ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

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Tome XXIII, nº 1 (2014), p. 95-114.

<http://afst.cedram.org/item?id=AFST_2014_6_23_1_95_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXIII, n° 1, 2014
pp. 95-114

Polyhedral Realization of a Thurston Compactification

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Abstract. — Let Σ_3^- be the connected sum of three real projective planes. We realize the Thurston compactification of the Teichmüller space $\mathsf{Teich}(\Sigma_3^-)$ as a simplex in $\mathbf{P}(\mathbf{R}^4)$.

Résumé. — Soit Σ_3^- la somme connexe de trois plans projectifs réels. Nous réalisons la compactification de Thurston de l'espace de Teichmüller $\mathsf{Teich}(\Sigma_3^-)$ comme un simplexe de $\mathbf{P}(\mathbf{R}^4)$.

1. Introduction

In order to classify diffeomorphisms of a given compact surface Σ, W. Thurston built a compactification of the Teichmüller space $\mathsf{Teich}(\Sigma)$ consisting of a closed ball lying in an infinite dimensional projective space.

(*) Reçu le 03/03/2011, accepté le 23/04/2013

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The first author was fully supported by the Fonds National Suisse de la Recherche Scientifique (SNF projects no. 200020-121506/1 and no. 200021-131967/1). The second author was partially supported by Grant-in-Aid for Scientific Research (19540194), Ministry of Education, Science and Culture of Japan.

Article proposé par Jean-Pierre Otal.

This *Thurston compactification* is simply defined as the closure of the image of the *geodesic-length functions embedding*

$$\begin{array}{rcl} \mathsf{Teich}(\Sigma) & \longrightarrow & \mathbf{P}(\mathbf{R}^{\mathcal{S}}) \\ X & \longmapsto & (\ell_s(X))_{s\in\mathcal{S}}, \end{array}$$

where $\mathcal{S}$ denotes the set of isotopy classes of nontrivial simple closed curves on Σ. Although the Thurston compactification is not a polytope, its boundary carries a piecewise integral projective structure, whose interest stands in the identification of the Thurston boundary with the projective space of measured foliations. One would simplify the complicated combinatorial structure of the Thurston boundary and, at the same time, preserve its piecewise integral projective structure. In this direction, we address the problem of realizing the Thurston compactification as a finite convex integral polytope of a projective space of dimension $\dim \mathsf{Teich}(\Sigma)$.

The first idea is to look at projections $\pi_F : \mathbf{P}(\mathbf{R}^{\mathcal{S}}) \to \mathbf{P}(\mathbf{R}^{F})$ where F is a subset of $\mathcal{S}$ of cardinal $\dim \mathsf{Teich}(\Sigma) + 1$. It is in general very hard to decide whether such a projection defines an embedding of the Thurston compactification. Nevertheless, we know that for some F the projection π_F gives an embedding of the interior of the Thurston compactification (P. Schmutz [7, 6]), and for some other F an embedding of the Thurston boundary (U. Hamenstädt [4]).

In this note we study the particular case of Σ_3^-, the connected sum of three real projective planes. Its Teichmüller space $\mathsf{Teich}(\Sigma_3^-)$ is of dimension three and quite easy to handle. We present an explicit embedding of its Thurston compactification into $\mathbf{P}(\mathbf{R}^4)$ and describe its image. More precisely, if $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ in $\mathcal{S}$ satisfy a precise topological configuration defined in §3, then

Theorem. — *The restriction of the following map induces an embedding of the Thurston compactification of* $\mathsf{Teich}(\Sigma_3^-)$ *into* $\mathbf{P}(\mathbf{R}^4)$ *whose image is a projective simplex,*

$$\begin{array}{rcl} L_2 : \mathbf{P}(\mathbf{R}^{\mathcal{S}}) & \longrightarrow & \mathbf{P}(\mathbf{R}^4) \\ (x_s)_{s\in\mathcal{S}} & \longmapsto & (x_\alpha : x_\beta : x_\gamma : x_{\alpha'} + x_{\beta'} + x_{\gamma'}). \end{array}$$

To prove that L_2 is projectively injective on the Teichmüller space, we introduce the *similar triangle flow* on $\mathsf{Teich}(\Sigma_3^-)$. A trajectory of this flow preserves the ratios between the length functions ℓ_α, ℓ_β and ℓ_γ. Accurate estimates on the derivatives of $\ell_{\alpha'}$, $\ell_{\beta'}$ and $\ell_{\gamma'}$ along these trajectories enable us to conclude that L_2 is injective.

We also consider a projection $L_1 : (x_s)_{s\in\mathcal{S}} \mapsto (x_\alpha : x_\beta : x_\gamma : x_\sigma)$ whose restriction does not define an embedding of the Thurston compactification of $\mathsf{Teich}(\Sigma_3^-)$. We show that in fact L_1 gives an embedding of another compactification, which is an interesting mixture of the Thurston compactification and the Teichmüller space of hyperbolic structures on Σ_3^- pinched at σ (§6).

The text is organized as follows: in §2 we recall some basic facts about Teichmüller spaces and their Thurston compactification, in §3 we describe the topology and geometry of Σ_3^-, in §4 we introduce the similar triangle flow and study the behaviour of some length functions along its trajectories, in §5 we show that the images of the embeddings L_1 and L_2 are interiors of projective polyhedra, and finaly in §6 we interpret these projective polyhedra as compactifications of $\mathsf{Teich}(\Sigma_3^-)$.

Acknowledgement. — Both authors are grateful to Prof. Ruth Kellerhals who made possible this collaboration with the support of SNF projects no. 200020-121506/1 and no. 200021-131967/1. The second author wishes to thank her for her hospitality during his stay at the university of Fribourg. We also thank the referee for his remarks and suggestions.

2. Preliminaries

In this section, we describe the Thurston compactification of the Teichmüller space through measured foliations. The reader can consult the classical references [8] and [2] for more details.

In what follows, Σ is a compact connected surface of negative Euler-Poincaré characteristic. We denote by $\mathcal{S}$ the set of isotopy classes of nontrivial simple closed curves, that do not deform into a boundary component. A simple closed curve is *nontrivial* if it does not bound a disk or a Mœbius strip.

