ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

GUIZHEN CUI, WENJUAN PENG, LEI TAN

On a theorem of Rees-Shishikura

Tome XXI, nº S5 (2012), p. 981-993.

<http://afst.cedram.org/item?id=AFST_2012_6_21_S5_981_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXI, n° 5, 2012
pp. 981–993

On a theorem of Rees-Shishikura

Guizhen Cui[(1)], Wenjuan Peng[(2)] and Lei Tan[(3)]

Abstract. — Rees-Shishikura's theorem plays an important role in the study of matings of polynomials. It promotes Thurston's combinatorial equivalence into a semi-conjugacy. In this work we restate and reprove Rees-Shishikura's theorem in a more general form, which can then be applied to a wider class of postcritically finite branched coverings. We provide an application of the restated theorem.

Résumé. — Le théorème de Rees-Shishikura joue un rôle important dans l'étude des accouplements de polynômes. Il permet d'obtenir une semi-conjugaison à partir d'une equivalence combinatoire de Thurston. Dans ce travail, nous reformulons et redémontrons ce théorème dans un cadre plus général. Cette nouvelle version du théorème est applicable à une classe plus large de revêtements ramifiés postcritiquement finis. Nous en fournissons un exemple à la fin de notre article.

1. Introduction

Consider the mating of two polynomials (refer to [4, 10, 11, 12] for the definitions of mating). M. Rees and M. Shishikura [10, 11] proved that if the formal mating of two postcritically finite polynomials is Thurston equivalent to a rational map, then the topological mating is conjugate to the rational map. The main step of the proof is to show the existence of a semi-conjugacy

(1) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China.
gzcui@math.ac.cn
(2) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China.
wenjpeng@amss.ac.cn
(3) Département de Mathématiques Université d'Angers, Angers, 49045 France.
tanlei@math.univ-angers.fr

from the formal mating to the rational map (refer to Theorem 2.1 in [11] and the theorem below).

Theorem A. — *Suppose that the degenerate mating $F' = (f_1 \perp\!\!\!\perp f_2)'$ of polynomials f_1 and f_2 is Thurston equivalent to a rational map R mapping from the Riemann sphere $\widehat{\mathbb{C}}$ onto itself. Then there exists a continuous map $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, satisfying that*
(i) the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{F} & \widehat{\mathbb{C}} \\ h\downarrow & & \downarrow h \\ \widehat{\mathbb{C}} & \xrightarrow{R} & \widehat{\mathbb{C}}, \end{array}$$

where $F = f_1 \perp\!\!\!\perp f_2$ is the formal mating;
(ii) h is a uniform limit of orientation preserving homeomorphisms;
(iii) h is conformal in $\mathrm{int}K_{f_1} \sqcup \mathrm{int}K_{f_2}$ onto $\widehat{\mathbb{C}} \smallsetminus J_R$ and $h^{-1}(\widehat{\mathbb{C}} \smallsetminus J_R) = \mathrm{int}K_{f_1} \sqcup \mathrm{int}K_{f_2}$, where $\mathrm{int}K_{f_i}$ are the interior of the filled-in Julia sets of f_i for $i = 1, 2$ and J_R is the Julia set of R.

M. Rees ([10]) proved that there exists a semi-conjugacy from a general postcritically finite branched covering to a rational map if it is Thurston equivalent to the rational map by a pair of homeomorphisms (ϕ_0, ϕ_1) and $\phi_0 = \phi_1$ near the critical cycles. In fact, the pull-back sequence $\{\phi_n\}$ (see the definition below) of the Thurston equivalence converges uniformly to the semi-conjugacy.

In the proof of Theorem A, under the property that the degenerate mating F' is holomorphic in a neighborhood of the critical cycles, M. Shishikura modified the original Thurston equivalence (θ_0, θ_1) so that $\theta_0 = \theta_1$ near the critical cycles by using Dehn twist near those points.

In this note, we will show that if the Thurston equivalence (ϕ_0, ϕ_1) satisfies that ϕ_0 is a local conjugacy near the critical cycles, then the pull-back sequence $\{\phi_n\}$ of the Thurston equivalence converges uniformly to the semi-conjugacy. Under the assumption that a postcritically finite branched covering is Thurston equivalent to a rational map, when the branched covering is holomorphic in a neighborhood of the critical cycles, then it is easy to show that there exists a Thurston equivalence (ϕ_0, ϕ_1) such that ϕ_0 is a local conjugacy near the critical cycles. Note that in this case ϕ_0 needs not coincide with ϕ_1 near the critical cycles and we do not need Dehn twist as constructed in [11].

