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Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties

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Real Monge–Ampère equations and Kähler–Ricci solitons on toric log Fano varieties

ROBERT J. BERMAN, BO BERNDTSSON⁽¹⁾

ABSTRACT. — We show, using a direct variational approach, that the second boundary value problem for the Monge–Ampère equation in \mathbb{R}^n with exponential non-linearity and target a convex body P is solvable iff 0 is the barycenter of P . Combined with some toric geometry this confirms, in particular, the (generalized) Yau–Tian–Donaldson conjecture for toric log Fano varieties (X, Δ) saying that (X, Δ) admits a (singular) Kähler–Einstein metric iff it is K-stable in the algebro-geometric sense. We thus obtain a new proof and extend to the log Fano setting the seminal result of Wang–Zhou concerning the case when X is smooth and Δ is trivial. Li’s toric formula for the greatest lower bound on the Ricci curvature is also generalized. More generally, we obtain Kähler–Ricci solitons on any log Fano variety and show that they appear as the large time limit of the Kähler–Ricci flow. Furthermore, using duality, we also confirm a conjecture of Donaldson concerning solutions to Abreu’s boundary value problem on the convex body P in the case of a given canonical measure on the boundary of P .

RÉSUMÉ. — Nous montrons, grâce à une approche variationnelle directe, que le deuxième problème avec valeurs au bord pour l’équation de Monge–Ampère dans \mathbb{R}^n avec non-linéarité exponentielle, et ensemble cible un corps convexe P , admet une solution si et seulement si 0 est le barycentre de P . En combinant ce résultat avec de la géométrie torique, on obtient en particulier confirmation de la conjecture de Yau–Tian–Donaldson (généralisée) pour les variétés toriques log-Fano (X, Δ) ; à savoir que (X, Δ) admet une métrique de Kähler–Einstein (singulière) si et seulement si elle est K-stable au sens algèbro-géométrique. Nous obtenons donc une nouvelle démonstration, qui s’étend au cas log-Fano, du résultat fondateur de Wang–Zhou qui concerne le cas où X est lisse et Δ est trivial. Nous généralisons également la formule torique de Li pour la borne inférieure de la courbure de Ricci. Plus généralement, nous

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obtenons des solitons de Kähler-Ricci sur toute variété (singulière) log-Fano, et montrons qu'ils apparaissent comme la limite en temps grand du flot de Kähler-Ricci. De plus, en utilisant la dualité, nous confirmons aussi une conjecture de Donaldson sur les solutions du problème de valeurs au bord d'Abreu sur le corps convexe P dans le cas d'une mesure canonique donnée sur la frontière de P .

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1. Introduction

1.1. Monge-Ampère equations in \mathbb{R}^n

Let us start by recalling the setting for the second boundary value problem for the real Monge-Ampère operator in the entire space \mathbb{R}^n [4]. A convex function ϕ on \mathbb{R}^n is said to be a (classical) solution for the latter problem if it is smooth and satisfies the following two conditions:

$$(i) \quad \det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = F(\phi, d\phi),$$

where F is a given positive smooth function on \mathbb{R}^{n+1} and

$$(ii) \quad d\phi(\mathbb{R}^n) = \Omega$$

where Ω is a (necessarily convex) given domain in \mathbb{R}^n . We will be concerned with the case when the domain Ω is *bounded*, i.e. its closure $P := \bar{\Omega}$ is a convex body and

$$F(t, p) = e^{-\gamma t} g(p)^{-1}$$

for $\gamma \in \mathbb{R}$, where g is a positive smooth function on \mathbb{R}^n . After a trivial scaling we may as well assume that $\gamma = \pm 1$. As is well-known, the positive case is, by far, most challenging one and the equation does then usually not admit any solutions. Our main result gives the general structure of the solutions:

THEOREM 1.1. — *Let P be a convex body containing 0 in its interior. Then there is a smooth convex function ϕ such that*

$$g(d\phi) \det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = e^{-\phi}$$

and such that its gradient induces a diffeomorphism

$$d\phi : \mathbb{R}^n \rightarrow \text{int}(P)$$

iff 0 is the barycenter for the measure $g(p)dp$ on P .

Moreover, ϕ is then uniquely determined up to the action of the additive group \mathbb{R}^n by translations and

- $\phi(x) - \sup_{p \in P} \langle x, p \rangle$ *is globally bounded on \mathbb{R}^n*
- *the Legendre transform ϕ^* of ϕ is Hölder continuous up to the boundary of P for any Hölder exponent in $[0, 1[$.*

The proof uses a variational approach to construct a solution ϕ as a maximizer of the functional

$$\mathcal{G}(\phi) := \log \int_{\mathbb{R}^n} e^{-\phi} dx - \int_P \phi^* g dp$$

on the space of all convex functions whose gradient image is P . The main point of the argument is to establish a direct coercivity estimate for the latter functional of independent interest, which can be seen as a refined Moser-Trudinger type inequality (see Theorem 2.16). In fact, the argument shows that *any* asymptotically minimizing sequence of the functional above converges – up to normalization – to a solution ϕ as in the previous theorem. This extra flexibility will be used when establishing the convergence of the Kähler-Ricci flow below.

1.2. Toric Kähler-Einstein geometry

We will mainly focus on the case when

$$g(p) = e^{\langle a, p \rangle}$$

for a given vector $a \in \mathbb{R}^n$. The main differential-geometrical motivation comes from the study of *Kähler-Einstein metrics* or more generally *Kähler-Ricci solitons* on toric varieties.

1.2.1. Kähler-Einstein metrics

Recall that a Kähler form ω on a compact complex manifold X is a closed positive two-form, which equivalently means that, locally,

$$\omega = i\partial\bar{\partial}\phi \tag{1.1}$$

for a local function ϕ , called the Kähler potential. The Kähler metric ω is said to be *Kähler-Einstein* if the Riemannian metric defined by its real part has constant Ricci curvature, which in form notation is written as

$$\text{Ric } \omega = \gamma\omega$$

for some $\gamma = 0, -1$ or $+1$. Since the Ricci form $\text{Ric } \omega$ represents the first Chern class of X :

$$c_1(X) := c_1(-K_X),$$

where $-K_X := \Lambda^n(TX)$, is the anti-canonical line bundle on X , it follows that, in the case $\gamma = \pm 1$, the Kähler potential ϕ in 1.1 represents a positively curved metric on the line bundle $-\gamma K_X$. Hence, if X admits a Kähler-Einstein metric then $c_1(X)$ is non-positive if $\gamma \leq 0$ and positive if $\gamma > 0$. Conversely, as shown in the fundamental works of Yau and Aubin (when $\gamma < 0$) any complex manifold X with $c_1(X)$ non-positive admits a Kähler-Einstein metric. However, in the case when $c_1(X)$ is positive, i.e. X is a *Fano manifold*, there are well-known obstructions to the existence of Kähler-Einstein metrics and the fundamental Yau-Tian-Donaldson conjecture expresses all the obstructions in terms of a suitable notion of algebro-geometric stability (see section 4.3):

CONJECTURE (Yau-Tian-Donaldson). — *A Fano manifold X admits a Kähler-Einstein metric iff it is K -polystable.*

More generally, from the point of view of current birational algebraic geometry, or more precisely the Minimal Model Program (MMP), it is natural to allow X to be a *singular* Fano variety or more generally to consider the category of *log Fano varieties* (X, Δ) , where X is a normal algebraic variety and Δ is a \mathbb{Q} -divisor on X such that the log anti-canonical line bundle $-(K_X + \Delta)$ is an ample \mathbb{Q} -line bundle. Here we will also assume, as usual, that the coefficients of Δ are < 1 , but we do allow negative coefficients (see section 3.1). The notion of K -stability still makes sense for X singular (see [46, 43] for recent developments) and its log version was recently considered in [32, 41, 47]. As for the notion of a Kähler-Einstein metric ω associated to a log Fano variety (X, Δ) it was recently studied in [12]: by definition ω is a Kähler current in $c_1(-(K_X + \Delta))$ with continuous potentials, satisfying

the following equation of currents on X :

$$\text{Ric } \omega - [\Delta] = \omega \tag{1.2}$$

By the regularity result in [12] such a (singular) metric ω restricts to a bona fide Kähler-Einstein metric on the Zariski open set X_0 defined as the complement of Δ in the regular locus of X (X_0 is sometimes called the log regular locus of X). See also section 1.4 below for relations to the theory of Kähler-Einstein metrics on Fano manifolds with edge-cone singularities, where there has been great progress recently.

The present paper concerns the case of *toric log Fano varieties* (X, Δ) . In particular, X is a toric variety, i.e. a compact projective algebraic variety with an action of the complex torus

$$T_c := \mathbb{C}^{*n} \cong T \times \mathbb{R}^n,$$

(where T is the real torus) such that (X, T_c) is an equivariant compactification of T_c with its standard action on itself and the divisor Δ is supported “at infinity”, i.e. in $X - T_c$. As explained in section 3) there is a correspondence

$$(X, \Delta) \longleftrightarrow P$$

between n -dimensional toric log Fano varieties (X, Δ) and rational convex polytopes P in \mathbb{R}^n containing 0 in their interior. Briefly, if P is written as the intersection of affine half-spaces $\langle l_F, \cdot \rangle \geq -a_F$, where the index F runs over all facets of P and l_F denotes the inward primitive lattice vector, normal to the facet F , then

$$\Delta = \sum_F (1 - a_F) D_F,$$

where D_F is the toric invariant prime divisor defined by the facet F . Applying Theorem 1.1 to such a polytope P , with $g = 1$, we then deduce the following

THEOREM 1.2. — *Let X be a toric Fano variety. Then the following is equivalent:*

- (X, Δ) admits a toric log Kähler-Einstein metric
- 0 is the barycenter of the canonical polytope $P_{(X, \Delta)}$ associated to X
- The log Futaki invariants of (X, Δ) vanish
- (X, Δ) is log K -polystable with respect to toric degenerations

This confirms the (generalized) Yau-Tian-Donaldson conjecture in the category of toric log Fano varieties. Of course, it is natural to ask if log K-polystability wrt toric degenerations implies log K-polystability wrt *any* test configuration? In fact, as shown very recently in [8], in a general non-toric setting, the existence of a log Kähler-Einstein metric does imply log K-polystability and hence the full Yau-Tian-Donaldson conjecture holds for any toric log Fano variety.

In the case when X is smooth and Δ is the trivial divisor the previous theorem was first shown in the seminal work [60] by Zhou-Wang, except for the last point, proven in [61]. One of our motivations for considering Kähler-Einstein metrics on *singular* toric varieties X comes from our recent work on the Ehrhart volume conjecture for polytopes [10]. Another motivation comes from the fact that, while there exist only a finite number of smooth Fano varieties of dimension n , there exists an infinite number of singular ones. On the other hand it is well-known that the number becomes finite if the Gorenstein index of X is fixed. The most well-studied class of toric Fano varieties are those of Gorenstein index one, which correspond to *reflexive* lattice polytopes P (i.e. the dual P^* is also a lattice polytope). This is a huge class of lattice polytopes which plays an important role in string theory, as they give rise to many examples of mirror symmetric Calabi-Yau manifolds [6]. Already in dimension three there are 4319 isomorphism classes of such polytopes [39], while there are only 105 families of *smooth* Fano threefolds, all in all.

For a general log Fano variety (X, Δ) we also obtain a generalization of recent results of Székelyhidi [54] and Li [40] concerning greatest lower bounds on the Ricci curvature of metrics in $c_1(-(K_X + \Delta))$ (see Theorems 3.8, 3.7).

1.2.2. Kähler-Ricci solitons

In the case when X is smooth it was furthermore shown in [60] that any toric Fano manifold admits a (shrinking) *Kähler-Ricci soliton*, i.e. a Kähler metric ω and an associated holomorphic vector field V on X such that

$$\text{Ric } \omega = \omega + L_V \omega, \tag{1.3}$$

where L_V denotes the Lie derivative of ω wrt (the real part of) V . In the case when (X, Δ) is a log Fano variety we will say that ω is a *log Kähler-Ricci soliton* associated to (X, Δ, V) if ω is a Kähler current in $c_1(-(K_X + \Delta))$ with continuous potentials satisfying the equation 1.3, with $\text{Ric } \omega$ replaced by the log Ricci curvature $\text{Ric } \omega - [\Delta]$ and such that ω is smooth on X_0 . One motivation for studying Kähler-Ricci solitons on a singular toric variety (even when Δ is trivial) is a conjecture of Tian [57] saying that on any

Fano manifold the Kähler-Ricci flow converges, modulo automorphisms, to a Kähler-Ricci soliton on a Zariski open set of codimension at least two (the complex structure is allowed to jump in the limit; see also [52] for the corresponding conjecture for general Fano varieties).

THEOREM 1.3. — *Any toric log Fano variety (X, Δ) admits a (singular) toric log Kähler-Ricci soliton (ω, V) , where the metric ω is unique up to toric automorphisms and the vector field V is uniquely determined by the vanishing of the modified log Futaki invariants associated to (X, Δ, V) . Concretely, V is the invariant holomorphic vector field with components a_i , where the vector a is the unique critical point of the Laplace transform of the measure $1_{P(X, \Delta)} dp$.*

We briefly remark that, given a log Fano variety (X, Δ) , it seems natural to expect that one can obtain a *complete* Kähler-Ricci soliton on the quasi-projective variety $X - \Delta$ (i.e. the complement in X of the support of Δ) by taking suitable limits of log Kähler-Ricci solitons (see section 3.9). This is in line with the discussion in [32] concerning limits of Kähler-Einstein metrics with edge-cone singularities.

It is interesting to compare the uniqueness property for toric Kähler-Einstein metrics contained in the previous theorem with the general results in [14, 12] saying that any two log Kähler-Einstein metrics associated to a given log Fano variety (X, Δ) coincide up to the action of the automorphism group of (X, Δ) , when Δ has *positive* coefficients. However, when negative coefficients are present it is well-known that this uniqueness property fails in general and hence the uniqueness property in the toric category – in the case when the divisor Δ has negative coefficients – appears to be rather surprising (compare the discussion in example 3.5).

We will also show, building on [12], that on any Fano variety X the Kähler-Ricci flow converges weakly, modulo automorphisms, to a Kähler-Ricci soliton on X (Theorem 5.1). This gives a (weak) confirmation of the toric case of the conjecture in [52] (which asks for the stronger notion of Gromov-Hausdorff convergence). We recall that in the case of a smooth Fano variety, not necessary toric, the (strong) convergence towards a Kähler-Ricci soliton – when one exists – was shown by Tian-Zhou [59], using Perelman’s estimates.

We next turn to a dual version of Theorem 1.1 formulated directly on the convex body P . It concerns the “Kähler-Einstein case” when $g = 1$ and is motivated by the works of Abreu [1] and Donaldson [31]. First recall that a Kähler metric ω on a complex manifold X satisfies the Kähler-Einstein equation precisely when ω is in $c_1(X)$ and its scalar curvature S_ω

is constant and equal to one with appropriate normalizations. Moreover, as shown by Bando-Mabuchi ω is then a minimizer of the Mabuchi K-energy functional.

1.3. Abreu’s equation on a convex body

As shown by Abreu [1] in the toric setting the scalar curvature of the Kähler metric on T_c induced by the Hessian of a smooth and strictly convex function ϕ on \mathbb{R}^n may be written in term of the Legendre transform u of ϕ as $S(u)$, where S is the following fourth order fully non-linear operator:

$$S(u) := - \sum_{i=1}^n \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}, \tag{1.4}$$

where (u^{ij}) denotes the inverse of the Hessian matrix $(u_{ij}) = (\frac{\partial^2 u}{\partial x_i \partial x_j})$. As a consequence any smooth solution ϕ as in Theorem 1.1 (for $g = 1$) yields a solution to an equation in the interior of P involving $S(u)$:

$$S(u) = 1$$

But there may be many very different solutions to the latter equation since the boundary behavior of u at ∂P has to be taken into account. To make this precise we note that there is canonical measure σ_P defined on the boundary of P . It may be defined in terms of the normal variations of the domain P (see formula 4.3). Following Donaldson [29] any measure σ , absolutely continuous wrt the induced Euclidean measure $\lambda_{\partial P}$ on the boundary, defines a functional \mathcal{F}_σ on the space C^∞ of all strictly convex functions u on P which are smooth in the interior and continuous up to the boundary:

$$\mathcal{F}_\sigma(u) := - \int_P \log \det(u_{ij}) dp + \mathcal{L}_\sigma(u),$$

where \mathcal{L}_σ is the linear functional

$$\mathcal{L}_\sigma(u) := \left(\int_{\partial P} u \sigma - a \int_P u dp \right), \quad a := \int_{\partial P} \sigma / \int_P dp \tag{1.5}$$

As explained in section 4.2 the functional \mathcal{F}_{σ_P} may, in the case when $P = P_{(X, \Delta)}$ for a log Fano variety (X, Δ) be identified with the log version of the Mabuchi K-energy functional.

THEOREM 1.4. — *Let P be a convex body containing 0 in its interior. Then the functional \mathcal{F}_{σ_P} admits a minimizer u in C^∞ iff 0 is the barycenter*

of P . Moreover, the minimizer is then unique modulo the addition of affine functions and satisfies Abreu's equation

$$S(u) = 1$$

in the interior of P .

Donaldson conjectured (see Conjecture 7.2.2 in [29]) that, given any measure σ as above there is a corresponding minimizer under the following condition:

$$\mathcal{L}_\sigma(u) > 0$$

for any non-affine convex function. In our “canonical case” where the measure in question is σ_P the latter condition is satisfied precisely when 0 is the barycenter of P (see Lemma 4.9) and the previous corollary thus confirms Donaldson's conjecture in this case. The case when P is a two-dimensional polytope and σ coincides with a multiple of $\lambda_{\partial P}$ on each facet was settled by Donaldson in a series of papers leading up to [31]. As emphasized in [29] the main motivation for Donaldson's conjecture comes from the toric version of the general Yau-Tian-Donaldson conjecture concerning constant scalar curvature metrics in $c_1(L)$ for a given polarized manifold (X, L) , which, as explained by Donaldson, corresponds to a certain measure on the boundary of the lattice polytope $P_{(X,L)}$ determined by the integral structure. As it turns out this latter measure coincides with our measure σ_P precisely when (X, L) is equal to $(X, -K_X)$ for a toric Fano variety (see section 4.2). The point – from our point of view – is that any toric line bundle $L \rightarrow X$ can always be written as $L = -(K_X + \Delta)$, where (X, Δ) is a toric log Fano variety and hence Theorem 1.3 furnishes, under the corresponding barycenter condition, a Kähler current in $c_1(L)$ with constant Ricci curvature on $X - \Delta$ and where the singularities along Δ are encoded by the measure σ_P (compare Cor 3.9).

1.4. Further comparison with previous results and methods

In terms of toric geometry the key ingredient in our approach is a direct convexity argument showing that the Ding type functional $\mathcal{G}_{(X,\Delta,V)}$ associated to a toric log Fano variety (X, Δ) with a toric vector field V is *relatively proper* (in the sense of [62, 63]) and even *relatively coercive* on the space of T -invariant metrics, if the appropriate assumption on the barycenter (Futaki invariant) holds (see Theorems 2.16, 4.5)). Given this relative properness we can adapt the variational approach in [16, 7, 12] to our setting to deduce the existence of a maximizer satisfying the corresponding Kähler-Ricci solution equation. The coercivity of $\mathcal{G}_{(X,\Delta,V)}$ implies in particular that the corresponding Mabuchi K-energy type functional $\mathcal{M}_{(X,\Delta,V)}$ is also relatively coercive. It should be pointed out that in the general setting of a

smooth Fano manifold X , not necessarily toric, but with $\Delta = 0$, the properness of the corresponding functionals – a priori assuming the existence of a Kähler-Einstein metric – was shown by Tian [57], who also conjectured its coercivity, eventually proved in [48]. For the corresponding results in the presence of a Kähler-Ricci soliton, see [24].

Another variational approach approach, in the more general setting of constant scalar curvature Kähler metrics in $c_1(L)$, for (X, L) smooth and toric, has been developed in [62, 61, 63] building on [29]. In particular, it is shown in [63], that if the corresponding Mabuchi functional $\mathcal{M}_{(X, L)}$ is relatively proper, then it admits a weak minimizer. However, the question of its regularity and whether it satisfies the constant scalar curvature equation was left open. One virtue of the present approach is thus that, when $L = -K_X$, the minimizer can indeed be shown to satisfy the Kähler-Einstein equation, even in the general setting of log Kähler-Einstein metrics and Kähler-Ricci solitons. On the other hand, our methods are closely tied to the Monge-Ampère operator and it does not seem clear, at this point, how to extend them to the general setting of constant scalar curvature metrics.

Log Fano varieties (X, Δ) with X smooth and $\Delta = (1-t)D$ for a smooth divisor D and $t \in [0, 1[$ have recently been studied in depth in [32, 37] from the point of view of *edge-cone singularities*. In particular, assuming that the corresponding Mabuchi functional $\mathcal{M}_{(X, \Delta)}$ is proper it was shown in [37] how to use a continuity method to obtain Kähler-Einstein metrics on $X - D$ which have cone singularities with an angle $2\pi t$ transversely to D (and in particular the metrics satisfy the equation 1.2 on X). More precisely, the metrics admit a polyhomogenous expansion along D in the sense of the “edge calculus”. It seems likely that, using these latter results, it can be shown that when $(X, (1-t)D)$ is moreover toric the Kähler-Einstein metrics constructed here also have cone singularities etc. However, there is a technical problem coming from the fact that in the toric setting $\mathcal{M}_{(X, \Delta)}$ is only *relatively* proper due to the presence of holomorphic vector fields. It should also be pointed that, under the assumption that $t \in]0, 1/2[$ (and similarly in the log smooth case where X is smooth and D has simple normal crossings) it is shown in [23] that *any* log Kähler-Einstein metrics has cone singularities. Of course, it would also be very interesting to understand the relations to edge-cone type singularities in the case when the variety X itself is singular. The present approach only gives regularity of the metrics on the log regular locus of (X, D) and it would of course be very interesting to be able to describe the type of singularities appearing on the singular locus of X and along Δ . However, at this point it does not even seem clear what the appropriate local models should be, even if $\Delta = 0$.

