ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

ADAM EPSTEIN, THOMAS SHARLAND

*A classification of bicritical rational maps with a pair of period two superattracting cycles*

Tome XXI, n° S5 (2012), p. 907-934.

<http://afst.cedram.org/item?id=AFST_2012_6_21_S5_907_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXI, n° 5, 2012
pp. 907–934

# A classification of bicritical rational maps with a pair of period two superattracting cycles

Adam Epstein[(1)], Thomas Sharland[(2)]

**Abstract.** — We give a Thurston classification of those bicritical rational maps which have two period two superattracting cycles. We also show that all such maps are constructed by the mating of two unicritical degree $d$ polynomials.

**Résumé.** — Nous donnons une classification de Thurston des fractions rationnelles bicritiques possédant deux cycles superattractifs de période deux. Nous montrons également que toutes les fractions rationnelles de ce type sont construites par accouplement de deux polynômes unicritiques de degré $d$.

## 1. Introduction

In this note we study bicritical rational maps whose critical points lie in distinct period two cycles. These maps are completely classified by a natural combinatorial invariant. A common strategy used in the classification of rational maps is to invoke Thurston's Theorem [1]. This will not be necessary in this note, due to the availability of a purely algebraic classification of the maps in question. However, we will still tackle the question of Thurston equivalence at the end of the paper, since it motivates a discussion of the combinatorial description of these maps, which is a characteristic that will be much more useful than the algebraic classification in general. We refer the reader to [5] for background material on the dynamics of rational maps.

(1) University of Warwick, Coventry, CV4 7AL, UK
A.L.Epstein@warwick.ac.uk
(2) State University of New York at Stony Brook, Stony Brook, New York, USA
tjshar@math.sunysb.edu

### 1.1. Thurston's Criterion

Let $F\colon \Sigma \to \Sigma$ be an orientation-preserving branched self-covering of a topological 2-sphere. We denote by $\Gamma_F$ the critical set of $F$ and define

$$P_F = \bigcup_{n>0} F^{\circ n}(\Gamma_F)$$

to be the postcritical set of $F$. We say that $F$ is postcritically finite if $|P_F| < \infty$.

Definition 1.1. — *Let $F\colon \Sigma \to \Sigma$ and $\widehat{F}\colon \widehat{\Sigma} \to \widehat{\Sigma}$ be postcritically finite orientation-preserving branched self-coverings of topological 2-spheres. A* combinatorial equivalence *(or Thurston equivalence) is given by a pair of orientation-preserving homeomorphisms $(\Phi, \Psi)$ from $\Sigma$ to $\widehat{\Sigma}$ such that*

- $\Phi|_{P_F} = \Psi|_{P_F}$
- *The following diagram commutes:*

$$\begin{array}{ccc} (\Sigma, P_F) & \xrightarrow{\Psi} & (\widehat{\Sigma}, P_{\widehat{F}}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \widehat{F}} \\ (\Sigma, P_F) & \xrightarrow[\Psi]{} & (\widehat{\Sigma}, P_{\widehat{F}}) \end{array}$$

- *$\Phi$ and $\Psi$ are isotopic via a family of homeomorphisms $t \mapsto \Phi_t$ which is constant on $P_F$.*

We say $F$ and $\widehat{F}$ are equivalent if there exists an equivalence as above. Given a branched covering $F$, we want to know when it is equivalent to a rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Let $\Gamma = \{\gamma_i : i = 1, \dots, n\}$ be a multicurve; that is, each $\gamma_i \in \Gamma$ is simple, closed, non-peripheral[1], disjoint and non-homotopic to all the other $\gamma_j$ relative to $P_F$. $\Gamma$ is *$F$-stable* if for all $\gamma_i \in \Gamma$, all the non-peripheral components of $F^{-1}(\gamma_i)$ are homotopic rel $\Sigma \setminus P_F$ to elements of $\Gamma$. In this case, we define $F_\Gamma = (f_{ij})_{n\times n}$ to be the non-negative matrix defined as follows. Let $\gamma_{i,j,\alpha}$ be the components of $F^{-1}(\gamma_j)$ which are homotopic to $\gamma_i$ in $\Sigma \setminus P_F$. Now define

$$F_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{\deg F|_{\gamma_{i,j,\alpha}}\colon \gamma_{i,j,\alpha} \to \gamma_j} \gamma_i$$

(1) *non-peripheral* means that each component of the complement of $\gamma$ contains at least two elements of $P_F$

where deg denotes the degree of the map. By standard results on non-negative matrices (see [2] or [7]), this matrix $(f_{ij})$ will have a leading non-negative eigenvalue $\lambda$. We write $\lambda(\Gamma)$ for the leading eigenvalue associated to the multicurve $\Gamma$.

The significance of these definition lies in the following rigidity theorem of Thurston.

THEOREM 1.2 (THURSTON). —

*(1) A postcritically finite branched covering $F\colon \Sigma \to \Sigma$ of degree $d \geqslant 2$ with hyperbolic orbifold is equivalent to a rational map $R\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if and only if there are no $F$-stable multicurves with $\lambda(\Gamma) \geqslant 1$[2].*

*(2) Any Thurston equivalence of rational maps $F$ and $\widehat{F}$ with hyperbolic orbifolds is represented by a Möbius conjugacy.*

*Remark 1.3. —*

- *The condition that the maps have hyperbolic orbifolds is purely combinatorial and is readily ascertained by inspection of the postcritical set. The condition will be satisfied by all the maps in this paper. For further details see [1].*
- *For bicritical maps, any such conjugacy (indeed, any equivalence) either fixes the two critical points or interchanges them. In our setting, the post-critical set has (at least) four elements, and so any Möbius conjugacy fixing the two critical points must be the identity.*

Before moving on, we need to introduce a special kind of multicurve, called a (good) Levy cycle.

DEFINITION 1.4 *A multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a Levy cycle if for each $i = 1, \ldots, n$, the curve $\gamma_{i-1}$ (or $\gamma_n$ if $i = 1$) is homotopic to some component $\gamma_i'$ of $F^{-1}(\gamma_i)$ (rel $P_F$) and the map $F\colon \gamma_i' \to \gamma_i$ is a homeomorphism. A Levy cycle is a good Levy cycle if the connected components of $\Sigma \setminus \bigcup_{i=1}^n \gamma_i$ are $D_1, \ldots, D_m, C$, with the $D_j$ all being disks. When $n = 1$, we have $C = \varnothing$ and $F|_{\gamma_1'}\colon \gamma_1' \to \gamma_1$ reverses the orientation. For $n > 1$, there exists a component $C'$ of $F^{-1}(C)$ which is isotopic to $C$ and such that $F|_{C'}\colon C' \to C$ is a homeomorphism.*

It is clear that a good Levy cycle $L$ is a Thurston obstruction, since $\lambda(L) = 1$.

[2] Such a multicurve is known as a *Thurston obstruction*

### 1.2. Matings

In this section, we review the basic terminology of matings, following [4, 11, 13]. Let $f$ and $g$ be monic degree $d$ polynomials. In this paper, $f$ and $g$ will be unicritical (that is, taking the form $z \mapsto z^d + c$ for some $c$) but this is not needed in general. We define

$$\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i t} : t \in \mathbb{R}/\mathbb{Z}\},$$

the complex plane compactified by the circle at infinity. We then continuously extend the two polynomials to the circle at infinity by defining

$$f(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2d\pi i t} \quad \text{and} \quad g(\infty \cdot e^{2\pi i t}) = \infty \cdot e^{2d\pi i t}.$$

Label this extended dynamical plane of $f$ (respectively $g$) by $\widetilde{\mathbb{C}}_f$ (respectively $\widetilde{\mathbb{C}}_g$). We create a topological sphere $\Sigma_{f,g}$ by gluing the two extended planes together along the circle at infinity:

$$\Sigma_{f,g} = \widetilde{\mathbb{C}}_f \uplus \widetilde{\mathbb{C}}_g / \sim$$

where $\sim$ is the relation which identifies the point $\infty \cdot e^{2\pi i t} \in \widetilde{\mathbb{C}}_f$ with the point $\infty \cdot e^{-2\pi i t} \in \widetilde{\mathbb{C}}_g$. The *formal mating* is then defined to be the branched covering $f \uplus g \colon \Sigma_{f,g} \to \Sigma_{f,g}$ such that

$$f \uplus g|_{\widetilde{\mathbb{C}}_f} = f \quad \text{and}$$
$$f \uplus g|_{\widetilde{\mathbb{C}}_g} = g.$$

The definition of the topological mating requires discussion of external rays. Suppose the (filled) Julia set $K(f)$ of the degree $d \geqslant 2$ monic polynomial $f \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is connected (this will always be the case in this article). By Böttcher's theorem, there is a conformal isomorphism

$$\phi = \phi_f \colon \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus K(f)$$

which can be chosen so that it conjugates $z \mapsto z^d$ on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with the map $f$ on $\widehat{\mathbb{C}} \setminus K(f)$. There is a unique such conjugacy which is tangent to the identity at infinity.