Statements: Let F be a branched covering of the Riemann sphere $\widehat{\mathbb{C}}$. We always assume $\deg F \geqslant 2$ in this paper. Denote by Ω_F the set of critical points of F. The *postcritical set* of F is defined by

$$\mathcal{P}_F = \overline{\bigcup_{n\geqslant 0} F^n(\Omega_F)}.$$

The map F is called *postcritically finite* if $\mathcal{P}_F$ is a finite set. Let f be a rational map. We denote by $\mathcal{F}_f$ and $\mathcal{J}_f$ the Fatou set and Julia set of f respectively.

Two postcritically finite branched coverings F and G are called *Thurston equivalent* through a pair of orientation preserving homeomorphisms $(\phi_0, \phi_1) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if ϕ_1 is isotopic to ϕ_0 rel $\mathcal{P}_F$ and $\phi_0 \circ F \circ \phi_1^{-1} = G$. The *pull-back* sequence $\{\phi_n\}_{n\geqslant 1}$ of the Thurston equivalence means that $\{\phi_n\}$ is a sequence of homeomorphisms of $\widehat{\mathbb{C}}$ such that ϕ_{n+1} is isotopic to ϕ_n rel $\mathcal{P}_F$ and $\phi_n \circ F = G \circ \phi_{n+1}$.

A *continuum* is a connected compact subset of $\widehat{\mathbb{C}}$.

Theorem 1.1. — *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite branched covering. Suppose that F is Thurston equivalent to a rational map f through a pair of homeomorphisms (ϕ_0, ϕ_1) such that $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F. Let $\{\phi_n\}$ $(n \geqslant 1)$ be a sequence of homeomorphisms of $\widehat{\mathbb{C}}$ such that $\phi_n \circ F = f \circ \phi_{n+1}$ and ϕ_{n+1} is isotopic to ϕ_n rel $\mathcal{P}_F$. Then $\{\phi_n\}$ converges uniformly to a continuous onto map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as $n \to \infty$. Moreover,*
(1) $h \circ F = f \circ h$.
(2) $h^{-1}(w)$ is a single point for $w \in \mathcal{F}_f$ and a full continuum for $w \in \mathcal{J}_f$.
(3) For points $x, y \in \widehat{\mathbb{C}}$ with $f(x) = y$, $h^{-1}(x)$ is a connected component of $F^{-1}(h^{-1}(y))$ and $F(h^{-1}(x)) = h^{-1}(y)$. Moreover, the degree of the map $F : h^{-1}(x) \to h^{-1}(y)$ is equal to $\deg_x f$; precisely speaking, for any $w \in h^{-1}(y)$,

$$\sum_{z\in F^{-1}(w)\cap h^{-1}(x)} \deg_z F = \deg_x f,$$

where $\deg_x f, \deg_z F$ are the local degrees of f, F at x, z respectively.
(4) $h^{-1}(E)$ is a continuum if $E \subset \widehat{\mathbb{C}}$ is a continuum.
(5) $h(F^{-1}(E)) = f^{-1}(h(E))$ for any $E \subset \widehat{\mathbb{C}}$.
(6) $F^{-1}(\widehat{E}) = \widehat{F^{-1}(E)}$ for any $E \subset \widehat{\mathbb{C}}$, where $\widehat{E} = h^{-1}(h(E))$.

Corollary 1.2. — *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite branched covering which is holomorphic in a neighborhood of the critical cycles. Suppose that F is Thurston equivalent to a rational map f through a pair of* *homeomorphisms* (ϕ_0, ϕ_1). *Then there exists a semi-conjugacy* h *from* F *to* f *in the homotopy class of* ϕ_0 *such that it satisfies the above conditions (1)-(6).*

As in [10, 11], the main idea of the proof is that the rational map f is expanding under the orbifold metric. The only new observation is that the homotopic length of the isotopy for any point is bounded if $\phi_0 \circ F = f \circ \phi_0$ near critical cycles.

Points (4)-(6) are also new but they are not difficult to prove. They are applied in our work [3].

2. Homotopic length of the isotopy

In this section we assume that the reader is familiar with the theory of orbifolds.

Let f be a postcritically finite rational map of $\widehat{\mathbb{C}}$. Denote by $\rho(z)|dz|$ the orbifold metric of f ([5]). Then $||f'|| > 1$ on $\widehat{\mathbb{C}}\backslash\mathcal{P}_f$ with respect to the orbifold metric $\rho(z)|dz|$, and on any compact subset $E \subset \widehat{\mathbb{C}}\backslash\mathcal{P}_f$, there is a constant $\lambda > 1$ such that $||f'|| > \lambda$. Define the *homotopic length* of a path $\alpha : [0, 1] \to \widehat{\mathbb{C}}\backslash\mathcal{P}_f$ by

$$\text{h-length}(\alpha) = \inf\{\text{length of } \alpha' \text{ with metric } \rho\},$$

where the infimum is taken over all the paths α' from $\alpha(0)$ to $\alpha(1)$ and homotopic to α in $\widehat{\mathbb{C}}\backslash\mathcal{P}_f$.

Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite branched covering. Suppose that F is Thurston equivalent to a rational map f via a pair of homeomorphisms (ϕ_0, ϕ_1), i.e., $\phi_0 \circ F = f \circ \phi_1$, and ϕ_1 is isotopic to ϕ_0 rel $\mathcal{P}_F$, that is, there is a continuous map $H_0 : \widehat{\mathbb{C}} \times [0, 1] \to \widehat{\mathbb{C}}$ such that $H_0(\cdot, 0) = \phi_0, H_0(\cdot, 1) = \phi_1$, $H_0(\cdot, t)$ is a homeomorphism for any $t \in (0, 1)$ and $H_0(z, t) = \phi_0(z)$ for $z \in \mathcal{P}_F, t \in [0, 1]$.

Lemma 2.1. — *If* $\phi_0 \circ F = f \circ \phi_0$ *in a neighborhood of the critical cycles of* F, *then the homotopic length of* $\{H_0(z, t), 0 \leqslant t \leqslant 1\}$ *is bounded by a constant* $M < \infty$ *for any point* $z \in \widehat{\mathbb{C}}\backslash\mathcal{P}_F$.

Proof. — We only need to show that the homotopic length of $\gamma := \{H_0(z, t), 0 \leqslant t \leqslant 1\}$ is bounded in a neighborhood of each critical cycle of f. Let x be a point in a critical cycle of f. Define the winding angle of the path γ around the point x by:

$$w_x(\gamma) = \frac{1}{2\pi i}\int_{\zeta\in B(\gamma)} \frac{d\zeta}{\zeta},$$

where B is the Böttcher map and ζ is Böttcher's coordinate of f at the point x. It is continuous. On the other hand, since $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F, we have $\phi_1 \circ \phi_0^{-1}$ is a rotation in Böttcher's coordinates of f at the point x, with angles $2k\pi/d$, where k is an integer and $d = \deg_x f$. Thus $w_x(\gamma) \equiv k/d \pmod 1$. It follows that $w_x(\gamma)$ is a constant in a neighborhood of x. This implies that the homotopic length of γ is bounded in a neighborhood of the point x. $\square$

LEMMA 2.2. — *If $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F, then the pull-back sequence $\{\phi_n\}$ converges uniformly to a continuous onto map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as $n \to \infty$.*

Proof. — By lifting the map H_0, for each $n \geqslant 1$, we get a continuous map $H_n : \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$ satisfying that $H_n(\cdot, t)$ is a homeomorphism for any $t \in [0,1]$, $H_n(\cdot, 0) = \phi_n$, $H_n(\cdot, 1) = \phi_{n+1}$, $H_n(z,t) = \phi_n(z)$ for $z \in \mathcal{P}_F, t \in [0,1]$ and $H_n(F(z),t) = f(H_{n+1}(z,t))$ for $z \in \widehat{\mathbb{C}}, t \in [0,1]$.

Let U be an open set containing critical cycles of F such that $\phi_0 \circ F = f \circ \phi_0$ in U, $F(\overline{U}) \subset U$ and every component of U contains exactly one point in the critical cycles of F.

CLAIM. — *For each $n \geqslant 1$, $\phi_n \circ \phi_0^{-1}$ is a rotation in Böttcher coordinates of the critical cycles of f.*

Proof. — Let x be a point in a critical cycle of f. By Böttcher's Theorem, there is a Jordan domain $U_x \subset \phi_0(U)$, $x \in U_x$ and a conformal map $u_x : U_x \to D_x = \{z \in \mathbb{C} : |z| < r_x < 1\}$ such that $f(U_x)$ is compactly contained in $U_{f(x)}$ (denote by $f(U_x) \subset\subset U_{f(x)}$), $u_x(x) = 0$ and

$$u_{f(x)} \circ F \circ u_x^{-1}(z) = z^{d_x},$$

where $d_x = \deg_x f$. In fact u_x is the Böttcher's coordinate of f at the cycle through the point x.

Fix $n \geqslant 1$. We may assume that $f^n(U_x) \subset\subset U_{f^n(x)}$ and $\phi_n\phi_0^{-1}(U_x) \subset\subset U_x$. Since $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F and $\phi_0 \circ F^n = f^n \circ \phi_n$ on $\widehat{\mathbb{C}}$, we have the following commutative diagrams.

