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Equivalence classes of Latin squares and nets in $\mathbb{CP}^2$

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Abstract. — The fundamental combinatorial structure of a net in $\mathbb{CP}^2$ is its associated set of mutually orthogonal Latin squares. We define equivalence classes of sets of orthogonal Latin squares by label equivalences of the lines of the corresponding net in $\mathbb{CP}^2$. Then we count these equivalence classes for small cases. Finally we prove that the realization spaces of these classes in $\mathbb{CP}^2$ are empty to show some non-existence results for 4-nets in $\mathbb{CP}^2$.

Résumé. — La structure combinatoire fondamentale d'un filet dans $\mathbb{CP}^2$ est donnée par l'ensemble des carrés latins orthogonaux associé. Nous définissons des classes d'équivalence de carrés latins orthogonaux a l'aide de classes d'équivalence des lignes apparaisant dans le filet de $\mathbb{CP}^2$. Nous comptons le nombre de classes d'équivalence pour certains exemples de carrés petits. Finalement, nous montrons que les espaces de réalisations de ces classes pour $n = 4$ et $k = 4, 5, 6$ sont vides et nous en déduisons que les filets correspondants n'existent pas.

1. Introduction

Nets have appeared in several different areas of mathematics over the last century. Reidemeister was one of the first to examine nets in his research on webs and their relationship to groups (see [15]). The existence of nets has

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also been shown to effect the existence of finite projective planes (see [8]). More recently there has been a surge of interest in nets due to their connection with resonance and characteristic varieties. This began with the work of Libgober and Yuzvinsky in [12] where they showed that nets supported positive-dimensional characteristic varieties (see [12]). Then Yuzvinsky in [21] studied nets exclusively, classified certain classes of nets in $\mathbb{CP}^2$, and posed numerous open problems concerning nets. Problem 1 of [21] is exactly the focus of this note.

Since Yuzvinsky's work in [21], the work of Falk and Yuzvinsky in [10] showed an even stronger connection between resonance and nets. In particular they showed that an arrangement supports a nontrivial resonance component if and only if the arrangement supports a multinet. Then in [14] Pereira and Yuzvinsky view nets as the collection of irreducible special fibers of a pencil of hypersurfaces and use methods from the theory of foliations to bound the number of special fibers. They then use this to bound the dimensions of essential components of resonance and characteristic varieties.

The recent work of Artal-Bartolo, Cogolludo-Agustin and Libgober [2, 3] demonstrates another application of the existence of nets and pencils of curves with irreducible special fibers. They show, among other things, that nets influence the characters of the fundamental group and the characteristic varieties of the complement of a plane curve. Then Denham and Suciu in [9] use multinets to exhibit non-trivial components of characteristic varieties and generate arrangements with torsion in the homology of the Milnor fiber. Still the exact parameters for the existence of nets are unknown. We next give our definition of a net which is essentially taken from [21].

Definition 1.1. — *A* (n,k)-net in $\mathbb{CP}^2$ *for* $n \geqslant 3$ *consists of a set of lines* $\mathcal{A} \subset \mathbb{CP}^2$ *and a finite set of points* $\chi \subset \mathcal{A}$ *such that* $\mathcal{A}$ *can be partitioned into* n *subsets* $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$ *where* $|\mathcal{A}_i| = k$ *for all* $i = 1, \dots, n$ *subject to the following conditions*

1. *If* $\ell_1 \in \mathcal{A}_i$ *and* $\ell_2 \in \mathcal{A}_j$ *then* $\ell_1 \cap \ell_2 \in \chi$ *whenever* $i \neq j$.
2. *For every* $X \in \chi$ *and every* $i \in \{1, \dots, n\}$ *there is exactly one line* $\ell \in \mathcal{A}_i$ *such that* $X \in \ell$.

In [16], Stipins makes significant progress on classifying nets with $n = 4, 5$. Then in [22] Yuzvinsky extends Stipins work to show no $n = 5$ nets exist. In [20] and [19] Urzua has furthered the classification of $(3,k)$-nets in $\mathbb{CP}^2$ by describing all possible realizations of $(3,6)$-nets and their associated moduli spaces. Yet for $n = 4$ there are still many open questions. In this note we take the first few steps towards showing that the only $n = 4$ net is the Hessian. We define equivalence classes of pairs of orthogonal Latin squares and use them to derive non-existence results for $(4, k)$-nets in $\mathbb{CP}^2$.