2.1. Teichmüller spaces

The *Teichmüller space* $\mathsf{Teich}(\Sigma)$ is the space of isotopy classes of hyperbolic metrics on Σ. If Σ has boundary, we assume that the lengths of the boundary components are fixed. It is well known that $\mathsf{Teich}(\Sigma)$ is a smooth manifold diffeomorphic to an open ball of dimension $-3\chi(\Sigma) - n$, where n is the number of boundary components.

Given a hyperbolic metric on Σ, each isotopy class s in $\mathcal{S}$ has a unique geodesic representative. The map that associates to a hyperbolic metric X the collection of lengths $(\ell_s(X))_{s\in\mathcal{S}}$ defines differential embeddings of $\mathsf{Teich}(\Sigma)$ into $\mathbf{R}^{\mathcal{S}}$ and $\mathbf{P}(\mathbf{R}^{\mathcal{S}})$.

The *modular group* $\mathsf{Mod}(\Sigma)$ is the group of isotopy classes of diffeomorphisms. If Σ has boundary, we only consider diffeomorphisms that stabilize each boundary component. The modular group acts properly and discontinuoulsy on $\mathsf{Teich}(\Sigma)$.

2.2. Measured foliations

A *measured foliation* is a foliation F equipped with a transverse measure μ. We assume that F and μ satisfy some specific properties. The foliation F has only a finite number of singular points, and the singularities are p-pronged saddles ($p \geqslant 3$). Unless otherwise is stated, each possible boundary component consists of a cycle of leaves with at least one singularity. The transverse measure μ is invariant under leaf-preserving homotopy, and regular with respect to the Lebesgue measure.

We denote by $\mathsf{MF}(\Sigma)$ the set of measured foliations up to Whitehead equivalence. To a measured foliation F corresponds the collection of its intersection numbers $(\mathsf{i}_s(F))_{s\in\mathcal{S}}$, where $\mathsf{i}_s(F)$ is the minimal transverse measure of a representative of s. This correspondence defines an injective map from $\mathsf{MF}(\Sigma)$ to $\mathbf{R}^{\mathcal{S}}$. We endow $\mathsf{MF}(\Sigma)$ with the topology induced by $\mathbf{R}^{\mathcal{S}}$.

2.3. Thurston compactification

The Thurston compactification consists of the union $\mathsf{Teich}(\Sigma)\cup\mathsf{PMF}(\Sigma)$, where $\mathsf{PMF}(\Sigma)$ is the projectivised space of measured foliations. This compactification is realized in the projective space $\mathbf{P}(\mathbf{R}^{\mathcal{S}})$.

The Thurston compactification is similar in many ways to the compactification of the hyperbolic space. In the affine space $\mathbf{R}^{\mathcal{S}}$, the Teichmüller space plays the role of the upper-sheet of the hyperboloid, and the space of measured foliations plays the role of the isotropic cone. The *Thurston boundary* $\partial\mathsf{Teich}(\Sigma)$ is the boundary of the Teichmüller space in $\mathbf{P}(\mathbf{R}^{\mathcal{S}})$. It is a topological sphere identified with $\mathsf{PMF}(\Sigma)$. The *Thurston compactification* is the closure of the Teichmüller space in $\mathbf{P}(\mathbf{R}^{\mathcal{S}})$, which turns out to be a topological closed ball. For a complete analogy between the two compactifications, we refer to the work of F. Bonahon ([1]).

2.4. Integral points on the Thurston boundary

The space $\mathsf{MF}(\Sigma)$ of measured foliations admits a piecewise integral linear structure, defined through train tracks. The set of integral points corresponds exactly to integral multi-curves.

As a consequence, the Thurston boundary possesses a piecewise integral projective structure. Although $\partial\mathsf{Teich}(\Sigma)$ has no canonical triangulation, the *curve complex* $\mathsf{C}(\Sigma)$ is a simplicial complex which injects canonically into $\partial\mathsf{Teich}(\Sigma)$. The set of vertices of $\mathsf{C}(\Sigma)$ is $\mathcal{S}$, and a collection $\{c_1, \ldots, c_n\} \subset \mathcal{S}$ defines an $(n-1)$-simplex if these curves have zero intersection numbers.

2.5. Notations and conventions

We denote by $\Sigma_{g,n}$ (resp. $\Sigma^-_{g,n}$) the orientable (resp. the non-orientable) compact surface of genus g with n boundary components. A *hyperbolic metric* is a metric of constant curvature -1, with geodesic boundary. Unless otherwise is stated, a *geodesic* means a nontrivial simple closed geodesic that is not a boundary component. With this convention, the set of geodesics is in bijection with $\mathcal{S}$.

Instead of *two-sided* (resp. *one-sided*), we prefer to say that a simple closed curve is *orientable* (resp. *non-orientable*) if it is transversely orientable (resp. if it is not transversely orientable). Two isotopic non-orientable curves always intersect, but their intersection number as measured foliations is zero, hence we set $\mathsf{i}(c, c) = 0$ for any element c of $\mathcal{S}$.

When we want to specify the lengths $b_1, \ldots, b_n$ of the boundary components, we will use the following notation: $\mathsf{Teich}_{b_1,\ldots,b_n}(\Sigma^{\pm}_{g,n})$.

3. Geometry and topology of the non-orientable surface of genus 3

In this part, we classify simple closed curves on Σ^-_3, and look at a configuration that gives a nice affine embedding of $\mathsf{Teich}(\Sigma^-_3)$. The last paragraphs are devoted to projective measured foliations of Σ^-_3.

3.1. Simple closed geodesics

Consider a hyperbolic metric X on the connected sum of three projective planes. The proposition below shows that the geometry of X is essentially the geometry of a one-holed torus $\mathsf{T_X}$. This proposition is due to M. Scharlemann ([5]).

Proposition 3.1. — *There is a unique simple closed geodesic σ in X that produces a one-holed torus $\mathsf{T_X}$ after cutting.*

As a consequence, there is a canonical bijection between the Teichmüller space $\mathsf{Teich}(\Sigma^-_3)$ and the union $\cup_{b>0}\mathsf{Teich}_b(\Sigma_{1,1})$. There is also a canonical isomorphism between the modular groups $\mathsf{Mod}(\Sigma^-_3)$ and $\mathsf{Mod}(\Sigma_{1,1})$.