Definition 1.5. — *The external ray of angle $t$ is*

$$R_f(t) = \phi_f(r_t)$$

*where $r_t = \{r \exp(2\pi i t) : r > 1\} \subset \mathbb{C} \setminus \mathbb{D}$ is the corresponding radial line.*

Recall that $K(f)$ (equivalently $J(f)$) is locally connected if and only if $\phi$ extends continuously to the unit circle, thereby inducing a semi-conjugacy $\gamma_f\colon \mathbb{R}/\mathbb{Z} \to J(f)$. The point $\gamma_f(t) \in K(f)$ is called the *landing point* of the external ray $R_f(t)$. The points $k/(d-1)$ are fixed under the map $t \mapsto dt$ on $\mathbb{R}/\mathbb{Z}$ with images $\beta_k = \gamma_f(k/(d-1))$ fixed under $f$. We call these the $\beta$-fixed points of $f_c$. In the case that these $\beta_k$ are distinct, there exists at most one other fixed point, which we will call the $\alpha$-fixed point of $f_c$.

LEMMA 1.6. — *Suppose $f$ is a unicritical polynomial. Then the $\beta$-fixed points of $f$ are all distinct.*

*Proof.* — We consider $f$ as a map on $\widetilde{\mathbb{C}}$ and suppose that $\beta_j = \beta_k$ for some $j \neq k$. The external rays of angle $\hat{\beta}_j$ and $\hat{\beta}_k$ and their common landing point divide the disk into two sectors. Denote by $U$ the one which does not contain the critical point. The smallest possible angular width of the sector $U$ is evidently $1/(d-1) > 1/d$ whence the image of the sector under $f$ must be the whole of the disk $\widetilde{\mathbb{C}}$. However since the critical point is not contained in $U$, the critical value cannot be in its image $f(U)$. Hence all the $\beta_i$ must be distinct. □

We now define the topological mating of monic degree $d$ polynomials $f$ and $g$ with locally connected Julia sets. We first define the ray-equivalence relation $\approx$ on $\Sigma_{f,g}$. We denote by $\sim_f$ the smallest equivalence relation on $\widetilde{\mathbb{C}}_f$ such that $x \sim_f y$ if and only if $x, y \in \overline{R}_f(t)$ for some $t$; we denote by $\sim_g$ the corresponding equivalence relation on $\widetilde{\mathbb{C}}_g$. Then the equivalence relation $\approx$ is the smallest equivalence relation on $\Sigma_{f,g}$ generated by $\sim_f$ on $\widetilde{\mathbb{C}}_f$ and $\sim_g$ on $\widetilde{\mathbb{C}}_g$. We will denote the equivalence class of $x$ under $\approx$ by $[x]$ and by $[x]_J$ the intersection of the Julia sets of the two polynomials with $[x]$. We now observe that there is an evident surjection $K(f) \uplus K(g) \to \Sigma_{f,g}/\approx$ formed by the composition

$$K(f) \uplus K(g) \hookrightarrow \widetilde{\mathbb{C}}_f \uplus \widetilde{\mathbb{C}}_g \to \Sigma_{f,g} \to \Sigma_{f,g}/\approx .$$

Moreover, since $x \approx y$ implies $f \uplus g(x) \approx f \uplus g(y)$, there exists a unique map $f \perp\!\!\!\perp g$ such that the following diagram commutes:

$$\begin{array}{ccc} K(f) \uplus K(g) & \xrightarrow{f \uplus g} & K(f) \uplus K(g) \\ \downarrow & & \downarrow \\ \Sigma_{f,g}/\approx & \xrightarrow[f \perp\!\!\!\perp g]{} & \Sigma_{f,g}/\approx . \end{array}$$

We refer to this map $f \perp\!\!\!\perp g$ as the topological mating of $f$ and $g$. When circumstances are favorable, the quotient space $\Sigma_{f,g}/\approx$ is an oriented topological 2-sphere.

We say that a rational map $F$ is the geometric mating of $f$ and $g$ if $F$ is topologically conjugate via an orientation-preserving homeomorphism $\varphi\colon \Sigma_{f,g}/\approx \to \widehat{\mathbb{C}}$ which is holomorphic on the interior of $K_f \uplus K_g$. In this situation, we will write $F \cong f \perp\!\!\!\perp g$. It is important to realize that the same rational map may arise as a mating in several different ways; as will in fact be the case in this paper. This state of affairs is referred to as a *shared mating*. The following key result on the matings of unicritical polynomials will be of use in this article.

THEOREM 1.7 (Tan Lei, [12]). — *Let $f_1$ and $f_2$ be unicritical postcritically finite polynomials with $\alpha$-fixed points $\alpha_1$ and $\alpha_2$ respectively. Then the mating $f_1 \perp\!\!\!\perp f_2$ is equivalent to a rational map if and only if $[\alpha_1] \neq [\alpha_2]$.*

### 1.3. Structure

Our classification will go as follows. First, we will consider all matings $f \perp\!\!\!\perp g$ where $f$ and $g$ both are monic unicritical polynomials with a critical orbit of period two. We then show that such matings, when unobstructed, yield rational maps whose Julia sets admit a simple combinatorial description. Finally, by using a simple counting argument, we show that these rational maps account for all the rational maps that have a pair of period two superattracting cycles. Our main result is the following.

THEOREM A. — *A degree $d$ bicritical rational map with labeled critical points and with two period two superattracting cycles may be realized as a mating in precisely $d-1$ ways. Furthermore, the rational map is completely defined (up to Möbius conjugacy) by one piece of combinatorial data.*

Despite the relative simplicity of the result, perhaps the most striking observation is that the rational maps with a pair of period two superattracting cycles have a common feature to their Julia sets, as will be seen in Section 3. It was this observation that lead to the conjecture of Theorem A.

*Remark 1.8. — The rational maps in question actually have* clusters*, as studied in the second author's thesis [8]. More general results on maps with cluster cycles can be found in [9, 10].*

At the end of this paper, we will show that the result could equally well be obtained by demonstrating the Thurston equivalence of the maps with the same combinatorial datum, which would allow us to avoid an explicit algebraic description as in Section 2. This second approach more readily adapts to the general setting of higher degree maps with cluster cycles, hence it is included here. Indeed, it should be noted that in general, clustering is not an algebraic condition; however, in the setting of this paper, it

is a fortunate and perhaps surprising fact that the algebraic classification suffices, hence there is no requirement to check the Thurston equivalence of these maps.

## 2. Algebraic Classification

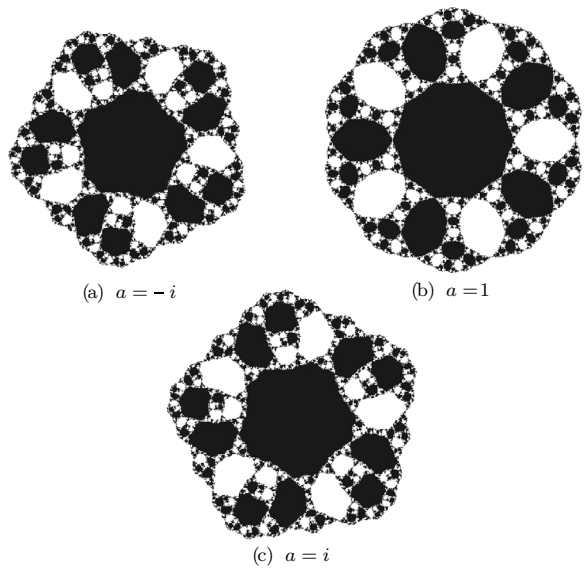
We remark that it is simple to classify the degree $d$ rational maps with two superattracting period two cycles in algebraic terms. Setting the two critical points to be 0 and $\infty$ and defining $F(\infty) = 1$, we obtain the following normal form:

$$F_a(z) = \frac{z^d - a}{z^d - 1}. \tag{2.1}$$

Note that $F_a(0) = a$, so the critical values of $F_a$ are 1 and $a$. For there to be a second period two cycle, we require

$$0 = F_a(a) = \frac{a^d - a}{a^d - 1} = \frac{a(a^{d-1} - 1)}{a^d - 1}.$$

It follows that in order for the map to have a second period two superattracting cycle, we need $a \neq 1$ and for $a$ to be a $(d-1)$th root of unity. Hence there are precisely $d-2$ parameters $a$ for which $F_a$ has a pair of period two superattracting cycles. The three period 5 examples are shown in Figure 1.



(a) $a = -i$ (b) $a = 1$ (c) $a = i$

Figure 1. — The three period 5 bicritical rational maps for which both critical points are of period two. In each diagram the basin of the periodic cycle containing 0 is shaded black, the basin of the periodic cycle containing $\infty$ is white.

## 3. Matings Classification

### 3.1. Structure of the polynomials

Let $\mathcal{M}_d$ be the degree $d$ multibrot set. That is, for $\mathrm{P}_c\colon z \mapsto z^d + c$,

$$\mathcal{M}_d = \{c \in \mathbb{C} : J(\mathrm{P}_c) \text{ is connected}\} = \{c \in \mathbb{C} : (\mathrm{P}_c^{\circ n}(0))_{n=0}^{\infty} \text{ is bounded}\}.$$

For notation, we will write

$$\hat{\beta}_j = \frac{j}{d-1} \quad \text{and} \quad \theta_j^r = \hat{\beta}_j + \frac{r}{d^2-1},$$

with subscripts taken modulo $d-1$ and $r \in \{1, \dots, d\}$. Note that the angles $\hat{\beta}_j$ are fixed under the map $z \mapsto dz$ on $S^1$; indeed, these angles are those of the rays which land on the $\beta$-fixed points of the degree $d$ polynomials under consideration, so that $\beta_j = \gamma_f(\hat{\beta}_j)$. Moreover, the angles $\theta_j^r$ all have period 2. By results of [6] (or by adapting the results of [3] to the degree $d$ case), there is a unique $c_j \in \widehat{\mathbb{C}}$ such that $f_j = \mathrm{P}_{c_j}$ has a period two superattracting cycle and such that the external rays of angles $\theta_{j-1}^1$ and $\theta_{j-1}^d$ land on the $\alpha$-fixed point $\alpha_j$ of $f_j$. Figure 2 shows the location of these maps in parameter space. Note that the $\alpha$-fixed point of $f_j$ is also the principal root point of both the critical point and critical value Fatou components. We prove some elementary facts about the $f_j$.

