points of ∂A .

If B is any other bounded Fatou component of p_c , then there is a minimal integer m such that $p_c^{om}(B) = A$. The map p_c^{om} is then a homeomorphism between the closure of B and the closure of A . Using this homeomorphism, we define internal angles for the boundary points of B . Thus any point on the boundary of any bounded Fatou component of p_c has a well-defined internal angle with respect to this bounded Fatou component.

On the other hand, any point z of J_c has the form $\gamma_c(\theta)$ for some angle $\theta \in \mathbb{R}/\mathbb{Z}$. This angle is called an *external angle* of z . The same point can have several different external angles. If the point z is on the boundary of

some bounded Fatou component of p_c , then there are two ways of identifying the point z : 1) we can just specify the external angle of z , and 2) we can specify the Fatou component, whose boundary contains the point z and the internal angle of z with respect to this component.

2.5. Captures

Capture is an operation making polynomials into (models of) rational functions. It was first introduced in the thesis of B. Wittner [19] in 1988. Similar to matings, captures can be defined in terms of topological models. However, an easier definition (due to M. Rees [13]) uses combinatorial equivalence classes. Combinatorial equivalence is a certain equivalence relation on the set of orientation preserving topological branched self-coverings of the sphere that are *critically finite*, i.e. every critical point is eventually mapped to a periodic cycle (such coverings are called *Thurston maps*). The post-critical set P_f of a Thurston map f is defined as the minimal forward invariant set containing all critical values. Two Thurston maps f and g are called *combinatorially equivalent* (or *Thurston equivalent*) if f is homotopic, relative to the set P_f , to a Thurston map $h \circ g \circ h^{-1}$ topologically conjugate to g (i.e. h is an orientation preserving self-homeomorphism of the sphere, and the homotopy connecting f to $h \circ g \circ h^{-1}$ consists of Thurston maps with the same post-critical set P_f).

Thurston's rigidity theorem claims that, with few exceptions that can be explicitly described, any combinatorial equivalence class of Thurston maps contains at most one rational function. In particular, this is true for *hyperbolic Thurston maps*, i.e. Thurston maps such that every critical point is eventually mapped to a cycle containing a critical point. Thus a combinatorial equivalence class of hyperbolic Thurston maps either contains exactly one rational function, or contains no rational functions at all. *Thurston's characterization theorem* [2] provides a topological criterion distinguishing these two cases.

Consider a critically finite polynomial $p_c(z) = z^2 + c$ such that 0 is a periodic point of p_c of some minimal period k . Let v be some strictly preperiodic point of p_c that is eventually mapped to 0. Then v lies in some interior component V of K_c . Let $O(v)$ denote the forward orbit of v . By our assumption, $0 \in O(v)$ but $O(v)$ is different from the orbit of 0. Let us choose a point b on the boundary of V . The operation of capture is almost determined by the choice of the two points v and b . Let $\beta : [0, 1] \rightarrow \mathbb{CP}^1$ be a simple path with the following properties: $\beta(0) = \infty$, $\beta(1/2) = b$, $\beta(1) = v$, and the intersection $\beta[0, 1] \cap J_c$ consists of only one point b . A path with these properties is called a *capture path*. What is really enough

to know to define a capture is the pair of points v, b plus the homotopy class of the path $\beta : [0, 1/2] \rightarrow \Omega_c \cup \{b\}$ with fixed endpoints. Note that this homotopy class is determined by the choice of an external angle of b . This angle will also be called the *external angle of the capture path* β . Define a *path homeomorphism* σ_β as a self-homeomorphism of the sphere that is equal to the identity outside a small neighborhood of $\beta[0, 1]$ (i.e. outside a “narrow tube” around $\beta[0, 1]$) and such that $\sigma_\beta(\infty) = v$.

Note that $\sigma_\beta \circ p_c$ is a topological branched covering, whose homotopy class relative to $O(v)$ is well defined (provided that the neighborhood of $\beta[0, 1]$, in which σ_β is different from the identity, is small enough so that it does not intersect $O(v)$)¹. Note also that this covering is critically finite, with post-critical set $O(v)$ (the critical point ∞ is mapped to v and then is eventually mapped to 0). Thus the combinatorial class of $\sigma_\beta \circ p_c$ is well defined. The map $\sigma_\beta \circ p_c$ (or rather its combinatorial class) is called a *formal capture* of p_c . An explicit description of all paths β , for which the formal capture is Thurston equivalent to a rational function is known by a result of Mary Rees and Tan Lei. We will state this result later. A rational function that is combinatorially equivalent to the formal capture is called a *conformal capture*.

2.6. Capture paths define matings

We now start describing topological models for captures. These models, due to M. Rees [13], reveal a close connection between captures and matings. Consider a quadratic polynomial p_c such that $p_c^{\circ k}(0) = 0$, a capture path β and points $v = \beta(1)$ and $b = \beta(1/2)$. Let ϑ be the external angle of the capture path β .

We now form the critical leaf lamination \mathcal{L}_ϑ . In our case, the angle ϑ cannot be periodic:

LEMMA 2.2. — *If $\vartheta \in \mathbb{R}/\mathbb{Z}$ is a periodic angle (with respect to the angle doubling map), then the point $\gamma_c(\vartheta)$ cannot belong to the boundary of a strictly preperiodic Fatou component of p_c .*

Proof. — The polynomial p_c is modeled by some quadratic invariant lamination \mathcal{L} in \mathbb{D} . If the point $\gamma_c(\vartheta)$ belongs to the boundary of some strictly preperiodic component of p_c , then the point $\bar{\vartheta}$ belongs to the boundary of some strictly preperiodic gap G of \mathcal{L} . Let m be the period length of $\bar{\vartheta}$. A certain iterate s of the map $\sigma_2^{\circ m}$ takes G to a periodic gap $s(G)$ (this

⁽¹⁾ Two branched coverings f_0 and f_1 are homotopic relative to a set O if there is a homotopy $\phi_t, t \in [0, 1]$ consisting of homeomorphisms such that $\phi_t = id$ on O for all t , and $\phi_0 \circ f_0 = f_1 \circ \phi_1$

means that the basis G' of the gap G is mapped under s to the basis of some periodic gap, which is denoted by $s(G)$). On the other hand, we have $s(\bar{\vartheta}) = \bar{\vartheta}$. Since $\bar{\vartheta}$ belongs to the boundaries of two different gaps G and $s(G)$, there must be a leaf of \mathcal{L} having $\bar{\vartheta}$ as an endpoint. Let g be this leaf (if it is not a part of a finite gap) or a finite gap containing this leaf. *Edges* of the gap (or leaf) g are defined as leaves on the boundary of g . *Vertices* of g are defined as points of $g \cap \mathbb{S}^1$. Then $\bar{\vartheta}$ is one of the vertices of g . Consider *co-oriented edges* of g , i.e. edges of g equipped with a choice of an outer side of g , a side on which there are no vertices of g apart from the endpoints of the given edge. If g is not a leaf, then co-oriented edges are the same as edges: for every edge, there is only one outer side. If g is a leaf, then g has two different co-oriented edges, one for each side of g .

Every gap of \mathcal{L} adjacent to g defines a co-oriented edge of g . Let ℓ_G be the co-oriented edge of g defined by the gap G . Then ℓ_G is mapped to $\ell_{s(G)}$ but $\ell_{s(G)}$ never maps back to ℓ_G under the iterates of s . It follows that g is eventually mapped under σ_2 to a finite critical gap of \mathcal{L} . However, a finite critical gap is always strictly preperiodic, as follows from [17]. A contradiction. \square

Since ϑ is not periodic, the endpoints of the critical geodesic chord $\ell_0 = \frac{\vartheta}{2} \frac{\vartheta+1}{2}$ are not periodic either. Therefore, \mathcal{L}_ϑ is either a clean lamination with the critical leaf ℓ_0 , or an unclean lamination, whose cleaning has a finite gap containing ℓ_0 . Consider the lamination $\mathcal{L}_\vartheta(\Omega_c)$. The lamination $\mathcal{L}_\vartheta(\Omega_c)$ defines an equivalence relation $\approx_{c,\beta}$ on \mathbb{CP}^1 . This is the minimal equivalence relation such that every leaf and every finite gap of $\mathcal{L}_\vartheta(\Omega_c)$ belongs to some equivalence class. The quotient space of \mathbb{CP}^1 by the equivalence relation $\approx_{c,\beta}$ (together with a natural map defined on this space) is the mating $p_{c_1} \sqcup \mathcal{L}_\vartheta$ of the polynomial p_{c_1} and the lamination \mathcal{L}_ϑ .