It should also be pointed out that in the case when X is a Fano variety with quotient singularities, i.e. X is an orbifold (which corresponds to the polytope P being simple) the existence of a Kähler-Ricci soliton was obtained recently in [51], building on [60]. The orbifold situation was further studied in [38].

When the first draft of the present paper had been completed two new preprints of Song-Wang [53] and Li-Sun [42] appeared which are relevant for the discussion on edge-cone Kähler-Einstein metrics above. In particular, in [53] certain *toric* edge-cone Kähler-Einstein metrics are obtained on any given smooth toric Fano variety X , by a method of continuity. We have included a discussion on the more precisely relations to [53, 42] in section 3.11.

1.5. Organization

In section 2 we start by setting up a variational approach to solving Monge-Ampère equations in \mathbb{R}^n with target a convex body. The core of the section is a direct proof of a coercive Moser-Trudinger type inequality, which is the basis of the proof of Theorem 1.1 stated in the introduction. In the following section 3 we give a fairly detailed exposition of toric varieties emphasizing analytical aspects of toric log Fano varieties, which in particular allows us to rephrase the results in the previous section in terms of toric Kähler-Einstein geometry. Then in section 4 we explore the relations to the Mabuchi K-energy functional, Futaki invariant and K-stability. Finally, in section 5 we show that the (weak) Kähler-Ricci flow on any toric Fano variety converges weakly to a (singular) Kähler-Ricci soliton.

At least part of the length of the paper is explained by our effort to make the paper readable for the reader with a background in convex analysis, as well as for the complex geometers.

2. Monge-Ampère equations in \mathbb{R}^n and Convex bodies

In this section we will adopt a direct variational approach to solve the Monge-Ampère equation in Theorem 1.1, stated in the introduction. This means that the solutions will be obtained as the maximizers of a certain functional \mathcal{G} on a space $\mathcal{E}_P^1(\mathbb{R}^n)$ of convex functions on \mathbb{R}^n of “finite energy”. At least formally the solutions are critical points of \mathcal{G} and according to the usual scheme of the calculus of variations the existence proof is thus divided into two distinct parts:

- A coercivity (properness) estimate for \mathcal{G} , which yields the existence of a maximizer ϕ (when the barycenter condition on P holds)

- An argument showing that any maximizer indeed satisfies the equation in question

The key new ingredient in our approach is the accomplishment of the first point using a direct convexity argument. As for the second point we will develop a real analog of the Kähler geometry setting considered in [11, 16, 12] by introducing appropriate finite energy spaces of convex functions and establishing a crucial differentiability result for the “energy of convexification” (Prop 2.13).

Of course, our comparison with the Kähler geometry setting may appear as an anachronism: the variational approach to real Monge-Ampère equations, originating in Alexandrov’s seminal work on the Minkowski problem on the n -sphere (see the book [4] and references therein) certainly precedes its complex analog. On the other hand, as far as we know the precise convex analytical setting in \mathbb{R}^n (as opposed to the n -sphere) that we need does not appear to have been developed in the literature¹. Moreover, the analogy between the real and complex settings gives a useful testing ground for conjectures in the Kähler geometry setting.

It is also interesting to see that our variational approach is closely related to the variational principles (based on Kantorovich duality) appearing in the theory of optimal transport (see section 2.11).

2.1. Setup

Let P be a *convex body* in an affine space of real dimension n , i.e. P is a compact convex subset with non-empty interior. Identifying the affine space with the vector space \mathbb{R}^n , with linear coordinates $p(= (p_1, \dots, p_n))$, we may as well assume that the origin 0 is contained in the interior of P . We will identify the dual vector space with \mathbb{R}^n , with linear coordinates $x(= (x_1, \dots, x_n))$,

A convex functions $\phi(x)$ on \mathbb{R}^n is, by definition, convex along affine lines, i.e. $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$ and takes values in $]\infty, \infty]$ and we will exclude the case when ϕ is identically ∞ . Note that such functions are called *proper convex functions* in [50], while we will, to conform to more standard general terminology, say that a function $\phi(x)$ is *proper* if $|\phi| \rightarrow \infty$, as $|x| \rightarrow \infty$.

⁽¹⁾ Added in proof: a closely related direct variational approach to solving real Monge-Ampère equations has been implented by Gangbo in the context of optimal transport theory and was previously outlined by Caffarelli, who attributes it to Varadhan. See Gangbo, W: An elementary proof of the polar factorization of vector-valued functions. Arch. Rational Mech. Anal. 128 (1994), no. 4, 381-399 and Caffarelli, L. A: Boundary regularity of maps with convex potentials. Comm. Pure Appl. Math. 45 (1992), no. 9, 1141-1151.

The *subdifferential* $d\phi|_x$ of ϕ at x is the closed set of \mathbb{R}^n consisting of all points p such that $f(y) \geq f(x) + \langle p, y - x \rangle$ for $y \in \mathbb{R}^n$. In particular, $d\phi|_x$ is equal to a point (the usual differential of ϕ at x) if ϕ is differentiable at x . The Monge-Ampère measure $MA_{\phi}(\phi)$ of a finite convex function ϕ is the (Borel) measure, which with our normalization convention is defined by

$$(MA(\phi)(E) := n! \int_{d\phi(E)} dp \tag{2.1}$$

for any Borel set E (see [49, 36]); this is sometimes also called the *Monge-Ampère measure in the sense of Alexandrov*. More generally, given any function g in $L^1\mathbb{R}^n$) we can define the “ g -Monge-Ampère measure” $MA_g(\phi)$ by replacing the measure dp in the definition 2.1 by gdp , so that $MA_g(\phi) = g(d\phi)MA(\phi)$ if ϕ is smooth. In fact, essentially all the results (a part from those concerned with regularity properties) below for the operator MA generalize word for word to this more general setting, but for clarity of exposition we will mainly stick to the case when $g = 1$.

Remark 2.1. — The reason that $MA(\phi)$ (and more generally MA_g) indeed defines a bona fide measure is that the multivalued map from $\mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $x \mapsto d\phi|_x$ (often called the “normal mapping” in the literature) is invertible almost everywhere on its image (wrt Lebesgue measure dp). This is a consequence of the almost everywhere differentiability of the Legendre transform (compare Lemma 1.1.12 in [36] or Lemma 2.7 below).

Let now P be a given convex body in \mathbb{R}^n containing 0 in its interior and of volume

$$V(P) := \text{Vol}(P)$$

and denote by $\mathcal{P}(\mathbb{R}^n)$ be the space of all convex functions $\phi(x)$ on \mathbb{R}^n such that

$$\phi(x) \leq \phi_P(x) + C,$$

where ϕ_P is the *support function of P* , i.e.

$$\phi_P(x) := \sup_{p \in P} \langle x, p \rangle$$

We let $\mathcal{P}_+(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ of elements of “maximal growth”:

$$-C + \phi_P(x) \leq \phi(x) \leq \phi_P(x) + C$$

for some constant C (depending on ϕ). In particular, any ϕ in $\mathcal{P}_+(\mathbb{R}^n)$ is *proper*. Standard examples of strictly convex and smooth elements in $\mathcal{P}_+(\mathbb{R}^n)$ are obtained by setting

$$\phi_{P,k} := \frac{1}{k} \log \int_P e^{k\langle x,p \rangle} \frac{dp}{V} \tag{2.2}$$

for a given positive integer k (note that $\phi_P = \phi_{P,\infty}$).

We equip the space $\mathcal{P}(\mathbb{R}^n)$ with the topology defined by point-wise convergence. Thanks to the uniform Lipschitz bound on the elements in $\mathcal{P}(\mathbb{R}^n)$ (coming from the boundedness of P) this coincides with the topology defined by local *uniform* convergence.

LEMMA 2.2 (*regularization*). — For any ϕ in $\mathcal{P}(\mathbb{R}^n)$ there is a sequence of strictly convex smooth functions ϕ_j in $\mathcal{P}_+(\mathbb{R}^n)$ decreasing to ϕ .

Proof. — Given $t \in \mathbb{R}^n$ and ϕ in $\mathcal{P}(\mathbb{R}^n)$ we have that $\phi(\cdot + t)$ is also in $\mathcal{P}(\mathbb{R}^n)$ and hence we may first define ψ_j by convolutions in the usual way so that ψ_j is smooth and decreases to ϕ . Finally, we may then set $\phi_j := \max_\epsilon(\psi_j, \phi_P - j)$ where \max_ϵ denotes a regularized max, which has the required properties a part from the *strict* convexity. But this may be achieved by taking suitable convex combinations of ϕ_j and ϕ_0 , where ϕ_0 is any fixed smooth and strictly convex function in $\mathcal{P}_+(\mathbb{R}^n)$ such that $\phi \leq \phi_0$, for example $\phi_0 = \phi_{P,1} + C$ for a sufficiently large constant C . \square

2.2. Relation to the complex setting: the Log map

Let Log be the map from \mathbb{C}^{*n} to \mathbb{R}^n defined by $\text{Log}(z) := x := (\log(|z_1|^2), \dots, \log(|z_n|^2))$, so that the real torus T acts transitively on its fibers. We will refer to x as the (real) *logarithmic coordinates* on \mathbb{C}^{*n} . Given a psh T -invariant bounded function $\phi(z)$ on \mathbb{C}^{*n} we will, abusing notation slightly, write $\phi(x)$ for the corresponding convex function on \mathbb{R}^n , i.e. $\phi(x) := \phi(z)$ for any $z \in (\text{Log})^{-1}\{x\}$. The normalizing constant $n!$ in the definition of $MA(\phi)$ has been chosen so that

$$(\text{Log})_* MA_{\mathbb{C}}(\phi) = MA(\phi)$$

where $MA_{\mathbb{C}}(\phi)$ is the Monge-Ampère measure on \mathbb{C}^{*n} of the locally bounded psh function $\phi(z)$, as defined in pluripotential theory (compare [49] and section 3.3). Note however, that MA_g for $g \neq 1$ does not have any immediate pluripotential analog.

2.3. The Legendre transform from \mathbb{R}^n to the convex body P

Recall that the *Legendre(-Fenchel) transform* (also called the *conjugate* function in [50]) of a convex function $\phi(x)$ is defined by

$$\phi^*(p) := \sup_{x \in \mathbb{R}^n} \langle p, x \rangle - \phi(x)$$

which is a convex function on \mathbb{R}^n with values in $] -\infty, \infty]$. Since the Legendre transform is an involution, i.e. $\phi^{**} = \phi$, we have

$$\phi_P^*(x) = 0 \text{ on } P, \quad \phi_P^*(x) = \infty \text{ on } \mathbb{R}^n - P$$

and

$$\phi(x) \leq \psi(x) \Leftrightarrow \phi^*(p) \geq \psi^*(p) \tag{2.3}$$

It follows immediately that the following proposition holds:

PROPOSITION 2.3. — *If ϕ is in $\mathcal{P}(\mathbb{R}^n)$ then $\phi^* = \infty$ on the complement of P . Moreover, the Legendre transform induces a bijection between $\mathcal{P}_+(\mathbb{R}^n)$ and the space $\mathcal{H}(P)$ of bounded convex functions on P . More precisely,*

$$\sup_{\mathbb{R}^n}(\phi - \phi_P) = -\inf_P \phi^*, \quad \inf_{\mathbb{R}^n}(\phi - \phi_P) = -\sup \phi^*$$

and

$$\|\phi - \phi_P\|_{L^\infty(\mathbb{R}^n)} = \|\phi^*\|_{L^\infty(P)}$$

2.3.1. Functions with full Monge-Ampère mass

We will say that an element ϕ in $\mathcal{P}(\mathbb{R}^n)$ has *full Monge-Ampère mass* if the total mass of $MA(\phi)$ on \mathbb{R}^n coincides with $n!$ times the volume $V(P)$ of P . Following the terminology in [16] in the Kähler geometry setting we will denote this subclass of $\mathcal{P}(\mathbb{R}^n)$ by $\mathcal{E}_P(\mathbb{R}^n)$ (compare Remark 2.12 below).

PROPOSITION 2.4. — *If ϕ_j converges to ϕ in $\mathcal{E}_P(\mathbb{R}^n)$, then $\int vMA(\phi_j) \rightarrow \int vMA(\phi)$ for any bounded continuous function v on \mathbb{R}^n .*

Proof. — If v has compact support this is well-known to hold for any sequence ϕ_j of convex functions converging locally uniformly to ϕ [49, 36]. Moreover, if ϕ_j converges ϕ in $\mathcal{E}_P(\mathbb{R}^n)$, then by assumption $\int MA(\phi_j) = \int MA(\phi)$. Hence, writing v as $v(\chi + (1-\chi))$ for χ a cut-off function supported on a sufficiently large ball proves the proposition. \square

According to the following basic lemma any $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ has full Monge-Ampère mass, i.e. it is in the class $\mathcal{E}_P(\mathbb{R}^n)$.

LEMMA 2.5. — *The following properties of the image of the subgradients of convex functions hold:*

- *If ϕ is a finite convex function ϕ on \mathbb{R}^n then $d\phi(\mathbb{R}^n) \subset \{\phi^* < \infty\}$. In particular $d\phi(\mathbb{R}^n) \subset P$ if $\phi \in \mathcal{P}(\mathbb{R}^n)$ and $\phi \in \mathcal{E}_P(\mathbb{R}^n)$ iff $d\phi(\mathbb{R}^n) = P$ up to a set of measure zero.*
- *If $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ then the interior of P is contained in $\{\phi^* < \infty\}$ and hence*

$$\phi \in \mathcal{P}_+(\mathbb{R}^n) \int_{\mathbb{R}^n} MA_{\mathbb{R}}(\phi) = n!V(P) \tag{2.4}$$

Proof. — The first point follows immediately from the definition of a subgradient and the second point follows from the fact that $\langle p, x \rangle - \phi(x)$ is clearly proper if p is an interior point of P and $\phi \in \mathcal{P}_+(\mathbb{R}^n)$. Indeed, then the sup defining $\phi^*(p)$ is attained, say at x_p , and it follows that $p \in d\phi|_{x_p}$. The final statement 2.4 then follows from the well-known fact that the topological boundary of P is a nullset for Lebesgue measure. \square

Before continuing it will be convenient to record the following property:

LEMMA 2.6. — *Any ϕ in $\mathcal{P}(\mathbb{R}^n)$ with full Monge-Ampère mass is proper. More precisely, there exists a constant $C > 0$ such that $\phi(x) \geq |x|/C - C$.*

Proof. — First note that if ϕ is an element in $\mathcal{E}(\mathbb{R}^n)$ then $\{\phi^* < \infty\}$ is the closure of the interior of P (the converse is trivial). Indeed, by the first point in 2.5 (and since the topological boundary of P is a nullset) the interior of P has full measure in $\{\phi^* < \infty\}$ and in particular is dense in the convex set $\{\phi^* < \infty\}$. But then it follows from a simple argument, using convexity, that all of the interior has to be contained in $\{\phi^* < \infty\}$. Finally, we note that if ϕ is a convex function (finite) convex function ϕ on \mathbb{R}^n such that 0 is contained in the interior of $\{\phi^* < \infty\}$ then ϕ is proper. Indeed, by assumption $u := \phi^*$ is finite on a closed small ball B_ϵ of radius ϵ centered at 0 and since ϕ^* is continuous there it follows that $|\phi^*| \geq C$ on B_ϵ . Hence, $\phi(x) = u^*(x) \geq \sup_{p \in B_\epsilon} \langle p, x \rangle - C = \epsilon|x| - C$ which concludes the proof. \square

We will also have great use for the following variational properties of the Legendre transform:

LEMMA 2.7. — *Let $\phi \in \mathcal{P}(\mathbb{R}^n)$ and p an element in the convex set $\{\phi^* < \infty\}(\subset P)$*

- ϕ^* is differentiable at p iff the sup defining ϕ^* is attained a unique point x_p and the differential at p is then given by $x_p = d\phi^*|_p$
- If ϕ has full Monge-Ampère mass and v is a bounded continuous function on \mathbb{R}^n and ϕ^* is differentiable at p then

$$\frac{d(\phi + tv)^*}{dt} \Big|_{t=0} = -v(d\phi^*|_p)$$

- Moreover we then have, for any non-negative continuous function v ,

$$\int_{\mathbb{R}^n} MA(\phi)v = \int_P v(d\phi^*|_p)dp$$

(where the rhs is well-defined since the derivative of a convex function exists a.e. wrt dp)

Since the content of the lemma above appears to be mostly well-known in the case when ϕ is smooth and strictly convex, we have, for completeness, provided a proof of the general case in the appendix.

2.4. Compactness, normalization and the action of the group \mathbb{R}^n

We let $\mathcal{P}(\mathbb{R}^n)_0$ be the subspace of all *sup-normalized* ϕ :

$$\sup_{\mathbb{R}^n}(\phi - \phi_P) = 0$$

PROPOSITION 2.8. — If ϕ is in $\mathcal{P}(\mathbb{R}^n)$ then

$$\sup_{\mathbb{R}^n}(\phi - \phi_P) = 0 \iff \phi(0) = 0$$

and hence the space $\mathcal{P}(\mathbb{R}^n)_0$ is compact.

Proof. — Since, by definition, the gradient image of ϕ is in P it follows from the convexity of ϕ along the affine line $t \rightarrow tx$ that $\phi(x) \leq \phi(0) + \phi_P(x)$, i.e. $\sup(\phi - \phi_P) \leq \phi(0)$. Since trivially, $\phi(0) = \phi(0) - \phi_P(0) \leq \sup_{\mathbb{R}^n}(\phi - \phi_P)$ this proves the equivalence in the proposition. In particular, if ϕ_j is a sequence in $\mathcal{P}(\mathbb{R}^n)_0$ then $\phi_j(0) = 0$. Hence, since that gradient image of ϕ is in P (and in particular bounded) we deduce that $\sup_K |\phi(x)| \leq C_K$ on any given compact subset of K and that ϕ is Lipschitz continuous on K with a uniform Lipschitz constant. Applying the Arzel–Ascoli theorem on K thus concludes the proof of the compactness. \square

We will say that ϕ is *normalized* if it is sup-normalized and $\phi \geq 0$, i.e.

$$0 = \phi(0) = \inf_{\mathbb{R}^n} \phi \tag{2.5}$$

Given a strictly convex function $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ we define its *normalization* $\tilde{\phi}$ by

$$\tilde{\phi}(a) := \phi_a - \phi(a) \tag{2.6}$$

where a is the point where the infimum of $\tilde{\phi}$ is attained and $\phi_a(x) := \phi(x+a)$ defines the action of the group \mathbb{R}^n on $\mathcal{P}_+(\mathbb{R}^n)$ by translations. Note that even if $\tilde{\phi}$ is not strictly convex we may always define its normalization $\tilde{\phi}$ by taking some point a where the infimum of $\tilde{\phi}$ is attained (but a may not be uniquely determined). Also note that under the Legendre transform

$$(\phi_a)^*(p) = \phi^*(p) - \langle a, p \rangle \tag{2.7}$$

for any $a \in \mathbb{R}^n$.

2.5. The functional \mathcal{E} and the finite energy class $\mathcal{E}_P^1(\mathbb{R}^n)$

Fix a reference element ϕ_0 in $\mathcal{P}_+(\mathbb{R}^n)$. Then there is a unique functional $\mathcal{E} := \mathcal{E}(\cdot, \phi_0)$ on $\mathcal{P}_+(\mathbb{R}^n)$ such that

$$d\mathcal{E}|_\phi = MA(\phi) \tag{2.8}$$

normalized so that $\mathcal{E}(\phi_0, \phi_0) = 0$. To see this we may first define

$$\mathcal{E}(\phi, \phi_0) := \int_0^1 (\phi - \phi_0)MA(\phi_0(1-t) + t\phi)dt \tag{2.9}$$

and then verify that 2.8 indeed holds. This could be shown using integration by parts, but we will give a different proof in the course of the proof of the following proposition. We can first extend the functional \mathcal{E} to be defined on $\mathcal{E}_P(\mathbb{R}^n)$ by the formula 2.9 and define the class of all ϕ in $\mathcal{P}(\mathbb{R}^n)$ of *finite energy* by

$$\mathcal{E}_P^1(\mathbb{R}^n) := \{\phi \in \mathcal{E}_P(\mathbb{R}^n) : \mathcal{E}(\phi) > -\infty\}$$

Note that the various spaces are related as follows:

$$\mathcal{P}_+(\mathbb{R}^n) \subset \mathcal{E}_P^1(\mathbb{R}^n) \subset \mathcal{E}_P(\mathbb{R}^n) \subset \mathcal{P}(\mathbb{R}^n) \tag{2.10}$$

More generally, given a bounded function g on P replacing MA with MA_g we obtain a corresponding functional \mathcal{E}_g , precisely as before. Anyway, since g is bounded the corresponding finite energy space is independent of g .