Using the hyperelliptic involution, which is the reflection along σ, one can easily prove the following proposition (M. Gendulphe [3]):

PROPOSITION 3.2. — *Let γ be a simple closed geodesic of X distinct from σ.*

1. *If γ is orientable, then γ is disjoint from σ.*
2. *If γ is non-orientable, then γ intersects σ in exactly one point.*
3. *There exists a unique simple closed geodesic $\gamma' \neq \sigma$ disjoint from γ which has opposite orientability. We say that γ and γ' are* duals.

The duality defines an involution of $\mathcal{S}$ whose unique fixed point is σ. The lengths of an orientable simple closed geodesic γ and its dual γ' are related by

$$\cosh \frac{\ell(\gamma)}{2} \quad = \quad \sinh \frac{\ell(\gamma')}{2} \ \sinh \frac{\ell(\sigma)}{2}. \tag{3.1}$$

The corresponding identity on the intersection numbers is

$$\mathsf{i}(\gamma, \cdot) \quad = \quad \mathsf{i}(\gamma', \cdot) + \mathsf{i}(\sigma, \cdot) \text{ on } \mathcal{S} \setminus \{\gamma', \sigma\}. \tag{3.2}$$

3.2. Triangle embedding

DEFINITION 3.3. — *A* triangle *is a triple (α, β, γ) of orientable simple closed geodesics with all intersection numbers equal to one.*

Remark 3.4. — A triple (α, β, γ) is a triangle if and only if its dual triple $(\alpha', \beta', \gamma')$ consists of three disjoint simple closed geodesics. The complement in Σ_3^- of these dual curves is a pair of pants.

Any triangle satisfies the geometric inequality

$$\ell(\alpha) + \ell(\beta) + \ell(\gamma) \quad > \quad 2\ \ell(\sigma), \tag{3.3}$$

and also the following identity

$$\cosh^2 \frac{\ell(\sigma)}{2} = \left[\cosh \frac{\ell(\alpha) + \ell(\beta)}{2} - \cosh \frac{\ell(\gamma)}{2}\right] \left[\cosh \frac{\ell(\gamma)}{2} - \cosh \frac{\ell(\alpha) - \ell(\beta)}{2}\right] \tag{3.4}$$

which comes from hyperbolic trigonometry in a right-angled hexagon.

Proposition 3.5. — *Let (α,β,γ) be a triangle. The following map is an embedding*

$$\begin{array}{rcl} L : \mathsf{Teich}(\Sigma_3^-) & \longrightarrow & \mathbf{R}^3 \\ X & \longmapsto & (\ell_\alpha(X), \ell_\beta(X), \ell_\gamma(X)) \end{array}$$

and its image is the following unbounded domain

$$\Delta = \Big\{ (a,b,c) \in \mathbf{R}^3 \quad | \quad b+c > a,\ c+a > b,\ a+b > c \text{ and } \left[\cosh\frac{a+b}{2} - \cosh\frac{c}{2}\right]\left[\cosh\frac{c}{2} - \cosh\frac{a-b}{2}\right] > 1 \Big\}.$$

It is well known that L is smooth and injective (see P. Schmutz [6]), one can easily show that L is in fact a differential embedding with image Δ.

Remarks 3.6. —

- *The triangle inequalities imply that a, b and c are positive.*
- *The last inequality can be replaced by any symmetric one in a, b, c.*
- *Δ is invariant under multiplication by a scalar $t \geqslant 1$.*

3.3. Scharlemann's description of the curve complex

M. Scharlemann gave in [5] a nice description of the inclusion $\mathsf{C}(\Sigma_3^-) \subset \mathsf{PMF}(\Sigma_3^-)$, based on the classical relation between $\mathsf{C}(\Sigma_{1,1})$ and the Farey tesselation.

After a choice of a symplectic basis of the homology, simple closed geodesics in the one-holed torus correspond bijectively to pairs $\pm(p,q)$ of relatively prime integers, and so to rational numbers p/q in $\mathbf{R} \cup \{\infty\}$. The intersection number between two pairs (p,q) and (r,s) is equal to the absolute value $|ps-rq|$ of the intersection form. Thus three geodesics correspond to the vertices of a triangle of the Farey tesselation if and only if they form a triangle in the one-holed torus.

The topological circle $\partial\mathsf{Teich}(\Sigma_{1,1})$ embeds piecewise linearly and canonically into the topological sphere $\partial\mathsf{Teich}(\Sigma_3^-)$, and divides it into two open hemispheres. One hemisphere contains the vertex of $\mathsf{C}(\Sigma_3^-)$ corresponding to σ. This vertex is related by an edge to each vertex of $\mathsf{C}(\Sigma_3^-)$ contained in $\partial\mathsf{Teich}(\Sigma_{1,1})$. The other hemisphere contains all vertices corresponding to non-orientable curves distinct from σ. The configuration of the curve complex in this hemisphere is a slightly modified version of the Farey tesselation. Let us look at the hemisphere as the unit disk $\mathbf{D}$ in the complex plane. If the point $z(p,q) \in \partial\mathbf{D}$ represents the orientable geodesic with coordinates (p,q), then the point $z'(p,q) = z(p,q)/(1+1/q)$ represents its dual geodesic. The segment $[z(p,q), z'(p,q)]$ is an edge of the curve complex, the other edges are between points $z'(p,q)$ and $z'(r,s)$ with $|ps - rq| = 1$. See Figure 1 for a picture where (α, β, γ) is a triangle.

More generally, we know that non-orientable geodesics are isolated in $\mathsf{PMF}(\Sigma)$, but any orientable geodesic is a limit of non-orientable geodesics in $\mathsf{PMF}(\Sigma)$. This is true for any compact non-orientable surface Σ with $\chi(\Sigma) < 0$. These results were proved by M. Scharlemann in [5].