Our approach towards this question uses the well known fact that one can associate $n-2$ mutually orthogonal $k \times k$ Latin squares to a (n, k)-net (see e.g. [8]), and we describe that association here. Recall that a Latin square is a $k \times k$ array containing the numbers $1, \ldots, k$ so that there are no repetitions of any number within the same row or column. Given a (n, k)-net we label the n sets of lines by the numbers $1, 2, \ldots, n$ and label the lines in each of the sets by the numbers $1, 2, \ldots, k$. With this labeling define $k \times k$-arrays L^m (for $m = 1, \ldots, n-2$) by setting the i^{th} entry in the j^{th} column of L^m to be $L^m_{ij} = \ell$, if the i^{th} line of the first set and the j^{th} line of the second set meet the ℓ line of the m^{th} set in their intersection point. Every intersection of a line in the first set with a line in the second set is contained by exactly one line from the m^{th} set. Thus, L^m is a Latin square. Recall also that two $k \times k$ Latin squares L^1 and L^2 are orthogonal if for each pair $(\ell, m) \in \{1, \ldots k\}^2$ there exists exactly one pair $(i, j) \in \{1, \ldots k\}^2$ such that $L^1_{ij} = \ell$ and $L^2_{ij} = m$. When $n \geqslant 4$, every intersection of a line in the first set with a line in the second set is contained in exactly one line from the m^{th} set, and one line from the $(m')^{\text{th}}$ set. Thus, the Latin squares L^m and $L^{m'}$ are orthogonal. This labeling of the lines is not unique; permuting the labels of the lines in one of the sets, or the numbering of the sets can lead to different sets of orthogonal Latin squares.

Example 1.2. — The following is a standard example of a pair of (4×4) orthogonal Latin squares. We will refer to these squares subsequently. Set

$$L_1 = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&1&4&3\\\hline 3&4&1&2\\\hline 4&3&2&1\\\hline\end{array}\ . \qquad (1.1)$$

It is easy to check that the following Latin square L_2 is orthogonal to L_1.

$$L_2 = \begin{array}{|c|c|c|c|}\hline 1&3&4&2\\\hline 2&4&3&1\\\hline 3&1&2&4\\\hline 4&2&1&3\\\hline\end{array}\ . \qquad (1.2)$$

In this paper we propose the following program to classify (n, k)-nets:

(1) Define an equivalence relation on the set of $(n-2)$-tuples of mutually orthogonal $k \times k$-Latin squares identifying those tuples which are obtained from the same net.

(2) Choose a representative for each equivalence class and investigate whether it can be realized as a net in $\mathbb{CP}^2$. One obtains a system of equations whose solutions describe the moduli space of isomorphism classes of nets in $\mathbb{CP}^2$.

In the present paper we will apply this approach to the case $n = 4$. Denote the set of all pairs of orthogonal $k \times k$ Latin squares by OLS_k. In this paper, if $(L, M) \in OLS_k$, we may refer to M as the *orthogonal mate* of L or vice versa. We define the equivalence relation $\sim'$ in the next section. We will call the set of these equivalence classes corresponding to Step (1) of our program $OLS_k/\sim'$, and express the equivalence class of (L, M) in $OLS_k/\sim'$ as $[(L, M)]'$. We obtain the following result (which is broken up later into the Theorems 3.4, 3.5, 3.6).

Theorem 1.3. — *Using the notation established above, we have*

1. $|OLS_3/\sim'| = 1$.
2. $|OLS_4/\sim'| = 1$.
3. $1 \leqslant |OLS_5/\sim'| \leqslant 2$.

Then we calculate the realization space, as a net in $\mathbb{CP}^2$, of a representative of each equivalence class in Theorem 1.3 to prove the following theorem.

Theorem 1.4. — *The following is a complete classification of* $(4, k)$*-nets in* $\mathbb{CP}^2$ *up to projective isomorphism, for* $k = 3, 4, 5, 6$.

1. *The Hessian arrangement is the only* $(4, 3)$*-net in* $\mathbb{CP}^2$ *up to projective isomorphism.*
2. *There are no* $(4, k)$*-nets in* $\mathbb{CP}^2$ *for* $k = 4, 5, 6$.

The solution to Euler's well-known "36 Officer Problem" says precisely $|OLS_6| = 0$ (see [18]), and so the case $k = 6$ in Assertion 2 of Theorem 1.4 is obvious.

Remark 1.5. — Other authors have studied different classes of orthogonal Latin squares (see [7, 8] for example). Though the literature is vast, the equivalence classes described herein appear in [11] where the author refers to these relations as *isotopy of Latin hypercubes* (see Section 3 of [11]). Some of our results overlap with previously known results, but we include our own proofs in this new context to provide a more complete and self-contained treatment of this subject.

2. Preliminaries

We will use the following notation throughout the paper. Let Sym_k be the standard permutation group of k objects. We denote permutations in Sym_k using the following standard notation: for any subset $\{i_{q_1} i_{q_2} \dots i_{q_r}\} \subset \{1, 2, \dots, k\}$ the expression $(i_{q_1} i_{q_2} \dots i_{q_r})$ denotes the r-cycle in Sym_k which maps i_{q_h} to $i_{q_{h+1}}$ and fixes its complement $\{1, 2, \dots, n\} \setminus \{i_{q_1}, i_{q_2}, \dots i_{q_r}\}$. It is well known that any permutation can be written as a product of such cycles. Let $\mathcal{L}_k$ be the set of all Latin squares of size k. An element $L = \{L_{ij}\}_{i,j=1}^k \in \mathcal{L}_k$ is an $k \times k$ array containing the numbers $1, \dots, k$ so that there are no repetitions of any number within the same row or column. Thus, we can associate to each Latin square and each pair $i \neq j$ the permutation $\sigma_{ij}^L \in \mathrm{Sym}_k$ defined by $\sigma_{ij}^L(L_{pi}) = L_{pj}$ for $p = 1, \dots, k$. That means σ_{ij}^L is the permutation that sends the p^{th} entry of the i^{th} column to the p^{th} entry of the j^{th} column; we will suppress the superscript except in cases where there could be ambiguity (see, for example, Proposition 3.3). For convenience, we may express σ_{ij}^L as $\sigma_{i,j}^L$ in times where a comma would be appropriate. A Latin square is therefore uniquely described by its first column and the permutations $\sigma_{1,2}, \sigma_{2,3}, \dots, \sigma_{k-1,k}$. For this paper, we may refer to a Latin square by these permutations as $L(\sigma_{1,2}, \dots, \sigma_{k-1,k})$, and when we do, we assume the first column reads downward as $1, 2, \dots, k$. Since there are no repetitions in any row, the σ_{ij} must be fixed point free.

