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Snakes and articulated arms in an Hilbert space

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Abstract. — The purpose of this paper is to give an illustration of results on integrability of distributions and orbits of vector fields on Banach manifolds obtained in [5] and [4]. Using arguments and results of these papers, in the context of a separable Hilbert space, we give a generalization of a Theorem of accessibility contained in [3] and [6] for articulated arms and snakes in a finite dimensional Hilbert space.

Résumé. — Le but de ce travail est de donner un exemple qui illustre les résultats d'intégrabilité des distributions et des orbites de champs de vecteurs sur des variétés banachiques établis dans [5] et [4] respectivement. Ces travaux permettent, dans le cadre des espaces de Hilbert séparables, de donner une généralisation des théorèmes d'accessibilité contenus dans [3] et [6], en dimension finie, pour des bras articulés et des serpents.

1. Introduction

In finite dimension, a snake (of length L) is a continuous piecewise C^1-curve $S : [0, L] \to \mathbb{R}^d$, arc-length parameterized, such that S(0) = 0. According to [6], "*Charming a snake*" consists in finding a 1-parameter deformation S_t of S so that the corresponding *head* $S_t(L)$ describes a given C^1-curve $c : [0, 1] \to \mathbb{R}^d$: that is $S_t(L) = c(t)$. More precisely, each snake S

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of length L in $\mathbb{R}^d$ can be given by a piecewise C^0-curve $u : [0, L] \to \mathbb{S}^{d-1}$ such that $S(t) = \int_0^t u(\tau)d\tau$. The set $Conf$ of snakes can be parametrized by such type piecewise C^0-curves and so can be provided with a Banach manifold structure. So we have to build some C^1-curve $t \to u_t$ in $Conf$ such that the associated family S_t of snakes satisfies $S_t(L) = c(t)$ for all $t \in [0, 1]$. The "*snake charmer algorithm*" proposed in [6] consists in building an appropriate "*horizontal* " distribution $\mathcal{D}$ in $Conf$ so that $t \to u_t$ is tangent to $\mathcal{D}$. In fact, this approach remains to build a lift $\tilde{c}$ of c in $Conf$ so that the "infinitesimal kinematic energy" $\dfrac{1}{2}||\dot{\tilde{c}}(t)||^2_{L^2}$ is minimal for all $t \in [0, 1]$ (see subsection 4.1). Note that, for articulated arms (i.e. when S is affine by parts), this algorithm is nothing but a generalization of an analogue approach developed in [3].

The purpose of this paper is to give a generalization of this problem in the context of separable Hilbert spaces. In more details, given a separable Hilbert space $\mathbb{H}$, we consider the smooth hypersurface $\mathbb{S}^\infty$ of elements of norm 1. As previously, a Hilbert snake of length L is a continuous piecewise C^1-curve $S : [0, L] \to \mathbb{H}$, arc-length parameterized, such that $S(0) = 0$. An articulated arm corresponds to the particular case where u is affine in each part. Then a snake is also given by a piecewise C^0-curve $u : [0, L] \to \mathbb{S}^\infty$ such that $S(t) = \int_0^t u(\tau)d\tau$. If we fix a partition $\mathcal{P}$ of $[0, L]$, the set $\mathcal{C}^L_\mathcal{P}$ of such curves will be called the configuration set and carries a natural structure of Banach manifold. For articulated arms, the configuration space is the subset $\mathcal{A}^L_\mathcal{P}$ of u which are constant on each subinterval associated to the partition. In fact, $\mathcal{A}^L_\mathcal{P}$ is a weak Hilbert submanifold of $\mathcal{C}^L_\mathcal{P}$. To any "configuration" $u \in \mathcal{C}^L_\mathcal{P}$ is naturally associated the "end map" $\mathcal{E}(u) = \displaystyle\int_0^L u(s)ds$. This map is smooth and we put on $\mathcal{C}^L_\mathcal{P}$ a natural (weak) Riemannian metric G. The orthogonal of $\ker T\mathcal{E}$ gives rise to a closed distribution $\mathcal{D}$ on $\mathcal{C}^L_\mathcal{P}$. As in finite dimension, for any curve $t \to u_t$ in $\mathcal{C}^L_\mathcal{P}$, defined on $[0, 1]$, we can associate a family S_t of snakes whose head $S_t(L)$ describes a curve $c : t \to S_t(L)$ for $t \in [0, 1]$. The curve $\tilde{c} : t \to u_t$ is called a "lift " of c in $\mathcal{C}^L_\mathcal{P}$. When $\tilde{c}$ is tangent to $\mathcal{D}$, it is called a "horizontal lift".

So the problem for the head of the Hilbert snake to join an initial state x_0 to a final state x_1 gives rise to the following "accessibility problem":

Given an initial (resp. final) configuration u_0 (resp. u_1) in $\mathcal{C}^L_\mathcal{P}$, such that $\mathcal{E}(u_i) = x_i$, $i = 0, 1$, find a horizontal piecewise C^1-curve $\gamma : [0, T] \to \mathcal{C}^L_\mathcal{P}$ (i.e. γ is tangent to $\mathcal{D}$) and which joins u_0 to u_1.

Now, given any configuration $u \in \mathcal{C}^L_{\mathcal{P}}$ we look for the accessibility set $\mathcal{A}(u)$ of all configurations $v \in \mathcal{C}^L_{\mathcal{P}}$ which can be joined from u by a horizontal piecewise C^1-curve. In the context of finite dimension, in [3] and [6], by using arguments about the action of the Möbius group on $\mathcal{C}^L_{\mathcal{P}}$, it can be shown that $\mathcal{A}(u)$ is the maximal integral manifold of a finite dimensional distribution on $\mathcal{A}^L_{\mathcal{P}}$ and $\mathcal{C}^L_{\mathcal{P}}$. Unfortunately, in our context, the same argument does not work. Moreover, as we are in the context of infinite dimension for $\mathbb{S}^\infty$, we cannot hope to get a finite dimensional distribution whose maximal integral manifold is $\mathcal{A}(u)$.

However, our principal result is to construct a canonical distribution $\bar{\mathcal{D}}$ modeled on a Hilbert space, which is integrable and such that the accessibility set $\mathcal{A}(u)$ is a dense subset of the maximal integral manifold through u of $\bar{\mathcal{D}}$. Moreover this distribution is minimal in some natural sense (see Remark 4.2). In fact, when $\mathbb{H}$ is finite dimensional, $\bar{\mathcal{D}}$ is exactly the finite distribution obtained in [6] whose leaves are the accessibility sets.

The arguments used in our proof can be found in [5] and [4]. Moreover, this Theorem of accessibility can be seen as an application of results obtained in [4]; it also gives rise to an illustration of the *"almost Banach Lie algebroid structures"* developed in [1] (see Appendix 5).

This paper is organized as follows. Section 2 contains all definitions and results of [5] and [4] which are used in the proof about the accessibility sets. *At first, the reader can go directly to the section 3 , and he may refer to this paragraph 2 only for a more detailed reading.* In the first subsection of paragraph 4 we expose the relation between "horizontal lift" and minimizing the "infinitesimal kinematic energy". But essentially, this paragraph contains the principal result of accessibility (Theorem 4.1 in subsection 4.2). The proof of this Theorem, which needs all definitions and results recalled in section 2, is developed in subsections 4.3 and 4.4. We end this paper by an appendix which gives an interpretation of the previous results in terms of "almost Banach Lie algebroid" (cf. [1]). In particular, this appendix is used as reference for an example developed in [1].

2. Preliminaries

2.1. Weak distributions on a Banach manifold

In this subsection, from [5] we recall all definitions, properties and results we shall use later.

Let M be a connected Banach manifold modeled on a Banach space E. We denote by $\mathcal{X}(M)$ the set of local vector fields on M. The flow of any

$X \in \mathcal{X}(M)$ will be denoted by ϕ_t^X. We then have the following definitions and properties:

• A **weak submanifold** of M is a pair (N, f) where N is a connected Banach manifold (modeled on a Banach space F) and $f : N \to M$ is a smooth map such that :

– there exists a continuous injective linear map $i : F \to E$ between these two Banach spaces;

– f is injective and the tangent map $T_x f : T_x N \to T_{f(x)} M$ is injective for all $x \in N$.

Note that for a weak submanifold $f : N \to M$, on the subset $f(N)$ of M we have two topologies:

– the induced topology from M;

– the topology for which f is a homeomorphism from N to $f(N)$.

With this last topology, via f, we get a structure of Banach manifold modeled on F. Moreover, the inclusion from $f(N)$ into M is continuous as a map from the Banach manifold $f(N)$ to M. In particular, if U is an open set of M, then $f(N) \cap U$ is an open set for the topology of the Banach manifold on $f(N)$.

• According to [5], a **weak distribution** on M is an assignment $\mathcal{D} : x \mapsto \mathcal{D}_x$ which, to every $x \in M$, associates a vector subspace $\mathcal{D}_x$ in $T_x M$ (not necessarily closed) endowed with a norm $|| \; ||_x$ such that $(\mathcal{D}_x, || \; ||_x)$ is a Banach space (denoted by $\tilde{\mathcal{D}}_x$) and such that the natural inclusion $i_x : \tilde{\mathcal{D}}_x \to T_x M$ is continuous. Moreover, if the Banach structure on $\mathcal{D}_x$ is a Hilbert structure, we say that $\mathcal{D}$ is a **weak Hilbert distribution**.

When $\mathcal{D}_x$ is closed, we have a natural Banach structure on $\tilde{\mathcal{D}}_x$, induced by the Banach structure on $T_x M$, and so we get the classical definition of a distribution; in this case we will say that $\mathcal{D}$ is **closed**.

A (local) vector field Z on M is **tangent** to $\mathcal{D}$, if for all $x \in \mathrm{Dom}(Z)$[(1)], $Z(x)$ belongs to $\mathcal{D}_x$. The set of local vector fields tangent to $\mathcal{D}$ will be denoted by $\mathcal{X}_{\mathcal{D}}$.

• We say that $\mathcal{D}$ is **generated by a subset** $\mathcal{X} \subset \mathcal{X}(M)$ if, for every $x \in M$, the vector space $\mathcal{D}_x$ is the linear hull of the set $\{Y(x), Y \in \mathcal{X}, x \in \mathrm{Dom}(Y)\}$.

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(1) $\mathrm{Dom}(Z)$ is the maximal open set on which Z is defined

For a weak distribution $\mathcal{D}$ on M we have the following definitions:

• an **integral manifold** of $\mathcal{D}$ through x is a weak submanifold $f : N \to M$ such that there exists $u_0 \in N$ such that $f(u_0) = x$ and $T_u f(T_u N) = \mathcal{D}_{f(u)}$ for all $u \in N$.

• $\mathcal{D}$ is called **integrable** if there exists an integral manifold N of $\mathcal{D}$ through any $x \in M$.

• if $\mathcal{D}$ is generated by a set $\mathcal{X}$ of local vector fields, then $\mathcal{D}$ is called **$\mathcal{X}$-invariant** if for any $X \in \mathcal{X}$, the tangent map $T_x\phi_t^X$ sends $\mathcal{D}_x$ onto $\mathcal{D}_{\phi_t^X(x)}$ for all $(x,t) \in \Omega_X$. $\mathcal{D}$ is **invariant** if $\mathcal{D}$ is $\mathcal{X}_{\mathcal{D}}-$ invariant.

Now we introduce essential properties of "local triviality" which will play an essential role through this paper:

• $\mathcal{D}$ is **lower (locally) trivial** if for each $x \in M$, there exists an open neighborhood V of x, a smooth map $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$ (called **lower trivialization**) such that :

(i) $\Theta(\tilde{\mathcal{D}}_x \times \{y\}) \subset \mathcal{D}_y$ for each $y \in V$

(ii) for each $y \in V$, $\Theta_y \equiv \Theta(\ ,y) : \tilde{\mathcal{D}}_x \to T_yM$ is a continuous operator and $\Theta_x : \tilde{\mathcal{D}}_x \to T_xM$ is the natural inclusion i_x

(iii) there exists a continuous operator $\tilde{\Theta}_y : \tilde{\mathcal{D}}_x \to \tilde{\mathcal{D}}_y$ such that $i_y \circ \tilde{\Theta}_y = \Theta_y$, $\tilde{\Theta}_y$ is an isomorphism from $\tilde{\mathcal{D}}_x$ onto $\Theta_y(\tilde{\mathcal{D}}_x)$ and $\tilde{\Theta}_x$ is the identity of $\tilde{\mathcal{D}}_x$

• $\mathcal{D}$ is **(locally) upper trivial** if, for each $x \in M$, there exists an open neighbourhood V of x, a Banach space F and a smooth map $\Psi : F \times V \to TM$ (called **upper trivialization**) such that :

(i) for each $y \in V$, $\Psi_y \equiv \Psi(\ ,y) : F \to T_yM$ is a continuous operator with $\Psi_y(F) = \mathcal{D}_y$;

(ii) $\ker \Psi_x$ complemented in F;

(iii) if $F = \ker \Psi_x \oplus S$, the restriction θ_y of Ψ_y to S is injective for any $y \in V$;

(iv) $\Theta(u,y) = (\theta_y \circ [\theta_x]^{-1}(u), y)$ is a lower trivialization of $\mathcal{D}$.

In this case the map Θ is called the **associated lower trivialization**.

An upper trivial weak distribution $\mathcal{D}$ is called **Lie bracket invariant** if, for any $x \in M$, there exists an upper trivialization $\Phi : F \times V \to TM$

such that for any $u \in F$, there exists $\varepsilon > 0$, such that, for all $0 < \tau < \varepsilon$, we have a smooth field of operators $C : [-\tau, \tau] \to L(F, F)$ with the following property

$$[X_u, Z_v](\gamma(t)) = \Phi(C(t)[v], \gamma(t)) \text{ for any } Z_v = \Phi(v,\ ) \text{ and any } v \in F \quad (2.1)$$

along the integral curve $\gamma : t \mapsto \phi_t^{X_u}(x)$ on $[-\tau, \tau]$ of the lower section $X_u = \Theta(\Phi(u, x),\ )$.