Recall from [17] that \mathcal{L}_ϑ has a unique finite invariant gap or non-degenerate leaf. We will call this gap or leaf the *central gap* of \mathcal{L}_ϑ . We can now state the result of M. Rees and Tan Lei [16]:

THEOREM 2.3. — *The Thurston map $\sigma_\beta \circ p_c$ is combinatorially equivalent to a rational function if and only if the image of the central gap of \mathcal{L}_ϑ in $\mathcal{L}_\vartheta(\Omega_c)$ does not separate the sphere.*

We will need the following lemma:

LEMMA 2.4. — *Suppose that the image of the central gap of \mathcal{L}_ϑ in $\mathcal{L}_\vartheta(\Omega_c)$ does not separate the sphere. Then no periodic leaf of $\mathcal{L}_\vartheta(\Omega_c)$ is a closed curve.*

The statement of the lemma follows from topological models for captures described in [13], see Subsection 3.1. However, we give a more direct proof here.

Proof. — We first recall a general statement about orbit portraits. An *orbit portrait* can be defined as a σ_2 -periodic cycle of geodesic chords of \mathbb{D} that have no intersection points in \mathbb{D} . Every orbit portrait \mathcal{O} defines a *wake* in the parameter plane of complex polynomials. The wake corresponding to an orbit portrait \mathcal{O} consists of all parameter values c such that, for every $\overline{ab} \in \mathcal{O}$, the external rays of angles a and b in the dynamical plane of the polynomial p_c land at the same point. It is proved in [9] that every wake is bounded by two external parameter rays that land at the same point (this point is called the *root point* of the wake). We say that two orbit portraits *co-exist* if they either coincide or have no intersection points in \mathbb{D} . Note that every finite gap or leaf, whose vertices are permuted by σ_2 preserving their cyclic order, defines an orbit portrait. Orbit portraits obtained in this way are called *principal orbit portraits*. Every orbit portrait co-exists with exactly one principal orbit portrait. This classical statement can be easily proved either by methods of [9] or using the minor leaf theory of [17]. In terms of the parameter plane of complex polynomials, this statement means that every wake lies in a *principal wake*, whose root point belongs to the main cardioid of the Mandelbrot set.

Suppose now that some periodic leaf ℓ_{Ω_c} of $\mathcal{L}_\vartheta(\Omega_c)$ is a closed curve. The corresponding periodic leaf ℓ of \mathcal{L}_ϑ defines an orbit portrait \mathcal{O} . Since ℓ_{Ω_c} is a closed curve, the geodesic chord ℓ^* that is obtained from ℓ by complex conjugation must belong to the lamination \mathcal{L} that models the polynomial p_c . Therefore, the lamination \mathcal{L} contains the conjugate orbit portrait \mathcal{O}^* . Let G be the central gap of \mathcal{L}_ϑ . It defines a principal orbit portrait, for which we will use the same letter G . Obviously, G co-exists with \mathcal{O} . It follows that the complex conjugate principal orbit portrait G^* co-exists with \mathcal{O}^* . Since there is only one principal orbit portrait co-existing with \mathcal{O}^* , the central gap of \mathcal{L} must coincide with G^* . It follows that the image of the central gap G of \mathcal{L}_ϑ in Ω_c disconnects the sphere. \square

We will now assume that the Thurston map $\sigma_\beta \circ p_c$ is combinatorially equivalent to a rational function. In the next section, we describe a topological model for this conformal capture due to M. Rees. We will need the following property of the mating $p_c \sqcup \mathcal{L}_\vartheta$:

PROPOSITION 2.5. — *Let π be the canonical projection from the filled Julia set K_c to the mating space of $p_c \sqcup \mathcal{L}_\vartheta$. If A is the Fatou component of p_c containing the critical point 0, then the restriction of π to the closed curve ∂A is a homeomorphism.*

It follows that the Fatou component $\pi(A)$ of the mating $p_c \sqcup \mathcal{L}_\vartheta$ has a Jordan curve boundary. This statement can also be deduced from a theorem of K. Pilgrim [11].

Proof. — Suppose that the restriction of π to ∂A is not one-to-one.

Step 1: the critical gap. Let G be the critical gap of the lamination \mathcal{L} that models p_c , i.e. the gap of \mathcal{L} containing the origin. Then G is periodic of period k and symmetric with respect to the origin. Note that $\gamma_c(G') = \partial A$. There is an edge M of G such that $\sigma_2^{\circ k}[M] = M$. Moreover, the *hole* of G behind M , i.e. the component of $\mathbb{S}^1 - G$ bounded by the endpoints of M and disjoint from G , is the longest among all holes of G (the opposite leaf $-M$ is also on the boundary of G ; its hole has the same length; all other holes have strictly smaller length). These facts are among the basic properties of *minor leaf laminations*; they are discussed in [17]. The leaf M is called the *major leaf* of G (and of \mathcal{L}), or simply the *major*.

Step 2: a new invariant lamination. Set G^* to be the set obtained from G by complex conjugation (if we want to place both G and \mathcal{L}_ϑ to the same disk, then we must take complex conjugation of something — either of G or of \mathcal{L}_ϑ). Let M^* denote the geodesic chord obtained from the major M of G by complex conjugation. Our assumption that the restriction of π to ∂A is not one-to-one translates as follows: there is a leaf or a finite gap of \mathcal{L}_ϑ such that two different points of ∂G^* are among its vertices. There is a natural monotone map $\xi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ that collapses the closures of all holes of G^* and that semi-conjugates the map $\sigma_2^{\circ k} : \partial G^* \rightarrow \partial G^*$ (i.e. the map $\sigma_2^{\circ k}$ restricted to the basis of G^* and extended over all edges of G^* in a monotone continuous way) with the map $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. For every leaf $\ell = \overline{ab}$, we set $\xi(\ell) = \overline{\xi(a)\xi(b)}$, thus the ξ -images of leaves are well defined. Consider the set of ξ -images of all leaves of \mathcal{L}_ϑ . Denote this set by \mathcal{L}_G . It is not hard to verify that the collection of leaves thus obtained is a quadratic invariant lamination. It follows from our assumption that the lamination \mathcal{L}_G is non-trivial, i.e. it contains non-degenerate leaves.

Step 3: the critical leaf. No leaf of \mathcal{L}_G can intersect the ξ -image of the critical geodesic chord $\ell_0 = \frac{\vartheta+1}{2}$ in \mathbb{D} . Note that the geodesic chord $\xi(\ell_0)$ is also a critical chord (i.e. a diameter of the unit circle), whose endpoints are eventually mapped to the fixed point $\overline{0}$, which is the ξ -image of M^* . It follows that \mathcal{L}_G is the critical leaf lamination generated by the critical leaf $\xi(\ell_0)$. Indeed, the chord $\xi(\ell_0)$ must eventually map to a geodesic chord containing $\overline{0}$. However, if an invariant lamination contains any leaf having $\overline{0}$ as an endpoint, then it must contain the leaf $\overline{0\frac{1}{2}}$.² We now need to consider

(2) Among all leaves $\overline{0a}$ choose the one, for which \overline{a} is the closest to $\overline{0}$. Then the leaf

two cases: either the leaf $\xi(\ell_0)$ coincides with $\overline{0\frac{1}{2}}$ or no endpoint of this leaf is periodic under the angle doubling map.

Case 1: the leaf $\xi(\ell_0)$ coincides with $\overline{0\frac{1}{2}}$. In this case, the critical leaf lamination generated by $\overline{0\frac{1}{2}}$ is not clean, and all leaves of this lamination are pullbacks of the critical leaf. It follows that every leaf of \mathcal{L}_G connects two points that both are eventually mapped to $\overline{0}$, but not simultaneously. In particular, there is a leaf ℓ of \mathcal{L}_ϑ , whose ξ -image coincides with $\overline{0\frac{1}{2}}$. An endpoint of ℓ that projects to $\overline{0}$ under ξ must lie in the closure of the complementary arc to G^* bounded by M^* . At the same time, this endpoint belongs to G^* by our assumption. Therefore, it coincides with an endpoint of M^* . Hence the leaf ℓ shares endpoints with M^* and with $-M^*$ (the centrally symmetric to M^* leaf with respect to the origin). Thus there is a quadrilateral such that two edges of it are M^* and $-M^*$, and the other two edges ℓ and $-\ell$ are leaves of \mathcal{L}_ϑ . Both ℓ and $-\ell$ map to $\sigma_2(\ell)$, and both M^* and $-M^*$ map to $\sigma_2(M^*)$. Hence $\sigma_2(\ell) = \sigma_2(M^*)$. The image of ℓ in $\mathcal{L}_\vartheta(\Omega_c)$ is a closed curve, since both endpoints of M map to the same point under γ_c . Moreover, this closed curve is a periodic leaf of $\mathcal{L}_\vartheta(\Omega_c)$, since the endpoints of M are periodic. This contradicts Lemma 2.4.

