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Effective bounds for Faltings's delta function

Jay Jorgenson[(1)], Jürg Kramer[(2)]

Dedicated to Christophe Soulé at his sixtieth birthday

Abstract. — In his seminal paper on arithmetic surfaces Faltings introduced a new invariant associated to compact Riemann surfaces X, nowadays called Faltings's delta function and here denoted by $\delta_{\mathrm{Fal}}(X)$. For a given compact Riemann surface X of genus $g_X = g$, the invariant $\delta_{\mathrm{Fal}}(X)$ is roughly given as minus the logarithm of the distance with respect to the Weil-Petersson metric of the point in the moduli space $\mathcal{M}_g$ of genus g curves determined by X to its boundary $\partial\mathcal{M}_g$. In this paper we begin by revisiting a formula derived in [14], which gives $\delta_{\mathrm{Fal}}(X)$ in purely hyperbolic terms provided that $g > 1$. This formula then enables us to deduce *effective* bounds for $\delta_{\mathrm{Fal}}(X)$ in terms of the smallest non-zero eigenvalue of the hyperbolic Laplacian acting on smooth functions on X as well as the length of the shortest closed geodesic on X. The article ends with a discussion of an application of our results to Parshin's covering construction.

Résumé. — Dans son article fondateur sur les surfaces arithmétiques Faltings a introduit un nouvel invariant des surfaces de Riemann compactes, que l'on appelle de nos jours l'invariant delta de Faltings et que l'on note $\delta_{\mathrm{Fal}}(\cdot)$. Pour une surface de Riemann compacte X de genre $g_X = g$, l'invariant $\delta_{\mathrm{Fal}}(X)$ est donné à peu de choses près par l'opposé du logarithme de la distance, pour la métrique de Weil-Petersson, du point sur l'espace de modules $\mathcal{M}_g$ des courbes de genre g déterminé par X à

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son bord $\partial\mathcal{M}_g$. Dans le présent article nous commençons par un nouvel examen de la formule obtenue dans [14], qui décrit $\delta_{\mathrm{Fal}}(X)$ en termes purement hyperboliques, tout au moins si $g > 1$. Cette formule nous permet ensuite de déduire des bornes effectives pour $\delta_{\mathrm{Fal}}(X)$ en termes de la plus petite valeur propre non-nulle du Laplacien hyperbolique agissant sur les fonctions lisses sur X et du minimum des longueurs des géodésiques fermées sur X. L'article se termine par une discussion d'une application de nos résultats à la construction du recouvrement de Parshin.

1. Introduction

In [1] and [2], S. J. Arakelov introduced Green's functions on compact Riemann surfaces in order to define an intersection theory on arithmetic surfaces, thus initiating a far-reaching mathematical program which bears his name. G. Faltings extended the pioneering work of Arakelov in [8] by defining metrics on determinant line bundles arising from the cohomology of algebraic curves, from which he derived arithmetic versions of the Riemann-Roch theorem, Noether's formula, and the Hodge index theorem. Although Faltings employed the classical Riemann theta function to define metrics on these determinant line bundles, he does refer to the emerging idea of D. Quillen to use Ray–Singer analytic torsion to define the metrics on these determinant line bundles as being "more direct". One of the many aspects of the mathematical legacy of Christophe Soulé is the central role he played in developing higher dimensional Arakelov theory where Quillen metrics are fully utilized; see [22] and the references therein.

From Faltings's theory [8], there appears a naturally defined analytic quantity associated to any compact Riemann surface X. The new invariant in [8] became known as Faltings's delta function, which we denote by $\delta_{\mathrm{Fal}}(X)$. Many of the fundamental arithmetic theorems and formulas in [8], such as those listed above, amount to statements which involve $\delta_{\mathrm{Fal}}(X)$. By comparing the Riemann-Roch theorem from [8] and the arithmetic Riemann-Roch theorem, Soulé expressed in [21] the Faltings's delta function in terms of the analytic torsion of the trivial line bundle on X when given the Arakelov metric; see equation (3.1) below, as well as [21] and, more recently, [24].

The Polyakov formula allows one to relate values of the analytic torsion for conformally equivalent metrics. As a result, one can use Soulé's formula for the Faltings's delta function and obtain an identity which expresses $\delta_{\mathrm{Fal}}(X)$ in terms of the hyperbolic geometry of X; see Theorem 3.2 below, which comes from [14]. In [14], we used the relation between the Faltings's delta function and the hyperbolic geometry in order to study $\delta_{\mathrm{Fal}}(X)$

through covers. As an arithmetic application of the analytic bounds obtained for $\delta_{\mathrm{Fal}}(X)$, we derived in [14] an improved estimate for the Faltings height of the Jacobian of the modular curve $X_0(N)$ for square-free N which is not divisible by 6.

After the completion of [14], A. N. Parshin posed the following question to the second named author: Can one derive an *effective* bound for the Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ in terms of basic information associated to the hyperbolic geometry of X? The purpose of the present article is to provide an affirmative answer to Parshin's question. More specifically, our main results, given in Theorem 6.1 and Corollaries 6.3, 6.4, explicitly bound $\delta_{\mathrm{Fal}}(X)$ in the case when X is a finite degree covering of a compact Riemann surface X_0 of genus bigger than 1, where the bound for $\delta_{\mathrm{Fal}}(X)$ is effectively computable once knowing the genera of X_0 and X, the smallest non-zero eigenvalues of the hyperbolic Laplacian acting on X_0 and X, and the length of the shortest closed geodesic on X_0 (as well as some ramification data in case the covering is ramified).

An important ingredient in the analysis of the present paper is the algorithm from [9], which provides effective means by which one can bound the Huber constant on X, a quantity associated to the error term in the prime geodesic theorem; see [9] and the references therein. As with the main result in [9], it is possible that the effective bound we obtain here may not be optimal, perhaps even far from it. However, the existence of an effective bound for the Faltings's delta function $\delta_{\mathrm{Fal}}(X)$, albeit a sub-optimal bound, may be a tool by which one can further investigate the application of Arakelov theory to diophantine problems, as originally intended.

The paper is organized as follows: After recalling basic notations in section 2, we express Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ in hyperbolic terms of X in section 3. Section 4 is devoted to derive effective bounds for the ratio $\mu_{\mathrm{can}}(z)/\mu_{\mathrm{hyp}}(z)$ of the canonical by the hyperbolic metric on X and section 5 gives effective bounds for the Huber constant $C_{\mathrm{Hub},X}$ on X. In section 6, we combine the results of the sections 3, 4, 5 to derive effective bounds for $\delta_{\mathrm{Fal}}(X)$. In section 7, we discuss an application of our results to an idea of A. N. Parshin for an attempt giving effective bounds for the height of rational points on smooth projective curves defined over number fields.

Acknowledgements. — We would like to use this opportunity to thank Christophe Soulé for having introduced us into the theory of arithmetic intersections by generously sharing his broad knowledge and deep insights on the subject with us. Furthermore, we would like to thank Alexei Parshin for his interest in our results and for having pointed out to us an application to his work. Finally, we would like to thank the referee for some of his/her comments.

2. Basic notations

2.1. Hyperbolic and canonical metrics

In this note X will denote a compact Riemann surface of genus $g_X > 1$. By the uniformization theorem, X is isomorphic to the quotient space $\Gamma\backslash\mathbb{H}$, where Γ is a cocompact and torsionfree Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, z = x + iy,\, y > 0\}$. In the sequel, we will identify X locally with its universal cover $\mathbb{H}$.

We denote by μ_{hyp} the $(1,1)$-form corresponding to the hyperbolic metric on X, which is compatible with the complex structure of X and has constant negative curvature equal to -1. Locally, we have

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\mathrm{Im}(z)^2} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}\,.$$

We write $\mathrm{vol}_{\mathrm{hyp}}(X)$ for the hyperbolic volume of X; recall that $\mathrm{vol}_{\mathrm{hyp}}(X)$ is given by $4\pi(g_X - 1)$. By μ_{shyp}, we denote the $(1,1)$-form corresponding to the rescaled hyperbolic metric, which measures the volume of X to be 1. We write $\mathrm{dist}_{\mathrm{hyp}}(z,w)$ for the hyperbolic distance between two points $z, w \in \mathbb{H}$. We recall the formula

$$\mathrm{dist}_{\mathrm{hyp}}(z,w) = \cosh^{-1}\left(1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\mathrm{Im}(w)}\right).$$

We denote the hyperbolic Laplacian on X by Δ_{hyp}; locally, we have

$$\Delta_{\mathrm{hyp}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The discrete spectrum of Δ_{hyp} is given by the increasing sequence of eigenvalues

$$0 = \lambda_{X,0} < \lambda_{X,1} \leqslant \lambda_{X,2} \leqslant \ldots$$

The $(1,1)$-form μ_{can} associated to the canonical metric is defined as follows. Let $\{\omega_1, \ldots, \omega_{g_X}\}$ denote an orthonormal basis of the space $\Gamma(X, \Omega^1_X)$ of holomorphic 1-forms on X. Then, μ_{can} is locally given by

$$\mu_{\mathrm{can}}(z) = \frac{1}{g_X} \cdot \frac{i}{2} \sum_{j=1}^{g_X} \omega_j(z) \wedge \overline{\omega}_j(z).$$

We recall that the Arakelov metric on X is induced by means of the residual canonical metric $\|\cdot\|_{\mathrm{Ar}}$ on Ω^1_X, which turns the residue map into an isometry.

2.2. Hyperbolic heat kernel for functions

The hyperbolic heat kernel $K_{\mathbb{H}}(t;z,w)$ ($t\in\mathbb{R}_{>0}$; $z,w\in\mathbb{H}$) for functions on $\mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t;z,w) := K_{\mathbb{H}}(t;\rho) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_{\rho}^{\infty}\frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r\,,$$

where $\rho=\mathrm{dist}_{\mathrm{hyp}}(z,w)$. The hyperbolic heat kernel $K_X(t;z,w)$ ($t\in\mathbb{R}_{>0}$; $z,w\in X$) for functions on X is obtained by averaging over the elements of Γ, namely

$$K_X(t;z,w) := \sum_{\gamma\in\Gamma} K_{\mathbb{H}}(t;z,\gamma w).$$

The heat kernel $K_X(t;z,w)$ satisfies the equations

$$\left(\frac{\partial}{\partial t}+\Delta_{\mathrm{hyp},z}\right)K_X(t;z,w)=0 \qquad (z,w\in X),$$

$$\lim_{t\to 0}\int\limits_X K_X(t;z,w)\,f(w)\,\mu_{\mathrm{hyp}}(w)=f(z)\quad (z\in X)$$

for all C^∞-functions f on X. As a shorthand, we use in the sequel the notation

$$HK_X(t;z) := \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma z).$$

2.3. Selberg zeta function

Let $\mathcal{H}(\Gamma)$ denote the set of conjugacy classes of primitive, hyperbolic elements in Γ. We denote by ℓ_γ the hyperbolic length of the closed geodesic determined by $\gamma\in\mathcal{H}(\Gamma)$ on X; it is well-known that the equality

$$|\mathrm{tr}(\gamma)| = 2\cosh(\ell_\gamma/2)$$

holds.