With these definitions we have the following criterion of integrability:

THEOREM 2.1. — *Let $\mathcal{D}$ be a upper trivial weak distribution. Then $\mathcal{D}$ is integrable if and only if $\mathcal{D}$ is Lie bracket invariant.*

2.2. Orbit of a family of vector fields

In this subsection we expose the results of [4] which will be used for the proof Theorem 4.1.

<u>*Notion of $\mathcal{X}$-orbit*</u>.

Let $\mathcal{X}$ be a set of local vector fields on M. Given $x \in M$, we say that $\mathcal{X}$ satisfies the condition (LBs) at x (Locally Bounded of order s), if there exists a chart (V_x, ϕ) centered at x and a constant $k > 0$ such that:
for any $X \in \mathcal{X}$, whose domain $\mathrm{Dom}(X)$ contains V_x, we have

$$\sup\{||J^s[\phi_* X](y)||,\ X \in \mathcal{X},\ y \in V_x\} \leqslant k. \quad (2.2)$$

For any finite or countable ordered set A of indexes, consider a family $\xi = \{X_\alpha\}_{\alpha \in A}$ where the X_α are defined on a same open set V and satisfies the condition (LBs) for $s \geqslant 1$. Given any bounded integrable map $u = (u_\alpha)_{\alpha \in A}$ from some interval I to $l^1(A) = \{\tau = (\tau_\alpha),\ \sum_{\alpha \in A} |\tau_\alpha| < \infty\}$ we can associate a time depending vector field of type

$$Z(x, t, u) = \sum_{\alpha \in A} u_\alpha(t) X_\alpha(x),$$

For such a vector field there exists a **flow** $\Phi_u^\xi(t,\ )$ (see Theorem 2 of [4]). Given some $\tau \in l^1(A)$, we set $||\tau||_1 = \sum_{\alpha \in A} |\tau_\alpha|$. On the corresponding interval $[0, ||\tau||_1]$, we consider the partition $(t_\alpha)_{\alpha \in A}$ of this interval defined by, $t_0 = 0$ and for $\alpha \in A$, $t_\alpha = \sum_{\beta=1}^{\alpha} |\tau_\beta|$. If we choose $u = \Gamma^\tau = (\Gamma_\alpha^\tau)$ where Γ_α^τ is the indicatrix function of $]t_\alpha, t_{\alpha+1}[$, we can associate to (ξ, τ) a time depending vector field $Z(x,t,u)$ as previously. Under appropriate assumptions, for such a Z, we get an associated flow, denoted by $\Phi^\xi_\tau(t,\ )$. Assume that the set of all Dom(X) for $X \in \mathcal{X}$ is a covering of M and is bounded at each point, i.e. the set of values $\{X(x),\ X \in \mathcal{X}\} \subset T_xM$ is bounded for any $x \in M$. We can enlarge $\mathcal{X}$ to the set $\hat{\mathcal{X}}$ given by

$$\hat{\mathcal{X}} = \{Z = \Phi_*(\nu Y),\ Y \in \mathcal{X},\ \Phi = \phi^{X_p}_{t_p} \circ \cdots \circ \phi^{X_1}_{t_1} \text{ for } X_1, \cdots, X_p \in \mathcal{X}; \tag{2.3}$$
$$\text{and appropriate } \nu \in \mathbb{R}\}$$

(see subsection 3.1 of [4]). Then $\hat{\mathcal{X}}$ satisfies the same previous properties as $\mathcal{X}$. From this set $\hat{\mathcal{X}}$, we associate an appropriate pseudo-group $\mathcal{G}_{\mathcal{X}}$ of local diffeomorphisms which are finite compositions of flows of type ϕ^X_t with $X \in \mathcal{X}$ and of type $\Phi^\xi_u(||\tau||_1, .)$ (as we have seen previously) or its inverse for $\xi \subset \hat{\mathcal{X}}$.

To $\mathcal{G}_{\mathcal{X}}$ is naturally associated the following equivalence relation on M:

$$x \equiv y \text{ if and only if there exists } \Phi \in \mathcal{G}_{\mathcal{X}} \text{ such that } \Phi(x) = y$$

An equivalence class is called a $\mathcal{X}$-orbit.

PROPOSITION 2.2 *[4]. — For each pair (x, y) in the same $\mathcal{X}$-orbit either we have a continuous piecewise smooth curve which joins x to y and whose each smooth part is tangent to X or $-X$ for some $X \in \mathcal{X}$, or there exists a sequence γ_k of such continuous piecewise smooth curves whose origin is x (for all curves) and whose sequence of ends converges to y.*

Integrability of weak distribution associated to an $\mathcal{X}$-orbit.

Consider any set $\mathcal{Y}$ of local vector fields which contains $\hat{\mathcal{X}}$. Assume that there exists a weak distribution $\triangle$ generated by $\mathcal{Y}$ which is integrable on M and, for each $x \in M$, there exists a lower trivialization $\Theta : F \times V \to TM$ for some Banach space F (which depends of x) and for some neighborhood V of x in M. Let N be the union of all integral manifolds $i_L : L \to M$ through x_0. Then $i_N : N \to M$ is the maximal integral manifold of $\triangle$ through x_0(see Lemma 2.14 [5]).

PROPOSITION 2.3 *(see [4]). — As previously, let $f : N \to M$ be the maximal integral manifold of $\triangle$ through x.*

1. *Let $Z \in \mathcal{X}(M)$ be such that $Dom(Z) \cap f(N) \neq \emptyset$ and Z is tangent to $\triangle$. Set $\tilde{V}_Z = f^{-1}(Dom(Z) \cap f(N))$. Then $\tilde{V}_Z$ is an open set in N*

and there exists a vector field $\tilde{Z}$ *on* N *such that* $Dom(\tilde{Z}) = \tilde{V}_Z$ *and* $f_*\tilde{Z} = Z \circ f$.

Moreover, if $]a_x, b_x[$ *is the maximal interval on which the integral curve* $\gamma : t \mapsto \phi^Z(t,x)$ *is defined in* M*, then the integral curve* $\tilde{\gamma} : t \to \phi^{\tilde{Z}}(t,\tilde{x})$ *is also defined on* $]a_x, b_x[$ *and we have*

$$\gamma = f \circ \tilde{\gamma} \tag{2.4}$$

2. *Let be* $\xi = \{X_\beta,\ \beta \in B\} \subset \hat{\mathcal{X}} \subset \mathcal{Y}$ *which satisfies the conditions (LBs) on a chart domain* V *centered at* $x \in f(N)$ *and consider the associated flow* Φ^ξ_τ. *For some* $\tau \in l^1(B)$ *let* γ *be the curve on* $[0, ||\tau||_1]$ *defined by* $\gamma(t) = \Phi^\xi_\tau(t,x)$. *Then there exists a curve* $\tilde{\gamma} : [0, ||\tau||_1[\to N$ *such that*

$$f \circ \tilde{\gamma} = \gamma \text{ on } [0, ||\tau||_1[\tag{2.5}$$

According to the properties of $\mathcal{X}$ we can associate to this set a weak distribution $\mathcal{D}$ in the following way:

$\mathcal{D}_x = \{Y = \sum_{X \in \mathcal{X}} \lambda_X X(x)\}$ for any absolutely summable family $\{\lambda_X,\ X \in \mathcal{X},\ x \in \mathrm{Dom}(X)\}$

In the same way we can also associate to $\hat{\mathcal{X}}$ a weak distribution $\hat{\mathcal{D}}$ which contains $\mathcal{X}$ and which is $\mathcal{X}$-invariant. Moreover, for a set $\mathcal{Y}$ of local vector fields which contains $\mathcal{X}$ and which is bounded at each point, we can also associate a weak distribution $\triangle$ of the previous type. If $\triangle$ is $\mathcal{X}$ invariant, then $\hat{\mathcal{D}}_x \subset \triangle_x$ for any $x \in M$.

On the other hand, to the set $\mathcal{X}$ we can associate the sequences of families

$$\begin{aligned} \mathcal{X} &= \mathcal{X}^1 \subset \mathcal{X}^2 = \mathcal{X} \cup \{[X,Y],\ X,Y \in \mathcal{X}\} \subset \cdots \subset \mathcal{X}^k \\ &= \mathcal{X}^{k-1} \cup \{[X,Y],\ X \in \mathcal{X}, Y \in \mathcal{X}^{k-1}\} \subset \cdots \end{aligned}$$

When $\mathcal{X}^k$ is bounded at each point, as previously, we can also associate a weak distribution $\mathcal{D}^k$ generated by $\mathcal{X}^k$.

Consider an ordered finite or countable set of indexes A and assume that we have $\hat{\mathcal{D}}$ fulfilling the following conditions

1. for any $x \in M$ there exists a upper trivialization $\Phi : l^1(A) \times V \to TM$ such that $\Phi(e_\alpha, .) = Y_\alpha(.)$ for each $\alpha \in A$ where $\{e_\alpha\}_{\lambda \in A}$ is the canonical basis of $l^1(A)$;

2. for any $x \in M$ there exists a neighborhood V of x such that, $V \subset \cap_{\alpha \in A} \mathrm{Dom}(Y_\alpha)$, and a constant $C > 0$ such that we have
$$[Y_\alpha, Y_\beta](y) = \sum_{\nu \in A} C^\nu_{\alpha\beta}(y) Y_\nu(y) \text{ for any } \alpha, \beta \in A \tag{2.6}$$
where each $C^\nu_{\alpha\beta}$ is a smooth function on V, for any $\alpha, \beta, \nu \in A$ and we have
$$\sum_{\alpha,\beta,\nu \in A} |C^\nu_{\alpha\beta}(y)| \leqslant C$$
for any $y \in V$.

Then we have:

THEOREM 2.4 *([4]).* —

1. *Under the previous assumptions, the distribution $\hat{\mathcal{D}}$ is integrable and each $\mathcal{X}$-orbit $\mathcal{O}$ is the union of the maximal integral manifolds which meet $\mathcal{O}$ and such an integral manifold is dense in $\mathcal{O}$.*
2. *If $\mathcal{D}^k$ is defined and satisfies the previous assumptions for some $k \geqslant 2$, then we have $\mathcal{D}^k = \hat{\mathcal{D}}$ and $\mathcal{D}^k$ is integrable.*

3. Hilbert snakes and Hilbert articulated arms

3.1. The configuration space

Let $\mathbb{H}$ be a separable Hilbert space and $< \, , \, >$ (resp. $|| \; ||$) the inner product (resp. the norm) on $\mathbb{H}$. We consider a fixed hilbertian basis $\{e_i\}_{i \in \mathbb{N}}$ in $\mathbb{H}$. Any $x \in \mathbb{H}$ will be written as a serie $x = \sum_{i \in \mathbb{N}} x_i e_i$ where $x_i = \langle x, e_i \rangle$ is the i^{th}coordinate of x. We denote by $\mathbb{S}^\infty = \{x \in \mathbb{H} \; : \; \|x\| = 1\}$ the unit sphere in $\mathbb{H}$. Note that $\mathbb{S}^\infty$ is a codimension one hypersurface in $\mathbb{H}$.

A curve $\gamma : [a, b] \to M$ (not necessary continuous) is called C^k -piecewise if there exists a finite set $\mathcal{P}$=$\{a = s_0 < s_1 < ... < s_N = b\}$ such that, for all $i = 0, ..., N-1$, the restriction of γ to the interval $]s_i, s_{i+1}[$ can be extended to a curve of class C^k on the closed interval $[s_i, s_{i+1}]$.

Given any metric space (X, d), we denote by $\mathcal{C}([a, b], X)$ the set of continuous curves $u : [a, b] \to X$. Recall that on $\mathcal{C}([a, b], X)$ we have the usual distance d_∞ defined by
$$d_\infty(u_1, u_2) = \sup_{t \in [a,b]} d(u_1(t), u_2(t))$$
and $\mathcal{C}([a, b], X), d_\infty)$ is a complete metric space.

For a given partition $\mathcal{P}$=$\{a = s_0 < s_1 < ... < s_N = b\}$ of $[a, b]$, let be $\mathcal{C}^k_{\mathcal{P}}([a, b], \mathbb{S}^\infty)$ (resp. $\mathcal{C}^k_{\mathcal{P}}([a, b], \mathbb{H})$ the set of curves $u \in \mathcal{C}([a, b], \mathbb{S}^\infty)$ (resp. $u \in \mathcal{C}([a, b], \mathbb{H})$) which are C^k-piecewise relatively to $\mathcal{P}$ for $k \in \mathbb{N}$.

Throughout this paper, we fix a real number $L > 0$ and $\mathcal{P}$ is a given fixed partition of $[0, L]$.

A Hilbert snake is a continuous piecewise C^1-curve $S : [0, L] \to \mathbb{H}$, such that $||\dot{S}(t)|| = 1$ and S(0) = 0. When S is affine by part, we call this snake **an affine snake or a Hilbert articulated arm**.

In fact, a snake is characterized by $u(t) = \dot{S}(t)$ and of course we have $S(t) = \int_0^t u(s)ds$ where $u : [0, L] \to \mathbb{S}^\infty$ is a piecewise C^0-curve associated to the partition $\mathcal{P}$. Moreover, this snake is affine if and only if u is constant on each subinterval of $\mathcal{P}$.

The set $\mathcal{C}^L_{\mathcal{P}} = \mathcal{C}^0_{\mathcal{P}}([0, L], \mathbb{S}^\infty)$ is called the **configuration space of the snakes** in $\mathbb{H}$ of length L relative to the partition $\mathcal{P}$. We can also put on $\mathcal{C}^L_{\mathcal{P}}$ the distance d_∞ defined by

$$d_\infty(u_1, u_2) = \sup_{t\in[a,b]} ||(u_1(t), u_2(t))||$$

Note that the subset

$\mathcal{A}^L_{\mathcal{P}} = \{u \in \mathcal{C}^L_{\mathcal{P}}$, such that u is constant on each subinterval $[s_{i-1}, s_i[, i = 1, \cdots N\}$

is the configuration space of Hilbert articulated arms in $\mathbb{H}$ of length L relative to the partition $\mathcal{P}$.