Case 2: no endpoint of $\xi(\ell_0)$ is periodic under the angle doubling map. In this case, the lamination \mathcal{L}_G is clean or becomes clean after removal of the critical leaf and all its pullbacks. It has no infinite gaps, i.e. the entire disk $\overline{\mathbb{D}}$ is the union of leaves and finite gaps of \mathcal{L}_G . It follows that the quotient of \mathbb{S}^1 by the equivalence relation generated by \mathcal{L}_G is a *dendrite* (i.e. is locally connected and homeomorphic to the complement of an open dense topological disk in $\mathbb{C}\mathbb{P}^1$). It follows that $\pi(\partial A)$ must also be a dendrite. This is a contradiction, because the complement to $\pi(A)$ contains some other Fatou components of the mating $p_c \sqcup \mathcal{L}_\vartheta$. \square

3. Regluing and topological captures

In this section, we recall the basic properties of regluing, a topological surgery on rational functions introduced in [18]. We will also relate regluing to topological models for captures.

3.1. Topological models for captures

We first describe topological models for captures given in [13]. Let p_c be a quadratic polynomial such that $p_c^{\circ k}(0) = 0$, and β a capture path for p_c of external angle ϑ . A topological model for the conformal capture of p_c

$\overline{0(2a)}$ will intersect the leaf $\overline{\frac{1}{2}(a + \frac{1}{2})}$ unless $a = 1/2$.

corresponding to the capture path β is perhaps easier to describe in terms of regluing of the mating $p_c \sqcup \mathcal{L}_\vartheta$. However, we need to know in advance that the corresponding mating space is homeomorphic to the sphere. We will prove this statement later using the topological models from [13].

Let \mathcal{L} be the quadratic invariant lamination that models p_c , and G the infinite gap of \mathcal{L} that corresponds to the Fatou component of p_c containing the point $\beta(1)$. The map $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ extends to a Thurston map s_c such that $s_c(\ell) = \sigma_2[\ell]$ for every leaf $\ell \in \mathcal{L}$. Moreover, we can set $s_c(z) = z^2$ outside the unit disk, and arrange that 0 be a critical point of s_c such that $s_c^{\circ k}(0) = 0$. The map s_c is combinatorially equivalent to p_c . Indeed, the process of collapsing leaves and finite gaps of \mathcal{L} can be performed continuously, so that there is a homotopy between s_c and p_c consisting of Thurston maps. Moreover, the size of the postcritical set does not change during this homotopy.

We define the point $w \in G$ as the *center* of G , i.e. the unique point in G that is eventually mapped to 0 under s_c . Let γ be a simple path that intersects the unit circle exactly once at the point $\bar{\vartheta} = \gamma(1/2)$ and such that $\gamma(0) = \infty$, $\gamma(1) = w$. The Thurston maps $\sigma_\gamma \circ s_c$ and $\sigma_\beta \circ p_c$ are also combinatorially equivalent. We can assume that the path homeomorphism σ_γ maps some narrow tube T around the curve $\gamma[0, 1/2]$ inside the gap G . Consider $T_1 = p_c^{-1}(T)$. This is a strip (a “tunnel”) connecting two gaps of \mathcal{L} outside the unit disk. Note that the image of T_1 under the map $\sigma_\gamma \circ s_c$ is in the unit disk, hence is disjoint from T_1 . Taking pullbacks of T_1 under the iterates of $\sigma_\gamma \circ s_c$, we obtain several disjoint tunnels in $\mathbb{CP}^1 - \mathbb{D}$. It is easy to see that the tunnels are arranged in the same way as the pullbacks of the critical leaves in the lamination $\mathcal{L}_\vartheta^{-1}$, i.e. the tunnels can be realized as slightly fattened leaves (we take only finitely many leaves at a time and use that the tunnels are narrow enough).

We can now formalize the picture with the tunnels. The mating $p_c \sqcup \mathcal{L}_\vartheta$ is modeled by the union $\mathcal{L} \cup \mathcal{L}_\vartheta^{-1}$ in the sense that, to obtain the mating space, we collapse all leaves and finite gaps in this union. We now modify the “two-sided lamination” $\mathcal{L} \cup \mathcal{L}_\vartheta^{-1}$ in the following way. The critical leaf ℓ_0^{-1} of $\mathcal{L}_\vartheta^{-1}$ gets “fattened”, i.e. it is transformed into a quadrilateral, whose sides are two geodesic chords of $\mathbb{CP}^1 - \mathbb{D}$ and two circle arcs (this quadrilateral serves to model the tunnel T_1). To this end, we need to blow up the endpoints $-\frac{\vartheta}{2}$ and $-\frac{\vartheta+1}{2}$ of ℓ_0^{-1} to circle arcs. We do the same operation with all the pullbacks of ℓ_0 . As a result, we obtain a geodesic lamination $\mathcal{L}_{(\infty)}^{-1}$ in the complement of the unit disk. The gaps of $\mathcal{L}_{(\infty)}^{-1}$ are *ideal quadrilaterals* (whose two sides are geodesic chords and two other sides are circle arcs) or finite geodesic polygons. Every leaf of $\mathcal{L}_\vartheta^{-1}$ that is not a pullback of ℓ_0 gives

rise to a leaf of $\mathcal{L}_{(\infty)}^{-1}$. It follows that every finite gap of $\mathcal{L}_{\vartheta}^{-1}$ gives rise to a finite gap of $\mathcal{L}_{(\infty)}^{-1}$.

The process of blowing up certain points of the unit circle into arcs can be formalized in the following way. There exists a two-fold orientation-preserving covering $s : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and a monotone continuous projection $\xi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with the following properties:

- the points \bar{a} that have non-trivial fibers $\xi^{-1}(\bar{a})$ are exactly those with $\sigma_2^m(\bar{a}) = \bar{\vartheta}$ for some $m > 0$;
- the projection ξ semi-conjugates s with σ_2 , i.e. $\xi \circ s = \sigma_2 \circ \xi$.

Let K be the Cantor set obtained as the closure of the complement in \mathbb{S}^1 of all non-trivial fibers of ξ . Then K is invariant under s .

With every leaf \overline{ab} of \mathcal{L}_{ϑ} , we associate one or two geodesic chords of $\mathbb{CP}^1 - \overline{\mathbb{D}}$. If the fibers of ξ over a and b are singletons $\{a'\}$ and $\{b'\}$, respectively, then we associate the chord $\overline{a'b'}$ with \overline{ab} . Otherwise, we associate two *disjoint* chords $\overline{a'b'}$ and $\overline{a''b''}$ with \overline{ab} , where a', a'' are the endpoints of $\xi^{-1}(a)$, and b'', b' are the endpoints of $\xi^{-1}(b)$. We can now define a geodesic lamination $\mathcal{L}_{(\infty)}^{-1}$ in $\mathbb{CP}^1 - \overline{\mathbb{D}}$ as the set of all geodesic chords associated with leaves of \mathcal{L}_{ϑ} in the way just described. The lamination $\mathcal{L}_{(\infty)}^{-1}$ is s -invariant in the sense of Subsection 2.1.

We can also modify the lamination \mathcal{L} so that to make it into an s -invariant lamination $\mathcal{L}_{(0)}$. The lamination $\mathcal{L}_{(0)}$ is uniquely defined by the following properties:

- for every leaf \overline{ab} of $\mathcal{L}_{(0)}$, the geodesic chord $\overline{\xi(a)\xi(b)}$ is a leaf of \mathcal{L} ;
- let G^1, G^2 be two gaps of \mathcal{L} , whose bases map to G' under σ_2 ; the geodesic convex hull of $\xi^{-1}(G^i \cap \mathbb{S}^1)$, $i = 1, 2$, is a gap of $\mathcal{L}_{(0)}$.

In other words, as we blow up a point of the unit circle into an arc, this arc is inserted to the boundary of an infinite gap of \mathcal{L} .

Consider the union of $\mathcal{L}_{(0)}$ and $\mathcal{L}_{(\infty)}$. It defines an equivalence relation \approx on \mathbb{CP}^1 : namely, the minimal equivalence relation containing the equivalence relation generated by $\mathcal{L}_{(0)}$ and the equivalence relation generated by $\mathcal{L}_{(\infty)}$. The map s descends to a self-map g of the subset $K/\approx \subset \mathbb{CP}^1/\approx$. There is a continuous extension of g to all components of the complement of K/\approx in \mathbb{CP}^1/\approx such that g is a Thurston map. Components of the complement of K/\approx in \mathbb{CP}^1/\approx will be referred to as *Fatou components* of g . We

can arrange that every Fatou component be mapped to a periodic Fatou component, and that the periodic Fatou components be super-attracting domains.