For $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$, the Selberg zeta function $Z_X(s)$ associated to X is defined via the Euler product expansion

$$Z_X(s) := \prod_{\gamma\in\mathcal{H}(\Gamma)} Z_\gamma(s),$$

where the local factors $Z_\gamma(s)$ are given by

$$Z_\gamma(s) := \prod_{n=0}^{\infty} \big(1 - e^{-(s+n)\ell_\gamma}\big).$$

The Selberg zeta function $Z_X(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian; furthermore, $Z_X(s)$ satisfies a functional equation. For our purposes, it suffices to know that the Selberg zeta function $Z_X(s)$ has a simple zero at $s = 1$, so that the quantity

$$\lim_{s\to 1} \left(\frac{Z_X'}{Z_X}(s) - \frac{1}{s-1} \right)$$

is well-defined.

2.4. Prime geodesic theorem

For any small eigenvalue $\lambda_{X,j} \in [0, 1/4)$, we define

$$s_{X,j} := \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,j}},$$

and note that $1/2 < s_{X,j} \leqslant 1$. For $u \in \mathbb{R}_{>1}$, we recall the prime geodesic counting function

$$\pi_X(u) := \#\big\{\gamma \in \mathcal{H}(\Gamma) \,|\, e^{\ell_\gamma} < u\big\}.$$

Introducing the logarithmic integral

$$\mathrm{li}(u) := \int_2^u \frac{\mathrm{d}\xi}{\log(\xi)}\,,$$

the prime geodesic theorem states

$$\pi_X(u) = \sum_{0\leqslant\lambda_{X,j}<1/4} \mathrm{li}(u^{s_{X,j}}) + O_X\big(u^{3/4}\log(u)^{-1/2}\big) \tag{2.1}$$

for $u > 1$, where the implied constant for all $u > 1$, not just asymptotically, depends solely on X. We call the infimum of all possible implied constants the Huber constant and denote it by $C_{\mathrm{Hub},X}$.

3. Faltings's delta function in hyperbolic terms

3.1. Faltings's delta function

Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ was introduced in [8], where also some of its basic properties were given. In [10], Faltings's delta function is expressed in terms of Riemann theta functions, and its asymptotic behavior is investigated; see also [23]. As a by-product of the analytic part of the arithmetic Riemann-Roch theorem for arithmetic surfaces, C. Soulé has shown in [21] that

$$\delta_{\mathrm{Fal}}(X) = -6D_{\mathrm{Ar}}(X) + a(g_X)\,, \tag{3.1}$$

where

$$D_{\mathrm{Ar}}(X) := \log\left(\frac{\det^*(\Delta_{\mathrm{Ar}})}{\mathrm{vol}_{\mathrm{Ar}}(X)}\right)$$

with $\det^*(\Delta_{\mathrm{Ar}})$ the regularized determinant of the Laplacian, $\mathrm{vol}_{\mathrm{Ar}}(X)$ the volume with respect to the Arakelov metric $\|\cdot\|_{\mathrm{Ar}}$, and

$$a(g_X) := -2g_X\log(\pi) + 4g_X\log(2) + (g_X-1)(-24\zeta'_{\mathbb{Q}}(-1)+1).$$

It has been shown in [14] how Faltings's delta function can be expressed solely in hyperbolic terms. Theorem 3.8 therein states:

3.2. Theorem

For X with genus $g_X > 1$, let

$$F(z) := \int_0^\infty \left(HK_X(t;z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t.$$

Then, we have

$$\begin{aligned}\delta_{\mathrm{Fal}}(X) = {}& \\ & 2\pi\left(1-\frac{1}{g_X}\right)\int_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z) - 6\log\left(Z'_X(1)\right) \\ & \qquad +2\lim_{s\to 1}\left(\frac{Z'_X}{Z_X}(s) - \frac{1}{s-1}\right) + c(g_X),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned} c(g_X) &:= a(g_X) - 6b(g_X) + 2(g_X-1)\log(4) + 6\log\left(\mathrm{vol}_{\mathrm{hyp}}(X)\right) - 2 \\ &= 2g_X\big(-24\zeta'_{\mathbb{Q}}(-1) - 4\log(\pi) + \log(2) + 2\big) + 6\log\left(\mathrm{vol}_{\mathrm{hyp}}(X)\right) \\ &\quad +\big(48\zeta'_{\mathbb{Q}}(-1) + 6\log(2\pi) - 2\log(4) - 6\big)\end{aligned}$$

with $a(g_X)$ *as above and* $b(g_X)$ *given by*

$$b(g_X) := (g_X - 1)\big(4\zeta'_{\mathbb{Q}}(-1) - 1/2 + \log(2\pi)\big).$$

Proof. — The proof is given in [14]. Here we present only a short outline of the proof, which consists of the following three main ingredients:

(i) One starts by using the Polyakov formula to relate the regularized determinants with respect to the Arakelov and the hyperbolic metric, namely

$$D_{\mathrm{Ar}}(X) = D_{\mathrm{hyp}}(X) + \frac{g_X - 1}{6} \int\limits_X \phi_{\mathrm{Ar}}(z)\big(\mu_{\mathrm{can}}(z) + \mu_{\mathrm{hyp}}(z)\big),$$

where $\phi_{\mathrm{Ar}}(z)$ is the conformal factor describing the change from the Arakelov to the hyperbolic metric.

(ii) In a second step, one uses the result [20] by P. Sarnak describing the hyperbolic regularized determinant in terms of the Selberg zeta function, namely

$$D_{\mathrm{hyp}}(X) = \log\left(\frac{Z'_X(1)}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) + b(g_X).$$

(iii) In order to express the conformal factor $\phi_{\mathrm{Ar}}(z)$ and the canonical metric form $\mu_{\mathrm{can}}(z)$ in hyperbolic terms, we make use of the fundamental relation

$$\mu_{\mathrm{can}}(z) = \mu_{\mathrm{shyp}}(z) + \frac{1}{2g_X}\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}} K_X(t;z)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(z), \tag{3.3}$$

which has been proven in Appendix 1 of [14]. □

3.3. Remark

We note that formula (3.3) has meanwhile been generalized to cofinite Fuchsian subgroups of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ without torsion elements in [16], and, as a relation of $(1,1)$-currents, to cofinite Fuchsian subgroups of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ allowing torsion elements in [3].

Based on formula (3.2), the following bound can be derived for $\delta_{\mathrm{Fal}}(X)$ in terms of basic hyperbolic invariants of X. For this we introduce the following notations

$$\lambda_X := \frac{1}{2}\min\left\{\lambda_{X,1}, \frac{7}{64}\right\},$$

$$N_{\mathrm{ev},X}^{[0,1/4)} := \#\{\lambda_{X,j} \,|\, 0 \leqslant \lambda_{X,j} < 1/4\},$$
$$N_{\mathrm{geo},X}^{(0,5)} := \#\{\gamma \in \mathcal{H}(\Gamma) \,|\, 0 < \ell_\gamma < 5\},$$
$$S_X := \sup_{z \in X}\left(\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}\right),$$

where $\lambda_{X,1}$ is the smallest non-zero eigenvalue of Δ_{hyp}, and we recall that ℓ_X denotes the length of the shortest closed geodesic on X and $C_{\mathrm{Hub},X}$ is the Huber constant introduced in subsection 2.4.

3.4. Corollary

With the above notations, we have the bound

$$\delta_{\mathrm{Fal}}(X) \leqslant D_1\left(g_X + \frac{1}{\lambda_X}\left(g_X(S_X+1)^2 + C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\right) + \left(1+\frac{1}{\ell_X}\right)N_{\mathrm{geo},X}^{(0,5)}\right)$$

with an absolute constant $D_1 > 0$, which can be taken to be 10^3.

Proof. — The proof is straightforward using Theorem 3.2 in combination with the estimates given in Propositions 4.1, 4.2, 4.3, and Lemma 4.4 in [14]. For the convenience of the reader, we give now a more detailed derivation of the proof.

Using Proposition 4.1 of [14] in combination with the inequalities $\lambda_{X,1} \geqslant \lambda_X$ and $\mathrm{vol}_{\mathrm{hyp}}(X) \leqslant 4\pi g_X$, the integral in (3.2) can be bounded as

$$\left|\int_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z)\right| \leqslant \frac{(S_X+1)^2\,\mathrm{vol}_{\mathrm{hyp}}(X)}{\lambda_{X,1}} \leqslant \frac{4\pi g_X}{\lambda_X}(S_X+1)^2. \quad (3.4)$$

In order to bound the absolute value of the second summand in (3.2), we first observe that we have to take the second bound in Proposition 4.3 of [14], since the first one being logarithmic in g_X is too small; choosing $\varepsilon = \lambda_X$, we obtain

$$\left|\log\left(Z'_X(1)\right)\right| \leqslant -\sum_{\substack{\gamma\in\mathcal{H}(\Gamma)\\ \ell_\gamma<5}} \log\left(Z_\gamma(1)\right) + 12\left(5+\frac{1}{\lambda_X}\right)\left(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)} + 1\right).$$

Using Lemma 4.4 (i) of [14], we derive from this the bound

$$\begin{aligned}\left|\log\left(Z'_X(1)\right)\right| &\leqslant \sum_{\substack{\gamma\in\mathcal{H}(\Gamma)\\ \ell_\gamma<5}} \frac{\pi^2}{6\ell_\gamma} + \frac{72}{\lambda_X}\left(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)} + 1\right)\\ &\leqslant \frac{\pi^2}{6\ell_X}N_{\mathrm{geo},X}^{(0,5)} + \frac{144}{\lambda_X}\left(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\right). \end{aligned} \quad (3.5)$$

Finally, in order to bound the absolute value of the third summand in (3.2), we again observe that we have to take the second bound in Proposition 4.2 of [14], since the first one being logarithmic in g_X is too small; choosing again $\varepsilon = \lambda_X$, we obtain

$$\left| \lim_{s\to 1}\left(\frac{Z_X'}{Z_X}(s) - \frac{1}{s-1}\right)\right| \leqslant \sum_{\substack{\gamma\in\mathcal{H}(\Gamma)\\ \ell_\gamma<5}} \frac{Z_\gamma'}{Z_\gamma}(1) + \frac{6}{\lambda_X}\big(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\big) + 2\,.$$