Example 2.1. — Consider again the Latin square L_1 from Example 1.2. In the Latin square L_1, we have $\sigma_{12}^{L_1} = \sigma_{34}^{L_1} = (12)(34) = \tau_1$ and $\sigma_{23}^{L_1} = (14)(23) = \tau_2$. So, we could unambiguously express $L_1 = L(\tau_1, \tau_2, \tau_1)$.

There are several operations on Latin squares that preserve the property of being Latin, i.e., they define bijective maps from $\mathcal{L}_k$ to itself. Let $L \in \mathcal{L}_k$.

- (S1) Exchange row i with row j.
- (S2) Exchange column i with column j.
- (S3) Exchange two of the symbols in L.

These relations appear in Section 3 of [11].

Definition 2.2. — *We define the relation $\sim$ on $\mathcal{L}_k$ by setting $L \sim L'$ if and only if one can transform L into L' by applying finitely many exchanges of type (S1), (S2), or (S3). We express the equivalence class of L in $\mathcal{L}/\sim$ as $[L]$.*

The following set, which is just the set of Latin squares that are multiplication tables of cyclic groups, will be of principal use in this note:

$$\begin{aligned} \mathcal{G}'_k &:= \{L \in \mathcal{L}_k \,|\, \sigma_{i,i+1}^L = \sigma_{1,2}^L \text{ for all } i, \text{ and } \sigma_{1,2}^L \text{ is a } k-\text{cycle}\}, \\ \mathcal{G}_k &:= \{L \in \mathcal{L}_k \,|\, \exists H \in \mathcal{G}'_k \text{ so that } L \sim H\}. \end{aligned}$$

In the event that $L \in \mathcal{G}'_k$, and the first column of L proceeds downward as $1, 2, \ldots, k$, we write $L = L_\sigma$ instead of $L(\sigma, \ldots, \sigma)$ in reference to our earlier convention.

It is easy to see that the Latin squares L and L' are orthogonal if and only if the map $(i, j) \mapsto (L_{ij}, L'_{ij})$ is surjective (and hence bijective) as a function from $\{1, \ldots, k\} \times \{1, \ldots, k\}$ to itself. Denote the set of all ordered pairs of orthogonal Latin squares of size k as OLS_k. A *transversal* on a $k \times k$ Latin square is a collection of k entries in L subject to two conditions: no two entries are in the same row or column, and there is no repetition of the values of the entries. The following useful theorem is well known and gives an equivalent condition to a pair of Latin squares being orthogonal (see [8]).

Theorem 2.3 ([8]). — *If $L \in \mathcal{L}_k$, then there exists an $L' \in \mathcal{L}_k$ so that $(L, L') \in OLS_k$ if and only if there exist k disjoint transversals on L.*

As with Latin squares, there are several operations one can perform on pairs of orthogonal Latin squares which preserve the property of being orthogonal. We list some of them below. Suppose $(L, L') \in OLS_k$.

- (R1) Exchange row i with row j in both L and L'.
- (R2) Exchange column i with column j in both L and L'.
- (R3) Exchange two symbols of $\{1, \ldots, k\}$ in L.
- (R4) Exchange two symbols of $\{1, \ldots, k\}$ in L'.
- (R5) Transpose either L or L'.
- (R6) Apply the map $(L, L') \mapsto (L', L)$.

Definition 2.4. — *We define the relation $\sim'$ on OLS_k by setting $(L_1, L_2) \sim' (L'_1, L'_2)$ if and only if one can transform (L_1, L_2) into (L'_1, L'_2) by applying finitely many operations (R1)–(R6).*

The next corollary is an obvious consequence of Theorem 2.3.

Corollary 2.5. — *If $(L, H) \in OLS_k$ and $L \sim L'$, then there exists an $H' \in \mathcal{L}$ so that $(L, H) \sim' (L', H')$.*

Most authors have attempted to determine $|OLS_k|$ for various k using design theory; see, for example, [1, 6] and [17]. In addition to computing their size the authors wish to understand these sets more explicitly. We would like to compute the size of the set $OLS_k / \sim'$ and to exhibit sets of representatives for these sets.

3. Equivalence Classes of Orthogonal Latin Squares

In this section we explicitly describe the structure of the sets of representatives of $OLS_k/\sim'$ for $k = 3, 4$ and 5 in Theorems 3.4, 3.5, 3.6. In order to prove these results we first have to establish some elementary results about the sets $\mathcal{L}_k/\sim$ and $\mathcal{G}_k/\sim$.

