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ALEXANDRE SUKHOV, ALEXANDER TUMANOV

Gluing complex discs to Lagrangian manifolds by Gromov's method

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Gluing complex discs to Lagrangian manifolds by Gromov’s method

ALEXANDRE SUKHOV⁽¹⁾, ALEXANDER TUMANOV⁽²⁾

ABSTRACT. — The paper discusses some aspects of Gromov’s theory of gluing complex discs to Lagrangian manifolds.

RÉSUMÉ. — L’article discute certains aspects de la théorie d’attachement des disques complexes aux variétés Lagrangiennes par la méthode de Gromov.

Contents

1	Introduction	812
2	Preliminaries	813
3	Gromov-Hartogs Lemma in complex dimension 2	822
	3.1. Proof of Theorem 3.1	813
	3.2. Comments and remarks	823
4	Gromov’s Non-Squeezing Theorem	827
5	Gromov-Hartogs Lemma in higher dimension	828
	5.1. Manifolds of discs, Fredholm maps and the Sard-Smale theorem	829
	5.2. Structure lift and symplectic area	832
	5.3. Separation	833
	5.4. Proof of Proposition 5.3	835
6	Gluing a complex disc to a Lagrangian submanifold of \mathbb{C}^n	836

⁽¹⁾ Université des Sciences et Technologies de Lille, Laboratoire Paul Painlevé, U.F.R. de Mathématique, 59655 Villeneuve d’Ascq, Cedex, France
sukhov@math.univ-lille1.fr

⁽²⁾ University of Illinois, Department of Mathematics 1409 West Green Street, Urbana, IL 61801, USA
tumanov@math.uiuc.edu

6.1.	Adapted manifolds of discs	836
6.2.	Non-linear Fredholm alternative	837
6.3.	Exotic symplectic structures	839
7	Appendix: Fredholm property	840
	Bibliography	841

1. Introduction

The present paper discusses some aspects of the work of M.Gromov [13] where the method of pseudo-holomorphic curves was introduced and successfully applied to several fundamental problems of symplectic topology; “...the most striking results in symplectic and contact topology have been so far obtained only by this method...” [2]. The concentration of ideas in Gromov’s paper is high and some of them are only sketched. Detailed proofs, additional technical ingredients and far reaching generalizations have been elaborated by many authors enlarging an impressive area of applications. At present there exist several excellent introductions to the theory of pseudo-holomorphic curves focused on its different aspects and various applications, see for instance [3, 9, 11, 15, 17] (this list is, of course, highly incomplete even in the category of monographs and expository articles). A brief but deep description of Gromov’s ideas is given in [7]. Our modest goal is to present some of the results of original Gromov’s work mainly from the point of view of Complex Analysis and PDE theory . This paper is not a survey so the references list is quite short; an interested reader can consult the above mentioned monographs and the exponentially growing literature.

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2. Preliminaries

2.1. Almost complex manifolds and their maps

Let M be a smooth (C^∞) manifold of dimension $2n$. An almost complex structure J on M is a map (of appropriate regularity class C^k or C^∞) which associates to every point $p \in M$ a linear isomorphism $J(p) : T_p M \rightarrow T_p M$ of the tangent space $T_p M$ satisfying $J(p)^2 = -I$, I being the identity map. A couple (M, J) is called an almost complex manifold of complex dimension n .

Let (M, J) and (M', J') be smooth almost complex manifolds. A C^1 -map $f : M' \rightarrow M$ is called (J', J) -complex or (J', J) -holomorphic if it satisfies the *Cauchy-Riemann equations*

$$df \circ J' = J \circ df. \tag{2.1}$$

Gromov's theory is devoted to the case where M' has the complex dimension 1; hence the structure J' is necessarily integrable (see, for instance, [3]) and M' is a Riemann surface. In this special case holomorphic maps are called J -complex (or J -holomorphic) curves. We use the notation \mathbb{D} for the unit disc in \mathbb{C} and J_{st} for the standard complex structure of \mathbb{C}^n ; the value of n will be clear from the context. If in the above definition we have $M' = \mathbb{D}$ and $J' = J_{st}$, we call such a map f a J -complex disc or a pseudo-holomorphic disc or just a holomorphic disc if J is fixed. Similarly, if M' is the Riemann sphere, f is called a J -complex sphere.

Let (M, J) be an almost complex manifold and $E \subset M$ be a real submanifold of M . Suppose that a J -complex disc $f : \mathbb{D} \rightarrow M$ is continuous on $\overline{\mathbb{D}}$ and satisfies $f(b\mathbb{D}) \subset E$. Then we say that (the boundary of) the disc f is *glued* or *attached* to E or simply that f is attached to E . Sometimes such maps are called *Bishop discs* for E and we employ this terminology. Of course, if p is a point of E , then the constant map $f \equiv p$ always satisfies this definition. Often it is of interest to prove an existence (or non-existence) of a *non-constant* J -complex disc attached to E . Gromov's theory provides a powerful tool for these studies.

2.2. Cauchy-Riemann equations in coordinates

In local coordinates $Z \in \mathbb{C}^n$, an almost complex structure J is represented by a \mathbb{R} -linear operator $J(Z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $Z \in \mathbb{C}^n$ such that $J(Z)^2 = -I$. We will use the notation $\zeta = \xi + i\eta \in \mathbb{D}$. Then the Cauchy-Riemann equations (2.1) for a J -complex disc $Z : \mathbb{D} \rightarrow \mathbb{C}^n$, $Z : \mathbb{D} \ni \zeta \mapsto Z(\zeta)$ have the form $Z_\eta = J(Z)Z_\zeta$. Similarly to [3], we represent J by a complex $n \times n$

matrix function $A = A(Z)$ so that the Cauchy-Riemann equations have the form

$$Z_{\bar{\zeta}} = A(Z)\bar{Z}_{\bar{\zeta}}, \quad \zeta \in \mathbb{D}. \quad (2.2)$$

We first discuss the relation between J and A for fixed Z . Let $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an \mathbb{R} -linear map so that $\det(J_{st} + J) \neq 0$. Put $Q = (J_{st} + J)^{-1}(J_{st} - J)$. Then $J^2 = -I$ if and only if $QJ_{st} + J_{st}Q = 0$, that is, Q is complex anti-linear.

We introduce

$$\mathcal{J} = \{J : \mathbb{C}^n \rightarrow \mathbb{C}^n : J \text{ is } \mathbb{R}\text{-linear, } J^2 = -I, \det(J_{st} + J) \neq 0\}$$

$$\mathcal{A} = \{A \in \text{Mat}(n, \mathbb{C}) : \det(I - A\bar{A}) \neq 0\}$$

Let $J \in \mathcal{J}$. Then the defined above map Q is anti-linear, hence, there is a unique matrix $A \in \text{Mat}(n, \mathbb{C})$ such that $Av = Q\bar{v}$, $v \in \mathbb{C}^n$. It is proved in [3] (see also [22]) that the map $J \mapsto A$ is a birational homeomorphism $\mathcal{J} \rightarrow \mathcal{A}$. We sum up. Let J be an almost complex structure in a domain $\Omega \subset \mathbb{C}^n$. Suppose $J(Z) \in \mathcal{J}$, $Z \in \Omega$. Then J defines a unique complex matrix function A in Ω such that $A(Z) \in \mathcal{A}$, $Z \in \Omega$. We call A the *complex matrix* of J avoiding an employment the general Kodaira deformation theory terminology. The matrix A has the same regularity properties as J . Therefore, the notation J_A or A_J is appropriate according to the sense in which the correspondence $A \leftrightarrow J$ is viewed.

Let M be an almost complex manifold of complex dimension n . Locally every almost complex structure J on M admits the complex matrix in a suitable coordinate chart. Denote by \mathbb{B}_n the euclidean unit ball of \mathbb{C}^n . For every point $p \in M$, every $k \geq 1$ and every $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $Z : U \rightarrow \mathbb{B}_n$ such that

$$Z(p) = 0, \quad dZ(p) \circ J(p) \circ dZ^{-1}(0) = J_{st} \quad (2.3)$$

and the direct image $Z_*(J) := dZ \circ J \circ dZ^{-1}$ satisfies

$$\|Z_*(J) - J_{st}\|_{C^k(\overline{\mathbb{B}_n})} \leq \lambda_0. \quad (2.4)$$

Indeed, first consider a diffeomorphism Z between a neighborhood U' of $p \in M$ and \mathbb{B}_n satisfying (2.3). Then for $\lambda > 0$ introduce the isotropic dilation $d_\lambda : t \mapsto \lambda^{-1}t$ in \mathbb{C}^n and the composition $Z_\lambda = d_\lambda \circ Z$. Clearly $\|(Z_\lambda)_*(J) - J_{st}\|_{C^k(\overline{\mathbb{B}_n})} \rightarrow 0$ as $\lambda \rightarrow 0$. Setting $U = Z_\lambda^{-1}(\mathbb{B}_n)$ for $\lambda > 0$ small enough, we obtain a coordinate chart satisfying (2.3), (2.4). This elementary observation is often used in the local theory of J -complex curves.

2.3. Analytic tools

The main analytic tool in the theory of J -complex curves is the Cauchy-Green integral

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(\tau)}{\tau - \zeta} d\tau \wedge d\bar{\tau} \quad (2.5)$$

Denote by $C^{k,\alpha}(\mathbb{D})$, $k \geq 0$, $0 < \alpha < 1$, the usual Hölder space of $C^{k,\alpha}$ -functions in \mathbb{D} . A classical property of T is its regularity asserting that $T : C^{k,\alpha}(\mathbb{D}) \rightarrow C^{k+1,\alpha}(\mathbb{D})$ is a linear bounded operator. The importance of the Cauchy-Green operator comes from the fundamental fact that T gives a solution for the $\bar{\partial}$ -equation in \mathbb{D} : we have $(Tf)_{\bar{\zeta}} = f$ for every $f \in C^{k,\alpha}(\mathbb{D})$. As an example, consider the result of Nijenhuis and Woolf (see for instance [3]) which lies in the very foundation of theory. It states that for a given point $p \in M$ and a tangent vector $v \in T_p M$ there exists a J -complex disc $f : \mathbb{D} \rightarrow M$ such that $f(0) = p$ and $df(0)(\frac{\partial}{\partial \bar{\zeta}}) = \lambda v$ for some $\lambda > 0$. The disc f can be chosen smoothly depending on the initial data (p, v) and the structure J . The above-mentioned regularity of T allows to prove this theorem quite similarly to the Cauchy existence theorem for ODE's. Indeed, we replace the Cauchy-Riemann equations (2.2) by the integral equation

$$Z - T(A(Z)\bar{Z}_{\bar{\zeta}}) = W \quad (2.6)$$

where W is a usual holomorphic vector-valued function in \mathbb{D} . One can assume that $A(0) = 0$ i.e. $J(0) = J_{st}$; as shown above, after isotropic dilations of coordinates the norm of A is small. But then the implicit function theorem establishes a one-to-one correspondence between the solutions Z of the integral equation (2.6) and usual holomorphic discs W in a sufficiently small neighborhood of the origin. This implies the theorem.

