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On the Configuration Spaces of Grassmannian Manifolds

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Abstract. — Let $\mathcal{F}_h^i(k,n)$ be the i-th ordered configuration space of all distinct points $H_1,\dots,H_h$ in the Grassmannian $Gr(k,n)$ of k-dimensional subspaces of $\mathbb{C}^n$, whose sum is a subspace of dimension i. We prove that $\mathcal{F}_h^i(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ of dimension $i(n-i)+hk(i-k)$ and its fundamental group is trivial if $i=min(n,hk)$, $hk\neq n$ and $n>2$ and equal to the braid group of the sphere $\mathbb{C}P^1$ if $n=2$. Eventually we compute the fundamental group in the special case of hyperplane arrangements, i.e. $k=n-1$.

Résumé. — Soit $\mathcal{F}_h^i(k,n)$ le i-ème espace de configuration ordonnée de tous les points distincts $H_1,\dots,H_h$ dans la Grassmannienne $Gr(k,n)$ de sous-espaces de dimension k de $\mathbb{C}^n$, dont la somme est un sous-espace de dimension i. Nous prouvons que $\mathcal{F}_h^i(k,n)$ est (si non vide) une sous-variété complexe de $Gr(k,n)^h$ de dimension $i(n-i)+hk(i-k)$ et que son groupe fondamental est trivial si $i=min(n,hk)$, $hk\neq n$ et $n>2$ et égal au groupe de tresses de la sphère $\mathbb{C}P^1$ si $n=2$. Finalement, nous calculons le groupe fondamental dans le cas particulier des arrangements d'hyperplans, c'est-à-dire $k=n-1$.

1. Introduction

Let M be a manifold. The *ordered configuration space*

$$\mathcal{F}_h(M)=\{(x_1,\dots,x_h)\in M^h | x_i\neq x_j,\ i\neq j\}$$

of h distinct points in M has been widely studied after it has been introduced by Fadell and Neuwirth [5] and Fadell [3] in the sixties. It is well known that

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for a simply connected manifold M of dimension greater or equal than 3, the *pure braid group* $\pi_1(\mathcal{F}_h(M))$ on h strings of M is trivial. This is not the case when the dimension of M is lower than 3 as, for example, the pure braid group of the sphere $S^2 \approx \mathbb{C}P^1$ with presentation:

$$\pi_1(\mathcal{F}_h(\mathbb{C}P^1)) \cong \left\langle \alpha_{ij}, 1 \leqslant i < j \leqslant h-1 \,\middle|\, (YB\,3)_{h-1}, (YB\,4)_{h-1}, D_{h-1}^2 = 1 \right\rangle$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})(\alpha_{14}\alpha_{24}\alpha_{34})\cdots(\alpha_{1k}\alpha_{2k}\cdots\alpha_{k-1\ k})$ and $(YB\,3)_n$ and $(YB\,4)_n$ are the Yang-Baxter relations (see [2] and [4]):

$$\begin{array}{ll} (YB\,3)_n: & \alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{ik}\alpha_{jk}\alpha_{ij} = \alpha_{jk}\alpha_{ij}\alpha_{ik},\ 1 \leqslant i < j < k \leqslant n, \\ (YB\,4)_n: & [\alpha_{kl}, \alpha_{ij}] = [\alpha_{il}, \alpha_{jk}] = [\alpha_{jl}, \alpha_{jk}^{-1}\alpha_{ik}\alpha_{jk}] = [\alpha_{jl}, \alpha_{kl}\alpha_{ik}\alpha_{kl}^{-1}] = 1, \\ & 1 \leqslant i < j < k < l \leqslant n. \end{array}$$

In a recent paper ([1]) Berceanu and Parveen introduced new configuration spaces. They stratify the classical configuration spaces $\mathcal{F}_h(\mathbb{CP}^n)$ with complex submanifolds $\mathcal{F}_h^i(\mathbb{CP}^n)$ defined as the ordered configuration spaces of all h points in $\mathbb{CP}^n$ generating a projective subspace of dimension i. They prove that the fundamental groups $\pi_1(\mathcal{F}_h^i(\mathbb{CP}^n))$ of these submanifolds are trivial except when $i = 1$ providing, in this last case, a presentation similar to those of the pure braid group of the sphere.

In a subsequent paper ([6]), authors apply similar techniques to the affine case, that is to the ordered configuration space $\mathcal{F}_h^{i,n} = \mathcal{F}_h^i(\mathbb{C}^n)$ of all h points in $\mathbb{C}^n$ generating an affine subspace of dimension i. They prove that the spaces $\mathcal{F}_h^{i,n}$ are simply connected except for $i = 1$ or $i = n = h-1$ and, in the last cases, they provide a presentation of the fundamental groups $\pi_1(\mathcal{F}_h^{i,n})$.