LEMMA 3.1. — *Adopt the notation given in Section 1. Fix an integer $k \geqslant 3$. Let $L \in \mathcal{G}_k$, and let $L \sim L_\sigma \in \mathcal{G}'_k$.*

1. *$L_\sigma \sim L_{(1\cdots k)}$. Thus, $\mathcal{G}_k/\sim$ contains only one element.*
2. *Let $L_\sigma \in \mathcal{G}_k$. There exists one transversal to L_σ if and only if there exists k disjoint transversals to L_σ.*
3. *If k is odd, and $L \in \mathcal{G}_k$ is given, then there exists an orthogonal mate to L.*
4. *If k is even, and $L \in \mathcal{G}_k$, then there does not exist an orthogonal mate to L.*

Proof. — Let $L_\sigma \in \mathcal{G}_k$ be given. Since σ is a $k-$cycle, it acts transitively on the set $\{1, \ldots, k\}$. Thus by a rearrangement of rows, we may order the first column from top to bottom as $\sigma^1(1), \sigma^2(1), \ldots, \sigma^k(1)$. Since the second column is produced by application of σ to each element of the first column, the second column must now read $\sigma^2(1), \ldots, \sigma^k(1), \sigma^{k+1}(1) = \sigma^1(1)$. Now relabel the symbols according to the rule $\sigma^p(1) \mapsto p$. Assertion (1) is now established.

We now prove assertion (2). It is enough to prove the assertion for $L_\sigma \in \mathcal{G}'_k$. Since permuting rows preserves σ, we may assume without loss of generality that the transversal is the main diagonal $\{(L_\sigma)_{ii}\}_{i=1}^k$. Now we create a new transversal by applying σ and shifting each entry to the right except that the right most entry moves to column 1. The new transversal has $\sigma((L_\sigma)_{ii})$ in the $i, i+1$-entry written modulo k. Then we repeat this process to obtain the remaining transversals. Since σ is a k-cycle each entry of the transversal are different and by construction we have exactly one entry on each row and column. This proves Assertion (2).

Because of assertion (1) and Corollary 2.5 it is enough to produce a transversal on $L_{(1\cdots k)}$ when k is odd to prove assertion (3). The entries of the diagonal are $(L_{(1\cdots k)})_{ii} = 2i - 1$ considered modulo k. Since k is odd on the we get all different entries. Assertion (3) is now established.

We finish the proof of Lemma 3.1 by establishing Assertion (4). By Corollary 2.5, it suffices to show that there does not exist a single transversal on $L_{(1\cdots k)}$. Suppose to the contrary that there exists a transversal. Make a new Latin square L' by rearranging the columns so that the transversal is the main diagonal. For convenience let $\xi = (1\cdots k)$. Note that σ_{1j} in L' is the same as ξ^{i_j} where, $\{i_j\}$ is some ordering of the numbers $0,\ldots,k-1$. Recall that σ_{1j} is the permutation sending column 1 to column j. Notice that $i_1 = 0$ and the numbers down the new first column are numbered $p, \xi(p), \ldots, \xi^{k-1}(p)$. Thus, the numbers in the transversal are as follows:

$$p, \xi^{1+i_2}(p), \xi^{2+i_3}(p), \ldots, \xi^{k-1+i_k}(p)\,.$$

But $\xi^{1+i_2}(p) = \xi^{i_2}(\xi^1(p)) = \xi^{i_2}([p+1]_k) = [p+1+i_2]_k$, where $[\cdot]_k$ denotes reduction mod k. Similar to above, we have $\xi^{j-1+i_j}(p) = [p+j-1+i_j]_k$. Since this is a transversal, the collection of these numbers (mod k) must be in set bijection with $\mathbb{Z}_k$. Thus the sum of all of these elements must be congruent to $k(k-1)/2k$.

We compute:

$$\sum_{j=1}^{k} [p+j-1+i_j]_k = [kp]_k + \left[\sum_{j=1}^{k-1} j\right]_k + \left[\sum_{j=1}^{k} i_j\right]_k$$

$$= \left[\frac{k(k-1)}{2}\right]_k + \left[\sum_{j=1}^{k} i_j\right]_k$$

But the numbers i_j are just reordering of the integers $1,\ldots,k$ so this becomes

$$= 2\left[\frac{k(k-1)}{2}\right]_k = [0]_k.$$

Now suppose, since k is even, $k = 2t$. Then above, as stated above, should be congruent to

$$\left[\frac{2t(2t-1)}{2}\right]_k$$

which means that it should be $[0]_k$. If $\left[\frac{2t(2t-1)}{2}\right]_k = [0]_k$ then there exists s such that $t(2t-1) = 2ts$. This means that $t(2(t-s)-1) = 0$ which implies that $t = 0$ hence we have a contradiction. $\square$

THEOREM 3.2. — *Let $k \geqslant 3$.*

1. *The set $\mathcal{G}_k/\sim$ contains exactly one element.*
2. *An element of $\mathcal{G}_k$ has an orthogonal mate if and only if k is odd.*