This simple principle is behind many local properties of J -complex curves. It allows to develop their local theory explicitly, employing the classical properties of the Cauchy-Green integral and related singular integrals without the general elliptic PDE machinery. This is due to the well-known particularity of the theory of first order elliptic PDE with two independent variables and its interactions with Complex Analysis (cf. [4, 20, 25]). As a consequence, the local properties of J -complex curves are similar to the properties of usual complex curves in complex manifolds (though complete proofs sometimes require substantial technical efforts). For example, the set of critical points of a non-constant J -complex curve is discrete and the intersection set of two J -complex curves with distinct images is also discrete. Further important consequence is *the positivity of intersections property* and *the adjunction formula* for J -complex curves, see [3, 17, 18].

Using the generalized Cauchy formula and adding into the equation (2.6) suitable terms containing the usual Cauchy integral (over $b\mathbb{D}$), one can use such a modified integral equation to obtain solutions to some boundary value problems for the Cauchy-Riemann equations (2.2). This allows to construct J -complex discs with boundary data only locally. The global case is the subject of Gromov's theory and, as we will see, requires more advanced methods of non-linear analysis. Nevertheless, it is useful to keep in mind that in coordinates we are dealing with boundary value problems for the equations (2.2).

2.4. Interaction with symplectic and metric structures

Let M be a smooth real manifold of dimension $2n$. A closed non-degenerate exterior 2-form ω on M is called a *symplectic form* on M . A couple (M, ω) is called a symplectic manifold. As an example, consider \mathbb{C}^n with the coordinates $z_j = x_j + iy_j$. The form $\omega_{st} = \sum_{j=1}^n dx_j \wedge dy_j = (i/2) \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ is called *the standard symplectic form*. According to the classical Darboux's theorem [3], every symplectic form ω on M is locally conjugated (or symplectomorphic) to ω_{st} , i.e. there exists a local coordinate diffeomorphism ϕ satisfying $\phi^* \omega_{st} = \omega$. One of the consequence of Gromov's theory is that globally (on the whole \mathbb{R}^{2n}) this property fails.

Let (M, J) be an almost complex manifold. A J -hermitian metric on M is a real bilinear form $h : TM \times TM \rightarrow \mathbb{C}$ such that

- (a) $h(Ju, v) = ih(u, v) = i\bar{h}(v, u)$.
- (b) $h(u, u) > 0, \forall u \neq 0$.

Given hermitian metric h , we have the decomposition on its real and imaginary parts: $h(u, v) = g_h(u, v) - i\omega_h(u, v)$. Then g_h is a Riemannian metric on M and ω_h is an exterior 2-form. Furthermore, $g_h(u, v) = \omega_h(u, Jv)$ and $\omega_h(Ju, Jv) = \omega_h(u, v)$. In particular $h(u, v) = \omega_h(u, Jv) - i\omega_h(u, v)$. The form ω_h is called *the 2-form associated with h* . An exterior 2-form on (M, J) is called *J -calibrated* if

- (a) $\omega(Ju, Jv) = \omega(u, v)$
- (b) $\omega(u, Ju) > 0, \forall u \neq 0$

Each calibrated form defines a hermitian form if we set $h(u, v) := \omega(u, Jv) - i\omega(u, v)$. We say that a 2-form ω *tames* an almost complex structure J if

$$\omega(u, Ju) > 0, \forall u \neq 0$$

i.e. only the above assumption (b) is imposed. It is known that calibrating (resp. tamed) almost complex structures on a given symplectic manifold form a non-empty contractible space [3, 13, 17]. A model example is provided by the standard symplectic form ω_{st} and the standard complex structure J_{st} of \mathbb{C}^n . Clearly, ω_{st} is J_{st} -calibrated. We mention also a useful characterization of almost complex structures J on \mathbb{C}^n tamed by ω_{st} in terms of their complex matrices, see [3]. Namely, J is ω_{st} -tamed if and only if for every $Z \in \mathbb{C}^n$ one has $\|A_J(Z)\| < 1$; here the operator norm is induced by the Euclidean inner product. Also, J is calibrating if in addition its complex matrix A_J is symmetric.

Suppose now that J is tamed by ω . Then we can define the Riemannian metric $g(u, v) = \frac{1}{2}[\omega(u, Jv) + \omega(v, Ju)]$. This metric is called *the canonical Riemannian metric* associated with ω and J . In the case where ω is calibrated by J , it coincides with the metric $\omega(\bullet, J\bullet)$.

Let (M, ω, J) be an almost complex manifold with J -calibrated symplectic structure. Let X be a J -complex submanifold of (M, ω, J) (i.e. at every point the tangent space of X is J -invariant) of complex dimension k with the canonical orientation. Denote respectively by dV_X the volume form and by $vol_{2k}X$ the volume of X induced by the canonical metric g . Then $dV_X = (1/k!)\omega^k|_X$; furthermore, if X is an oriented real $2k$ -dimensional submanifold in (M, ω, J) , then $(1/k!)\int_X \omega^k \leq vol_{2k}X$ and the equality (in the case of a finite volume) holds if and only if X is J -complex. This volume estimate is called *the Wirtinger inequality*. As a consequence, J -complex submanifolds are minimal and their volume with respect to the metric g is given by

$$vol_{2k}X = \frac{1}{k!} \int_X \omega^k$$

Let (M, ω, J) be a tamed almost complex manifold. Consider a Riemann surface (S, J_S) and a J -complex curve $f : (S, J_S) \rightarrow (M, \omega, J)$. Its ω -area (or *symplectic area*) is defined by

$$area(f) = \int_S f^* \omega \tag{2.7}$$

If additionally J calibrates ω , this is precisely the area defined by the metric g associated with ω and J and $f(S)$ is a minimal surface for this metric. In the tamed case the minimality in general fails. Consider the special case where $S = \mathbb{D}$ i.e. f is a J -complex disc in a tamed almost complex manifold. Then the expression

$$E(f) := \frac{1}{2} \int_{\mathbb{D}} \left(\left\| \frac{\partial f}{\partial \xi} \right\|_g^2 + \left\| \frac{\partial f}{\partial \eta} \right\|_g^2 \right) d\xi \wedge d\eta$$

where the norm $\|\bullet\|_g$ is taken with respect g , is called *the energy* of f . We have

$$E(f) = \int_{\mathbb{D}} f^* \omega$$

This equality is called the energy identity, see for instance [17].

Example. — As a consequence every non-constant J -complex curve has a strictly positive symplectic area. For instance, suppose that ω is globally exact on M . Then for every almost complex structure tamed by ω the manifold (M, ω, J) does not contain non-constant J -complex spheres. Indeed, by Stokes' formula the symplectic area of such a sphere is equal to 0.

2.5. Real submanifolds

Let (M, ω, J) be a tamed manifold of complex dimension n . A real submanifold E of real dimension n in M is called

- (i) *Lagrangian* if $\omega|_E = 0$.
- (ii) *Totally real* if $T_p E \cap J(p)(T_p E) = \{0\}$ for every $p \in E$.

It is well-known that every Lagrangian submanifold is totally real as well as that the inverse in general fails. For example, the standard torus $\Lambda = b\mathbb{D} \times \dots \times b\mathbb{D} = (b\mathbb{D})^n$ in $(\mathbb{C}^n, \omega_{st}, J)$ is Lagrangian and totally real for every J tamed by ω_{st} . This example will be in the focus of our study.

Let E be a Lagrangian or totally real submanifold in M . Suppose that E is the zero set $E = \rho^{-1}(0)$ of a smooth vector function $\rho : M \rightarrow \mathbb{R}^n$. If f is a Bishop disc for E , then

$$\rho \circ f(\zeta) = 0, \zeta \in b\mathbb{D} \tag{2.8}$$

In local coordinates f satisfies the Cauchy-Riemann equations (2.2) which together with (2.8) form a non-linear elliptic boundary value problem. An appropriate tool of the non-linear analysis here is the continuity method. The strategy is the following. Given boundary value problem one associates a homotopy in the suitably chosen spaces of PDE operators (i.e. essentially the complex matrices A in our case) and the boundary value data (i.e. the above functions ρ) joining the initial problem with a simpler one for which a solution can be constructed. Next one constructs a homotopy in the space of solutions in order to go back to the initial problem and to obtain its solution. This procedure is based on two main technical ingredients.