THEOREM 3.1. — *The map g defined above is topologically conjugate to the conformal capture of p_c corresponding to the capture path β . In particular, the topological space $\mathbb{C}\mathbb{P}^1/\approx$ is homeomorphic to the sphere.*

The proof of this theorem is organized as follows. We can extend the map s to a Thurston map such that $s(\ell) = \sigma_2[\ell]$ for every $\ell \in \mathcal{L}_{(0)}$ and $s(\ell^{-1}) = \sigma_2[\ell^{-1}]$ for every $\ell^{-1} \in \mathcal{L}_{(\infty)}^{-1}$. Then it can be shown that s is combinatorially equivalent to $\sigma_\gamma \circ s_c$, hence also to $\sigma_\beta \circ p_c$. A rough geometric reason for that is the picture with the tunnels discussed above. Next, we use that a Thurston map combinatorially equivalent to a hyperbolic rational function is semi-conjugate to this function. This general theorem proved in [13] is in fact a version of Thurston’s rigidity principle. Thus there is a semi-conjugacy between s and a conformal capture. Finally, the fibers of the semi-conjugacy can be studied, and it can be proved that the fibers are precisely leaves and finite gaps of the two-sided lamination $\mathcal{L}_{(0)} \cup \mathcal{L}_{(\infty)}^{-1}$.

Theorem 3.1 has the following immediate corollary.

COROLLARY 3.2. — *The mating space of $p_c \sqcup \mathcal{L}_\partial$ is homeomorphic to the sphere.*

Proof. — The mating space is obtained from the space $\mathbb{C}\mathbb{P}^1/\approx$ by collapsing the images of all ideal quadrilaterals. Hence the result follows from the theorem of Moore [6] that characterizes topological quotients of the sphere that are homeomorphic to the sphere. \square

3.2. Regluing

Consider a countable set \mathcal{Z} of disjoint simple curves in the sphere S^2 . Recall that \mathcal{Z} is said to be a *null-set* if, for every $\varepsilon > 0$, there exist only finitely many curves from \mathcal{Z} , whose diameter is bigger than ε . To measure diameter, we can use any metric compatible with the topology of the sphere. It is easy to see that the notion of a null-set does not depend on the choice of a metric. In fact, the notion of a null-set can be stated in purely topological terms. Namely, \mathcal{Z} is a null-set if, for every open covering \mathcal{U} of S^2 , there exist only finitely many curves from \mathcal{Z} that are not entirely covered by an element of \mathcal{U} .

Regluing data on \mathcal{Z} are the choice of an equivalence relation on each curve $Z \in \mathcal{Z}$ such that there exists a homeomorphism $h : Z \rightarrow [-1, 1]$

that transforms this equivalence relation into the equivalence relation on $[-1, 1]$, whose classes are of the form $\{\pm x\}$, $x \in [0, 1]$. To define regluing, we need a null-set of disjoint simple curves \mathcal{Z} and a choice of regluing data on them. We first cut along the curves in \mathcal{Z} , and then reglue these curves in a different way. To cut along finitely many curves Z_1, \dots, Z_n means to consider the Caratheodory compactification of the set $U_n = S^2 - \bigcup_{i=1}^n Z_i$, i.e. the union of the set U_n and the set of all prime ends of U_n , equipped with a suitable topology. This is a formalization of an intuitively obvious process: as we cut a surface with boundary along a curve disjoint from the boundary, we obtain a new piece of the boundary, which is a simple closed curve. Thus cutting along finitely many disjoint simple curves in the sphere leads to a compact surface with boundary. In fact the definition works even in the case, where the curves are not disjoint.

We need to be careful when defining a sphere with countably many cuts. Suppose that \mathcal{Z} consists of curves Z_1, \dots, Z_n, \dots . Let Y_n be the result of cutting along the curves Z_1, \dots, Z_n , i.e. the Caratheodory compactification of $U_n = S^2 - \bigcup_{i=1}^n Z_i$. The natural inclusion $\iota_n : U_{n+1} \rightarrow U_n$ gives rise to the continuous map $\iota_{n*} : Y_{n+1} \rightarrow Y_n$ (which is not an inclusion). Let Y be the inverse limit of the topological spaces Y_n and the continuous maps ι_{n*} . The space Y is called *the sphere with cuts* (made along the set \mathcal{Z} of curves). We will sometimes use the notation $S^2 \ominus \mathcal{Z}$ for Y . In fact, we never used in the definition of Y that curves are disjoint and that they form a null-sequence. Thus Y is well defined even without these assumptions. We will need these assumptions to glue the cuts. We will also need the regluing data.

Every curve $Z \in \mathcal{Z}$ gives rise to a simple closed curve Z^\ominus obtained by cutting along Z . Recall that the regluing data contains an equivalence relation on Z such that there is a homeomorphism between Z and $[-1, 1]$ mapping every equivalence class onto $\{\pm x\}$ for some $x \in [-1, 1]$. There are two marked points in Z^\ominus that project to the endpoints of Z . We can now define an equivalence relation on Z^\ominus as follows: two points of Z^\ominus are equivalent if their projections in Z are equivalent, and they are not separated by the marked points of Z^\ominus . Thus the sphere with cuts Y comes equipped with equivalence relations on all the cuts. These equivalence relations extend trivially to an equivalence relation on the entire space Y . The quotient Y^* of Y by this equivalence relation is called the *regluing* of the sphere S^2 along the null-set of disjoint curves \mathcal{Z} , equipped with regluing data. We will sometimes use the notation $S^2 \# \mathcal{Z}$ for this regluing.

It can be shown using Moore's characterization [5] of a topological sphere that the topological space Y^* is homeomorphic to the sphere (see [18]). Thus we reglued a topological sphere and obtained another topological sphere.

This operation becomes useful, however, when we have a geometric structure on the sphere. Then the regluing may produce a different geometric structure.

A continuous map $f : S^2 \rightarrow S^2$ acting on the sphere can be thought of as a geometric structure. Geometrically, we can think that the sphere is equipped with arrows connecting every point $z \in S^2$ with the point $f(z)$. We now assume that f is an orientation preserving topological branched covering, and see what happens with arrows when we reglue. Cutting along a curve creates problems as we cut through the tips of some arrows. These arrows get doubled, and we obtain two different arrows originating at the same point. To rectify this issue, we also need to cut along the pullbacks of the curve, i.e. along components of its full preimage under the map f . Hence, if we cut along a curve, we also need to cut along all pullbacks.

Suppose that $f : S^2 \rightarrow S^2$ is a topological branched covering, and $\alpha_0 : [0, 1] \rightarrow S^2$ a simple path such that $\alpha_0(0)$ is a critical value of multiplicity one, and there are no other critical values in $\alpha_0[0, 1]$. Then there is a simple path $\alpha_1 : [-1, 1] \rightarrow S^2$ such that $f \circ \alpha_1(t) = \alpha_0(t^2)$. The point $\alpha_1(0)$ must be a critical point of f . The path α_1 defines the curve $Z_1 = \alpha_1[-1, 1]$ together with an equivalence relation on Z_1 identifying $\alpha_1(t)$ with $\alpha_1(-t)$ (in other words, two points in Z_1 are equivalent if they are mapped to the same point under f). Let \mathcal{Z} be the set of all pullbacks of Z_1 . We will write $\mathcal{Z} = [\alpha_0]$.

Suppose that every element $Z \in \mathcal{Z}$ is mapped to Z_1 one-to-one under a suitable iterate of f . Suppose also that \mathcal{Z} is a null-set consisting of disjoint curves. Under these assumptions, the space Y^* defined above makes sense, and there is a natural map $F : Y^* \rightarrow Y^*$ called the *regluing of f* . We started with a topological dynamical system on the sphere, performed a regluing, and obtained another topological dynamical system on the sphere. We say that the dynamical system $F : Y^* \rightarrow Y^*$ is obtained from $f : S^2 \rightarrow S^2$ by regluing the pullbacks of the path α_0 . As we frequently think of the map $f : S^2 \rightarrow S^2$ as a geometric structure on the sphere, we write (S^2, f) to indicate that S^2 is equipped with this structure, and we sometimes write $(S^2, f) \# \mathcal{Z}$ or $(S^2, f) \# [\alpha_0]$ for (Y^*, F) , the space Y^* equipped with the map F .

3.3. Captures vs. regluing

Let p_c be a quadratic polynomial such that $p_c^{o^k}(0) = 0$, and β a capture path for p_c of external angle ϑ . We assume that the Thurston map $\sigma_\beta \circ p_c$ is combinatorially equivalent to a rational function H , equivalently, the central gap of $\mathcal{L}_\vartheta(\Omega_c)$ does not disconnect the sphere. We know by Corollary

3.2 that the mating space of $p_c \sqcup \mathcal{L}_\vartheta$ is homeomorphic to the sphere. Recall that the *Julia set* of the mating $p_c \sqcup \mathcal{L}_\vartheta$ is defined as the projection of J_c under the quotient map π collapsing all leaves and finite gaps of $\mathcal{L}_\vartheta(\Omega_c)$. The complement of the Julia set is called the *Fatou set*. Connected components of the Fatou set are called *Fatou components*. We will now describe a topological model for the rational function H in terms of the mating $p_c \sqcup \mathcal{L}_\vartheta$ and regluing.