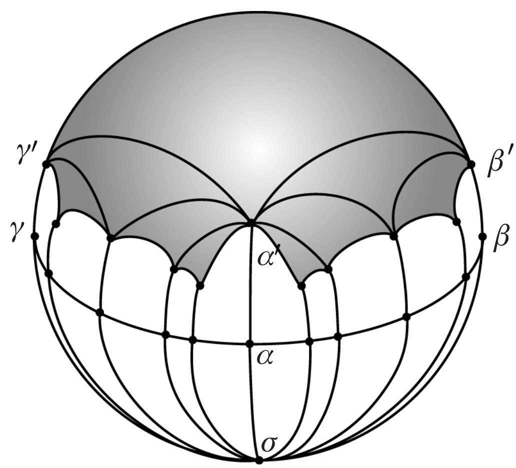


Figure 1. — Scharlemann's picture

3.4. A triangulation of $\mathsf{PMF}(\Sigma_3^-)$

Up to isotopy and multiplication by a positive constant, there are only two measured foliations of the Mœbius band: foliations with closed leaves, and foliations with leaves transverse to the boundary. Consider α', β' and γ' three nontrivial simple closed curves that bound a pair of pants in Σ_3^-. Any measured foliation on Σ_3^- is isotopic to a measured foliation such that each of these curves is a leave, a cycle of leaves, or is transverse to the leaves. If one of these curves is a leave, then it admits a maximal Mœbius neighborhood foliated by closed leaves.

The complement of these maximal Mœbius neighborhoods is a pair of pants P, or eventually a graph if the support of the measure is contained in the maximal Mœbius neighborhoods. In the first case, none of the boundaries of P is a leave. Then, using the classification of foliations of P given in [2] exposé 6 § II, we classify measured foliations on Σ_3^- (Figure 3). The measures of the maximal Mœbius neighborhoods can change the dimension of simplices, for instance compare simplices (4), (5) and (6) in Figure 4 of [2] exposé 6 with their corresponding simplices in Figure 3. In the case where P is a graph, the measured foliation is isotopic to the foliation F of the Figure 3, it gives a new 2-simplex. We have described a triangulation of $\mathsf{PMF}(\Sigma_3^-)$, we draw a global picture of it in Figure 2.

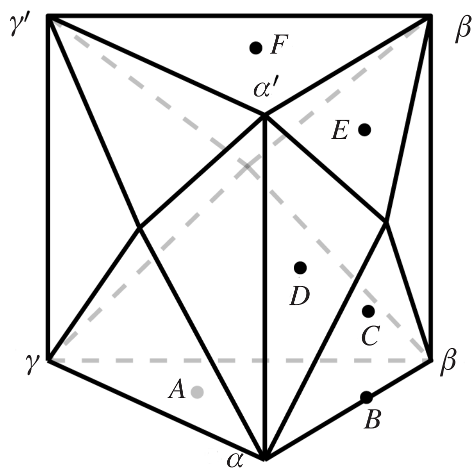


Figure 2. — Triangulation

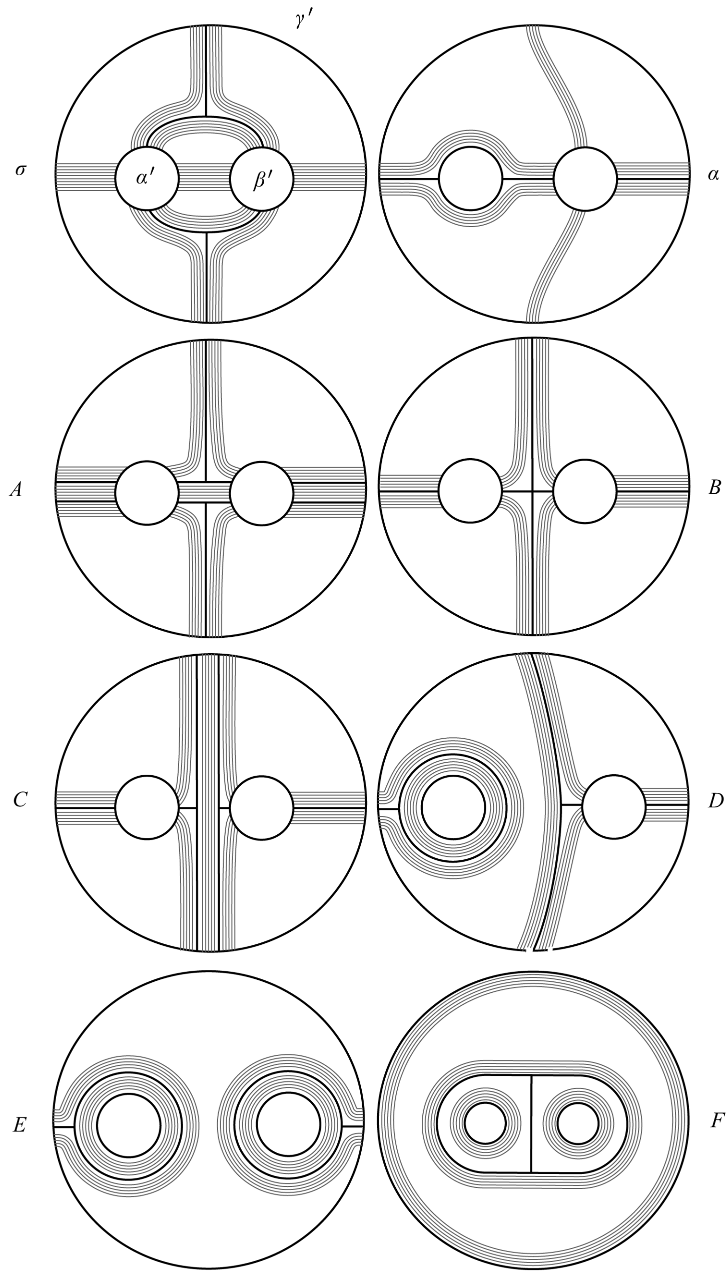


Figure 3

4. Monotonicity of length functions under the similar triangle flow

In §5 we will use the similar triangle flow to show the injectivity of some map from $\mathsf{Teich}(\Sigma_3^-)$ into $\mathbf{P}(\mathbf{R}^4)$. Two points of $\mathsf{Teich}(\Sigma_3^-)$ with same image with respect to these maps belong to the same trajectory of the similar triangle flow. So, we have to study the behaviour of certain length functions along these trajectories to prove that distinct points have distinct images.

4.1. Similar triangle flow

Let us consider the smooth homomorphism

$$\phi : \begin{array}{ccc} \mathbf{R}_+^* & \longrightarrow & \mathsf{Diff}(\mathbf{R}^3) \\ t & \longmapsto & \phi_t \end{array} \text{ defined by } \phi_t : \begin{array}{ccc} \mathbf{R}^3 & \longrightarrow & \mathbf{R}^3 \\ x & \longmapsto & tx \end{array} .$$

Let $x = (a, b, c)$ be a point in the positive cone. It follows from remark 3.6 that the trace of the trajectory $\{\phi_t(x)\}_{t>0}$ on Δ is the half-line $\{tx\ ; t > t_x\}$ where t_x is the unique positive solution of

$$\left(\cosh \frac{a+b}{2} t - \cosh \frac{c}{2} t\right)\left(\cosh \frac{c}{2} t - \cosh \frac{a-b}{2} t\right) = 1.$$

Let us fix a triangle (α, β, γ).