PROPOSITION 2.9. — *We have that in the space $\mathcal{E}_P^1(\mathbb{R}^n)$ the functional $\mathcal{E}(\cdot, \phi_0)$ defined by 2.9 (which by definition is finite) satisfies*

$$d\mathcal{E}|_\phi = MA(\phi)$$

In general the functional $\mathcal{E}(\cdot, \phi_P)$ may be uniquely extended to an increasing (wrt the usual order relation) and upper semi-continuous functional $\mathcal{P}(\mathbb{R}^n) \rightarrow [-\infty, \infty[$ by setting

$$\mathcal{E}(\phi, \phi_P) = -n! \int_P \phi^* dp \tag{2.11}$$

In particular, an element ϕ in $\mathcal{P}(\mathbb{R}^n)$ is in $\mathcal{E}_P^1(\mathbb{R}^n)$ iff ϕ^ is in $L^1(P, dp)$. For the energy functional associated to a function g on P the formula 2.11 holds with dp replaced by gdp .*

Proof. — Denote by $\mathcal{H}(P)$ the space of all convex functions on P and denote by L the map from $\mathcal{E}_P(\mathbb{R}^n)$ to $\mathcal{H}(P)$ induced by the Legendre transform. i.e $(L\phi)(p) := \phi^*(p)$. Let λ be the linear functional on $\mathcal{H}(P)$ defined

by integration against dp on P and let MA be the one-form on $\mathcal{E}_P(\mathbb{R}^n)$ such that the linear functional $MA|_\phi$ is defined by integration on \mathbb{R}^n against the Monge-Ampère measure $MA(\phi)$. Then we have the following key relation

$$L^*(-n!\lambda) = MA \tag{2.12}$$

Indeed, this follows immediately from the second point in Lemma 2.7, since by definition $\langle L^*, v \rangle_\phi = -n! \langle \lambda, d(L(\phi + tv))/dt_{t=0} \rangle$. Since λ clearly defines a *closed* one-form on $\mathcal{H}(P)$ (which is even exact with primitive $I_P := \int_P(\cdot)dp$), it follows that the pull-back MA is also closed and exact. In particular it has a primitive \mathcal{E} (unique up to the normalization $\mathcal{E}(\phi_0) = 0$) such that

$$\mathcal{E}(\phi, \phi_0) = \int_{[0,1]} \gamma^* MA, \quad \gamma : [0, 1] \rightarrow \mathcal{H}(\mathbb{R}^n)$$

where γ is any smooth curve connecting ϕ_0 and ϕ . In particular, taking γ to be an affine curve in $\mathcal{H}(\mathbb{R}^n)$ gives the formula 2.9, while taking γ to be the pull-back under L of an affine curve in $\mathcal{H}(P)$ gives the formula 2.11 on $\mathcal{E}_P^1(\mathbb{R}^n)$ (using that $L\phi_P = 0$).

Finally, we note that it follows immediately from the fact that the Legendre transform is decreasing together with Fatou’s lemma that the functional on $\mathcal{P}(\mathbb{R}^n)$ defined by 2.11 is increasing and upper-semicontinuous. Indeed, it follows immediately from the variational definition of the Legendre transform that if ϕ_i is a sequence in $\mathcal{P}(\mathbb{R}^n)$ converging point-wise to ϕ , then

$$\liminf_{i \rightarrow \infty} \phi_i^* \geq \phi^*, \quad \phi_i^* \geq -C$$

for some constant C . □

We will often omit the explicit dependence on a reference ϕ_0 in the definition of $\mathcal{E}(\cdot, \phi_0)$. Anyway, as a consequence of the property 2.8 *differences* $\mathcal{E}(\phi) - \mathcal{E}(\psi)$ are independent of the choice of ϕ_0 .

Remark 2.10. — Since any ϕ in $\mathcal{P}(\mathbb{R}^n)$ may be written as a decreasing limit of elements ϕ_j in $\mathcal{P}_+(\mathbb{R}^n)$ (just set $\phi_j = \max\{\phi, \phi_p - j\}$) we could, by the monotonicity and upper semi-continuity of \mathcal{E} in the previous proposition, equivalently have defined \mathcal{E} by $\mathcal{P}(\mathbb{R}^n)$ by setting

$$\mathcal{E}(\phi) = \inf \{ \mathcal{E}(\psi) : \psi \in \mathcal{P}_+(\mathbb{R}^n), \psi \geq \phi \} \tag{2.13}$$

using formula 2.9 and then let $\mathcal{E}_P^1(\mathbb{R}^n) := \{ \mathcal{E} > -\infty \}$. More concretely, we could even assume that ψ above is smooth (by the approximation Lemma 2.2). Compare [11] for the Kähler geometry setting.

Example 2.11. — The function $\phi(x) = 0$ is *not* in the space $\mathcal{E}_P^1(\mathbb{R}^n)$. Indeed, its Legendre transform is identically equal to infinity on the complement of 0 and hence ϕ^* is not in $L^1(P, dp)$. This also shows that the formula 2.9 is not valid in general on the complement of $\mathcal{E}_P(\mathbb{R}^n)$.

Remark 2.12. — The reason to use the notation $\mathcal{E}_P(\mathbb{R}^n)$ for the space of all ϕ with full Monge-Ampère mass is that, just as in the Kähler geometry setting [16], $\phi \in \mathcal{E}_P(\mathbb{R}^n)$ iff ϕ has finite χ -weighted energy for some convex positive weight χ , i.e. ϕ is in the class $\mathcal{E}_P^\chi(\mathbb{R}^n)$ defined as in 2.13, but with $\mathcal{E}(\psi)$ replaced by

$$\int_0^1 \chi(\phi - \phi_0) MA(\phi_0(1-t) + t\phi) dt \tag{2.14}$$

2.6. The projection Pr on the convexification

Fix an element ϕ in $\mathcal{P}(\mathbb{R}^n)$ and a bounded continuous function v , i.e. $v \in \mathcal{C}_b^0(\mathbb{R}^n)$. If ϕ is in the “boundary” of $\mathcal{P}(\mathbb{R}^n)$, i.e. ϕ is not strictly convex, then some perturbation $\phi + v$ will leave the space $\mathcal{P}(\mathbb{R}^n)$. As a remedy for this we define the projection operator Pr from $\{\phi\} + \mathcal{C}_b^0(\mathbb{R}^n)$ onto $\mathcal{P}(\mathbb{R}^n)$ by

$$Pr(\phi + v)(x) := \sup_{\psi \in \mathcal{P}(\mathbb{R}^n)} \{\psi(x) : \psi \leq \phi + v\}$$

Noting that $Pr(\phi + v)^* = (\phi + v)^*$ we could also use

$$Pr(\phi + v) = (\phi + v)^{**}$$

as the definition of $Pr(\phi + v)$.

PROPOSITION 2.13. — *Fix an element ϕ in $\mathcal{E}_P^1(\mathbb{R}^n)$. Then the functional*

$$\mathcal{C}_b^0(\mathbb{R}^n) : v \mapsto (\mathcal{E} \circ Pr)(\phi + v) (= \int_P (\phi + v)^* dp)$$

is Gateaux differentiable and its differential at v is given by $MA(Pr(\phi + v))$.

Proof. — Let us first consider the differential of the functional at $v = 0$. Observe that, for p fixed, $t \mapsto (\phi + tv)^*(p)$ is convex and hence its right and left derivatives $d_\pm(p)$ exist everywhere, defining functions which are in $L_{loc}^\infty(P)$ and $d_+(p) = d_-(p)$ for almost every p (by the first point of Prop 2.7). Moreover, by convexity $d_\pm(p)$ are both defined as monotone limit. Hence it follows from the Lebesgue monotone convergence theorem that

$$d\left(\int_P (\phi + tv)^* dp\right)/dt_{t=0^\pm} = \int_P d_\pm(p) dp$$

and since $d_+(p) = d_-(p)$ a.e. wrt dp this gives the desired differentiability property of the functional in question. Moreover, by Lemma 2.7 we have, setting $u := \phi^*$, that

$$d\left(\int_P (\phi + tv)^* dp\right)/dt_{t=0^\pm} = - \int_P v(du_p) dp$$

and the proof is concluded by invoking the formula in the third point of Lemma 2.7. Finally, to obtain the differential at any v we note that the previous argument gives that

$$d\left(\int_P (\phi + tv)^* dp\right)/dt_{t=1^\pm} = - \int_P v(d(\phi + v)^*(p)) dp.$$

But since $(\phi + v)^* = \psi^*$ for $\psi = Pr(\phi + v)$ we can now apply the formula in the third point of Lemma 2.7 to ψ . \square

2.7. Geodesics

Given two strictly convex and smooth functions ϕ_0 and ϕ_1 in $\mathcal{P}_+(\mathbb{R}^n)$ the geodesic ϕ_t in $\mathcal{P}_+(\mathbb{R}^n)$ from ϕ_0 to ϕ_1 is defined as the the map $[0, 1] \rightarrow \mathcal{P}_+(\mathbb{R}^n)$ defined by

$$\phi_t = ((1 - t)\phi_0^* + t\phi_1^*)^*$$

i.e. under the Legendre transform (for t fixed) ϕ_t corresponds to an affine curve in $\mathcal{H}(P)$. In particular, $\phi_t(x)$ is smooth and convex in (t, x) .

2.8. Variational principles and a coercive Moser-Trudinger type inequality

Given (P, g) we consider the following Moser-Trudinger type functional on $\mathcal{P}_+(\mathbb{R}^n)$:

$$\mathcal{G}_{(P,g)}(\phi) := \frac{1}{V(P,g)} \mathcal{E}_g(\phi, \phi_P) - \mathcal{I}(\phi), \quad \mathcal{I}(\phi) := - \log \int e^{-\phi} dx$$

To simplify the notation we will set $g = 1$ and write $\mathcal{G}_{(P,g)} = \mathcal{G}$, but the proofs in the general case are the same. Note that \mathcal{G} is a well-defined as a functional on $\mathcal{E}_P^1(\mathbb{R}^n)$ taking values in $] - \infty, \infty]$. The normalization of the energy term above is made so that $\mathcal{G}(\phi + c) = \mathcal{G}(\phi)$. Moreover, we have the following simple, but crucial

LEMMA 2.14. — *The functional \mathcal{G} is invariant under the action of \mathbb{R}^n by translations (i.e. under $\phi \mapsto \phi_a$ for $a \in \mathbb{R}^n$) iff 0 is the barycenter of P . Moreover, if $\mathcal{G}(\phi)$ is bounded from above, then 0 is the barycenter of P .*

Proof. — Since the volume form dx is invariant under translations so is the functional \mathcal{I} and hence \mathcal{G} is invariant under translations iff the function $\mathcal{E}(\cdot, \phi_P)$ is. The proof of the first statement is thus concluded by noting that

$$\mathcal{E}(\phi_a, \phi_P) - \mathcal{E}(\phi, \phi_P) = - \int_P ((\phi_a)^* - \phi^*) dp = \int_P \langle a, p \rangle dp,$$

where we have used formula 2.7 in the last equality. Finally, if 0 is not the barycenter of P we can take a curve ϕ_t in $\mathcal{P}_+(\mathbb{R}^n)$ such that $\phi_t^*(0) = tp_{i_0}$ where p is the barycenter of P and $p_{i_0} \neq 0$. Then it follows as above that

$$\mathcal{G}(\phi_t) = \log \int e^{-\phi_P} dx + \mathcal{E}(\phi_t, \phi_P) = \log \int e^{-\phi_P} dx - tp_{i_0}$$

which is unbounded from above when either $t \rightarrow \infty$ or $t \rightarrow -\infty$. \square

Next, we have the following key concavity property:

PROPOSITION 2.15. — *The functional \mathcal{G} is concave along geodesics in $\mathcal{P}_+(\mathbb{R}^n)$ and strictly concave modulo the action of \mathbb{R}^n by translations. In particular, any solution ϕ as in the statement of Theorem 1.1 maximizes \mathcal{G} on the space $\mathcal{P}_+(\mathbb{R}^n)$.*

Proof. — Let ϕ_t be a geodesic in $\mathcal{P}_+(\mathbb{R}^n)$. By the Prekopa-Leindler inequality

$$t \mapsto -\log \int_{\mathbb{R}^n} e^{-\phi_t} dx \tag{2.15}$$

is convex, since ϕ_t is convex in (x, t) (see [10] for complex geometric generalizations of this inequality). Moreover, by Prop 2.9 $\mathcal{E}(\phi_t, \phi_0)$ is affine wrt t and hence $\mathcal{G}(\phi_t)$ is convex as desired. The statement about strict convexity follows from the equality case for the Prekopa-Leindler inequality giving that if the function in 2.15 is affine in t then $\phi_t(x) = \phi(x + ta)$ for some vector a . The final statement of the proposition now follows by connecting a given element ϕ (which by approximation may be assumed smooth) and the solution ϕ_0 with a geodesic and using that the differential of \mathcal{G} vanishes at ϕ_0 . \square

We are now in the position to prove one of the main results in the present paper:

THEOREM 2.16. — *Let P be a convex body containing 0 in its interior. For any $\delta > 0$ there is a constant C_δ such that*

$$\mathcal{G}(\phi) \leq (1 - \delta)\mathcal{E}(\phi, \phi_P) + C_\delta$$

for any normalized $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ (i.e. $\phi(0) = 0$ and $\phi \geq 0$). Moreover, \mathcal{G} is bounded from above on $\mathcal{P}_+(\mathbb{R}^n)$ iff it is invariant under the action of \mathbb{R}^n by translations iff 0 is the barycenter of P .

Proof. — *Step one (a crude M-T type inequality):* there is a positive constant C such that

$$\log \int e^{-\phi} dx \leq -C\mathcal{E}(\phi, \phi_P) + C$$

for any $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ such that $\phi(0) = 0$ (or equivalently such that $\sup(\phi - \phi_P) = 0$).

We first fix a reference ϕ_0 in $\mathcal{P}_+(\mathbb{R}^n)$ such that $\int e^{-\phi_0} dx = 1$. Given ϕ in $\mathcal{P}_+(\mathbb{R}^n)$ there is a geodesic ϕ_t in $\mathcal{P}_+(\mathbb{R}^n)$ starting at ϕ_0 such that $\phi_1 = \phi$. By the previous proposition $\mathcal{G}(\phi_t)$ is concave giving $\mathcal{G}(\phi_1) \leq \mathcal{G}(\phi_0) + d\mathcal{G}(\phi_t)/dt_{t=0}$, where

$$d\mathcal{G}(\phi_t)/dt_{t=0} = \int_{\mathbb{R}^n} (-d\phi_t/dt_{t=0})(e^{-\phi_0} dx - MA(u_0))$$

By the invariance of \mathcal{G} under $\phi \mapsto \phi + c$ we may assume that $\sup(\phi_1 - \phi_0) = 0$ and hence by the convexity of ϕ_t wrt t we have $d\phi_t/dt_{t=0} \leq 0$. Next we note that we may take the fixed reference ϕ_0 so that

$$e^{-\phi_0} \leq C \cdot MA(\phi_0) \tag{2.16}$$

for some constant C . Accepting, for the moment, the existence of such a ϕ_0 we deduce that there is a constant C such that

$$\mathcal{G}(\phi) - \mathcal{G}(\phi_0) \leq C \int -d\phi_t/dt_{t=0} MA(\phi_0) = -C\mathcal{E}(\phi, \phi_0),$$

using 2.8 and that $\mathcal{E}(\phi_t, \phi_0)$ is affine wrt t . Finally, since $\mathcal{E}(\phi, \phi_0) = \mathcal{E}(\phi, \phi_P) + \mathcal{E}(\phi_P, \phi_0)$ this concludes the proof up to the existence of $\phi_0 \in \mathcal{P}_+(\mathbb{R}^n)$ satisfying 2.16. An explicit choice of ϕ_0 may be obtained by setting $\phi_0 = \phi_{P,1}$ as in 2.2. The property 2.16 can then be checked by straightforward, but somewhat tedious calculations. Alternatively, we may set $\phi_0 = \phi$ for a solution $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ of the inhomogeneous Monge-Ampère equation $MA(\phi) = e^{-\psi} dx$, where ψ is any given element in $\mathcal{P}_+(\mathbb{R}^n)$, e.g. $\psi = \phi_P$ or even in $\mathcal{E}_P(\mathbb{R}^n)$ (see Cor 2.20). Since, $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ we have $\psi \leq \phi + A$ and hence the property 2.16 follows with $C = e^{-A}$.

Step two: refinement by scaling. Let now ϕ be a normalized function in $\mathcal{P}_+(\mathbb{R}^n)$ and in particular $\phi \geq 0$. We will improve the inequality in the

previous step by a scaling argument. To this end fix $t \in]0, 1[$. Since $\phi \geq t\phi$ we have that

$$\log \int e^{-\phi} dx \leq \log \int e^{-t\phi} dx = \log \int e^{-\phi_t} d(x/t)$$

where $\phi_t(x) := t\phi(x/t)$. Note that $\phi_t = (tu)^*$, where $u = \phi^*$ and in particular $\phi_t \in \mathcal{P}_+(\mathbb{R}^n)_0$. Hence, applying the previous step gives

$$\log \int e^{-\phi_t} dx \leq -C\mathcal{E}(\phi_t, \phi_P) + C = Ct \int_P u + C$$

All in all this means that

$$\log \int e^{-\phi} dx \leq Ct \int_P u dp + C + n \log t$$

and hence setting $\delta = t/C$ concludes the proof of the first statement of the theorem.

Finally, if 0 is the barycenter of P then we have, by the previous lemma, and the definition of the normalization $\tilde{\phi}$ that

$$\mathcal{G}(\phi) = \mathcal{G}(\tilde{\phi}) \leq 0 + C$$

for any $\mathcal{P}_+(\mathbb{R}^n)$. The converse was proved in the previous lemma. \square

Remark 2.17. — The scaling argument in the previous proof is somewhat analogous to a scaling argument used by Donaldson [29] for the Mabuchi functional on a toric manifold.

Interestingly, the boundedness of \mathcal{G} under the moment condition on (P, g) , say for $g = 1$, is also a consequence of the functional form of the Santalo inequality [3]. Indeed, the latter inequality says that, for any convex function $\phi(x)$ in \mathbb{R}^n the following inequality holds after perhaps replacing ϕ by ϕ_a for some $a \in \mathbb{R}^n$: $\int e^{-\phi(x)} dx \int e^{-\phi^*(p)} dp \leq (2\pi)^n$ and hence the boundedness of \mathcal{G} follows from Jensen's inequality. However, for the proof of Theorem 1.1 we do need the stronger coercivity inequality obtained in the previous theorem.

2.9. Proof of Theorem 1.1

Step 1: the sup of \mathcal{G} is attained, i.e. there exists a maximizer ϕ of finite energy.

Proof. — By Prop 2.9 \mathcal{E} is upper semi-continuous (usc) on the space $\mathcal{P}(\mathbb{R}^n)$. By the invariance under translations (see Lemma 2.14) we can take

a sequence of normalized functions ϕ_j in $\mathcal{P}_+(\mathbb{R}^n)$ such that $\mathcal{G}(\phi_j) \rightarrow \sup \mathcal{G}$. But then it follows from the coercivity inequality in Theorem 2.16 that there is constant C such that $\mathcal{E}(\phi_j) \geq -C$. By the compactness of $\mathcal{P}_0(\mathbb{R}^n)$ we may, after perhaps passing to a subsequence, assume that $\phi_j \rightarrow \phi$ in $\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{E}(\phi) \geq -C$, since $\mathcal{E}(\phi)$ is usc. The proof of step 1 is now concluded by noting that the functional $-\mathcal{I}$ is usc along ϕ_j , or more precisely that

$$\int e^{-\phi_j} dx = \lim_{j \rightarrow \infty} \int e^{-\phi_j} dx \tag{2.17}$$

Indeed, since $\mathcal{E}(\phi_j) \geq -C$ it follows from the coercivity inequality in Theorem 2.16 (and a simple scaling argument) that

$$\int e^{-p(\phi_j - \phi_P)} \mu_P \leq C_p, \quad \mu_P := e^{-\phi_P} dx$$

for some positive number $p > 1$. But since $\phi_j \rightarrow \phi$ uniformly on any compact set the desired upper semi-continuity of $-\mathcal{I}$ then follows from Hölder's inequality. Indeed, integrating the lhs in 2.17 over the complement of a ball B_R of radius R gives

$$\int_{\mathbb{R}^n - B_R} e^{-(\phi_j - \phi_P)} \mu_P \leq \left(\int_{\mathbb{R}^n - B_R} e^{-p(\phi_j - \phi_P)} \mu_P \right)^{1/p} \mu_P(\mathbb{R}^n - B_R)$$

and since μ_P has finite mass on \mathbb{R}^n the rhs above can be made arbitrary small by taking R sufficiently large.

Step 2: the maximizer ϕ satisfies the equation $MA(\phi) = e^{-\phi} dx / \int e^{-\phi} dx$ in the weak sense

Fix a smooth function v of compact support and consider the following functional on the real line: $t \mapsto f(t) := \frac{1}{V} \mathcal{E}(Pr(\phi + tv) - \mathcal{I}(\phi))$. Since \mathcal{I} is increasing we have that $f(t) \leq \mathcal{G}(Pr(\phi + tv)) \leq \mathcal{G}(\phi)$ (since ϕ is a maximizer), i.e. $f(t) \leq f(0)$. But by Prop 2.13 $f(t)$ is differentiable and hence $df/dt = 0$ at $t = 0$, which by Prop 2.13 gives the desired equation, since $Pr(\phi) = \phi$.

Step 3: local regularity of solutions

We will give the argument for the more general case when ϕ is a finite energy solution of $MA_g(\phi) = Ce^{-F_1(\phi)}$ where $g(p) = e^{-F_2(p)}$ for F_1 a continuous function on \mathbb{R}^n and F_2 a bounded function on P (see Remark 2.1 for the definition of MA_g). By Lemma 2.6 ϕ is proper and hence the sublevel sets $\Omega_R := \{\phi < R\}$ are bounded convex domains exhausting \mathbb{R}^n . Fixing R , writing $\Omega := \Omega_R$ and replacing ϕ with $\phi - R$ we then have that $\phi = 0$ on $\partial\Omega$ and $1/C dx \leq MA(\phi) \leq C dx$ on Ω for some positive constant C .