Let V be the interior component of K_c containing the point $v = \beta(1)$. The restriction of π to V is a homeomorphism. Moreover, $\pi(V)$ is a Fatou component of the mating $p_c \sqcup \mathcal{L}_\vartheta$ containing the critical value $\pi(b)$ on its boundary, where $b = \beta(1/2)$ as above. Let $\alpha_0 : [0, 1] \rightarrow \overline{\pi(V)}$ be any simple path such that $\alpha_0(0) = \pi(b)$ (the critical value of the mating), $\alpha_0(1) = \pi(v)$ (the *center* of the Fatou component $\pi(V)$), and $\alpha_0(0, 1) \subset \pi(V)$. E.g. we can define α_0 as a suitably reparameterized restriction of $\pi \circ \beta$ to $[1/2, 1]$. Consider the corresponding topological dynamical system $F : Y^* \rightarrow Y^*$ obtained by regluing the pullbacks of α_0 (up to topological conjugacy, it does not depend on the choice of α_0 , provided that α_0 satisfies the requirements listed above).

THEOREM 3.3. — *The conformal capture H is topologically conjugate to the regluing F of the mating $p_c \sqcup \mathcal{L}_\vartheta$.*

The proof is straightforward and can be easily performed by a detailed comparison of the two topological models. We only give a sketch here.

Sketch of a proof. — First we replace the capture H with its topological model g described in Subsection 3.1. Note that there is a semi-conjugacy between the dynamics of g on the set K/\approx and the dynamics of the mating $p_c \sqcup \mathcal{L}_\vartheta$ on its Julia set. Let h denote this semi-conjugacy. The fiber $h^{-1}(x)$ of h over any point x in the Julia set of the mating is either a singleton or a pair of points. Namely, $h^{-1}(x)$ is a pair of points precisely if x is mapped to the critical value $\pi(b)$ of the mating under some strictly positive iterate of the mating map. Thus the topological model for $g : K/\approx \rightarrow K/\approx$ is easy to describe in terms of the action of the mating map on its Julia set by doubling certain points. Note that the regluing does exactly the same thing with the dynamics on the Julia set. Thus $g : K/\approx \rightarrow K/\approx$ is topologically conjugate to the restriction of the regluing to the image of the Julia set under the regluing. It remains to extend the topological conjugacy over all components of $\mathbb{C}\mathbb{P}^1/\approx \approx -K/\approx$, which is straightforward. \square

Thus the map F obtained from $p_c \sqcup \mathcal{L}_\vartheta$ by regluing is a topological model for the capture. Note that many different paths give rise to the same capture. Hence we obtain many topological models for the same capture. It

is not at all obvious that these models are topologically conjugate unless we use that they are all conjugate to the capture. We would be interested to know a direct proof of the topological conjugacy between the models.

3.4. Reversed regluing

The operation of regluing is reversible. Consider a null-set \mathcal{Z} of simple disjoint curves in S^2 , equipped with regluing data. Regluing of this set yields a topological space $Y^* = S^2 \# \mathcal{Z}$ homeomorphic to the sphere. Actually, it yields more than that. We also obtain a null-set $\mathcal{Z}^\#$ of simple disjoint curves in the space Y^* equipped with regluing data. Namely, we define $\mathcal{Z}^\#$ as the set of images of all cuts under the natural projection from Y to Y^* . Thus the curves in $\mathcal{Z}^\#$ are in one-to-one correspondence with the curves in \mathcal{Z} . Given any $Z \in \mathcal{Z}$, we first cut it to obtain a simple closed curve Z^\ominus and then glue it back in a different way to obtain the corresponding curve $Z^\# \in \mathcal{Z}^\#$. There are natural projections from Z^\ominus to Z and to $Z^\#$. We can now define an equivalence relation on $\mathcal{Z}^\#$ in terms of these projections. Namely, two points of $\mathcal{Z}^\#$ are equivalent if they are projections of points, whose images in Z coincide. These equivalence relations define regluing data on the set of curves $\mathcal{Z}^\#$. In particular, we can consider the regluing $(S^2 \# \mathcal{Z}) \# \mathcal{Z}^\#$. This topological space is *canonically* homeomorphic to S^2 .

Suppose now that the sphere S^2 is equipped with a topological branched covering $f : S^2 \rightarrow S^2$, and $\mathcal{Z} = [\alpha_0]$, where a simple path α_0 is as in Subsection 3.2. Then, after regluing, we have $\mathcal{Z}^\# = [\alpha_0^\#]$, where the path $\alpha_0^\#$ is obtained as the image of the path α_0 in the space $S^2 \# \mathcal{Z}$ (the multivalued correspondence between points of S^2 and points of $S^2 \# \mathcal{Z}$ is in fact single valued on $\alpha_0[0, 1]$ since this set is disjoint from all the cuts). We have

$$(S^2, f) = ((S^2, f) \# [\alpha_0]) \# [\alpha_0^\#].$$

This means that, to recover (S^2, f) from the regluing $(S^2, f) \# \mathcal{Z}$, we only need to reglue the set of curves $[\alpha_0^\#]$.

4. Captures and matings in parameter slices

We now consider some natural complex one-dimensional parameter spaces of rational functions, and discuss parameter values that correspond to captures and matings. Recall now that any rational function $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ of degree at least two defines the splitting of the Riemann sphere into the Fatou set and the Julia set. However, these sets are not defined in the same way as for polynomials, because infinity is no better than any other point of the sphere, when a non-polynomial rational function acts. By definition,

the Fatou set of f consists of all points, at which the map f is Lyapunov stable, equivalently, the sequence f^{on} is equicontinuous. The Julia set is by definition the complement to the Fatou set.

4.1. The slices $Per_k(0)$

Consider the space Rat_2 of conformal conjugacy classes of all quadratic rational functions with marked critical points. This space is complex two-dimensional: a quadratic rational function is determined by five coefficients, and the group $Aut(\mathbb{C}P^1)$ of conformal automorphisms of the Riemann sphere has complex dimension three. Thus the dimension of the space Rat_2 equals the number of critical points a quadratic rational function has, and this is not a mere coincidence. It is a general observation in holomorphic dynamics that the behavior of critical orbits is to a large extent responsible for the dynamical behavior of the whole map. There are a number of theorems to this effect saying roughly that if all critical orbits behave nicely, then the function itself is nice from the dynamical viewpoint. For example, if all critical orbits are attracted by attracting cycles, then the function is *hyperbolic*, i.e. 1) there exists a neighborhood of the Julia set and a Riemannian metric on this neighborhood such that the restriction of the function to the Julia set is strictly expanding with respect to this metric, 2) there exists a Riemannian metric on the Fatou set, in which the function is strictly contracting.

To simplify the model (and to make nice pictures) people consider complex one-dimensional slices of Rat_2 . To define the slices, one fixes a particular nice behavior of one critical point, so there remains only one “free” critical point. E.g. one can impose that one critical point is periodic of period k . We let $Per_k(0)$ denote the corresponding slice, following J. Milnor [8] (0 in the notation stands for the multiplier of a k -periodic point: having a periodic point of multiplier 0 is the same as having a periodic critical point). More precisely, the space $Per_k(0)$ is defined as the set of all conformal conjugacy classes of rational functions f with marked critical points c_1, c_2 such that $f^{ok}(c_1) = c_1$, and k is minimal with this property. Clearly, each $Per_k(0)$ is an algebraic curve in the algebraic surface Rat_2 . For $k = 1, 2, 3$ and 4 , the genus of this curve is equal to zero, i.e. there is a rational parameterization.

If $k = 1$, then one critical point must be fixed. By a conformal coordinate change, we can map this point to infinity. A rational function, for which the infinity is a fixed critical point, is necessarily a quadratic polynomial. By an affine change of variables, the coefficient with z can be killed, so that every quadratic polynomial reduces to the form $z^2 + c$. Thus $Per_1(0)$ can be identified with the standard quadratic family $\{z^2 + c\}$.

Consider the case $k = 2$. Any conjugacy class from $Per_2(0)$ that does not contain the map $z \mapsto 1/z^2$ has a unique representative of the form $z \mapsto a/(z^2 + 2z)$. Thus $Per_2(0)$ can be identified with the a -plane punctured at 0, to which we need to add a single point at infinity corresponding to the class of the map $1/z^2$. Rational parameterizations for $Per_3(0)$ and $Per_4(0)$ are also easy to obtain.