Using Lemma 4.4 (ii) of [14], we derive from this the bound

$$\begin{aligned}\left| \lim_{s\to 1}\left(\frac{Z_X'}{Z_X}(s) - \frac{1}{s-1}\right)\right| &\leqslant \sum_{\substack{\gamma\in\mathcal{H}(\Gamma)\\ \ell_\gamma<5}} \left(3+\log\left(\frac{1}{\ell_\gamma}\right)\right) \\ &\qquad + \frac{6}{\lambda_X}\big(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\big) + 2 \\ &\leqslant \sum_{\substack{\gamma\in\mathcal{H}(\Gamma)\\ \ell_\gamma<5}} \left(3+\frac{1}{\ell_\gamma}\right) + \frac{6}{\lambda_X}\big(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\big) + 2 \\ &\leqslant \left(3+\frac{1}{\ell_X}\right) N_{\mathrm{geo},X}^{(0,5)} \\ &\qquad + \frac{6}{\lambda_X}\big(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\big) + 2\,. \end{aligned} \tag{3.6}$$

The quantity $c(g_X)$ in (3.2) is easily bounded as

$$c(g_X) \leqslant 11 g_X + 10. \tag{3.7}$$

Adding up the bounds (3.4)–(3.7), using that $g_X > 1$, and by crudely estimating the arising integral constants by $D_1 = 10^3$, yields the claimed bound. Note that, estimating more rigorously, D_1 can in fact be taken to be 876. □

4. Effective bounds for the sup-norm

4.1. Hyperbolic heat kernel for forms

In addition to the hyperbolic heat kernel on $\mathbb{H}$, resp. X, introduced in subsection 2.2, we also need the hyperbolic heat kernel for forms of weight 1 on $\mathbb{H}$, resp. X. The hyperbolic heat kernel for forms of weight 1 on $\mathbb{H}$ is defined as in [13], namely we have

$$K_{\mathbb{H}}^{(1)}(t;z,w) := K_{\mathbb{H}}^{(1)}(t;\rho) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{r e^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}} T_2\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r,$$

where T_2 is the Chebyshev polynomial given by $T_2(r) := 2r^2 - 1$. The hyperbolic heat kernel for forms of weight 1 on X on the diagonal is then given as

$$K_X^{(1)}(t;z) := \sum_{\gamma\in\Gamma} c(\gamma;z)\, K_{\mathbb{H}}^{(1)}(t;z,\gamma z),$$

where $c(\gamma,z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is defined as

$$c(\gamma,z) := \frac{c\bar{z}+d}{cz+d}\cdot\frac{z-\gamma\bar{z}}{\gamma z-\bar{z}}\,.$$

We note that $|c(\gamma,z)| = 1$. From [13], we recall the crucial relation

$$\lim_{t\to\infty} K_X^{(1)}(t;z) = \frac{g_X\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}\,. \tag{4.1}$$

4.2. Lemma

With the above notations, we have the bound

$$\begin{aligned} K_{\mathbb{H}}^{(1)}(t;\rho) \;\leqslant\;& \frac{17\sqrt{2}\,e^{-t/4}}{(4\pi t)^{3/2}}\,\frac{(\rho+\log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)} + \frac{4\sqrt{2}\,e^{-(\rho/(2\sqrt{t})+\sqrt{t}/2)^2}}{\pi^{3/2}\sqrt{t}} \\ &+ \frac{4\sqrt{2}\,e^{-\rho}}{\pi^{3/2}} \int\limits_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} e^{-r'^2}\,\mathrm{d}r' \end{aligned} \tag{4.2}$$

for any $t>0$ and $\rho>0$.

Proof. — Starting with the defining formula

$$K_{\mathbb{H}}^{(1)}(t;\rho) := \frac{\sqrt{2}\,e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_{\rho}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\, T_2\!\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r\,,$$

we decompose the integral under consideration as

$$\int\limits_{\rho}^{\infty} \ldots \;=\; \int\limits_{\rho}^{\rho+\log(4)} \ldots \;+\; \int\limits_{\rho+\log(4)}^{\infty} \ldots \tag{4.3}$$

We start by estimating the first integral on the right-hand side of (4.3). Using the mean value theorem for the function $\cosh(r)$ with $r\in[\rho,\rho+\log(4)]$, we obtain the bound

$$\cosh(r)-\cosh(\rho) = (r-\rho)\sinh(r_*) \geqslant (r-\rho)\sinh(\rho),$$

where $r_* \in [\rho, \rho+\log(4)]$. With this in mind, we have the estimate

$$\begin{aligned}
&\int\limits_{\rho}^{\rho+\log(4)} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\, T_2\biggl(\frac{\cosh(r/2)}{\cosh(\rho/2)}\biggr)\,\mathrm{d}r \\
&\leqslant \frac{(\rho+\log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)} T_2\biggl(\frac{\cosh((\rho+\log(4))/2)}{\cosh(\rho/2)}\biggr) \int\limits_{\rho}^{\rho+\log(4)} (r-\rho)^{-1/2}\,\mathrm{d}r \\
&\leqslant \frac{2\log(4)^{1/2}(\rho+\log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)} T_2\biggl(\frac{\cosh((\rho+\log(4))/2)}{\cosh(\rho/2)}\biggr).
\end{aligned}$$

Since, for any $r_1, r_2 \in \mathbb{R}_{>0}$, we have

$$\begin{aligned}
\frac{\cosh(r_1+r_2)}{\cosh(r_1)} &= \frac{\cosh(r_1)\cosh(r_2)}{\cosh(r_1)} + \frac{\sinh(r_1)\sinh(r_2)}{\cosh(r_1)} \\
&\leqslant \cosh(r_2)+\sinh(r_2) = e^{r_2},
\end{aligned}$$

we can estimate the Tshebyshev polynomial contribution as

$$T_2\biggl(\frac{\cosh((\rho+\log(4))/2)}{\cosh(\rho/2)}\biggr) \leqslant T_2\bigl(e^{\log(4)/2}\bigr) = 7.$$

In summary, we find the following bound for the integral in question

$$\int\limits_{\rho}^{\rho+\log(4)} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}} T_2\biggl(\frac{\cosh(r/2)}{\cosh(\rho/2)}\biggr)\,\mathrm{d}r \leqslant \frac{17(\rho+\log(4))e^{-\rho^2/(4t)}}{\sinh^{1/2}(\rho)}. \tag{4.4}$$

We now estimate the second integral on the right-hand side of (4.3). Since $r \geqslant \rho + \log(4)$, we have

$$\frac{\cosh(r)}{2} \geqslant \frac{\cosh(\rho+\log(4))}{2} \geqslant \frac{\cosh(\rho)\cosh(\log(4))}{2} \geqslant \cosh(\rho),$$

whence

$$\cosh(r)-\cosh(\rho) \geqslant \frac{\cosh(r)}{2} \geqslant \frac{e^r}{4}\,.$$

Therefore, using the estimate $T_2(r) \leqslant 2r^2$ in combination with

$$\cosh(r/2) \leqslant e^{r/2} \quad \text{and} \quad \cosh(\rho/2) \geqslant \frac{e^{\rho/2}}{2}\,,$$

we derive the bound

$$\int\limits_{\rho+\log(4)}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\, T_2\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r \leqslant$$
$$\int\limits_{\rho+\log(4)}^{\infty} \frac{2\,re^{-r^2/(4t)}}{e^{r/2}} \frac{8\,e^{r}}{e^{\rho}}\, \mathrm{d}r = 16\, e^{-\rho} \int\limits_{\rho+\log(4)}^{\infty} r\, e^{r/2}\, e^{-r^2/(4t)}\, \mathrm{d}r\,. \qquad (4.5)$$

In order to complete the proof, we will further estimate the integral in (4.5). Keeping in mind that we finally have to multiply (4.5) by the factor $e^{-t/4}$, we estimate the quantity

$$e^{-t/4} \int\limits_{\rho+\log(4)}^{\infty} r\, e^{r/2}\, e^{-r^2/(4t)}\, \mathrm{d}r \leqslant \int\limits_{\rho}^{\infty} r\, e^{-(r/(2\sqrt{t})-\sqrt{t}/2)^2}\, \mathrm{d}r$$
$$= 2\sqrt{t} \int\limits_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} \big(2\sqrt{t}r' + t\big) e^{-r'^2}\, \mathrm{d}r'$$
$$= 2\, t\, e^{-(\rho/(2\sqrt{t})-\sqrt{t}/2)^2} + 2\, t^{3/2} \int\limits_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} e^{-r'^2}\, \mathrm{d}r'\,.$$

Multiplying by the remaining factor

$$\frac{16\sqrt{2}\, e^{-\rho}}{(4\pi t)^{3/2}} = \frac{2\sqrt{2}\, e^{-\rho}}{(\pi t)^{3/2}}\,,$$

yields the following bound involving the second integral

$$\frac{\sqrt{2}\, e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_{\rho+\log(4)}^{\infty} \frac{re^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\, T_2\left(\frac{\cosh(r/2)}{\cosh(\rho/2)}\right) \mathrm{d}r \leqslant$$
$$\frac{4\sqrt{2}\, e^{-(\rho/(2\sqrt{t})+\sqrt{t}/2)^2}}{\pi^{3/2}\sqrt{t}} + \frac{4\sqrt{2}\, e^{-\rho}}{\pi^{3/2}} \int\limits_{\rho/(2\sqrt{t})-\sqrt{t}/2}^{\infty} e^{-r'^2}\, \mathrm{d}r'\,. \qquad (4.6)$$

Adding up the bounds (4.4) and (4.6) yields the claimed upper bound for $K_{\mathbb{H}}^{(1)}(t;\rho)$. □

4.3. Lemma

Let $X \longrightarrow X_0$ be an unramified covering of finite degree with $X_0 := \Gamma_0\backslash\mathbb{H}$ a compact Riemann surface of genus $g_{X_0} > 1$, and let ℓ_{X_0} denote the length of the shortest closed geodesic on X_0. Then, the quantity S_X can be bounded as

$$S_X \leqslant 4\pi \int\limits_{\ell_{X_0}/4}^{\infty} K_{\mathbb{H}}^{(1)}(t_0;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho$$
$$+4\pi\, K_{\mathbb{H}}^{(1)}(t_0;\ell_{X_0}/4)\left(\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}-1\right) \tag{4.7}$$

for any $t_0 > 0$.