Proof. — Theorem 3.2 (1) follows from Assertion (1) of Lemma 3.1, since $L_\sigma \sim L_{(1\dots k)}$ immediately yields that $\mathcal{G}_k/\sim$ contains exactly one element as asserted. Assertions (3) and (4) of Lemma 3.1 establish Assertion (2) of Theorem 3.2. □

There is a stronger version of Assertion 1 of Lemma 3.1 that will be of use. Recall that we can associate to each permutation $\sigma \in \mathrm{Sym}_k$, a multi-index $I(\sigma)$ that describes the size and number of disjoint cycles in σ when it is written uniquely as a product of disjoint cycles (excluding the fixed points of σ). For instance, the permutation $(15)(236)(49) \in S_9$ corresponds to the multi-index $I((15)(236)(49)) = (2, 2, 3)$ since there are two disjoint 2-cycles, and one 3-cycle. The following result shows that this multi-index is the only relevant information when considering permutations in equivalence classes of Latin squares.

PROPOSITION 3.3. — *Let $L \in \mathcal{L}_k$, and fix any $i \neq j$, $i, j \in 1, \dots, k$. Let $\sigma \in Sym_k$ be any permutation with $I(\sigma^L_{ij}) = I(\sigma)$. There exists $W \in \mathcal{L}_k$ such that $L \sim W$ and $\sigma^W_{12} = \sigma$. Moreover, one can choose W such that $W_{j1} = j$.*

Proof. — We produce L' by exchanging columns $i \leftrightarrow 1$, and $j \leftrightarrow 2$ in L so that $\sigma^L_{ij} = \sigma^{L'}_{12}$. It is easy to verify that if $\tau \in \mathrm{Sym}_k$ is a global relabeling of the entries of L' and we produce the equivalent square W' from such a relabeling, then $\sigma^{W'}_{12} = \tau\sigma^{L'}_{12}\tau^{-1}$. Since the conjugation action of S_k on itself is a transitive action among permutations of the same type, we may choose a τ so that $\sigma = \tau\sigma^{L'}_{12}\tau^{-1}$, and produce W' by enforcing this global relabeling on L'. This proves the first part of the proposition. Exchanging rows does not change the permutation $\sigma^{W'}_{12}$, thus we may reorder the rows of W' to produce $W \sim W'$ so that the first column reads downward as $1, 2 \dots, k$. □

We can now lay the framework for our study of 4-nets in $\mathbb{CP}^2$ below by separately establishing Assertions 1, 2, and 3 of Theorem 1.3.

THEOREM 3.4. — *The set $OLS_3/\sim' = \{[(L_{(123)}, L_{(132)})]\}$.*

Proof. — It is a basic fact [5] that the maximum number of mutually orthogonal Latin squares of order 3 is 2. The only Latin squares of size 3 must belong to $\mathcal{G}_3$. The only 3-cycles to generate these squares are (123) and (132). □

For the next theorem, set $\tau_1 = (12)(34), \tau_2 = (14)(23)$, and $\tau_3 = (13)(24)$ as elements of Sym_4, and set $L_1 := L(\tau_1, \tau_2, \tau_1)$, $L_2 := L(\tau_2, \tau_3, \tau_2)$, and $L_3 := L(\tau_3, \tau_1, \tau_3)$ as 4×4 Latin squares. We have seen already that the Latin square L_1 from Example 2.1 satisfies $L_1 = L(\tau_1, \tau_2, \tau_1)$.

THEOREM 3.5. — *Let L_i and τ_i be as above.*

1. *L_1, L_2, and L_3 are mutually orthogonal.*
2. *Suppose that $i \neq j$, and $k \neq \ell$, where all of $i, j, k, \ell \in \{1, 2, 3\}$. Then $(L_i, L_j) \sim' (L_k, L_\ell)$. In other words, any pair of squares using the L_i is equivalent under $\sim'$ to any other pair of squares using the L_i for $i = 1, 2, 3$.*
3. *$OLS_4/\sim' = \{[(L_1, L_2)]\}$ contains only one element.*

Proof. — One can easily verify Assertion (1). The relation of orthogonality is symmetric. Thus, to establish Assertion (2), we produce a sequence of steps showing $(L_1, L_2) \sim' (L_2, L_3) \sim' (L_3, L_1)$. To show each relation, simply use (R2) and cycle (column 2 $\to$ column 3 $\to$ column 4 $\to$ column 2) in both squares.

We now prove Assertion 3. Let $(L, L') \in OLS_4$, as in the proof of Theorem 3.4, express L in terms of its associated fixed point free permutations $\sigma_{12}, \sigma_{23}, \sigma_{34}$, and σ_{41}. The fixed point free permutations in Sym_4 are exactly τ_1, τ_2 and τ_3, and the 4-cycles. We consider two cases: either one of σ_{12}, σ_{23}, or σ_{34} is a 4-cycle σ, or none of them are.

Assume without loss of generality that σ_{12} is the $4-$cycle $\sigma = (1234)$. By Proposition 3.3, we may assume the entries $L_{j1} = j$. After filling in the rest of the square by imposing the condition of it being Latin, we see $L \sim L_{(1234)}$. By Theorem 3.2 and Corollary 2.5, we know that L must not have an orthogonal mate, contradicting our assumption that $(L, L') \in OLS_4$. We conclude that each of σ_{12}, σ_{23}, and σ_{34} are the permutations τ_1, τ_2, and τ_3.