Hence, it follows from the first point in Theorem 2.24 below that ϕ is in the Hölder class $\mathcal{C}_{loc}^{1,\alpha}$ for some $\alpha > 1$. In particular, the gradient $d\phi$ is a single-valued continuous function and hence $MA(\phi) = f$ for a continuous function f such that $1/C' \leq f \leq C'$ in Ω . Applying the second point in Theorem 2.24 thus shows that ϕ is in the Sobolev space $W_{loc}^{2,p}(\Omega)$ for any $p > 1$. Finally, by general Evans-Krylov theory for non-linear PDEs we deduce that $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Step 4: global regularity

Applying Cor 2.20 below with $\psi = \phi$ shows that $\phi - \phi_P$ is globally bounded on \mathbb{R}^n . More precisely, the Legendre transform of ϕ is Hölder continuous up to the boundary of P for any Hölder exponent $\gamma < 1$.

Uniqueness: Let ϕ_0 and ϕ_1 be two solutions and let ϕ_t be the geodesic segment connecting them. By the strict concavity in Prop 2.15 $\phi_t = \phi_0(x + ta)$ for some vector a which concludes the proof. \square

2.10. The invariant R of a convex body

Let P be a convex body containing 0 and define the following invariant $R_P \in]0, 1]$, which is a measure of the failure of P having the property that its barycenter b coincides with 0 :

$$R_P := \frac{\|q\|}{\|q - b\|}, \tag{2.18}$$

where q is the point in ∂P where the line segment starting at b and passing through 0 meets ∂P . In the case when P is the canonical polytope associated to a smooth Fano variety the invariant R_P was introduced in [40], where it was shown to coincide with another invariant of an analytical nature. A slight modification of the proof of Theorem 1.1 gives the following theorem, which – when translated to toric geometry – generalizes the main result of [40] (see section 3.10):

THEOREM 2.18. — *Let P be a convex body containing 0 in its interior and fix an element $\phi_0 \in \mathcal{P}_+(\mathbb{R}^n)$. Then the invariant R_P coincides with the following two numbers, defined as the sup over all $r \in [0, 1[$ such that*

- *there is a solution $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ to the equation*

$$\det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = e^{-r\phi} e^{-(1-r)\phi_0} \tag{2.19}$$

- *The following functional on $\mathcal{P}_+(\mathbb{R}^n)$ is bounded from above:*

$$G_{r,\phi_0}(\phi) := \frac{1}{V(P)}\mathcal{E}(\phi, \phi_P) + \frac{1}{r} \log \int e^{-r\phi} e^{-(1-r)\phi_0} dx$$

Proof. — First observe that R_P is the sup over all $r \in [0, 1]$ such that

$$(1-r)\phi_P(x) + r \langle b, x \rangle \geq 0 \tag{2.20}$$

Accepting this for the moment we will start by showing that if r is a positive number such that 2.20 holds for $r+\delta$, for some $\delta > 0$, then $G_{r+\delta}$ is bounded from above. By assumption $-(1-r)\phi_P(x) \leq r \langle b, x \rangle$ and hence

$$G_{r,\phi_P}(\phi) \leq \frac{1}{V(P)}\mathcal{E}(\phi, \phi_P) + \frac{1}{r} \log \int e^{-r\phi} e^{r\langle b, x \rangle} dx$$

Applying the boundedness statement in Theorem 2.16 to the translated and scaled convex body $rP - \{b\}$ (which has its barycenter at 0) thus shows that $G_{r,\phi_P}(\phi) \leq C_r$ and, by the same argument, $G_{r+\delta,\phi_P}(\phi) \leq C_{r+\delta}$. Moreover, since $\phi_0 - \phi_P$ is bounded the corresponding inequalities also hold when ϕ_P is replaced by ϕ_0 .

Next, we show that the inequality $G_{r+\delta,\phi_0}(\phi) \leq C_{r+\delta}$ implies the existence of a solution to equation 2.19 for the parameter r . First, by a simple scaling argument, it follows that the functional G_{r,ϕ_0} is coercive, i.e. there exists positive numbers δ and C_δ such that

$$G_{r,\phi_0}(\phi) \leq \delta \mathcal{E}(\phi, \phi_P) + C_\delta$$

for any sup-normalized ϕ . But then it follows, exactly as in the proof of Theorem 1.1, that any sup-normalized maximizing sequence of G_{r,ϕ_0} converges to a solution to the equation 2.19.

Conversely, let r be such that there is solution to the equation 2.19. It then follows, just like in the proof of Prop 2.15 (now using the Prekopa inequality for the convex function $(t, x) \mapsto r\phi_t(x) + (1-r)\phi_0(x)$) that G_{r,ϕ_P} (and hence also G_{r,ϕ_0}) is bounded from above. Now fix a point $a \in \mathbb{R}^n$. By definition $\phi_P(x+a) \leq \phi_P(x) + \phi_P(a)$ and hence, for any $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ we have

$$-\frac{(1-r)}{r}\phi_P(a) + \frac{1}{r} \log \int e^{-r\phi} e^{-(1-r)\phi_P} dx \leq \frac{1}{r} \log \int e^{-r\phi(x)} e^{-(1-r)\phi_P(x+a)} dx$$

Making the change of variables $x \mapsto x+a$ in the latter integral and applying the boundedness of G_{r,ϕ_P} thus gives that

$$-\frac{(1-r)}{r}\phi_P(a) + \frac{1}{r} \log \int e^{-r\phi} e^{-(1-r)\phi_P} dx \leq -\frac{1}{V(P)} \int (\phi - a)^* dp + C$$

But, using 2.7 and rearranging we deduce that

$$G_{r,\phi_P}(\phi) - C \leq (1-r)\phi_P(a) + r \langle b, a \rangle$$

Assume for a contradiction that the rhs above is negative ($= -\delta$). Then replacing a with ta for t a positive number t shows that $G_{r,\phi_P}(\phi) \leq -\delta t + C$ and hence letting $t \rightarrow \infty$ yields a contradiction.

Finally, we come back to the first claim concerning the inequality 2.20. First observe that the inequality holds for $r \leq R_P$. Indeed, by definition, the lhs in 2.20 is equal to the sup over $\langle (1-r)p + rb, x \rangle$, as p ranges over all points in P . In particular, for $r = R_P$ and $p = q$ we have by definition $(1-r)p + rb = 0$ and hence 2.20 holds for $r \leq R_P$. Conversely, suppose that $r > R_P$ and let a be the vector defining a supporting hyperplane of P at q , i.e. $\langle p, a \rangle \leq \langle q, a \rangle$ for any $p \in P$. Hence, the lhs in 2.20 for $x = a$ is bounded from above by $\langle (1-r)q + rb, a \rangle := f(r)$. Finally, by assumption $f(r) = 0$ at $r = R_P$ and $df(r)/dr = \langle b - q, a \rangle < 0$, since b is in the interior of P and hence $f(r) < 0$ for any $r > R_P$. \square

2.11. The inhomogeneous Monge-Ampère equation

In this section we will establish some local and global regularity properties for the “inhomogeneous” Monge-Ampère equation used above.

THEOREM 2.19. — *Let P be a convex body in \mathbb{R}^n and let μ be measure on \mathbb{R}^n of total mass $V(P)$. Then there exists a unique (mod \mathbb{R}) convex function ϕ on \mathbb{R}^n such that*

$$MA(\phi) = \mu \tag{2.21}$$

with $\phi \in \mathcal{E}_P(\mathbb{R}^n)$, i.e. the closure of the subgradient image is P :

$$\overline{d\phi(\mathbb{R}^n)} = P$$

- If moreover

$$\int_{\mathbb{R}^n} |x|^q \mu < \infty$$

for some number $q > n$ then $\phi - \phi_P$ is bounded and if the finiteness holds for any $q > 0$ then the Legendre transform ϕ^* of ϕ is Hölder continuous up to the boundary of P for any Hölder exponent in $[0, 1[$.

- If $\mu = f dx$ for f smooth and strictly positive a solution ϕ is unique modulo constants and smooth. In particular, the gradient $d\phi$ then maps \mathbb{R}^n diffeomorphically onto the image of the interior of P .

Proof. — Uniqueness modulo constant follows from a comparison principle argument as in section 16.2 in [4].

Existence: First observe that it will be enough to prove the result when μ has finite first moments, i.e.

$$\int \mu|x| < \infty. \tag{2.22}$$

Indeed, any measure μ can be written as $\mu = f\nu$ where ν has finite first moments (e.g. take $f = (1 + |x|)$ and $(1 + |x|)^{-1}\mu$) and we can solve the $MA(\phi_i) = \mu_i$ where $\mu_i = \max(f, i)\nu$ with $\phi_i \in \mathcal{P}(\mathbb{R}^n)_0$. Finally, by compactness we have, after perhaps passing to a subsequence, that $\phi_i \rightarrow \phi \in \mathcal{P}(\mathbb{R}^n)_0$ and by the local continuity of MA acting on $\mathcal{P}(\mathbb{R}^n)_0$ (see section 2.3) $MA(\phi) = \mu$

Assume now that μ has finite first moments. Since, the gradient image of any ϕ in $\mathcal{P}(\mathbb{R}^n)_0$ is uniformly bounded there is a constant C such that $|\phi(x)| \leq C|x|$ and hence the functional

$$I_\mu(\phi) := \int \phi\mu$$

is finite on $\mathcal{P}(\mathbb{R}^n)_0$. In fact, it is even *continuous*. Indeed, if $\phi_j \rightarrow \phi$ in $\mathcal{P}(\mathbb{R}^n)_0$ then the convergence is uniform on any large ball B_R of radius R , so that the desired continuity is obtained by decomposing $\mu = 1_{B_R}\mu + 1_{\mathbb{R}^n - B_R}$ for a large ball B_R of radius R and using the uniform bound $|\phi(x)| \leq C|x|$ on $\mathbb{R}^n - B_R$ together with 2.22 and finally letting $R \rightarrow \infty$. As a consequence the functional

$$\mathcal{G}_\mu(\phi) := \frac{1}{V} \mathcal{E}(\phi, \phi_P) - I_\mu(\phi)$$

is upper semi-continuous on the compact space $\mathcal{P}(\mathbb{R}^n)_0$. In particular it has a maximizer ϕ_μ of finite energy and the proof is concluded by noting that ϕ_μ satisfies the equation 2.21. Indeed, this is shown precisely as in the end of the proof of Theorem 1.1, using the projection operator Pr .

Regularity: this is proved exactly as in the proof of Theorem 1.1, using Caffarelli’s interior regularity results (see below). For the global C^0 -bound and the Hölder regularity see Prop 2.22. \square

It should be emphasized that the existence of a (weak) solution ϕ for a general measure μ in the previous theorem is essentially well-known (for example, this is shown in [4] when μ has an L^1 -density). The result is also closely related to the theory of *optimal transportation of measures* (and can be deduced from the results of Brenier [16] and McCann [45]). Briefly,

the problem, in the original formulation of Monge, is to find, given two probability measure measure μ and ν on \mathbb{R}^n , a map T such that $T_*\mu = \nu$ where T minimizes a certain cost function $c(x, p)$. In the present setting $\nu = 1_P g dp$ and $c(x, p) = -\langle x, p \rangle$ and then $T = d\phi$, for ϕ the solution in the previous theorem, under suitable regularity assumptions. Interestingly, the variational principle used in our proof is equivalent to the duality formula for the minimum of the Kantorovich problem for optimal transport (see Theorem 6.1.1 in [2] and references therein).

COROLLARY 2.20. — *Let P be a convex body in \mathbb{R}^n . For any given $\psi \in \mathcal{E}_P(\mathbb{R}^n)$ there is unique (mod \mathbb{R}) function $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ such that*

$$MA(\phi) = V(P)e^{-\psi} dx / \int e^{-\psi} dx$$

Proof. — We may assume that 0 is contained in the interior of P . By lemma 2.6 the moment condition in the previous theorem is satisfied, so that the previous theorem furnishes the desired solution. \square

Remark 2.21. — In the Kähler geometry setting the analog of the previous corollary is known to hold for a Kähler class $[\omega]$ on a smooth manifold X . The point is that if $v \in \mathcal{E}(X, \omega)$ then v has no Lelong numbers and hence $e^{-pv} \in L^1(X, dV)$ for any $p > 0$, so that v is bounded by Kolodziej’s estimate.

2.11.1. A global C_0 -estimate

In the arguments above we used the following

PROPOSITION 2.22. — *If the q :th moment of μ is finite for some $q > n$, then any solution $\phi \in \mathcal{P}(\mathbb{R}^n)$ to equation 2.21 is in $\mathcal{P}_+(\mathbb{R}^n)$. More precisely, the following inequality holds for any $\phi \in \mathcal{P}_0(\mathbb{R}^n)$ with full Monge-Ampere mass:*

$$\|\phi - \phi_P\|_{C^0(\mathbb{R}^n)} \leq C_q(-\mathcal{E}(\phi, \phi_P) + (\int_{\mathbb{R}^n} MA(\phi)|x|^q))^{1/q}$$

for any $q > n$. More generally, the Legendre transform $u := \phi^*$ is in the Hölder space $C^\gamma(P)$ for $\gamma = 1 - n/q$ if the q th moments of $MA(\phi)$ are finite.

Proof. — By assumption P is the closure of an open convex domain D and in particular the boundary of P is Lipschitz (cf. [33] Sec. V.4.1). We will deduce the desired inequality from the Sobolev inequality on the

Lipschitz domain D which says that $W^{q,1}$ injects in $C^0(D)$ in a continuous manner if $q > n$ and

$$\|u\|_{C^0(D)} \leq C_q \left(\int_D |u| dp + \left(\int_D |du|^q \right)^{1/q} \right)$$

Indeed, setting $f(x) := |x|^q$ and taking $\phi \in \mathcal{P}_0(\mathbb{R}^n)$ we let $u := \phi^*$ so that $u \geq 0$ (since $\phi(0) = 0$). But then the inequality to be proved follows immediately from combining Prop 2.9, Prop 2.3 and the last point in Lemma 2.7. The last claim follows from the general formulation of the Sobolev embedding theorem for Hölder spaces. \square

Remark 2.23. — The moment condition on μ in the previous proposition may also equivalently be formulated in the following way: $\int \mu f_\delta(\phi - \phi_P) < \infty$ for any $\phi \in \mathcal{P}(\mathbb{R}^n)$ and some $\delta > 0$, where $f_\delta(x) := x^{n+\delta}$. In the Kähler geometry setting it is known, as a consequence of Kolodziej’s estimates that a measure μ with the stronger integrability property obtained by setting $f(x) = e^{\delta x}$ for any $\delta > 0$, has a bounded Monge-Ampère potential v_μ , i.e. a bounded solution to $MA(v) = \mu$. Moreover, it has been conjectured [27] that the latter property is equivalent to v_μ being Hölder continuous. Comparing with the real setting it seems hence natural to ask if boundedness of v_μ holds for $f(x) = x^{n+\delta}$?

2.12. Caffarelli’s interior regularity results

Let Ω be a bounded open convex set in \mathbb{R}^n and f a function on Ω . Consider the following boundary value problem for a convex function ϕ in Ω , continuous up to the boundary:

$$MA(\phi) = f dx \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega$$

THEOREM 2.24 (*Caffarelli [20, 18]*). — *Assume that $f > 0$. Then any (convex) solution ϕ on Ω of the previous equation is*

- *strictly convex and locally $C^{1,\alpha}$ for some $\alpha > 0$ if there exists a constant C such that $1/C \leq f \leq C$.*
- *in the class $W_{loc}^{2,p}$ for any $p > 1$, if f is continuous*
- *smooth if f is smooth*

Proof. — For the proof of the first point we first recall the following special case of Cor 2 in [20]: if the following holds in the viscosity sense;

$$1/C dx \leq MA(\phi) \leq C dx, \quad \phi = 0 \text{ on } \partial\Omega$$

in a bounded set Ω , then ϕ is strictly convex in Ω . Moreover, as pointed out in [20] if the previous inequalities hold weakly (i.e. in the sense of Alexandrov) then they hold in the viscosity sense (see also Prop 1.3.4 in [36] where it is assumed that f is continuous, which anyway will always be the case in this paper). Hence ϕ is strictly convex in our case. But then, as shown in [19], it follows from the previous differential inequalities that ϕ is in fact locally $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. As for the second point it is contained in the main result in [18] and the final point then follows from Evans-Krylov theory for fully non-linear elliptic operators follows by standard linear bootstrapping. \square

3. Toric log Fano varieties, polytopes and Kähler-Ricci solitons

3.1. Log Fano varieties

Let X be an n -dimensional normal compact projective variety. Recall that a (Weil-) divisor D on X is a formal sum of prime divisors, i.e. codimension one irreducible subvarieties. As usual we will often identify divisors up to linear equivalence: $D \sim D'$ if $D - D'$ is principal, i.e. equal to the zero divisor of a rational function on X . A divisor D is a *Cartier divisor* if it is locally principal and we can hence identify Cartier divisors on X with line bundles on X in the standard manner. In case X is regular these two notions of divisors coincide. We will use additive notation for tensor products of line bundles on X .

When X is smooth the *canonical line bundle* K_X is defined as the top-exterior power of the cotangent bundle of X . When X is singular K_X is well-defined on the regular locus X_{reg} of X and K_X is said to be \mathbb{Q} -*Cartier* (also called a \mathbb{Q} -line bundle) if there is a positive number r such that rK_X extends from X_{reg} to a line bundle defined on all of X . The minimal such integer r is called the *Gorenstein index* of X and X is said to be Gorenstein if r is equal to one.

A normal variety X is said to be *Fano* if $-K_X$ is \mathbb{Q} -*Cartier* and ample (in the literature such a variety is sometimes said to be a \mathbb{Q} -Fano variety). When X is toric any Fano variety has *log-terminal* singularities (also called *Kawamata log-terminal*, or *klt* for short); see Remark 3.6. Such singularities play a key role in the Minimal Model Program (MMP); see [12] and references therein. From the algebro-geometric point of view klt singularities are defined in terms of discrepancies on resolutions of X , but there is also a direct analytical definition on X that is the one that we will use here (see below).

A *log pair* (X, Δ) (in the sense of MMP) consists of a normal variety X and a \mathbb{Q} -divisor Δ on X such that the log-canonical line bundle

$$K_{(X, \Delta)} := K_X + \Delta$$

is \mathbb{Q} -line bundle. Here will also assume that Δ has coefficients < 1 (in particular we do allow negative coefficients). We will write X_0 for the complement of Δ in X_{reg} . A log pair (X, Δ) is said to be a *log Fano variety* if $-K_{(X, \Delta)}$ is ample. We will also fix a section s_Δ on X_{reg} whose zero divisor is $r\Delta$ for some integer r . There is also a notion of klt singularities for log pairs (X, Δ) (see below).

3.2. Metrics on line bundles, ω -psh functions and the klt condition

Given a holomorphic $L \rightarrow X$ we let $\mathcal{H}(X, L)$ be the space of all (possibly singular) metrics on L with positive curvature current and denote by $\mathcal{H}_b(X, L)$ the subspace consisting of the locally bounded metrics (see below). We will use additive notation for (Hermitian) metrics on L . This means that a metric $\|\cdot\|$ on L is represented by a collection of local functions $\phi(= \{\phi_U\})$ defined as follows: given a local generator s of L on an open subset $U \subset X$ we define ϕ_U by the relation

$$\|s\|^2 = e^{-\phi_U},$$

where ϕ_U is upper semi-continuous (usc). It will convenient to identify the additive object ϕ with the metric it represents. Of course, ϕ_U depends on s but the curvature current

$$dd^c \phi := \frac{i}{2\pi} \partial \bar{\partial} \phi_U$$

is globally well-defined on X and represents the first Chern class $c_1(L)$, which with our normalizations lies in the integer lattice of $H^2(X, \mathbb{R})$. By definition the metric ϕ is *smooth* if ϕ_U can be chosen smooth, i.e. it is the restriction of a smooth function for some local embedding $U \rightarrow \mathbb{C}^m$. When X is smooth a smooth metric ϕ is said to be *strictly positively curved* if $dd^c \phi > 0$ (as a $(1, 1)$ -form) and for a general variety X the metric ϕ is said to be smooth if it is locally the restriction of a positively curved metric on some ambient space, i.e. $dd^c \phi$ is a Kähler form on X . Continuous and bounded (also called *locally bounded*) metrics etc are defined in a similar manner and then $dd^c \phi$ is a well-defined positive *current* on X . Fixing $\phi_0 \in \mathcal{H}_b(X, L)$ and setting $\omega_0 := dd^c \phi_0$ the map $\phi \mapsto v := \phi - \phi_0$ thus gives an isomorphism between the space of all metrics on L with positive current and the space $PSH(X, \omega_0)$ of all ω_0 -psh functions, i.e. the space of all usc functions on X such that $dd^c v + \omega_0 \geq 0$ [35].