Definition 4.1. — *The pull back of ϕ by L is called a* similar triangle flow *with respect to (α, β, γ). We denote by $\triangle(X, t)$, or simply by $\triangle(t)$ when X is fixed, the trajectory of the similar triangle flow passing through X at $t = 1$: $\Delta(X, t) = L^{-1}(\phi_t(L(X)))$.*

The frontier of Δ can be canonically identified with the Teichmüller space of cusped tori, or equivalently with the Teichmüller space of hyperbolic structures on Σ_3^- noded at σ. The triangle flow can be extended to this space.

We now fix a point X in $\mathsf{Teich}(\Sigma_3^-)$. The trajectory $\triangle(t)$ is defined on an open interval $]t_X, +\infty[\subset \mathbf{R}_+^*$ containing 1. The geometrical meaning of the similar triangle flow can be expressed as follows: for any $Y \in \mathsf{Teich}(\Sigma_3^-)$ and any $t \in]t_X, +\infty[$

$$\triangle(t) = Y \iff \begin{cases} \ell_\alpha(Y) = t\ell_\alpha(X) \\ \ell_\beta(Y) = t\ell_\beta(X) \\ \ell_\gamma(Y) = t\ell_\gamma(X) \end{cases} .$$

We will denote by $\eta(t)$ the length of any geodesic η at the point $\triangle(t)$.

4.2. Monotonicity of $\sigma(t)$

PROPOSITION 4.2. — *For any $t > t_X$ we have*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) > \frac{\alpha(1)+\beta(1)+\gamma(1)}{2}.$$

Proof. — For simplicity we pose $a = \alpha(1)$, $b = \beta(1)$ and $c = \gamma(1)$. In view of (3.4)

$$\sigma(t) \quad = \quad 2\cosh^{-1}\sqrt{f(t)},$$

with

$$f(t) \quad = \quad \left(\cosh\frac{a+b}{2}t - \cosh\frac{c}{2}t\right)\left(\cosh\frac{c}{2}t - \cosh\frac{a-b}{2}t\right).$$

By the following simple estimation

$$\frac{\mathrm{d}}{\mathrm{d}t}2\cosh^{-1}\sqrt{f(t)} = \frac{f'(t)}{\sqrt{f(t)}\sqrt{f(t)-1}} > \frac{f'(t)}{f(t)},$$

it suffices to show that

$$\frac{f'(t)}{f(t)} > \frac{a+b+c}{2} = \frac{\alpha(1)+\beta(1)+\gamma(1)}{2}.$$

In practice

$$\frac{f'(t)}{f(t)} \quad = \quad \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\cosh\frac{a+b}{2}t - \cosh\frac{c}{2}t)}{\cosh\frac{a+b}{2}t - \cosh\frac{c}{2}t} + \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\cosh\frac{c}{2}t - \cosh\frac{a-b}{2}t)}{\cosh\frac{c}{2}t - \cosh\frac{a-b}{2}t}. \quad (4.1)$$

But for any $q > p > 0$ and any $t > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\cosh qt - \cosh pt) \quad &= \quad q\sinh qt - p\sinh pt \\ &> \quad q(\sinh qt - \sinh pt) = q\int_{pt}^{qt}\cosh(s)\mathrm{d}s \\ &> \quad q(\cosh qt - \cosh pt) = q\int_{pt}^{qt}\sinh(s)\mathrm{d}s.\end{aligned}$$

Thus, from (4.1) we conclude that

$$\frac{f'(t)}{f(t)} \quad > \quad \frac{a+b}{2} + \frac{c}{2} = \frac{a+b+c}{2}.$$

□

4.3. Monotonicity of $\alpha'(t)$, $\beta'(t)$ and $\gamma'(t)$.

PROPOSITION 4.3. — *For any $t > t_X$ we have*

$$\frac{\mathrm{d}\alpha'}{\mathrm{d}t}(t), \frac{\mathrm{d}\beta'}{\mathrm{d}t}(t), \frac{\mathrm{d}\gamma'}{\mathrm{d}t}(t) < 0.$$

Proof. — Let us consider the case of α'. From (3.1) we have

$$\sinh(\frac{\alpha'(t)}{2})\sinh(\frac{\sigma(t)}{2}) = \cosh(\frac{\alpha(t)}{2}) = \cosh(\frac{t\alpha(1)}{2}) \quad (t > t_X),$$

taking the derivative with respect to t we obtain

$$\frac{\mathrm{d}\alpha'}{\mathrm{d}t}(t)\cosh(\frac{\alpha'(t)}{2})\sinh(\frac{\sigma(t)}{2}) + \frac{\mathrm{d}\sigma}{\mathrm{d}t}(t)\sinh(\frac{\alpha'(t)}{2})\cosh(\frac{\sigma(t)}{2}) = \alpha(1)\sinh(\frac{t\alpha(1)}{2}).$$

Because of Proposition 4.2 and triangle inequality

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) > \frac{\alpha(1)+\beta(1)+\gamma(1)}{2} > \alpha(1).$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}\alpha'}{\mathrm{d}t}(t)\cosh(\frac{\alpha'(t)}{2})\sinh(\frac{\sigma(t)}{2}) &= \alpha(1)\sinh(\frac{t\alpha(1)}{2}) - \frac{\mathrm{d}\sigma}{\mathrm{d}t}(t)\sinh(\frac{\alpha'(t)}{2})\cosh(\frac{\sigma(t)}{2}) \\
&< \alpha(1)\sinh(\frac{t\alpha(1)}{2}) - \alpha(1)\sinh(\frac{\alpha'(t)}{2})\cosh(\frac{\sigma(t)}{2}) \\
&< \alpha(1)\sinh(\frac{t\alpha(1)}{2}) - \alpha(1)\sinh(\frac{\alpha'(t)}{2})\sinh(\frac{\sigma(t)}{2}) \\
&< \alpha(1)\sinh(\frac{t\alpha(1)}{2}) - \alpha(1)\cosh(\frac{t\alpha}{2}) \\
&< 0
\end{aligned}$$

which implies $\frac{\mathrm{d}\alpha'}{\mathrm{d}t}(t) < 0$. Identical proofs work for β' and γ'. □

5. Two polyhedral realizations of $\mathsf{Teich}(\Sigma_3^-)$ in $\mathbf{P}(\mathbf{R}^4)$

In this part we give two embeddings of $\mathsf{Teich}(\Sigma_3^-)$ into $\mathbf{P}(\mathbf{R}^4)$. The injectivity of these embeddings comes from the monotonicity of ℓ_σ, $\ell_{\alpha'}$, $\ell_{\beta'}$ and $\ell_{\gamma'}$ along the similar triangle flow. We fix a triangle (α, β, γ) for all this part.