The first one is an analysis of the linearized boundary value problem which allows to extend slightly by the implicit function theorem the homotopy path in the space of solutions. Usually the Fredholm properties of the

linearized problem are useful here. In our case they again follow essentially from the regularity properties of the Cauchy-Green integral. This is one of the central result of the classical theory of linear singular integral equations developed from 50-s to 70-s (first for a scalar equation on the plane, and then for vector-valued dependent variables). The book [26] contains a rather complete survey of this theory. For reader's convenience we include to Section 7 a short proof of the Fredholm property for the model boundary value problem with the usual $\bar{\partial}$ -operator. Substantially more general operators are considered in [26]. An application of standard methods based on the Cauchy integral theory requires a global coordinate neighborhood for a prescribed J -complex disc. If such a disc is not embedded or immersed, one can consider its graph and a suitable lift of an almost complex structure. In [23] this approach is used in order to construct the deformation theory for J -complex discs with free boundaries. However, a study of the Bishop discs requires a deformation theory with Lagrangian or totally real boundary data. In the case of complex dimension 2 this is rather simple. Given J -complex disc glued to a Lagrangian or totally real manifold, one can associate an integer invariant under homotopy: *the Maslov index*, see for instance [17]. An existence of nearby discs (under a perturbation of an almost complex structure and boundary data), as well as the maximal number of real variables parametrizing the perturbed discs is completely determined by this index, see for instance [10, 17, 14]. Essentially this is a direct consequence of the classical theory of the linear Riemann-Hilbert boundary value problem in the unit disc for usual (or generalized) scalar analytic functions. The number of parameters in the general solution is completely determined by the winding number of the coefficient from the boundary value condition, see [25]. Another approach is especially fruitful in the case of compact J -complex curves. One considers the pull-back of the tangent bundle of M by a given J -complex map. This gives rise to a complex vector bundle over the source Riemann surface and the Cauchy-Riemann equations can be expressed intrinsically in terms of associated connections and metric structures. In this way the well-elaborated machinery of elliptic operators on vector bundles can be applied. This general approach is employed by many authors, see [11, 13, 17]. In the case of complex bundle of rang 2 (corresponding to the complex dimension 2 of the target almost complex manifold) a deformation of a given compact J -complex curve is again determined by a single homotopy invariant: the first Chern class of the bundle. Unfortunately, both in the compact or Lagrangian boundary data case the situation changes seriously when the complex dimension of M is higher than 2. Though the first Chern class or, respectively, the Maslov index are still defined, a possibility of deformation of a given J -complex curve depends on a finite number of additional characteristics which in general are not stable under homotopy. For

instance, in the model case of the linear Riemann-Hilbert boundary value problem for usual vector-valued analytic functions the solvability depends on the so called partial indices which are not homotopically stable, see for instance [19]. A similar problem occurs in the compact case. This difficulty was overcome in Gromov's theory by geometrization of the problem using the Sard-Smale theorem. This explains the substantial difference between the theory in complex dimension 2 and higher dimensions.

The second ingredient is a priori estimates often coming from geometric considerations. In our case they are incorporated into Gromov's compactness theorem giving a very strong convergence of sequences of J -complex curves with uniformly bounded areas.

2.6. Gromov's Compactness theorem

Let S be a compact Riemann surface with (possibly empty) smooth boundary bS . We use the canonical identification of the complex plane \mathbb{C} with $\mathbb{C} \setminus \{\infty\}$. Let (M, ω, J) be a symplectic manifold with a tamed almost complex structure; as above, g is the associated Riemann metric. We assume that M has bounded geometry with respect to g i.e. satisfies the standard assumptions on the completeness, curvature and injectivity radius, see [3], p.178. Let E be a smooth compact totally real submanifold of maximal dimension in M .

Consider a sequence $f^n : S \rightarrow M$ of J -complex maps such that $f^n(bS) \subset E$.

Let $\psi : \mathbb{C} \rightarrow M$ be a *non-constant* J -complex map. We say that ψ occurs as a *spherical bubble* for the sequence (f^n) if there exists a sequence of holomorphic charts $\phi^n : R_n\mathbb{D} \rightarrow S$ with $R_n \rightarrow \infty$ converging uniformly on compact subsets of \mathbb{C} to a point $p \in S$ and such that $f^n \circ \phi^n \rightarrow \psi$ uniformly on compact subsets of \mathbb{C} .

Let $\psi : \mathbb{D} \rightarrow M$ be a *non-constant* J -complex map, continuous on $\overline{\mathbb{D}}$, with $\psi(b\mathbb{D}) \subset E$. We say that ψ occurs as a *disc bubble* for the sequence (f^n) if there exists a sequence of holomorphic charts $\phi^n : \mathbb{D} \setminus (-1 + \delta_n\mathbb{D}) \rightarrow S \cup bS$, smooth on $\mathbb{D} \setminus (-1 + \delta_n\mathbb{D})$ with $\phi^n(b\mathbb{D} \setminus (-1 + \delta_n\mathbb{D})) \subset E$ and $\delta_n \rightarrow 0$, such that (ϕ^n) converge uniformly on compact subsets of $\mathbb{D} \setminus \{-1\}$ to a point $p \in bS$ and $f^n \circ \phi^n \rightarrow \psi$ uniformly on compact subsets of $\mathbb{D} \setminus \{-1\}$.

One of the simplest versions of Gromov's Compactness Theorem is the following:

PROPOSITION 2.1. — Let $(f^k) : S \rightarrow M$ be a sequence of J -complex maps continuous on $S \cup bS$, $f^k(bS) \subset E$, intersecting a fixed compact subset $K \in M$ and such that

$$\text{area}(f^k) \leq C$$

where $C > 0$ is a constant. Then there exists a finite set Σ in $S \cup bS$, possibly empty, such that after extraction a subsequence we have:

- (i) (f^k) converges uniformly on compact subsets of $(S \cup bS) \setminus \Sigma$ to a J -complex map $f^\infty : S \rightarrow M$. Furthermore, the convergence is in every C^r -norm.
- (ii) A spherical bubble occurs at every point in $\Sigma \cap S$,
- (iii) A disc occurs at every point in $\Sigma \cap bS$.

This result is sufficient for our goals. Much more advance versions and detailed proofs are contained in [3, 12, 15, 17]. We conclude by some remarks.

1. It follows from the above definition of a spherical or disc bubble that if they arise, then they are *non-constant*. Therefore, if for some topological reasons non-constant bubbles can not arise, a sequence of J -complex maps uniformly converges in the closed unit disc. This is often used in order to prove a convergence of sequences of J -complex curves.

2. "Preservation of energy". As above, $\Sigma = \{p_1, \dots, p_l\}$ denotes the set of points where bubbles arise. Given $\varepsilon > 0$ and $p_j \in \Sigma$ set

$$m_\varepsilon(p_j) = \lim_{k \rightarrow \infty} E(f^k|_{p_j + \varepsilon \mathbb{B}_n})$$

and

$$m(p_j) = \lim_{\varepsilon \rightarrow 0} m_\varepsilon(p_j)$$

Then

$$E(f^\infty) + \sum_{j=1}^l m(p_j) = \lim_{k \rightarrow \infty} E(f^k)$$

3. Consider the special case $S = \mathbb{D}$. The sequence of sets $f^k(\bar{\mathbb{D}})$ converges to a finite connected union X of J -complex spheres and discs (a cusp-curve) in the Hausdorff distance. The above definition of bubbles depend on the initial parametrization of f^n i.e. the above version of Gromov's theorem describes the convergence of maps, not the sets. For instance, after a suitable reparametrization of S by a sequence of conformal isomorphisms, we can

obtain a sequence (\tilde{f}^n) converging outside the finite bubbling set to a single point in E . Then all limit spheres and discs in X will occur as bubbles for (\tilde{f}^n) .

4. The above theorem still holds, of course, if we vary the structure J together with maps, i.e. J is the limit in an appropriate C^s -norm of the sequence (J^n) of almost complex structure and every f^n is a J^n -complex map.

The proof of the above theorem is based on two elementary tricks: the covering argument due to Sacks-Uhlenbeck and the renormalization argument essentially appearing in the definition of a bubble. This allows to show an arising of a finite number of finite area bubbles defined on the punctured Riemann sphere or on the disc with punctured boundary respectively. In order to remove these isolated singularities, the standard elliptic estimates and the bootspraping argument can be applied. In the case of the disc bubble, these arguments are related to a suitable non-analytic version of the reflection principle.

3. Gromov-Hartogs Lemma in complex dimension 2

In this section we solve the model boundary value problem for J -complex discs attached to a Lagrangian torus in \mathbb{C}^2 . The small dimension allows to control effectively the geometric properties of solutions.

We use the notation $Z = (z, w) \in \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ for the standard coordinates in \mathbb{C}^2 . Set $\omega_1 = \frac{i}{2}dz \wedge d\bar{z}$ and $\omega_2 = \frac{i}{2}dw \wedge d\bar{w}$ and denote by $\omega = \omega_1 + \omega_2$ the standard symplectic form on \mathbb{C}^2 .

Let M_2 denotes the vector space of complex (2×2) -matrix functions A defined on \mathbb{C}^2 and of class $C^\infty(\mathbb{C}^2)$. Consider the maps $A \in M_2$ satisfying the following assumptions:

(i) The *strong taming assumption* consists of two parts. First, we suppose that there exists a real $0 \leq a_0 < 1$ such that

$$\|A(Z)\| < a_0, \forall Z \in \mathbb{C}^2 \tag{3.9}$$

where the matrix norm is induced by the Euclidean norm of \mathbb{R}^4 . Second, we suppose that *the map $Z \mapsto A(Z)$ is uniformly continuous on \mathbb{C}^2 .*

Recall that the defined in the previous section map $J_A \leftrightarrow A$ establishes a one-to-one correspondence between matrix functions A satisfying (3.9) and almost complex structures J_A on \mathbb{C}^2 tamed by the standard symplectic form ω . In what follows we simply denote J_A by J . Our assumptions on A

and the explicit formula expressing J_A in terms of A , see [22], imply that J is uniformly continuous on \mathbb{C}^2 . This guarantees that (M, ω, J) has the bounded geometry and allows to employ Gromov's compactness theorem.

The second assumption is

(ii) The map $\mathbb{C} \ni z \mapsto (z, 0)$ is J -complex. Writing explicitly

$$A = \begin{pmatrix} a & d \\ b & c \end{pmatrix} \tag{3.10}$$

we see that this assumption is equivalent to the condition $a(z, 0) = b(z, 0) = 0$ for every $z \in \mathbb{C}$.

Introduce the real 2-torus $\Lambda^t = b\mathbb{D} \times tb\mathbb{D}$ where $t > 0$. Then every torus Λ^t is Lagrangian with respect to ω . Therefore, Λ^t is totally real with respect to each almost complex structure tamed by ω .

THEOREM 3.1. — *Fix $t = T$. Under the above assumptions (i),(ii) the following holds.*

- (a) *For every point $p = (1, q) \in \Lambda^T$ there exists a J -complex disc $f : \mathbb{D} \rightarrow \mathbb{C}^2$ of class $C^\infty(\overline{\mathbb{D}})$ such that $f(1) = p$, f is an embedding, $f(b\mathbb{D}) \subset \Lambda^T$, and $f(\overline{\mathbb{D}})$ does not meet $\overline{\mathbb{D}} \times \{0\}$. Furthermore, $\text{area}(f) = \pi$.*
- (b) *When q runs over the unit circle, the discs in (a) form a C^∞ -smooth one-parameter family. They are disjoint and fill a smooth Levi-flat (with respect to J) hypersurface $\Gamma \subset \mathbb{C}^2$ with boundary Λ^T . Furthermore, they depend continuously on J and t .*

Since the proof is short, we directly present it. Then we discuss relations of Theorem 3.1 with other results.