Lemma 3.1. —

*(1) A point of $J(f_j)$ is biaccessible if and only if it is in the backward orbit of $\alpha_j$.*

*(2) The only biaccessible periodic point of $f_j$ is $\alpha_j$.*

*Proof.* — For the first claim, if $z$ is biaccessible then it must eventually map onto the Hubbard tree of $f_j$. In this case, the Hubbard tree consists of two internal rays meeting at the $\alpha$-fixed point, whence the only point of the Hubbard tree that belongs to $J(f_j)$ is $\alpha_j$, and so there exists some $\ell$ such that $f^{\circ \ell}(z) = \alpha_j$. Conversely, since $\alpha_j$ and the critical point belong to distinct grand orbits and since $\alpha_j$ is biaccessible, every point on the grand orbit of $\alpha_j$ is biaccessible. The second claim follows immediately from the first. □

We now discuss which periodic rays land on the periodic components of $J(f_j)$.

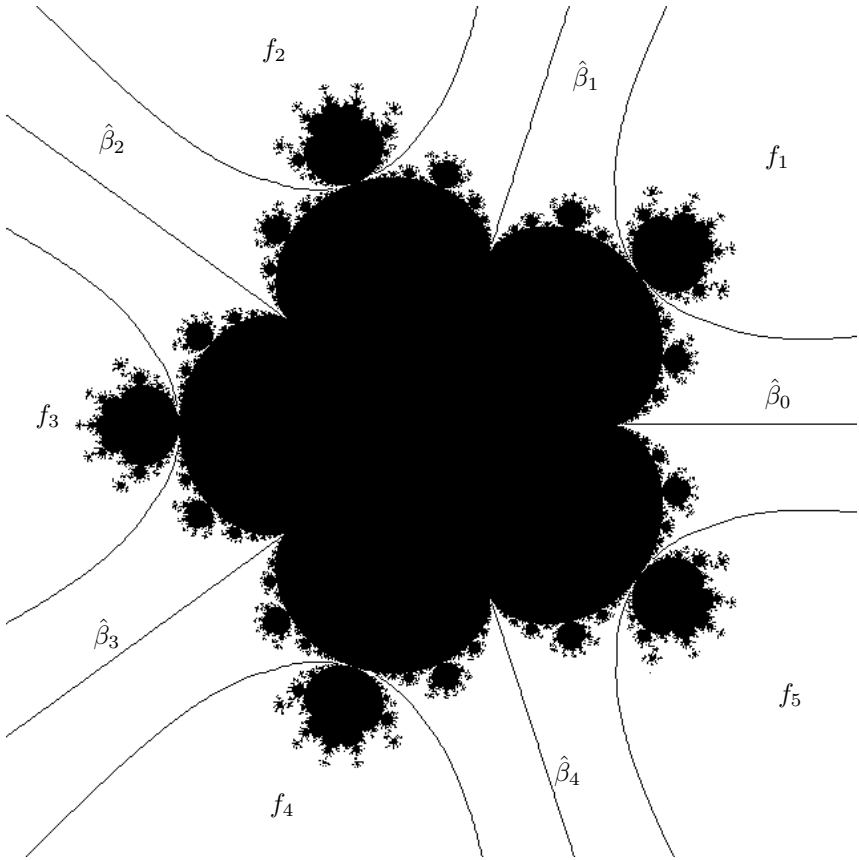



Figure 2. — The maps $f_j$ in the parameter space $\mathcal{M}_6$. The $\hat{\beta}_j$ are labeling the rays of angle $\hat{\beta}_j$ and the $f_i$ labels are contained in the wake of the period two hyperbolic component which contains $f_i$.

LEMMA 3.2. —

*(1) The external rays of angles*

$$\theta_{j-1}^2, \theta_{j-1}^3, \ldots, \theta_{j-1}^r, \ldots, \theta_{j-1}^{d-1} \tag{3.1}$$

*land on the critical value component of $f_j$.*

*(2) The external rays of angles*

$$\theta_j^{d-1}, \theta_{j+1}^{d-2}, \ldots, \theta_{j+r-2}^{d-(r-1)}, \ldots, \theta_{j-1+(d-2)}^2 = \theta_{j-2}^2 \tag{3.2}$$

*land on the critical point component of $f_j$.*

*Proof.* —

(1) The map $f_j^{\circ 2}$ is a degree $d$ map from the critical value component to itself, and so there must exist precisely $d-1$ fixed points of $f_j^{\circ 2}$ on the boundary of this component. One of these is the $\alpha$-fixed point and the others must be of period two, since the only fixed points of $f_j$ are the $\alpha_j$ and the $\beta_\ell$. These points must be the landing points of external rays whose period is divisible by two, and since a fixed ray cannot land on a period 2 point, the periods of the rays landing there is also exactly two. Furthermore, the angles of these rays must lie in the arc $(\theta_{j-1}^1, \theta_{j-1}^d) \subset S^1$, and the only period two angles in this arc are

$$\theta_{j-1}^2, \theta_{j-1}^3, \dots, \theta_{j-1}^r, \dots, \theta_{j-1}^{d-1}$$

as required.

(2) The images under the map $z \mapsto dz$ of the angles in (3.1) are precisely those angles in (3.2), since

$$\begin{aligned} d(\theta_{j-1}^r) &= d\hat{\beta}_{j-1} + \frac{dr}{d^2-1} \\ &= \hat{\beta}_{j-1} + \frac{dr}{d^2-1} \\ &= \hat{\beta}_{j-1} + \frac{r-1}{d-1} + \frac{d-(r-1)}{d^2-1} \\ &= \hat{\beta}_{j+r-2} + \frac{d-(r-1)}{d^2-1} \\ &= \theta_{j+r-2}^{d-(r-1)}. \end{aligned}$$

Since the rays of the angles given in (3.1) land on the critical value component, the rays in (3.2) must land on the critical point component.

□

Following custom, we refer to the fixed points of the first return map to a periodic Fatou component as the *root* points of the component. If the root point is the landing point of more than one external ray, we will call it the *principal* root point. The other root points will be called *co-roots*. Here we see that $\alpha_j$ is the principal root point of both the critical value and critical point components, and that the landing points of the rays of angles given in (3.1) and (3.2) will land at the co-roots of the respective components. Roots of preperiodic components are accordingly defined as the preimages of roots of periodic components. See Figure 3 for an example of the rays landing on the Julia set in the case $d = 6$ and $j = 2$.

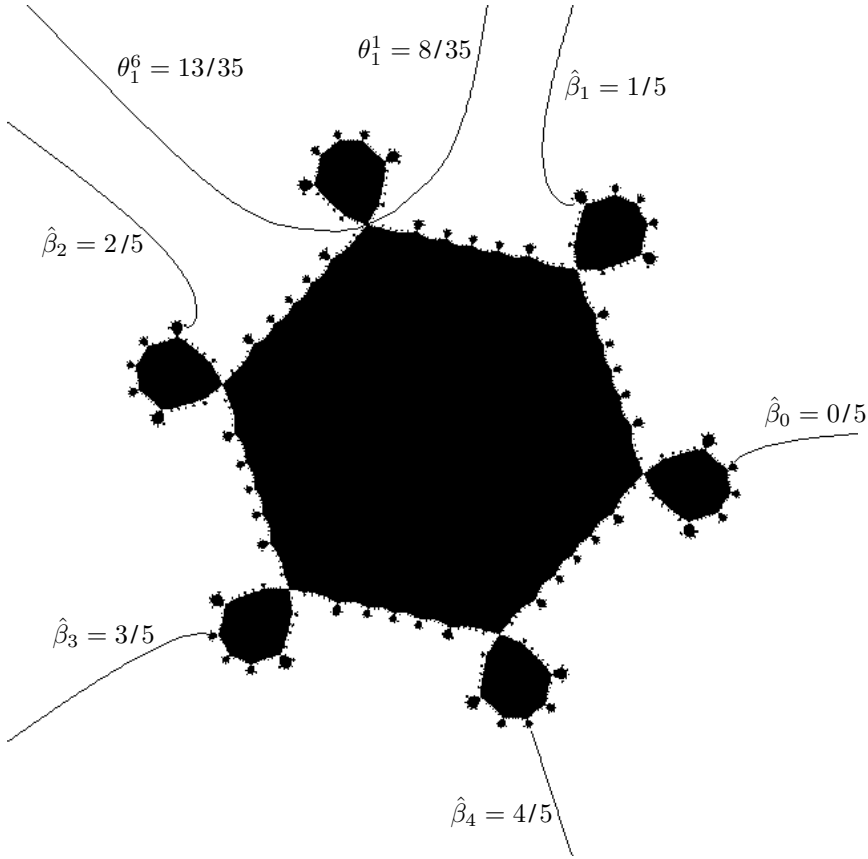



Figure 3. — The Julia set of the map $f_2$ in degree 6 with some important dynamical rays labeled.