By considering the distinct permutations σ_{12}, σ_{13}, and σ_{14} instead (which must collectively be, in some order, the permutations τ_1, τ_2 and τ_3), we see that up to a change of columns that $L \sim L_1$. It is an easy exercise to show that there are two possibilities for orthogonal mates: (L_1, L_2) and $(L_1, L_3) \in OLS_4$. Assertion 3 now follows by Assertion 2. $\square$

We conclude our study of orthogonal Latin squares with the following result.

THEOREM 3.6. — *Let $L \in \mathcal{L}_5$.*

1. *If $\sigma_{12} = (12)(345)$, then there do not exist 5 disjoint transversals.*
2. *If L has an orthogonal mate, then $L \sim L_{(12345)}$.*
3. *The possible orthogonal mates to $L_{(12345)}$ are $L_{(15432)}, L_{(14253)}$, and $L_{(13524)}$.*
4. *The set $OLS_5/\sim'$ contains at most two elements: $[(L_{(12345)}, L_{(15432)})]'$ and $[(L_{(12345)}, L_{(14253)})]'$.*

Proof. — Assertion 1 follows from a case-by-case check that there do not exist 5 disjoint transversals in each of the possibilities for L where $\sigma_{12}^L = (12)(345)$. For all cases, we arrange the rows of L to read downward as $1, 2, 3, 4, 5$, and rearrange the final 3 columns of L so that the first row reads to the right as $1, 2, 3, 4, 5$ (the $(1, 2)$ position of this square is determined as 2 since $\sigma_{12}^L(1) = 2$). These adjustments shift representatives within the equivalence class of L, but do not change the permutation changing column 1 to column 2. Note that the second column now reads downward as $2, 1, 4, 5, 3$. In what follows to prove Assertion (1), we simply investigate each case and show that there is can not be a 5 disjoint transversals.

There are 4 possibilities for L subject to the conditions outlined above, and each is completely determined by specifying a certain value in each of two locations in addition to the conditions above. Firs let's examine the case where L is the Latin square with $\sigma_{12}^L = (12)(345)$ and has entries $(L)_{23} = 4$, and $(L)_{43} = 1$. This uniquely determines L:

1	2	3	4	5
2	1	4	5	3
3	4	5	2	1
4	5	1	3	2
5	3	2	1	4

Now we show that there does not exist a transversal for this L. We start with the $(1, 1)$ entry and try to build a transversal that includes it. In doing so there are three cases that are determined by either choosing 3,4, or 5 as the element of the transversal in the second row. Then this choice determines the remainder of the entries except that the last entry fails to make a transversal. We illustrate this with in the following pictures by circling the entries which are determined and would be part of the transversal and then boxing the last entry which breaks the transversal.

(1)	2	3	4	5
2	1	(4)	5	3
3	4	5	(2)	1
4	5	1	3	[2]
5	(3)	2	1	4

(1)	2	3	4	5
2	1	4	(5)	3
3	(4)	5	2	1
4	5	1	3	(2)
5	3	[2]	1	4

The two cases where $(L)_{23} = 4$, and $(L)_{43} = 2$, and $(L)_{23} = 5$, and $(L)_{43} = 1$ have exactly the same arguments. The case where $(L)_{23} = 5$, and $(L)_{43} = 2$ is more interesting. In this case there are three different transversals that all contain the 1 upper left entry. With each of these transversals there are two possibilities for a second disjoint transversal. However in all of these cases a second transversal does not exist. We illustrate this again with a diagram where the first transversal is circled and the second attempt at a transversal is boxed. Then we put an "X" over the entry or entries where the second transversal fails. (In the diagram below the cases with the same 1st circled transversal are stacked on top of each other.)

(1)	[2]	3	4	5
2	1	[5]	3	(4)
[3]	4	1	(5)	2
4	5	(2)	[1]	3
5	(3)	4	2	~~1~~

(1)	[2]	3	4	5
2	1	(5)	[3]	4
3	(4)	[1]	5	2
[4]	5	2	1	(3)
5	3	4	(2)	~~1~~

(1)	[2]	3	4	5
2	1	[5]	(3)	4
[3]	4	1	5	(2)
4	(5)	2	[1]	3
5	3	(4)	2	~~1~~

(1)	[2]	3	4	5
2	1	5	[3]	(4)
3	4	[1]	(5)	2
4	5	(2)	1	~~3~~
5	(3)	4	2	~~1~~

(1)	[2]	3	4	5
2	1	(5)	3	[4]
[3]	(4)	1	5	2
4	5	2	[1]	(3)
5	3	~~4~~	(2)	1

(1)	[2]	3	4	5
2	1	5	(3)	[4]
3	4	~~1~~	5	(2)
4	(5)	2	[1]	3
[5]	3	(4)	2	1

For Assertion (2), we note that by Proposition 3.3, we need to only consider two situations: either there is one permutation among $\sigma_{i,i+1}$ that is not a $5-$cycle (and therefore has multi-index $(2,3)$ since otherwise such a permutation would not be fixed-point free), or all permutations are $5-$cycles. The assumption that L has an orthogonal mate excludes the first possibility since we could then assume via Proposition 3.3 that $\sigma_{12} = (12)(345)$, and Assertion (1) violates the condition that there exists an orthogonal mate. Therefore the only such square to fit our hypotheses can be arranged to have $\sigma_{12} = (12345)$, and $1, 2, 3, 4, 5$ both down the first column (by rearranging rows) and across the first row to the right (by rearranging columns 3, 4, and