The natural map

$$\begin{aligned} h : \quad & \mathcal{C}^L_{\mathcal{P}} \to \prod_{i=0}^{N-1} \mathcal{C}^0([s_i, s_{i+1}], \mathbb{S}^\infty) \\ & u \mapsto (u\,|_{[s_0,s_1]}, ..., u\,|_{[s_i,s_{i+1}]}, u\,|_{[s_{N-1},s_N]}) \end{aligned} \tag{3.1}$$

is a homeomorphism. In particular, $(\mathcal{C}^L_{\mathcal{P}}, d_\infty)$ is a complete metric space. Note that the restriction of h to $\mathcal{A}^L_{\mathcal{P}}$ is a homeomorphism onto $[\mathbb{S}^\infty]^N$. Moreover as in finite dimension we have (see [6])

Proposition 3.1. — *$\mathcal{C}^L_{\mathcal{P}}$ has a structure of Banach manifold and according to (3.1) the natural map*

$$h : \mathcal{C}^L_{\mathcal{P}} \to \prod_{i=0}^{N-1} \mathcal{C}^0([s_i, s_{i+1}], \mathbb{S}^\infty)$$

is a diffeomorphism. Moreover $\mathcal{A}^L_{\mathcal{P}}$ is a weak Hilbert submanifold diffeomorphic to $[\mathbb{S}^\infty]^N$ and the topology associated to this structure and the topology induced by $\mathcal{C}^L_{\mathcal{P}}$ coincide.

The tangent space $T_u\mathcal{C}^L_{\mathcal{P}}$ can be identified with the set

$$\{v \in \mathcal{C}^0_{\mathcal{P}}([0,L],\mathbb{H}) \text{ such that } < u(s), v(s) >= 0 \text{ for all } s \in [0,L]\}$$

This space is naturally provided with the induced norm $|| \,||_\infty$. On the other hand, note that any $v \in \mathcal{C}^0_{\mathcal{P}}([0,L],\mathbb{H})$ is integrable on $[0,L]$ and so we get a inner product on this space given by:

$$< v, w >_{L^2}= \int_0^L < v(s), w(s) > ds \tag{3.2}$$

This inner product induces a natural norm $|| \,||_{L^2}$ on $T_u\mathcal{C}^L_{\mathcal{P}}$ given by:

$$||v||_{L^2} = [\int_0^L < v(s), v(s) > ds]^{\frac{1}{2}}$$

and we have the following inequality

$$||u||_{L^2} \leqslant \sqrt{L}||u||_\infty. \tag{3.3}$$

In the same way the tangent space $T_u\mathcal{A}^L_{\mathcal{P}}$ can be identified with the set $v = (v_1, \cdots, v_N) \in \mathbb{H}^N$ such that $< v_i, u_i >= 0$ for $i = 1 \cdots N$ if $u = (u_1, \cdots, u_N)$. Of course, this vector space can be also considered as a subspace of $T_u\mathcal{C}^L_{\mathcal{P}}$. Note that this subspace in closed in $T_u\mathcal{C}^L_{\mathcal{P}}$.

Remark 3.2. — As $(T_u\mathcal{C}^L_{\mathcal{P}}, || \,||_{L^2})$ is not complete, the norm $|| \,||_\infty$ and $|| \,||_{L^2}$ are not equivalent (on each $T_u\mathcal{C}^L_{\mathcal{P}}$). So the inner product defined by (3.2) gives rise only to a weak Riemannian G metric on $T\mathcal{C}^L_{\mathcal{P}}$.

As $\mathcal{A}^L_{\mathcal{P}}$ is diffeomorphic to $[\mathbb{S}^\infty]^N$, the tangent $T_u\mathcal{A}^L_{\mathcal{P}}$ can be identified with

$$T_{x_1}\mathbb{S}^\infty \times \cdots \times T_{x_N}\mathbb{S}^\infty$$

for $u = (x_1, \cdots, x_N) \in [\mathbb{S}^\infty]^N$. So the canonical inner product on $\mathbb{H}^N$, induces an natural inner product on $T_u\mathcal{A}^L_{\mathcal{P}}$.
On the other hand, the inner product $< \, , \, >_{L^2}$ on $T_u\mathcal{C}^L_{\mathcal{P}}$ induces an inner product on $T_u\mathcal{A}^L_{\mathcal{P}}$ as subspace of $T_u\mathcal{C}^L_{\mathcal{P}}$. In fact these inner products are proportional and moreover, the norm $|| \,||_\infty$ and $|| \,||_{L^2}$ induce equivalent norm on $T_u\mathcal{A}^L_{\mathcal{P}}$.

3.2. The horizontal distribution associated to a Hilbert snake

For any $u \in \mathcal{C}^L_{\mathcal{P}}$ consider the map $S_u : [0, L] \to \mathbb{H}$ given by:

$$S_u(t) = \int_0^t u(s)ds \tag{3.4}$$

called the **Hilbert snake** associated to u. On the other hand, to each configuration $u \in \mathcal{C}^L_{\mathcal{P}}$ we can associate the **endpoint map**:

$$\begin{aligned} \mathcal{E} : \qquad & \mathcal{C}^L_{\mathcal{P}} \to \mathbb{H} \\ & u \to S_u(L) \end{aligned} \tag{3.5}$$

As $\mathcal{E}$ is the restriction to $\mathcal{C}^L_{\mathcal{P}}$ of the linear map $u \to \displaystyle\int_0^L u(s)ds$ defined on $\mathcal{C}^0_{\mathcal{P}}([0, L], \mathbb{H})$ it follows that $\mathcal{E}$ is smooth and we have:

$$T_u\mathcal{E}(v) = \int_0^L v(s)ds \tag{3.6}$$

Note that $\mathcal{E}[\mathcal{C}^L_{\mathcal{P}}]$ is the closed ball $\mathbb{B}_L = \{x \in \mathbb{H} \text{ such that } ||x|| \leqslant L\}$.

Lemma 3.3. —

1. *The subspace* $\ker T_u\mathcal{E} \subset T_u\mathcal{C}^L_{\mathcal{P}}$ *is a Banach space for each induced norm* $||\ ||_\infty$ *and* $||\ ||_{L^2}$*.*
2. *The orthogonal of* $\ker T_u\mathcal{E}$ *(for the inner product* $<\ ,|||>_{L^2}$ *on* $T_u\mathcal{C}^L_{\mathcal{P}}$*), denoted* $\mathcal{D}_u$*, is a closed space in each normed spaces* $(T_u\mathcal{C}^L_{\mathcal{P}}, ||.||_{L^2})$ *and* $(T_u\mathcal{C}^L_{\mathcal{P}}, ||\ ||_\infty)$ *and we have the decomposition*
$$T_u\mathcal{C}^L_{\mathcal{P}} = \mathcal{D}_u \oplus \ker T_u\mathcal{E}. \tag{3.7}$$
3. *In the Banach space* $(T_u\mathcal{C}^L_{\mathcal{P}}, ||\ ||_\infty)$*, the restriction of* $T_u\mathcal{E}$ *to* $\mathcal{D}_u$ *is a continuous injective morphism into* $\mathbb{H}$

The proof of this Lemma used the same argument as in [6].

Definition 3.4. —

1. *The family* $u \mapsto \mathcal{D}_u$ *is a (closed) distribution on* $\mathcal{C}^L_{\mathcal{P}}$ *called the* **horizontal distribution***.*
2. *Each vector field* X *on* $\mathcal{C}^L_{\mathcal{P}}$ *which is tangent to* $\mathcal{D}$ *is called a* **horizontal vector field***.*

On $\mathcal{A}^L_{\mathcal{P}}$, the intersection $\mathcal{D}_u \cap T_u\mathcal{A}^L_{\mathcal{P}}$ gives rise to a (closed) Hilbert distribution $\mathcal{D}^{\mathcal{A}}$. Note that we can also define $\mathcal{D}^{\mathcal{A}}$ directly as the orthogonal of $\ker T_u\mathcal{E} \cap T_u\mathcal{A}^L_{\mathcal{P}}$ relatively to one of the equivalent inner products defined on $T_u\mathcal{A}^L_{\mathcal{P}}$ (see Remark 3.2). When no confusion is possible, this distribution $\mathcal{D}^{\mathcal{A}}$ on $\mathcal{A}^L_{\mathcal{P}}$ will be also denoted by $\mathcal{D}$ and also called the **horizontal distribution on $\mathcal{A}^L_{\mathcal{P}}$**.

The inner product on $\mathbb{H}$ gives rise to a Riemannian metric g on $T\mathbb{H} \equiv \mathbb{H} \times \mathbb{H}$ given by $g_x(u,v) =< u, v >$. Let $\phi : \mathbb{H} \to \mathbb{R}$ be a smooth function. The usual gradient of ϕ on $\mathbb{H}$ is the vector field

$$\mathrm{grad}(\phi) = (g^\flat)^{-1}(d\phi)$$

where $g^\flat$ is the canonical isomorphism of bundle from $T\mathbb{H}$ to its dual bundle $T^*\mathbb{H}$, corresponding to the Riesz representation i.e. $g^\flat(v)(w) =< v, w >$. So $\mathrm{grad}(\phi)$ is characterized by:

$$g(\mathrm{grad}(\phi), v) =< \mathrm{grad}(\phi), v >= d\phi(v) \tag{3.8}$$

for any $v \in \mathbb{H}$.

On the opposite, on $T\mathcal{C}^L_{\mathcal{P}}$, the Riemannian metric G is *only weak* (see Remark 3.2) and we cannot define in the same way the gradient of any smooth function on $\mathcal{C}^L_{\mathcal{P}}$. However let be
$G^\flat : T\mathcal{C}^L_{\mathcal{P}} \to T^*\mathcal{C}^L_{\mathcal{P}}$ the morphism bundle defined by:

$$G^\flat_u(v)(w) = G_u(v, w)$$

for any v and w in $T_u\mathcal{C}^L_{\mathcal{P}}$. Then we have

Lemma 3.5. — *Let $\phi : \mathbb{H} \to \mathbb{R}$ be a smooth function. Then* $\ker d(\phi \circ \mathcal{E})$ *contains* $\ker T\mathcal{E}$ *and belongs to $G^\flat_u(T_u\mathcal{C}^L_{\mathcal{P}})$. Moreover,*

$$\nabla\phi = (G^\flat)^{-1}(d(\phi \circ \mathcal{E})) \tag{3.9}$$

is tangent to $\mathcal{D}_u$, and we have

$$\nabla\phi(u)(s) = \mathrm{grad}(\phi)(\mathcal{E}(u)) - < \mathrm{grad}(\phi)(\mathcal{E}(u)), u(s) > u(s) \tag{3.10}$$

Remark 3.6. — When $\mathbb{H}$ is finite dimensional, the relation (3.10) is exactly the definition of $\nabla\phi$ given in [6].

Definition 3.7. — *For any smooth function $\phi : \mathbb{H} \to \mathbb{R}$, the vector field $\nabla\phi$ is called* **horizontal gradient** *of ϕ.*

To each vector $x \in \mathbb{H}$, we can associate the linear form x^* such that $x^*(z) =< z, x >$. So from Lemma 3.5 the horizontal gradient ∇x^* is well defined. In particular, to each vector e_i, $i \in \mathbb{N}$, of the Hilbert basis, we can associate the horizontal vector field $E_i = \nabla e_i^*$. Then as in [6] we have:

LEMMA 3.8. — *The family $\{E_i\}_{i\in\mathbb{N}}$ of vector fields generates the distribution $\mathcal{D}$.*

Proof. — Let $u \in \mathcal{C}^L_{\mathcal{P}}$ be; we can write

$$u(s) = \sum_{i\in\mathbb{N}} u_i(s)e_i$$

Denote by $\triangle_u$ the closed subspace generated by the family $\{E_i(u)\}_{i\in\mathbb{N}}$ in the normed space $(T_u\mathcal{C}^L_{\mathcal{P}}, || \ ||_{L^2})$. A vector $v \in T_u\mathcal{C}^L_{\mathcal{P}}$ belongs to the orthogonal of $\triangle_u$ (relatively to G) if and only if $G(v, E_i(u)) = 0$ for all $i \in N$. But as $< v(s), u(s) >= 0$ we have:

$$G(v, E_i) = \int_0^L < v(s), e_i - u_i(s)u(s) > ds =< \int_0^L v(s), e_i > \text{ for all } i \in \mathbb{N} \quad (3.11)$$

According to (3.11) v is orthogonal to $\triangle_u$ if and only if $v \in \ker T_u\mathcal{E}$. As $\mathcal{D}_u$ is also closed in $(T_u\mathcal{C}^L_{\mathcal{P}}, || \ ||_{L^2})$, we get $\triangle_u = \mathcal{D}_u$. □

Remark 3.9. —

1. *As in finite dimension (see [6]), for $\phi = e_i^*$ using the left member of (3.10), for any $i \in \mathbb{N}$ we have*

$$E_i(s) = e_i- < e_i, u(s) > u(s)$$

So, each $E_i(u)$ can be considered as a vector field on $\mathbb{S}^\infty$ along $u : [0, L] \to \mathbb{S}^\infty$. In this way, $E_i(u)$ is nothing but the orthogonal projection of e_i onto the tangent space to S^∞ along $u([0, L])$.