3.2.1. The measure μ_ϕ and the klt condition

In the particular case when $L = -K_X$ a locally bounded metric ϕ determines a measure μ_ϕ on X defined as follows on X_0 (and extended by zero to all of X) :

$$\mu_\phi = i^{n^2} e^{-\phi_U} \chi^{1/r} \wedge \bar{\chi}^{1/r} \tag{3.1}$$

where χ is the local $(n, 0)$ -form which is dual to a given local generator s of $-rK_X$, i.e. $\chi = s^{-1}$ and $\|s\|_\phi^2 = e^{-r\phi_U}$. In particular, if $\chi^{1/r}$ is taken as $dz_1 \wedge \cdots \wedge dz_n$ wrt some local coordinates z_i then we will, abusing notation slightly, write $\mu_\phi = 1_{X_{reg}} e^{-\phi} dz \wedge d\bar{z}$. Now the analytical definition of X having klt singularities amounts to μ_ϕ having finite total mass for some (and hence any) locally bounded metric ϕ . It is sometimes convenient to represent μ_ϕ globally as follows: if $s \in H^0(X, -rK_X)$ is such that X_0 is contained in the Zariski open set $Y := \{s \neq 0\}$ we can write

$$\mu_\phi = 1_Y i^{n^2} s^{-1/r} \wedge \overline{s^{-1/r}} \|s\|_\phi^{2/r}$$

More generally, if (X, Δ) is a log pair then any metric ϕ on $-(K_X + \Delta)$ determines a measure μ_ϕ defined as above, but replacing s with a local generator of $-r(K_X + \Delta)$ and then taking the tensor product with $(s_\Delta^* \otimes \overline{s_\Delta^*})^{1/r}$, where s_Δ is a section with zero-divisor Δ on X_{reg} . As before, we can also write $\mu_\phi = 1_{X_{reg}} e^{-(\phi + \psi_\Delta)} dz \wedge d\bar{z}$, where $\psi_\Delta := \log(|s_\Delta|^2)/r$ is the singular metric defined by Δ on the line bundle $L_\Delta \rightarrow X_{reg}$. The log pair (X, Δ) is then said to be *klt* if μ_ϕ has finite mass for any locally bounded metric ϕ . See [12] for the equivalence with the algebro-geometric definition.

3.3. Pluricomplex energy and the Monge-Ampère measure

Let us first recall the definition of the energy type-functional \mathcal{E} on $\mathcal{H}_b(X, L)$ for a given ample line bundle $L \rightarrow X$ over a variety X [12]. It depends on the choice of a reference metric ϕ_0 in $\mathcal{H}_b(X, L)$:

$$\mathcal{E}(\phi, \phi_0) := \frac{1}{(n+1)} \sum_{j=0}^n \int_X (\phi - \phi_0) (dd^c \phi)^{n-j} \wedge (dd^c \phi_0)^j$$

where the integration pairing \int_X refers, as usual, to integration along the regular locus X_0 of X and the wedge products are defined in the usual sense of pluripotential theory a la Bedford-Taylor (see [12] and references therein). In particular, we will write

$$MA(\phi) := (dd^c \phi)^n$$

for the Monge-Ampère measure of $\phi \in \mathcal{H}_b(X, L)$. We will often omit the explicit dependence of \mathcal{E} on the reference ϕ_0 . The functional \mathcal{E} is, up to

an additive normalizing constant, uniquely determined by the variational property

$$d\mathcal{E}|_\phi = MA(\phi)$$

(viewed as one-forms on $\mathcal{H}_b(X, L)$). As a consequence the differences $\mathcal{E}(\phi) - \mathcal{E}(\psi)$ are independent of the choice of fixed reference metric. Now for any arbitrary positively curved singular metric ϕ on L we define, following [12],

$$\mathcal{E}(\phi) = \inf \{ \mathcal{E}(\psi) : \psi \in \mathcal{H}_b(X, L), \psi \geq \phi \}$$

and let $\mathcal{E}^1(X, L) := \{ \mathcal{E} > -\infty \}$. We point out, even though this fact will not be used here, that the Monge-Ampère measure can be defined for any $\phi \in \mathcal{H}(X, L)$ in terms of non-pluripolar products and one then denotes by $\mathcal{E}(X, L)$ the space of all ϕ with *full Monge-Ampère mass*, i.e. $\int_X MA(\phi) = c_1(L)^n$. In particular, we then have the relations

$$\mathcal{H}_b(X, L) \subset \mathcal{E}^1(X, L) \subset \mathcal{E}(X, L) \subset \mathcal{H}(X, L)$$

(see [16, 12])

3.4. Kähler-Einstein metrics on log Fano varieties

Following [12] an element $\phi \in \mathcal{E}^1(X, -K_X)$ is said to be a (*singular*) *Kähler-Einstein metric* if

$$(dd^c \phi)^n = C \mu_\phi \tag{3.2}$$

for some constant C , where μ_ϕ is the canonical measure 3.1 associated to ϕ . Similarly, on a (log) Kähler-Einstein metric $\phi \in \mathcal{E}^1(X, -(K_X + \Delta))$ on the log Fano variety on (X, Δ) is defined by the same equation as above, using the corresponding measure μ_ϕ . By the regularity result in [12] ϕ is in fact automatically smooth on the complement X_0 of Δ in the regular locus of X and continuous on all of X . In particular, the curvature current $\omega := dd^c \phi$ is a bona fide Kähler-Einstein metric on X_0 , i.e. $\text{Ric } \omega = \omega$ on X_0 and globally on X the equation $\text{Ric } \omega = \omega + [\Delta]$ holds in the sense of currents.

3.5. Geodesics, convexity and Ding type functionals

As explained in [9] any two metrics ϕ_0 and ϕ_1 in $\mathcal{H}_b(X, L)$ can be connected by a *geodesic* ϕ_t defined as the point-wise sup over all *subgeodesics* ψ_t connecting ϕ_0 and ϕ_1 , where such a curve of metrics ψ_t on L is defined as follows: complexifying t to take values in the strip $T := [0, 1] + i\mathbb{R}$ the corresponding metric $\psi(z, t) := \psi_t(z)$ is an $i\mathbb{R}$ -invariant continuous semi-positively curved metric ψ on the pull-back of L to $X \times T$. This is a weak

analog of bona fide geodesics defined wrt the Mabuchi metric on the space of Kähler metrics (see [9] and references therein). We recall the complex version of the Prekopa theorem [13, 14, 12] in this context: If X is a Fano variety and ψ_t a subgeodesic in $\mathcal{H}(X, -K_X)$, then the functional

$$t \mapsto -\log \int_X \mu_{\psi_t} \tag{3.3}$$

is convex in t . We note that, since $\psi_t + \psi_\Delta$ is a subgeodesic in $\mathcal{H}(X, -K_X)$ the result also applies if (X, Δ) is a Fano variety and ψ_t is a subgeodesic in $\mathcal{H}(X, -(K_X + \Delta))$. Following [12] (up to a sign difference) we define the Ding type functional.

$$\mathcal{G}_{(X, \Delta)} := \mathcal{E}(\phi) + \log \int_X \mu_{\psi_t} \tag{3.4}$$

PROPOSITION 3.1 [14, 12]. — *Let (X, Δ) be a log Fano variety. Then any Kähler-Einstein metric ϕ_{KE} for (X, Δ) maximizes the functional $\mathcal{G}_{(X, \Delta)}$ on $\mathcal{E}(X, -(K_X + \Delta))$.*

For future reference we also note that there is a “twisted” variant of the previous setting obtained by replacing μ_ϕ with $\mu_{r\phi+(1-r)\phi_0}$ for any given $\phi_0 \in \mathcal{H}_b(X, -(K_X + \Delta))$ and $r \in [0, 1]$. Then the previous proposition still holds (with the same proof) when $\mathcal{G}_{(X, \Delta)}$ is replaced by the corresponding functional $\mathcal{G}_{(X, \Delta, \phi_0, r)}$ and ϕ_{KE} with the corresponding twisted Kähler-Einstein metric (see also the even more general setting of mean field type equations in [7]).

3.6. Toric varieties and polytopes

Let T be the unit-torus in \mathbb{C}^n of real dimension n and denote by $T_c := (\mathbb{C}^*)^n$ its complexification, with its standard group structure. A n -dimensional algebraic variety X is said to be *toric* if it admits an effective holomorphic action of the complex torus T_c with an open dense orbit. In practice we will fix such an embedding and identify T_c with its image in X .

We will next briefly recall the well-known correspondence between T_c -equivariant polarizations (X, L) and convex lattice polytopes P . In fact, using the scaling $L \mapsto rL$ and $P \mapsto rP$ this will give a correspondence between polarizations by \mathbb{Q} -line bundles and rational polytopes. First recall that there are two equivalent ways of defining a polytope P in a vector space, say in \mathbb{R}^n :

1. P is the convex hull of a finite set of points A (and P is called a *lattice (rational) polytope* if $A \in \mathbb{Z}^n$ (\mathbb{Q}^n)).

2. P is the intersection of a finite number of half spaces $\langle l_F, \cdot \rangle \geq -a_F$, where l_F is a vector in the dual vector space and the label F thus runs over the facets F of P .

In the following all polytopes P will be assumed to be full-dimensional. Let us first consider the correspondence referred to above using the *first description* of a lattice polytope above. Starting with a T_c -equivariant ample line bundle L on X one considers the induced action of the group T_c on the space $H^0(X, kL)$ of global holomorphic sections of $kL \rightarrow X$ (for k a given positive integer). Decomposing the action of T_c according to the corresponding one-dimensional representations e^m , labeled by $m \in \mathbb{Z}^n$:

$$H^0(X, kL) = \bigoplus_{m \in B_k} \mathbb{C}e^m$$

one then defines the lattice polytope $P_{(X,L)}$ as the convex hull of B_k in \mathbb{R}^n . Note that, from an abstract point of view, \mathbb{R}^n thus arises as $M \otimes_{\mathbb{Z}} \mathbb{R}$, where M is the character lattice of the group T_c (compare [25]).

Conversely, given a convex lattice polytope P one obtains a pair (X_P, kL_P) by letting X_P be the closure of the image of X_P under the following map:

$$T_c \rightarrow \mathbb{P}(\bigoplus_{m \in kP \cap \mathbb{Z}^n} \mathbb{C}e^m), \quad z \mapsto [z^{m_1} : \dots : z^{m_N}] \tag{3.5}$$

equipped with its standard action of T_c , taking kL as the restriction of the line bundle $\mathcal{O}(1)$ on \mathbb{P}^{N-1} (it is well-known [25] that this is an embedding for k sufficiently large, where in fact $k = 1$ will do if X is smooth).

Next, we will briefly recall how the *second description* of $P_{(X,L)}$ above arises from the toric point of view. After perhaps twisting the action on T_c (which corresponds to translating the polytope) we may as well assume that 0 is an interior point of $P_{(X,L)}$. Now any rational polytope P containing zero in its interior may be uniquely represented as

$$P = \{p \in M_{\mathbb{R}} : \langle l_F, p \rangle \geq -a_F\} \tag{3.6}$$

for primitive dual lattice vectors l_F and strictly positive rational numbers a_F , where the index F runs over all facets of P . Let s_0 be the equivariant element in $H^0(X, L)$ corresponding to 0 and denote by D_0 its zero-divisor. By the orbit-cone correspondence (or rather orbit-*face* correspondence) [25] the facets F of $P_{(X,L)}$ correspond to the T_c -invariant prime divisors D_F on X and hence any T_c -invariant divisor D on X can be written uniquely as

$$D = \sum_F c_F D_F \tag{3.7}$$

for some integers c_F . When $D = D_0$ the integers c_F are precisely the positive numbers a_F appearing in 3.6 (note that the divisor D_0 is referred to as D_P

in [25]). In other words, $a_F = \nu_{D_F}(s_0)$; the order of vanishing of s_0 along the corresponding prime divisor D_F :

$$D_0 = \sum_F a_F D_F$$

As a consequence, if s_m is an arbitrary equivariant element in $H^0(X, L)$ then

$$\nu_{D_F}(s_m) = \nu_{D_F}(\chi^m s_0) = \nu_{D_F}(\chi^m) + \nu_{D_F}(s_0) = \langle l_F, m \rangle + \alpha_F \geq 0,$$

where χ^m is the character corresponding to m which may be identified with a rational function on T_c and where we have used the basic fact that $\nu_{D_F}(\chi^m) = \langle l_F, m \rangle$. Hence m is a lattice point in $P_{(X,L)}$ (which was the starting point of the previous correspondence).

3.6.1. The canonical divisor and toric (log) Fano varieties

Let X be a toric variety. Then $\pm K_X$ exists as a divisor on X (but in general not as \mathbb{Q} -line bundles) and

$$-K_X \sim \sum_F D_F, \tag{3.8}$$

where the index F ranges over all T_c -invariant prime divisors of X . As we will next explain there is a correspondence between *toric log Fano varieties* (X, Δ) on one hand and *rational convex polytopes P containing 0 in the interior*. This is conceptually a bit different than the more standard correspondence, referred to above, between polarized toric varieties (X, L) with a fixed T_c -action and rational convex polytopes. As we will see the point is simply that writing $L = -(K_X + \Delta)$ for a fixed divisor Δ corresponds to fixing a particular lift of the T_c -action to L . To see this first note that in the particular case when X is a *Fano* variety there is a canonical lift of T_c to the line bundle rK_X , which thus, as explained above, gives rise to a lattice polytope rP_X . Similarly, if (X, Δ) is a log Fano variety then we can get a canonical rational polytope $P_{(X,\Delta)}$ by setting $P_{(X,\Delta)} = P_{(X,L)}$ for $L = -(K_X + \Delta)$ with the T_c action induced by the one from $-K_X$, i.e. the action is compatible with the natural isomorphism between L and $-K_X$ on the embedding of T_c (using that Δ has a canonical trivialization on T_c).

PROPOSITION 3.2. — *Let (X, Δ) be a log-Fano variety and let $P_{(X,\Delta)}$ be the corresponding rational polytope in \mathbb{R}^n . Then $P_{(X,\Delta)}$ is a rational polytope containing 0 in its interior and the coefficients c_F of Δ in 3.7 are given by $1 - a_F$:*

$$\Delta = \sum_F (1 - a_F) D_F$$

Conversely, if P is a rational polytope containing zero in its interior then $P = P_{(X,\Delta)}$ for a log Fano variety (X, Δ) . In particular, Δ is effective iff $a_F \leq 1$ and $X \mapsto P_{(X,0)}$ gives a correspondence between Fano varieties and polytopes P as above with $a_F = 1$.

Proof. — Let us start with the case when $\Delta = 0$, where a proof can be found in [25] and hence we just sketch the proof. First, assume that X is a Fano variety and $-rK_X$ is an ample line bundle, for r large. Then there is a section $s \in H^0(X, -rK_X)$ with zero divisor $r \sum D_F$ (as follows from the linear equivalence 3.8). Under an equivariant embedding of T_c in X the section s pulls back to a multiple of the r th tensor power of the holomorphic n -vector field $z_1 \frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n \frac{\partial}{\partial z_n}$ on T_c (using that the latter n -vector field is naturally defined on X_{reg} and that its zero divisor there is given by $r \sum D_F$). As a consequence, s is invariant under the T_c -action, i.e. in the notation above $s = s_0$ and hence $a_F(rP_X) = \nu_{D_F}(s_0) = r$, i.e. $a_F(P_X) = 1$ as desired. Conversely, if $a_F(P) = 1$ and we take r such that rP is a lattice polytope. Then $s_0 \in H^0(X, rL)$ is such that $\nu_{D_F}(s_0) = a_F(rP) = r \cdot 1$, i.e. the zero divisor of $s_0 \in H^0(X, rL_P)$ is equal to $r \sum D_F$ and hence (by 3.8) $-K_X \sim L_P$ is ample as desired. The proof for a general Δ is similar: if $-r(K_X + \Delta)$ is an ample line bundle, for r large, we can take $s \in H^0(X, -r(K_X + \Delta))$ with zero divisor $r(\sum_F D_F - \Delta)$ which is indeed effective iff Δ has coefficients < 1 . We then deduce that $s = s_0$ as before (since Δ is trivial on T_c) and hence $a_F(P_{(X,\Delta)}) = \nu_{D_F}(s_0)/r$ is a rational positive number. The converse is then obtained as before. \square

3.7. Toric metrics as convex functions on \mathbb{R}^n and Legendre transforms

Let now (X, L) be a toric variety with corresponding polytope P and assume that $0 \in P$. As before we denote by s_0 the corresponding element in $H^0(X, L)$. Given any metric $\|\cdot\|$ on L we obtain a function $\phi(x)$ on \mathbb{R}^n by setting

$$\phi(x) := -\log \|s_0\|^2(z), \tag{3.9}$$

where $x = \log z$, coordinate-wise (see section 2.2) wrt the fixed embedding of T_c in X , where s_0 is non-vanishing.

PROPOSITION 3.3. — *The correspondence 3.9 gives a bijection between the space of $\mathcal{H}_b(X, L)^T$ of T -invariant locally bounded metrics on $L \rightarrow X$ with positive curvature current and the space $\mathcal{P}_+(\mathbb{R}^n)$. In particular, the Legendre transform induces a bijection between $\mathcal{H}_b(X, L)^T$ and the space $\mathcal{H}_b(P)$ of bounded convex functions on P .*

Proof. — First note that, by definition, $\|\cdot\|$ has positive curvature iff $\phi(z)$ is psh iff $\phi(x)$ is convex. Next, let $h_0(=\|\cdot\|^2)$ be a fixed element in $\mathcal{H}_b(X, L)^T$ with curvature current ω_0 . Writing an arbitrary metric on L as $h = e^{-v}h_0$ gives a bijection, $h \mapsto v$, between $\mathcal{H}_b(X, L)$ and the space $PSH_b(X, \omega_0)^T$. Moreover, since T_c is embedded as a Zariski open set in X it follows from the basic fact that any psh function, which is bounded from above, extends uniquely over an analytic set, that we may as well replace $PSH_b(X, \omega_0)^T$ with its restriction to T_c . Now, by definition, the space of all $\phi(x)$ in $\mathcal{P}_+(\mathbb{R}^n)$ may be identified with $PSH_b(T_c, \omega_P)^T$, where $\omega_P := dd^c \phi_P$. To conclude the proof it will thus be enough to show that $\omega_P = F^* \omega_0$ for some $h_0 \in \mathcal{H}_b(X, L)$, where F is the embedding of T_c in X . To this end we fix $k > 0$ such that kL is very ample, i.e the map 3.5 is an embedding. Let h_0 be the locally bounded (in fact continuous) metric with positive curvature on $\mathcal{O}(1) \rightarrow \mathbb{P}^{N-1}$ induced by the continuous two-homogenous psh function $\Phi(Z) := \log \max_{i=1, \dots, N} |Z_i|^2$ on $\mathbb{C}^N - \{0\}$ (the total space of $\mathcal{O}(1)^* \rightarrow \mathbb{P}^{N-1}$). By definition the restriction of h_0 to the image of X in \mathbb{P}^{N-1} is an element in $\mathcal{H}_b(X, L)$ and $-\log h_0(s_0)(z) = \phi_P(z)$ and hence $F^* \omega_0 = dd^c \phi_P$ as desired. \square

Remark 3.4. — As shown by Guillemin smooth strictly positively metrics correspond, under the Legendre transform, to smooth functions on the interior of P with a particular boundary singularity (see [17] for the extension to singular toric varieties).

3.8. Toric Kähler-Einstein metrics and solitons (proofs of Theorems 1.2, 1.3)

Here we will prove Theorems 1.2 and 1.3 apart from the statements concerning K-stability and Futaki invariants which will be considered in section 4.3.

Let X be a toric log Fano variety (X, Δ) and denote by $P(= P_{(X, \Delta)})$ the corresponding rational polytope. As explained above P contains 0 as an interior point and the corresponding invariant element $s_0 \in H^0(X, -r(K_X + \Delta))$ is such that $T_c = \{s_0 \neq 0\}$. Moreover, under the canonical identification of K_X with $K_X + \Delta$ on T_c we may identify the dual of an r th root of s_0 with the standard invariant $(n, 0)$ -form dz on T_c and hence under the correspondence in the previous section the canonical measure on X defined by a metric ϕ on $-(K_X + \Delta)$ satisfies

$$\text{Log } * \mu_\phi = e^{-\phi(x)} dx \tag{3.10}$$

Hence, the Kähler-Einstein equation 3.2 is equivalent to the equation

$$MA_{\mathbb{R}}(\phi) = Ce^{-\phi} dx \tag{3.11}$$

for a convex function $\phi \in \mathcal{E}_P^1(\mathbb{R}^n)$, where dx denotes the usual Euclidean measure. Recall that geometrically $e^{-\phi}$ is the point-wise norm of $s_0^{1/r}$ for the given metric on $-(K_X + \Delta)$. We can now deduce the equivalence between the first two points in Theorem 1.2 from Theorem 1.1 with $g = 1$ (the regularity is shown in [12]).

More generally, given a toric holomorphic vector field $V = \sum a_i \frac{\partial}{\partial z_i}$ we may define the corresponding (singular) Kähler-Ricci soliton metric $\phi \in \mathcal{E}(X, -(K_X + \Delta))$ by the equation

$$MA_{\mathbb{R}}(\phi) = Ce^{-\phi + \langle a, d\phi \rangle} dx \tag{3.12}$$

for $\phi \in \mathcal{E}_P^1(\mathbb{R}^n)$. By Theorem 1.3 ϕ is in fact automatically smooth on \mathbb{R}^n , i.e. the corresponding metric on $-(K_X + \Delta)$ is smooth on the complex torus T_c in X . Note that, for any smooth ϕ in $\mathcal{P}(\mathbb{R}^n)$ the function

$$H_V(\phi) := \langle a, d\phi \rangle \tag{3.13}$$

is globally bounded on \mathbb{R}^n . Indeed, $d\phi$ takes values in the bounded set P and hence

$$|H_V(\phi)| \leq C \tag{3.14}$$

for a constant C independent of ϕ . To see the relation to the usual Kähler-Ricci soliton equation 1.3 we note that a simple computation gives,

$$dd^c H_V(\phi) := -d(i_V \omega) \tag{3.15}$$

where $\omega = dd^c \phi$ and where the rhs above, by Cartan's formula, equals $-L_V \omega$. Hence ω indeed satisfies the equation 1.3 on the complex torus T_c . Finally, applying Theorem 1.1 with $g(p) = e^{\langle a, p \rangle}$ and using that 0 is the barycenter of $(P, e^{\langle a, p \rangle})$ iff a is the unique critical point of the strictly convex function $a \mapsto \log \int_P e^{\langle a, p \rangle} dp$, proves Theorem 1.3 up to the global regularity statement on X . The global continuity of the metric on $-(K_X + \Delta) \rightarrow X$ defined by ϕ follows from the bound 3.14, which implies that the Monge-Ampère measure of the finite energy metric has a density in $L^p(X, \mu_P)$ for any $p > 1$ and hence the continuity follows immediately as in the case $a = 0$ considered in [12]. As for the global smoothness on the complement of Δ in the regular locus of X it will be established in section 5.1.