3.1. Proof of Theorem 3.1

The proof is based on the continuity method discussed above. We proceed in several steps. Without loss of generality assume $T = 1$ and write $\Lambda = b\mathbb{D} \times b\mathbb{D}$.

(1) Since the torus Λ is fixed, with some abuse of notation we denote again by t the parameter which determines a homotopy of almost complex structures. Namely, for $t \in [0, 1]$ consider the matrix tA and the corresponding almost complex structure $J_t := J_{tA}$. As a consequence of the assumption (ii) the line $\mathbb{C} \times \{0\}$ remains J_t -complex for all t . Next, $J_0 = J_{st}$ and for

$t = 0$ we have the Levi-flat hypersurface $\mathbb{D} \times b\mathbb{D}$ foliated by the embedded J_{st} -complex discs of the form $h_c : \mathbb{D} \ni \zeta \mapsto (\zeta, c)$ where $c \in b\mathbb{D}$ is a constant. This provides for $t = 0$ the discs f with required properties. Suppose that the family of J_t -complex discs is defined on $[0, t_0[$ with $0 \leq t_0 < 1$. Our goal is to extend this family with respect to the parameter t on the whole interval $[0, 1]$.

(2) Using the notation $z = x + iy$ and $w = u + iv$, set $\lambda_1 = (1/2)(xdy - ydx)$, $\lambda_2 = (1/2)(udv - vdu)$ and $\lambda = \lambda_1 + \lambda_2$. Hence $\omega = d\lambda$. By continuity in t , the restrictions of z - and w - components of the constructed above discs $f : b\mathbb{D} \ni \zeta \mapsto f(\zeta) = (z(\zeta), w(\zeta))$ have the winding numbers about the origin equal to 1 and 0 respectively. Since the components of f take $b\mathbb{D}$ to the circles around the origin, by Stokes' formula we obtain

$$area(f) = \int_{b\mathbb{D}} f^* \lambda = \pi.$$

Furthermore, the z -component vanishes somewhere in \mathbb{D} since the winding number is equal to 1. Reparametrizing the disc f by a conformal automorphism of \mathbb{D} , we can assume that $z(0) = 0$ for all t . Next choose a point $p = (1, q) \in \Lambda$; one can also assume that $f(1) = p$. These *normalization conditions* define uniquely a family of J_t -complex discs (f_t) depending continuously on $t \in [0, t_0[$.

(3) The key argument is the following

LEMMA 3.2. — *Suppose that every disc $f_t : \zeta \mapsto (z_t(\zeta), w_t(\zeta))$, $t \in [0, t_0[$ does not intersect the axis $\mathbb{C} \times \{0\}$. Then there exists $\eta > 0$ such that*

$$|w_t(\zeta)| \geq \eta, \forall \zeta \in \bar{\mathbb{D}}, \forall t \in [0, t_0[$$

Proof. — Suppose by absurd that there exists a sequence $f^k = (z^k, w^k)$, $f^k = f^{t(k)}$, $t(k) \rightarrow t_0$, such that $\inf_{\mathbb{D}} |w^k| \rightarrow 0$ as $k \rightarrow \infty$. The area of all discs are equal to π , the structures J_t are tamed by assumption (i) and the torus Λ is totally real. Gromov's compactness theorem implies (after extracting a subsequence) that the images $f^k(\mathbb{D})$ converge in the Hausdorff metric to a finite union of J_{t_0} -complex discs with boundaries glued to the torus Λ^{t_0} . Notice here that ω is globally exact on \mathbb{C}^2 so every J -complex sphere in $(\mathbb{C}^2, \omega, J)$ is constant. Hence, spherical bubbles do not occur. Given such a limit disc, after a suitable reparametrization by a sequence of conformal isomorphisms, the convergence is in every $C^l(K)$ -norm on each compact subset K of $\bar{\mathbb{D}} \setminus \Sigma$ where Σ is a finite subset of $b\mathbb{D}$. By assumption, one of the limit discs touches the J_{t_0} - complex line $\mathbb{C} \times \{0\}$. Since the boundaries of

discs is attached to Λ^{t_0} , the disc is not contained in this line. Positivity and stability of the intersection indices of J -complex curves imply that $f^k(\mathbb{D})$ also intersects the line $\mathbb{C} \times \{0\}$ for k big enough. This contradiction proves the lemma. \square

(4) As a consequence we obtain that the above sequence $f^k = (z^k, w^k)$ converges to a single J_{t_0} -complex disc in every C^l -norm on $\bar{\mathbb{D}}$. Indeed, consider the "principal" limit disc $f^\infty = (z^\infty, w^\infty)$ of this sequence defined as the limit of the sequence of maps f^k converging on $\bar{\mathbb{D}}$ (without any additional reparametrization) off at most a finite subset of $b\mathbb{D}$ where bubbles arise. The disc f^∞ has a non-constant z -component because of the normalization condition imposed above. In particular, its area is positive. Since the winding number of its w -component is equal to zero by Lemma, we conclude that $area(f^\infty) = \pi$. But the total area of limit discs is bounded π . Therefore the limit set consists of a single disc and there are no bubbles. Gromov's compactness theorem implies the convergence in every $C^l(\bar{\mathbb{D}})$ -norm. In particular, the winding numbers of the z - and w - components of the limit disc f^∞ again are equal to 1 and 0 respectively. The discs under consideration are embeddings for $t \in [0, t_0[$. Suppose that the limit disc f^∞ is multiply covered that is $f^\infty = \tilde{f} \circ \Pi$ where \tilde{f} is a J_{t_0} -complex disc, $\tilde{f}(b\mathbb{D}) \subset \Lambda$, and Π is the Blaschke product of degree $d \geq 2$. Then the winding number of the z -component of f^∞ is an integer multiple of d and cannot be equal to 1. Thus the disc f^∞ is not multiply covered and remains an embedding by the adjunction formula for J -complex curves. Then constructed discs remain embeddings for all t . Our families of discs and almost complex structures are homotopic to the above J_{st} -complex disc h_c glued to the standard tori and by continuity the Maslov index of every disc has the same value as for h_c and so is equal to 0. By the implicit function theorem (see, for instance [10, 14]) the disc f_{t_0} generates a real 1-parameter family of J_t -complex discs with boundaries glued to Λ for $t \in [0, t_0 + \varepsilon[$ for some $\varepsilon > 0$ and satisfying the normalization condition (the discs f_t already defined for $t < t_0$ belong to this generated family by the uniqueness part of the implicit function theorem). This proves the part (a) of Theorem 3.1.

(5) Consider another point $p' \in \Lambda^1$ and corresponding family of discs constructed as above. The discs of the two constructed families do not intersect for t close to 0 and hence for all t because of the positivity and stability of intersection indices of J -complex curves. This implies (b) and concludes the proof of Theorem 3.1. \square

3.2. Comments and remarks

1. J.Duval and D.Gayet [8] recently constructed an example of a totally real 2-torus in the unit sphere S^3 of \mathbb{C}^2 , isotopic to the standard torus (i.e. unknotted) and such that there does not exist a J_{st} -complex disc with boundary attached to this torus. They proved that to every torus of this class one can attach the boundary of J_{st} -complex disc or the boundary of a J_{st} -complex annulus.

2. Theorem 3.1 implies Gromov's non-squeezing theorem [13] in \mathbb{C}^2 . It suffices to use the discs provided by Theorem 3.1 instead of J -complex spheres in classical Gromov's argument. We give the details in the next section.

3. Theorem 3.1 can be used for studying holomorphically convex hulls (of course, this question is of interest only if the structure J is integrable). For such application it is appropriate to give a slightly different construction.

We replace (ii) by a stronger assumption:

(ii') There exists $r_0 > 0$ such that A is a lower triangular complex matrix function i.e. $d = 0$ in (3.10) on $\mathbb{D} \times r_0\mathbb{D}$ and $\|a\|_\infty \leq a_0$, $\|c\|_\infty \leq a_0$ for some $a_0 < 1$. Furthermore, we still assume that the map $\mathbb{C} \ni z \mapsto (z, 0)$ is J -complex. Consider the tori $\Lambda^t = b\mathbb{D} \times tb\mathbb{D}$. We will construct a homotopy of J_A -complex discs starting from small t and then extend it for all $t \in [0, 1]$ without deformation of the almost complex structure J_A .

Reparametrizing our homotopy of tori in t if necessary, we can choose $r_1 > 0$ very small with respect to r_0 and assume that $\Lambda^0 = b\mathbb{D} \times r_1b\mathbb{D}$ and $T = 1$. Then in view of the assumption (ii') it follows by [24] that there exists an open neighborhood W of 0 in \mathbb{C} such that the set $(\overline{\mathbb{D}} \times \overline{W}) \setminus (\overline{\mathbb{D}} \times \{0\})$ is foliated by smooth Levi-flat (with respect to J_A) hypersurfaces whose boundaries coincide with the tori Λ^t , $t \in [0, r_2]$, $r_1 < r_2 < r_0$. Every hypersurface in turn is foliated by J -complex discs f homotopic to the J_{st} -complex disc h_c as above. Now we extend the homotopy in t precisely as in the proof of Theorem.

4. Denote by H the unbounded "Hartogs type" figure $H = (\{1 - \delta < |z| < 1\} \times \mathbb{C}) \cup (\mathbb{D} \times \varepsilon\mathbb{D})$ with $0 < \delta < 1$, $\varepsilon > 0$. Assume that the Levi form (with respect to J_A) of the boundary $\Pi = b\mathbb{D} \times \mathbb{C}$ of the domain $\mathbb{D} \times \mathbb{C}$ is non-negative definite at every point. As a consequence of Remark 3 we obtain that this domain coincides with the holomorphic hull of H . We point out that the Levi flat hypersurfaces constructed in Theorem 3.1 cannot touch the hypersurface Π because of the Levi non-negative definiteness of

II (see [6]). Furthermore, since J is uniformly continuous by (i) and the area of discs are bounded, there exists an upper bound on diameters of discs with boundaries glued to a fixed torus (see, for instance, [3]). This implies that the constructed Levi flat hypersurfaces sweep out the domain $\mathbb{D} \times \mathbb{C}$ i.e. the holomorphic envelope of H . For this reason we call Theorem 3.1 “Gromov-Hartogs lemma”.