2. *On $\mathcal{A}^L_{\mathcal{P}}$ the induced inner product $< \ , \ >_{L^2}$ induces a (strong) Riemannian metric on the horizontal distribution $\mathcal{D}$. In the same way, to the Hilbert basis $\{e_i, \ i \in \mathbb{N}\}$ of $\mathbb{H}$ we can associate a family of global vector fields (again denoted) $\{E_i, \ i \in \mathbb{N}\}$ on $\mathcal{A}^L_{\mathcal{P}}$. In fact these vector fields are only the restriction to $\mathcal{A}^L_{\mathcal{P}}$ of the family defined on the whole manifold $\mathcal{C}^L_{\mathcal{P}}$. if we identify $T\mathcal{A}^L_{\mathcal{P}}$ with $[T\mathbb{S}^\infty]^N$ (see Remark 3.2), the vector field E_i at $u = (x_1, \cdots, x_N)$ is*

$$(e_i- < x_1, e_i > x_1, \cdots e_i- < x_N, e_1 > x_N).$$

If there is no ambiguity, we also denote these family in the same way. Of course, on $\mathcal{A}^L_{\mathcal{P}}$, the distribution $\mathcal{D}$ is also generated by this family of vector fields.

3.3. Set of critical values and set of singular points of the endpoint map

As the continuous linear map $T_u\mathcal{E} : T_u\mathcal{C}^L_{\mathcal{P}} \to T_{\mathcal{E}(u)}\mathbb{H} \equiv \mathbb{H}$ is closed (see [2] section 8.7), it follows that $\rho_u = T_u\mathcal{E}_{|\mathcal{D}_u}$ is an isomorphism from $\mathcal{D}_u$ to the closed subset $\rho_u(\mathcal{D}_u)$ of $\mathbb{H}$. Consider a point $u \in \mathcal{C}^L_{\mathcal{P}}$. According to remarks 3.2 *1.*, the annulator of $\rho_u(\mathcal{D}_u) = T_u\mathcal{E}(\mathcal{D}_u)$ is

$$[\rho_u(\mathcal{D}_u)]^0 = \{z \in T_{\mathcal{E}(u)}\mathbb{H} \equiv \mathbb{H} \text{ such that } < z, \rho_u(v) >= 0, \ \forall v \in \mathcal{D}_u\}$$

So, u is a **singular point** of $\mathcal{E}$ if and only if $[\rho_u(\mathcal{D}_u)]^0 \neq \{0\}$.

On the other hand, as the family $\{E_i(u)\}_{i\in\mathbb{N}}$ generates $\mathcal{D}_u$, any $z \in \mathbb{H}$ belongs to $[\rho_u(\mathcal{D}_u)]^0$ if and only if we have

$$< z, \int_0^L E_i(u)(s)ds >= 0, \ \forall i \in \mathbb{N} \tag{3.12}$$

Consider the decompositions $z = \sum_{i\in\mathbb{N}} z_i e_i$ and $u(s) = \sum_{i\in\mathbb{N}} u_i(s)e_i$. Then (3.12) is equivalent to

$$Lz_i = \sum_{j\in\mathbb{N}} \int_0^L u_i(s)u_j(s)z_j ds \ \forall i \in \mathbb{N} \tag{3.13}$$

Let Γ_u be the endomorphism defined by matrix of general term $(\int_0^L u_i(s)u_j(s)ds)$. Note that Γ_u is self-adjoint. The endomorphism $A_u = L.Id - \Gamma_u$ is also self-adjoint and, in fact, its matrix in the basis $\{e_i\}_{i\in\mathbb{N}}$ is $(L\delta_{ij} - \int_0^L u_i(s)u_j(s)ds)$. So (3.13) is equivalent to

$$A_u(z) = 0 \tag{3.14}$$

So u is a singular point if and only if L is an eigenvalue of Γ_u.

The proof of the following result is an adaptation of the argument used in finite dimension (see [6])

LEMMA 3.10. — *A point $u \in \mathcal{C}^L_{\mathcal{P}}$ is a singular point of $\mathcal{E}$ in and only if the vector space generated by $u([0, L])$ is 1-dimensional.*

Summarized proof. — At first, note that for any unitary automorphism U of $\mathbb{H}$ we have

$$U\Gamma_u U^* = \Gamma_{U(u)} \tag{3.15}$$

On the other hand, we have

$$\mathcal{E} \circ U(u) = U(\mathcal{E}(u)) \tag{3.16}$$

If $u([0, L])$ generates a 1-dimensional space then we have $u(s) = \pm x \in \mathbb{S}^\infty$. Using (3.15), without loss of generality, we can suppose that $u(s) = \pm e_1$ for any $s \in [0, L]$. In this case, using the relation obtained by derivation of (3.16) we show that e_1 is an eigenvector associated to the eigenvalue L of Γ_u and so $\ker(L.Id - \Gamma_u) = \ker A_u \neq \{0\}$.

On the other hand, if u is a singular point of $\mathcal{E}$, there exists a vector $x \in \mathbb{S}^\infty$ which is an eigenvector associated to the eigenvalue L of Γ_u. If U is an unitary automorphism such that $U(x) = e_1$ then e_1 is an eigenvector associated to L for $U\Gamma_u U^* = \Gamma_{U(u)}$. If we set $\bar{u} = U(u)$ then we get $\Gamma_{\bar{u}}(e_1) = Le_1$. So, for the decomposition $\bar{u}(s) = \sum_{i \in \mathbb{N}} \bar{u}_i(s) e_i$, we get $\int_0^L [\bar{u}_1]^2 = L$ and $\bar{u}_i(s) \equiv 0$ for all $i > 1$. It follows that $\bar{u}(s) = \pm e_1$ and so $u(s) = \pm x$. $\square$

According to Lemma 3.10, a point $u \in \mathcal{C}^L_{\mathcal{P}}$ is singular if and only if the restriction to $[s_{i-1}, s_i]$ is equal to $\pm x$ for some $x \in \mathbb{S}^\infty$. It follows that the **set of singular points** $\Sigma(\mathcal{E})$ of $\mathcal{E}$ is diffeomorphic to the projective space a $\mathbb{P}^\infty$ of $\mathbb{H}$.

On the other hand, let $u \in \Sigma(\mathcal{E})$ be with $u(s) \equiv x \in \mathbb{S}^\infty$. For any $v \in \mathcal{C}^0_{\mathcal{P}}([0, L], \mathbb{H})$ such that
$< u(s), v(s) >= 0$ for all $s \in [0, L]$ we consider

$$\bar{u}_n = \frac{1}{n} v(s) + x$$

As $||x|| = 1$, for n large enough, we have $||\bar{u}_n(s)|| \geqslant \frac{1}{2}$ and $u_n(s) = \frac{\bar{u}_n}{||\bar{u}_n(s)||}$ belongs to $\mathcal{C}^L_{\mathcal{P}} \setminus \Sigma(\mathcal{E})$. Moreover we have

$$\lim_{n \to \infty} u_n = u$$

So the set $\mathcal{C}^L_{\mathcal{P}} \setminus \Sigma(\mathcal{E})$ of **regular points** of $\mathcal{E}$ is an open dense subset of $\mathcal{C}^L_{\mathcal{P}}$.

Recall that the image of $\mathcal{E}$ is the closed ball $B(0, L)$ in $\mathbb{H}$. As in finite dimension, when $\mathcal{P} = \{0, L\}$ the set of critical values of $\mathcal{E}$ is then the boundary of $B(0, L)$ i.e. the sphere $S(0, L)$ and $\{0\}$. In the general case, $\mathcal{P} = \{a = s_0 < s_1 < ... < s_N = b\}$, the same argument applied to each subinterval $[s_{i-1}, s_i]$ gives that the **set of critical values** of $\mathcal{E}$ is the union of spheres $S(0, L_j)$ for $j = 1, \cdots n$ with $0 \leqslant L_j \leqslant L$.

Remark 3.11. —

1. *Recall that* $\rho_u = T_u\mathcal{E}_{|\mathcal{D}_u}$ *is an isomorphism from* $\mathcal{D}_u$ *to* $\rho_u(\mathcal{D}_u)$*, which is a closed subspace of* $\mathbb{H}$*. So on* $\mathcal{D}_u$*, the norm induced by* $|| \ ||_\infty$ *is equivalent the norm* $|| \ ||_{\mathbb{H}}$*; moreover,* ρ_u *is an isometry between* $\mathcal{D}_u$ *and* $\rho_u(\mathcal{D}_u)$ *endowed with the Hilbert induced norm. In particular, for any regular point* u*, the inverse of* ρ_u *is given by* $\dfrac{1}{L}\nabla v^*$ *and according to (3.11) we have* $\rho_u(\dfrac{1}{L}\nabla v^*) = v$*. So* $\{E_i(u),\ i \in \mathbb{N}\}$ *is then a Hilbert basis of* $\mathcal{D}_u$ *according to this isometry. If now,* u *is a singular point of* $\mathcal{E}$*, according to the previous proof, there exists a Hilbert basis* $\{e'_i,\ i \in \mathbb{N}\}$ *of* $\mathbb{H}$ *such that* $e'_1 = u(s)$ *for all* $s \in [0, L]$*. So,* ρ_u *is an isomorphism from* $\mathcal{D}_u$ *to* $\{e'_1\}^\perp$*. It follows that, on* $\mathcal{D}_u$*, the norm induced by* $|| \ ||_\infty$ *is equivalent to the norm* $|| \ ||_{\mathbb{H}}$ *so that* ρ_u *is an isometry between* $\mathcal{D}_u$ *and* $\{e'_1\}^\perp$*. Then, the family* $\{E'_i(u) = \nabla(e'_i)^*(u),\ i > 1\}$ *is a Hilbert basis of* $\mathcal{D}_u$*.*

2. *According to the beginning of this section, as* $G(E_i(u), E_j(u)) = L\delta_{ij} - \int_0^L u_i(s)u_j(s)ds$*, the matrix of* G *in the basis* $\{E_i(u),\ i \in \mathbb{N}\}$ *is the matrix of* $A_u = L.Id - \Gamma_u$*. But* A_u *is a self-adjoint endomorphism of* $\mathbb{H}$ *which is compact. So the sequence* $\{\lambda_i,\ i \in \mathbb{N}\}$ *of eigenvalues of* A_u *is bounded and converges to* 0 *and there exists a Hilbert basis* $\{e'_i,\ i \in \mathbb{N}\}$ *of eigenvectors of* A_u*. In this basis, the matrix of* A_u *is diagonal and equal to* $(L - \lambda_i\delta_{ij})$*. So for the associated family* $\{E'_i(u),\ i \in \mathbb{N}\}$ *of generators of* $\mathcal{D}_u$ *we have:*

   (1) if u *is regular, the matrix of* G *in the basis* $\{E'_i(u),\ i \in \mathbb{N}\}$ *is* $(L - \lambda_i)\delta_{ij}$*. Note that* 0 *is not an eigenvalue of* A_u *otherwise, it would mean that* u *is an eigenvector of* Γ_u *associated to the eigenvalue* L*. As the sequence* $\{\lambda_i,\ i \in \mathbb{N}\}$ *is bounded and converges to* 0*, there exists* $K > 0$ *so that* $\frac{1}{K} \leqslant L - \lambda_i \leqslant K$ *for any* $i \in \mathbb{N}$*. It follows that the norm associated to* G *and the norm associated to the isometry* ρ_u *are equivalents.*

   (2) if u *is singular, according to the proof of Lemma 3.10, we can choose* e'_1 *so that* $u = \pm e'_1$ *and then, by the same arguments as the ones used in (1) but applied to the restriction of* A_u *to* $\{e'_1\}^\perp$ *we again obtain that the norm associated to* G *and the norm associated to the isometry* ρ_u *are equivalent.*

Finally we obtain the following result :

Proposition 3.12. —

1. *The set $\mathcal{R}(\mathcal{E})$ (resp. $\mathcal{V}(\mathcal{E})$) of regular values (resp. points) of $\mathcal{E}$ is an open dense subset of $\mathcal{C}^L_{\mathcal{P}}$ (resp. $\mathbb{H}$).*
2. *For any $u \in \mathcal{R}(\mathcal{E})$ the linear map $\rho_u : \mathcal{D}_u \to \{\mathcal{E}(u)\} \times \mathbb{H}$ is an isomorphism and on $\mathcal{D}_u$, the inner product induced by $< \, , \, >_{L^2}$) and the inner product defined ρ_u from $\mathbb{H}$ are equivalent. Moreover the distribution $\mathcal{D}_{|\mathcal{R}(\mathcal{E})}$ is a weal Hilbert distribution which is a trivial Hilbert bundle over $\mathcal{R}(\mathcal{E})$*
3. *The distribution $\mathcal{D}_{|\Sigma(\mathcal{E})}$ is a weak Hilbert distribution which is isometrically isomorphic to $T\mathbb{P}^\infty$*

Proof. — We have to prove only the properties of $\mathcal{D}_{|\mathcal{R}(\mathcal{E})}$ and $\mathcal{D}_{|\Sigma(\mathcal{E})}$. Consider the map $F : \mathcal{C}^L_{\mathcal{P}} \times l^2(\mathbb{N}) \to \mathcal{C}^L_{\mathcal{P}} \times \mathcal{C}^0_{\mathcal{P}}([0, L], \mathbb{H})$ defined by

$$F(u, \sigma) = \sum_{i \in \mathbb{N}} \sigma_i E_i(u)$$

It is easy to see that the family of smooth vector fields $\{E_i, \ i \in \mathbb{N}\}$ satisfies the condition (LBs) for any $s \in \mathbb{N}$ at any point and as F is linear in σ, it follows that F is a smooth map. According to Lemma 3.8 and Remark 3.11, the range of F is $\mathcal{D}$. Again, from Remark 3.11, $\{E_i(u), \ i \in \mathbb{N}\}$ is an Hilbert basis of $\mathcal{D}_u$ for $u \in \mathcal{R}(\mathcal{E})$. So, the restriction of F to $\mathcal{R}(\mathcal{E}) \times l^2(\mathbb{N})$ is a global trivialization of $\mathcal{D}_{|\mathcal{R}(\mathcal{E})}$. The same argument can be used for the restriction of F to $\Sigma(\mathcal{E})$. $\square$

4. An control problem for a Hilbert snake

4.1. A problem of optimality and control

Recall that given any continuous piecewise C^k-curve $c : [0, T] \to \mathbb{H}$, a lift of c is a continuous piecewise C^k-curve $\gamma : [0, T] \to \mathcal{C}^L_{\mathcal{P}}$ such that $\mathcal{E}(\gamma(t)) = c(t)$. When such a lift γ is tangent to $\mathcal{D}$ we say that γ is a horizontal lift. By construction of $\mathcal{D}$ and the weak Riemannian metric G on $T_u\mathcal{C}^L_{\mathcal{P}}$ (cf subsection 3.2), if $v \in T_u\mathcal{C}^L_{\mathcal{P}}$ is such that $T\mathcal{E}(v) = \dot{c}(t)$ then the the quantity $\frac{1}{2}||v||^2_{L^2} = \frac{1}{2}\int_0^L ||v(s)||^2 ds = \frac{1}{2}G(v, v)$ is minimal if and only if v belongs to $\mathcal{D}_u$. It follows that a lift γ of c is horizontal, if and only γ for each $t \in [0, 1]$, the quantity $\frac{1}{2}G(\dot{\gamma}(t), \dot{\gamma}(t))$ is minimal in the set

$\{\frac{1}{2}G(v,v):\ v \in T_{\gamma(t)}\mathcal{C}^L_{\mathcal{P}}\}$. So we can consider a kind of **optimal problem** for a Hilbert snake which can be formulated in the following way:

given any continuous piecewise C^k-curve $c : [0,T] \to \mathbb{H}$, we look for a lift $\gamma : [0,1] \to \mathcal{C}^L_{\mathcal{P}}$, say $t \to u_t$, such that, for all $t \in [0,1]$,

– the associated family $S_t = \displaystyle\int_0^L u_t(s)ds$ of snakes satisfies $S_t(L) = c(t)$ for all $t \in [0,1]$:

– the **infinitesimal kinematic energy**: $\frac{1}{2}||\dot{\gamma}(t)||_{L^2} = \frac{1}{2}G(\dot{\gamma}(t),\dot{\gamma}(t))$ is minimal.