5). To complete the square, we find that the $(5, 3)$ position is either 2 or 4. In the event it is 2, this square is nothing other than $L_{(12345)}$ and Assertion (2) is complete. There are 5 different Latin squares where the $(5, 3)$ position is 4.

1	2	3	4	5
2	3	5	1	4
3	4	1	5	2
4	5	2	3	1
5	1	4	2	3

1	2	3	4	5
2	3	5	1	4
3	4	2	5	1
4	5	1	3	2
5	1	4	2	3

1	2	3	4	5
2	3	1	5	4
3	4	5	1	2
4	5	2	3	1
5	1	4	2	3

1	2	3	4	5
2	3	1	5	4
3	4	5	2	1
4	5	2	1	3
5	1	4	3	2

1	2	3	4	5
2	3	5	1	4
3	4	2	5	1
4	5	1	2	3
5	1	4	3	2

In each case there is one permutation of successive columns that has index $(2, 3)$ which falls outside our hypotheses.

Assertion (3) now follows since orthogonality is symmetric, and by using the above argument one reveals that any Latin square of size 5 that has only 5−cycles between consecutive columns is equivalent to L_σ for a 5−cycle σ.

To prove the last assertion, we simply note that by permuting columns in the pair $(L_{(12345)}, L_{(14253)})$ and relabeling each square separately, we see that

$$(L_{(12345)}, L_{(14253)}) \sim' (L_{(12345)}, L_{(13524)}).$$

This completes the proof of Assertion 4. □

Remark 3.7. — Let $L = L(\sigma_{12}, \ldots, \sigma_{k-1,k}) \in \mathcal{L}_k$. We have proved in this section that if $k = 3, 4$, or 5, and if L is to have an orthogonal mate, then each of the associated permutations to L must be even (i.e., expressible as a product of an even number of transpositions). However, for higher k there are examples (of size $k = 10$, note below that $10 = 24$) where (at least) one of the associated permutations is odd and it has an orthogonal mate (see [8]). There are also examples where all the permutations are even, but there is no orthogonal mate. It would be interesting to know how the parity of the σ_{ij} effects the existence of an orthogonal mate. We conjecture the following: If $(L, L') \in OLS_k$ and $k \neq 24$, then every permutation σ_{ij}^L and $\sigma_{ij}^{L'}$ is even.

4. Realization Spaces

As discussed in the introduction, there is a relationship between $(4,k)$-nets and $OLS_k/\sim'$. Given a $(4,k)$-net, one can construct a unique element of $OLS_k/\sim'$ that represents the underlying combinatorial structure. Nets in $\mathbb{CP}^2$ that are projectively isomorphic will produce equivalent combinatorial structures. Conversely, however, given an element of $OLS_k/\sim'$ there need not exist a $(4,k)$-net in $\mathbb{CP}^2$ with the given structure.

Now we construct a variety that represents possible net realizations in $\mathbb{CP}^2$ of a pair of orthogonal Latin squares (see [4]). A pair of orthogonal Latin squares (L_1, L_2) combinatorially defines the points of χ for the combinatorial structure of a $(4,k)$-net. Let $M_{(L_1,L_2)}$ be a $4k\times 3$ matrix of complex numbers, defined by 4 blocks of k rows where the i^{th} row in the j^{th} block is $a_{j,i}\ b_{j,i}\ c_{j,i}$. Let the rows of $M_{(L_1,L_2)}$ be the coefficients of the linear forms defining the lines of the alleged $(4,k)$-net in $\mathbb{CP}^2$. Then for each point of χ, the corresponding $\binom{4}{3} = 4$ minors of $M(L_1,L_2)$ should be zero. For example, if $(L_1)_{ij} = s$ and $(L_2)_{ij} = t$ then the $\binom{4}{3}$ minors are given by choosing 3 of the rows following four: $(a_{1,i}, b_{1,i}, c_{1,i})$, $(a_{2,j}, b_{2,j}, c_{2,j})$, $(a_{3,s}, b_{3,s}, c_{3,s})$, and $(a_{4,t}, b_{4,t}, c_{4,t})$. Thus, the variety of representations of the pair of orthogonal Latin squares (L_1, L_2) is the variety defined by the vanishing of all of the minors associated to χ; we denote this ideal of minors $I(L_1, L_2)$ and the associated variety $R(L_1, L_2)$. Each line of the net to be realized is labeled by a distinct element of $\{1, \ldots, 4k\}$; hence the points of χ are given by 4-tuples of distinct elements of this set. In each case we use the lexicographic ordering of the 4-tuples to compute the minors consecutively. The next proposition shows that we only need to consider one representative of each equivalence class. This proposition is a consequence of the fact that the relations (R1)-(R6) preserve the ideal $I(L_1, L_2)$.

