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A supplementary proof of $L^p$–logarithmic Sobolev inequality


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A supplementary proof of $L^p$–logarithmic Sobolev inequality

Yasuhiro Fujita

1. Introduction

Let $n \in \mathbb{N}$. For a smooth enough function $f \geq 0$ on $\mathbb{R}^n$, we define the entropy of $f$ with respect to the Lebesgue measure by

$$
\text{Ent}(f) = \int f(x) \log f(x) dx - \int f(x) dx \log \int f(x) dx.
$$

In this paper, the integral without its domain is always understood as the one over $\mathbb{R}^n$, and we interpret that $0 \log 0 = 0$. 

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Let $p \geq 1$. We denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions $f$ on $\mathbb{R}^n$ such that $f$ and $|Df|$ (the Euclidean length of the gradient $Df$ of $f$) are in $L^p(\mathbb{R}^n)$. For $f \in W^{1,p}(\mathbb{R}^n)$, the following $L^p$–logarithmic Sobolev inequality was shown for $p=2$ by [10], $p=1$ by [9], and $1 < p < n$ by [6]:

$$\text{Ent}(|f|^p) \leq \frac{n}{p} \int |f(x)|^p dx \log \left( \frac{\int |Df(x)|^p \, dx}{\int |f(x)|^p \, dx} \right). \quad (1.1)$$

Here,

$$L_p = \begin{cases} 
\frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left( \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( n \frac{p-1}{p} + 1 \right)} \right)^{p/n}, & p > 1, \\
\frac{1}{n} \pi^{-1/2} \left[ \Gamma \left( \frac{n}{2} + 1 \right) \right]^{1/n}, & p = 1.
\end{cases} \quad (1.2)$$

This is the best possible constant satisfying (1.1) for $1 \leq p < n$ (cf. [1, 6]).

For a general $p > 1$, with a deep insight, Gentil [8, Theorem 1.1] tried to give inequality (1.1) in the following way: First, he gave a hypercontractivity inequality for the unique viscosity solution to the Cauchy problem of the Hamilton-Jacobi equation

$$u_t(x,t) + \frac{1}{p} |Du(x,t)|^p = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \quad (1.3)$$

$$u(\cdot,0) = \phi \quad \text{in } \mathbb{R}^n. \quad (1.4)$$

Here, $\phi \in \text{Lip}(\mathbb{R}^n)$. He showed that if there is a constant $\alpha > 0$ such that $e^\phi \in L^\alpha(\mathbb{R}^n)$, then $e^{u(\cdot,t)} \in L^\beta(\mathbb{R}^n)$ for any $\beta > \alpha$ and $t > 0$ and

$$\|e^{u(\cdot,t)}\|_\beta \leq \|e^\phi\|_\alpha \left( \frac{nL_p e^{p-1}(\beta - \alpha)}{p^\gamma t} \right)^{\frac{n}{p} \frac{\beta - \alpha}{\alpha p}} \frac{\alpha^{\frac{n}{p} + (p-1)\beta}}{\beta^{\frac{n}{p} + (p-1)\alpha}}, \quad (1.5)$$

where $L_p$ is the constant of (1.2) and

$$\|f\|_\gamma = \left( \int |f(x)|^\gamma \, dx \right)^{1/\gamma}, \quad \gamma > 0.$$

For completeness, we prove (1.5) in Section 2 for $\alpha = 1$ and $\beta > 1$; this case is sufficient to prove (1.1). Gentil [8, Theorem 1.1] tried to derive inequality (1.1) from inequality (1.5).
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However, his proof for inequality (1.1) seems to be valid only when $f \in W^{1,p}(\mathbb{R}^n)$ has the form $f = e^{\frac{1}{p}\phi}$ for $\phi \in \text{Lip}(\mathbb{R}^n)$ of (1.4) with

$$
\lim \inf_{s \to 0^+} \frac{1}{s} \int [e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)}] \, dx \geq -\frac{1}{p} \int e^{\phi(x)} |D\phi(x)|^p \, dx
$$

(1.6)

for any $k > 0$, where $u$ is a viscosity solution to Cauchy problem (1.3) with (1.4). So, his paper proves (1.1) for a special class of functions $f \in W^{1,p}(\mathbb{R}^n)$.

Our aim in this paper is to bridge this gap in the proof of [8, Theorem 1.1] and provide a supplementary proof of inequality (1.1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. The strategy of our proof is the following: First, we show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ such that

$$
f \in C^1(\mathbb{R}^n), \ 0 < f \leq 1 \text{ in } \mathbb{R}^n, \text{ and } D(\log f) \text{ is bounded on } \mathbb{R}^n.
$$

(1.7)

The point is that, under (1.7) for $f \in W^{1,p}(\mathbb{R}^n)$, inequality (1.6) is fulfilled for letting $\phi(\cdot) = p \log f(\cdot)$ (see the proof of Lemma 3.1 below). Such an argument was used in [3].