$$\begin{array}{ccccccccc}
D_x & \xleftarrow{u_x} & \phi_n(\phi_0^{-1}(U_x)) & \xleftarrow{\phi_n} & \phi_0^{-1}(U_x) & \xrightarrow{\phi_0} & U_x & \xrightarrow{u_x} & D_x \\
P\downarrow & & f^n\downarrow & & F^n\downarrow & & f^n\downarrow & & \downarrow P \\
D_{f^n(x)} & \xleftarrow{u_{f^n(x)}} & U_{f^n(x)} & \xleftarrow{\phi_0} & \phi_0^{-1}(U_{f^n(x)}) & \xrightarrow{\phi_0} & U_{f^n(x)} & \xrightarrow{u_{f^n(x)}} & D_{f^n(x)},
\end{array}$$

where $P(z) = z^{d_x d_{f(x)} \cdots d_{f^n(x)}}$. It follows easily that $\phi_n \circ \phi_0^{-1}$ is a rotation in Böttcher coordinates of the critical cycles of f. $\square$

By the claim, we may take a compact subset $E \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ such that $\widehat{\mathbb{C}} \backslash \phi_n(U) \subset E$ for all $n \geqslant 0$. Then there exists a constant $\lambda > 1$ such that $||f'|| > \lambda$ on E. Let $d(\cdot, \cdot)$ denote the spherical metric of $\widehat{\mathbb{C}}$.

Fix $n \geqslant 1$.

If $z \in \widehat{\mathbb{C}} \backslash F^{-n}(U \cup \mathcal{P}_F)$, then the path $\{H_n(z,t), 0 \leqslant t \leqslant 1\} \subset \widehat{\mathbb{C}} \backslash (\phi_n(F^{-n}(U)) \cup \mathcal{P}_f) \subset \widehat{\mathbb{C}} \backslash (\phi_n(U) \cup \mathcal{P}_f) \subset E$. Thus $F(z) \in \widehat{\mathbb{C}} \backslash F^{-(n-1)}(U \cup \mathcal{P}_F)$ and

$$\begin{aligned} \text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \frac{1}{\lambda} \text{h-length}\big(f(\{H_n(z,t), 0 \leqslant t \leqslant 1\})\big) \\ &= \frac{1}{\lambda} \text{h-length}(\{H_{n-1}(F(z),t), 0 \leqslant t \leqslant 1\}). \end{aligned}$$

Note that by Lemma 2.1, for all $z \in \widehat{\mathbb{C}} \backslash \mathcal{P}_F$,

$$\text{h-length}(\{H_0(z,t), 0 \leqslant t \leqslant 1\}) \leqslant M.$$

Hence for $z \in \widehat{\mathbb{C}} \backslash F^{-n}(U \cup \mathcal{P}_F)$,

$$\begin{aligned} d(\phi_n(z), \phi_{n+1}(z)) &= d(H_n(z,0), H_n(z,1)) \\ &\leqslant \text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) \\ &\leqslant \frac{1}{\lambda^n} \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) \\ &\leqslant M\lambda^{-n}. \end{aligned}$$

If $z \in F^{-n}(\mathcal{P}_F)$, then it follows from the relation $H_n(F(z),t) = f(H_{n+1}(z,t))$ that $d(\phi_n(z), \phi_{n+1}(z)) = 0$.

If $z \in F^{-n}(U) \backslash F^{-n}(\mathcal{P}_F)$, then

$$f^n(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) = \{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}$$

and $F^n(z) \in U \backslash \mathcal{P}_F$. Let p be the least common multiple of the periods of all critical cycles of F, l be the minimal of $\frac{p}{p'}$, where p' is the period of a critical cycle of F, and D be the minimal of the product of local degrees of all critical points in C, where C is a critical cycle of F.

We may assume $n \geqslant p$. If $z, F(z), \cdots, F^n(z) \in U$, then there is a critical cycle of F such that $F^m(z) \in U_0$, $\forall m \geqslant 0$, where U_0 is the union of components of U containing that cycle. Let p_0 be the period of that cycle, $l_0 := \frac{p}{p_0}$, D_0 be the product of the local degrees of all critical points in that cycle.

First we consider the case that $p_0 = 1$, that is U_0 contains a critical fixed point q and $D_0 = \deg_q F$. Since $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F, the point $\phi_0(q)$ is a critical fixed point of f and $\deg_{\phi_0(q)} f = \deg_q F$. Let B be the Böttcher map f at the point $\phi_0(q)$ and we define $w_{\phi_0(q)}(\{H_m(\alpha,t), 0 \leqslant t \leqslant 1\})$ as in Lemma 2.1 for all $0 \leqslant m \leqslant n$ and $\alpha \in U_0$. Fix $0 \leqslant m \leqslant n-1$. Set $\gamma_{m+1} := \{H_{m+1}(z,t), 0 \leqslant t \leqslant 1\}$ and $\gamma_m := \{H_m(F(z),t), 0 \leqslant t \leqslant 1\}$. Then

$$w_{\phi_0(q)}(\gamma_{m+1}) = \frac{1}{2\pi i}\int_{\xi \in B(\gamma_{m+1})} \frac{d\xi}{\xi}$$

and

$$w_{\phi_0(q)}(\gamma_m) = \frac{1}{2\pi i}\int_{\eta \in B(\gamma_m)} \frac{d\eta}{\eta},$$

where $\eta = \xi^{D_0}$. An easy calculation shows that

$$w_{\phi_0(q)}(\gamma_m) = D_0 \cdot w_{\phi_0(q)}(\gamma_{m+1}).$$

This implies that

$$\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) \leqslant \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) D_0^{-n}.$$