5. M.Gromov [13] proved (together with many other things) a result similar to Theorem 3.1 in a more general setting of elliptic structures (an almost complex structure can be viewed as a special case). He assumes that a Lagrangian torus is contained in a strictly pseudoconvex hypersurface (this notion can be defined in the elliptic category). It is not obvious how to deform such a hypersurface keeping the strict pseudoconvexity through a deformation of almost complex (or elliptic) structure. However, if we apply the construction described in Remark 4, we do not need to deform a complex structure. This leads to the following bounded version of the above result. Suppose additionally that under the assumptions of Remark 3 the torus $\Lambda = b\mathbb{D} \times b\mathbb{D}$ is contained in a smooth hypersurface which bounds a domain Ω and whose Levi form with respect to J_A is non-negative definite. Then the torus Λ bounds the Levi-flat hypersurface (produced by the above construction) contained in Ω . In this case it suffices to require that A and J are defined only in a neighborhood of the closure $\bar{\Omega}$.

4. Gromov's Non-Squeezing Theorem

As above, let ω denote the standard symplectic form of \mathbb{C}^2 .

THEOREM 4.1. — *Let G be a relatively compact domain in $R\mathbb{D} \times \mathbb{C}$ where $R > 0$. Suppose that $r > 0$ and there exists a diffeomorphism $\Phi : r\mathbb{B} \rightarrow G$ with $\Phi^*\omega = \omega$. Then $r \leq R$.*

Proof. — Performing a translation in the w -direction one can assume that the disc $R\mathbb{D} \times \{0\}$ does not meet \bar{G} . Consider an increasing sequence $r_n \rightarrow r$. The almost complex structure $J := \Phi_*(J_{st})$ is tamed by ω . Multiplying the complex matrix of J by suitable smooth cut-off functions, we obtain for every n a smooth almost complex structure J_n on \mathbb{C}^2 such that $J_n = J$ on $\Phi(r_n\mathbb{B}_2)$ and $J_n = J_{st}$ on $\mathbb{C}^2 \setminus \bar{G}$. Consider the point $p = \Phi(0)$. According to previous section, for every n there exists a J_n -complex disc f^n such that

- (i) $f^n(0) = p$,
- (ii) $f^n(b\mathbb{D}) \subset Rb\mathbb{D} \times tb\mathbb{D}$ for some $t > 0$,
- (iii) $area(f^n) = \pi R^2$

Then $X^n = \Phi^{-1}(f^n(\mathbb{D}) \cap \Phi(r_n \mathbb{B}_2))$ is a closed J_{st} -complex curve in $r_n \mathbb{B}_2$. Furthermore, $0 \in X^n$ and $area(X^n) \leq \pi R^2$. Passing to the limit as $n \rightarrow \infty$, we obtain by Bishop's convergence theorem [5] that there exists a closed complex curve X in $r \mathbb{B}_2$, containing the origin and with $area(X) \leq \pi R^2$. On the other hand, since $0 \in X$, it follows by the classical results [5] that $area(X) \geq \pi r^2$ and theorem follows. \square

5. Gromov-Hartogs Lemma in higher dimension

An attempt to generalize directly the previous argument to higher dimensions meets difficulties. For instance, the positivity of intersections does not make sense. Another problem concerns a possibility of deformation of a J -complex disc with a Lagrangian (or totally real) boundary data. As it was discussed above, such a deformation can not be described in terms of a single index invariant under a homotopy. So more advanced tools are needed.

We use the notation $Z = (z, w) = (z, w_2, \dots, w_n) \in \mathbb{C} \times \mathbb{C}^{n-1}$ for the standard coordinates in \mathbb{C}^n . Set $\omega_1 = \frac{i}{2} dz \wedge d\bar{z}$ and $\omega_j = \frac{i}{2} dw_j \wedge d\bar{w}_j$. Let $\omega = \sum_{j=1}^n \omega_j$ denotes the standard symplectic form on \mathbb{C}^n . Let G be an open set in \mathbb{R}^N , let also $0 < \alpha < 1$ and $k \geq 0$ be an integer. We denote by $C^{k,\alpha}(G)$ the class of functions $u : G \rightarrow \mathbb{R}$ admitting the partial derivatives $D^s u$, $|s| \leq k$ in G which are α -Holder continuous on G when $|s| = k$. This is a Banach space with respect to the standard norm. For simplicity of notations we keep the same notation for the space of vector-valued functions $u : G \rightarrow \mathbb{R}^{N'}$ with components of class $C^{k,\alpha}(G)$.

Denote by M_n the vector space of complex $(n \times n)$ -matrix functions A defined on \mathbb{C}^n and of class $C^{k,\alpha}(\mathbb{C}^n)$. Consider the maps $A \in M_n$ satisfying the following assumptions:

- (i) The *strong taming assumption* is precisely the same as in dimension 2.

The next condition we impose is

- (ii) (*Support assumption*) The support $K := \text{supp} A$ is separated from the union of hyperplanes $\{(z, w) \in \mathbb{C}^n : w_j = 0\}$, $j = 2, \dots, n$. Therefore, the almost complex structure J_A coincides with the standard complex structure J_{st} of \mathbb{C}^n in a neighborhood of this union. More precisely, there exists constant $r_0 > 0$ such that the following separation property holds:

$$\inf\{|w_j| : (z, w) \in K, j = 2, \dots, n\} \geq 2r_0 \tag{5.11}$$

We point out that $\text{supp} A$ in general is not supposed to be a compact subset in \mathbb{C}^n . For example, it can be unbounded and we do not require that the

structure J_A could be extended to an almost complex structure on the complex projective space.

We study maps $Z : \zeta \mapsto Z(\zeta) = (z(\zeta), w(\zeta))$, $Z : \mathbb{D} \rightarrow \mathbb{C}^n$ of class $C^{k+1,\alpha}(\mathbb{D})$, $k \geq 0$, satisfying the following elliptic PDE system:

$$Z_{\bar{\zeta}} - A(Z)\bar{Z}_{\bar{\zeta}} = 0, \zeta \in \mathbb{D} \tag{5.12}$$

with the non-linear boundary value condition

$$|z(\zeta)|^2 = R, |w_j(\zeta)|^2 = t_j, j = 2, \dots, n, \zeta \in b\mathbb{D} \tag{5.13}$$

Here $R > 0$, $t_j > 0$ are prescribed constants and the matrix function $A \in C^{k,\alpha}(\mathbb{C}^n)$ satisfies the assumptions (i), (ii). A C^1 -map $Z : \mathbb{D} \rightarrow \mathbb{C}^n$ is a J_A -complex disc if and only if it satisfies the system (5.12) which represents the Cauchy-Riemann equations corresponding to the structure J_A .

The main result of this section is the following :

THEOREM 5.1. — *For every point $(a, b) \in \mathbb{D} \times \mathbb{C}^{n-1}$, $b = (b_2, \dots, b_n)$, $b_j \neq 0$, there exists $t \in (R_+^*)^{n-1}$ and a solution $Z = (z, w) \in C^{k+1,\alpha}(\mathbb{D})$ of the boundary value problem (5.12), (5.13) satisfying $Z(0) = (a, b)$. Furthermore, the winding number of the z -component of Z is equal to 1 and the winding number of every w_j -component is equal to 0.*

This result has several applications. We mention some of them.

1. An obvious consequence is Gromov's non-squeezing theorem in any dimension.

2. In the case where the complex structure J_A is integrable Theorem 5.1 can be used as a version of "Hartogs lemma" in order to study the holomorphic convexity properties.

3. Gromov proved the above result in a slightly different form. For a fixed torus he establishes an existence of a non-constant disc whose boundary contains a given point of this torus. This does not give an information about the behavior of interior points of the discs. Therefore, it does not allow to determine what subset of \mathbb{C}^n is swept by discs when one varies the boundary tori. For this reason we present here a modified version.

5.1. Manifolds of discs, Fredholm maps and the Sard-Smale theorem

In this section we outline the proof.

Fix a smooth map γ from the unit circle to the space of matrix functions satisfying the above assumptions (i), (ii) such that $\gamma(e^{i0}) = 0$ and $\gamma(e^{i\pi}) = A$. The image by γ of the unit circle is denoted by \mathcal{M} . Without loss of generality assume $R = 1$. Denote by Λ^t the torus

$$\Lambda^t = \{(z, w) \in \mathbb{C}^n : |z| = 1, |w_j|^2 = t_j, j = 2, \dots, n\} = b\mathbb{D} \times t_2^{1/2}b\mathbb{D} \times \dots \times t_n^{1/2}b\mathbb{D}$$

Since Λ^t is Lagrangian for ω and J is tamed by ω , this torus is totally real with respect to J .

Denote by X the set (Z, t) of maps $Z : \mathbb{D} \rightarrow \mathbb{C}^n$ of class $C^{k+1, \alpha}(\mathbb{D})$ and $t = (t_2, \dots, t_n)$, $t_j > 0$ satisfying the following assumption:

(iii) (*Boundary data condition*) For every (Z, t) the boundary condition (5.13) holds. Geometrically this means that X is formed by the smooth discs with boundaries attached to the tori Λ^t when t runs over $(\mathbb{R}_+^*)^{n-1}$.

Denote by $(Z^0, t^0) \in X$ the map (z^0, w^0) where $w^0 = b$ is a constant map and z^0 is a conformal automorphism of \mathbb{D} satisfying $z^0(0) = a$. Put $t^0 = (t_2^0, \dots, t_n^0)$ with $t_j^0 = |b_j|^2$. Then $Z^0(b\mathbb{D}) \subset \Lambda^{t^0}$ and the disc Z^0 satisfies the Cauchy-Riemann equations (5.12) with $A = 0$.