Then such a type of optimal problem has a solution if and only if the curve c has a horizontal lift. We shall say that such a horizontal lift is an **optimal control**.

On the other hand, we can also ask when two positions x_0 and x_1 of the "head" of the snake can be joined by a continuous piecewise smooth curve c which has an optimal control γ as lift. As in finite dimension, the **accessibility set** $\mathcal{A}(u)$, for some $u \in \mathcal{C}^L_{\mathcal{P}}$, is the set of endpoints $\gamma(T)$ for any piecewise smooth horizontal curve $\gamma : [0,T] \to \mathcal{C}^L_{\mathcal{P}}$ such that $\gamma(0) = u$. In this case if $x_0 = S_u(L)$ then any $z = S_{u'}(L)$ can be joined from x_0 by an absolutely continuous curve c which has an optimal control when u' belongs to $\mathcal{A}(u)$.

4.2. Properties of the accessibility sets

In finite dimension, given any horizontal distribution $\mathcal{D}$ on a finite dimension manifold M, the famous *Sussmann's Theorem* (see [7]) asserts that each accessibility set is a smooth immersed manifold which is an integral manifold of a distribution $\hat{\mathcal{D}}$ which contains $\mathcal{D}$ (i.e. $\mathcal{D}_x \subset \hat{\mathcal{D}}_x$ for any $x \in M$) and characterized by:

$\hat{\mathcal{D}}$ is the smallest distribution which contains $\mathcal{D}$ and which is invariant by the flow of any (local) vector field tangent to $\mathcal{D}$.

In the context of Banach manifolds the reader can find some generalization of this result in [4]. However, in our context, we will use the results of this paper to give some density results on accessibility sets, with analogue construction as in finite dimension case.

Precisely, according to subsection 2.2, we will associate to each Hilbert basis $\{e_i,\ i \in \mathbb{N}\}$ of $\mathbb{H}$, the family $\mathcal{X} = \{E_i,\ i \in \mathbb{N}\}$ of (global) vector fields on $\mathcal{C}^L_{\mathcal{P}}$ (cf Lemma 3.8). According to the definition of a $\mathcal{X}$-orbit (cf subsection 2.2) each accessibility set $\mathcal{A}(u)$ is contained in the $\mathcal{X}$-orbit through u. On the other hand, we can enlarge $\mathcal{X}$ into a set fields $\hat{\mathcal{X}}$ (cf (2.3)), and under some assumptions, we can show that the the distribution $\hat{\mathcal{D}}$ generated by $\hat{\mathcal{X}}$ is integrable (see [4] section 4) and each $\mathcal{X}$-orbit is dense in a maximal integral manifold of $\hat{\mathcal{D}}$. Unfortunately, in our context in infinite dimension these assumptions are not satisfied.

However, instead of $\hat{\mathcal{D}}$, we shall build a weak distribution $\bar{\mathcal{D}}$ which will give some analogue properties for the accessibility sets. More precisely, we extend this family $\mathcal{X} = \{E_i,\ i \in \mathbb{N}\}$ to a family $\{E_i, [E_j, E_k],\ i, j, k \in \mathbb{N}, k < l\}$ which generates a weak Hilbert distribution $\bar{\mathcal{D}}$ on $\mathcal{C}^L_{\mathcal{P}}$ with the following properties:

(i) $\bar{\mathcal{D}}$ does not depend on the choice of the basis $\{e_i,\ i \in \mathbb{N}\}$;

(ii) $\hat{\mathcal{D}}_x$ is dense in $\bar{\mathcal{D}}_x$ for all $x \in M$;

(iii) $\bar{\mathcal{D}}$ is integrable and each maximal integral manifold of $\bar{\mathcal{D}}$ contains the $\{E_i,\ i \in \mathbb{N}\}$-orbit for any choice of basis $\{e_i,\ i \in \mathbb{N}\}$ of $\mathbb{H}$;

(iv) the accessibility set of a point in any maximal integral manifold N of $\bar{\mathcal{D}}$ is a dense subset of N.

In this way we obtain:

Theorem 4.1. — *Let $\{e_i,\ i \in \mathbb{N}\}$ be a Hilbert basis of $\mathbb{H}$ and $\{E_i,\ i \in \mathbb{N}\}$ the associate family of vector fields on $\mathcal{C}^L_{\mathcal{P}}$. The vector space*

$$\bar{\mathcal{D}}_u = \{\sum_{i\in\mathbb{N}} x_i E_i(u) + \sum_{j,l\in\mathbb{N}, j<l} \xi_{ij}[E_i, E_j](u) : \ \sum (x_i)^2 < \infty,\ \sum (\xi_{ij})^2 < \infty\}$$

is a well defined subspace of $T_u\mathcal{C}^L_{\mathcal{P}}$ and carries a natural structure of Hilbert space such that the inclusion of $\bar{\mathcal{D}}_u$ in $T_u\mathcal{C}^L_{\mathcal{P}}$ is continuous and gives rise to a weak Hilbert distribution on $\mathcal{C}^L_{\mathcal{P}}$. This distribution has the following properties:

(1) $\bar{\mathcal{D}}$ does not depend on the choice of the Hilbert basis $\{e_i\}$ of $\mathbb{H}$.

(2) The distribution $\bar{\mathcal{D}}$ is integrable. Moreover, for each $u \in \mathcal{C}^L_{\mathcal{P}}$, the accessibility set $\mathcal{A}(u)$ is a dense subset of the maximal integral manifold N of $\bar{\mathcal{D}}$ through u

(3) on the manifold $\mathcal{A}^L_P$, each subspace $\bar{\mathcal{D}}_u \cap T_u\mathcal{A}^L_P$ induces a closed distribution (again denoted by $\bar{\mathcal{D}}$) which satisfies the two previous properties and moreover, in this case, each integral maximal manifold of this distribution is a Hilbert submanifold of $\mathcal{A}^L_P$.

Remark 4.2. — Recall that a horizontal curve γ is an absolutely continuous curve in $\mathcal{C}^L_{\mathcal{P}}$ which is almost everywhere tangent to $\mathcal{D}$. Given $u \in \mathcal{C}^L_{\mathcal{P}}$, we denote by $H_u \subset T_u\mathcal{C}^L_{\mathcal{P}}$ the set of tangent vectors at u of a horizontal curve through u which has a tangent vector at u. If X and Y are vector fields on $\mathcal{C}^L_{\mathcal{P}}$ whose domain contains u, the curve

$$t \to \Phi^X_t \circ \Phi^Y_t \circ \Phi^X_{-t} \circ \Phi^Y_{-t}(u)$$

is a horizontal curve and it is well known that its tangent vector at u is $[X,Y](u)$. So, if we look for the smallest (weak) manifold of $\mathcal{C}^L_{\mathcal{P}}$ which contains the accessibility set $\mathcal{A}(u)$, its tangent space must contain H_u. In particular, this tangent space must contain the family $\{E_i(u), [E_j,E_l](u), i,j,l \in \mathbb{N}\}$. Note that from Theorem 4.1, it follows that $\bar{\mathcal{D}}_u$ contains H_u. On one hand, if we consider the closed distribution generated by $\mathcal{X} = \{E_i, [E_j,E_l], i,j,l \in \mathbb{N}\}$, we can show that this distribution satisfies properties (i) and (ii) before Theorem 4.1 and also some property of type "upper local triviality" but not in terms in our definition (see section 2.1), in particular we can not prove that this distribution is integrable. In fact we do not know if this last distribution is integrable or not.

On the other hand, according to the following subsection, the l^1- weak distribution $\triangle^1$ generated by $\mathcal{X}$ satisfies property (2), but not property (1) and so $\triangle^1_u$ does not contain H_u. So, in this sense the distribution $\bar{\mathcal{D}}$ is the "smallest" weak distribution which is integrable and such that the maximal integral manifold through u contains $\mathcal{A}(u)$. Moreover, from property (1) of Theorem 4.1, any maximal integral manifold N, for any family $\mathcal{X}' = \{E'_i,\ i \in \mathbb{N}\}$ (associated to any t basis $\{e'_i,\ i \in \mathbb{N}\}$), the $\mathcal{X}'$-orbit of u is contained in N.

Finally, when $\mathbb{H}$ is a finite dimensional space, we have $\hat{\mathcal{D}} = \bar{\mathcal{D}}$, this distribution is closed, and $\bar{\mathcal{D}}$ is the "smallest" distribution whose leaves are the accessibility sets, as it is proved in [4] (see Example 4.5). We get another proof of the result of [6].

According to our problem of optimality for the head of the snake, we know that if u is a configuration, and N is the maximal integral manifold of $\bar{\mathcal{D}}$ through u, for all other configuration $v \in N$ there exists a sequences (γ_n) of horizontal curves in N whose origin u and whose sequence extremities converges to v. So, if $\mathcal{E}(u) = x$ and $\mathcal{E}(v) = y$, the family of curves $c_n = \mathcal{E} \circ \gamma_n$ are optimal (in the sense of subsection 4.1). The origin of each such curve c_n is x , and, the sequence of extremities y_n of c_n converges to y.

So, for each maximal integral manifold N of $\bar{\mathcal{D}}$, denote by N' the range $N' = \mathcal{E}(N)$. Then for each pair $(x,y) \in N'$ there exists a family of optimal curves c_n which have x for origin, and, the sequence of extremities y_n of c_n converges to y.

4.3. Construction of the distribution $\bar{\mathcal{D}}$

For the construction of $\bar{\mathcal{D}}$ we need the following result whose proof is the same as in the case of a finite dimensional Hilbert space $\mathbb{H}$ (see [6]). According to Remark 3.9, each E_i can be considered as a vector field on $\mathbb{S}^\infty$. In these way, we have

Lemma 4.3. — *The brackets of vector fields of the family $\{E_i\}_{i\in\mathbb{N}}$ satisfy the following relations:*

$[E_i, E_j](u) =< e_j, u > E_i(u)- < e_i, u > E_j(u)$ *for any* $u \in \mathcal{C}^L_{\mathcal{P}}$ *and any* $i, j \in \mathbb{N}$;

$[E_i[E_j, E_k]] = \delta_{ij}E_k - \delta_{ik}Ej$ *for any* $i, j, k \in \mathbb{N}$

$[[E_i, E_j], [E_k, E_l]] = \delta_{il}[E_j, E_k]+\delta_{jk}[E_i, E_l]-\delta_{ik}[E_j, E_l]-\delta_{jl}[E_i, E_k]$ *for any* $i, j, k, l \in \mathbb{N}$.

We consider the countable set of indexes $\Lambda = \{(i, j),\ i, j \in \mathbb{N},\ i < j\}$ and let $\mathbb{G}^1$ (resp $\mathbb{G}^2$) be the Banach space $l^1(\mathbb{N})\oplus l^1(\Lambda)$ (resp. $l^2(\mathbb{N})\oplus l^2(\Lambda)$). We then have the following result:

Lemma 4.4. —

1. *For $p = 1, 2$, the map Ψ^p from the trivial bundle $\mathcal{C}^L_{\mathcal{P}} \times \mathbb{G}^p$ to $T\mathcal{C}^L_{\mathcal{P}}$ characterized by:*
$$\Psi^p_u(\sigma, \xi) = \sum_{i\in\mathbb{N}} \sigma_i E_i(u) + \sum_{(i,j)\in\Lambda} \xi_{ij}[E_i, E_j](u), \tag{4.1}$$
$$\sigma = (\sigma_i) \in l^p(\mathbb{N}),\ \xi = (\xi_{ij}) \in l^p(\Lambda)$$
is well defined and each Ψ^p_u is a continuous linear map.