For the general case, the assumption $n \geqslant p$ implies that there is an integer $k \geqslant 1$ such that $kl_0p_0 \leqslant n \leqslant (k+1)l_0p_0$. Then

$$\begin{aligned}\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) D_0^{-(l_0k)} \\ &\leqslant MD^{-(lk)},\end{aligned}$$

where M is the constant obtained as in Lemma 2.1. Note that as $n \to \infty$, k tends to infinity linearly with l, in particular the bound $MD^{-(lk)}$ has a finite sum over n.

Now we suppose $z \notin U, F(z) \notin U, \cdots, F^{i-1}(z) \notin U, F^i(z) \in U, \cdots, F^n(z) \in U$ for some $i \geqslant 1$. Then similarly to the previous case, there is a critical cycle of F such that $F^m(z) \in U_1$, $\forall m \geqslant n$, where U_1 is the union of components of U containing that cycle. Let p_1 be the period of that cycle, $p = l_1p_1$, D_1 be the product of the local degrees of all critical points in that cycle.

If $n - i < p = l_1p_1$, then there is some integer $0 \leqslant j \leqslant l_1 - 1$, such that $jp_1 \leqslant n - i \leqslant (j+1)p_1$ and

$$\begin{aligned}\text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) D_1^{-j} \\ &\leqslant M.\end{aligned}$$

Thus

$$\begin{aligned}\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\})\lambda^{-i}\\ &\leqslant M\lambda^{-i}.\end{aligned}$$

Noticing that $n - i < p$, we have as $n \to \infty$, the bound $M\lambda^{-i}$ has a finite sum over n.

Otherwise, there is some $s \geqslant 1$ such that $sp \leqslant n - i \leqslant (s+1)p$. Then

$$\text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\}) \leqslant MD_1^{-(l_1 s)} \leqslant MD^{-(ls)}$$

So

$$\begin{aligned}\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\})\lambda^{-i}\\ &\leqslant M\lambda^{-i}D^{-(ls)}.\end{aligned}$$

As $n \to \infty$, either i or s tends to infinity.

Combining the conclusions of the above paragraphs together, we get the uniform convergence of ϕ_n with respect to the spherical metric of $\widehat{\mathbb{C}}$. The continuity and surjectivity of h follow directly from the property that it is a uniform limit of a sequence of homeomorphisms. □

Proof of Corollary 1.2. — By Böttcher's theorem, we may modify the Thurston equivalence (ϕ_0, ϕ_1) such that $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F. Now it follows by Theorem 1.1. □

3. Quotient maps

Let $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous onto map. We call it a *quotient map* if $h^{-1}(y)$ is a full continuum for any point $y \in \widehat{\mathbb{C}}$, i.e. $\widehat{\mathbb{C}} \backslash h^{-1}(y)$ is a simply connected domain.

LEMMA 3.1. — *Let $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous onto map. Then the following conditions are equivalent.*
(a) The map h is a quotient map.
(b) $h^{-1}(E)$ is a continuum if $E \subset \widehat{\mathbb{C}}$ is a continuum.
(c) $h^{-1}(E)$ is a full continuum if $E \subset \widehat{\mathbb{C}}$ is a full continuum.
(d) There exists a sequence of homeomorphisms $h_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\{h_n\}$ converges uniformly to h.

There is a similar statement in [8], see Lemma 2.3 and Theorem 2.12 in [8]. In the following, we will first prove (a), (b) and (c) are equivalent and then prove $(d) \Rightarrow (b)$. For $(a) \Rightarrow (d)$, the reader may refer to [8] for its proof. In the proof of Theorem 1.1, we will not use (a), (b) or $(c) \Rightarrow (d)$, but $(d) \Rightarrow (a)$, (b) and (c).

Proof of Lemma 3.1. — $(a) \Rightarrow (b)$. Let $E \subset \widehat{\mathbb{C}}$ be a continuum. If $h^{-1}(E)$ is not connected, then there are two disjoint open sets U and V in $\widehat{\mathbb{C}}$ such that $h^{-1}(E) \subset U \cup V$ and both $K_1 = U \cap h^{-1}(E)$ and $K_2 = V \cap h^{-1}(E)$ are not empty. Note that both K_1 and K_2 are closed since $h^{-1}(E)$ is closed. Thus both $h(K_1)$ and $h(K_2)$ are closed. On the other hand, $h(K_1)$ and $h(K_2)$ are disjoint by (a). This contradicts the condition that E is connected.