Proof. — From the spectral expansion, one immediately sees that the function $K_X^{(1)}(t;z)$ is monotone decreasing in t. Using relation (4.1) together with the triangle inequality, we then obtain for any $t_0 > 0$, the bound

$$\frac{g_X\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)} \leqslant \sum_{\gamma\in\Gamma} K_{\mathbb{H}}^{(1)}\big(t_0;z,\gamma z\big) \leqslant \sum_{\gamma\in\Gamma_0} K_{\mathbb{H}}^{(1)}\big(t_0;z,\gamma z\big).$$

Using the counting function

$$N_{X_0}(\rho;z) := \#\big\{\gamma\in\Gamma_0 \,|\, \mathrm{dist}_{\mathrm{hyp}}(z,\gamma z) < \rho\big\},$$

we can express the latter bound in terms of the Stieltjes integral

$$\frac{g_X\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)} \leqslant \int\limits_{\ell_{X_0}/4}^{\infty} K_{\mathbb{H}}^{(1)}(t_0;\rho)\,\mathrm{d}N_{X_0}(\rho;z)\,.$$

With the notation of Lemma 4.6 of [11], we put $u := \rho$, $a := \ell_{X_0}/4$, and further

$$\begin{aligned} F(u) &:= K_{\mathbb{H}}^{(1)}(t_0;\rho),\\ g_1(u) &:= N_{X_0}(\rho;z),\\ g_2(u) &:= \frac{\sinh^2\big((\rho+2r)/2\big)-\sinh^2\big((\rho_0-2r)/2\big)}{\sinh^2(r/2)} + N_{X_0}(\rho_0;z), \end{aligned}$$

where $r := \ell_{X_0}/4$ and $\rho_0 := 3\ell_{X_0}/4$. By the latter choices for r and ρ_0, the inequalities

$$2r < \ell_{X_0}\,, \quad 2r < \rho_0 < \ell_{X_0}$$

hold, which enables us to apply Lemma 2.3 (a) of [17] to derive the inequality

$$g_1(u) \leqslant g_2(u).$$

This in turn allows us to apply Lemma 4.6 of [11], in particular taking into account that $K_{\mathbb{H}}^{(1)}(t_0;\rho)$ is strictly monotone decreasing in ρ by Proposition A.2, namely the inequality of Stieltjes integrals

$$\int_a^\infty F(u)\,\mathrm{d}g_1(u) + F(a)\,g_1(a) \leqslant \int_a^\infty F(u)\,\mathrm{d}g_2(u) + F(a)\,g_2(a). \tag{4.8}$$

Using the above notation, we get

$$F(a)\,g_1(a) = K_{\mathbb{H}}^{(1)}(t_0;\ell_{X_0}/4)\,N_{X_0}(\ell_{X_0}/4;z) = K_{\mathbb{H}}^{(1)}(t_0;\ell_{X_0}/4),$$
$$F(a)\,g_2(a) = K_{\mathbb{H}}^{(1)}(t_0;\ell_{X_0}/4)\,\frac{\sinh^2(3\ell_{X_0}/8)-\sinh^2(\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} + K_{\mathbb{H}}^{(1)}(t_0;\ell_{X_0}/4).$$

Furthermore, we compute

$$\begin{aligned} g_2(u) &= \frac{\sinh^2(\rho/2+\ell_{X_0}/4)-\sinh^2(\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}+1 \\ &= \frac{\frac{1}{4}\big(e^{\rho+\ell_{X_0}/2}-2+e^{-\rho-\ell_{X_0}/2}\big)-\sinh^2(\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}+1, \end{aligned}$$

hence

$$\begin{aligned} \frac{\mathrm{d}g_2(u)}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}\rho}\,\frac{\frac{1}{4}\big(e^{\rho+\ell_{X_0}/2}-2+e^{-\rho-\ell_{X_0}/2}\big)-\sinh^2(\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} \\ &= \frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,. \end{aligned}$$

Inserting all of the above into (4.8), we arrive at the bound

$$\begin{aligned} \int_{\ell_{X_0}/4}^\infty K_{\mathbb{H}}^{(1)}(t_0;\rho)\,\mathrm{d}N_{X_0}(\rho;z) \;&\leqslant \int_{\ell_{X_0}/4}^\infty K_{\mathbb{H}}^{(1)}(t_0;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho \\ &\quad + K_{\mathbb{H}}^{(1)}(t_0;\ell_{X_0}/4)\bigg(\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}-1\bigg). \end{aligned}$$

Observing the inequality

$$\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)} \leqslant 4\pi\,\frac{g_X\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}$$

proves the claimed bound. □

4.4. Proposition

Let $X \longrightarrow X_0$ be an unramified covering of finite degree with $X_0 := \Gamma_0\backslash\mathbb{H}$ a compact Riemann surface of genus $g_{X_0} > 1$, and let ℓ_{X_0} denote the length of the shortest closed geodesic on X_0. Then, the quantity S_X can be bounded as

$$S_X \leqslant \frac{D_2\, e^{\ell_{X_0}/2}}{(1 - e^{-\ell_{X_0}/4})^{5/2}}$$

with an absolute constant $D_2 > 0$, which can be taken to be $1.2 \cdot 10^3$.

Proof. — We work from the estimate (4.7) for S_X given in Lemma 4.3 and insert therein the bound (4.2) for $K_{\mathbb{H}}^{(1)}(t_0;\rho)$ obtained in Lemma 4.2, which we rewrite as

$$K_{\mathbb{H}}^{(1)}(t_0;\rho) \leqslant A_1(t_0;\rho) + A_2(t_0;\rho) + A_3(t_0;\rho),$$

where

$$A_1(t_0;\rho) := \frac{17\sqrt{2}e^{-t_0/4}}{(4\pi t_0)^{3/2}}\,\frac{(\rho + \log(4))e^{-\rho^2/(4t_0)}}{\sinh^{1/2}(\rho)}\,,$$

$$A_2(t_0;\rho) := \frac{4\sqrt{2}\,e^{-(\rho/(2\sqrt{t_0})+\sqrt{t_0}/2)^2}}{\pi^{3/2}\sqrt{t_0}}\,,$$

$$A_3(t_0;\rho) := \frac{4\sqrt{2}\,e^{-\rho}}{\pi^{3/2}} \int\limits_{\rho/(2\sqrt{t_0})-\sqrt{t_0}/2}^{\infty} e^{-r'^2}\,\mathrm{d}r'\,.$$

With this notation and keeping in mind that our bounds are valid for all $t_0 > 0$, we can rewrite (4.7) in the form

$$S_X \leqslant B_1(t_0;\ell_{X_0}) + B_2(t_0;\ell_{X_0}) + B_3(t_0;\ell_{X_0}),$$

where

$$B_j(t_0;\ell_{X_0}) := 4\pi \int\limits_{\ell_{X_0}/4}^{\infty} A_j(t_0;\rho)\,\frac{\sinh(\rho + \ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho$$
$$+4\pi\, A_j(t_0;\ell_{X_0}/4)\left(\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} - 1\right)$$

for $j = 1, 2, 3$. In order to obtain a precise, effective upper bound for S_X, we will evaluate the expression under consideration at $t_0 = 10$; there is no particular reason for this choice of t_0 except to derive an explicit bound for S_X.

For the first summand of $B_1(t_0;\ell_{X_0})$ involving the integral, since $\sinh(\rho+\ell_{X_0}/2)\leqslant e^{\rho+\ell_{X_0}/2}$ and

$$\frac{1}{\sinh^{1/2}(\rho)}=\frac{\sqrt{2}\,e^{-\rho/2}}{(1-e^{-2\rho})^{1/2}}\leqslant\frac{\sqrt{2}\,e^{-\rho/2}}{(1-e^{-\ell_{X_0}/2})^{1/2}}$$

for $\rho\geqslant\ell_{X_0}/4$, we have the bound

$$\int\limits_{\ell_{X_0}/4}^{\infty}A_1(t_0;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho$$

$$\leqslant\frac{17\,e^{\ell_{X_0}/2}}{(4\pi t_0)^{3/2}\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\int\limits_{\ell_{X_0}/4}^{\infty}(\rho+\log(4))e^{-(\rho/(2\sqrt{t_0})-\sqrt{t_0}/2)^2}\,\mathrm{d}\rho$$

$$\leqslant\frac{34\,e^{\ell_{X_0}/2}}{(4\pi t_0)^{3/2}\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\int\limits_{-\infty}^{\infty}\big(t_0^{3/2}+2t_0|\rho'|+\log(4)t_0^{1/2}\big)e^{-\rho'^2}\,\mathrm{d}\rho'$$

$$=\frac{34\,e^{\ell_{X_0}/2}}{(4\pi)^{3/2}\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\left(\left(1+\frac{\log(4)}{t_0}\right)\sqrt{\pi}+\frac{2}{\sqrt{t_0}}\right),$$

hence we obtain for $t_0=10$

$$\int\limits_{\ell_{X_0}/4}^{\infty}A_1(10;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho\leqslant\frac{3\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\,.$$

We thus get the bound

$$\begin{aligned}\frac{B_1(10,\ell_{X_0})}{4\pi}\quad&\leqslant\quad\frac{3\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\\&\qquad+A_1(10;\ell_{X_0}/4)\left(\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}-1\right)\\&\leqslant\quad\frac{3\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\\&\qquad+\frac{17\,\sqrt{2}\,(1+\log(4))}{(40\,\pi)^{3/2}\sinh^{1/2}(\ell_{X_0}/4)}\,\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}\\&\leqslant\quad\frac{3\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\\&\qquad+\frac{e^{5\ell_{X_0}/8}}{\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\,.\end{aligned}\tag{4.9}$$

For the first summand of $B_2(t_0;\ell_{X_0})$ involving the integral, we have the bound

$$\int_{\ell_{X_0}/4}^{\infty} A_2(t_0;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho \leqslant \frac{2\sqrt{2}\,e^{\ell_{X_0}/2}}{\pi^{3/2}\sqrt{t_0}\,\sinh^2(\ell_{X_0}/8)}\int_{\ell_{X_0}/4}^{\infty} e^{-(\rho/(2\sqrt{t_0})-\sqrt{t_0}/2)^2}\,\mathrm{d}\rho\,,$$

hence we obtain for $t_0=10$

$$\int_{\ell_{X_0}/4}^{\infty} A_2(10;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho \leqslant \frac{4\sqrt{2}\,e^{\ell_{X_0}/2}}{\pi\,\sinh^2(\ell_{X_0}/8)} \leqslant \frac{2\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)}\,.$$

We thus get the bound

$$\begin{aligned}\frac{B_2(10,\ell_{X_0})}{4\pi} &\leqslant \frac{2\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)} + A_2(10;\ell_{X_0}/4)\left(\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)}-1\right)\\ &\leqslant \frac{2\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)} + \frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} \leqslant \frac{3\,e^{3\ell_{X_0}/4}}{\sinh^2(\ell_{X_0}/8)}\,. \qquad (4.10)\end{aligned}$$