(iv) Denote by X_0 a subset of X of maps satisfying the *normalization condition*

$$Z(0) = (z(0), w(0)) = (a, b), \quad z(1) = 1 \tag{5.14}$$

Consider the subset $Y \subset X_0 \times \mathcal{M} \times C^{k, \alpha}(\mathbb{D})$ which consists of all $(Z, t, A, h) \in X_0 \times \mathcal{M} \times C^{k+1, \alpha}(\mathbb{D})$ satisfying on \mathbb{D} the non-homogeneous Cauchy-Riemann equations

$$Z_{\bar{c}} - A(Z)\bar{Z}_{\bar{c}} - h = 0 \tag{5.15}$$

with the boundary conditions (5.13). Finally, denote by Y_0 a subset of Y formed by (Z, t, A, h) homotopic to $(Z^0, t^0, 0, 0)$ through Y . In what follows we assume that h belongs to an open neighborhood Ω of the origin in $C^{k, \alpha}(\mathbb{D})$ which can be shrunk and assumed to be small enough during the proof.

Recall that a bounded linear map $u : E \mapsto E'$ between two Banach spaces is called a *Fredholm operator* if $\ker u$ and $\operatorname{coker} u$ are finite dimensional. The number $\operatorname{ind}(u) = \dim \ker u - \dim \operatorname{coker} u$ is called *the index* of u . It is stable under small perturbations of u and a homotopy in the space of Fredholm operators.

A C^1 -map $F : M \rightarrow M'$ between two Banach manifolds is called a *Fredholm map* if for every point $q \in M$ the tangent map $dF_q : T_q M \rightarrow T_{F(q)} M'$ is Fredholm. The index of every tangent map is called the index of F ; it is denoted $ind(F)$.

A point $q \in M$ is called *regular* if the tangent map $dF(q)$ at this point is surjective. A point $q' \in M'$ is called a *regular value* of F if the preimage $F^{-1}(q')$ is empty or consists of regular points.

Consider now the natural projection

$$F : Y_0 \rightarrow \mathcal{M} \times \Omega$$

defined by

$$F : (Z, t, A, h) \mapsto (A, h)$$

The first technical step is

PROPOSITION 5.2. —

- (i) Y_0 is a Banach manifold.
- (ii) The projection $F : Y_0 \rightarrow \mathcal{M} \times \Omega$ is a Fredholm map with $ind(F) = 0$.

Since this statement is a variation of well-known results [1, 3, 13] which follow from the classical theory of linear integral equations [19, 26], we drop the proof. For reader's convenience, we include in Appendix a proof for the model case of the standard complex structure used in the next section. Here we only point out that the index is invariant with respect to a homotopy and it suffices to compute it for the above disc Z^0 . But it is easy to check that $ind(dF(Z^0)) = 0$.

The main step of the proof of Theorem is the following

PROPOSITION 5.3. — *There exists an open neighborhood of the origin Ω in $C^{k,\alpha}(\mathbb{D})$ such that the restriction $F : Y_0 \cap F^{-1}(\mathcal{M} \times \Omega) \rightarrow \mathcal{M} \times \Omega$ is a proper map.*

Admitting for a moment Proposition 5.3, we prove Theorem 5.1. The key ingredient is provided by the following general topological principle due to Smale [21].

PROPOSITION 5.4. — (*Sard-Smale's theorem.*) *Let $\tau : M_1 \rightarrow M_2$ be a proper Fredholm map between two Banach manifolds. Then the set of its*

regular values is dense in M_2 . For every regular value $p \in M_2$ the preimage $\tau^{-1}(p)$ is a manifold of dimension equal to the Fredholm index of $d\tau_q$ (or empty), $q \in \tau^{-1}(p)$. Furthermore, for any two regular values p_1 and p_2 the manifolds $\tau^{-1}(p_1)$ and $\tau^{-1}(p_2)$ are (non-orientedly) cobordant.

The cobordance here means that the union $\tau^{-1}(p_1) \cup \tau^{-1}(p_2)$ is the (non-oriented) boundary ∂N (here we prefer to use the homological notation) of a submanifold $N \subset M_1$.

Proof of Theorem 5.1. — The point $(0, 0) \in \mathcal{M} \times \Omega$ is a regular value of F and the preimage $F^{-1}(0, 0)$ consists of the single point $\{(Z^0, t^0, 0, 0)\}$. Let (A, h) be another regular value of F . Since the set $F^{-1}(0, 0)$ consists of a single point, it can not be cobordant to the empty set. Therefore the preimage $F^{-1}(A, h)$ is not empty. Hence the image of F contains a dense subset of $\mathcal{M} \times \Omega$. Since F is proper, we conclude that F is surjective. In particular $F^{-1}(A, 0)$ is not empty. \square

The remainder of the section is devoted to the proof of Proposition 5.3.

5.2. Structure lift and symplectic area

When A is prescribed, the solutions to the non-homogeneous equations (5.15) can be viewed as complex discs for a suitable structure $J_{A,h}$ determined by A and h (we will drop A and write just J_h when A is fixed). The structure J_h is defined on $\mathbb{D} \times \mathbb{C}^n \subset \mathbb{C}^{n+1} = \mathbb{C}_{z_0} \times \mathbb{C}_Z^n$. Setting $z_0(\zeta) = \zeta$, we see that (5.15) can be written in the form

$$\begin{cases} (z_0)_{\bar{\zeta}} = 0, \\ Z_{\bar{\zeta}} - A(Z)Z_{\bar{\zeta}} - h(z_0)(\bar{z}_0)_{\bar{\zeta}} = 0 \end{cases} \quad (5.16)$$

We view this PDE system as the Cauchy-Riemann equations for a J_h -complex disc $\hat{Z} : \zeta \mapsto (z_0(\zeta), Z(\zeta)) = (\zeta, Z(\zeta))$. This defines the complex matrix of some almost complex structure which we denote by J_h . Hence $Z = Z(\zeta)$ is a solution of (5.15) if and only if \hat{Z} is a J_h -complex disc.

If $Z \in Y$ then its lift \hat{Z} is glued to the torus $\hat{\Lambda}^t := b\mathbb{D} \times \Lambda^t$. The symplectic form ω lifts to \mathbb{C}^{n+1} as $\hat{\omega} = (1/2i)dz_0 \wedge d\bar{z}_0 + \omega$ i.e. as a standard symplectic form on \mathbb{C}^{n+1} . The torus $\hat{\Lambda}^t$ remains Lagrangian with respect to $\hat{\omega}$ and totally real with respect to J_h . We note (see below) that we will consider only h which are close to the zero-function in the $C^{k,\alpha}$ -norm, so the almost complex structure J_h remains tamed by $\hat{\omega}$.

We use the notation $z = x + iy$ and $w_j = u_j + iv_j$. Set $\lambda_1 = (1/2)(xdy - ydx)$ and $\lambda_j = (1/2)(u_j dv_j - v_j du_j)$. Finally put $\lambda = \sum_{j=1}^n \lambda_j$. Then $\omega = d\lambda$.

Let $Z \in Y_0$ be a disc homotopic to Z^0 . By Stokes' formula

$$\text{area}(Z) = \int_{b\mathbb{D}} Z^* \lambda = \pi$$

because the winding numbers of the complex functions z and w_j , $j = 2, \dots, n$ are equal to 1 and 0 respectively. In particular, we have the following

PROPOSITION 5.5. — *The ω -area of every disc $Z \in Y_0$ is equal to π .*

The area of every lift \hat{Z} is equal to 2π since an additional integral of the z_0 -component arises. We obtain the following

PROPOSITION 5.6. — *Let $(Z, A, h) \in Y_0$. Then the $\hat{\omega}$ area of \hat{Z} is equal to 2π .*

The first consequence is

LEMMA 5.7. — *Let (A^m, h^m) be a sequence converging in $\mathcal{M} \times \Omega$ and let (Z^m, t^m, A^m, h^m) be in $Y_0 \cap F^{-1}(A^m, h^m)$. The sequence (t^m) is bounded.*

Proof. — Suppose by absurd that after extracting a subsequence we have $t_m \rightarrow \infty$. Then the lifts \hat{Z}^m have a bounded area and unbounded diameters because $\hat{Z}^m(0) = (0, a, b)$; by the diameter we mean here the maximum on the disc of the distance to the boundary of this disc. But this is impossible, since the taming assumption (i) implies an upper bound on the diameter of every disc in terms of its area, see [3] \square

5.3. Separation

The key statement is

PROPOSITION 5.8. — *Fix $\eta > 0$. There exists $\varepsilon_0 > 0$ and a neighborhood Ω of the origin in $C^{k,\alpha}(\mathbb{D})$ such that for all $(Z, t, A, h) \in Y_0 \cap F^{-1}(\mathcal{M} \times \Omega)$ with $t_j \geq \eta$, $j = 2, \dots, n$ the following estimate holds*

$$|w_j(\zeta)| \geq \varepsilon_0, j = 2, \dots, n$$

for every $\zeta \in \overline{\mathbb{D}}$.

Proof. — The assertion holds for $A = 0$ and $h = 0$. Using the homotopy assumption in the definition of Y_0 and arguing by absurd, assume that there exists a sequence $(Z^m, t^m, A^m, h^m) \in Y_0 \cap F^{-1}(\mathcal{M} \times \Omega)$ such that $t_j \geq \eta$, $j = 2, \dots, n$, $A^m \rightarrow A^\infty$ (recall that the loop \mathcal{M} is compact) and $h^m \rightarrow 0$, but

for some j one has $0 < \inf_{\mathbb{D}} |w_j^m|$ for all m and $\inf_{\mathbb{D}} |w_j^m| \rightarrow 0$ as $m \rightarrow \infty$. Recall that the standard symplectic form ω is exact and spherical bubbles can not arise. By Gromov's compactness theorem (extracting a subsequence) the sequence of lifts $\hat{Z}^m(\mathbb{D})$ converges in the Hausdorff distance to a finite union of $J_{A^\infty, 0}$ -complex discs of class $C^{k, \alpha}(\bar{\mathbb{D}})$ with boundaries glued to the torus Λ^{t^∞} , $t^\infty = \lim_{m \rightarrow \infty} t^m$. Given the limit disc, after a suitable reparametrization by a sequence of conformal automorphisms, the convergence is in every $C^l(Q)$ -norm on every compact subset Q of $\bar{\mathbb{D}} \setminus \Sigma$, where Σ is at most a finite subset of $b\mathbb{D}$. Then the projection of one of the limit discs on $\mathbb{C}^n(z, w)$, say, Z , touches the J_{st} -complex hyperplane $P = \{(z, w) : w_j = 0\}$ at some point q .