Example 3.5. — When $n = 1$ we have $P = [a_1, a_2]$ and X is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with $\Delta = (1 - a_1)[0] + (1 - a_2)[\infty]$. The barycenter condition for the existence of a log Kähler-Einstein metric forces $a_1 = a_2 = t$ for some positive number t . For any t a direct calculation reveals that $\phi(x) = \log(e^{tx} + e^{-tx})$ gives a solution to equation 3.11 and hence, by the uniqueness statement in Theorem 1.3, any other solution is given by

$\log(e^{t(x+s)} + e^{-t(x+s)})$ for some $s \in \mathbb{R}$. Geometrically, the corresponding Kähler metrics ω_a on the two-sphere are thus “foot-balls” with a cone angle $2\pi t$:

$$\omega_t := dd^c \phi(z)/t = \frac{1}{2\pi} \frac{|z|^{2(t-1)}}{(1 + |z|^{2t})^2} dz \wedge d\bar{z} \tag{3.16}$$

and the parameter s comes from the action of the automorphism group of (X, Δ) , which does not change the isometry class of the corresponding Riemannian metrics on the two-sphere. However, for some $t > 1$, there may be different mutually non-isometric Kähler metrics ω solving the log Kähler-Einstein equation for (X, Δ) . In fact, as shown in [55], this happens precisely when Δ has negative integer coefficients.

Remark 3.6. — Any toric log Fano variety (X, Δ) in fact has klt singularities See [25] for an algebraic proof, but from the analytical point this follows almost immediately. Indeed, letting ϕ be a locally bounded metric on X represented by the function $\phi \in \mathcal{P}_+(\mathbb{R}^n)$ the mass of μ_ϕ on X coincides, according to formula 3.10 with $\int_P e^{-\phi(x)} dx$. But, since 0 is in the interior of P and $\phi - \phi_P$ is bounded we have $\phi(x) \geq |x|/C - C$ and hence the integral is indeed finite.

3.9. Relations to complete Kähler-Ricci solitons

Now assume for simplicity that X is a *smooth* toric variety and consider a family of toric \mathbb{Q} -divisors Δ_t with coefficients < 1 for $t \in]0, 1]$, such that Δ_t is affine wrt t and converges to a reduced divisor Δ_0 as $t \rightarrow 0$, i.e. $\Delta_t = \Delta_0 + O(t)$. More precisely, the coefficients $c_F(t)$ are assumed to tend to either zero or one, as $t \rightarrow 0$. Let ω_t be the corresponding curve of log Kähler-Ricci solitons associated to (X, Δ_t) , which, by Theorem 1.3, is uniquely determined modulo toric automorphisms. It seems natural to conjecture that, as $t \rightarrow 0$, the scaled metrics $\tilde{\omega}_t := \omega_t/t$ converge towards a *complete* (translating) Kähler-Ricci soliton on the quasi-projective variety $Y := X - \Delta_0$, i.e.

$$\text{Ric } \omega = L_V \omega \tag{3.17}$$

on Y for some holomorphic vector field V on Y , which is the limit of $\tilde{V}_t := tV_t$. Of course, the notion of convergence needs to be made precise, but the least one could ask for is that – modulo toric automorphisms – the convergence holds in the weak topology of currents on Y . The rationale for this conjecture is that, on X , we have $\text{Ric } \tilde{\omega}_t = t\tilde{\omega}_t + L_{\tilde{V}_t} \tilde{\omega}_t + [\Delta_t]$ and hence, when $t \rightarrow 0$, at least heuristically, one obtains a limiting Kähler current ω such that $\text{Ric } \omega = L_V \omega + [\Delta_0]$, which indicates that ω is asymptotic to a Euclidean cylinder in the normal directions close to the “boundary” Δ_0 of Y . For example, taking X to be \mathbb{P}^n and Δ_0 the hyperplane at infinity, so

that $Y = \mathbb{C}^n$, should give the the complete Kähler-Ricci soliton on \mathbb{C}^n constructed by Cao [22], generalizing Hamilton’s “cigar soliton” in \mathbb{C} . Similarly, taking X to be the total space of the bundle $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}(0)) \rightarrow \mathbb{P}^{n-1}$ and Δ_0 the “section at infinity” should give the complete Kähler-Ricci soliton in the total space of $\mathcal{O}(k) \rightarrow \mathbb{P}^{n-1}$ found in [22]. For similar limit considerations, with very precise converge results, see [34].

3.10. The invariant $R(X, \Delta)$ of a log Fano variety and lower bounds on the Ricci curvature

Let (X, Δ) be a log Fano variety and fix a smooth semi-positive form $\omega_0 \in c_1(-K_X + \Delta)$. Given $r \in [0, 1]$ we consider the following “twisted Kähler-Einstein equation” for a Kähler current $\omega \in c_1(-K_X + \Delta)$:

$$\text{Ric } \omega - \Delta = r\omega + (1 - r)\omega_0 \tag{3.18}$$

for ω smooth on $X_0(= X_{reg} - \Delta)$ and with continuous local potentials on X . In the case when X is smooth and $\Delta = 0$ the equation was introduced by Aubin as a continuity method to produce Kähler-Einstein metrics. The following theorem generalizes the main result of [40] (which concerned the case when $\Delta = 0$ and X is smooth):

THEOREM 3.7. — *Let (X, Δ) be a toric log Fano variety and ω_0 a smooth semi-positive form in $c_1(-K_X + \Delta)$. Then the supremum over all r such that the equation 3.18 admits a solution coincides with the invariant R_P (formula 2.18) of the canonical polytope P associated to (X, Δ) .*

Proof. — Given the “toric dictionary” above the theorem follows immediately from Theorem 2.18, apart from the global regularity of the solutions, which in turn follows from Theorem 1.5 in [12]. □

As shown in [54] when X is any smooth (and not necessarily toric) Fano manifold and $\Delta = 0$ the sup over all $r \in [0, 1[$ such that the equation 3.18 admits a solution coincides with the geometric invariant $R(X)$ defined as the the sup of all numbers $r \in [0, 1[$ such that there exists a Kähler metric $\omega \in c_1(-K_X)$ with $\text{Ric } \omega \geq r\omega$. Here we note that one can similarly define an invariant $R(X, \Delta)$ of any log Fano variety (X, Δ) , as the sup over all $r \in [0, 1[$ such that there exists a Kähler current $\omega \in c_1(-K_X + \Delta)$, smooth on X_0 and such that $\text{Ric } \omega - \Delta - r\omega$ is a smooth positive form. Then the following generalization of the main result of [54] holds:

THEOREM 3.8. — *Let (X, Δ) be a log Fano variety with klt singularities. If Δ is an effective divisor and ω_0 a given semi-positive form in $c_1(-K_X + \Delta)$, then the invariant $R(X, \Delta)$ coincides with the sup over all r such that*

the equation 3.18 admits a solution. Moreover, in the case when (X, Δ) is toric the divisor Δ need not be effective.

Proof. — Let us start by noting that the sup over all r such that the equation 3.18 admits a solution coincides with the sup over all $r \in [0, 1[$ such that the Ding type functional $\mathcal{G}_{(X, \Delta, \phi_0, r)} (=:\mathcal{G}_{r, \phi_0})$ is bounded from above. First, if $\mathcal{G}_{r+\delta, \phi_0} \leq C$ then a simple scaling argument gives that \mathcal{G}_{r, ϕ_0} is coercive and by the variational approach in [12] there hence exists a solution ω to the equation 3.18. Conversely, if the latter equation admits a solution, then it follows from Prop 3.1 and the subsequent discussion that the functional \mathcal{G}_{r, ϕ_0} is bounded from above (note that in the toric case the convexity argument uses Prekopa’s theorem in \mathbb{R}^n , or its generalization in [10], and hence does not rely on the positivity of the current). Finally, we note that since $\phi_0 - \phi'_0$ is bounded the upper boundedness of \mathcal{G}_{r, ϕ_0} holds for one choice of ϕ_0 precisely one it holds for *any* choice of ϕ_0 and hence the invariants above both coincide with $R(X, \Delta)$. \square

3.11. Relations to the work of Song-Wang and Li-Sun

We start by rephrasing the existence results in Theorem 1.2 in terms of a given polarized toric variety (X, L) , where L is thus an ample toric \mathbb{Q} -line bundle over X . As explained in section 3.6 the rational polytope $P := P_{(X, L)}$ may be written as $P_{(X, \Delta_L)}$ for a toric (Weil) \mathbb{Q} -divisor Δ_L such that L is linearly equivalent to $-(K_X + \Delta_L)$. Next, after replacing P by $P' := P - \{b\}$, where b is the barycenter of P , we obtain a new polytope P' with barycenter in the origin. This amounts to replacing Δ_L with another toric divisor Δ , linearly equivalent to Δ_L , such that $P' = P_{(X, \Delta)}$. Hence applying Theorem 1.2 we deduce the following

COROLLARY 3.9. — *Let X be a toric variety and L an ample toric \mathbb{Q} -line bundle over X . Then there exists a toric \mathbb{Q} -divisor Δ with coefficients in $] -\infty, 1[$ and a Kähler current $\omega \in c_1(L)$ with continuous potentials on X , such that ω is Kähler-Einstein on $X - \Delta$, satisfying $\text{Ric } \omega - [\Delta] = \omega$ in the sense of currents on X .*

Note however that the divisor Δ may not be effective, i.e. its coefficients may be negative. In particular if X is a Fano variety and $L = -rK_X$ for some $r < 1$ then it is natural to ask for which r the corresponding divisor Δ above is effective? We next observe that for $r \leq R_P$, where R_P defined as in section 2.10, the corresponding divisor Δ is indeed effective. To see this we take Δ_L to be the canonical divisor scaled by r so that $P_{(X, L)} = rP_X$, where we recall that P_X is the set where $\langle l_F, \cdot \rangle \geq -1$. Accordingly, P' is the set where $\langle l_F, \cdot \rangle \geq -r(-1 - \langle l_F, b \rangle) (=:-a_F(P'))$. Since the coefficients of

Δ are given by $c_F = 1 - a_F = 1 - r - \langle rl_F, b \rangle$ it follows that Δ is effective iff

$$1 - r - r \langle l_F, b \rangle \geq 0,$$

for any F . But when $r = R_P$ the previous inequality follows immediately from relation 2.20 applied to $x := -l_F$ and we thus deduce the following

COROLLARY 3.10. — *Let X be a toric Fano variety and denote by P the canonical rational polytope associated to X . Then, for any $r \in]0, R_P]$, there exists an effective toric \mathbb{Q} -divisor D_r , linearly equivalent to $-K_X$ and a singular Kähler metric $\omega_r \in c_1(-K_X)$ with continuous potentials, such that*

$$\text{Ric } \omega_r = r\omega_r + (1 - r)[D_r],$$

More precisely, the coefficient c_F of D_r along the invariant divisor D_F defined by the facet F of P , is given by $c_F = 1 - \langle l_F, b \rangle r / (1 - r)$, where l_F is the primitive lattice vector defining an inward normal of the facet F .

In the case when X is smooth it is shown in [53], using a method of continuity, that the metric ω_r in the previous corollary in fact has edge-cone singularities along the divisor D_r . As explained in [53] this latter result is closely related to a conjecture of Donaldson [32] concerning the invariant $R(X)$ (i.e. the greatest lower bound on the Ricci curvature) of a smooth Fano variety. According to Donaldson’s conjecture, if one replaces the metric ω_0 in equation 3.18 with a current $[D]$, where D is a given smooth divisor linearly equivalent to $-K_X$, then the corresponding equation is still solvable for any $r \in]0, R_X[$. In other words, for any such r there exists a log Kähler-Einstein metric ω_r associated to the pair $(X, (1 - r)D)$. It was moreover conjectured by Donaldson that the metric has edge-cone singularities. Very recently, Li-Sun [42] confirmed a variant of this conjecture on a smooth toric Fano variety, by using the result of Song-Wang (see Cor 3.10 and the subsequent discussion). More precisely, it was shown that for a “generic” divisor D_λ linearly equivalent to $-\lambda K_X$, for λ a sufficiently divisible integer, Donaldson’s conjecture holds for $D := D_\lambda / \lambda$. Let us briefly recall their elegant argument. By the result of Song-Wang the equation in question can be solved for $r = R_P$ if one replaces D with D_{R_P} . Next, Li-Sun use a \mathbb{C}^* -action to produce a deformation of D_λ to D_r and deduce, by the convexity results in [14] (compare Prop 3.1), that the log Ding functional of $(X, (1 - R_P)D_\lambda)$ is bounded from below. To conclude the proof they then need to show that the log Ding functional of $(X, (1 - r)D_\lambda)$ is *proper* for any $r < R_P$ (so that the existence results in [37] can be invoked). To this end Li-Song use a result from [7] which gives that the properness holds for r sufficiently small and then finally conclude by an interpolation argument. It may be worth comparing with the singular situation considered here. In

case X is a singular Fano variety then Cor 3.10 can be used as a starting point and by the generalized convexity results in [12] the same argument as in the smooth case gives that the log Ding functionals of $(X, (1 - r)D)$ are bounded for any $r \leq R_P$. However, to deduce the properness (so that the existence results in [12] can be invoked) one would need to further study the regularity properties of the log pairs (X, D_λ) .

4. K-energy type functionals and K-stability

Let us start by recalling the definition of the Mabuchi K-energy functional \mathcal{M} in Kähler geometry. This functional was first introduced in the case when X is smooth and $L \rightarrow X$ is an ample line bundle. Then \mathcal{M} is defined by the property that its differential at $\phi \in \mathcal{H}(X, L)$ is equal to $-(S_\phi - \bar{S})(dd^c\phi)^n$, where S_ϕ is the (suitably normalized) scalar curvature of the Kähler metric $dd^c\phi$. In the case when $L = -K_X$ and X is a Fano variety with log-terminal (klt) singularities it was shown in [7, 12] how to extend the definition of \mathcal{M} to a singular setting (see also [28] for related results). In case ϕ is smooth and positively curved the formula reads

$$\mathcal{M}(\phi) = F(MA(\phi)), \quad F(\mu) := -E(\mu) + D(\mu, \mu_{\phi_0}) \tag{4.1}$$

where $E(\mu)$ is the *pluricomplex energy* of the measure μ (relative to $dd^c\phi_0$) and $D(\mu, \mu')$ denotes the classical *relative entropy* of μ wrt to μ' :

$$D(\mu, \mu') =: D_{\mu'}(\mu) := \int_X \log(\mu/\mu')\mu(\geq 0)$$

if μ is absolutely continuous wrt μ' and $D(\mu, \mu') = \infty$ otherwise. When $\mu = MA(\phi)$ for $\phi \in \mathcal{H}(X, L)$ we have, by definition, that

$$E(MA(\phi)) = \mathcal{E}(\phi, \phi_0) - \int_X (\phi - \phi_0)MA(\phi)$$

It should be pointed out that in the case when X is smooth the corresponding formula 4.1 is equivalent to a previous formula of Tian and Chen [56].

We next come back to the setting of convex functions in \mathbb{R}^n associated to a convex body P , taking $\phi_P = \phi_P$ as the reference. We also equip P with a smooth positive density $g(p)$. For any function ϕ in $\mathcal{P}_+(\mathbb{R}^n)$ we define the following Mabuchi type functional associated to (P, g) :

$$\mathcal{M}_{(P,g)}(\phi) = \frac{1}{V(P,g)} \left(-\mathcal{E}_g(\phi, \phi_P) + \int \phi MA_g(\phi) + D(MA_g(\phi), dx) \right)$$

Note that $\mathcal{M}(\phi + c) = \mathcal{M}(\phi)$ and hence \mathcal{M} is determined by its restriction to the subspace of all sup-normalized elements. In this case when P is the canonical rational polytope associated to Fano variety X the functional $\mathcal{M}_P(\phi)$ coincides with the Mabuchi functional of the T -invariant metric on $-K_X$ corresponding to ϕ . Indeed, the push-forward from T_c to \mathbb{R}^n of μ_{ϕ_P} may be written as $e^{-\phi_P} dx$ and hence $D(MA(\phi), \mu_P) = D(MA(\phi), dx) - \int \phi_P MA(\phi)$.

More generally, in the setting of a log Fano variety (X, Δ) with canonical rational polytope P , with ϕ denoting a positively curved metric on $-(K_X + \Delta)$, we will write $\mathcal{M}_{(X, \Delta, V)}$ for the Mabuchi type functional corresponding to $\mathcal{M}_{(P, g)}$ for $g(p) = e^{\langle a, p \rangle}$, where V is the holomorphic toric vector field V with components a_i . In the case when $\Delta = 0$ and X is a Fano manifold the functional $\mathcal{M}_{(X, \Delta, V)}$ essentially coincides with the “modified Mabuchi functional” appearing in [59]. Similarly, we will write $\mathcal{G}_{(X, \Delta, V)}$ for the functional corresponding to \mathcal{G}_g .

4.1. Variational principles and coercivity

We will say that a functional \mathcal{F} on $\mathcal{P}_+(\mathbb{R}^n)$ is *relatively coercive* if there exists a positive constant C such that

$$\mathcal{F}(\phi) \geq -\mathcal{E}(\phi, \phi_P)/C - C$$

on the subspace of all normalized ϕ . In particular, \mathcal{F} is then bounded from below on the latter subspace. In order to relate this notion to other equivalent notions of coercivity (sometimes also called strong properness) in Kähler geometry we recall the definition of Aubin’s J -functional, which is the scale invariant analog of $-\mathcal{E}$:

$$J(\phi, \phi_0) := -\mathcal{E}(\phi, \phi_0) + \int (\phi - \phi_0) MA(\phi_0)$$

In particular, in the toric setting, $J(\phi, \phi_P) = -\mathcal{E}(\phi, \phi_P)$ if ϕ is sup-normalized, since $\phi - \phi_P = 0$ on the support of $MA(\phi_P)$. Fixing a *smooth* positively curved metric ϕ_0 we will simply write $J(\phi) := J(\phi, \phi_0)$.

LEMMA 4.1. — *Let $L \rightarrow X$ be a semi-positive line bundle over a projective variety X and let \mathcal{H}_0 denote the space of all smooth positively curved metrics on L such that $\sup_X(\phi - \phi_0) = 0$ for a fixed reference $\phi_0 \in \mathcal{H}_0$. Then there is a constant C (only depending on the reference ϕ_0) such that*

$$|J(\phi, \phi_0) - |\mathcal{E}(\phi, \phi_0)|| \leq C$$

for any $\phi \in \mathcal{H}_0$.

Proof. — The lemma follows immediately from the following estimate: there is a constant C such that, if $\mu_0 := MA(\phi_0)$

$$\int (\phi - \phi_0) MA(\phi_0) \leq C$$

When X is smooth the lemma is well-known [35] and holds more generally for any measure μ_0 such that $\phi - \phi_0$ is in $L^1(X, \mu)$ for any $\phi \in \mathcal{H}$. Taking a smooth resolution $Y \rightarrow X$ and pulling back L thus proves the general case. \square

The following proposition reveals the close connections between the two functionals \mathcal{G}_P and \mathcal{M}_P :

PROPOSITION 4.2. — *Let P be a convex body containing 0 in its interior. Then*

$$\inf_{\mathcal{P}_+(\mathbb{R}^n)} -\mathcal{G}_{(P,g)} = \inf_{\mathcal{P}_+(\mathbb{R}^n)} \mathcal{M}_{(P,g)}. \quad (4.2)$$

Moreover, the minimizers of the two functionals coincide and $-\mathcal{G}_P$ is relatively coercive iff \mathcal{M}_P is relatively coercive.