4.2. Hyperbolic components

The set of elements in $Per_k(0)$ representing hyperbolic functions is open. Connected components of this set are called *hyperbolic components*. J. Milnor [8] gave a classification of hyperbolic components in $Per_k(0)$ into four types: A, B, C and D. Hyperbolic elements in $Per_k(0)$ of type A are classes of rational functions such that both critical points c_2 and c_1 are in the same super-attracting domain (i.e. c_2 lies in the Fatou component containing c_1). It can be proved that there are no type A components for $k > 1$, and the space $Per_1(0)$ has just one type A component that is identified with the complement of the Mandelbrot set. Hyperbolic elements in $Per_k(0)$ of type B are classes of rational functions such that the free critical point c_2 lies in a periodic Fatou component, whose cycle contains c_1 but which itself does not contain c_1 . Every slice $Per_k(0)$ contains a finite number of type B hyperbolic components (this number is nonzero unless $k = 1$). Hyperbolic elements of $Per_k(0)$ of type C are classes of rational functions such that the free critical point c_2 lies in a strictly preperiodic Fatou component that is eventually mapped to the component containing c_1 . For all $k > 1$, the slices $Per_k(0)$ contain infinitely many hyperbolic type C components. Finally, hyperbolic elements of $Per_k(0)$ of type D are classes of rational functions such that c_1 and c_2 lie in disjoint periodic cycles of Fatou components. All slices $Per_k(0)$ contain infinitely many hyperbolic type D components.

It follows from [4] that every hyperbolic component in $Per_k(0)$ of type B, C or D has a unique *center*, i.e. a critically finite conjugacy class. A conformal mating of two hyperbolic critically finite quadratic polynomials represents the center of some type D hyperbolic component in $Per_k(0)$. Similarly, a conformal capture of a hyperbolic critically finite quadratic polynomial represents the center of some type C hyperbolic component. However, the converse is not true in general. It is true for $Per_2(0)$ but, in the slice $Per_3(0)$, there are type D components, whose centers are not matings, and there are type C components, whose centers are not captures. Examples are given in [8, Appendix F by J. Milnor and Tan Lei], [14], respectively. We say that a hyperbolic component of type C is a *capture component* if its center is a conformal capture.

4.3. Regluing and type C boundaries

In [18], topological models were given for classes in $Per_k(0)$ that lie on the boundaries of type C hyperbolic components. These models were defined in terms of regluing. We now cite the result:

THEOREM 4.1. — *Suppose that the class of a rational function f belongs to the boundary of a type C hyperbolic component \mathcal{H} in $Per_k(0)$ but does not belong to the boundary of a type B component. Then f is topologically conjugate to a map obtained from the center of \mathcal{H} by regluing.*

We now make this statement more precise. Suppose that a rational function f represents a point in the parameter slice $Per_k(0)$ lying on the boundary of \mathcal{H} but not lying on the boundary of a type B hyperbolic component. Then it can be shown ([18, Subsection 3.3], which imitates an argument from [1]) that the critical point c_2 of f belongs to the boundary of some Fatou component W that is eventually mapped to a Fatou component containing c_1 . The critical value $f(c_2)$ belongs to the boundary of $f(W)$. There are two components of $f^{-1}(f(W))$, say, W and \tilde{W} . The boundary of each of the two components contains the critical point c_2 . It may also happen that there are more than two Fatou components, whose boundary contains c_2 . Equivalently, there may be more than one Fatou component, whose boundary contains $f(c_2)$. However, the choice of the component $f(W)$ is determined by the choice of a hyperbolic component \mathcal{H} , as is explained in [18, Subsection 3.3].

Let m be the minimal positive integer such that $f^{om}(W) \ni c_1$. The Fatou component W has a unique center, i.e. a point z_W such that $f^{om}(z_W) = c_1$. We now consider a simple path $\alpha_0 : [0, 1] \rightarrow f(W)$ such that $\alpha_0(0) = f(c_2)$, $\alpha_0(1) = f(z_W)$, and $\alpha_0(0, 1) \subset f(W)$. Such a path exists because the boundary of $f(W)$ is locally connected (see [1, 18]). There is a unique simple path $\alpha_1 : [-1, 1] \rightarrow \mathbb{CP}^1$ such that $f \circ \alpha_1(t) = \alpha_0(t^2)$ for all $t \in [-1, 1]$. Then $\alpha_1(0)$ must coincide with the critical point c_2 . We can now reglue the path α_1 and all its pullbacks according to the construction given above. Let H be a hyperbolic critically finite rational function representing the center of \mathcal{H} . Then (\mathbb{CP}^1, H) is topologically conjugate to $(\mathbb{CP}^1, f) \# [\alpha_0]$. Since regluing is reversible, we can also obtain the topological dynamical system (\mathbb{CP}^1, f) as a regluing $(\mathbb{CP}^1, H) \# [\alpha_0^\#]$. Here the simple path $\alpha_0^\# : [0, 1] \rightarrow \mathbb{CP}^1$ connects the non-periodic critical value of H to a boundary point of the Fatou component containing it. This statement is a more precise form of Theorem 4.1 cited above.

4.4. Angles on the boundaries of capture components

Let \mathcal{H} be a capture component, i.e. a type C hyperbolic component in $Per_k(0)$, whose center is a capture. Every point of $\partial\mathcal{H}$ that is not on the boundary of a type B component is determined by its *angle*. Let f be a rational function representing this point of the parameter slice. Then the critical value $f(c_2)$ of f belongs to the boundary of a Fatou component $f(W)$ that is eventually mapped to the Fatou component containing c_1 . Moreover, as was mentioned above, the choice of the Fatou component $f(W)$ is determined by the choice of the type C component \mathcal{H} , whose boundary contains the class of f .

Angles of points on the boundary of $f(W)$ are defined similarly to internal angles in polynomial case. By the Böttcher theorem, there exists a bi-holomorphic map $\psi : \mathbb{D} \rightarrow f^{om}(W)$ that conjugates the map $z \mapsto z^2$ with the restriction of the map f^{ok} to $f^{om}(W)$. Here m is the minimal integer such that $f^{om}(W) \ni c_1$. Since the boundary of $f^{om}(W)$ is locally connected, there is a continuous extension $\overline{\psi} : \mathbb{D} \rightarrow \overline{f^{om}(W)}$. The point of angle $\varkappa \in \mathbb{R}/\mathbb{Z}$ on the boundary of $f^{om}(W)$ is by definition the point $\overline{\psi}(\overline{\varkappa})$. The point of angle $\varkappa \in \mathbb{R}/\mathbb{Z}$ on the boundary of $f(W)$ is by definition the point $z \in \partial f(W)$ such that $f^{om-1}(z)$ is the point of angle \varkappa on the boundary of $f^{om}(W)$. Since the boundary of $f(W)$ maps one-to-one onto the boundary of $f^{om}(W)$ under f^{om-1} , the point of angle \varkappa on the boundary of $f(W)$ is well defined.

Recall that a set $A_\lambda \subset \mathbb{C}P^1$ depending on a parameter λ (taking values in a Riemann surface Λ) *moves holomorphically* with λ if there is a subset $A \subset \mathbb{C}P^1$ and a map $(a, \lambda) \mapsto \iota_\lambda(a)$ (a *holomorphic motion*) from $A \times \Lambda$ to $\mathbb{C}P^1$ that is holomorphic with respect to λ for every fixed $a \in A$, injective with respect to a for every fixed $\lambda \in \Lambda$, and such that $\iota_\lambda(A) = A_\lambda$ for every $\lambda \in \Lambda$. A theorem, sometimes called the λ -*lemma*, of Mañé, Sud and Sullivan [7] claims that if A_λ moves holomorphically with λ , and ι_{λ_0} is a quasi-symmetric embedding for some $\lambda_0 \in \Lambda$, then all ι_λ are quasi-symmetric embeddings; moreover, the closure \overline{A}_λ also moves holomorphically with λ .

Let $\lambda \in Per_k(0)$ be a parameter value, and f_λ a rational function representing λ . Suppose that $f = f_{\lambda_0}$. At least for the values of λ that are close to λ_0 , we can choose representatives so that f_λ depends holomorphically on λ . There is a holomorphic motion that includes the sets $V_\lambda = f_\lambda(W_\lambda)$ such that $V_{\lambda_0} = f(W)$, and V_λ is a Fatou component of f_λ (see the proof of Proposition 4.2 that follows for more detail on this holomorphic motion). By the λ -lemma, the boundaries ∂V_λ also move holomorphically. We can continue this holomorphic motion all the way up to the center λ_1 of the hyperbolic component \mathcal{H} . As follows from the topological model for captures,

the boundary of $f_{\lambda_1}(W_{\lambda_1})$ is homeomorphic to the boundary of a periodic Fatou component in the mating $p_c \sqcup \mathcal{L}_\theta$. On the other hand, by the holomorphic motion argument, the boundary of $f_{\lambda_1}(W_{\lambda_1})$ is homeomorphic to the boundary of $f(W)$. Therefore, by Proposition 2.5, the boundary of $f(W)$ is a Jordan curve. It follows that different angles cannot correspond to the same point on the boundary of $f(W)$, i.e. every point on the boundary of $f(W)$ has a well-defined angle.