5.1. Embeddings defined *via* length functions

Theorem 5.1. — *The following maps are embeddings of* $\mathsf{Teich}(\Sigma_3^-)$ *into* $\mathbf{P}(\mathbf{R}^4)$*:*

$$\begin{array}{rcl} L_1 : \mathsf{Teich}(\Sigma_3^-) & \longrightarrow & \mathbf{P}(\mathbf{R}^4) \\ X & \longmapsto & (\ell_\alpha(X) : \ell_\beta(X) : \ell_\gamma(X) : \ell_\sigma(X)), \\ L_2 : \mathsf{Teich}(\Sigma_3^-) & \longrightarrow & \mathbf{P}(\mathbf{R}^4) \\ X & \longmapsto & (\ell_\alpha(X) : \ell_\beta(X) : \ell_\gamma(X) : \ell_{\alpha'+\beta'+\gamma'}(X)), \end{array}$$

with the abuse of notations $\ell_{\alpha'+\beta'+\gamma'} := \ell_{\alpha'} + \ell_{\beta'} + \ell_{\gamma'}$.

Proof. — Let us prove the injectivity of L_1. We consider two points X and Y in $\mathsf{Teich}(\Sigma_3^-)$ with same image $L_1(X) = L_1(Y)$. We will show that X and Y have same lengths with respect to α, β and γ. This suffices to conclude as the map L is injective (Proposition 3.5).

These two points belong to the same trajectory $t \mapsto \Delta(t)$ of the similar triangle flow. Without loss of generality, we assume $X = \Delta(1)$ and $Y = \triangle(t)$ for some $t \geqslant 1$. On one hand, by definition of the similar triangle flow, we have

$$\ell_\alpha(Y) = \alpha(t) = t\alpha(1) = t\ell_\alpha(X),$$

which implies by proportionality of lengths

$$\sigma(t) = \ell_\sigma(Y) = t\ell_\sigma(X) = t\sigma(1).$$

On the other hand, if $t > 1$, Proposition 4.2 and (3.3) imply that

$$\begin{array}{rcl} t\sigma(1) & = & \sigma(t), \\ & = & \sigma(1) + \displaystyle\int_1^t \frac{\mathrm{d}\sigma}{\mathrm{d}s}(s)\mathrm{d}s, \\ & > & \sigma(1) + (t-1)\dfrac{\alpha(1)+\beta(1)+\gamma(1)}{2}, \\ & > & \sigma(1) + (t-1)\sigma(1). \end{array}$$

By this contradiction $t = 1$.

The same proof also works for L_2. We just have to make few modifications: we consider $(\alpha'+\beta'+\gamma')(t)$ instead of $\sigma(t)$, and we use Proposition 4.3 instead of Proposition 4.2.

It remains to show that the maps L_i are submersions, or equivalently that the maps $(\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\sigma)$ and $(\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_{\alpha'+\beta'+\gamma'})$ are transverse to lines of $\mathbf{R}^4$ passing through the origin. Let us consider a germ of curve $c = (c_\alpha, c_\beta, c_\gamma, c_\sigma)$ in the image of $(\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\sigma)$, which is tangent to a line of $\mathbf{R}^4$ passing through the origin. Then $(c_\alpha, c_\beta, c_\gamma)$ is a germ of curve in $L(\mathsf{Teich}(\Sigma_3^-))$ tangent to a line of $\mathbf{R}^3$ passing through the origin, and so tangent to the image $L(\triangle(t))$ of a trajectory of the similar triangle flow. As L is an embedding (Proposition 3.5) the germ of curve c is tangent to the image of $\triangle(t)$ by $(\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_\sigma)$. This is not possible according to Proposition 4.2 and inequality (3.3), so L_1 is a submersion. The same argument works also for L_2. □

5.2. Images as convex projective polyhedra

Corollary 5.2. — *The images of L_1 and L_2 are interiors of convex polyhedra in $\mathbf{P}(\mathbf{R}^4)$:*

- *the image $L_1(\mathsf{Teich}(\Sigma_3^-))$ is the interior of the convex projective polyhedron*

$$\Delta_1 := \Big\{(a:b:c:d) \in \mathbf{P}(\mathbf{R}^4) \mid a+b>c,\ b+c>a,\ c+a>b \text{ and } d>0 \text{ and } a+b+c>2d\Big\}.$$

- *the image $L_2(\mathsf{Teich}(\Sigma_3^-))$ is the interior of the simplex*

$$\Delta_2 := \Big\{(a:b:c:d) \in \mathbf{P}(\mathbf{R}^4) \mid b+c>a,\ c+a>b,\ a+b>c \text{ and } d>0\Big\}.$$

Remark 5.3. — Triangle inequalities imply that a, b and c are nonnegative.

Proof. — We clearly have $L_2(\mathsf{Teich}(\Sigma_3^-)) \subset \Delta_2$, and also $L_1(\mathsf{Teich}(\Sigma_3^-)) \subset \Delta_1$ by means of (3.3). So it remains to show that $\Delta_i \subset L_i(\mathsf{Teich}(\Sigma_3^-))$ for $i = 1, 2$.