Proof. — This is proved using Legendre transforms in infinite dimension, following the argument in [7]. First note that, up to a trivial scaling, we may assume that $V(P, g) = 1$ and to simplify the notation we will omit the subindex g in the following. To conform to the sign conventions for the Legendre transforms used in the present paper it is convenient to introduce the functional $\mathcal{I}_-(v) := \log \int_{\mathbb{R}^n} e^{-v} \mu_P$ and $\mathcal{E}_-(v) := -(\mathcal{E} \circ Pr)(\phi_P + v)$ and set $\mathcal{G}_- = \mathcal{E}_- - \mathcal{I}_-$ which is thus a difference of two *convex* functionals defined on the vector space $\mathcal{C}_b(\mathbb{R}^n)$ of all bounded continuous functions v on \mathbb{R}^n . By definition, $-\mathcal{G}(\phi) = \mathcal{G}_-(\phi_P + v)$, for $v := \phi - \phi_P$ if $\phi \in \mathcal{P}_+(\mathbb{R}^n)$. Then, just as in Step 2 in the proof of Thm 1.1, the infimum of $-\mathcal{G}$ over $\mathcal{P}_+(\mathbb{R}^n)$ coincides with the infimum of \mathcal{G}_- over $\mathcal{C}_b(\mathbb{R}^n)$. We will also use the pairing $(v, \mu) := -\int_{\mathbb{R}^n} v \mu$ between $\mathcal{C}_b(\mathbb{R}^n)$ and the space $\mathcal{M}(\mathbb{R}^n)$ of all signed measures on \mathbb{R}^n . The sign conventions have been chosen so that, if μ is a probability measure, then we can write the energy of a measure μ as a Legendre transform:

$$E(\mu) = (\mathcal{E}_-)^*(\mu),$$

where the Legendre transform of a functional \mathcal{F} on the vector space $\mathcal{M}(\mathbb{R}^n)$ is defined by $\mathcal{F}^*(\mu) := \sup_{v \in \mathcal{C}_b(\mathbb{R}^n)} ((v, \mu) - \mathcal{F}(v))$. Next, one notes that, since, by Prop 2.13, the gradient of \mathcal{E}_- takes values in the subspace $\mathcal{M}_1(\mathbb{R}^n)$ of all probability measures in $\mathcal{M}(\mathbb{R}^n)$, it follows that $(\mathcal{E}_-)^*(\mu) = \infty$, unless μ is a probability measure. Similarly, it well-known that $D(\mu) = \mathcal{I}_-^*(\mu)$. Hence, $\mathcal{M}(\phi) = -(\mathcal{E}_-)^*(\mu) + \mathcal{I}_-^*(\mu)$ for $\mu = MA(\phi)$. With these preparations in place the proof of the equality 4.2 follows immediately from the

monotonicity of the Legendre transform and the fact that it is an involution (compare formula 2.3). Finally, the last two statements are proved exactly as in [7]. \square

From the results in section 2.8 concerning $\mathcal{G}_{(P,g)}$ we then deduce the following variational principle:

PROPOSITION 4.3. — *The following is equivalent for $\phi \in \mathcal{P}_+(\mathbb{R}^n)$:*

- $MA_g(\phi) = e^{-\phi}$
- ϕ minimizes the functional $-\mathcal{G}_{(P,g)}$
- ϕ minimizes the functional $\mathcal{M}_{(P,g)}$

Combining Proposition 4.2 with Theorem 2.16 also immediately gives the following analog of the latter theorem (and Theorem 1.1):

THEOREM 4.4. — *Let P be a convex body such that 0 is in the interior of P . Then $\mathcal{M}_{(P,g)}(\phi)$ is relatively coercive. Moreover, $\mathcal{M}_{(P,g)}$ is bounded from below on all of $\mathcal{P}_+(\mathbb{R}^n)$ iff 0 is the barycenter of (P, g) iff $\mathcal{M}_{(P,g)}$ admits an absolute minimizer ϕ solving the Monge-Ampère equation in Theorem 1.1.*

In the setting of toric varieties the previous results give the following

THEOREM 4.5. — *Let (X, Δ) be a toric log Fano variety with canonical polytope P and V a toric holomorphic vector field on X with components a_i . Then the following is equivalent:*

- *For any $\delta > 0$ there is a constant C_δ such that for any T -invariant locally bounded metric on $-(K_X + \Delta)$ with positive curvature*

$$\mathcal{M}_{(X,\Delta,V)}(\phi) \geq (1 - \delta) \inf_{t \in T_c} J(t^*\phi) - C_\delta$$

(and similarly for the functional $\mathcal{G}_{(X,\Delta,V)}$)

- *0 is the barycenter of $(P, e^{(a,P)})$ (i.e. a is the critical point of the Laplace transform of $1_P dp$)*
- *(X, Δ) admits a (singular) Kähler-Ricci soliton with vector field V*

Proof. — By Prop 4.2 it will be enough to consider the functional $\mathcal{G}_{(X,\Delta,V)}$. If the inequality in the first point holds then $\mathcal{G}_{(X,\Delta,V)}(\phi)$ is bounded from below and hence the second point holds, by the previous theorem. Conversely, if the second point above holds, then $\mathcal{G}_{(X,\Delta,V)}(\phi)$ is

invariant under normalizations $\phi \mapsto \tilde{\phi}$ (by Lemma 2.14) and hence by the previous theorem 4.1

$$\mathcal{G}_{(X,\Delta,V)}(\phi) \geq (1 - \delta)(-\mathcal{E}(\tilde{\phi}, \phi_0)) - C_\delta$$

Now, by the definition of normalization $-\mathcal{E}(\tilde{\phi}, \phi_0) \geq \inf_{t \in T_c} (-\mathcal{E})(t^*\phi, \phi_0)$. Finally, using that $\mathcal{G}_{(X,\Delta,V)}(\phi)$ is invariant under $\phi \mapsto \phi + c$ and invoking Lemma 4.1 concludes the proof of the equivalence between the first and the second point (which we already know is equivalent to the third point). \square

4.2. The Mabuchi functional expressed in terms of the Legendre transform on P

In this section we will consider the “Kähler-Einstein case” when $g = 1$. Following Donaldson [29] we denote by \mathcal{C}^∞ the space of all functions u on P which are smooth and strictly convex in the interior and continuous up to the boundary and set

$$\mathcal{F}(u) := \mathcal{M}_P(u^*)V(P)/n!$$

(note that if $u \in \mathcal{C}^\infty$ then $\phi := u^*$ is a smooth and strictly convex function in $\mathcal{P}(\mathbb{R}^n)_+$). We will show how to express the functional \mathcal{F} in terms of the following linear functional:

$$\mathcal{L}_{\sigma_P}(u) := \int_{\partial P} u \sigma_P - n \int_P u dp,$$

where σ_P is the canonical measure on ∂P defined by

$$\sigma_P := \frac{d}{dt} \Big|_{t=1^+} (1_{tP} dp) \tag{4.3}$$

Equivalently, a simple argument shows that P is absolutely continuous wrt the standard measure $\lambda_{\partial P}$ on ∂P induced by the Euclidean structure \mathbb{R}^n and

$$\sigma_P = \lambda_{\partial P} / \|d\rho\| \tag{4.4}$$

a.e. on ∂P , where ρ is the Minkowski functional of P , i.e. the one-homogenous defining convex function such that $P = \{\rho < 1\}$. The next proposition can be seen as a generalization of a formula of Donaldson [29] concerning the case when P is a rational simple polytope (which by definition means that there are precisely n facets meeting any given vertex). One virtue of the present approach is that it avoids any integration by parts on P (which seem rather complicated in the case of a non-simple polytope). See section 4.2 for a comparison with Donaldson’s notation.

PROPOSITION 4.6. — *The following formula holds:*

$$\mathcal{F}(u) := - \int_P \log \det(u_{ij}) dp + \mathcal{L}_{\sigma_P}(u) \tag{4.5}$$

Proof. — We start by noting that

$$D(MA(\phi), dx) = -n! \int_P \log \det(u_{ij}) dp,$$

which follows from making the change of variables $p = d\phi|_x$ in the integral defining the lhs above and using that, by duality, $\det(u_{ij}) \det(\phi_{ij}) = 1$. The rest of the proof then follow from combining formula 2.11 with the following lemma. \square

LEMMA 4.7. — *Let $\phi \in \mathcal{P}_+(\mathbb{R}^n)$. Then*

$$\frac{1}{n!} \int \phi MA(\phi) = \int_{\partial P} u d\sigma_P - (n+1) \int_P u dp$$

where $u = \phi^*$, the Legendre transform of ϕ .

Proof. — By definition $\phi(x) = \langle p, x \rangle - v(p)$ for $x = dv(p)$. Hence, making the change of variables $p = d\phi(x)$ in the integral

$$\frac{1}{n!} \int \phi MA(\phi) = \left(\int_P \langle p, dv \rangle - v(p) dp \right) - \int_P (n+1) \int_P v dp,$$

where we have rearranged the rhs in order to identify the first integral with $\int_{\partial P} v d\sigma$. To see this set $\sigma(t) := \int_{tP} v dp$ for $t > 0$. On one hand, by definition, $d\sigma(t)/dt|_{t=1} = \int_{\partial P} v \sigma_P$. On the other making the change of variables $p \rightarrow tp$ in the integral defining $\sigma(t)$ and using Leibniz rule gives an integral over P which is precisely the one in the bracket above. \square

THEOREM 4.8. — *Let P be a convex body containing 0 in its interior. Then the following is equivalent:*

- *The functional \mathcal{F} (formula 4.5) admits a minimizer u in \mathcal{C}^∞*
- *0 is the barycenter of P*
- *For any convex function on P we have $\mathcal{L}_{\sigma_P}(v) \geq 0$ with equality iff v is linear.*

Moreover, the minimizer (when it exists) is unique modulo the addition of affine functions and satisfies Abreu's equation

$$S(u) = 1 \tag{4.6}$$

Proof. — As explained above, we have, up to normalization, that $\mathcal{F}(u) = \mathcal{M}_P(\phi)$ for $\phi = u^*$ and hence the equivalence between the first two points follows from Theorem 4.4 combined with Proposition 4.5. The equivalence between the second and third point follows from Lemma 4.9 below. To see that equation 4.6 holds we recall that if u minimizes \mathcal{F} then, by Prop 4.3, ϕ satisfies the corresponding real Monge-Ampère equation with $g = 1$. But then the corresponding Kähler metric on T_c has constant Ricci curvature and in particular constant scalar curvature so that the equation 4.6 follows from Abreu’s formula [1]. \square

As explained in the introduction of the paper the previous theorem confirms a special case of a conjecture of Donaldson in [29]. In the proof of the previous theorem we used the following

LEMMA 4.9. — *Let P be a convex body containing 0 in its interior. Then*

$$\mathcal{L}_{\sigma_P}(u) = \int_{\partial P} u \sigma_P - n \int_P u dp \geq \int_P u dp$$

for any convex function v on P such that $u(0) = 0$. Moreover, equality holds iff u is linear. In particular, $\mathcal{L}_{\sigma_P}(u) > 0$ for any non-affine convex function v iff 0 is the barycenter of P .

Proof. — The lemma could be proved exactly as in Lemma 4.1 and Lemma 4.2 in [62], which applies to any convex polytope. But for completeness we give a simple alternative proof which works direct for any convex body P . Fix a convex function u on P (by a simple approximation argument we may assume that u is smooth on \bar{P}) and set $\sigma(t) := \int_{tP} u dp$ for $t > 0$. By definition $d\sigma(t)/dt_{t=1} = \int_{\partial P} u d\sigma_P$. Since $u(0) = 0$ and u is convex $u(tp)/t$ is increasing in t , Hence, $\sigma(t)/t^{n+1}$ is also increasing in t (using the change of variables $p \rightarrow tp$ in the integral), i.e. $d(\sigma(t)/t^{n+1})/dt \geq 0$. Evaluating the previous derivative at $t = 1$ then proves the desired inequality (using Leibniz rule) and the equality case also follows since $u(tp)/t$ is constant if u is linear. \square

4.2.1. Comparison with Donaldson’s setting

In [29] Donaldson associates to any rational polytope P a measure on ∂P , that we will here denote by σ'_P and which in general is different than the measure σ_P introduced above (which is defined for any convex body P). Donaldson’s measure is induced from the integral lattice in \mathbb{R}^n and defined as $\sigma'_P := d\lambda / \|d\rho\|$, where now ρ is given by $\rho(p) := \max_F(-\langle l_F, p \rangle - a_F)$ of P (compare formula 3.6), i.e. ρ is a defining one-homogenous function of

P such that $d\rho$ is a primitive integral vector on any facet. Hence, on any facet F of P

$$\sigma_P = \sigma'_P/a_F \tag{4.7}$$

and $\sigma_P = \sigma'_P$ iff P is the canonical polytope of a Fano variety. As shown by Donaldson, when P is a Delzant polytope, i.e. P corresponds to a polarized toric manifold (X, L) and the boundary of P is equipped with the measure σ_P , the solutions u as in Theorem 1.4 (which moreover satisfy Guillemin’s boundary conditions) are precisely the Legendre transforms of toric metrics on L whose curvature form $\omega \in c_1(L)$ has constant scalar curvature on X . On the other hand, writing, as in section 3.6,

$$L = -(K_X + \Delta), \quad \Delta = \sum_F (1 - a_F)D_F$$

the solutions obtained here, i.e. those induced by the measure σ_P , satisfy the following equation on X :

$$\text{Ric } \omega = \omega + \sum_F (1 - a_F)D_F, \tag{4.8}$$

Accordingly they have constant scalar curvature on the complement of the toric divisor “at infinity” D with singularities along the components D_F of D determined by the numbers a_F .

For future reference we also record the following consequence of the relation 4.7:

$$\mathcal{L}_{\sigma_P}(u) - \mathcal{L}_{\sigma'_P}(u) = \sum_F (1 - a_F)(b_F \int_P u dp - \int_F u \sigma'_P), \quad b_F = \int_F \sigma'_P / \int_P dp, \tag{4.9}$$

where $\mathcal{L}_\sigma(u)$ is defined by Donaldson’s general formula 1.5.

4.3. Futaki invariants and K -stability

4.3.1. Futaki invariants

The Futaki invariant was originally defined for X a smooth Fano manifold as a Lie algebra character. Here we will follow the approach of Ding-Tian [28] which applies to any irreducible normal Fano variety X . Given a holomorphic vector field W on the regular locus X_0 of X the corresponding Futaki invariant $f(W) \in \mathbb{R}$ may be defined as

$$f_X(W) := \frac{d}{dt} \mathcal{M}_X(\phi_t)$$

where ϕ_0 is a fixed metric, invariant under the corresponding S^1 -action and ϕ_t is the curve obtained by pull-back ϕ_0 under the flow of W (strictly, speaking in [28] there is an extra extension condition on V but as observed in [12] the condition is always satisfied). As shown in [28] $f_X(W)$ thus defined is independent of the reference ϕ_0 and the time t . More generally, given a log Fano variety (X, Δ) and a holomorphic vector field W whose flow preserves the log regular locus $X_0(=: X_{reg} - \Delta)$ we may define the log Futaki invariant $f_{(X, \Delta)}(W)$ as above, by replacing \mathcal{M}_X with $\mathcal{M}_{(X, \Delta)}$. Even more generally, given holomorphic vector fields V and W as above we define the modified log Futaki invariant $f_{(X, \Delta, V)}(W)$ as above, by replacing \mathcal{M}_X with $\mathcal{M}_{(X, \Delta, V)}$. The independence on ϕ_0 and t can then be checked as before.

In the toric log Fano case we have the following result, well-known in the smooth case [45, 29] (when Δ is trivial):

PROPOSITION 4.10. — *Let (X, Δ) be a toric log Fano variety and W the invariant vector field on X with components $a \in \mathbb{R}^n$. Then*

$$f_{(X, \Delta)}(W) := -\mathcal{L}_{\sigma_P}(\langle a, p \rangle)$$

In particular, $f_{(X, \Delta)}(W) = 0$ for all W iff 0 is the barycenter in the corresponding polytope $P_{(X, \Delta)}$.

Proof. — Letting ϕ_0 be a T -invariant metric we note that $\phi_t(x) = \phi(x + at)$. Setting $u_t := (\phi_t)^*$ this means that $u_t = u_0 - \langle a, p \rangle t$ and hence the previous formula follows immediately from Lemma 4.9. \square

4.3.2. K-stability

Let us start by recalling Donaldson's general definition [29] of K-stability of a polarized variety (X, L) , generalizing the original definition of Tian [57]. First, a *test configuration* for (X, L) consists of a polarized projective scheme $\mathcal{L} \rightarrow \mathcal{X}$ with a \mathbb{C}^* -action and a \mathbb{C}^* -equivariant map π from \mathcal{X} to \mathbb{C} (equipped with its standard \mathbb{C}^* -action) such that any polarized fiber (X_t, L_t) is isomorphic to (X, rL) for $t \neq 0$, for some integer r . The corresponding *Donaldson-Futaki invariant* $f(\mathcal{X}, \mathcal{L})$ is defined as follows: consider the N_k -dimensional space $H^0(X_0, kL_0)$ over the central fiber X_0 and let w_k be the weight of the \mathbb{C}^* -action on the complex line $\det H^0(X_0, kL_0)$. Then the Donaldson-Futaki invariant of $f(\mathcal{X}, \mathcal{L})$ is defined as the sub-leading coefficient in the expansion of w_k/kN_k in powers of $1/k$. More precisely, expanding

$$w_k = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}), \quad N_k := b_0 k^n + O(k^{n-1})$$

gives

$$f(\mathcal{X}, \mathcal{L}) = \frac{1}{b_0^2}(a_1 b_0 - a_0 b_1)$$

The polarized variety (X, L) is said to be *K-polystable* if, for any test configuration, $f(\mathcal{X}, \mathcal{L}) \leq 0$ with equality iff $(\mathcal{X}, \mathcal{L})$ is a product test configuration. Following [43] we will also assume that the total space \mathcal{X} of the test configuration is normal, to exclude some pathological phenomena observed in [43].

Similarly, if one also fixes a \mathbb{Q} -divisor Δ on X , with normal crossings, one can more generally define the *log K-polystability* of (X, L) wrt Δ as before [32, 41, 47], but phrased in terms of the corresponding *log Donaldson-Futaki invariants* defined by

$$f(\mathcal{X}, \mathcal{L}, \Delta) := f(\mathcal{X}, \mathcal{L}) + a_0 \frac{\tilde{b}_0}{b_0} - \tilde{a}_0,$$

where \tilde{a}_0 is the leading coefficient of the weight of $\det H^0(\Delta_0, kL_0)$ and a_0 is the leading coefficient of the dimension of $H^0(\Delta_0, kL_0)$ (in the definition we first assume that Δ is an irreducible divisor and then extend by linearity). In particular, if (X, Δ) is a log Fano variety then we say that (X, Δ) is log K-stable if $L := -(K_X + \Delta)$ is log K-stable wrt Δ .

Remark 4.11. — As explained in [29], in the case when X smooth, the equivariant Riemann-Roch theorem shows that the Donaldson-Futaki invariant $f(\mathcal{X}, \mathcal{L})$ is proportional (with a sign difference) to the Futaki-invariant $f_{X_0}(W)$, where W is the generator of the induced \mathbb{C}^* -action on X_0 (and a similar relation holds in the log setting [41]).

In the case when X is a general polarized toric variety it was shown by Donaldson [29] how to obtain toric test configurations from a convex piecewise linear rational function u on a polytope (called *toric degenerations*). Briefly, $(\mathcal{X}, \mathcal{L})$ is the polarized toric variety such that the corresponding rational polytope Q is defined as one side of the graph of u over P with the projection π defined so that the “roof” of Q corresponds to the central fiber X_0 .

PROPOSITION 4.12 (*Donaldson [29]*). — *Let (X, L) be a polarized toric variety, P the corresponding polytope and u a piece-wise affine convex function on P . Then u determines a test configuration such that the corresponding Donaldson-Futaki invariant is given by $\mathcal{L}_{\sigma_P}(u)$ (up to a numerical factor).*

Combining the previous proposition with formula 4.9 we then arrive at the following

PROPOSITION 4.13 (*same notation as in the previous proposition*). — Write $L = -(K_X + \Delta)$ for a toric divisor Δ . Then the log Donaldson-Futaki invariant of (X, L, Δ) is given by $\mathcal{L}_{\sigma_P}(u)$.

Proof. — By linearity we may as well assume that $\Delta = D_F$ for a fixed facet F of P . As explained in the proof of Prop 4.12 in [29] the formula $a_0 = -\int_P u dp$ holds and since we may apply the same result to the polarized toric variety $(D_F, L|_{D_F})$ we also have $\tilde{a}_0 = -\int_F u d\sigma'_P$. Moreover, since $b_0 = \int_P dp$ (and similarly $\tilde{b}_0 = \int_F \sigma'_P$) combining the previous proposition with formula 4.9 concludes the proof. \square

4.3.3. End of proof of Theorem 1.2

By Theorem 1.2 we just have to verify the equivalence between the last three points above. But this follows immediately from Lemma 4.9 combined with Prop 4.10 and Prop 4.13, at least for Futaki invariants defined with respect to toric vector fields V . Finally, if there is a Kähler-Einstein metric on (X, Δ) and Δ is *effective*, then, by Prop 3.1 the corresponding Ding type functional is bounded from above and hence so is the corresponding Mabuchi type functional $\mathcal{M}_{(X, \Delta)}$ (by the analogue of Prop 4.2; see [7, 12]). But $\mathcal{M}_{(X, \Delta)}(\phi_t)$ is linear in t if ϕ_t comes from the flow of V and hence it must be that it is actually constant, i.e. its derivative $f_{(X, \Delta)}(V)$ vanishes.

5. Convergence of the Kähler-Ricci flow

Recall that the Kähler-Ricci flow on a Fano manifold X is defined by

$$\frac{d\omega_t}{dt} = -\text{Ric } \omega_t + \omega_t \tag{5.1}$$

for a given initial Kähler form ω_0 . It may be equivalently formulated as the following flow of positively curved metrics ϕ_t on $-K_X$:

$$\frac{d\phi_t}{dt} = \log \frac{MA(\phi_t)}{\tilde{\mu}_{\phi_t}},$$

where $\tilde{\mu}_{\phi_t}$ is the measure defined by the metric ϕ_t (formula 3.1), normalized by its mass. As shown by Song-Tian [52] the latter flow can also be given a meaning on any Fano variety X with log terminal singularities. In particular, the corresponding flow of currents $\omega_t := dd^c \phi_t$ restricts to the usual Kähler-Ricci flow 5.1 on the regular locus X_0 . In fact, all the constructions and results in this section carry over immediately to the general setting of a log Fano variety (X, Δ) with klt singularities, with X_0 denoting the complement of Δ in the regular locus of X , but to simplify the notation we will assume that $\Delta = 0$.