LEMMA 3.3. — *Let $U$ be the critical point component of $f_j$ and let $U'$ be a component of $f^{-1}(U)$ that is not the critical value component. Then the external rays landing at the principal root point of $U'$ separate the critical point 0 from precisely one of the fixed points $\beta_r$.*

*Proof.* — Let $A \subset \mathbb{C}$ be the region which is bounded by the external rays landing at the principal root point, namely $\alpha_j$, of the critical value component of $f_j$ and the level $2d$ equipotential curve, but which does not contain the critical value component itself. Let $g$ be the local inverse of $f_j$ which takes $U$ to $U'$. Then $g$ is a homeomorphism and $\overline{g(A)} \subset A$, since $g(A)$ is bounded by the rays landing at the preimage of $\alpha_j$ on the boundary

of $U'$ and the level 2 equipotential curve. Hence by the Schwarz Lemma, $g$ has a unique fixed point in $g(A)$. This fixed point is not $\alpha_j$, since clearly $\alpha_j \notin g(A)$ and it is not $\infty$ since $g(A)$ is bounded. $\square$

Note that the above result is equivalent to saying that the rays landing at the principal root point of $U'$ separate the critical point 0 from (precisely) one the rays of angle $\hat{\beta}_r$.

### 3.2. Matings and ray classes

In this section we want to consider the ray classes of matings which are not obstructed. For ease of notation, we will write $\gamma_+$ (respectively $\gamma_-$) for the semi-conjugacy $\gamma_{f_{i_+}}$ (respectively $\gamma_{f_{i_-}}$).

LEMMA 3.4. — *Suppose the mating $f_j \perp\!\!\!\perp f_k$ is not obstructed and $z \in J(f_j) \cup J(f_k)$. Then the ray class $[z]_J$ has one of the following forms.*

- *If $z$ is in the backward orbit of $\alpha_j$ then $[z]_J = \{z, \gamma_k(-t_1), \gamma_k(-t_2)\}$ where $t_1$ and $t_2$ are the angles of the external rays landing at $z$.*
- *If $z$ is in the backward orbit of $\alpha_k$ then $[z]_J = \{z, \gamma_j(-\tau_1), \gamma_j(-\tau_2)\}$ where $\tau_1$ and $\tau_2$ are the angles of the external rays landing at $z$.*
- *If $z$ is not in the backward orbit of $\alpha_j$ or $\alpha_k$ then $[z]_J = \{\gamma_j(t), \gamma_k(-t)\}$ for some $t$.*

*Proof.* — Note that $[\alpha_j] \neq [\alpha_k]$ by Theorem 1.7. We claim that if $z = \alpha_j$ then the ray class $[z]_J$ contains three points. Indeed, all points in $[\alpha_j]_J$ must be periodic of period dividing two and by Lemma 3.1 the only biaccessible periodic points of $J(f_j) \cup J(f_k)$ are $\alpha_j$ and $\alpha_k$, whence the points $\gamma_k(-t_1)$ and $\gamma_k(-t_2)$ are not biaccessible. Thus $[\alpha_j]_J = \{\alpha_j, \gamma_k(-t_1), \gamma_k(-t_2)\}$ where $t_1$ and $t_2$ are the angles of the external rays which land at $\alpha_j$. The case where $z = \alpha_k$ is analogous. Moreover, if $z$ is in the backward orbit of $\alpha_j$ or $\alpha_k$ then the ray class $[z]_J$ also contains three points since neither $\alpha_J$ or $\alpha_k$ belong to the grand orbit of a critical point. If $z$ is not in the backward orbit of $\alpha_j$ or $\alpha_k$, then by Lemma 3.1, none of the points of $[z]_J$ will be biaccessible and so $[z]_J = \{\gamma_j(t), \gamma_k(-t)\}$ for some $t$. $\square$

LEMMA 3.5. — *The mating $f_j \perp\!\!\!\perp f_k$ is obstructed if and only if $k = d - j$.*

*Proof.* — If $k = d - j$ then clearly $[\alpha_j] = [\alpha_k]$. Conversely, by Lemma 3.4,

$$[\alpha_j]_J = \{\alpha_j, \gamma_k(-\theta^1_{j-1}), \gamma_k(-\theta^d_{j-1})\}.$$

Now $\alpha_k \notin \{\gamma_k(-\theta^1_{j-1}), \gamma_k(-\theta^d_{j-1})\}$ since $k \neq d-j$. In particular $[\alpha_j] \neq [\alpha_k]$ and the proof of the lemma follows from Tan Lei's theorem (Theorem 1.7). □

We can say more: the only periodic ray classes which contains more than two points of $J(f_j) \cup J(f_k)$ are $[\alpha_j]$ and $[\alpha_k]$.

PROPOSITION 3.6. — *Suppose $j + k \neq d$. Then the rational map $F \cong f_j \amalg f_k$ has the following properties.*

- *The closures of the two critical value components are disjoint.*
- *The closures of the two critical point components are disjoint.*

*Proof.* — Note that the angles of the rays landing on the critical value component of $f_j$ lie in the arc

$$(\theta^1_{j-1}, \theta^d_{j-1})$$

and the angles of the rays landing on the critical value component of $f_k$ lie in the arc

$$(\theta^1_{k-1}, \theta^d_{k-1}).$$

Since $j + k \neq d$ and $-\theta^r_{k-1} = \theta^{d-r+1}_{d-k-1}$, it follows that

$$(\theta^1_{j-1}, \theta^d_{j-1}) \cap (-\theta^d_{k-1}, -\theta^1_{k-1}) = \varnothing. \tag{3.3}$$

By Lemma 3.4, all ray classes $[z]_J$ contain at most 3 points, and the result follows since any ray class corresponding to a meeting point between the two critical value components would contain a ray with angle in the intersection in (3.3), which is empty. □

LEMMA 3.7. — *Let $F \cong f_j \amalg f_k$. Then the following conditions hold.*

- *The only ray class which meets the boundary of the critical point component of $f_j$ and the critical value component of $f_j$ is $[\alpha_j]$.*
- *The only ray class which meets the boundary of the critical point component of $f_k$ and the critical value component of $f_k$ is $[\alpha_k]$.*

*Proof.* — We will prove the first claim; the proof of the second claim is entirely analogous. Since the critical point component and the critical value component of $f_j$ have closures intersecting at $\alpha_j$, the corresponding Fatou components of $F$, $C_j$ (which contains the critical point) and $V_j$ (which contains the critical value) meet at $[\alpha_j] \in J(F)$. We claim that this is the only meeting point. Indeed, any such point arises from a ray class which

contains one boundary point from $C_j$ and one boundary point from $V_j$. If the ray class is not $[\alpha_j]$, then it contains a biaccessible point of $J(f_k)$ and so, by Lemma 3.4, eventually maps onto $[\alpha_k]$. It follows that $[\alpha_k]$ intersects both component boundaries. Moreover, since $[\alpha_k]$ is fixed, such intersection points must be periodic and by considering the denominators of the angles of the rays landing at $\alpha_k$, we see that these periodic points must have periods dividing 2. However, the relevant periodic rays landing on $V_j$ are given in (3.1) and those landing on $C_j$ are given in (3.2). These two sets of angles are separated by the angles $\hat{\beta}_{j-1}$ and $\hat{\beta}_j$, whence the rays landing on $\alpha_k$ (namely the rays of angle $\theta^1_{k-1}$ and $\theta^d_{k-1}$) would have to be separated by the rays of angles $-\hat{\beta}_{(j-1)}$ and $-\hat{\beta}_j$, which is a contradiction. □

Now let us consider the ray classes which connect the critical point component on one side with the critical value component on the other.

Lemma 3.8. —

- *There exists a ray class $[z]$ of period 2 which contains a point on the boundary of the critical value component of $f_j$ and a point on the critical point component of $f_k$.*
- *There exists a ray class $[z']$ of period 2 which contains a point on the boundary of the critical value component of $f_k$ and a point on the critical point component of $f_j$.*

*Proof.* — Observe that $\hat{\beta}_r = -\hat{\beta}_{d-1-r}$ and $\theta^r_{k-1} = -\theta^{d+1-r}_{d-k}$ and so

$$-\theta^r_{j-1} = \theta^{d-r+1}_{d-j-1} \tag{3.4}$$

In view of 3.1) and (3.2) if the two sets

$$\{-\theta^2_{j-1}, -\theta^3_{j-1}, \ldots, -\theta^r_{j-1}, \ldots, -\theta^{d-1}_{j-1}\}$$

and

$$\{\theta^{d-1}_k, \theta^{d-2}_{k+1}, \ldots, \theta^{d-(r-1)}_{k+r-2}, \ldots, \theta^2_{k-1+(d-2)}\}$$

intersect, then the intersections would correspond to points where the critical point component of $f_k$ meet the critical value component of $f_j$ at period 2 orbits. By (3.4) we see that the angle $\theta^{j+k}_{d-j-1}$ is the unique element in both these sets and so the associated ray class will provide a periodic point of intersection on the boundaries of the two components. This point is evidently of period 2, and the other member of the period 2 cycle is the ray class that suffices to prove the second claim. □

We refer to the points corresponding to the ray classes of Lemma 3.8 as the periodic meeting points. We need to show that the periodic ray classes

found in Lemma 3.8 are in fact the only ray classes which contain a point on the boundary of the critical value component of one polynomial and a point on the boundary of the critical point component of the other.