We can now define the angle of the point in $\partial\mathcal{H}$ represented by f as the angle of the critical value $f(c_2)$ in $\partial f(W)$ (in the sense just described). If \mathcal{B} denotes the union of all type B components, then the angle is a function on $\partial\mathcal{H} - \overline{\mathcal{B}}$ with values in \mathbb{R}/\mathbb{Z} .

PROPOSITION 4.2. — *The angle is an injective continuous function on $\partial\mathcal{H} - \overline{\mathcal{B}}$.*

We need the following simple lemma (cf. also [1]):

LEMMA 4.3. — *Let $z_\lambda \in \mathbb{C}\mathbb{P}^1$ be a point, and $R_\lambda \subset \mathbb{C}\mathbb{P}^1$ be a set. Suppose that both z_λ and R_λ are moving holomorphically with $\lambda \in \Lambda$. If $z_{\lambda_0} \in \overline{R_{\lambda_0}}$ for some $\lambda_0 \in \Lambda$, then we either have $z_\lambda \in R_\lambda$ for values of λ arbitrarily close to λ_0 , or $z_\lambda \in \overline{R_\lambda} - R_\lambda$ for all λ sufficiently close to λ_0 .*

Proof. — Let $\iota : R \times \Lambda \rightarrow \mathbb{C}\mathbb{P}^1$ be the holomorphic motion of R_λ so that $R_\lambda = \iota_\lambda(R)$ for all $\lambda \in \Lambda$. We assume that $R \subset \mathbb{C}\mathbb{P}^1$ and ι_{λ_0} is a homeomorphic embedding. By the λ -lemma, ι extends to a holomorphic motion $\bar{\iota} : \overline{R} \times \Lambda \rightarrow \mathbb{C}\mathbb{P}^1$. Consider the holomorphic functions z_λ and $w_\lambda = \bar{\iota}_\lambda(r_0)$ of λ , where $r_0 \in \overline{R}$ is the point such that $\bar{\iota}_{\lambda_0}(r_0) = z_{\lambda_0}$. By definition, the holomorphic function $z_\lambda - w_\lambda$ of λ vanishes at $\lambda = \lambda_0$. As λ goes around the circle $|\lambda - \lambda_0| = \varepsilon$, the point $z_\lambda - w_\lambda$ makes at least one loop around 0, unless $z_\lambda = w_\lambda$ identically for λ in some neighborhood of λ_0 . Therefore, for $r \in R$ very close to r_0 , the point $z_\lambda - \iota_\lambda(r)$ also makes at least one loop around 0. We conclude that the function $z_\lambda - \iota_\lambda(r)$ vanishes for some λ in the disk $|\lambda - \lambda_0| < \varepsilon$. \square

Proof of Proposition 4.2. — Let $\lambda \in \text{Per}_k(0)$ be a class of rational functions, and f_λ a rational function representing λ . We can choose representatives f_λ , at least locally, so that they depend holomorphically on λ . We let A_λ denote the immediate basin of the periodic critical point c_1^λ of f_λ , i.e. the Fatou component of f_λ containing c_1^λ . We know that A_λ moves holomorphically with λ . More precisely, the bi-holomorphic isomorphism $\psi_\lambda : \mathbb{D} \rightarrow A_\lambda$ conjugating the map $z \mapsto z^2$ with the restriction of $f_\lambda^{\circ k}$ to A_λ is a holomorphic motion.

Define the *ray of angle* θ in A_λ as the set of all points $R_\lambda(t, \theta) = \psi_\lambda(e^{-t+2\pi i\theta})$, where t runs through $(0, \infty)$ (this ray will sometimes be denoted by $R_\lambda(\theta)$). If f_λ is hyperbolic, then every ray *lands*, i.e. there exists a limit of $R_\lambda(t, \theta)$ as $t \rightarrow 0$. By the λ -lemma, the closure $\overline{R_\lambda(\theta)}$ of every ray moves holomorphically with $\lambda \in \underline{\Lambda} = \text{Per}_k(0) - \overline{\mathcal{B}}$. We denote the corresponding holomorphic motion by $\overline{\psi}_\lambda$. It follows that the ray $R_\lambda(\theta)$ always lands at the point $\overline{\psi}_\lambda(\overline{\theta})$.

Suppose that, for every $\lambda \in \mathcal{H}$, the non-periodic critical point c_2^λ belongs to some Fatou component W_λ such that $f_\lambda^{om}(W_\lambda) = A_\lambda$, and m is the smallest positive integer with this property (clearly, the number m does not depend on the choice of λ in \mathcal{H}). Consider the map $\lambda \mapsto \psi_\lambda^{-1}(f_\lambda^{om}(c_2^\lambda))$. This is a holomorphic map from \mathcal{H} to \mathbb{D} . It is proved in [12] that this map is actually a bi-holomorphic isomorphism between \mathcal{H} and \mathbb{D} . Let $\Psi_{\mathcal{H}}$ denote the inverse of this map.

Define the *parameter ray* $\mathcal{R}_{\mathcal{H}}(\theta)$ in \mathcal{H} as the set of all points of the form $\mathcal{R}_{\mathcal{H}}(t, \theta) = \Psi_{\mathcal{H}}(e^{-t+2\pi i\theta})$, where $t \in (0, \infty)$. We will now prove that every point $\lambda_0 \in \partial\mathcal{H} - \overline{\mathcal{B}}$ is the landing point of exactly one parameter ray. Indeed, the non-periodic critical point $c_2^{\lambda_0}$ of f_{λ_0} has a well-defined angle θ , i.e. $f_{\lambda_0}^{om}(c_2^{\lambda_0}) \in \overline{R_{\lambda_0}(\theta)}$. Both the point $f_\lambda^{om}(c_2^\lambda)$ and the set $\overline{R_\lambda(\theta)}$ move holomorphically with λ . By Lemma 4.3, the point λ_0 is an accumulation point of the parameter ray $\mathcal{R}_{\mathcal{H}}(\theta)$, i.e. a partial limit of $\mathcal{R}_{\mathcal{H}}(t, \theta)$ as $t \rightarrow 0$. A point λ_0 in $\partial\mathcal{H} - \overline{\mathcal{B}}$ of angle θ is a zero of the holomorphic function

$$\lambda \mapsto f_\lambda^{om}(c_2^\lambda) - \overline{\psi}_\lambda(e^{2\pi i\theta}).$$

Since this function is not constant, its zeros must be isolated. Therefore, there are no other points of angle θ in a neighborhood of λ_0 in Λ . On the other hand, by [18, Proposition 5], all accumulation points of the parameter ray $\mathcal{R}_{\mathcal{H}}(\theta)$ must have angle θ . Since the set of accumulation points is connected, the ray $\mathcal{R}_{\mathcal{H}}(\theta)$ must land at λ_0 . Since any parameter ray can only land at one point, we obtain that the angle is an injective function on $\partial\mathcal{H} - \overline{\mathcal{B}}$. The continuity of this function follows from the fact that a zero of a holomorphic function depending continuously on parameters moves continuously with respect to parameters. \square

COROLLARY 4.4. — *Suppose that \mathcal{H} is a capture hyperbolic component, whose closure is disjoint from $\overline{\mathcal{B}}$. Then the angles establish a homeomorphism between $\partial\mathcal{H}$ and \mathbb{R}/\mathbb{Z} .*

4.5. Identification of matings

As above, let \mathcal{H} be a capture hyperbolic component in $Per_k(0)$ with center λ_1 (we will write $H = f_{\lambda_1}$ for the corresponding capture), and $f = f_{\lambda_0}$ a map representing a point λ_0 on the boundary of \mathcal{H} . Suppose that this point is not on the boundary of a type B component, and suppose also that this point has angle \varkappa in $\partial\mathcal{H}$. We know that the dynamical system $(\mathbb{C}\mathbb{P}^1, f)$ is obtained from the dynamical system $(\mathbb{C}\mathbb{P}^1, H)$ by regluing the system $[\alpha_0^\#]$ of curves, where the simple path $\alpha_0^\#$ connects the non-periodic critical value $H(c_2^{\lambda_1})$ of H lying in some Fatou component $H(W_{\lambda_1})$ with the boundary point of this Fatou component of angle \varkappa .