For the first summand of $B_3(t_0;\ell_{X_0})$ involving the integral, we have the bound

$$\begin{aligned}&\int_{\ell_{X_0}/4}^{\infty} A_3(t_0;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho\\ &\leqslant \frac{2\sqrt{2}\,e^{\ell_{X_0}/2}}{\pi^{3/2}\,\sinh^2(\ell_{X_0}/8)}\int_{\ell_{X_0}/4}^{\infty}\int_{\rho/(2\sqrt{t_0})-\sqrt{t_0}/2}^{\infty} e^{-r'^2}\,\mathrm{d}r'\,\mathrm{d}\rho\\ &= \frac{2\sqrt{2}\,e^{\ell_{X_0}/2}}{\pi^{3/2}\,\sinh^2(\ell_{X_0}/8)}\int_{\ell_{X_0}/(8\sqrt{t_0})-\sqrt{t_0}/2}^{\infty}\int_{\ell_{X_0}/4}^{2\sqrt{t_0}r'+t_0} e^{-r'^2}\,\mathrm{d}\rho\,\mathrm{d}r'\\ &= \frac{2\sqrt{2}\,e^{\ell_{X_0}/2}}{\pi^{3/2}\,\sinh^2(\ell_{X_0}/8)}\int_{\ell_{X_0}/(8\sqrt{t_0})-\sqrt{t_0}/2}^{\infty}\left(2\sqrt{t_0}r'+t_0-\frac{\ell_{X_0}}{4}\right)e^{-r'^2}\,\mathrm{d}r'\\ &\leqslant \frac{2\sqrt{2}\,e^{\ell_{X_0}/2}}{\pi^{3/2}\,\sinh^2(\ell_{X_0}/8)}\left(2\sqrt{t_0}+\sqrt{\pi}\,t_0\right),\end{aligned}$$

hence we obtain for $t_0 = 10$

$$\int_{\ell_{X_0}/4}^{\infty} A_3(10;\rho)\,\frac{\sinh(\rho+\ell_{X_0}/2)}{2\,\sinh^2(\ell_{X_0}/8)}\,\mathrm{d}\rho \leqslant \frac{13\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)}\,.$$

We thus get the bound

$$\begin{aligned}\frac{B_3(10,\ell_{X_0})}{4\pi} &\leqslant \frac{13\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)} + A_3(10;\ell_{X_0}/4)\left(\frac{\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} - 1\right)\\ &\leqslant \frac{13\,e^{\ell_{X_0}/2}}{\sinh^2(\ell_{X_0}/8)} + \frac{2\,\sinh^2(3\ell_{X_0}/8)}{\sinh^2(\ell_{X_0}/8)} \leqslant \frac{15\,e^{3\ell_{X_0}/4}}{\sinh^2(\ell_{X_0}/8)}\,. \qquad (4.11)\end{aligned}$$

Adding up the bounds (4.9) – (4.11), we obtain

$$\begin{aligned}S_X &\leqslant \frac{88\,\pi\,e^{3\ell_{X_0}/4}}{\sinh^2(\ell_{X_0}/8)(1-e^{-\ell_{X_0}/2})^{1/2}}\\ &\leqslant \frac{352\,\pi\,e^{\ell_{X_0}/2}}{(1-e^{-\ell_{X_0}/4})^{5/2}}\,,\end{aligned}$$

which proves the claim. □

4.5. Remark

In addition to the cartesian coordinates x, y, we introduce the euclidean polar coordinates $\rho = \rho(z)$, $\theta = \theta(z)$ of the point z centered at the origin. These are related to x, y by the formulae

$$x := e^{\rho}\cos(\theta)\,, \quad y := e^{\rho}\sin(\theta). \qquad (4.12)$$

Given $\gamma \in \mathcal{H}(\Gamma)$, then there exists $\sigma_\gamma \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\sigma_\gamma^{-1}\gamma\,\sigma_\gamma = \begin{pmatrix} e^{\ell_\gamma/2} & 0 \\ 0 & e^{-\ell_\gamma/2}\end{pmatrix} \iff \gamma = \sigma_\gamma\begin{pmatrix} e^{\ell_\gamma/2} & 0 \\ 0 & e^{-\ell_\gamma/2}\end{pmatrix}\sigma_\gamma^{-1}\,.$$

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, the hyperbolic Eisenstein series $\mathcal{E}_{\mathrm{hyp},\gamma}(z,s)$ associated to γ is defined by the series

$$\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) := \sum_{\eta\in\langle\gamma\rangle\backslash\Gamma} \sin\left(\theta(\sigma_\gamma^{-1}\eta z)\right)^s \qquad (4.13)$$

using the polar coordinates (4.12). The hyperbolic Eisenstein series (4.13) is absolutely and locally uniformly convergent for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with

$\mathrm{Re}(s) > 1$; it is invariant under the action of Γ and satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp}} - s(1-s)\big)\mathcal{E}_{\mathrm{hyp},\gamma}(z,s) = s^2\,\mathcal{E}_{\mathrm{hyp},\gamma}(z,s+2).$$

For proofs of theses facts and further details, we refer to [18].

By means of the hyperbolic Eisenstein series the following alternative bound for the quantity S_X, namely

$$S_X \leqslant 8\Bigg[\sum_{\gamma\in\mathcal{H}(\Gamma_0)}\Bigg(\sup_{z\in X_0}\Big|\Delta_{\mathrm{hyp}}\mathcal{E}_{\mathrm{hyp},\gamma}(z,2)\Big|e^{-\ell_\gamma}+\frac{24}{(1-e^{-\ell_{X_0}})^3}\sup_{z\in X_0}\Big|\mathcal{E}_{\mathrm{hyp},\gamma}(z,2)\Big|\,e^{-2\ell_\gamma}\Bigg)+170\Bigg],$$

has been obtained in [15]. This upper bound for S_X involves special values of hyperbolic Eisenstein series in the half-plane of convergence of the series. As such, it is possible to use various counting function arguments, as above, to complete this approach to obtaining an upper bound for the quantity S_X analogous to the one given in Proposition 4.4.

5. Effective bounds for the Huber constant

5.1. Remark

In Table 2 of the recent joint work [9] with J. S. Friedman, an algorithm was given to bound the Huber constant $C_{\mathrm{Hub},X}$ for X effectively in terms of our basic quantities g_X, d_X, ℓ_X, $\lambda_{X,1}$, and $N^{[0,1/4)}_{\mathrm{ev},X}$; here the newly introduced quantity d_X denotes the diameter of X. In the subsequent proposition, we will summarize the result of this algorithm by utilizing convenient yet possibly crude estimates.

5.2. Proposition

The Huber constant $C_{\mathrm{Hub},X}$ for X can be bounded as

$$C_{\mathrm{Hub},X} \leqslant \frac{D_3\,g_X\,e^{8\pi g_X/\ell_X+\ell_X/2}}{(1-s_{X,1})(1-e^{-\ell_X/2})^2}\,;$$

here ℓ_X denotes the length of the shortest closed geodesic on X,

$$s_{X,1} := \frac{1}{2}+\sqrt{\frac{1}{4}-\lambda_{X,1}}$$

with $\lambda_{X,1}$ denoting the smallest non-zero eigenvalue of Δ_{hyp}, and $D_3 > 0$ is an absolute constant, which can be taken to be 10^{11}.

Proof. — As mentioned in 5.1, we follow the algorithm given in Table 2 of [9]. In the sequel we also use the definitions of the quantities A, B, C, C_j $(j = 1, 2, 3, \ldots)$ therein.

Recalling from [6] the bound for $N_{\mathrm{ev},X}^{[0,1/4)}$, we obtain for the quantity A the estimate

$$A := N_{\mathrm{ev},X}^{[0,1/4)} \leqslant 4g_X - 2 \leqslant 4g_X.$$

Using the inequality (2) from the main theorem of [7], namely

$$2\,\frac{\ell_X}{4}\,d_X \leqslant 2\sinh\left(\frac{\ell_X}{4}\right)d_X \leqslant 4\pi(g_X - 1),$$

we obtain the following bound for the diameter d_X of X

$$d_X \leqslant \frac{8\,\pi\,g_X}{\ell_X}.$$

Hence, the quantity B can be estimated by

$$B := \frac{2\pi e^{d_X}}{4\pi(g_X - 1)} \leqslant \frac{e^{8\pi g_X/\ell_X}}{2}\,.$$

For the quantity C, we have

$$C := 3\biggl(\frac{4\pi(g_X - 1)}{4\pi} + 745B\biggr) \leqslant 3\,g_X + 1118\,e^{8\pi g_X/\ell_X}.$$

Next, we have

$$C_1 := 2e - 2 \leqslant 4,$$

and

$$C_{10} := 8480\sqrt{\frac{e}{2\pi}} \leqslant 5578.$$

From this we derive

$$\begin{aligned} C_{12} &:= (A-1)\biggl(1 + 3C_1 + \frac{2}{1 - s_{X,1}}(1 + C_1)\biggr) + 2C_1 + 2 \\ &\leqslant 4\,g_X\biggl(13 + \frac{10}{1 - s_{X,1}}\biggr) + 10 \\ &\leqslant \frac{92\,g_X}{1 - s_{X,1}} + 10 \leqslant \frac{102\,g_X}{1 - s_{X,1}}\,, \end{aligned}$$

and

$$\begin{aligned} C_{13} &:= \frac{41}{6}\cdot C \cdot C_{10} \leqslant \frac{41\cdot 5578}{6}\bigl(3\,g_X + 1118\,e^{8\pi g_X/\ell_X}\bigr) \\ &\leqslant 114\,349\,g_X + 42\,614\,061\,e^{8\pi g_X/\ell_X}. \end{aligned}$$

From this we obtain

$$\begin{aligned} C_{16} &:= C_{12} + C_{13} + \frac{3}{2\pi} 4\pi(g_X - 1) C_{10} \\ &\leqslant \frac{102\, g_X}{1 - s_{X,1}} + 114\,349\, g_X + 42\,614\,061\, e^{8\pi g_X/\ell_X} + 6 \cdot 5578\, g_X \\ &\leqslant \frac{147\,919\, g_X}{1 - s_{X,1}} + 42\,614\,061\, e^{8\pi g_X/\ell_X} \leqslant \frac{42\,761\,980\, g_X\, e^{8\pi g_X/\ell_X}}{1 - s_{X,1}} \,. \end{aligned}$$

For notational convenience, let us keep the constant C_{16} without replacing it with the above bound for the next few computations. We further have

$$\begin{aligned} C_{17} &:= 4A + 4\, C_{16} \leqslant 16\, g_X + 4\, C_{16}, \\ C_{18} &:= 4A + 5\, C_{16} \leqslant 16\, g_X + 5\, C_{16}. \end{aligned}$$

The constant c must satisfy $1 < c < e^{\ell_X}$, so we may take $c := e^{\ell_X/2}$, and hence $\mu := \ell_X/2$. With this choice, we find

$$\begin{aligned} C_{19} &:= C_{18} + \frac{8A + 4\, C_{18}}{1 - 1/c} \leqslant 16\, g_X + 5\, C_{16} + \frac{96\, g_X + 20\, C_{16}}{1 - e^{-\ell_X/2}} \\ &\leqslant \frac{112\, g_X + 25\, C_{16}}{1 - e^{-\ell_X/2}} \,. \end{aligned}$$