2. *For each $u \in \mathcal{C}^L_{\mathcal{P}}$, let $\mathbb{V}_u$ be the Hilbert subspace of $\mathbb{H}$ generated by the set*
$$\{u(t) - u(0),\ t \in [0, L]\}$$
For $p = 1, 2$, if the kernel of Ψ^p_u is not $\{0\}$ then $\mathbb{V}_u \neq \mathbb{H}$

3. *For $p = 1, 2$, the distribution $\triangle^p$ defined by $\triangle^p_u = \Psi^p_u(\mathbb{G}^p)$ is a weak distribution and the map Ψ^p defines a (global) upper trivialization of $\triangle^p$.*

4. *The distribution $\triangle^2$ do not depend of the choice the Hilbert basis (e_i) in $\mathbb{H}$ and contains $\mathcal{D}$ (i.e. $\mathcal{D}_u \subset \triangle^2_u$)*

Proof of Lemma 4.4.— Proof of part 1: For any $\sigma \in l^p(\mathbb{N})$ the vector $\sum_{i\in\mathbb{N}} \sigma_i e_i$ belongs to $\mathbb{H}$ and for any $s \in [0, L]$ the vector $\sum_{i\in\mathbb{N}} \sigma_i E_i(u(s))$ is the orthogonal projection on $T_{u(s)}\mathbb{S}^\infty$ of $\sum_{i\in\mathbb{N}} \sigma_i e_i$. So we have

$$||\sum_{i\in\mathbb{N}} \sigma_i E_i(u)||_\infty \leqslant [\sum_{i\in\mathbb{N}}(\sigma_i)^2]^{1/2} = ||\sigma||_2 \tag{4.2}$$

If σ belongs to $l^1(\mathbb{N})$, as $||\sigma||_2 \leqslant |\sigma||_1$ in this case we get

$$||\sum_{i\in\mathbb{N}} \sigma_i E_i(u)||_\infty \leqslant ||\sigma||_1$$

On the other hand as $[E_k, E_l](u) = u_l E_k(u) - u_k E_l(u)$, in the same way, for any $s \in [0, L]$, the vector $\sum_{(k,l)\in\Lambda} \xi_{kl}[E_k, E_l](u(s))$ is the orthogonal projection of $\sum_{(k,l)\in\Lambda} \xi_{kl}(u_l(s)e_k - u_k(s)e_l)$ on $T_{u(s)}\mathbb{S}^\infty$.

But we have:

$$\sum_{(k,l)\in\Lambda} \xi_{kl} u_l(s) e_k = \sum_{k\in\mathbb{N}}[\sum_{l>k} \xi_{kl} u_l(s)] e_k]$$

So we get

$$||\sum_{(k,l)\in\Lambda} \xi_{kl} u_l(s) e_k||^2 = \sum_{k\in\mathbb{N}}[\sum_{l>k} \xi_{kl} u_l](s)]^2$$

Using the fact that $|u_j(s)| \leqslant ||u(s)|| = 1$, from Cauchy -Schwartz inequality we get:

$$|\sum_{l>k} \xi_{kl} u_l](s)| \leqslant [\sum_{l>k}(\xi_{kl})^2]^{1/2}$$

Finally we obtain

$$||\sum_{(k,l)\in\Lambda} \xi_{kl} u_l(s) e_k||^2 \leqslant \sum_{(k,l)\in\Lambda} (\xi_{kl})^2 = (||\xi||_2)^2$$

By same argument we get

$$||\sum_{(k,l)\in\Lambda} \xi_{kl} u_k(s) e_l||^2 \leqslant \sum_{(k,l)\in\Lambda} (\xi_{kj})^2 = (||\xi||_2)^2$$

So we obtain

$$||\sum_{(k,l)\in\Lambda}\xi_{kl}[E_k,E_l](u)||_\infty\leqslant 2||\xi||_2$$

If $\xi\in l^1(\Lambda)$ by same argument as previously we also get:

$$||\sum_{(k,l)\in\Lambda}\xi_{kl}[E_k,E_l](u)||_\infty\leqslant 2||\xi||_1$$

Finally we get :

$$||\Psi^p_u(\sigma,\xi)||_\infty\leqslant 2||(\sigma,\xi)||_p \text{ for } p=1,2 \tag{4.3}$$

It follows that Ψ^p is well defined. From its expression, it is easy to see that Ψ^p_u is linear and continuous from (4.3). This ends the proof of part 1.

Proof of part 2. — At first, note that as the natural inclusion $I:\mathbb{G}^1\hookrightarrow\mathbb{G}^2$ is continuous and with dense range, we have $\Psi^2_u\circ I=\Psi^1_u$, the closure of $\ker\Psi^1_u$ in $\mathbb{G}^2$ is equal to $\ker\Psi^2_u$. So $\ker\Psi^1_u\neq 0$ if and only if $\ker\Psi^2_u\neq 0$. Assume that $\ker\Psi^1_u\neq\{0\}$.

Let be $(\sigma,\xi)\in\ker\Psi^1_u$. According to (4.1), and Remark 3.9, we must have

$$\sum_{i\in\mathbb{N}}\sigma_iE_i(u(s))+\sum_{(i,j)\in\Lambda}\xi_{ij}[u_i(s)E_j(u(s))-u_j(s)E_i(s)]=0 \text{ for any } s\in[0,L] \tag{4.4}$$

We set $\bar{\xi}_{kj}=\dfrac{\xi_{kj}}{2}$ (resp. $\bar{\xi}_{kj}=-\dfrac{\xi_{kj}}{2}$) for $j<k$ (resp $j>k$) and $\bar{\xi}_{jj}=0$. Then (4.4) can be written:

$$\sum_{i\in\mathbb{N}}[\sum_{j\in\mathbb{N}}(\bar{\xi}_{ij}u_j(s)+\sigma_i]E_i(u(s))=0 \text{ for any } s\in[0,L] \tag{4.5}$$

Given any $\xi\in l^1(\Lambda)$, denote Ξ the endomorphism of $l^1(\mathbb{N})$ whose matrix in the canonical basis is precisely $(\bar{\xi}_{ij},\ i,j\in\mathbb{N})$. So, (4.5) is equivalent to

$$\Xi u(s)=-\sigma \text{ for any } s\in[0,L] \tag{4.6}$$

So, σ must belong to the range of Ξ.

According to the definition of $\mathbb{V}_u$, (4.6) is equivalent to

$\Xi u(0)=-\sigma$ and $\mathbb{V}_u\subset\ker\Xi$.

Proof of part 3. — From part 1, $\triangle^p_u = \Phi^p_u(\mathbb{G}^p)$ gives rise to a well defined distribution on $\mathcal{C}^L_{\mathcal{P}}$. On the other hand, denote by $\hat{\Psi}^p_u$ the canonical bijection induced by Ψ^p_u:

$$\hat{\Psi}^p_u : \mathbb{G}^p / \ker \Psi^p_u \to \triangle^p_u$$

So we can put on $\triangle^p_u$ the Banach structure so that $\hat{\Psi}^p_u$ is an isometry. In this way, $\triangle^p$ is then a weak distribution. On the other hand, the family of smooth vector field $\{E_i,\ i \in \mathbb{N}\}$ satisfies the condition (LBs) for any $s \in \mathbb{N}$ at any point, and as Ψ^1_u is linear with Lipschitz constant independent of u, the map $(u, (\sigma, \xi)) \mapsto \Psi^p_u(\sigma, \xi)$ is smooth.

It remains to show that $\ker \Psi^p_u$ is complemented in $\mathbb{G}^p$ for each $u \in \mathcal{C}^L_{\mathcal{P}}$. At first, for $p = 2$, as $\mathbb{G}^2$ is a Hilbert space, it is always true. In particular, the previous Banach structure on each $\triangle^2_u$ is a Hilbert structure. However, we shall show this result for each case $p = 1$ and $p = 2$.

Assume that $u \in \mathcal{R}(\mathcal{E})$

If $\ker \Psi^p_u = \{0\}$ there is nothing to prove. Now assume that $\ker \Psi^p_u \neq \{0\}$. At first, suppose that we have a partition $\mathbb{N} = A \cup B$ such that $\{e_a,\ a \in A\}$ (resp. $\{e_b,\ b \in B\}$) is a Hilbert basis of $[\mathbb{V}_u]^\perp$ (resp. $\mathbb{V}_u$). By construction, each component u_a is constant, for all $a \in A$. So the Lie brackets $[E_a, E_{a'}]$, for $a, a' \in A$, belongs to $\mathcal{D}_u$. Let be

$$\mathbb{K} = \{\xi \in l^p(\Lambda) \text{ such that } \xi_{ij} = 0 \text{ if } i \text{ or } j \in B\}$$

According to the notations of the proof of part 2, for any $\xi \in \mathbb{K}$ if we denote again by Ξ the associated endomorphism of $\mathbb{H}$, $\ker \Xi$ contains $\mathbb{V}_u$ and if $\sigma = -\Xi u(0)$ then (σ, ξ) belongs to $\ker \Psi^p_u$. So, the subspace

$$\hat{\mathbb{K}} = \{(\sigma, \xi) \in l^p(\mathbb{N}) \oplus l^p(\Lambda),\ \xi \in \mathbb{K}, \sigma = -\Xi u(0)\}$$

is contained in $\ker \Psi^p_u$.

On the other hand, if (σ, ξ) belongs to $\ker \Psi^p_u$, from the proof of part 2 and we have $\sigma = -\Xi u(0)$ and $\mathbb{V}_u \subset \ker \Xi$ and, as $\{e_b\}$ is a basis of $\mathbb{V}_u$, we then have $(\sigma, \xi) \in \hat{\mathbb{K}}$. It follows that $\ker \Psi^1_u$ is complemented:

if we denote by $\mathbb{L}$ is the subspace of $\{\xi \in l^1(\Lambda),\ \xi_{ij} = 0 \text{ for all } i, j \in A\}$, then the subspace $l^p(\mathbb{N}) \oplus \mathbb{L}$ is a complement subspace of $\ker \Psi^p_u$.

In the general case, choose a Hilbert basis $\{e'_a,\ a \in A\}$ (resp $e'_b,\ b \in B\}$) of $[\mathbb{V}_u]^\perp$ (resp. $\mathbb{V}_u$). There exists a linear isometry T of $\mathbb{H}$ such that $T(e'_a) = e_a$ for $a \in A$ and $T(e'_b) = e_b$ for $b \in B$. Denote by $E'_j = \nabla(e'_j)^*$ the associated vector field on $\mathcal{C}^L_{\mathcal{P}}$ (see Lemma 3.5). The map $\tilde{T} : (z, v) \to (z, T(v))$ is an isomorphism of $T\mathbb{H}$ such that $\tilde{T}(E'_j)(u) = E_j(u)$ for any $j \in \mathbb{N}$. Consider the map $\Psi' : \mathbb{G}^1 \times \mathcal{C}^L_{\mathcal{P}} \to T\mathcal{C}^L_{\mathcal{P}}$ defined by

$$\Psi'_u(\sigma,\xi) = \sum_{i\in\mathbb{N}} \sigma_i E'_i(u) + \sum_{(i,j)\in\Lambda} \xi_{ij}[E'_i, E'_j](u), \sigma = (\sigma_i) \in l^p(\mathbb{N}), \xi = (\xi_{ij}) \in l^p(\Lambda)$$

Of course, we have

$$\Psi^p_u = \Psi'_u \circ T$$

But in the new basis, for Ψ'_u we are in the previous situation. So, it follows that $\ker \Psi^p_u$ is complemented, which ends the proof of part 3.

Assume now that $u \in \Sigma(\mathcal{E})$.
According to the proof of Lemma 3.10, then $u(t) = \pm x \in \mathbb{S}^\infty$ for any $t \in [0, L]$, there exists an Hilbert basis $\{e'_i,\ i \in \mathbb{N}\}$ such that $x = e'_1$, the associated family $\{E'_i(u),\ i > 1\}$ is a basis of $\triangle_u$ and we have $E'_1(u) = 0$ (see Remark 3.8). Moreover, as the components of u are constant, from Lemma 4.3, all brackets $[E'_j, E'_l](u)$ belongs to $\mathcal{D}_u$ for $i > 1$ and $j > 1$ and $[E'_1, E'j] = -x_i E_j$ also belongs to $\triangle_u$. As previously, we can consider the map

$$\Psi'_u(\sigma,\xi) = \sum_{i\in\mathbb{N}} \sigma_i E'_i(u) + \sum_{(k,l)\in\Lambda} \xi_{kj}[E'_i, E'_j](u), \sigma = (\sigma_i) \in l^p(\mathbb{N}), \xi = (\xi_{ij}) \in l^p(\Lambda)$$

Its kernel is $\mathbb{R}e'_1 \oplus l^p(\Lambda)$. From the same argument as previously, we obtain that the $\ker \Psi^p_u$ is complemented. Moreover, the restriction of Ψ^p_u to $l^P(\mathbb{N})$ has a kernel of dimension 1 and the restriction of Ψ^p to the orthogonal $[\ker \Psi^p_u]$ in $l^1(\mathbb{N})$ is an isomorphism onto $\triangle^p_u$.

Proof of part 4. — Now, we must show that the range of Ψ^2_u does not depend on the choice of the Hilbert basis (e_i) of $\mathbb{H}$. So given any other Hilbert basis (e'_j) of $\mathbb{H}$, denote again by $E'_j = \nabla(e'_j)^*$ the associated vector field on $\mathcal{C}^L_{\mathcal{P}}$. So we have the decomposition:

$$E'_i = \sum_{j\in\mathbb{N}} a^j_i E_j \text{ and}$$

$$[E'_j, E'_k] = \sum_{l,m\in\mathbb{N}} a^l_j a^m_k [E_l, E_m] = \sum_{(l,m)\in\Lambda} (a^l_j a^m_k - a^m_j a^l_k)[E_l, E_m] \qquad (4.7)$$

Let T be the isometry of $\mathbb{H}$ defined by $T(e_i) = e'_i$. Let $l^2_{BA}(\mathbb{H})$ be the set of Hilbert-Schmidt bilinear antisymmetric maps. Then $\{e^*_i \wedge e^*_j,\ (i,j) \in \Lambda\}$ is a Hilbert basis of $l^2_{BA}(\mathbb{H})$. Denote by T^2 the isometry of $l^2_{BA}(\mathbb{H})$

induced by T on $l^2_{BA}(\mathbb{H})$. Then matrix of T^2 in this basis is precisely $[(a_i^k a_j^l - a_j^k a_i^l)]_{(i,j),(k,l)\in\Lambda}$. It follows that T (resp. T^2) is an isometry of $l^2(\mathbb{N})$ (resp. $l^2(\Lambda)$)

On the other hand, to the choice (e'_i) of a basis of $\mathbb{H}$ is naturally associated the map $\Psi'^2 : \mathcal{C}^L_{\mathcal{P}} \times \mathbb{G}^p \to T\mathcal{C}^L_{\mathcal{P}}$ characterized by:

$$\Psi'^2_u(\sigma,\xi) = \sum_{i\in\mathbb{N}} \sigma_i E'_a(u) + \sum_{(j,k)\in\Lambda} \xi_{jk}[E'_j, E'_k](u)$$

According to (4.7) we have:

$$\Psi^2_u(\sigma,\xi) = \Psi'^2_u(T\sigma, T^2\xi)$$

So we have $\Psi'^2_u(\mathbb{G}^2) = \triangle^2_u$ for any $u \in \mathcal{C}^L_{\mathcal{P}}$.