PROPOSITION 3.9. —

- *The critical orbit component of $f_j$ and critical value component of $f_k$ have exactly one boundary point in common under the topological mating.*
- *The critical orbit component of $f_k$ and critical value component of $f_j$ have exactly one boundary point in common under the topological mating.*

*In both cases, the meeting point corresponds to the periodic meeting points of Lemma 3.8.*

The proof of this proposition will be the goal of the remainder of this section. First we introduce some terminology.

DEFINITION 3.10. — *A component $U$ of the Fatou set of $f_j$ will be called a level-$n$ component if $n$ is the least natural number such that $f_j^{\circ n}(U)$ is the critical point component. A strictly preperiodic component will be called a primary component if its principal root point lies on the boundary of a periodic Fatou component. A level-$n$ sector is the sector of the sphere which is bounded by the rays landing at the root point of a level-$n$ component and which contains the component itself. A sector is primary if it is the sector of a primary component (see Figure 4).*

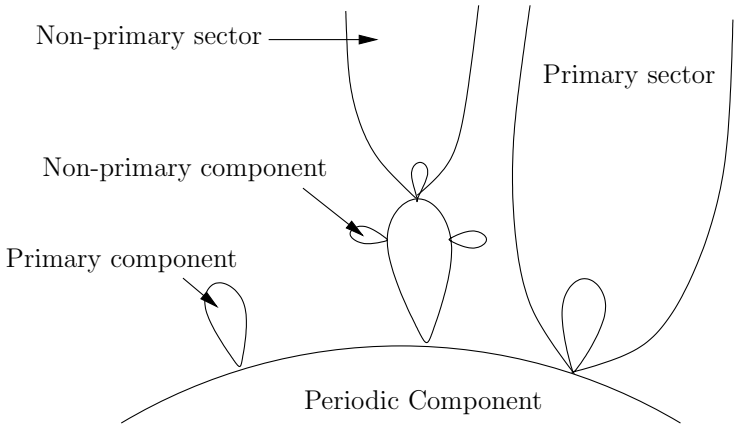



Figure 4. — The definitions of primary components and sectors.

Note that by definition the critical point component of $f_j$ is a level-0 component and that the critical value component is a level-1 component. Moreover, if $r$ is the principal root point of a level-$n$ component other than the critical value component, then $f^{\circ n}(r) = \alpha_j$. It is easy to see that all level-1 components are primary, and that their root points lie on the boundary of the critical point component. Also, by Lemma 3.3, each level-1 sector, save for the sector corresponding to the critical value component, contains exactly one ray of angle $\hat{\beta}_r$ for some $r$. We define the angular width of a sector to be $|\theta_1 - \theta_2|$, where $\theta_1, \theta_2$ are the angles of the rays landing at the root of the sector.

Let $U_1, U_2$ be any two primary level-$n$ components for the same polynomial $f$. $U_1$ and $U_2$ both have roots which lie on the boundary of the same critical orbit component $C$. Removing these two points will split $\partial C$ into two complementary Jordan arcs. We say that the primary level-$n$ component $U_1$ is *adjacent* (to the periodic meeting point of Lemma 3.8) if there exists a second primary level-$n$ component $U_2$ such that one of the complementary arcs $A \subset \partial C$ does not contain a root point of another level-$k$ primary component for all $k \leqslant n$ but does contain a point which belongs to the ray class which becomes one of the periodic meeting points under the mating. Note that in this case $U_2$ will also be adjacent. A primary level-$n$ sector will be referred to as being *adjacent* to the periodic meeting point if it corresponds to an adjacent component. Given two adjacent level-$n$ components, there are two rays landing at each of their root points. There is a unique such pair of rays (one landing at the root of each component) such that these rays and $A$ (where $A$ is the complementary arc referred to in the definition of adjacency) bound a region of $\mathbb{C}$ which does not contain either of the two adjacent components but does contain a point which belongs to the ray class which becomes the periodic meeting point on its boundary. These rays will be called the *closest* rays to the periodic meeting point.

Lemma 3.11. — *Suppose $U$ is a primary level-$n$ component. Then*

- *$n$ is even if and only if $U$ shares a boundary with the critical value component.*
- *$n$ is odd if and only if $U$ shares a boundary with the critical point component.*

*Proof.* — Note that if $U$ is not the critical value component, then its root point $r$ is such that $f^{\circ n}(r) = \alpha$. It suffices to prove only the first claim, so assume $n$ is even and, to obtain a contradiction, assume $U$ shares a boundary with the critical point component $C$. Let $r$ be the root point of $U$ and

consider sequences $(x_n)$, $(y_n) \to r$ such that $x_n \in U$ and $y_n \in C$ for all $n$. Then the sequences $(f^{\circ n}(x_n))$ and $(f^{\circ n}(y_n))$ both converge to $\alpha$, with both sequences contained in $C$. This would mean that for a small neighborhood $N \ni r$, the intersection of $f^{\circ n}(N)$ with the critical value component would be empty, which is impossible since $f^{\circ n}$ is locally a homeomorphism at $r$. □

Given an external ray $R_{f_j}(\theta)$, we will refer to the ray $R_{f_k}(-\theta)$ as its *conjugate ray*. Similarly, the conjugate ray to $R_{f_k}(\theta)$, will be $R_{f_j}(-\theta)$.

LEMMA 3.12. — *The conjugate rays to those landing on $\alpha_j$ land on the Julia set of $f_k$ in distinct level-1 sectors of $f_k$. Moreover these sectors are adjacent.*

*Proof.* — The rays landing on $\alpha_j$ have angles $\theta^1_{j-1}$ and $\theta^d_{j-1}$. If $\theta \in \{\theta^1_{j-1}, \theta^d_{j-1}\}$ then the conjugate ray of angle $-\theta$ lands on a periodic point of period dividing 2 on $J(f_k)$. As the only periodic points of period dividing 2 on the critical point component are $\alpha_k$ and points corresponding to the angles in (3.2), there exists $n > 1$ such that the ray of angle $-\theta$ lands inside a primary level-$n$ sector $S$. The root point of $S$ lies on the boundary of the critical point component of $f_k$, as it is blocked from the critical value component by the rays of angle $-\hat{\beta}_j$ and $-\hat{\beta}_{j-1}$. If $n > 1$, the image of $S$ is a level-$(n-1)$ sector $S'$ which shares a boundary point with the critical value component. However, $S'$ must contain the landing point of the other ray in the 2-cycle, but this is impossible since again this ray is separated from the critical value component by the rays of angles $-\hat{\beta}_j$ and $-\hat{\beta}_{j-1}$. Hence the sector must be a level-1 sector.

By Proposition 3.6, neither of the sectors correspond to the critical value component, since otherwise the closures of the two critical value components would meet. If the rays of angles $-\theta^1_{j-1}$ and $-\theta^d_{j-1}$ landed inside the same sector, then that sector would have angular width strictly greater than $1/(d+1)$. However, all level-1 sectors have angular width equal to $1/(d+1)$ and so the two sectors are distinct. Suppose that the two components were not adjacent. By Lemma 3.3, each level-1 sector must contain a ray of angle $\hat{\beta}_r$. However, if the components were not adjacent, then there would exist an angle of the form $-\hat{\beta}_r = -r/(d-1)$ between $-\theta^1_{j-1}$ and $-\theta^d_{j-1}$. This is a contradiction. □

The following lemma will allow us to locate the (primary) preimages of primary components.

LEMMA 3.13. — *The following hold for the polynomial $f_k$.*

*(1) Given a primary level-$n$ component $U$ which has its root point on the boundary of the critical point component, there exists a unique primary level-$(n+1)$ component $U'$ which is a preimage of $U$.*

*(2) Given a primary level-$n$ component $U$ which has its root point on the boundary of the critical value component, for each $j \neq k$ there exists a unique primary level-$(n+1)$ component $U'$ which is a preimage of $U$ and such that the angles of the rays landing at the root point of $U'$ lie in the arc $(-\theta^d_{j-1}, -\theta^1_{j-1})$.*

*Proof.* —

(1) If the level-$n$ component has its root point on boundary of the critical point component, then there exists a unique preimage which has a boundary point on the critical value component, and hence it is primary.

(2) If the level-$n$ component has its root point on the boundary of the critical value component, then every preimage component must share a boundary point with the critical point component and so will be primary. The rays of angles $-\theta^d_{j-1}$ and $-\theta^1_{j-1}$ land inside distinct adjacent (primary) level-1 sectors of $f_k$ by Lemma 3.12 and so there exists a unique preimage of $U'$ which have rays landing at its root point whose corresponding angles lie in $(-\theta^d_{j-1}, -\theta^1_{j-1})$.

□

We now use this lemma to study the adjacent components and sectors in the mating $f_j \amalg f_k$.

COROLLARY 3.14. — *Consider the unobstructed mating $f_j \amalg f_k$. Then for $U$ and $U'$ as in Lemma 3.13, $U$ is adjacent if and only if $U'$ is adjacent.*

*Proof.* — We remark that $f_k$ restricted to the boundary of the critical value component is a homeomorphism and so the proof for $U'$ and $U$ as in the first part of Lemma 3.13 follows.