Case (A). q is a boundary point of Z . Then $t_j^\infty = 0$. Let \hat{Z} be the ‘‘principal’’ limit disc i.e. the sequence \hat{Z}^m converges to this disc without additional reparametrization off a finite set. Outside this finite set, the boundary of the limit disc Z^∞ (the projection of \hat{Z} to \mathbb{C}^n) coincides with the circle $\{|z| = 1\} \times \{0\}$. Since the almost complex structure is standard near P , by the boundary uniqueness theorem for holomorphic functions, an open subset of Z^∞ is contained in the $J_{A^\infty, 0}$ -complex hyperplane P . Then this inclusion holds globally: a contradiction to the normalization condition (5.14).

Case (B). q is an interior point. Then we can assume by Case (A) that $t_j^\infty > 0$. Consider on $\mathbb{D}_{z_0} \times \mathbb{C}_{w_j} \subset \mathbb{C}^2$ the equations

$$\begin{cases} (z_0)_{\bar{z}} = 0, \\ (w_j)_{\bar{z}} - h^m(z_0)(\bar{z}_0)_{\bar{z}} = 0, \end{cases} \tag{5.17}$$

This is the Cauchy-Riemann equations (2.2) associated to an almost complex structure J^m on $\mathbb{D} \times \mathbb{C}$. The complex matrix of J^m is

$$\begin{pmatrix} 0 & 0 \\ h^m & 0 \end{pmatrix} \tag{5.18}$$

The sequence \hat{Z}^m converges to \hat{Z} , $q = Z(\zeta_0)$, $\hat{q} = (\zeta_0, q) = \hat{Z}(\zeta_0)$. Since $A^\infty = 0$ in a neighborhood of the hyperplane P , the projections $\hat{Z}_j^m := (z_0^m, w_j^m)$ satisfy the equations (5.17) near the point $(\zeta_0, 0) \in \mathbb{C}^2$, i.e. they are J^m -complex curves there. Fix m big enough. By the Nijenhuis-Woolf theorem a fixed neighborhood of the disc $\mathbb{D} \times \{0\}$ in \mathbb{C}^2 is foliated by a complex 1-parameter family of J^m -complex discs (small deformation of the family $w_j = \text{const}$ converging to this family when $m \rightarrow \infty$). Then \hat{Z}_j^m touches one of these discs: a contradiction to the positivity of intersections. \square

5.4. Proof of Proposition 5.3

Now we proceed quite similarly to the case of dimension 2. For reader's convenience we include details.

Arguing by absurd, assume that there exists a sequence Ω_j of open neighborhoods of the origin converging to the origin, such that for every j the map $F : Y_0 \cap F^{-1}(\mathcal{M} \times \Omega_j) \rightarrow \mathcal{M} \times \Omega_j$ is not proper. Then for every j , there exists a sequence $(A^{m,j}, h^{m,j})_m$ converging in $\mathcal{M} \times \Omega_j$ to some $(A^{\infty,j}, h^{\infty,j})$ as $m \rightarrow \infty$ and sequence $(Z^{m,j}, t^{m,j}, A^{m,j}, h^{m,j})_m$ in $Y_0 \cap F^{-1}(A^{m,j}, h^{m,j})$ which does not admit a converging subsequence. Therefore, by Gromov's compactness theorem [13], a boundary disc-bubble arises in every sequence $(\hat{Z}^{m,j})_m$ (recall that there are no spherical bubbles since ω is exact). One can assume that for every j the sequence $(\hat{Z}^{m,j}(\mathbb{D}))_m$ converges in the Hausdorff distance to a connected finite union of $J_{A^{\infty,j}, h^{\infty,j}}$ -complex discs with boundaries glued to the torus $\hat{\Lambda}^{t^{\infty,j}}$ where $t^{\infty,j}$ is the limit of $(t^{m,j})$.

Choose some j . We have the "principal" limit disc $\hat{Z}^{\infty,j}$ which is the limit of the sequence of maps $(\hat{Z}^{m,j})_m$ converging as $m \rightarrow \infty$ on \mathbb{D} off at most a finite number of boundary points where disc-bubbles arise. The disc $Z^{\infty,j}$ is centered at (a, b) , its boundary is attached to the torus $\Lambda^{t^{\infty,j}}$. Since the w -components of all discs of our sequence are uniformly separated from the origin by Proposition 5.8, the limit disc has the same property. Furthermore, its z_0 -component remains equal to ζ . Its z -component is a smooth function on \mathbb{D} , $z : b\mathbb{D} \rightarrow b\mathbb{D}$ and $z(0) = a$.

LEMMA 5.9. — *The winding number of the z -component of $\hat{Z}^{\infty,j}$ does not vanish for all j big enough.*

Proof. — Suppose by absurd that it does. Then the area of every disc $\hat{Z}^{\infty,j}$ is equal to π and we apply Gromov's compactness theorem to the sequence $(\hat{Z}^{\infty,j})_j$. Since \mathcal{M} is compact, one can assume that $A^{\infty,j}$ converges to A^∞ ; we also can assume that $(h^{\infty,j})_j$ converges to 0 and $(t^{\infty,j})_j$ converges to t^∞ . Again we consider the "principal" limit disc $\hat{Z}^\infty(\zeta) = (\zeta, Z^\infty(\zeta))$ being the limit of the sequence of maps $(\hat{Z}^{\infty,j})_j$ off at most a finite subset in the boundary. Then $area(\hat{Z}^\infty) = \pi$. By the compactness theorem the sum of this area and the areas of eventually arising bubbles is equal to π and every bubble has a non-zero area. Hence the bubbles do not arise. Therefore, passing to a subsequence, we have a convergence in the $C^{k+1,\alpha}$ norm on the closed disc. Then the disc Z^∞ is J_{A^∞} -complex (because $h^\infty = 0$), its boundary is glued to Λ^{t^∞} and its area is equal to zero. So it is the

constant map equal to some point of Λ^{t^∞} . This is a contradiction, since its z -component still satisfies $z(0) = a$. \square

Going back to the disc $\hat{Z}^{\infty,j}$, we conclude that the winding number of its z -component is strictly positive. Indeed, it can not be negative since the symplectic area of the non-constant $J_{A^{\infty,j}, h^{\infty,j}}$ -complex disc $\hat{Z}^{\infty,j}$ must be positive. This implies that $area(\hat{Z}^{\infty,j}) = 2\pi$. On the other hand, once again by Gromov's compactness theorem the sum of $area(\hat{Z}^{\infty,j})$ and the areas of all bubbles is equal to $area(\hat{Z}^{m,j}) = 2\pi$. We see that the area of every bubble which could arise in the sequence $(\hat{Z}^{m,j})_m$ must be equal to 0 implying that this bubble is constant. This is a contradiction since bubbles can not be constant. \square

6. Gluing a complex disc to a Lagrangian submanifold of \mathbb{C}^n

An adaptation of the method employed in previous section allows to prove the following result [13]:

THEOREM 6.1. — *Let E be a smooth compact Lagrangian submanifold in $(\mathbb{C}^n, \omega_{st})$. Then there exists a non-constant J_{st} -holomorphic disc attached to E .*

Our exposition here is inspired by [1, 11]. As above the standard symplectic form ω_{st} is denoted simply by ω . In the following we only slightly modify the notation of previous section.

6.1. Adapted manifolds of discs

Fix a point $a \in E$ and consider the constant holomorphic map $f^0(\zeta) \equiv a$. Fix $k \geq 1$ and $0 < \alpha < 1$. Consider the set $X \subset C^{k,\alpha}(\mathbb{D})$ consisting of maps $f : \mathbb{D} \rightarrow \mathbb{C}^n$ with $f(b\mathbb{D}) \subset E$ and $f(1) = a$. Assume in addition that $E = \{\rho = 0\}$ where $\rho = (\rho_1, \dots, \rho_n) : \mathbb{C}^n \rightarrow \mathbb{R}^n$ is a smooth map of maximal rank. Furthermore

$$\partial\rho_1 \wedge \dots \wedge \partial\rho_n \neq 0 \tag{6.19}$$

because E is totally real. Then a holomorphic map $f \in C^{k,\alpha}(\mathbb{D})$ is in X if and only if it is a solution of the following non-linear Riemann-Hilbert type boundary value problem:

$$(RH) : \begin{cases} \frac{\partial f(\zeta)}{\partial \bar{\zeta}} = 0, & \zeta \in \mathbb{D} \\ \rho(f)(\zeta) = 0, & \zeta \in b\mathbb{D} \\ f(1) = a \end{cases}$$

Denote by X_0 the subset of X formed by the discs homotopic in X to the constant disc f^0 . Set $G = C^{k,\alpha}(\mathbb{D})$. Consider the cartesian product $X_0 \times G$ and define a subset

$$Y \subset X_0 \times G = \{(f, h) : \frac{\partial f}{\partial \bar{\zeta}} = h\}$$

Denote by $F : X_0 \times G \rightarrow G$ the natural projection.

As in the previous section we have

PROPOSITION 6.2. —

(i) Y is a Banach manifold.

(ii) The projection $F : Y \rightarrow G$ is a Fredholm map with $\text{ind}(F) = 0$.

Proof. — Since X_0 consists of discs homotopic to the constant map $f^0 \equiv a \in E$, the Fredholm index of $dF_{(f,h)}$ is independent of (f, h) and coincides with the index of $dF_{(f^0,0)}$ which is equal to 0. \square

A crucial property of F is given by the following

LEMMA 6.3. — *The projection $F : Y \rightarrow G$ is not surjective.*

Proof. — Suppose by contradiction that F is surjective. Then for every $t > 0$ there exists $f^t \in X_0$ such that $(f^t, h^t) \in Y$ where $h^t(\zeta) := (t, 0, \dots, 0)$. On the other hand, $\partial f^t / \partial \bar{\zeta} = h^t$ and, in particular $\partial f_1^t / \partial \bar{\zeta} = t$. Hence $f_1^t = t\bar{\zeta} + q^t(\zeta)$ where q^t is a function holomorphic in \mathbb{D} . Since $f^t(b\mathbb{D}) \subset E$, the family (f^t) is bounded on $b\mathbb{D}$ by a constant $C > 0$ independent of t . Therefore $|\bar{\zeta} + t^{-1}q^t(\zeta)| \leq t^{-1}C$ for $\zeta \in b\mathbb{D}$. However the function $\zeta \mapsto \bar{\zeta} + t^{-1}q^t(\zeta)$ is harmonic in \mathbb{D} and by the maximum principle a similar estimate holds for all $\zeta \in \mathbb{D}$. Letting $t \rightarrow \infty$, we obtain that the function $\zeta \mapsto \bar{\zeta}$ can be uniformly approximated by holomorphic functions in \mathbb{D} : a contradiction. \square

6.2. Non-linear Fredholm alternative

In order to conclude the proof of Theorem 6.1, it suffices to establish the following

PROPOSITION 6.4. — *Suppose that there does not exist a non-constant Bishop disc for E . Then the projection $F : Y \rightarrow G$ is surjective.*

Then the theorem follows by contradiction. The remainder is devoted to the proof of Proposition.