Observing that

$$f(r) := \frac{r}{1 - e^{-r}} \geqslant 1$$

for $r \in \mathbb{R}_{\geqslant 0}$, we find

$$\frac{1}{\mu} = \frac{2}{\ell_X} \leqslant \frac{1}{1 - e^{-\ell_X/2}} \,.$$

Thus, we obtain

$$\begin{aligned} C_{20} &:= C_{19} + \frac{8A + 4\, C_{18}}{\mu} \leqslant \frac{112\, g_X + 25\, C_{16}}{1 - e^{-\ell_X/2}} + \frac{8A + 4\, C_{18}}{1 - e^{-\ell_X/2}} \\ &\leqslant \frac{112\, g_X + 25\, C_{16}}{1 - e^{-\ell_X/2}} + \frac{96\, g_X + 20\, C_{16}}{1 - e^{-\ell_X/2}} \\ &= \frac{208\, g_X + 45\, C_{16}}{1 - e^{-\ell_X/2}} \,. \end{aligned}$$

For the quantity C_{21}, we find the estimate

$$\begin{aligned} C_{21} &:= \frac{|c-2|}{\log(2)} + \frac{2|2 - \sqrt{c}|}{\log(c)} \leqslant \frac{c+2}{\log(2)} + \frac{2(\sqrt{c}+2)}{\log(c)} \\ &\leqslant \frac{e^{\ell_X/2} + 2}{\log(2)} + \frac{4(e^{\ell_X/4} + 2)}{\ell_X} \leqslant \frac{3\, e^{\ell_X/2}}{1/2} + \frac{4 \cdot 3\, e^{\ell_X/2}}{\ell_X} \\ &\leqslant 12\, e^{\ell_X/2} \left(1 + \frac{1}{\ell_X}\right) \leqslant \frac{18\, e^{\ell_X/2}}{1 - e^{-\ell_X/2}} \,. \end{aligned}$$

At this point, we have to correct the statement about the constant C_{22}, which comes from Lemma 4.14 in [9]. The correct assertion is that

$$C_{22} := \frac{1}{1 + 1/\log(2)}\,.$$

In fact, C_{22} has to be such that for any $r \geqslant 2$, we have the inequality

$$\mathrm{li}(r) \leqslant C_{22}\,\frac{r}{\log(r)}\,.$$

For a proof we consider the function

$$f(r) := \mathrm{li}(r) - d\,\frac{r}{\log(r)}$$

for some positive constant d, which we determine such that $f(r)$ is negative for $r \geqslant 2$. Obviously, $f(2) < 0$, so we have to determine d such that $f(r)$ becomes a decreasing function. We have

$$f'(r) = \frac{1}{\log(r)}\left(1 - d + \frac{d}{\log(r)}\right),$$

hence, we need to have

$$1 - d + \frac{d}{\log(r)} \leqslant 0 \qquad \Longleftrightarrow \qquad 1 - \frac{1}{d} \geqslant \frac{1}{\log(r)}$$

for $r \geqslant 2$, which holds for

$$1 - \frac{1}{d} \geqslant \frac{1}{\log(2)} \qquad \Longleftrightarrow \qquad d \geqslant \frac{1}{1 + 1/\log(2)}$$

giving the claimed value of C_{22}. (Note that the error in the proof of Lemma 4.14 of [9] arose by dividing by a constant which is negative, so then the inequality has to change directions.) Continuing with this value of C_{22}, we have

$$C_{22} = \frac{1}{1 + 1/\log(2)} \leqslant \frac{1}{2}.$$

Finally, we are in a position to compute C_u; we have

$$\begin{aligned} C_u :=& C_{21}A + C_{20}\frac{c^{3/4}}{\log(c)} + C_{20}(1 + C_{22}) + \frac{3}{4}C_{20}C_{21} \\ \leqslant& \frac{72\,g_X\,e^{\ell_X/2}}{1 - e^{-\ell_X/2}} + \frac{(208\,g_X + 45\,C_{16})e^{\ell_X/2}}{(1 - e^{-\ell_X/2})^2} \\ &+ \frac{312\,g_X + 69\,C_{16}}{1 - e^{-\ell_X/2}} + \frac{(3744\,g_X + 810\,C_{16})e^{\ell_X/2}}{(1 - e^{-\ell_X/2})^2}\,. \end{aligned}$$

Employing finally the bound for C_{16} yields the estimate

$$C_u \leqslant \frac{384\, g_X\, e^{\ell_X/2}}{1-e^{-\ell_X/2}} + \frac{69\, C_{16}}{1-e^{-\ell_X/2}} + \frac{(3952\, g_X + 855\, C_{16})e^{\ell_X/2}}{(1-e^{-\ell_X/2})^2}$$
$$\leqslant \frac{39\,512\,073\,856\, g_X\, e^{8\pi g_X/\ell_X + \ell_X/2}}{(1-s_{X,1})(1-e^{-\ell_X/2})^2}\,.$$

This completes the proof of the proposition. □

6. Effective bounds for Faltings's delta function

The main result proven in this paper consists in simplifying the bound obtained in Corollary 3.4 and making it effective.

6.1. Theorem

Let $X \longrightarrow X_0$ be an unramified covering of finite degree with $X_0 := \Gamma_0\backslash\mathbb{H}$ a compact Riemann surface of genus $g_{X_0} > 1$. Let ℓ_{X_0} denote the length of the shortest closed geodesic on X_0 and $\lambda_{X,1}$, $\lambda_{X_0,1}$ the smallest non-zero eigenvalues of Δ_{hyp} on X, X_0, respectively, and

$$\lambda_X = \frac{1}{2}\min\left\{\lambda_{X,1}, \frac{7}{64}\right\}, \qquad s_{X_0,1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X_0,1}}\,.$$

Then, we have the effective bound

$$\delta_{\mathrm{Fal}}(X) \leqslant \frac{D_4\, g_{X_0}\, e^{8\pi g_{X_0}/\ell_{X_0} + \ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5(1-s_{X_0,1})}\,\frac{g_X}{\lambda_X} \tag{6.1}$$

with an absolute constant $D_4 > 0$, which can be taken to be 10^{15}.

Proof. — We work from the bound

$$\delta_{\mathrm{Fal}}(X) \leqslant 876\left(g_X + \frac{1}{\lambda_X}\left(g_X(S_X+1)^2 + C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\right) + \left(1 + \frac{1}{\ell_X}\right)N_{\mathrm{geo},X}^{(0,5)}\right). \tag{6.2}$$

obtained in the proof of Corollary 3.4 (using the notation therein). We will next bound the quantities

$$\ell_X, \quad N_{\mathrm{ev},X}^{[0,1/4)}, \quad S_X, \quad C_{\mathrm{Hub},X}, \quad N_{\mathrm{geo},X}^{(0,5)}$$

in terms of the underlying compact Riemann surface X_0.

(i) We start by observing that the trivial inequality

$$\ell_X \geqslant \ell_{X_0} \tag{6.3}$$

holds true for the lengths of the shortest closed geodesics on X, X_0, respectively.

(ii) In order to estimate $N_{\mathrm{ev},X}^{[0,1/4)}$, we recall as in the proof of Proposition 5.2 from [6] the bound

$$1 \leqslant N_{\mathrm{ev},X}^{[0,1/4)} \leqslant 4g_X - 2 \leqslant 4g_X. \tag{6.4}$$

(iii) From Proposition 4.4, we recall the bound

$$S_X \leqslant \frac{1\,200\, e^{\ell_{X_0}/2}}{(1-e^{-\ell_{X_0}/4})^{5/2}} \tag{6.5}$$

with ℓ_{X_0} as in the statement of the theorem.

(iv) Next, we have to estimate $C_{\mathrm{Hub},X}$. We start by citing Theorem 3.4 of [12] and use the Artin formalism for the covering $X \longrightarrow X_0$, to derive the bound

$$C_{\mathrm{Hub},X} \leqslant [\Gamma_0 \colon \Gamma]\, C_{\mathrm{Hub},X_0}.$$

From the Riemann–Hurwitz formula we now easily derive the bound

$$[\Gamma_0 \colon \Gamma] \leqslant \frac{g_X - 1}{g_{X_0} - 1} \leqslant g_X,$$

from which we get

$$C_{\mathrm{Hub},X} \leqslant g_X\, C_{\mathrm{Hub},X_0}, \tag{6.6}$$

where the proof of Proposition 5.2 shows

$$C_{\mathrm{Hub},X_0} \leqslant \frac{39\,512\,073\,856\, g_{X_0}\, e^{8\pi g_{X_0}/\ell_{X_0} + \ell_{X_0}/2}}{(1-s_{X_0,1})(1-e^{-\ell_{X_0}/2})^2} \tag{6.7}$$

with ℓ_{X_0} and $s_{X_0,1}$ as in the statement of the theorem.

(v) Finally, we need to bound $N_{\mathrm{geo},X}^{(0,5)}$. With the above notation, using arguments from the proof of Theorem 4.11 in [11] (as well as the notation $r_{\Gamma_0,\Gamma}$ therein), we find (as above)

$$N_{\mathrm{geo},X}^{(0,5)} \leqslant \frac{5\, r_{\Gamma_0,\Gamma}}{\ell_{X_0}}\, N_{\mathrm{geo},X_0}^{(0,5)} \leqslant \frac{5\,[\Gamma_0 \colon \Gamma]}{\ell_{X_0}}\, N_{\mathrm{geo},X_0}^{(0,5)} \leqslant \frac{5\, g_X}{\ell_{X_0}}\, N_{\mathrm{geo},X_0}^{(0,5)}.$$

Applying the prime geodesic theorem (2.1) to X_0 and recalling the monotonicity of the logarithmic integral for $u > 0$, we find

$$\begin{aligned} N_{\mathrm{geo},X_0}^{(0,5)} = \pi_{X_0}\big(\log(5)\big) \quad &\leqslant N_{\mathrm{ev},X_0}^{[0,1/4)} \mathrm{li}\big(\log(5)\big) + C_{\mathrm{Hub},X_0} \frac{\log(5)^{3/4}}{\log\big(\log(5)\big)^{1/2}} \\ &\leqslant N_{\mathrm{ev},X_0}^{[0,1/4)} + 3C_{\mathrm{Hub},X_0} \leqslant 4g_{X_0} + 3C_{\mathrm{Hub},X_0}, \end{aligned} \tag{6.8}$$

where C_{Hub,X_0} can be effectively bounded using Proposition 5.2.