On the other hand, according to Lemma 3.8, it is clear that $\mathcal{D}_u$ is contained in $\triangle^2_u$. According to Remark 3.11, We can note that $\psi^2_u(l^2(\mathbb{N}) = \mathcal{D}_u$ and from the proof of part 3, $\Psi^2(l^2(\Lambda))$ is a complemented space of $\mathcal{D}_u$ in $\triangle^2_u$ $\square$

4.4. Proof of Theorem 4.1

Claim 1.— *The distributions $\triangle^1$ and $\triangle^2 = \bar{\mathcal{D}}$ are integrable.*

Proof.— According to Lemma 4.4 , we have $\bar{\mathcal{D}}_u = \triangle^2_u$ and so $\bar{\mathcal{D}}$ is a well defined weak Hilbert distribution on $\mathcal{C}^L_{\mathcal{P}}$ which does not depend on the choice of the Hilbert basis $\{e_i,\ i\in\mathbb{N}\}$ of $\mathbb{H}$.
We take place in the context of the proof of Lemma 4.4. According to Lemma 4.3, Lemma 4.4 and Theorem 2.1, it follows that the distribution $\triangle^1$ is integrable. On the other hand, according to Lemma 4.3, Lemma 4.4 and Theorem 2.1 we also get that $\bar{\mathcal{D}} = \triangle^2$ is integrable. We again denote by $\mathcal{X}$ the family $\{E_i,\ i\in\mathbb{N}\}$ of vector fields.

Claim 2.— *Any $\mathcal{X}$-orbit is contained in a maximal integral manifold of $\bar{\mathcal{D}}$.*

Proof.— For the sake of simplicity, we only denote by $\mathbb{G}$ the previous Hilbert space $\mathbb{G}^2$. As the distribution $\bar{\mathcal{D}}$ is integrable, let $f : N \to \mathcal{C}^L_{\mathcal{P}}$ be any maximal integral manifold of $\bar{\mathcal{D}}$. Without loss of generality, we can identify N with $f(N)$ and take $f = i_N$ the natural inclusion of N (with its Hilbert manifold structure) into $\mathcal{C}^L_{\mathcal{P}}$. Consider the pull-back bundle $f^*(\mathcal{C}^L_{\mathcal{P}}\times\mathbb{G})$ over N. Note that $f^*(\mathcal{C}^L_{\mathcal{P}}\times\mathbb{G})$ can be identified with $N\times\mathbb{G}$. As the range of Ψ_u is $\bar{\mathcal{D}}_u$, for any u, the bundle morphism $\Psi : \mathcal{C}^L_{\mathcal{P}}\times\mathbb{G} \to T\mathcal{C}^L_{\mathcal{P}}$ induces a bundle morphism $\tilde{\Psi}$ from $N \times \mathbb{G}$ to TN which is onto. Moreover, the orthogonal of $\ker \tilde{\Psi}_u$ in $\{u\} \times \mathbb{G}$, gives rises to a Hilbert sub-bundle of $N \times \mathbb{G}$. Denote by $\mathcal{N}$ this sub-bundle and by Π the natural orthogonal projection of $N \times \mathbb{G}$ on $\mathcal{N}$. Now, we have $\Pi \circ \tilde{\Psi} = \tilde{\Psi}$ and the restriction of $\tilde{\Psi}$ to $\mathcal{N}$ is an isomorphism from $\mathcal{N}$ onto TN and we have

$$Tf \circ \tilde{\Psi} = \Psi \circ (Id \times f) \tag{4.8}$$

Now, given the canonical Hilbert basis $\{\epsilon_i, \omega_{jl},\ i \in \mathbb{N}, (j,l) \in \Lambda\}$ of $\mathbb{G}$, we set $\hat{E}_i(u) = \tilde{\Psi}_u(\epsilon_i)$ and $\hat{E}_{jl}(u) = \tilde{\Psi}_u(\omega_{jl})$. In fact, $\hat{E}_i$ and $\hat{E}_{jl}$ are smooth global vector fields on N. On the other hand, we have $\Psi_v(\epsilon_i)(u) = E_i(u)$ and $\Psi_v(\omega_{jl}) = [E_j, E_l](v)$ for any $v \in f(N)$.
According to proposition 2.3, there exist (global) vector fields $\tilde{E}_i$ on N such that $f_* \tilde{E}_i = E_i$ and so $f_*[\tilde{E}_j, \tilde{E}_l] = [E_j, E_l]$. It follows from (4.8) that $\hat{E}_i = \tilde{E}_i$ and $\hat{E}_{jl} = [\tilde{E}_j, \tilde{E}_l]$.

Let $\tilde{\mathcal{X}}$ be the induced family $\{\tilde{E}_i, [\tilde{E}_j, \tilde{E}_l],\ i, j, l \in \mathbb{N},\ j < l\}$. As Ψ (resp. $\tilde{\Psi}$) is a (global) upper trivialization for $\bar{\mathcal{D}}$ on $\mathcal{C}^L_{\mathcal{P}}$ (resp TN on N), it follows that, for any $u \in \mathcal{C}^L_{\mathcal{P}}$ (resp.$u \in N$), there exists an open neighborhood $U \subset \mathcal{C}^L_{\mathcal{P}}$ (resp. $\tilde{U} \subset N$) of u such that $\mathcal{X}$ (resp. $\tilde{\mathcal{X}}$) satisfies the condition (LBs) on U (resp. $\tilde{U}$) for $s > 3$ (see [4], proof of Theorem 6, part 2).
Consider any family $\xi = \{X_\alpha,\ \alpha \in A\} \subset \mathcal{X}$ and let $\tilde{\xi} = \{\tilde{X}_\alpha,\ \alpha \in A\}$ be the corresponding family on a maximal integral manifold N. Given $u \in f(N)$, consider some flow Φ^ξ_τ associated to ξ and let $\gamma(t) = \Phi^\xi_\tau(t, u)$ be the integral curve defined on $[0, ||\tau||_1]$. From proposition 2.3, there exists a curve $\tilde{\gamma}$: $[0, ||\tau||_1[\to N$ such that $f \circ \tilde{\gamma} = \gamma$ on $[0, ||\tau||_1[$. We set $v = \gamma(||\tau||_1)$.

We shall show that v also belongs to $f(N)$, or equivalently, $\Phi^\xi_\tau(||\tau||_1, u) = \phi^\xi_\tau(u)$ belongs to N.

Consider a maximal integral manifold $g \equiv i_M : M \to \mathcal{C}^L_{\mathcal{P}}$ of $\bar{\mathcal{D}}$ through v and set $\tilde{v} = (i_M)^{-1}(v)$. As, we have already seen, if $\mathcal{X}'$ is the family of vector fields $\{E'_i, [E'_i, E'_l],\ i, j, l, \in \mathbb{N}, j < l\}$ on M such that $g_* E'_i = E_i$, then $\mathcal{X}'$ satisfies the condition (LBs). On N, we also have a family $\xi' = \{X'_\alpha, \alpha \in A\}$ defined on a neighborhood of v and so $g_* X'_\alpha = X_\alpha$. Then ξ' also satisfies the condition (LBs) for $s > 3$. So, from Theorem 2 of [4], there exists $\eta > 0$ such that, for $\tau' \in l^1(A)$ with $||\tau'||_1 \leqslant \eta$, the corresponding flow $\Phi^{\xi'}_{\tau'}(.,.)$ is defined on a neighborhood $\tilde{V}$ of $\tilde{v} = g^{-1}(v)$ in M. Now coming back to the original flow Φ^ξ_τ on $\mathcal{C}^L_{\mathcal{P}}$, if $\tau = (\tau_\alpha)_{\alpha \in A}$, there exists α_0 such that $\displaystyle\sum_{\alpha \geqslant \alpha_0} |\tau_\alpha| < \eta$. Then, given any $a \in A$ with $a \geqslant \alpha_0$, we set $\tau_a = (\tau'_\alpha)$ with $\tau'_\alpha = 0$ for $\alpha < a$ and $\tau'_\alpha = \tau_\alpha$ for $\alpha \geqslant a$. The corresponding flows $\Phi^{\xi'}_{\tau_a}$ and

$\hat{\Phi}^{\xi'}_{\tau_a}$ are defined on M. Moreover, we have

$$g \circ \Phi^{\xi'}_{\tau_a}(t, \tilde{z}) = \Phi^{\xi}_{\tau_a}(t, g(\tilde{z})) \text{ and } g \circ \hat{\Phi}^{\xi'}_{\tau_a}(t, \tilde{z}) = \hat{\Phi}^{\xi}_{\tau_a}(t, g(\tilde{z})) \tag{4.9}$$

for any $\tilde{z} \in \tilde{V}$.
By construction of the flow Φ^{ξ}_{τ} we have

$$\Phi^{\xi}_{\tau_a}(||\tau_a||_1, \gamma(\tau_a)) = v \text{ and so } \hat{\Phi}^{\xi}_{\tau_a}(||\tau_a||_1, v) = \gamma(\tau_a)$$

For any $a \geqslant \alpha_0$, in $\mathcal{C}^L_{\mathcal{P}}$ consider the curve $\hat{\gamma}_a(s) = \hat{\Phi}^{\xi}_{t_a}(||\tau_a||_1 - s, v)$. This curve is defined on $[0, ||\tau_a||_1]$ and joins v to $\gamma(\tau_a)$. In the same way, in M, consider the curve $\hat{\gamma}'_a(s) = \hat{\Phi}^{\xi'}_{\tau_a}(||\tau_a||_1 - s, v)$. This curve is also defined on $[0, ||\tau_a||_1]$ and joins $\tilde{v}$ to $\tilde{v}_a$ in N. According to (4.9) we have

$$g \circ \hat{\gamma}'_a = \hat{\gamma}_a.$$

In particular, we get $g(\tilde{v}) = \gamma(\tau_a)$. But $\gamma(\tau_a)$ belongs to $f(N) \equiv N$ and to $g(M) \equiv M$ as subsets of $\mathcal{C}^L_{\mathcal{P}}$. But, $(N, f \equiv i_N)$ and $(M, g \equiv i_M)$ are maximal integral manifolds of $\bar{\mathcal{D}}$. So, as $N \cap M \neq \emptyset$, we must have $N = M$ and so we can extend $\tilde{\gamma}$ to the closed interval $[0, ||\tau||_1]$ and, in particular, $\phi^{\xi}_{\tau}(u) = \Phi^{\xi}_{\tau}(||\tau||_1, u)$ belongs to N.

Now if we have $v = \Phi(u)$ for some $\Phi \in \mathcal{G}_{\mathcal{X}}$ (see subsection 2.2), then Φ is a finite composition of local diffeomorphisms of type ϕ^{ξ}_{τ} or $[\phi^{\xi}_{\tau}]^{-1}$ or of type Φ^X_t for some $X \in \mathcal{X}$. From the previous argument, if $u \in N$, then $\phi^{\xi}_{\tau}(u)$ and $[\phi^{\xi}_{\tau}]^{-1}(u)$ belong to N and from Lemma 2.3, part 1, $\Phi^X_t(u)$ also belongs to N. By induction we obtain that $v = \Phi(u)$ belongs to N. So the $\mathcal{X}$-orbit $\mathcal{O}(u)$ of u is contained in N.

Claim 3. — *If N is a maximal integral manifold of $\bar{\mathcal{D}}$, the accessibility set $\mathcal{A}(u)$ of $u \in N$ is a dense subset of N.*

Proof. — From Theorem 2.4, $\mathcal{O}(u)$ contains the maximal integral manifold N^1 of $\triangle^1$ through u and N^1 is dense in $\mathcal{O}(u)$ (for the topology of $\mathcal{C}^L_{\mathcal{P}}$), and so N^1 and $\mathcal{O}(u)$ have the same closure in $\mathcal{C}^L_{\mathcal{P}}$. But we have $N^1 \subset \mathcal{O}(u) \subset N$. As the inclusion of N in $\mathcal{C}^L_{\mathcal{P}}$ is continuous we obtain that $\mathcal{O}(u)$ is dense in N.

From proposition 2.2 the set $\mathcal{A}(u)$ is a dense subset of $\mathcal{O}(u)$. On the other hand, according to proposition 2.3, we see that $\mathcal{A}(u)$ is contained in N. So we obtain that $\mathcal{A}(u)$ is dense in N.

End of the proof of Theorem 4.4.. — Now, it remains to show that the same results are true on $\mathcal{A}^L_{\mathcal{P}}$. It is easy to see (and it is left to the reader) that all the proofs of Lemma 4.4 work in the same way on the manifold $\mathcal{A}^L_{\mathcal{P}}$. So the previous arguments work too in this context. But, in $\mathcal{A}^L_{\mathcal{P}}$, the corresponding distribution $\bar{\mathcal{D}}$ is closed. So each maximal integral manifold of $\bar{\mathcal{D}}$ is a weak Hilbert manifold whose topology is the topology induced by the topology of the Hilbert manifold $\mathcal{A}^L_{\mathcal{P}}$. So, such a manifold must be a Hilbert submanifold of $\mathcal{A}^L_{\mathcal{P}}$. $\square$

5. Appendix: Almost Lie algebroid structures

The purpose of this appendix is to construct some (almost) Lie Banach algebroid on $\mathcal{C}^L_{\mathcal{P}}$ and also on each maximal integral manifold of $\bar{\mathcal{D}}$ and some properties of horizontal lifts in this last structures. These result are used in [1].