Let us consider $(a, b, c, d) \in \mathbf{R}^4$ satisfying the conditions in the definition of Δ_i. Up to multiplication by a positive scalar, we can also assume that $(a, b, c) = L_i(X)$ for some X in $\mathsf{Teich}(\Sigma_3^-)$. We claim that there exists $t_0 \in]t_X, +\infty[$ such that

$$\begin{aligned}(\alpha:\beta:\gamma:\sigma)(t_0) &= (a:b:c:d) \quad \text{if } i=1\\ (\alpha:\beta:\gamma:\alpha'+\beta'+\gamma')(t_0) &= (a:b:c:d) \quad \text{if } i=2.\end{aligned}$$

This will prove that $(a:b:c:d) \in L_i(\mathsf{Teich}(\Sigma_3^-))$ and conclude the proof.

Case of L_1.— By definition of t_X we have $\sigma(t) \to 0$ when $t \to t_X$, so

$$dt \quad > \quad \sigma(t)$$

for t sufficiently close to t_X. Proposition 4.2 and $a+b+c > 2d$ leads to

$$\begin{aligned} 2\sigma(t) &\geqslant 2\sigma(1) + (a+b+c)(t-1) \\ &\geqslant 2\sigma(1) - 2d + (a+b+c-2d)(t-1) + 2dt \\ &\geqslant 2dt \end{aligned}$$

for t sufficiently large. As a consequence, there is a $t_0 > t_x$ such that $\sigma(t_0) = dt_0$, and $(\alpha, \beta, \gamma, \sigma)(t_0) = t_0(a, b, c, d)$ as we claimed.

Case of L_2.— Two intersecting geodesics can not be simultaneously arbitrary short thus

$$\sigma(t) \underset{t \to t_X}{\longrightarrow} 0 \quad \text{implies} \quad (\alpha' + \beta' + \gamma')(t) \underset{t \to t_X}{\longrightarrow} +\infty.$$

So $(\alpha' + \beta' + \gamma')(t) > dt$ for t sufficiently close to t_X. But $(\alpha' + \beta' + \gamma')(t)$ is a decreasing function (Proposition 4.3), thus $(\alpha' + \beta' + \gamma')(t) < dt$ for t sufficiently large. As a byproduct there exists $t_0 > t_X$ such that

$$(\alpha, \beta, \gamma, \alpha' + \beta' + \gamma')(t_0) = t_0(a, b, c, d),$$

as we claimed. □

6. Interpretation of the boundaries of Δ_1 and Δ_2

Let us fix a triangle (α, β, γ). There exists a unique curve $\eta \neq \alpha$ such that (η, β, γ) is a triangle, we denote $\alpha^\star$ the dual curve of η.

6.1. Description of the boundaries

The closure of Δ_2 is the simplex spanned by A', B', C' and S. Whereas the closure of Δ_1 is the truncated simplex with vertices A', B', C', and E, F, G. All these points are defined in Table 6.1.

If these configurations are not obvious, one can use the projective transformation

$$(a : b : c : d) \mapsto ((b+c) - a : (a+c) - b : (a+b) - c : a+b+c+d)$$

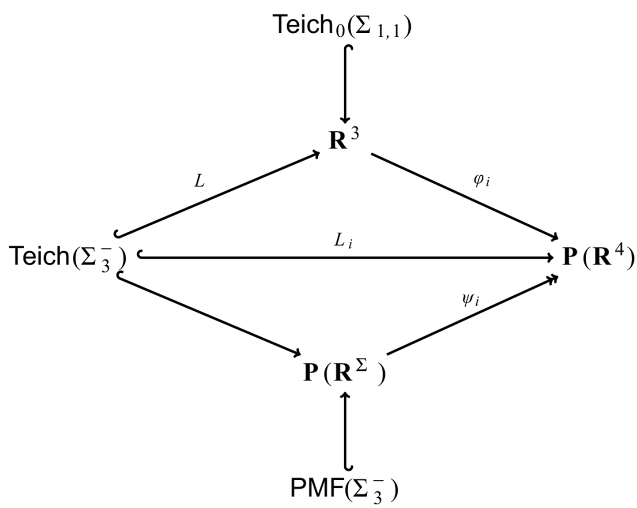
which sends respectively Δ_2 and Δ_1 on $\{(x : y : z : t) \mid x, y, z > 0 \text{ and } t > x+y+z\}$ and $\left\{(x : y : z : t) \mid x, y, z > 0 \text{ and } t > x+y+z > \frac{2}{3}t\right\}$. Then, everything becomes clear in the affine chart $\{t = 1\}$.

Table 6.1. — Some points in $\mathbf{P}(\mathbf{R}^4)$ and their coordinates

point	coordinates	point	coordinates	point	coordinates
A	$(0:1:1:2)$	A'	$(0:1:1:0)$	$A^\star$	$(2:1:1:1)$
B	$(1:0:1:2)$	B'	$(1:0:1:0)$	$B^\star$	$(1:2:1:1)$
C	$(1:1:0:2)$	C'	$(1:1:0:0)$	$C^\star$	$(1:1:2:1)$
E	$(0:1:1:1)$	F	$(1:0:1:1)$	G	$(1:1:0:1)$
S	$(0:0:0:1)$				

6.2. Interpretation

Each embedding L_i factors through L, and also through the canonical embedding of $\mathsf{Teich}(\Sigma_3^-)$ into $\mathbf{P}(\mathbf{R}^{\mathcal{S}})$. The situation is represented in the commutative diagram below, where the maps ϕ_i and ψ_i are quite obvious.



6.2.1. Case of Δ_1

The map ϕ_1 extends differentiably to the closure of Δ, and induces an embedding of the Teichmüller space of cusped tori into $\mathbf{P}(\mathbf{R}^4)$, which is simply $(\ell_\alpha : \ell_\beta : \ell_\gamma : 0)$. The image of this embedding is exactly the interior of the face $\langle A', B', C' \rangle$ of Δ_1.

The map ψ_1 extends coutinuously to the set of points in the Thurston boundary satisfying the condition $(\mathsf{i}_\alpha, \mathsf{i}_\beta, \mathsf{i}_\gamma, \mathsf{i}_\sigma) \neq 0$. This set turns out to be $\partial\mathsf{Teich}(\Sigma_3^-) \setminus \{\sigma\}$. The vertices of the triangulation of $\mathsf{PMF}(\Sigma_3^-) \simeq$

$\partial\mathsf{Teich}(\Sigma_3^-)$ studied in §3.4 are sent on vertices, and centers of faces, of Δ_1 (Figure 4). Thus, the extension of ψ_1 induces a piecewise integral projective isomorphism from the complement in $\mathsf{PMF}(\Sigma_3^-)$ of the interior of the simplex $\langle\alpha,\beta,\gamma\rangle$, to the complement in $\partial\Delta_1$ of the interior of the simplex $\langle A',B',C'\rangle$. The extension of ψ_1 mashes $\langle\alpha,\beta,\gamma\rangle\setminus\{\sigma\}$ onto the boundary of $\langle A',B',C'\rangle$. More precisely, a point X in $\langle\alpha,\beta,\gamma\rangle\setminus\{\sigma\}$ has same image as the point Y on the boundary of the simplex such that X belongs to $[\sigma,Y]$.