THEOREM 5.1. — *Let X be a toric Fano variety and let ω_t evolve according to the corresponding Kähler-Ricci flow. Then there exists a family A_t of toric automorphisms of X such that $A_t^*\omega_t$ converges weakly towards a (singular) Kähler-Ricci soliton ω on X .*

Given the coercivity estimate for the modified Mabuchi functional in Theorem 4.5 the proof of the previous theorem is a rather straight-forward adaptation of the proof in [12] of the convergence of Kähler-Ricci flow on a Fano manifold for which the ordinary Mabuchi K-energy functional is proper.

Turning to the details of the proof we let ψ_t be defined as the pull-back of the metric ϕ_t under the time t flow $\exp(tV)$ of the holomorphic vector field V , where V is the unique toric vector field with components a_i determined by the canonical polytope P corresponding to X . Then ψ_t satisfies the following modified Kähler-Ricci flow (compare [59]):

$$\frac{d\psi_t}{dt} = \log \frac{MA_g(\psi_t)}{\tilde{\mu}_{\psi_t}}, \tag{5.2}$$

where, $g(p) = e^{(a,p)}$ and MA_g is the corresponding Monge-Ampère type operator. Now, a direct computation reveals that, along the latter flow,

$$\frac{d\mathcal{G}(\psi_t)}{dt} = D(MA_g(\psi_t), \tilde{\mu}_{\psi_t}),$$

where we recall that D denotes, as before, the relative entropy. In particular, $\mathcal{G}_g(\psi_t)$ is increasing in t . Strictly, speaking the previous computation is only valid in the smooth setting, but it can easily be justified by regularizing, precisely as in the proof of Lemma 6.4 in [12].

Now, by Theorem 2.16, $\mathcal{G}_g(\psi_t)$ is bounded from above and hence there is a subsequence t_j such that the rhs above tends to zero. But then it follows from the Pinsker inequality that

$$\left\| \tilde{\mu}_{\psi_{t_j}} - MA_g(\psi_{t_j}) \right\| \rightarrow 0, \tag{5.3}$$

in the absolute variation norm of the measures (i.e. the L^1 -norm between the densities wrt any fixed background measure). Let now $\tilde{\psi}_t$ be the normalization of ψ_t , obtained by applying an appropriate toric automorphism B_t and denote by $\tilde{\psi}$ a weak limit point of ψ_{t_j} . By invariance the convergence 5.3 still holds when ψ_t is replaced with its normalization $\tilde{\psi}_t$ and $\mathcal{G}_g(\tilde{\psi}_t)$ is still increasing in t (by the invariance of \mathcal{G}_g under toric automorphism, which holds as in the proof of Theorem 1.1). It thus follows from the relative

coercivity inequality in Thm 2.16 that

$$\mathcal{E}(\tilde{\psi}_t) \geq -C \tag{5.4}$$

and hence $\mathcal{E}(\tilde{\psi}) \geq -C$. In particular, $\tilde{\psi}_t$ and $\tilde{\psi}$ have full Monge-Ampère mass and hence it follows from Prop 2.4 and 5.3 that

$$MA_g(\tilde{\psi}) = \tilde{\mu}_{\tilde{\psi}}, \tag{5.5}$$

All we have to do now is to verify the following

$$\text{Claim: } \mathcal{E}_g(\tilde{\psi}_{t_j}) \rightarrow \mathcal{E}_g(\tilde{\psi})$$

Indeed, accepting the latter claim for the moment we note that, since, $\tilde{\psi}$ satisfies the equation 5.5 and hence (by Prop 4.3) maximizes the functional \mathcal{G}_g it follows, using that $\mathcal{G}(\tilde{\psi}_t)$ is increasing in t , that any subsequence of $\tilde{\psi}_{t_j}$ is an asymptotic maximizer of the functional \mathcal{G}_g . Hence, by the proof of Theorem 1.1, it converges to the unique normalized finite energy minimizer of \mathcal{G}_g (which thus coincides with $\tilde{\psi}$).

All in all, setting $A_t = B_t \circ \exp(tV)$ concludes the proof of the theorem up to the claim above to whose proof we finally turn. First note that since the modified Mabuchi functional \mathcal{M}_g is bounded from below it follows from 5.4 that

$$D_{\mu_0}(MA_g(\tilde{\psi}_t)) \leq C'$$

At this point we can invoke the following crucial compactness property (see Theorem 3.10 in [12]):

LEMMA 5.2. — *Let μ_0 be a probability measure with locally Hölder potentials and let $\phi_j \rightarrow \phi$ be a weakly convergent sequence such that $\mathcal{E}(\phi_j) \geq -C$. For each probability measure ν with finite relative entropy, i.e. $D_{\mu_0}(\nu) < \infty$, we then have*

$$\int_X (\phi_j - \phi)\nu \rightarrow 0,$$

uniformly wrt $D_{\mu_0}(\nu)$.

Applying the previous lemma to $\phi_j := \tilde{\psi}_{t_j}$ and $\nu = MA_g(\tilde{\psi}_{t_j})$ gives, after perhaps passing to a subsequence, that

$$\int_X (\tilde{\psi}_{t_j} - \tilde{\psi})MA_g(\tilde{\psi}_{t_j}) \rightarrow 0$$

But then it follows, since $\tilde{\psi}_{t_j}$ is sup-normalized, that the convergence in the claim indeed holds (compare the proof of Lemma 2.4 in [12]).

5.1. Regularity of singular Kähler-Ricci solitons

Here we will use the Kähler-Ricci flow to show that any toric (singular) Kähler-Ricci soliton (ω, V) on a toric Fano variety X has the property that $\omega (= dd^c\psi)$ is smooth on the regular locus X_0 . As explained in section 3.8 we already know that ψ is continuous, viewed as a metric on $-K_X$. We take $\psi := \psi_0$ as the initial data for the modified Kähler-Ricci flow ψ_t (formula 5.2). By the work of Song-Tian [52] the usual Kähler-Ricci flow ϕ_t is smooth on X_0 for $t > 0$ and hence so is ψ_t , since the two flows coincide up to conjugation by the flow of V . Now, by Prop 4.3 ψ_0 is a maximizer for \mathcal{G}_g and, since, as explained above, the corresponding functional $\mathcal{G}_g(\psi_t)$ is increasing ψ_t is also a maximizer for \mathcal{G}_g for any $t > 0$ (more precisely, as explained above $\mathcal{G}_g(\psi_t)$ is increasing for $t > 0$ which is enough since it is also continuous up to $t = 0$). But then it follows from Prop 4.3 that for any $t > 0$, ψ_t satisfies the corresponding Kähler-Ricci soliton equation and is smooth on X_0 . By the uniqueness of solutions modulo automorphisms we deduce that ψ_0 is also smooth on X_0 . Actually, we do not need to use the uniqueness: since the time-derivative of the flow vanishes for $t > 0$ it follows, by continuity, that $\psi_0 = \psi_t$ for any $t > 0$ and hence ψ_0 is also smooth on X_0 , as desired.

6. Appendix: proof of Lemma 2.7

A proof of the first point in Lemma 2.7 can be found in [50]; but it is also a special case of the following slightly more general claim that we will use to prove the second point: *let G_0 be a proper upper semi-continuous function on \mathbb{R}^n with a unique maximizer x_0 and let $G_t(x) := G_0(x) + tv(x)$ for a bounded continuous function v . Then $g(t) := \sup_{x \in \mathbb{R}^n} G_t(x)$ is differentiable at $t = 0$ and*

$$\frac{dg(t)}{dt} \Big|_{t=0} = v(x_0)$$

This is without doubt a well-known fact but for completeness we include the proof. First note that G_0 is bounded from above (since it is usc and hence, by properness, $G_0(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$). Since v is bounded it then follows that, for t sufficiently small, the sup of G_t is attained at some (but not necessarily unique) point x_t . Hence, $g(t) - g(0) =$

$$= G_t(x_t) - G_0(x_0) = (G_t(x_0) - G_0(x_0)) + (G_0(x_t) - G_0(x_0)) + t(v(x_t) - v(x_0))$$

Next we will show that

$$v(x_t) - v(x_0) = o(t). \tag{6.1}$$

By the continuity of v it will be enough to establish that $x_t = x_0 + o(t)$. To this end we first note that since v is bounded and G_0 is proper it follows that the x_t stay in a compact subset K and $\limsup_{t \rightarrow 0} G(x_t) \geq G(x_0) (= \sup_{x \in \mathbb{R}^n} G_t(x))$. Hence, if x_* is a limit point of x_t then the upper-semicontinuity of G_0 implies that x_* is a maximizer for G_0 . By the uniqueness assumption this means that $x_* = x_0$ and hence $x_t = x_0 + o(t)$ as desired, thus proving 6.1.

If G_0 were differentiable at x_0 we could use the maximization property of x_0 to deduce that $(G_0(x_t) - G_0(x_0)) / t = o(t)$ and hence that $\frac{dg(t)}{dt} \Big|_{t=0} = v(x_0) + 0 + 0$. But in general we only know, a priori, that $(G_0(x_t) - G_0(x_0)) \leq 0$ with equality at $t = 0$, so that $\frac{dg(t)}{dt} \Big|_{t=0^+} \leq v(x_0)$. Moreover, by symmetry (i.e. replacing t by $-t$) we also have $\frac{dg(t)}{dt} \Big|_{t=0^-} \geq v(x_0)$. On the other hand g_t is convex in t (as it is defined as a sup of affine functions) and hence its right and left derivatives exist and satisfy the inequality $\frac{dg(t)}{dt} \Big|_{t=0^-} \leq \frac{dg(t)}{dt} \Big|_{t=0^+}$. Thus it must be that the right and left derivatives both coincide with $v(x_0)$ which concludes the proof of the claim above.

To prove the first point set $G_t(x) = \langle p + ta, x \rangle - \phi(x)$ for given vectors p and a and for the second point one set $G(x, t) := \langle p, x \rangle - (\phi(x) + tv(x))$. As for the last point we first assume that ϕ is smooth and strictly convex and that f is bounded. Making the change of variables $p = d\phi|_x$ then gives

$$\int vMA(\phi) = \int v(x)d(p(x)) = \int v(x_p)dp,$$

where x_p is uniquely determined by $p = d\phi|_{x_p}$, which by duality and the first point above means that $x_p = d\phi|_p^*$, proving the desired formula in the case. Finally, we take smooth and strictly ϕ_j decreasing to a given ϕ and hence $u_j := \phi_j^*$ increase to $u := \phi^*$. By convexity $du_{j|p} \rightarrow du|_p$ for any $p \in S$ where S is the set of points p where u is finite and differentiable. By assumption $S = P - N$ where N has measure zero. Finally, letting $j \rightarrow \infty$ and using Prop 2.4 in the lhs and dominated convergence in the rhs concludes the proof for v bounded. But writing v as an increasing limit v_j of non-negative bounded continuous functions and then using the Lebesgue monotone convergence theorem then proves the general case.

Bibliography

- [1] ABREU (M.). — Kähler geometry of toric manifolds in symplectic coordinates, *Symplectic and contact topology: interactions and perspectives* (Toronto, ON/Montreal, QC, 2001).
- [2] AMBROSIO (L.), GIGLI (N.), SAVAR (G.). — Gradient flows in metric spaces and in the space of probability measures. Second edition. *Lectures in Mathematics ETH Zürich*. Birkhäuser Verlag, Basel, 2008. x+334 pp.
- [3] ARTSTEIN-AVIDANA (S.), KLARTAG (B.), MILMAN (V. M.). — The Santal point of a function, and a functional form of the Santal inequality. *Mathematika*, 51, p. 33-48 (2004).
- [4] BAKEMAN (I.J.). — Convex analysis and nonlinear geometric elliptic equations. Springer-Verlag, Berlin, (1994).
- [5] BARTHE (F.). — On a reverse form of the Brascamp-Lieb inequality. *Invent. Math.* 134, no. 2, p. 335-361 (1998).
- [6] BATYREV (V.V.). — Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. *J. Algebr. Geom.* 3, p. 493-535 (1994).
- [7] BERMAN (R.J.). — A thermodynamical formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kahler-Einstein metrics. arXiv:1011.3976.
- [8] BERMAN (R.J.). — K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics. Preprint.
- [9] BERMAN (R.J.), BERNDTSSON (B.). — Moser-Trudinger type inequalities for complex Monge-Ampère operators and Aubin’s “hypothèse fondamentale”. Preprint in 2011 at arXiv:1109.1263.
- [10] BERMAN (R.J.), BERNDTSSON (B.). — The volume of Kähler-Einstein varieties and convex bodies. arXiv:1112.4445.
- [11] BERMAN (R.J), BOUCKSOM, (S.), GUEJ (V.), ZERIAHI (A.). — A variational approach to complex Monge-Ampère equations. *Publications Math. de l’IHES* 117, p. 179-245 (2013).
- [12] BERMAN (R.J), EYSSIDIEUX (P.), BOUCKSOM (S.), GUEJ (V.), ZERIAHI (A.). Convergence of the Kähler-Ricci flow and the Ricci iteration on Log-Fano varieties. arXiv:1111.7158.
- [13] BERNDTSSON (B.). — Curvature of vector bundles associated to holomorphic fibrations. *Annals of Math.* Vol. 169, p. 531-560 (2009).
- [14] BERNDTSSON (B.). — A Brunn-Minkowski type inequality for Fano manifolds and the Bando- Mabuchi uniqueness theorem , arXiv:1103.0923.
- [15] BOUCKSOM (S.), EYSSIDIEUX (P.), GUEJ (V.), ZERIAHI (A.). — Monge-Ampère equations in big cohomology classes. *Acta Math.* 205, no. 2, p. 199-262 (2010).
- [16] BRENIER (Y.). — Polar factorization and monotone rearrangement of vector valued functions. *Communications on pure and applied mathematics* (1991).
- [17] BURNS (D.), GUILLEMIN (Vi.), LERMAN (E.). — Kähler metrics on singular toric varieties. *Pacific J. Math.* 238, no. 1, p. 27-40 (2008).
- [18] CAFFARELLI (L. A.). — Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math.* (2) 131, no. 1, p. 135-150 (1990).
- [19] CAFFARELLI (L. A.). — Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math.* 44, no. 8-9, p. 965-969 (1991).
- [20] CAFFARELLI (L. A.). — A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math.* (2) 131, no. 1, p. 129-134 (1990).

- [21] CAFFARELLI (L. A.). — The regularity of mappings with a convex potential. *J. Amer. Math. Soc.* 5, no. 1, p. 99-104 (1992).
- [22] CAO (H.-D.). — Existence of gradient Kähler-Ricci solitons. *Elliptic and parabolic methods in geometry* (Minneapolis, MN, 1994), 1-16, A K Peters, Wellesley, MA, (1996).
- [23] CAMPANA (F.), GUENANCIA (H.), PĂUN (M.). — Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields. arXiv:1104.4879. To appear in *Annales Scientifiques de l'ENS*.
- [24] CAO (H.-D.), TIAN, GANG (T.), ZHU (X.). — Kähler-Ricci solitons on compact complex manifolds with $C_1(M) > 0$. *Geom. Funct. Anal.* 15, no. 3, p. 697-719 (2005).
- [25] COX (D. A.), LITTLE (J. B.), SCHENCK (H. K.). — Toric varieties. *Graduate Studies in Mathematics*, 124. American Mathematical Society, Providence, RI (2011).
- [26] DEBARRE (O.). — Fano varieties. *Higher dimensional varieties and rational points* (Budapest, 2001), 93-132, *Bolyai Soc. Math. Stud.*, 12, Springer, Berlin (2003).
- [27] DEMAILLY (J.-P.), DINEW (S.), GUEDJ (V.), PHAM (H.H.), KOŁODZIEJ (S.), ZERIAHI (A.). — Hölder continuous solutions to Monge-Ampère equations. arXiv:1112.1388.
- [28] DING (W.Y.), TIAN (G.). — Kähler-Einstein metrics and the generalized Futaki invariant. *Invent. Math.* 110, no. 2, p. 315-335 (1992).
- [29] DONALDSON (S. K.). — Scalar curvature and stability of toric varieties. *J. Diff. Geom.* 62, p. 289-349 (2002).
- [30] DONALDSON (S. K.). — Kähler geometry on toric manifolds, and some other manifolds with large symmetry. *Handbook of geometric analysis*. No. 1, 29-75, *Adv. Lect. Math. (ALM)*, 7, Int. Press, Somerville, MA (2008).
- [31] DONALDSON (S. K.). — Constant scalar curvature metrics on toric surfaces. *Geom. Funct. Anal.* 19, no. 1, 83-136 (2009).
- [32] DONALDSON (S. K.). — Kähler metrics with cone singularities along a divisor. arXiv:1102.1196, 2011 - arxiv.org
- [33] EDMUNDS (D. E.), EVANS (W. D.). — *Spectral Theory and Differential Operators*, Oxford University Press, New York (1987).
- [34] FELDMAN (M.), ILMANEN (T.), KNOPF (D.). — Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons. *J. Differential Geom.* 65, no. 2, p. 169-209 (2003).
- [35] GUEDJ (V.), ZERIAHI (A.). — Intrinsic capacities on compact Kähler manifolds. *J. Geom. Anal.* 15, no. 4, p. 607-639 (2005).
- [36] GUTIERREZ (C.E.). — *The Monge-Ampère equation*. *Progress in Nonlinear Differential Equations and their Applications*, 44. Birkhäuser Boston, Inc., Boston, MA, 2001. xii+127 pp. ISBN: 0-8176-4177-7
- [37] JEFFRES (T.D.), MAZZEO (R.), RUBINSTEIN (Y.A.). — Kähler-Einstein metrics with edge singularities. Preprint (2011) arXiv:1105.5216.
- [38] LEGENDRE (E.). — Toric Kähler-Einstein metrics and convex compact polytopes. arXiv:1112.3239
- [39] KREUZER (M.), SKARKE (H.). — PALP: A package for analyzing lattice polytopes with applications to toric geometry. *Computer Phys. Comm.*, 157, p. 87-106 (2004).
- [40] LI (C.). — Greatest lower bounds on Ricci curvature for toric Fano manifolds. *Adv. Math.* 226, no. 6, p. 4921-4932 (2011).
- [41] LI (C.). — Remarks on logarithmic K-stability. arXiv:1104.042

- [42] LI (C.), SUN (S.). — Conical Kahler-Einstein metric revisited. arXiv:1207.5011
- [43] LI (C.), XU (C.). — Special test configurations and K-stability of Fano varieties. arXiv:1111.5398
- [44] MABUCHI (T.). — Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties. *Osaka J. Math.* 24, no. 4, p. 705-737 (1987).
- [45] McCANN (R. J.). — Existence and uniqueness of monotone measure-preserving maps, *Duke Math. J.* 80, no. 2, p. 309-323 (1995).
- [46] ODAKA (Y.). — The GIT stability of Polarized Varieties via Discrepancy. arXiv:0807.1716.
- [47] ODAKA (Y.), SUN (S.). — Testing log K-stability by blowing up formalism. arXiv:1112.1353
- [48] PHONG (D. H.), SONG (J.), STURM (J.), WEINKOVE (B.). — The Moser-Trudinger inequality on Kähler-Einstein manifolds. *Amer. J. Math.* 130, no. 4, p. 1067-1085 (2008).
- [49] RAUCH (J.), TAYLOR (B.A.). — The dirichlet problem for the multidimensional monge-ampere equation, *Rocky Mountain J. Math.* Volume 7, Number 2, p. 345-364 (1977).
- [50] ROCKAFELLAR (R. T.). — Convex analysis. Reprint of the 1970 original. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ (1997).
- [51] SHI (Y.), ZHU (X.H.). — Kähler-Ricci solitons on toric Fano orbifolds. *Math. Z.* (to appear). preprint arXiv:math/1102.2764
- [52] SONG (J.), TIAN (G.). — The Kähler-Ricci flow through singularities. Preprint (2009) arXiv:0909.4898.
- [53] SONG (J.), WANG (X.). — The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality. arXiv:1207.4839
- [54] SZKELYHIDI (G.). — Greatest lower bounds on the Ricci curvature of Fano manifolds. *Compos. Math.* 147, no. 1, p. 319-331 (2011).
- [55] TROYANOV (M.). — Metrics of constant curvature on a sphere with two conical singularities. *Differential geometry (Pescola, 1988)*, 296-306, *Lecture Notes in Math.*, 1410.
- [56] TIAN (G.). — Canonical metrics in Kähler geometry. *Lectures in Mathematics ETH Zürich*. Birkhäuser Verlag, Basel, 2000. vi+101 pp.
- [57] TIAN (G.). — Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.* 130, no. 1, p. 1-37 (1997).
- [58] TIAN (G.), ZHU (X.). — Uniqueness of Kähler-Ricci solitons. *Acta Math.* 184, no. 2, p. 271-305 (2000).
- [59] TIAN (G.), ZHU (X.). — Convergence of Kähler-Ricci flow. *J. Amer. Math. Soc.* 20, no. 3, p. 675-699 (2007).
- [60] WANG (X.), ZHU (X.). — Kähler-Ricci solitons on toric manifolds with positive first Chern class, *Advances in Mathematics* 188, p. 87-103 (2004).
- [61] ZHOU (B.), ZHU (X.). — K-stability on toric manifolds. *Proc. Amer. Math. Soc.* 136, no. 9, p. 3301-3307 (2008).
- [62] ZHOU (B.), ZHU (X.). — Relative K-stability and modified K-energy on toric manifolds. *Adv. Math.* 219, no. 4 (2008).
- [63] ZHOU (B.), ZHU (X.). — Minimizing weak solutions for Calabi's extremal metrics on toric manifolds. *Calc. Var. Partial Differential Equations* 32, no. 2, p. 191-217 (2008).