We deal with the non-homogeneous Cauchy-Riemann equations

$$\frac{\partial f}{\partial \bar{\zeta}}(\zeta) = h \tag{6.20}$$

on the unit disc. In order to use Gromov's compactness theorem, we again view its solutions as J_h -complex discs for a suitably chosen almost complex structure J_h in \mathbb{C}^{n+1} . Quite similarly to previous section (cf. with (5.17)), given h consider the almost complex structure J_h on $\mathbb{D} \times \mathbb{C}^n \subset \mathbb{C}^{n+1}$. The equations (6.20) are equivalent to the fact that the lift $\hat{f} : \zeta \mapsto (\zeta, f(\zeta))$ of f is a J_h -complex disc.

LEMMA 6.5. — *The projection $F : Y \rightarrow G$ is proper.*

Proof. — We must show that for every sequence $h^k \rightarrow h^\infty$ in G and every sequence (f^k) such that $(f^k, h^k) \in Y$ there exists a subsequence of (f^k) converging in X_0 . The structures J_{h^k} converge to J_h and the discs $\hat{f}^k(\zeta) = (\zeta, f^k(\zeta))$ are J_{h^k} -complex. Their boundaries are attached to the manifold $\hat{E} := b\mathbb{D} \times E$. Since E is a Lagrangian manifold, it follows that \hat{E} is a Lagrangian manifold in $\mathbb{C} \times \mathbb{C}^n$ with respect to the symplectic form $\hat{\omega} = C \frac{i}{2} dz_0 \wedge d\bar{z}_0 + \omega$. The sequence (h^k) is bounded, which implies that there exists a constant $C > 0$ such that the structures J_{h^k} is tamed by $\hat{\omega}$ for all $k = 0, 1, \dots, \infty$.

Denote by λ a primitive of $\hat{\omega}$. Then

$$\int_{\hat{f}^k(\mathbb{D})} \hat{\omega} = \int_{\hat{f}^k(b\mathbb{D})} \lambda$$

On the other hand the boundaries $\hat{f}^k(b\mathbb{D})$ of our discs are homotopic (the discs are homotopic) and $d\lambda|_{\hat{E}} = \hat{\omega}|_{\hat{E}} = 0$. Then by Stokes' theorem the last integral is independent of k . Gromov's compactness theorem implies that there are only the following possibilities:

(a) The limit of some subsequence of (\hat{f}^k) contains a disc-bubble ψ . Since \hat{f}^k are the graphs over \mathbb{D} , it follows easily from the definition of disc-bubbles that ψ is "vertical", i.e. has the form $\psi : \zeta \mapsto (q, f(\zeta))$ where q is a point of the unit circle $b\mathbb{D}$. Indeed, one readily sees from the definition of a disc-bubble that the renormalizing sequence ϕ^n of conformal biholomorphisms is not compact (in the compact-open topology) and converges to a boundary point of the unit disc. Then it follows from the Cauchy-Riemann equations associated with J_{h^∞} that f is a usual holomorphic (with respect to J_{st}) disc attached to E (cf. with the equations (5.17) of previous section). The disc f is non-constant because the bubble ψ is non-constant. This contradicts the assumption of Proposition 6.4.

(b) The limit of some subsequence of (\hat{f}^k) contains a non-constant J_{h^∞} -complex sphere. This is impossible since $\hat{\omega}$ is exact.

Thus, only the last possibility realizes:

(c) there exists a subsequence converging in $C^{k+1,\alpha}(\mathbb{D})$ -norm. \square

Now using the Sard-Smale theorem we conclude as in previous section that F is surjective. This proves Proposition 6.4 and Theorem 6.1. \square

6.3. Exotic symplectic structures

As a consequence we obtain the existence of exotic symplectic structures on \mathbb{R}^{2n} . Denote by ω_{st} the standard symplectic form of \mathbb{C}^n , identifying \mathbb{C}^n with \mathbb{R}^{2n} . Set $\lambda_{st} = \sum_j x_j dy_j$ so that $d\lambda_{st} = \omega_{st}$.

A symplectic structure ω on \mathbb{R}^{2n} is called exotic if there is no global diffeomorphic map $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that $\phi^*\omega_{st} = \omega$.

COROLLARY 6.6. — *Let E be a compact Lagrangian submanifold in $(\mathbb{C}^n, \omega_{st})$. Then the restriction $\lambda_{st}|_E$ represents a non-zero class in $H^1(E, \mathbb{R})$.*

Proof. — Since $d\lambda_{st}|_E = 0$, then $\lambda_{st}|_E$ does represent a class in $H^1(E, \mathbb{R})$. If this class is zero, then for every closed smooth curve $\gamma \subset E$ we have $\int_\gamma \lambda_{st} = 0$.

Let f be a non-constant Bishop disc glued to E and let $\gamma = f(b\mathbb{D})$. Then by Stokes' formula

$$\int_{b\mathbb{D}} f^* \lambda_{st} = \int_{\mathbb{D}} f^* \omega_{st} = \text{area}[f(\mathbb{D})] > 0$$

which proves the corollary. \square

How to find an exotic structure? Consider the standard torus $\Lambda = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_j| = 1\}$ which is Lagrangian for ω_{st} in \mathbb{C}^2 .

LEMMA 6.7. — *Suppose that ω is a symplectic form such that for some 1-form λ we have $\omega = d\lambda$ on \mathbb{R}^4 and $\lambda|_\Lambda = 0$. Then ω is exotic.*

Proof. — Suppose that ϕ is a diffeomorphism of \mathbb{R}^4 satisfying $\omega = \phi^*\omega_{st}$. Then $\phi^*\lambda_{st} - \lambda$ is a closed 1-form and there exists a function h on \mathbb{R}^4 such that $dh = \phi^*\lambda_{st} - \lambda$. Then

$$\phi^*\lambda_{st}|_\Lambda = (\phi^*\lambda_{st} - \lambda)|_\Lambda = d(h|_\Lambda)$$

so that $\phi^* \lambda_{st}|_\Lambda$ is exact. Therefore $\lambda_{st}|_{\phi(\Lambda)}$ is exact and $\phi(\Lambda)$ is Lagrangian for ω_{st} : a contradiction. \square

It turns out that it is not difficult to write explicitly a symplectic structure satisfying the assumptions of the above Lemma, see [16].

7. Appendix: Fredholm property

Using notations of previous section, we prove here that $F : Y \rightarrow G$ is a Fredholm map. Let $(f_0, h_0) \in Y$. We follow [1]. The tangent space $T_{(f_0, h_0)}Y$ to Y at (f_0, h_0) is formed by the maps $(f, \dot{h}) \in C^{k+1, \alpha}(\mathbb{D}) \times C^{k, \alpha}(\mathbb{D})$ satisfying

$$\begin{cases} \frac{\partial \dot{f}}{\partial \bar{\zeta}} = 0, \zeta \in \mathbb{D}, \\ 2\operatorname{Re}P(\zeta)\dot{f}(\zeta) = 0, \zeta \in b\mathbb{D}, \\ \dot{f}(1) = 0 \end{cases}$$

Here $P(\zeta)$ is the Jacobian matrix

$$\left(\frac{\partial \rho}{\partial Z}(f_0(\zeta)) \right)$$

Since \dot{h} is arbitrary, we identify $T_{(f_0, h_0)}H$ with the space of maps $\dot{f} : \mathbb{D} \rightarrow \mathbb{C}^n$ satisfying $\dot{f}(\zeta) \in T_{f_0(\zeta)}(E)$ for $\zeta \in b\mathbb{D}$ and $\dot{f}(1) = 0$. We identify G with the tangent space $T_{h_0}G$. Then the tangent map $dF : T_{(f_0, h_0)}H \rightarrow G$ to F at (f_0, h_0) is

$$dF : \dot{f} \mapsto \dot{h} = \frac{\partial \dot{f}}{\partial \bar{\zeta}}$$

The condition that E is totally real is equivalent to the fact that $\det P(\zeta) \neq 0$, $\zeta \in b\mathbb{D}$ (“the Lopatinski condition”). Set $A = C^{k+1, \alpha}(\mathbb{D}, \mathbb{C}^n)$ and $B = C^{k, \alpha}(\mathbb{D}, \mathbb{C}^n) \times C^{k+1, \alpha}(b\mathbb{D}, \mathbb{C}^n)$. Consider the linear operator

$$L : A \rightarrow B,$$

$$L : \dot{f} \mapsto \left(\frac{\partial \dot{f}}{\partial \bar{\zeta}}, \operatorname{Re}P\dot{f}|_{b\mathbb{D}} \right)$$

According to [26], Th. 3.2.5., the operator L is Fredholm. Define the operator $L_1 : A \rightarrow B \times \mathbb{C}$ by $L_1(\dot{f}) = (L(\dot{f}), \dot{f}(1))$. Obviously L_1 is Fredholm. Let $I : T_{(f_0, h_0)}Y \rightarrow A$ be the inclusion map. Consider the map $I' : G \rightarrow B \times \mathbb{C}$ defined by $I' : \dot{h} \mapsto (\dot{h}, 0, 0)$. Then we have the commutative diagramm

$$I' \circ dF = L_1 \circ I$$

In particular, $I(\ker(dF)) \subset \ker L_1$ and so $\ker dF$ has a finite dimension. Furthermore, $I'(im(dF)) \subset im L_1$ and so the induced quotient map $\hat{I}' : G/im(dF) \rightarrow (B \times \mathbb{C})/im L_1$ is correctly defined. Obviously it is injective. Hence $\dim coker dF \leq \dim coker L_1 < +\infty$. This means that dF is a Fredholm map.

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