By Lemma 3.12, the rays of angle $-\theta^d_{j-1}$ and $-\theta^1_{j-1}$ land inside adjacent level-1 sectors. These root points divide the boundary of the critical point component into two Jordan arcs, one of which, $A$ contains the periodic meeting point and no root point of another level-1 component. If $U'$ is adjacent to the periodic meeting point on the critical point component of $f_k$, then its root point must lie on $A$. However, $f_k$ restricted to $A$ is a

homeomorphism. The existence of this homeomorphism (and its inverse) means that $U$ is adjacent if and only if $U'$ is adjacent in the second case of Lemma 3.13. □

We can now rephrase the previous two results for sectors, to get the following immediate corollary.

COROLLARY 3.15. — *Consider the unobstructed mating $f_j \amalg f_k$.*

*(1) Given a primary level-$n$ sector $S$ of $f_k$ which has its root point on the boundary of the critical point component, there exists a unique primary level-$(n+1)$ sector $S'$ which is a preimage of $S$. Moreover $S$ is adjacent if and only if $S'$ is adjacent.*

*(2) Given a primary level-$n$ sector $S$ which has its root point on the boundary of the critical value component, there exists a unique primary level-$(n+1)$ sector $S'$ which is a preimage of $S$ and such that the angles of the rays landing at the root point of $S'$ lie in the arc $(-\theta^d_{j-1}, -\theta^1_{j-1})$. Moreover $S$ is adjacent if and only if $S'$ is adjacent.*

LEMMA 3.16. — *Consider the unobstructed mating $f_j \amalg f_k$. Let $U$ be a primary level-$n$ component which is adjacent to the periodic meeting point on a critical orbit component of $f_k$. Then the conjugate ray to the closest ray of $U$ to the periodic meeting point lands inside a level-$(n+1)$ primary sector $S$ of $f_j$. Furthermore, $S$ is adjacent to the periodic meeting point on the critical orbit component of $f_j$.*

*Proof.* — We proceed by induction. By Lemma 3.12, the conjugate rays to those landing on $\alpha_j$ land inside the two primary level-1 sectors which are adjacent to the periodic meeting point on the Fatou component of $f_k$ which contains the critical point. There are two rays landing at the root points of each of these sectors, since these points belong to $f_k^{-1}(\alpha_k)$. Consider one of the closest rays to the periodic meeting point. The conjugate ray must land inside a level-2 sector of $f_j$ (by Lemma 3.12 and the fact that the preimage of a level-1 sector is a level-2 sector) and this sector must have its root point on the boundary of a preimage of the critical point component of $f_j$. It is clear that this component can only be the critical value component of $f_j$ (Figure 5) since all the level-1 (preperiodic) components of $f_j$ will be separated from this ray by the rays landing at $\alpha_j$, whence this sector is primary. Furthermore, this sector is adjacent by Corollary 3.15.

Now suppose that the statement holds for the two primary level-$n$ components which are adjacent to the periodic meeting point on a critical orbit component of $f_k$ and denote one of them by $U'$. Take the unique (primary)

preimage $U$ of $U'$ that satisfies the conditions of Lemma 3.13 (in the second case, take the preimage for which the rays landing at the root point lie in $(-\theta^d_{j-1}, -\theta^1_{j-1})$). By Corollary 3.14, the preimage $U$ will be an adjacent level-$(n+1)$ component. The conjugate ray to the closest ray landing at the root point of $U$ lands inside a level-$(n+2)$ sector, since it is the pre-image of a level-$(n+1)$ sector. Moreover, by Lemma 3.13 and Corollary 3.15, this sector is adjacent. □

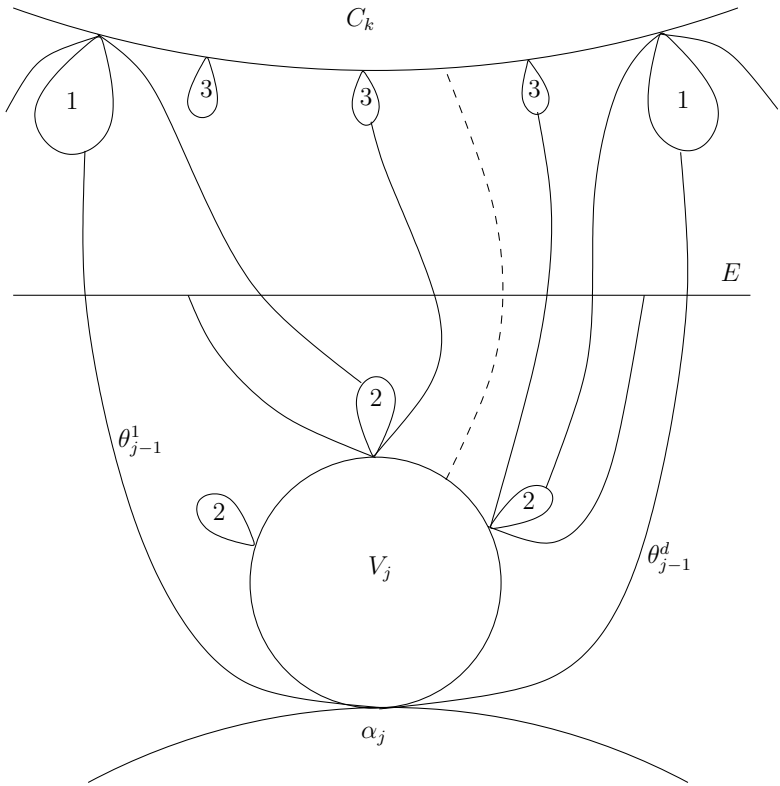



Figure 5. — The diagram for Lemma 3.16. The dotted line shows the position of the periodic (and by Proposition 3.9, unique) meeting point of the critical value component $V_j$ of $f_j$ and the critical point component $C_k$ of $f_k$. The numbers inside the components signify their level and $E$ is the equator. Note that any second meeting point would have to correspond to a ray class contained in the region containing the dotted line.

*Proof of Proposition 3.9.* — Again by symmetry it suffices to prove only the first claim. We begin with some notation. We call the *complete ray* of an external ray $R$ to be the union of the ray $R$ along with its landing point, the conjugate ray and the conjugate ray's landing point. Let $R^1_0$ and

$R_0^2$ be the complete rays landing at $\alpha_k$. By Lemma 3.12, $R_0^1$ and $R_0^2$ have end points lying inside level-1 adjacent sectors of $f_j$. We denote by $R_1^1$ and $R_1^2$ the complete rays of the closest rays which land at the root points of these two level-1 sectors. We inductively define pairs of rays $R_n^1$ and $R_n^2$ as follows. Given $R_{n-1}^1$ and $R_{n-1}^2$, the complete rays landing at the root points of adjacent level-$(n-1)$ sectors, then in light of Lemma 3.16 define $R_n^1$ and $R_n^2$ to be the complete closest rays of the adjacent level-$n$ sectors which include landing points of the complete rays $R_{n-1}^1$ and $R_{n-1}^2$; see Figure 5 for the early steps of this construction. Now denote by $\mathcal{C}_n$ the connected component of $\Sigma \setminus (R_n^1 \cup R_n^2 \cup K(f_j) \cup K(f_k))$ which contains the ray class which collapses to the periodic meeting point $p$.

If a second meeting point exists, it must correspond to a ray class that is contained in $\mathcal{C}_n$ for all $n$. To see this, notice that if for any point $z$ not contained in $\overline{\mathcal{C}}_n$, the union

$$R_{n-1}^1 \cup R_{n-1}^2 \cup R_n^1 \cup R_n^2 \cup K(f_j) \cup K(f_k)$$

separates $z$ from either $V_k$, the critical value component of $f_k$ or $C_j$, the critical point component of $f_j$, whence $z$ cannot belong to a ray class that becomes a second meeting point. Clearly $\overline{\mathcal{C}}_{n+1} \subset \overline{\mathcal{C}}_n$ for all $n$, so we define

$$\mathcal{C} = \bigcap_{i=0}^{\infty} \overline{\mathcal{C}}_n.$$

We claim $\mathcal{C} \cap J(f_k)$ is a singleton $\{p_k\}$, with $p_k$ being the point on the boundary of $V_k$ which becomes the periodic meeting point. Indeed, if $\mathcal{C} \cap J(f_k)$ were not singleton, then it would be an arc in $\partial V_k$ which contains no root points of any primary components. However, there would then exist a neighborhood $U$ of $p_k$ such that $\alpha_k \notin f^{\circ n}(U)$ for all $n$, which is impossible. Hence $\mathcal{C} \cap J(f_k)$ and similarly, $\mathcal{C} \cap J(f_j)$ are singletons and thus there is only one periodic meeting point. $\square$

### 3.3. Structure of the Julia sets of the Rational Maps

We discuss what the above results tell us about the Julia set of the rational map $F$. Since the critical value components meet the critical point components at precisely one point (as shown in Lemma 3.7 and Proposition 3.9), by taking preimages we see that the preimages of the critical point components meet the critical point components at a unique point. Furthermore, the two critical value components and two critical point components don't share boundary points (by Proposition 3.6. Since it must also have rotational symmetry, we see that the Julia set can be described by the schematic as in Figure 6, which shows a period 4 example.