Inserting the bounds (6.3) – (6.8) into the estimate (6.2) yields the following bound for $\delta_{\mathrm{Fal}}(X)$:

$$\begin{aligned}
&876\bigg(g_X+\frac{1}{\lambda_X}\bigg(g_X\bigg(\frac{1\,200\,e^{\ell_{X_0}/2}}{(1-e^{-\ell_{X_0}/4})^{5/2}}+1\bigg)^2+g_X C_{\mathrm{Hub},X_0}+4g_X\bigg)+\frac{5\,g_X}{\ell_{X_0}}\bigg(1+\frac{1}{\ell_{X_0}}\bigg)N^{(0,5)}_{\mathrm{geo},X_0}\bigg)\\
&\quad\leqslant 876\,g_X\bigg(1+\frac{1}{\lambda_X}\bigg(\frac{1\,201^2\,e^{\ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5}+C_{\mathrm{Hub},X_0}+4\bigg)+\frac{10\,N^{(0,5)}_{\mathrm{geo},X_0}}{\ell_{X_0}(1-e^{-\ell_{X_0}/2})}\bigg)\\
&\quad\leqslant 876\,g_X\bigg(\frac{1}{\lambda_X}\bigg(\frac{1\,442\,401\,e^{\ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5}+C_{\mathrm{Hub},X_0}+5\bigg)+\frac{20\,g_{X_0}+15\,C_{\mathrm{Hub},X_0}}{(1-e^{-\ell_{X_0}/2})^2}\bigg)\\
&\quad\leqslant 876\,\frac{g_X}{\lambda_X}\bigg(\frac{1\,442\,426\,g_{X_0}\,e^{\ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5}+\frac{16\,C_{\mathrm{Hub},X_0}}{(1-e^{-\ell_{X_0}/4})^2}\bigg)\\
&\quad\leqslant \frac{876}{(1-e^{-\ell_{X_0}/4})^5}\,\frac{g_X}{\lambda_X}\bigg(1\,442\,426\,g_{X_0}\,e^{\ell_{X_0}}+\frac{632\,193\,181\,696\,g_{X_0}\,e^{8\pi g_{X_0}/\ell_{X_0}+\ell_{X_0}/2}}{1-s_{X_0,1}}\bigg)\\
&\quad\leqslant 553\,802\,490\,730\,872\,\frac{g_{X_0}\,e^{8\pi g_{X_0}/\ell_{X_0}+\ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5(1-s_{X_0,1})}\,\frac{g_X}{\lambda_X}\,.
\end{aligned}$$

This completes the proof of the theorem. □

6.2. Remarks

(i) We can further refine the lower bound for ℓ_{X_0} provided that X_0 has a model defined over some number field. In fact, by Bélyi's theorem, we then have $X_0 \cong \overline{\Delta_0\backslash\mathbb{H}}$, where Δ_0 is a subgroup of finite index in $\Gamma(2)$. Therefore, we have the estimate

$$2\cosh(\ell_{X_0}/2)=|\mathrm{tr}(\delta_0)|\geqslant 4,$$

where $\delta_0\in\Delta_0$ is such that $\ell_{\delta_0}=\ell_{X_0}$; this gives $\ell_{X_0}\geqslant 2\,\mathrm{arcosh}(2)$. The factor depending on X_0 in (6.1) can thus be bounded as

$$\begin{aligned}
\frac{D_4\,g_{X_0}\,e^{8\pi g_{X_0}/\ell_{X_0}+\ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5(1-s_{X_0,1})}&\leqslant\frac{40\,D_4\,g_{X_0}\,e^{10\,g_{X_0}+\ell_{X_0}}}{1-s_{X_0,1}}\\
&\leqslant\frac{10^{17}\,g_{X_0}\,e^{10\,g_{X_0}+\ell_{X_0}}}{1-s_{X_0,1}}\,. \qquad (6.9)
\end{aligned}$$

(ii) On the other hand, if X_0 can be covered by a modular curve $\overline{\Gamma(N)\backslash\mathbb{H}}$ for the full congruence subgroup $\Gamma(N)$ for some $N\in\mathbb{N}$, a result of R. Brooks in [5] shows that $\lambda_{X_0,1}\geqslant 5/36$, which gives the estimate

$$\frac{1}{1-s_{X_0,1}}\leqslant 6.$$

In addition, assuming that X_0 has a model defined over some number field, case (i) above also applies and the bound (6.9) simplifies to

$$\frac{D_4\, g_{X_0} e^{8\pi g_{X_0}/\ell_{X_0}+\ell_{X_0}}}{(1-e^{-\ell_{X_0}/4})^5(1-s_{X_0,1})} \leqslant 10^{18} g_{X_0}\, e^{10\, g_{X_0}+\ell_{X_0}}.$$

6.3. Corollary

Let X be a compact Riemann surface of genus $g_X > 1$. Let ℓ_X denote the length of the shortest closed geodesic on X, $\lambda_{X,1}$ the smallest non-zero eigenvalue of Δ_{hyp} on X, and

$$\lambda_X = \frac{1}{2}\min\left\{\lambda_{X,1}, \frac{7}{64}\right\}, \quad s_{X,1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,1}}\,.$$

Then, we have the effective bound

$$\delta_{\mathrm{Fal}}(X) \leqslant \frac{D_4\, g_X\, e^{8\pi g_X/\ell_X+\ell_X}}{(1-e^{-\ell_X/4})^5}\, \frac{1}{\lambda_X(1-s_{X,1})} \tag{6.10}$$

with an absolute constant $D_4 > 0$, which can be taken to be 10^{15}.

Proof. — The proof follows immediately from an analysis of the proof of Theorem 6.1 for the trivial covering $X_0 = X$. □

Using Corollary 6.3, we can now also give a variant of the bound (6.1) in the case that X is a *ramified* covering of finite degree of a compact Riemann surface X_0 of genus $g_{X_0} > 1$. For this, we let $\mathrm{Ram}(X/X_0) \subset X_0$ denote the ramification locus of the given covering.

6.4. Corollary

Let $X \longrightarrow X_0$ be a ramified covering of finite degree of compact Riemann surfaces of genera $g_X, g_{X_0} > 1$, respectively. With ℓ_{X_0} denoting the length of the shortest closed geodesic on X_0, put

$$r_{X_0} := \min\left\{\ell_{X_0}\,, \min_{\substack{z,w\in\mathrm{Ram}(X/X_0)\\ z\neq w}} \mathrm{dist}_{\mathrm{hyp}}(z,w)\right\},$$

$$R_{X_0} := \max\left\{\ell_{X_0}\,, \max_{\substack{z,w\in\mathrm{Ram}(X/X_0)\\ z\neq w}} \mathrm{dist}_{\mathrm{hyp}}(z,w)\right\}.$$

Furthermore, with $\lambda_{X,1}$ denoting the smallest non-zero eigenvalue of Δ_{hyp} on X, put

$$\lambda_X = \frac{1}{2}\min\left\{\lambda_{X,1}, \frac{7}{64}\right\}, \quad s_{X,1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,1}}\,.$$

Then, we have the effective bound

$$\delta_{\mathrm{Fal}}(X) \leqslant \frac{D_4\, g_X\, e^{8\pi g_X/r_{X_0}+g_X R_{X_0}}}{(1-e^{-r_{X_0}/4})^5}\,\frac{1}{\lambda_X(1-s_{X,1})}$$

with an absolute constant $D_4>0$*, which can be taken to be* 10^{15}*.*

Proof. — We work from the effective bound obtained in Corollary 6.3 and estimate the length of the shortest closed geodesic ℓ_X from below and above by quantities depending on the base X_0.

In order to estimate ℓ_X from below, we observe that the length of closed geodesics on X, which do not pass through ramification points, can be bounded from below by ℓ_{X_0}; the same estimate holds true, if the closed geodesic passes through a single ramification point. However, if the closed geodesic happens to pass through at least two ramification points lying above two distinct points of $\mathrm{Ram}(X/X_0)$, we additionally have to take into account the distances between mutually distinct points of $\mathrm{Ram}(X/X_0)$ in our estimate. All in all this leads to the lower bound

$$\ell_X \geqslant r_{X_0}. \tag{6.11}$$

Similarly, we find that the length of closed geodesics on X, which do not pass through ramification points, can be bounded from above by $\deg(X/X_0)\,\ell_{X_0}$, and the same estimate holds true, if the closed geodesic passes through a single ramification point. Again, if the closed geodesic happens to pass through at least two ramification points lying above two distinct points of $\mathrm{Ram}(X/X_0)$, we additionally have to take into account the distances between mutually distinct points of $\mathrm{Ram}(X/X_0)$ in our estimate. This leads to the upper bound

$$\ell_X \leqslant \deg(X/X_0)\, R_{X_0} \leqslant g_X R_{X_0}. \tag{6.12}$$

Inserting the bounds (6.11) and (6.12) into the estimate (6.10) completes the proof of the corollary. □

7. Application to Parshin's covering construction

7.1. The set-up

Let K be a number field with ring of integers $\mathcal{O}_K$ and $S := \mathrm{Spec}\,(\mathcal{O}_K)$. In contrast to the previous sections, let X denote a smooth projective curve defined over K of genus $g_X>1$, and let $\mathcal{X}/S$ be a minimal regular model of X/K, which is semistable. Denote by $\overline{\omega}_{\mathcal{X}/S}$ the relative dualizing sheaf of

$\mathcal{X}/S$ equipped with the Arakelov metric. For $\mathfrak{p} \in S$, we let $\delta_{\mathfrak{p}}$ be the number of singular points in the fiber above $\mathfrak{p}$. For an archimedean place v, we put

$$X_v := X \times_v \mathbb{C},$$

whose complex points $X_v(\mathbb{C})$ constitute a compact Riemann surface of genus equal to g_X. In order to simplify our notation, we allow ourselves subsequently to write X_v instead of $X_v(\mathbb{C})$.

In his quest for an arithmetic version of the van de Ven–Bogomolov–Miyaoka–Yau inequality, A. N. Parshin proposed the following inequality (see [19])

$$\overline{\omega}^2_{\mathcal{X}/S} \leqslant c_1 \bigg(\sum_{\mathfrak{p}} \delta_{\mathfrak{p}} \log \big(N_{K/\mathbb{Q}}(\mathfrak{p})\big) + \sum_{v} \varepsilon_v \, \delta_{\mathrm{Fal}}\big(X_v\big) \bigg) \\ + c_2 \big(2g_X - 2\big) \log \big|\mathrm{disc}(K/\mathbb{Q})\big| + c_3 [K\colon \mathbb{Q}]\,; \qquad (7.1)$$

here c_j are positive constants depending solely on K ($j = 1, 2, 3$), $N_{K/\mathbb{Q}}(\mathfrak{p})$ denotes the absolute norm of $\mathfrak{p}$, and $\mathrm{disc}(K/\mathbb{Q})$ is the discriminant of the field extension $K/\mathbb{Q}$. As is well known by subsequent work of J.-B. Bost, J.-F. Mestre, and L. Moret-Bailly (see [4]), the inequality (7.1) does not hold true in general.

7.2. The covering construction

Assuming the validity of the inequality (7.1), A. N. Parshin proposed in [19], how to bound the height of K-rational points $P \in X(K)$ as effective as possible using the following ramified covering construction.