In this paper we generalize the result in [1] to the Grassmannian manifold $Gr(k,n)$ parametrizing k-dimensional subspaces of $\mathbb{C}^n$. We define the i-th ordered configuration space $\mathcal{F}_h^i(k,n)$ as the ordered configuration space of all distinct points $H_1, \ldots, H_h$ in the Grassmannian $Gr(k,n)$ such that the sum $(H_1 + \cdots + H_h)$ is an i-dimensional space.

We prove that the i-th ordered configuration space $\mathcal{F}_h^i(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ and we compute its dimension.

As a corollary, we prove that if $n \neq hk$ and $i = \min(n, hk)$ then the i-th ordered configuration space $\mathcal{F}_h^i(k,n)$ has trivial fundamental group except when $n = 2$, that is:

$$\begin{array}{ll} \pi_1(\mathcal{F}_h^{min(n,hk)}(k,n)) = 0 \quad \text{if } (k,n) \neq (1,2) & (1.1) \\ \pi_1(\mathcal{F}_1^1(1,2)) = \pi_1(\mathcal{F}_2(\mathbb{CP}^1)). & \end{array}$$

As a consequence, the fundamental group of the i-th ordered configuration space $\mathcal{F}_h^i(n-1,n)$ of hyperplane arrangements of cardinality h vanishes except when $n=2$.

Using a dual argument, we also get that the fundamental group of the ordered configuration space of all distinct k-dimensional subspaces $H_1,\ldots,H_h$ in $\mathbb{C}^n$ such that the intersection $(H_1\cap\cdots\cap H_h)$ is an i-dimensional subspace is a simply connected manifolds when $i=\max(0,n-hk)$, except when $n=2$.

We conjecture that similar results to that obtained in [1] for projective spaces holds also for Grassmannian manifolds and the fundamental group of the i-th ordered configuration space $\mathcal{F}_h^i(k,n)$ vanishes except for low values of i. This will be the object of forthcoming publications.

2. Main Section

Let $Gr(k,n)$ be the Grassmannian manifold parametrizing k-dimensional subspaces of the n-dimensional complex space $\mathbb{C}^n$, $0<k<n$, and $\mathcal{F}_h(Gr(k,n))$ be its ordered configuration spaces.

2.1. The spaces $\mathcal{F}_h^i(k,n)$

Let's define the i-th ordered configuration space $\mathcal{F}_h^i(k,n)$ as the space of all distinct points $H_1,\ldots,H_h$ in the Grassmannian $Gr(k,n)$ whose sum is an i-dimensional subspace of $\mathbb{C}^n$, i.e.

$$\mathcal{F}_h^i(k,n)=\{(H_1,\ldots,H_h)\in\mathcal{F}_h(Gr(k,n))\mid \dim(H_1+\cdots+H_h)=i\}.$$

It is easy to see that the following results hold:

1. if $h=1$ then $\mathcal{F}_1^i(k,n)$ is empty unless $i=k$, in which case $\mathcal{F}_1^k(k,n)=Gr(k,n)$;

2. if $i=1$ then $\mathcal{F}_1^i(k,n)$ is empty unless $k=h=1$ and we get $\mathcal{F}_1^1(1,n)=Gr(1,n)=\mathbb{CP}^{n-1}$;

3. for $h\geqslant 2$, $\mathcal{F}_h^i(k,n)\neq\emptyset$ if and only if $i\geqslant k+1$ and $i\leqslant\min(hk,n)$;

4. for $i=hk\leqslant n$, then the h subspaces giving a point of $\mathcal{F}_h^{hk}(k,n)$ form a direct sum;

5. for $h\geqslant 2$, $\mathcal{F}_h(Gr(k,n))=\coprod_{i=2}^{n}\mathcal{F}_h^i(k,n)$;

6. for $h \geqslant 2$, the adjacency of the strata is given by

$$\overline{\mathcal{F}_h^i(k,n)} = \mathcal{F}_h^i(k,n) \coprod \mathcal{F}_h^{i-1}(k,n) \coprod \dots \coprod \mathcal{F}_h^2(k,n).$$

By above remarks, it follows that the case $h = 1$ is trivial, hence from now on, we will consider $h > 1$ (and hence $i > k$).

We want to show that $\mathcal{F}_h^i(k,n)$ is (when non empty) a complex submanifold of $Gr(k,n)^h$ and compute its dimension. We need to briefly recall few easy facts and introduce some notations.

2.2. The determinantal variety

Let's recall that the determinantal variety $D_r(m,m')$ is the variety of $m \times m'$ matrices with complex entries of rank less than or equal to $r \leqslant \min(m,m')$. It is an analytic (algebraic, in fact) variety of dimension $r(m+m'-r)$ whose set of singular points is given by those matrices of rank less than r. From now on, $D_r(m,m')^*$ will denote the set of non-singular points of the determinantal variety $D_r(m,m')$, that is the set of $m \times m'$ matrices of rank equal to r.