$(b) \Rightarrow (c)$. We only need to show that $h^{-1}(E)$ is full. Otherwise, $\widehat{\mathbb{C}}\backslash h^{-1}(E)$ is disconnected. Thus there are two distinct points $x, y \in \widehat{\mathbb{C}}\backslash h^{-1}(E)$ such that they are contained in different domains in $\widehat{\mathbb{C}}\backslash h^{-1}(E)$. Since $h(x), h(y) \in \widehat{\mathbb{C}}\backslash E$ and E is full, there exists an arc $\alpha \subset \widehat{\mathbb{C}}\backslash E$ which connects $h(x)$ with $h(y)$. Thus $h^{-1}(\alpha) \subset \widehat{\mathbb{C}}\backslash h^{-1}(E)$ is a continuum which contains x with y. This is a contradiction.

$(c) \Rightarrow (a)$. This is obvious.

$(d) \Rightarrow (b)$. Suppose that there exists a sequence of homeomorphisms $h_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\{h_n\}$ converges uniformly to h. Then $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a continuous onto map. Thus $h^{-1}(E)$ is closed for any continuum $E \subset \widehat{\mathbb{C}}$. Now assume that $h^{-1}(E)$ is not connected, i.e., there are two disjoint open sets $U, V \subset \widehat{\mathbb{C}}$ such that $h^{-1}(E) \subset U \cup V$ and both U and V intersect with $h^{-1}(E)$. Then $K := h(\widehat{\mathbb{C}}\backslash(U \cup V))$ is a compact set disjoint from E. Let $W \supset E$ be a connected domain such that $\overline{W} \cap K = \emptyset$. Since h_n converges uniformly to h, there exists some $n > 0$ such that

$$d(h, h_n) = \sup_{z \in \widehat{\mathbb{C}}} d(h(z), h_n(z)) < \min\{d(E, \partial W), d(\overline{W}, K)\},$$

where $d(\cdot, \cdot)$ denotes the spherical distance. It follows that $h_n(\widehat{\mathbb{C}}\backslash(U \cup V)) \cap \overline{W} = \emptyset$, hence $h_n^{-1}(W) \subset U \cup V$. It follows from $d(h, h_n) < d(E, \partial W)$ that $h_n(h^{-1}(E)) \subset W$. Thus both U and V intersect with $h_n^{-1}(W)$. This contradicts the fact that $h_n^{-1}(W)$ is connected. □

Proof of Theorem 1.1. — The sequence $\{\phi_n\}$ converges uniformly to a continuous onto map h by Lemma 2.1 and Lemma 2.2. Point (1) follows easily from the fact that $f \circ \phi_{n+1} = \phi_n \circ F$ and h is a uniform limit of ϕ_n. Point (4) follows from Lemma 3.1. Now we want to show the remaining points.

(2) It follows directly from Lemma 3.1 that for any $w \in \widehat{\mathbb{C}}$, $h^{-1}(w)$ is a full continuum. Since $\phi_0 \circ F = f \circ \phi_0$ near the critical cycles of F, $\phi_n \circ \phi_0^{-1}$ is a rotation in the Böttcher coordinates of the the critical cycles of f. It follows that there is a neighbourhood U of critical cycles of f such that $h^{-1}(q)$ is a single point for any $q \in U$. For any $w \in \mathcal{F}_f$, there is an integer $n \geqslant 1$ such that $f^n(w) \in U$. Since $h^{-1} \circ f^n(w) = F^n \circ h^{-1}(w)$, $h^{-1}(f^n(w))$ is a single point and $h^{-1}(w)$ is connected, we get that $h^{-1}(w)$ is a single point.

(3) Clearly $h(F(h^{-1}(x))) = f(h(h^{-1}(x))) = f(x) = y$. So $F(h^{-1}(x)) \subset h^{-1}(y)$. By Point (2), $h^{-1}(x)$ is connected. Let L be the connected component of $F^{-1}(h^{-1}(y))$ containing $h^{-1}(x)$. Then $h(L)$ is connected and $f(h(L)) = h(F(L)) \subset h(h^{-1}(y)) = y$. So $h(L) \subset f^{-1}(y)$. Notice that $x \in h(h^{-1}(x) \cap L) \subset h(L)$, that $f^{-1}(y)$ is a finite set, and that $h(L)$ is connected. We have therefore $h(L) = \{x\}$ and $L \subset h^{-1}(x)$. Consequently $h^{-1}(x) = L$. Notice that $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering. It follows easily from a property of a branched covering that $F(h^{-1}(x)) = h^{-1}(y)$ (see a proof in [1] §5.4).