On the other hand, $(\mathbb{C}\mathbb{P}^1, H)$ is obtained by regluing from the topological mating $p_c \sqcup \mathcal{L}_\vartheta$ for various values of ϑ . The capture path determines both an external angle ϑ and the internal angle \varkappa of the point $b = \beta(1/2)$. We will say that β is a *capture path of internal angle \varkappa* . Note however, that, in contrast to external angle, knowing the internal angle and the Fatou component of p_c containing the point $v = \beta(1)$ may not uniquely determine a capture path. Suppose that the angle \varkappa is equal to the angle of the point $\lambda_0 \in \partial\mathcal{H}$. Then the path that was used to reglue the mating into $(\mathbb{C}\mathbb{P}^1, H)$ was a suitably reparameterized restriction $\beta : [1/2, 1] \rightarrow K_c$. The corresponding path in $(\mathbb{C}\mathbb{P}^1, H)$ that appears after the regluing is a simple path connecting the critical value $H(c_2^{\lambda_1})$ with the point on the boundary of $H(W_{\lambda_1})$ with angle \varkappa . Therefore, the reverse regluing of $(\mathbb{C}\mathbb{P}^1, H)$ into the mating $p_c \sqcup \mathcal{L}_\vartheta$ is the same as the regluing of $(\mathbb{C}\mathbb{P}^1, H)$ into $(\mathbb{C}\mathbb{P}^1, f)$! It follows that the results should also be the same, up to topological conjugacy. We have thus proved the following

THEOREM 4.5. — *Let p_c be a quadratic polynomial such that the critical point 0 of p_c is periodic of minimal period k , and $\beta : [0, 1] \rightarrow \mathbb{C}\mathbb{P}^1$ a capture path for p_c of external angle ϑ and internal angle \varkappa . Suppose that $\sigma_\beta \circ p_c$ is combinatorially equivalent to a rational function H . If \mathcal{H} is the hyperbolic component in $Per_k(0)$, whose center is represented by H , and a rational function f represents a boundary point of \mathcal{H} of angle \varkappa not lying on the boundary of a type B hyperbolic component, then f is topologically conjugate to the mating $p_c \sqcup \mathcal{L}_\vartheta$.*

We can make this theorem more precise.

PROPOSITION 4.6. — *The lamination \mathcal{L}_ϑ^c models some quadratic polynomial.*

Proof. — The statement will follow if we prove that, in the parameter plane of complex polynomials, the external ray of angle ϑ lands at a unique

point of the Mandelbrot set, and the quadratic polynomial corresponding to this point has locally connected Julia set. By the theorem of Yoccoz on local connectivity (see e.g. [10]), actually, by a simple version of it, this is true if the critical leaf or gap g of \mathcal{L}_g^c is non-recurrent, i.e. no iterated σ_2 -image of g' intersects a small neighborhood of g' . Recall from Subsection 2.1 that the basis of a gap g is defined as $g' = g \cap \mathbb{S}^1$.

If \mathcal{L}_g^c has a critical gap rather than a critical leaf, then this critical gap must be preperiodic, hence it is not recurrent. We now assume that \mathcal{L}_g^c has a critical leaf ℓ_0 , hence $\mathcal{L}_g^c = \mathcal{L}_g$. Let \mathcal{L} be the quadratic invariant lamination that models the polynomial p_c . We will write G^1 and G^2 for the strictly preperiodic gaps of \mathcal{L} that contain the endpoints of ℓ_0^* . The bases of the gaps G^1 and G^2 map to the basis of a strictly preperiodic gap G of \mathcal{L} . If ℓ_0^* is recurrent, then the gap G contains a recurrent point or a recurrent edge. Since the forward orbit of G contains only finitely many gaps and since two intersecting gaps have always an edge in common, there are no recurrent points in G that are not in edges of G . Every edge of G is eventually periodic. However, it follows from Lemma 2.2 that all edges are strictly preperiodic, hence non-recurrent. \square

It follows that the conclusion of Theorem 4.5 can be made stronger: *the rational function f is topologically conjugate to the mating of p_c with some quadratic polynomial!* The next question is: given a capture hyperbolic component \mathcal{H} in $Per_k(0)$, how many of the boundary points of \mathcal{H} correspond to matings? We will see that, in some cases, all boundary maps correspond to matings, and in some cases, there is a simple arc on the boundary of \mathcal{H} consisting of matings.

4.6. End-captures and cut-captures

We now discuss how much a capture depends on the choice of a capture path. Let p_c be a quadratic polynomial such that 0 is periodic of minimal period k , and $\beta : [0, 1] \rightarrow \mathbb{C}\mathbb{P}^1$ a capture path for p_c . Note that the combinatorial class of the Thurston map $\sigma_\beta \circ p_c$ depends only on the homotopy class of the path β relative to the forward orbit of the point $v = \beta(1)$ (this forward orbit is a finite set by definition of a capture path). Let V be the Fatou component of p_c containing the point v . Define *limbs* of V as the closures of the components of the complement of \overline{V} in the filled Julia set K_c . As was noted in [14], the iterated forward images of v under p_c are contained in only one or two limbs of V . In the first case, we say that β is an *end-capture path*. In the second case, we say that β is a *cut-capture path*. A rational function (if any) combinatorially equivalent to $\sigma_\beta \circ p_c$ is called an *end-capture* or a *cut-capture* according to whether β is an end-capture

path or a cut-capture path. Let us first consider a hyperbolic component in $Per_k(0)$ corresponding to an end-capture.

THEOREM 4.7. — *Let \mathcal{H} be a hyperbolic component in $Per_k(0)$, whose center is represented by an end-capture H of p_c . Then representatives of all points in $\partial\mathcal{H} - \overline{\mathcal{B}}$ are matings of p_c with certain quadratic polynomials.*

Proof. — Let λ_0 be a point in $\partial\mathcal{H} - \overline{\mathcal{B}}$ of angle \varkappa , and f a rational function representing λ_0 . Suppose that H is the capture of p_c corresponding to some capture path β_0 . Consider any capture path β for p_c such that $\beta(1) = \beta_0(1)$, and the point $b = \beta(1/2)$ has internal angle \varkappa with respect to the Fatou component of p_c containing the point $\beta_0(1)$. Since H is an end-capture, it is also the capture of p_c corresponding to the capture path β . Theorem 4.5 and Proposition 4.6 now imply that f is topologically conjugate to the mating of p_c with some quadratic polynomial. \square

We now consider cut-capture paths for p_c . Let V be a strictly preperiodic Fatou component of p_c . Suppose that the forward orbit of V is contained in two limbs of V . In this case, there are two homotopy classes of capture paths terminating in V . One class \mathcal{C}_0 contains capture paths, whose internal angles are in $(0, 1/2)$, and the other class \mathcal{C}_1 contains capture paths, whose internal angles are in $(1/2, 1)$. Capture paths with internal angles 0 and $1/2$ can belong to either class. There are, in general, two conformal captures H_0 and H_1 of p_c , up to conformal conjugacy. For every capture path in \mathcal{C}_0 , the corresponding capture is H_0 , and for every capture path in \mathcal{C}_1 , the corresponding capture is H_1 . Let \mathcal{H}_0 and \mathcal{H}_1 denote the hyperbolic components of $Per_k(0)$, whose centers are represented by H_0 and H_1 , respectively. Since H_0 corresponds to capture paths with internal angles from 0 to $1/2$, we call \mathcal{H}_0 a $[0, 1/2]$ -capture component associated with p_c . Similarly, we call \mathcal{H}_1 a $[1/2, 1]$ -capture component.

THEOREM 4.8. — *Let \mathcal{H}_0 be a $[0, 1/2]$ -capture hyperbolic component in $Per_k(0)$ associated with p_c . Then any point of $\partial\mathcal{H}_0 - \overline{\mathcal{B}}$, whose angle belongs to $[0, 1/2]$, is represented by a mating of p_c with some quadratic polynomial. Similarly, let \mathcal{H}_1 be a $[1/2, 1]$ -capture hyperbolic component in $Per_k(0)$ associated with p_c . Then any point of $\partial\mathcal{H}_1 - \overline{\mathcal{B}}$, whose angle belongs to $[1/2, 1]$, is represented by a mating of p_c with some quadratic polynomial.*

This theorem leads to the following question: *is it true that every $[0, 1/2]$ -capture component is simultaneously a $[1/2, 1]$ -capture component?* If this is true, then, for every hyperbolic component \mathcal{H} in $Per_k(0)$, all points of $\partial\mathcal{H} - \overline{\mathcal{B}}$ are represented by matings. Conjecturally, *all points of $\partial\mathcal{H}$ are represented by matings*, including those in $\overline{\mathcal{B}}$.

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