2.3. A system of local coordinates for $Gr(k,n)^h$

Let $V_0 \subset \mathbb{C}^n$ be a subspace of dimension $\dim V_0 = n-k$, then the set

$$U_{V_0} = \{H \in Gr(k,n) \mid H \oplus V_0 = \mathbb{C}^n\}$$

is an open dense subset of $Gr(k,n)$.

Let $B = \{w_1,\dots,w_k,v_1,\dots,v_{n-k}\}$ be a basis of $\mathbb{C}^n$ such that $\{v_1,\dots,v_{n-k}\}$ is a basis of V_0. We get a (complex) coordinate system on U_{V_0} as follows.

Let H be an element in U_{V_0}, then the affine subspace $V_0 + w_j$ intersects H in one point u_j for any $j = 1,\cdots,k$ and $\{u_1,\cdots,u_k\}$ form a basis of H. Hence H is uniquely determined by a $n \times k$ matrix of the form $\begin{pmatrix} I \\ A \end{pmatrix}$, where I is the $k \times k$ identity matrix and A is the $(n-k) \times k$ matrix of the coordinates of $u_1 - w_1,\dots,u_k - w_k$ with respect to vectors $\{v_1,\dots,v_{n-k}\}$. The coefficients of A give complex coordinates in $U_{V_0} \cong \mathbb{C}^{k(n-k)}$.

Let $(H_1,\dots,H_h)$ be a point in $Gr(k,n)^h$, the open sets $U_{H_1},\dots,U_{H_h}$ in the Grassmannian manifold $Gr(n-k,n)$ have non empty intersection, that is there exists an element $V_0 \in Gr(n-k,n)$ such that $V_0 \oplus H_j = \mathbb{C}^n$ for all

$j = 1, \ldots, h$. Thus, $Gr(k,n)^h$ is covered by the open sets $U^h_{V_0}$ as V_0 varies in $Gr(n-k,n)$. Taking a basis as defined above, each element in $U^h_{V_0}$ is uniquely determined by a $n \times hk$ matrix of the form $\begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix}$ and the coefficients of $\begin{pmatrix} A_1 & A_2 & \cdots & A_h \end{pmatrix}$ give complex coordinates in $U^h_{V_0} \cong \mathbb{C}^{hk(n-k)}$.

2.4. A system of local coordinates for $\mathcal{F}^i_h(k,n)$

In terms of the above coordinates, $(H_1, \ldots, H_h) \in U^h_{V_0}$ belongs to $\mathcal{F}^i_h(k,n)$ if and only if $A_j \neq A_l$ when $j \neq l$ and $\mathrm{rank}\begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix} = i$. Let us remark that

$$\begin{aligned} \mathrm{rank}\begin{pmatrix} I & I & \cdots & I \\ A_1 & A_2 & \cdots & A_h \end{pmatrix} &= \mathrm{rank}\begin{pmatrix} I & I & \cdots & I \\ 0 & A_2 - A_1 & \cdots & A_h - A_1 \end{pmatrix} \\ &= k + \mathrm{rank}\begin{pmatrix} A_2 - A_1 & \cdots & A_h - A_1 \end{pmatrix}. \end{aligned}$$

Then the coefficients of $B_j = A_j - A_1$ are new coordinates, in which the intersection $U_{V_0} \cap \mathcal{F}^i_h(k,n)$ corresponds, in $\mathbb{C}^{hk(n-k)}$, to the product $\mathbb{C}^{k(n-k)} \times D_{i-k}(n-k, hk-k)^*$ minus the closed sets given by $B_j = 0$ for $2 \leqslant j \leqslant h$ and by $B_j = B_l$ for $2 \leqslant j, l \leqslant h$, $j \neq l$. We get the following theorem.

THEOREM 2.1. — *The i-th ordered configuration space $\mathcal{F}^i_h(k,n)$ is a complex submanifold of the Grassmannian manifold $Gr(k,n)$ of dimension*

$$d^i_h(k,n) = i(n-i) + hk(i-k). \tag{2.2}$$

Equation (2.2) is an easy consequence of the equality:

$$k(n-k) + (i-k)(n-k+hk-k-(i-k)) = i(n-i) + hk(i-k).$$

Let us remark that the dimension $d^i_h(k,n)$ attains its maximum $hk(n-k)$ if and only if $i = n$ or $i = hk$. Hence $d^i_h(k,n)$ is a strictly increasing function of i when $i \leqslant \min(n, hk)$.