Structures on $\mathcal{C}^L_{\mathcal{P}}$

According to Lemma 4.4, for $p = 1, 2$, on $\mathbb{G}^p$ we define a Lie algebra structure in the following way:

let be $(\epsilon_i)_{i\in\mathbb{N}}$ (resp. $(\epsilon_{ij})_{(i,j)\in\Lambda}$ the canonical basis of $(l^p(\mathbb{N})$ (resp. $(l^p(\Lambda))$; according to Lemma 4.3, we then define:

$[\epsilon_i, \epsilon_j] = \omega_{ij}$, for all $i, j \in \mathbb{N}$

$[\epsilon_i, \omega_{jk}] = \delta_{ij}\epsilon_k - \delta_{ik}\epsilon_j$, for all $i \in \mathbb{N}$ and $(j, k) \in \Lambda$

$[\omega_{ij}, \omega_{kl}] = \delta_{il}\omega_{jk} + \delta_{jk}\omega_{il} - \delta_{ik}\omega_{jl} - \delta_{jl}\omega_{ik}$, for all $(i, j)(kl) \in \Lambda$.

For any $x = \sum x_i\alpha_i$, $y = \sum y_j\epsilon_j$ in $l^p(\mathbb{N})$ and $\xi = \sum \xi_{ij}\omega_{ij}$, $\eta = \sum \eta_{kl}\beta_{kl}$ in $l^p(\Lambda)$, naturally we can define:

$$[\sigma, \sigma'] = \sum_{i,j\in\mathbb{N}} \sigma_i\sigma'_j[\epsilon_i, \epsilon_j]$$

$$[\sigma, \eta] = \sum_{i\in\mathbb{N},(k,l)\in\Lambda} \sigma_i\eta_{kl}[\epsilon_i, \omega_{kl}]$$

$$[\xi, \eta] = \sum_{(i,j)\in\Lambda,(k,l)\in\Lambda} \xi_{ij}\eta_{kl}[\omega_{ij}, \omega_{kl}].$$

Now, according to Lemma 4.4, the map $\Psi^p : \mathcal{C}^L_{\mathcal{P}}\times\mathbb{G}^p \to T\mathcal{C}^L_{\mathcal{P}}$ is morphism bundle over $\mathcal{C}^L_{\mathcal{P}}$. Moreover, each section φ of the trivial bundle $\mathcal{C}^L_{\mathcal{P}} \times \mathbb{G} \to \mathcal{C}^L_{\mathcal{P}}$ can be identified with a map $\varphi : \mathcal{C}^L_{\mathcal{P}} \to \mathbb{G}^p$. So, on the set $\Gamma(\mathbb{G}^p)$ of section of this trivial bundle we can define a Lie bracket by:

$$[\varphi, \varphi'](u) = [\varphi(u), \varphi'(u)] + d\varphi(\Psi^p(u, \varphi'(u)) - d\varphi'(\Psi^p u, \varphi(u))$$

According to [5] section 4 or [1], it follows that $(\mathbb{G}^p \times \mathcal{C}^L_{\mathcal{P}}\, \Psi, \mathcal{C}^L_{\mathcal{P}}, [\ ,\ ])$ has a **Banach Lie algebroid structure** on $\mathcal{C}^L_{\mathcal{P}}$

In $\mathbb{G}^p$ let be $\pi : \mathbb{G}^p \to l^p(\mathbb{N})$ the canonical projection whose kernel is $l^p(\Lambda)$ and denote again by $\pi : \mathcal{C}^L_{\mathcal{P}} \times \mathbb{G}^p \to \mathcal{C}^L_{\mathcal{P}} \times l^p(\mathbb{N})$ the associated projection bundle. Again any section of the trivial bundle $\mathcal{C}^L_{\mathcal{P}} \times l^p(\mathbb{N}) \to \mathcal{C}^L_{\mathcal{P}}$ can be identified with a map from $\mathcal{C}^L_{\mathcal{P}}$ to $l^p(\mathbb{N})$. Of course the set $\Gamma(l^p(\mathbb{N}))$ of such sections is contained in $\Gamma(\mathbb{G}^p)$. So, according to [1], on $\Gamma(l^p(\mathbb{N}))$, we can define an almost Banach Lie bracket by:

$$[[\varphi, \varphi']](u) = \pi([\varphi, \varphi'](u)).$$

So, if we denote by θ^p the restriction of Ψ^p to $\mathcal{C}^L_{\mathcal{P}} \times l^p(\mathbb{N})$ we get an **almost Banach Lie algebroid structure** $(\mathcal{C}^L_{\mathcal{P}} \times l^p(\mathbb{N}), \Psi{\cdot}\mathcal{C}^L_{\mathcal{P}}, [\ ,\ ])$ on $\mathcal{C}^L_{\mathcal{P}}$.

Moreover, again, according to [1] subsection 4.3, the inner product on $l^2(\mathbb{N})$ gives rise to **strong Riemaniann metric** on $(\mathcal{C}^L_{\mathcal{P}} \times l^2(\mathbb{N}), \Psi^2, \mathcal{C}^L_{\mathcal{P}}, [\ ,\ ])$. Note that, according to Remark 3.11, the induced inner product on $\mathcal{D}_u$ is equivalent to the inner product associated to the Riemaniann metric G.

Structures on a maximal integral manifold N of $\bar{\mathcal{D}}$

Given a maximal integral manifold (f, N) of $\bar{\mathcal{D}}$, the pull back $f_*(\mathcal{C}^L_{\mathcal{P}} \times l^2(\mathbb{N}))$ and $f^*(\mathcal{C}^L_{\mathcal{P}} \times \mathbb{G}^2)$ can be identified with $N \times l^2(\mathbb{N})$ and $N \times \mathbb{G}$ respectively. Then, θ^2 and Ψ^2 induces anchors $\theta_N : N \times l^2(\mathbb{N}) \to TN$ and $\Psi_N : N \times \mathbb{G}^2 \to TN$ and the almost bracket $[[\ ,\ ]]$ induces an almost bracket again denoted $[[\ ,\ ]]$. So $(N \times l^2(\mathbb{N}), \theta_N, N, [[\ ,\ ]])$ is an **almost Banach Lie algebroid** on N and $(N \times \mathbb{G}^2, \Psi_N, N, [[\ ,\ ]])$ is a **Banach Lie algebroid** on N. Moreover the canonical scalar product on $l^2(\mathbb{N})$ (resp. $\mathbb{G}^2$) gives rise to a strong Riemannian metric on $(N \times l^2(\mathbb{N}), \theta_N, N, [[\ ,\ ]])$ (resp. on $(N \times \mathbb{G}^2, \Psi_N, N, [[\ ,\ ]])$).

Essentially from the proof of Lemma 4.4 we get the following result (used in [1])

PROPOSITION 5.1. — *Fix some $u \in N$. Then we have the following properties*

(1) Assume that $u \in \Sigma(\mathcal{E})$ then $N = \Sigma(\mathcal{E})$. Let be $\mathcal{L}_v$ the 1-codimensional Hilbert subspace $[\ker \theta_N]^{\perp}_v \subset \{v\} \times l^2(\mathbb{N})$ for any $v \in N$. Then $\mathcal{L} = \cup_{v \in N} \mathcal{L}_v$ is a 1-codimensional Hilbert sub-bundle of $N \times l^2(\mathbb{N})$ and the

restriction ψ_N *of* Ψ_N *to* $\mathcal{L}$ *is an isomorphism onto* TN *and we have* $\mathcal{D}_{|N} = TN$.

(2) Assume that $u \in \mathcal{R}(\mathcal{E})$. *Let be* $\mathbb{V}_u$ *the Hilbert subspace of* $\mathbb{H}$ *generated by the set*

$$\{u(t) - u(0),\ t \in [0, L]\}$$

and choose an Hilbert basis $\{e'_a,\ a \in A\}$ *(resp* $e'_b,\ b \in B\}$*) of* $[\mathbb{V}_u]^\perp$ *(resp.* $\mathbb{V}_u$*) (see the proof of part 3 of Lemma 4.4). If* Λ_u *is the set of pair* $(i,j) \in \Lambda$ *such that that* i *or* j *do not belongs to* A*, then* N *is an Hilbert manifold modeled on* $l^2(\mathbb{N}) \oplus l^2(\Lambda_u)$ *and is contained in* $\mathcal{R}(\mathcal{E})$ *.*
Let be $\mathcal{L}_v$ *the orthogonal of* $\ker[\Psi_N]_v \subset \{v\} \times \mathbb{G}^2$. *Then* $\mathcal{L} = \cup_{v \in N} \mathcal{L}_v$ *is a Hilbert sub-bundle of* $N \times \mathbb{G}^2$ *which contains* $N \times l^2(\mathbb{N})$ *and the restriction of* ψ_N *of* Ψ_N *to* $\mathcal{L}$ *is an isomorphism on* TN
Moreover, $\mathcal{L}$ *contains* $N \times l^2(N)$ *and the restriction of* θ_N *to* $N \times l^2(N)$ *is an isomorphism on* $\mathcal{D}_{|N}$

(3) Let be γ *an horizontal piecewise* C^k*-curve in* N*. Then there exists an unique piecewise* C^{k-1} *section* (γ, σ) *of* $\mathcal{L}$ *over* γ *such that* $\psi_N(\gamma(t)), \sigma(t)) = \dot{\gamma}(t)$ *for all* t.

Proof. — From the proof of part 3 of Lemma 4.4 for $p = 2$, we get that $\bar{\mathcal{D}}_u = \mathcal{D}_u$ for any $u \in \Sigma(\mathcal{E})$. So $\Sigma(\mathcal{E})$ is an integral manifold of $\bar{D}$. On the other hand, from the proof of part 3 of Lemma 4.4 for $p = 2$, taking $u \in \mathcal{R}(\mathcal{E})$ we get that the tangent space to N is modeled on $l^2(\mathbb{N}) \oplus l^2(\Lambda_u)$ (with the notations introduced in part (2)). As for $u \in \mathcal{R}(\mathcal{E})$, we have $\mathcal{D} \neq \bar{\mathcal{D}}$, it follows that $\Sigma(\mathcal{E})$ is a maximal integral manifold of $\bar{D}$, and, for $u \in \mathcal{R}(\mathcal{E})$, so N is contained in $\mathcal{R}(\mathcal{E})$.

As $\Psi_N : N \times \mathbb{G}^2 \to TN$ is a surjective Hilbert bundle morphism, the kernel of this morphism is an Hilbert sub-bundle $\mathcal{K}$ of $N \times \mathbb{G}^2$. It follows that $\mathcal{L}$ is also an Hilbert sub-bundle of $N \times \mathbb{G}^2$ and the restriction ψ_N of Ψ_N to each $\mathcal{L}$ is an isomorphism onto TN. Given $u \in N$, if $u \in \mathcal{R}(\mathcal{E})$, according to notations in (2), by the same arguments used in proof of part 4 of Lemma 4.4, we can show that $l^2(\mathbb{N})$ is contained in $\mathcal{L}_u$. Now, if we identify $\mathbb{H}$ with $l^2(\mathbb{N})$, for $u \in \Sigma(\mathcal{E})$, the kernel of Ψ^2_u in $\{u\} \times \mathbb{G}^2$ is $\mathbb{R}.\mathcal{E}(u) \oplus l^2(\Lambda)$. So the vector space $\mathcal{L}_u$ is the orthogonal of $\ker[\Psi_N]_u$ in $l^2(\mathbb{N})$ (see the proof of part 3 of Lemma 4.4).

Let be $\gamma : [0, T] \to N$ a horizontal piecewise C^k-curve for $k \geqslant 1$. Assume that $N \subset \mathcal{R}(\mathcal{E})$. On one hand, $\psi_N : \mathcal{L} \to TN$ is an isomorphism and on the other hand $\psi_N(N \times l^2(\mathbb{N})) = \mathcal{D}_{|N}$ for $u \in \mathcal{R}(\mathcal{E})$ (resp. $\psi_N(\mathcal{L} = \mathcal{D}_{|N}$ for $u \in \Sigma(\mathcal{E})$. So the restriction θ_N of ψ_N to $N \times l^2(\mathbb{N})$ is an isomorphism of bundle onto $\mathcal{D}_{|N}$. As γ is horizontal, the curve $(\gamma(t), x(t)) = (\gamma(t), [\theta_N]^{-1}(\dot{g}(t))$ is a well defined piecewise C^{k-1}-curve which satisfies the conclusion in (3). Finally, when $N = \Sigma(\mathcal{E})$, ψ_N is an isomorphism from $\mathcal{L}$ on TN (see part (1)) so conclusion (3) is clear in this case. $\square$

Bibliography

[1] Cabau (P.), Pelletier (F.). — Almost Lie algebroid structure on an anchored Banach bundle, Journal of Geometry and Physics Vol 62, p. 2147-2169 (2012).

[2] Dieudonné (J.). — Fondements de l'analyse moderne, Cahiers scientifiques Fasc XXVII, Gauthier-Villars, Paris (1967).

[3] Hausmann (J.-C.). — Contrôle des bras articulés et transformations de Moëbus, Enseignement Mathématique, t. 51, p. 87-115 (2005).

[4] Lathuille (A.), Pelletier (F.). — On Sussmann theorem for orbits of set of vector fields on Banach manifolds, Bulletin des Sciences Mathématiques Vol 136, p. 579-616 (2012).

[5] Pelletier (F.). — Integrability of weak distributions on Banach manifolds, Indagationes Mathematicae 23, p. 214-242 (2012).

[6] Rodriguez (E.). — L'algorithme du charmeur de serpents, PhD Thesis, University of Geneva,
http://www.unige.ch/cyberdocuments/theses2006/RodriguezE/these.pdf.

[7] Sussmann (H.-J.). — Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc., vol 80, p. 171-188 (1973).