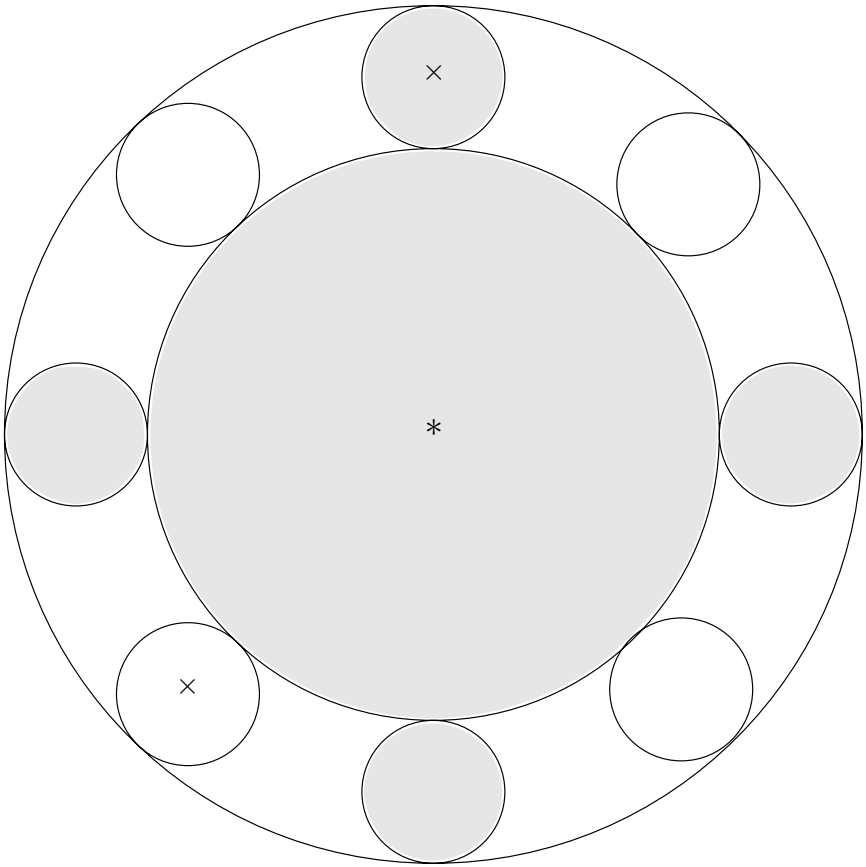



Figure 6. — The schematic for the Julia set of $F \cong f_j \amalg f_k$. The first critical point is labeled with $*$, the second critical point lies outside the larger circle. The two critical values are labeled by $\times$ and here the displacement is 3. The reader is encouraged to compare this schematic to the examples shown in Figure 1.

We now have a general picture of what the Julia set of $F \cong f_j \amalg f_k$ will look like. An intuitive picture is the following: between the two critical point components (which are neighborhoods of the north and south poles at 0 and $\infty$) there is a "belt" which contains all the preimages of the critical point components, including the critical value components themselves. Each of these preimage components meets each of the critical point components in precisely one place. We will refer to all maps who have a Julia set with this structure (whether or not they are matings) as type $\mathcal{A}$ maps.

The important piece of data that we require when studying type $\mathcal{A}$ maps will be the displacement $\delta$, which will measure the combinatorial distance

between the critical values round the "belt" which forms the space between the two critical point components of the rational map. Let the two critical points be $c_1$ and $c_2$ respectively, with respective critical values $v_1$ and $v_2$. Form the star which made up of the internal rays in the critical point component of $c_1$ which go from $c_1$ to the boundary point where the critical point meets one of the preimages of a critical point component. On the end points of the star thus formed, union the internal rays inside the preimage components which go from the boundary point to the preimage of a critical point contained in the component. Now label the arms which end with a point in the set $F^{-1}(c_2)$ by $\ell_1, \ell_2, \ldots, \ell_d$ in clockwise order, where $\ell_1$ is the arm which is immediately clockwise of the (unlabeled) arm whose endpoint is $v_1$.

Definition 3.17. — *Let $F$ be a rational map of type $\mathcal{A}$. Then the displacement of $F$ will be $\delta$, where $\ell_\delta$ is the arm of the star which has the critical value of the second critical point as an endpoint.*

See Figure 6 for an example where the displacement is 3. We note that if $F$ has displacement $\delta$ then the map with opposite critical marking $F$ also has displacement $\delta$. We will show that if we know the displacement of a rational map, we actually know the rational map up to Möbius conjugacy. Clearly, the displacement is well-defined; the cyclic ordering of the critical value components around the boundary of the critical point components is maintained under Möbius conjugacy. We now show that the two critical values can not be adjacent in the cyclic ordering.

Proposition 3.18. — *The displacements $\delta = 1$ and $\delta = d$ are not realized for rational maps.*

We will show that the displacements $\delta = 1$ and $\delta = d$ allow the existence of a Levy cycle. Suppose $F$ is a rational map where the two critical values are adjacent. Denote the Fatou component containing $c_1$ by $C_1$. The component $C_1$ shares a common fixed boundary point $\zeta$ with the Fatou component containing $v_1 = F(c_1)$ and has a common period two boundary point $\omega$ with the Fatou component containing $v_2 = F(c_2)$. Let $\lambda_1$ be the internal ray from $v_1$ towards $\zeta$ with the Fatou component $C_1$ containing $c_1$ but perturbed slightly so it does not pass through $\zeta$ and ends at a point on $\partial C_1$ near $\zeta$ so that it lies on the arc of $\partial C_1 \setminus \{\zeta, \omega\}$ which contains no boundary points of components which are preimages of the Fatou components containing $c_1$ or $c_2$. Let $\lambda_3$ be the internal ray from $v_2$ to $\omega$ and $\lambda_2$ the arc in $\partial C_1$ connecting $\lambda_1$ and $\lambda_3$ such that $\lambda_2$ does not contain any boundary points of pre-images of the two critical point components (the existence of $\lambda_2$ is guaranteed by the adjacency of the critical value components in the cyclic ordering), see

the part (a) of Figure 7. Define $\Lambda = \cup_{i=1}^{3} \lambda_i$. The set $F^{-1}(\Lambda)$ is made up of the union of $d = \deg F$ paths from $c_1$ to $c_2$ which go from $c_1$ along an internal ray towards a preimage of $\zeta$, then pass around this preimage to a point on the boundary of the a Fatou component containing an element of $F^{-1}(c_1)$. It then travels along the boundary of this component to the element in $F^{-1}(\omega)$ which is also on the boundary. It then travels along an internal ray to $c_2$, see part (b) of Figure 7 for an example of one of these preimages. It is clear that $\Lambda$ and the set $F^{-1}(\Lambda)$ have an empty intersection.

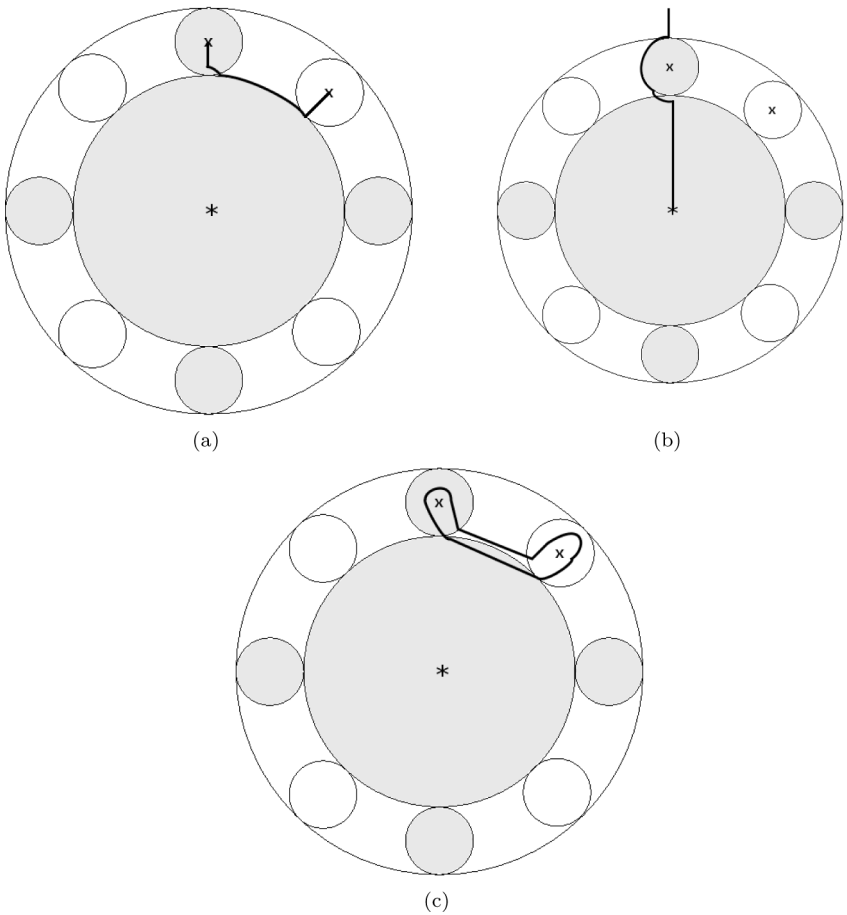



Figure 7. — The curve $\Lambda$ is shown in the bold line in (a), a branch of its preimage in (b) and the curve $\gamma$ (which is a Levy cycle) is shown in (c). Note that the curve in (b) continues to the second critical point, which lies off the diagram.