2.5. The fundamental group of $\mathcal{F}^{\min(n,hk)}_h(k,n)$

The space $\mathcal{F}^{min(n,hk)}_h(k,n)$ is an open subset of the ordered configuration space $\mathcal{F}_h(Gr(k,n))$ and all other (non void) $\mathcal{F}^j_h(k,n)$ have strictly lower dimension. Moreover, if $i = n$ the difference of dimensions $d^i_h(k,n) - d^{i-1}_h(k,n)$

equals $1+hk-n$ and if $i=hk$ it equals $1+n-hk$. Then if $n\neq hk$, all (non void) $\mathcal{F}_h^j(k,n)$ with $j<$min(n,hk) have real codimension at least 4 in $\mathcal{F}_h(Gr(k,n))$. Then, if $n\neq hk$ and $i=$min(n,hk), the fundamental group of $\mathcal{F}_h^i(k,n)=\mathcal{F}_h(Gr(k,n))\setminus\overline{\mathcal{F}_h^{i-1}(k,n)}$ is the same as the fundamental group of $\mathcal{F}_h(Gr(k,n))$ (since, by the adjacency of the strata, the closure $\overline{\mathcal{F}_h^{i-1}(k,n)}$ is the finite union of complex subvarieties of $\mathcal{F}_h(Gr(k,n))$ of real codimension at least 4).

Let us recall that the complex Grassmannian manifolds $Gr(k,n)$ are simply connected and have real dimension at least 4 except $Gr(1,2)=\mathbb{CP}^1$ and that for a simply connected manifold of real dimension at least 3 the pure braid groups vanish, i.e. $\pi_1(\mathcal{F}_h(Gr(k,n)))=0$ if $(k,n)\neq(1,2)$. We get the following corollary.

Corollary 2.2. — *The fundamental group of the i-th ordered configuration space $\mathcal{F}_h^i(k,n)$ vanishes if $n\neq hk$ and $i=\min(n,hk)$ except when $n=2$ in which it is the pure braid group of the sphere.*

2.6. The dual case

Let $Gr(k,n)^*$ be the Grassmannian manifold parametrizing k-dimensional subspaces in the dual space $(\mathbb{C}^n)^*$. Then we can define the i-th dual ordered configuration space $\mathcal{F}_h^i(k,n)^*$ as

$$\mathcal{F}_h^i(k,n)^*=\{(H_1,\ldots,H_h)\in\mathcal{F}_h(Gr(k,n)^*)\mid \dim(H_1\cap\cdots\cap H_h)=i\}.$$

The spaces $\mathcal{F}_h^i(k,n)^*$ stratify the ordered configuration space $\mathcal{F}_h(Gr(k,n)^*)$ of the Grassmannian manifold $Gr(k,n)^*$.
The annihilators define homeomorphisms Ann: $Gr(n-k,n)\to Gr(k,n)^*$ which induce homeomorphisms between the $(n-i)$th ordered configuration space $F_h^{n-i}(n-k,n)$ and the i-th dual ordered configuration space $F_h^i(k,n)^*$. As a consequence the spaces $F_h^{\max(0,n-hk)}(n-k,n)^*$ are simply connected manifolds except when $n=2$. In this case the fundamental group is the pure braid group of the sphere.

2.7. i-th ordered configuration spaces of hyperplane arrangements

If $k=n-1$ points in the ordered configuration space $\mathcal{F}_h(Gr(n-1,n))$ are h-uple of hyperplanes in $\mathbb{C}^n$, i.e. ordered arrangements of hyperplanes. In this case, $h=1$ implies $i=n-1$ and the i-th ordered configuration space is the Grassmannian manifold, i.e. $\mathcal{F}_1^{n-1}(n-1,n)=Gr(n-1,n)$. While $h>1$ implies $i=n$, since the sum of two (different) hyperplanes is the whole space $\mathbb{C}^n$, and the following equalities hold

$$\mathcal{F}_h^n(n-1,n) = \mathcal{F}_h(Gr(n-1,n)) = \mathcal{F}_h(\mathbb{CP}^{n-1}).$$

Hence, the fundamental group of the i-th ordered configuration space of hyperplane arrangements $\mathcal{F}_h^i(n-1,n)$ vanishes except when $n=2$. In this case it is the fundamental group of the sphere $\mathbb{CP}^1$.
In the dual case there are homeomorphisms $\mathcal{F}_h^i(n-1,n)^* \cong \mathcal{F}_h^{n-i}(1,n)$ and fundamental groups of $\mathcal{F}_h^{n-i}(1,n)$ are zero except if $i = n-1$ (see [1]). Hence the space of h-uples of distinct hyperplanes in $\mathbb{C}^n$ whose intersection has dimension equal to i is simply connected except if $i = n-1$.

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