Interpretation. — The polyhedon Δ_1 is a compactification of the union of Teichmüller spaces $\cup_{b>0}\mathsf{Teich}_b(\Sigma_{1,1})$. Its boundary decomposes into two pieces. One piece corresponds to the set of projective measured foliations of the one-holed torus, where leaves transverse to the boundary are allowed. The other piece corresponds exactly to the Teichmüller space of cusped tori. The frontier between these two pieces is the Thurston boundary of $\mathsf{Teich}_b(\Sigma_{1,1})$, which does not depend on the fixed length $b>0$.

We have to take care of the way we define measured foliations. If we work with $\mathsf{Teich}_b(\Sigma_{1,1})$ ($b\geqslant 0$ fixed) then we consider measured foliations such that the boundary is a cycle of leaves with at least one singularity (as in [8] and [2] exposé 11 §1). But if we work on $\cup_{b>0}\mathsf{Teich}_b(\Sigma_{1,1})$, then we allow measured foliations to have leaves transverse to the boundary. Nevertheless, we still assume that no closed leave is isotopic to the boundary, that's why the simplex $\langle\alpha,\beta,\gamma\rangle$ collapses.

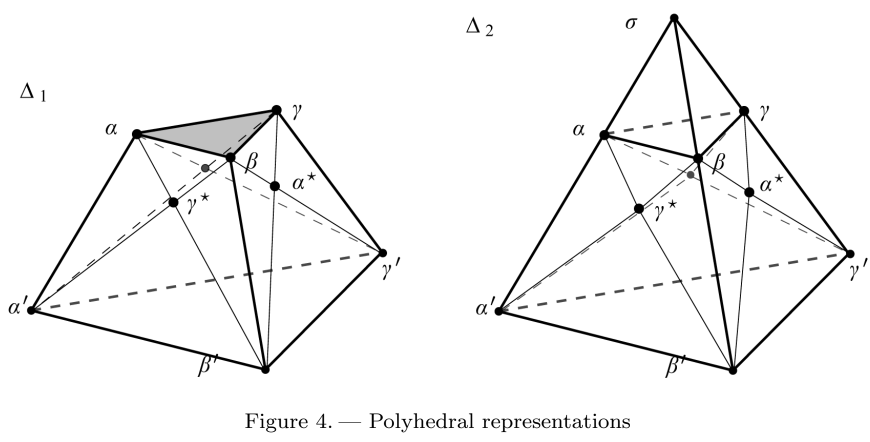


Figure 4. — Polyhedral representations

Table 6.2. — Intersection numbers

	i_α	i_β	i_γ	i_σ	$\mathsf{i}_{\alpha'+\beta'+\gamma'}$	in Δ_1	in Δ_2
α	0	1	1	0	2	A'	A
β	1	0	1	0	2	B'	B
γ	1	1	0	0	2	C'	C
σ	0	0	0	0	3		S
α'	0	1	1	1	0	E	A'
β'	1	0	1	1	0	F	B'
γ'	1	1	0	1	0	G	C'
$\alpha^\star$	2	1	1	1	1	$A^\star$	$A^\star$
$\beta^\star$	1	2	1	1	1	$B^\star$	$B^\star$
$\gamma^\star$	1	1	2	1	1	$C^\star$	$C^\star$

6.2.2. Case of Δ_2

The lengths of α', β' and γ' are infinite at any point on the boundary of Δ. Thus the map ϕ_2 is not defined on $\partial\Delta$. However it admits a continuous extension given by $\phi_2 \equiv (0:0:0:1)$ on $\partial\Delta$. The map ψ_2 also extends continuously to the whole Thurston boundary of $\mathsf{Teich}(\Sigma_3^-)$, for no point in the Thurston boundary satisfies $(\mathsf{i}_\alpha, \mathsf{i}_\beta, \mathsf{i}_\gamma, \mathsf{i}_{\alpha'+\beta'+\gamma'}) = 0$. The images of the vertices of the triangulation of $\mathsf{PMF}(\Sigma_3^-)$ are sent to points in Δ_2 as shown in Figure 4. So the extension of ψ_2 induces a piecewise integral projective isomorphism between the Thurston boundary of $\mathsf{Teich}(\Sigma_3^-)$ and the boundary of Δ_2. Note that the image $\psi_2(\sigma)$ coincide with $\phi_2(\partial\Delta)$.

Interpretation. — The polyhedron Δ_2 is a convex polyhedral realization of the Thurston compactification of $\mathsf{Teich}(\Sigma_3^-)$. The map $(\ell_\alpha, \ell_\beta, \ell_\gamma, \ell_{\alpha'+\beta'+\gamma'})$ is an embedding of $\mathsf{Teich}(\Sigma_3^-)$ onto the interior of Δ_2. It has a continuous extension which induces a piecewise integral projective isomorphism given by $(\mathsf{i}_\alpha, \mathsf{i}_\beta, \mathsf{i}_\gamma, \mathsf{i}_{\alpha'+\beta'+\gamma'})$ between the Thurston boundary of $\mathsf{Teich}(\Sigma_3^-)$ and the boundary of Δ_2 .

6.3. Few words about the action of the modular group

The modular group $\mathsf{Mod}(\Sigma_3^-)$ acts on the Thurston boundary of the Teichmüller space $\mathsf{Teich}(\Sigma_3^-)$, and consequently on the boundary of the projective simplex Δ_2. This action is not faithful as the hyperelliptic involution acts trivially. Each element of $\mathsf{Mod}(\Sigma_3^-)$ fixes the vertex associated to σ. Each element acting projectively stabilizes the set of curves $\{\alpha', \beta', \gamma'\}$. So the subgroup of $\mathsf{Mod}(\Sigma_3^-)$ acting projectively is isomorphic to D_{12} the dihedral group with 12 elements.

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