*Proof of Proposition 3.18.* — We use the $\Lambda$ from the previous paragraph to help us construct a multicurve $\Gamma$ which we will show is a Levy cycle. Let $\gamma$ be the boundary of an $\epsilon$-neighborhood of $\Lambda$ so that the component of $\widehat{\mathbb{C}} \setminus \gamma$ containing the two critical values does not contain $\zeta$, nor either of the critical points, see part (c) of Figure 7. By considering the details of $F^{-1}(\Lambda)$, we then see that $F^{-1}(\gamma)$ is made up of $d-1$ peripheral curves and one curve $\gamma'$ which is homotopic to $\gamma$ relative to $P_F$ (but with the orientation reversed). We then see that $F_{|\gamma'}: \gamma' \to \gamma$ is a homeomorphism and so $\Gamma = \{\gamma\}$ is a good Levy cycle. □

In the following proposition, we use $d-1$ as the representative of the integers equal to 0 modulo $d-1$.

Proposition 3.19. — *The degree $d$ rational map $F \cong f_j \amalg f_k$ has displacement $\delta \equiv j + k(d-1)$.*

*Proof.* — The displacement is equal to the number of preimage components of the critical point component of $f_j$ between the critical value component of $f_j$ and the critical value component of $f_k$ (inclusive of the first component). By the result of Lemma 3.3, this is equal to the number of angles of the form $\hat{\beta}_r$ between $\hat{\beta}_{j-1}$ (inclusive) and $-\hat{\beta}_k = \hat{\beta}_{d-k-1}$. This value is clearly $j+k$. □

*Remark 3.20.* — Notice that, using the convention above, the two non-realizable cases $\delta = 1$ and $\delta = d$ both correspond to $j + k \equiv 1d - 1$.

Corollary 3.21. — *Each displacement can be obtained by matings in precisely $d-1$ ways.*

*Proof.* — Suppose we want to get the displacement to be $\delta$. Given any $f_j$, the rational map $F \cong f_j \amalg f_{\delta - j}$ has displacement $\delta$. The map $f_{\delta-j}$ exists and is unique for each $j$, and since there are $d-1$ choices for $j$, the result follows. □

*Proof of Theorem A.* — By the algebraic classification in Section 2 there are exactly $d-2$ degree $d$ rational maps (up to Möbius conjugacy) with two period two superattracting cycles. Furthermore, by Proposition 3.19, we can obtain $d-2$ different displacements, each of which must correspond to a different rational map and by Corollary 3.21 each displacement is obtained in precisely $d-1$ ways. □

## 4. Thurston Classification

We now show that the critical displacement is enough to classify a rational map in the sense of Thurston. The advantage of this approach is that,

in general, "clustering" (as defined in [9, 8, 10]) is not an algebraic condition, and therefore the approach used above, by using the algebraic normal form of Section 2, will not work in the general case where the period of the critical orbits is greater than 2. In this section we will show how the result of Theorem A can be obtained without reference to this algebraic classification.

THEOREM 4.1. — *Suppose $F$ and $G$ are bicritical rational maps of type $\mathcal{A}$ and have the same displacement $\delta$. Then $F$ and $G$ are equivalent in the sense of Thurston.*

Recall that to satisfy the conditions of Thurston equivalence, we need to find two homeomorphisms $\Phi$ and $\widehat{\Phi}$ with the following properties:

(1) $\Phi \circ F = G \circ \widehat{\Phi}$.

(2) $\Phi|_{P_F} = \widehat{\Phi}|_{P_F}$.

(3) $\Phi$ and $\widehat{\Phi}$ are isotopic rel $P_F$.

We will first of all construct the homeomorphism $\Phi$, making use of the Alexander trick. We will then define the homeomorphism $\widehat{\Phi}$, making sure that each condition above in turn is satisfied. Since we are considering maps of type $\mathcal{A}$, we can take advantage of what we know about the Julia sets of these maps. Denote the critical points of $F$ by $c_1$ and $c_2$, with values $v_1$ and $v_2$, and the critical points of $G$ by $c_1'$ and $c_2'$, with values $v_1'$ and $v_2'$. Construct a closed curve $\gamma$ in the $F$-sphere as follows. Let $\gamma_1$ be the path through internal rays from $c_1$ to $v_1$, $\gamma_2$ the path through internal rays from $v_1$ to $c_2$, $\gamma_3$ the path through internal rays from $c_2$ to $v_2$ and $\gamma_4$ the path through internal rays from $v_2$ to $c_1$. Define the curve $\gamma'$ in the $G$-sphere in the analogous way.

We observe that $F\colon \gamma \to \gamma$ and $G\colon \gamma' \to \gamma'$ are homeomorphisms. It is easy to see that there exists $\phi\colon \gamma \to \gamma'$ which conjugates the dynamics of $F$ and $G$ on $\gamma$ and $\gamma'$ respectively. Then, by two applications of the Alexander trick, we can extend $\phi$ to a homeomorphism $\Phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Notice that the set $F^{-1}(\gamma)$ divides the $F$-sphere up into $2d$ regions and similarly, the set $G^{-1}(\gamma')$ divides the $G$-sphere up into $2d$ regions. Label the regions in the $F$-sphere $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{2d}$, starting with the region immediately anticlockwise of the curve $\gamma$ at $c_1$. Similarly label the regions in the $G$-sphere by $\mathcal{A}_1', \mathcal{A}_2', \ldots, \mathcal{A}_{2d}'$.

LEMMA 4.2. — *There exists a homeomorphism $\widehat{\Phi}$ so that $\Phi \circ F = G \circ \widehat{\Phi}$ and $\Phi|_\gamma = \widehat{\Phi}|_\gamma$.*

*Proof.* — We now use this labeling to define the second homeomorphism $\widehat{\Phi}$. To satisfy the Thurston criterion, we require $\Phi \circ F = G \circ \widehat{\Phi}$, so that for each $z$ we have $\widehat{\Phi}(z) \in G^{-1}(\Phi(F(z)))$. If $z \in \mathcal{A}_i$, then let $\widehat{\Phi}(z)$ be the unique point in $G^{-1}(\Phi(F(z))) \cap \mathcal{A}'_i$. We can extend the map $\widehat{\Phi}|_{\mathcal{A}_i}$ continuously to the boundary for each $i$, and in particular, since the displacements of $F$ and $G$ are the same, we have $\Phi|_\gamma = \widehat{\Phi}|_\gamma$. Finally, $\widehat{\Phi}$ is a homeomorphism since it is continuous and the sphere is a compact, Hausdorff space. □

LEMMA 4.3. — *$\widehat{\Phi}$ is isotopic to $\Phi$ rel $\gamma$.*

*Proof.* — We know that $\Phi|_\gamma = \phi = \widehat{\Phi}|_{\gamma'}$. The complement to $\gamma$ is two topological discs and so by the Alexander trick, $\Phi$ and $\widehat{\Phi}$ are isotopic to each other on both of these discs rel the boundary. Hence they are isotopic rel $\gamma$. □

*Proof of Theorem 4.1.* — By Lemma 4.2, we see that $\Phi \circ F = G \circ \widehat{\Phi}$ and $\Phi|_\gamma = \widehat{\Phi}|_\gamma$, and hence $\Phi|_{P_F} = \widehat{\Phi}|_{P_F}$. Then, by Lemma 4.3, the homeomorphisms $\Phi$ and $\widehat{\Phi}$ are isotopic rel $\gamma$ and hence rel $P_F$. Thus $F$ and $G$ are equivalent.

## Bibliography


[1] DOUADY (A.) and HUBBARD (J. H.). — A proof of Thurston's topological characterization of rational functions, Acta. Math. 171, p. 263-297 (1993).

[2] GANTMACHER (F. R.). — The Theory of Matrices, Chelsea (1959).

[3] MILNOR (J.). — Periodic orbits, external rays and the Mandelbrot set: An expository account, Astérisque 261, p. 277-333 (2000).

[4] MILNOR (J.). — Pasting together Julia sets: A worked out example of mating, Experimental Math. 13, p. 55-92 (2004).

[5] MILNOR (J.). — Dynamics in One Complex Variable, 3rd ed., Princeton University Press (2006).

[6] SCHLEICHER (D.). — On fibers and local connectivity of Mandelbrot and multibrot sets, Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Amer. Math. Soc., p. 477-517 (2004).

[7] SENETA (E.). — Nonnegative Matrices and Markov Chains, 3rd ed., Springer (1981).

[8] SHARLAND (T.). — Rational maps with clustering and the mating of polynomials, Ph.D. thesis, University of Warwick, 2011, Can be found at http://wrap.warwick.ac.uk/35776/.

[9] SHARLAND (T.). — Constructing rational maps with cluster points using the mating operation, J. Lond. Math. Soc., to appear (2012).

[10] SHARLAND (T.). — Thurston equivalence for rational maps with clusters, Ergodic Th. Dyn. Sys., to appear (2012).

[11] Shishikura (M.) and Lei (T.). — A family of cubic rational maps and matings of cubic polynomials, Experimental Math. 9, p. 29-53 (2000).

[12] Tan (Lei). — Accouplements des polynomes complexes, Ph. D. thesis, Université de Paris-Sud, Orsay (1987).

[13] Tan (Lei). — Matings of quadratic polynomials, Ergodic Th. Dyn. Sys. 12, p. 589-620 (1992).