Suppose $f^{-1}(y)$ has m preimages denoted by $x_1 := x, x_2, \cdots, x_m$. By the previous paragraph, we know that each $h^{-1}(x_i)$ is a connected component of $F^{-1}(h^{-1}(y))$ for $1 \leqslant i \leqslant m$. We claim that they are all the connected components of $F^{-1}(h^{-1}(y))$. In fact, let E be a connected component of $F^{-1}(h^{-1}(y))$. Since $f(h(E)) = h(F(E)) = h(h^{-1}(y)) = y$, we have $h(E) = x_j$ for some $1 \leqslant j \leqslant m$. Noticing that $E \subset h^{-1}(h(E)) = h^{-1}(x_j)$ and both E and $h^{-1}(x_j)$ are connected components of $F^{-1}(h^{-1}(y))$, we get $E = h^{-1}(x_j)$.

Since $\deg_q F = \deg_{\phi_1(q)} f$ for any critical point q of F and $h = \phi_n$ on $\mathcal{P}_F$ for all $n \geqslant 0$, we can conclude that for any critical point c of f, $h^{-1}(c)$ contains a critical point of F with local degree $\deg_c f$. Denote by $\deg F|_{h^{-1}(x_i)}$ the degree of the map $F : h^{-1}(x_i) \to h^{-1}(y)$. It follows that for each $1 \leqslant i \leqslant m$, $\deg F|_{h^{-1}(x_i)} \geqslant \deg_{x_i} f$. But $\sum_{i=1}^m \deg F|_{h^{-1}(x_i)} = \sum_{i=1}^m \deg_{x_i} f = d$, where d is the degree of F and f on $\widehat{\mathbb{C}}$. Thus $\deg F|_{h^{-1}(x_i)} = \deg_{x_i} f$.

(5) From $f \circ h(F^{-1}(E)) = h \circ F(F^{-1}(E)) = h(E)$, we have $h(F^{-1}(E)) \subset f^{-1}(h(E))$. Conversely, for any point $w \in f^{-1}(h(E))$, $f(w) \in h(E)$. So there is a point $z_0 \in E$ such that $f(w) = h(z_0)$. In Point (3), we have shown that $F(h^{-1}(w)) = h^{-1}(f(w))$. Noticing that $z_0 \in h^{-1}(f(w))$, there is a point $z_1 \in h^{-1}(w)$ such that $F(z_1) = z_0$. So $w = h(z_1) \in h(F^{-1}(z_0)) \subset h(F^{-1}(E))$. Therefore, $f^{-1}(h(E)) \subset h(F^{-1}(E))$.

(6) $F^{-1}(\widehat{E}) = F^{-1}(h^{-1}(h(E))) = h^{-1}(f^{-1}(h(E)))$. From Point (5), we obtain

$$F^{-1}(\widehat{E}) = h^{-1}(h(F^{-1}(E))) = \widehat{F^{-1}(E)}.$$

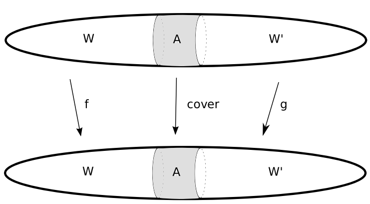
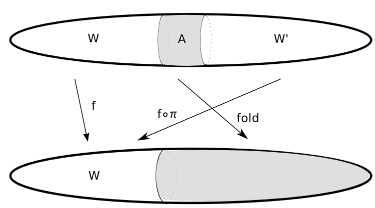
□

4. An application

In [3] a new type of surgery on polynomials, called 'foldings', is constructed. One can compare it with matings as follows: Set

- $\overline{W} = \mathbb{C} \cup \{\infty \cdot e^{2i\pi\theta}, \theta \in \mathbb{R}\}, \quad \overline{W'} = \mathbb{C}' \cup \{(\infty \cdot e^{2i\pi\theta})', \theta \in \mathbb{R}\}$,
- $A = [-1, 1] \times S^1$,
- $S = \overline{W} \sqcup A \sqcup \overline{W'} / \sim$,
  with $\infty \cdot e^{2\pi i\theta} \sim (-1, e^{2\pi i\theta})$ and $(+1, e^{2\pi i\theta}) \sim (\infty \cdot e^{-2\pi i\theta})'$,
- $\pi = id : \overline{W'} \to \overline{W}$.

Let f, g be monic postcritically finite polynomials of degree d. The mating M and a folding F are defined by :

f, g monic mating (f, g) : M :	folding $(f$, fold$)$ $F^2\,(\Omega_F \cap K_f)$ (pre)periodic
	

More precisely $M|_W = f$, $M|_{W'} = g$ and $M : A \to A$ is a degree d covering matching the boundary values. This M is automatically postcritically finite and its Thurston equivalence class is uniquely determined (if one does not introduce twist in A). On the other hand, $F|_W = f$, $F|_{W'} = f \circ \pi$ and $F : A \mapsto A \cup \overline{W'}$ is a branched covering matching the boundary values. In order for F to be postcritically finite, we also require that $F^2(\Omega_F \cap A)$ to be contained in the set of preperiodic points of f. The Thurston equivalence class of F depends on the choices of F on A.