Given the smooth projective curve X/K of genus $g_X > 1$, and $P \in X(K)$ a K-rational point, there exists a finite covering X_P/K_P over X with the following properties:

(i) The field extension K_P/K is a finite extension of degree effectively bounded as $O(g_X)$ with prescribed ramification.

(ii) The covering X_P/X is finite of degree effectively bounded as $O(g_X)$ and ramified only at P of ramification index effectively bounded as $O(g_X)$; by the Riemann–Hurwitz formula, the genus g_{X_P} of X_P is then also effectively bounded as $O(g_X)$.

(iii) For each archimedean place v of K and each archimedean place v' of K_P lying above v, there exists a smooth projective complex surface Y_v together with a smooth morphism $\varphi_v\colon Y_v \longrightarrow X_v$ such that

$$\varphi_v^{-1}(P) \cong X_{P,v'} := X_P \times_{v'} \mathbb{C}.$$

Denoting by $\mathcal{O}_{K_P}$ the ring of integers of K_P, setting $S_P := \operatorname{Spec}(\mathcal{O}_{K_P})$, letting $\mathcal{X}_P/S_P$ be a minimal regular model of X_P/K_P, which is semistable, and denoting by $\overline{\omega}_{\mathcal{X}_P/S_P}$ the relative dualizing sheaf of $\mathcal{X}_P/S_P$ equipped with the Arakelov metric, the height $h(P)$ of P can be bounded by the arithmetic self-intersection number

$$h(P) \ll \overline{\omega}^2_{\mathcal{X}_P/S_P}\,, \tag{7.2}$$

which, in turn, can then be bounded using (7.1), after replacing $\overline{\omega}_{\mathcal{X}/S}$ by $\overline{\omega}_{\mathcal{X}_P/S_P}$. In [19], the quantities $\delta_{\mathfrak{P}}$ ($\mathfrak{P} \in S_P$), $\operatorname{disc}(K_P/\mathbb{Q})$, and $[K_P\colon\mathbb{Q}]$ are then effectively bounded in terms of the genus g_X of X. The contribution from Faltings's delta function $\delta_{\mathrm{Fal}}\big(X_{P,v'}\big)$ $(v'|v)$ is bounded in terms of X by arguing that, as P is moving through the set of K-rational points $X(K)$, the function $\delta_{\mathrm{Fal}}\big(X_{P,v'}\big)$ can be viewed as the restriction of a real-analytic function on X_v, which takes its maximum on the compact Riemann surface X_v.

7.3. Parshin's question

After having presented our estimate (6.2) for Faltings's delta function obtained in Corollary 3.4, Parshin proposed to apply our bound to $\delta_{\mathrm{Fal}}\big(X_{P,v'}\big)$ in order to obtain a more explicit bound than his.

Indeed, applying the bound obtained in Corollary 6.4 to the ramified covering $X_{P,v'} \longrightarrow X_v$ of finite degree, observing that the ramification locus $\operatorname{Ram}(X_{P,v'}/X_v)$ consists of only one point, we are led to the bound

$$\delta_{\mathrm{Fal}}\big(X_{P,v'}\big) \leqslant \frac{D_4\, g_{X_P}\, e^{8\pi g_{X_P}/\ell_{X_v} + g_{X_P}\ell_{X_v}}}{(1-e^{-\ell_{X_v}/4})^5}\,\frac{1}{\lambda_{X_{P,v'}}(1-s_{X_{P,v'},1})}\,, \tag{7.3}$$

where

$$\lambda_{X_{P,v'}} = \frac{1}{2}\min\left\{\lambda_{X_{P,v'},1}, \frac{7}{64}\right\}, \qquad s_{X_{P,v'},1} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X_{P,v'},1}}$$

with ℓ_{X_v} denoting the length of the shortest closed geodesic on X_v and $\lambda_{X_{P,v'},1}$ denoting the smallest non-zero eigenvalue of Δ_{hyp} on $X_{P,v'}$.

As P is moving through the set of K-rational points $X(K)$ or, more generally, through the compact Riemann surface X_v, the Riemann surfaces $X_{P,v'}$ (or, rather their isomorphism classes) cover a compact region $\mathcal{D}$ in the moduli space $\mathcal{M}_{g_{X_P}}$ of curves of genus g_{X_P}. While P is ranging over X_v, the function

$$\lambda_{X_{P,v'}}(1-s_{X_{P,v'},1})$$

takes its minimum on $\mathcal{D}$, which we denote by $\lambda_{v,\min}$. Keeping in mind that X_v is defined over a number field, Remark 6.2 (i) allows us to simplify the bound (7.3) to

$$\delta_{\mathrm{Fal}}\big(X_{P,v'}\big) \leqslant \frac{10^{17}\, g_{X_P}\, e^{10 g_{X_P} + g_{X_P} \ell_{X_v}}}{\lambda_{v,\min}}\,;$$

here we recall that the genus g_{X_P} can be effectively bounded in terms of the genus g_X.

We conclude by emphasizing that our results do not lead to an *effective* bound for the height $h(P)$ of K-rational points $P \in X(K)$, since the bound (7.2) as well as the determination of the minimum $\lambda_{v,\min}$ are not effective.

A. Appendix

In order to apply the inequality of Stieltjes integrals (4.8), we need that the function $K_{\mathbb{H}}^{(1)}(t;\rho)$ is monotone decreasing in ρ. The purpose of this appendix is to provide a proof of this claim.

A.1. Lemma

For $t > 0$, $\rho > 0$, and $r \geqslant \rho$, let

$$F(t;\rho,r) := \frac{re^{-r^2/(4t)}}{\sinh(r)} T_2\bigg(\frac{\cosh(r/2)}{\cosh(\rho/2)}\bigg).$$

Then, for all values of t, ρ, r in the given range, we have

$$\sinh(r)\frac{\partial}{\partial\rho}F(t;\rho,r) + \sinh(\rho)\frac{\partial}{\partial r}F(t;\rho,r) < 0.$$

Proof. — We set

$$X := \frac{\cosh(r/2)}{\cosh(\rho/2)},$$

and compute

$$\frac{\partial}{\partial\rho}F(t;\rho,r) = -\frac{re^{-r^2/(4t)}}{\sinh(r)}\frac{2\cosh^2(r/2)\sinh(\rho/2)}{\cosh^3(\rho/2)} =$$

$$-\frac{re^{-r^2/(4t)}}{\sinh(r)} 2X^2\tanh(\rho/2) = -F(t;\rho,r)\frac{2X^2}{2X^2-1}\tanh(\rho/2),$$

and

$$\frac{\partial}{\partial r}F(t;\rho,r) = F(t;\rho,r)\left(\frac{1}{r} - \frac{r}{2t} + \frac{2X^2}{2X^2-1}\tanh(r/2) - \frac{\cosh(r)}{\sinh(r)}\right).$$

From this we deduce

$$\sinh(r)\frac{\partial}{\partial\rho}F(t;\rho,r) + \sinh(\rho)\frac{\partial}{\partial r}F(t;\rho,r) =$$

$$-F(t;\rho,r)\sinh(\rho)\left(\frac{r}{2t} + \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r}\right) - F(t;\rho,r)\frac{2X^2}{2X^2-1}h_\rho(r), \quad (7.4)$$

where

$$h_\rho(r) := \tanh(\rho/2)\sinh(r) - \sinh(\rho)\tanh(r/2).$$

For $r > 0$, we now have the estimate

$$\frac{r}{2t} + \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r} > \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r} = \frac{r\cosh(r) - \sinh(r)}{r\sinh(r)} > 0,$$

using the power series expansions for $\cosh(r)$ and $\sinh(r)$. Next, we compute and estimate for $r \geqslant \rho > 0$

$$\begin{aligned}
h'_\rho(r) &= \tanh(\rho/2)\cosh(r) - \frac{\sinh(\rho)}{2\cosh^2(r/2)} \\
&= \tanh(\rho/2)\big(2\cosh^2(r/2) - 1\big) - \frac{\sinh(\rho/2)\cosh(\rho/2)}{\cosh^2(r/2)} \\
&= \frac{\tanh(\rho/2)}{\cosh^2(r/2)}\big(2\cosh^4(r/2) - \cosh^2(r/2) - \cosh^2(\rho/2)\big) \\
&\geqslant \frac{2\tanh(\rho/2)}{\cosh^2(r/2)}\big(\cosh^4(r/2) - \cosh^2(r/2)\big) \\
&= 2\tanh(\rho/2)\big(\cosh^2(r/2) - 1\big) > 0.
\end{aligned}$$

Since $h_\rho(\rho) = 0$, this shows that $h_\rho(r) \geqslant 0$ for $r \geqslant \rho > 0$. Recalling (7.4) the claim of the lemma follows from the above estimates. □

A.2. Proposition

For any $t > 0$, the heat kernel $K_{\mathbb{H}}^{(1)}(t;\rho)$ for forms is strictly monotone decreasing for $\rho > 0$.

Proof. — We will prove that $\partial/\partial_\rho K_{\mathbb{H}}^{(1)}(t;\rho) < 0$ for $\rho > 0$. To simplify notations, we put

$$c(t) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}.$$

In the notation of Lemma A.1, we then have, using integration by parts,

$$\begin{aligned} K_{\mathbb{H}}^{(1)}(t;\rho) &= c(t)\int_{\rho}^{\infty} F(t;\rho,r)\frac{\sinh(r)}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r \\ &= -2c(t)\int_{\rho}^{\infty}\frac{\partial}{\partial r}F(t;\rho,r)\sqrt{\cosh(r)-\cosh(\rho)}\,\mathrm{d}r\,. \end{aligned}$$

We now apply the Leibniz rule of differentiation to write

$$\begin{aligned} \frac{\partial}{\partial\rho}K_{\mathbb{H}}^{(1)}(t;\rho) &= -2c(t)\int_{\rho}^{\infty}\frac{\partial^2}{\partial r\,\partial\rho}F(t;\rho,r)\sqrt{\cosh(r)-\cosh(\rho)}\,\mathrm{d}r \\ &\quad +c(t)\int_{\rho}^{\infty}\frac{\partial}{\partial r}F(t;\rho,r)\frac{\sinh(\rho)}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r\,. \end{aligned}$$

Using integration by parts on the first term once again, yields the identity

$$\frac{\partial}{\partial\rho}K_{\mathbb{H}}^{(1)}(t;\rho) = c(t)\int_{\rho}^{\infty}\left(\sinh(r)\frac{\partial}{\partial\rho}F(t;\rho,r)+\sinh(\rho)\frac{\partial}{\partial r}F(t;\rho,r)\right)\frac{\mathrm{d}r}{\sqrt{\cosh(r)-\cosh(\rho)}}\,.$$

From Lemma A.1, we conclude that $\partial/\partial_\rho K_{\mathbb{H}}^{(1)}(t;\rho)<0$ for $\rho>0$, which proves the claim. □

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