Proposition 4.1. — *If $(L_1, L_2) \sim' (L_1', L_2')$ then $R(L_1, L_2)$ and $R(L_1', L_2')$ are isomorphic as varieties.*

Remark 4.2. — Note that $R(L_1, L_2)$ is not necessarily the realization space a of matroid. It is missing extra conditions that might occur between the elements of a single class in the net. In [11] Kawahara discusses this non-trivial point from the motivation of non-vanishing cohomology in the Orlik-SOlomon algebra.

Remark 4.3. — Using Theorem 3.2(iii) in [21] we can assume that

$$B = \begin{bmatrix} a_{1,1} & b_{1,1} & c_{1,1} \\ a_{1,2} & b_{1,2} & c_{1,2} \\ a_{1,3} & b_{1,3} & c_{1,3} \\ a_{2,1} & b_{2,1} & c_{2,1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

4.1. Realization of (4,3)-nets/Proof of Theorem 1.3 (1)

Theorem 3.4 shows that the only combinatorial structure possible for a (4,3)-net is given by the pair $(L_{(123)}, L_{(132)}) \in OLS_3/\sim'$. By computing minors, we conclude

$$M_{(L_{(123)},L_{(132)})} = \begin{bmatrix} & B & \\ 1 & \omega^i & \omega^j \end{bmatrix}$$

where ω is a (primitive) root of x^3-1 and i and j range through the set $\{0,1,2\}$. This is the Hessian configuration, see Example 6.29 of [13] and Example 3.6 of [21]. This proves Theorem 1.4 for $k=3$. □

4.2. Realization of (4,4)-nets/Proof of Theorem 1.3 (2)

By Theorem 3.5 and Proposition 4.1, there is only one combinatorial structure for a (4,4)-net. We prove Theorem 1.4 in the case $k=4$ by attempting to realize the combinatorial structure given by $(L_1, L_2) \in OLS_4/\sim'$, where L_1 and L_2 are the squares given in Theorem 3.5. In this case $|\chi| = 16$. Using the first 13 points of χ given by the pair L_1 and L_2 we get that

$$M_{(L_1,L_2)} = \begin{bmatrix} & B & \\ r-t^{-1} & 1+r & 1-r \\ t & -t & 1 \\ t & 1 & -1 \\ -t^{-1} & 1 & t^{-1} \\ t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -1 \\ -1 & 1 & t^{-1} \\ -t^{-1} & 1 & 1 \\ -1 & -t & 1 \\ 1 & -1 & 1 \\ 1 & 1 & t^{-1} \end{bmatrix}.$$

Then using the last three points of χ we find that $r = \frac{1}{4}(1-t^{-2})$, $t = -2 \pm \sqrt{5}$, and $t^2+3=0$. This system of equations has no solution, so $R(L_1, L_2) = \emptyset$ proving Theorem 1.4 for $k=4$: there do not exist any (4,4)-nets in $\mathbb{CP}^2$. □

4.3. Realization of (4,5)-nets/Proof of Theorem 1.3 (3)

By Theorem 3.6 and Proposition 4.1, there are at most two possible combinatorial structures for a (4,5)-net. In this case $|\chi| = 25$. First, we consider the pair $(L_{(12345)}, L_{(15432)})$ as our combinatorial structure. Using only the first 17 points of χ, we find that for some number t, there is a line in class 2 corresponding to the line defined by the linear form $tx + y + t^2 z = 0$, and a line in class 3 that corresponds to the line defined by the linear form $x + y + tz = 0$. Then using one more point of χ we get that $t = 0$ or 1. This cannot happen, in both cases a line repeats. Hence the realization space is empty and there does not exist any $(4, 5)$-net in $\mathbb{CP}^2$ with Latin squares $(L_{(12345)}, L_{(15432)})$.

Now, we compute the realization space for the pair of Latin squares $(L_{(12345)}, L_{(14253)})$. Using the first 22 points of χ we compute that

$$M_{(L_{(12345)}, L_{(14253)})} = \begin{bmatrix} & B & \\ 1 - qt^{-3} & 1 - q & 1 - qt^{-1} \\ a & b & c \\ t^{-1} & t^{-3} & 1 \\ t^{-1} & 1 & t \\ t & 1 & t^{-1} \\ t & t^3 & 1 \\ t & 1 & 1 \\ 1 & t^{-3} & 1 \\ 1 & 1 & t \\ t^{-3} & 1 & t^{-1} \\ t^2 & t^3 & 1 \\ t^{-1} & 1 & 1 \\ t^{-2} & t^{-3} & 1 \\ t^3 & 1 & t \\ 1 & 1 & t^{-1} \\ 1 & t^3 & 1 \end{bmatrix}$$

where $t^5 - 1 = 0$ and $q = \frac{1+t^3}{2+t^3-t^2}$. Then using one more point of χ we get that $t^3 - 1 = 0$. Hence, $R(L_{(12345)}, L_{(14253)}) = \emptyset$ and there does not exist any $(4, 5)$-nets in $\mathbb{CP}^2$. This concludes the proof of Theorem 1.4. □

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