The multicurve consisting of the single Jordan curve $\gamma = \partial W$ behaves quite differently under the mating M and the folding F : the set $M^{-1}(\gamma)$ is again a single Jordan curve, and is homotopic to γ rel $\mathcal{P}_M$, whereas $F^{-1}(\gamma)$ has two connected components, and each of them are homotopic rel $\mathcal{P}_F$ to γ.

Just as in the mating case, we have shown in [3] cases of foldings that are Thurston equivalent to a rational map and cases of foldings that are not.

Assume that a folding F is Thurston equivalent to a rational map R. Then there is a pair of homeomorphisms (h_0, h_1) making the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow[\approx]{h_1} & \widehat{\mathbb{C}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle R} \\ S & \xrightarrow[h_0]{\approx} & \widehat{\mathbb{C}}\,. \end{array}$$

We may then apply Rees-Shishikura's theorem, in the form of Theorem 1.1 and Corollary 1.2, to promote this diagram into a semi-conjugacy diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & \widehat{\mathbb{C}} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle R} \\ S & \xrightarrow[h]{} & \widehat{\mathbb{C}}\,. \end{array}$$

Note that if F were a mating of polynomials, then h would reduce the annular space between K_f and K_g to a space with empty interior. The folding case is quite the opposite. We have actually proved, using Theorem 1.1 (see [3] for details) :

Proposition 4.1. — *In the above setting, the set $h(A)$ contains a non-empty annulus $\mathcal{A}$ s.t.*

- *$\mathcal{A}$ separates $h(\overline{W})$ and $h(\overline{W'})$,*
- *$\mathcal{A}$ contains two essential annuli A_1, A_2 satisfying that $R : A_1 \to \mathcal{A}$ and $R : A_2 \to \mathcal{A}$ are coverings, and $\partial\mathcal{A} \subset \partial(A_1 \cup A_2)$.*

An interesting consequence is that the folding rational map R has a polynomial renormalization. Moreover it has wandering continua in its Julia set (as in [9]). Such a phenomenon does not exist for polynomials ([2, 6, 13]).

Acknowledgements. — This work was supported by the NSF of China under grants No. 10831004, No. 11125106, No. 11101402 and No. 11231009, by the PSSF of China under grant No. 201003020 and by SRF for ROCS, SEM. The authors would like to thank the referee for carefully reading the paper and providing valuable suggestions.

Bibliography

[1] Beardon (A. F.). — Iteration of rational functions, Graduate text in Mathemathics, vol. 132, Springer-Verlag, New York (1993).

[2] Blokh (A.) and Levin (G.). — An inequality for laminations, Julia sets and 'growing trees', Erg. Th. and Dyn. Sys., 22, p. 63-97 (2002).

[3] Cui (G.), Peng (W.) and Tan (L.). — Renormalization and wandering continua of rational maps, arXiv: math/1105.2935.

[4] Douady (A.). — Systèmes dynamiques holomorphes, (Bourbaki seminar, Vol. 1982/83) Astérisque, p. 105-106, p. 39-63 (1983).

[5] Douady (A.) and Hubbard (J. H.). — Étude dynamique des polynômes complexes, I, II, Publ. Math. Orsay (1984-1985).

[6] Kiwi (J.). — Rational rays and critical portraits of complex polynomials, Preprint 1997/15, SUNY at Stony Brook and IMS.

[7] Levin (G.). — On backward stability of holomorphic dynamical systems, Fund. Math., 158, p. 97-107 (1998).

[8] Petersen (C. L.) and Meyer (D.). — On the notions of mating, to appear in Annales de la Faculté des Sciences de Toulouse.

[9] Pilgrim (K.) and Tan (L.). — Rational maps with disconnected Julia set, Astérisque 261, volume spécial en l'honneur d'A. Douady, p. 349-384 (2000).

[10] Rees (M.). — A partial description of parameter space of rational maps of degree two: Part I, Acta Math., 168, p. 11-87 (1992).

[11] Shishikura (M.). — On a theorem of M. Rees for matings of polynomials, in The Mandelbrot set, Theme and Variations, ed. Tan Lei, LMS Lecture Note Series 274, Cambridge Univ. Press, p. 289-305 (2000).

[12] Tan (L.). — Matings of quadratic polynomials, Erg. Th. and Dyn. Sys., 12, p. 589-620 (1992).

[13] Thurston (W.). — The combinatorics of iterated rational maps (1985), published in: "Complex dynamics: Families and Friends", ed. by D. Schleicher, A K Peters, p. 1-108 (2008).