Second, we approximate $f \in W^{1,p}(\mathbb{R}^n)$ by a sequence of functions satisfying (1.7) by several steps. This is the key point to derive (1.1) from (1.5) (see Theorem 3.3 below). An important estimate is the following Fatou–type inequality: if a family $\{f_\epsilon\}_{0 < \epsilon < 1}$ of nonnegative and measurable functions on $\mathbb{R}^n$ approximates a function $f$ in some sense, then

$$
\lim \inf_{\epsilon \to 0^+} \int f_\epsilon(x)^p \log f_\epsilon(x) \, dx \geq \int f(x)^p \log f(x) \, dx.
$$

(1.8)

We provide a sufficient condition on $\{f_\epsilon\}_{0 < \epsilon < 1}$ for (1.8) (see Lemmas 2.2 and 2.3 below). From this result, we provide a stability condition such that if $f_\epsilon$ satisfies (1.1), so does $f$.

Finally, by using these approximations, we show that $L^p$–logarithmic Sobolev inequality (1.1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$. This bridges a gap of the proof of [8, Theorem 1.1] for $L^p$–logarithmic Sobolev inequality (1.1) with $p > 1$.

The content of this paper is organized as follows: In Section 2, we provide preliminaries. In Section 3, we provide a supplementary proof of inequality (1.1) for all $f \in W^{1,p}(\mathbb{R}^n)$ and $p > 1$.

I express my hearty appreciation to Ivan Gentil for his encouragement.
2. Preliminaries

In this section, we provide preliminaries to the next section. In the following, we assume $p > 1$. Set $q = p/(p - 1)$. We assume that
\[
\phi \in \text{Lip}(\mathbb{R}^n), \phi \leq 0 \text{ in } \mathbb{R}^n \text{ and } e^\phi \in L^1(\mathbb{R}^n). \tag{2.1}
\]
We put
\[
L := \|D\phi\|_\infty. \tag{2.2}
\]
Here, $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{R}^n; \mathbb{R}^n)$–norm. Under (2.1), Cauchy problem (1.3) with (1.4) admits the unique viscosity solution $u \in C(\mathbb{R}^n \times [0, \infty))$ with the following properties:
\[
u(x, t) = \inf_{y \in \mathbb{R}^n} \left[ \phi(y) + \frac{1}{qt^{q-1}}|x - y|^q \right], \quad x \in \mathbb{R}^n, \ t > 0. \tag{2.3}
\]
\[
|u(x, t) - u(y, t)| \leq L|x - y|, \quad x, y \in \mathbb{R}^n, \ t \geq 0. \tag{2.4}
\]
\[
|u(x, t) - \phi(x)| \leq Mt, \quad x \in \mathbb{R}^n, \ t \geq 0 \tag{2.5}
\]
for some constant $M > 0$.

Hopf-Lax formula (2.3) is well-known for a viscosity solution to Cauchy problem (1.3) with (1.4). For inequalities (2.4) and (2.5), see [4, Theorem 1.3.2].

Next, under (2.1), we derive inequality (1.5) for completeness. Here, in (1.5), we take $\alpha = 1$ for simplicity, since this case is sufficient to prove (1.1). Following the idea due to Gentil [7, 8], we prove (1.5) by Prékopa–Leindler inequality. Note that we do not use (1.1) in this proof of (1.5).

Recall Prékopa–Leindler inequality (cf. [5, Theorem 2]): Let $h_0, h_1 : \mathbb{R}^n \to \mathbb{R}$ be Borel measurable and nonnegative functions, and $\theta \in (0, 1)$ a constant. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable and nonnegative function such that
\[
h_0(x_0)^{1-\theta}h_1(x_1)^{\theta} \leq h((1 - \theta)x_0 + \theta x_1), \quad x_0, x_1 \in \mathbb{R}^n. \tag{2.6}
\]
If $h_0, h_1, h \in L^1(\mathbb{R}^n)$, then
\[
\left( \int h_0(x)dx \right)^{1-\theta} \left( \int h_1(x)dx \right)^{\theta} \leq \int h(x)dx. \tag{2.7}
\]
Now, let $\beta > 1$ and $t > 0$. Under (2.1), we consider the functions $h_0, h_1, h$ defined by
\[
h_0(x) = \exp\{\beta u(x, t)\},
\]
\[
h_1(x) = \exp\left\{ -\beta (\beta - 1)^{q-1} \frac{|x|^q}{qt^{q-1}} \right\},
\]
\[
h(x) = \exp \{ \phi(\beta x) \}.
\]
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Since $u(x, t) \leq \phi(x) \leq 0$ for $(x, t) \in \mathbb{R}^n \times [0, \infty)$ by (2.3), we have

$$\beta u(x, t) \leq \beta \phi(x) \leq \phi(x), \quad (x, t) \in \mathbb{R}^n \times [0, \infty),$$

so that $h_0 \in L^1(\mathbb{R}^n)$ by (2.1). It is clear that $h_1 \in L^1(\mathbb{R}^n)$. Since $e^\phi \in L^1(\mathbb{R}^n)$, we have $h \in L^1(\mathbb{R}^n)$. Furthermore, let $\theta = (\beta - 1)/\beta \in (0, 1)$. By (2.3), we have, for $x_0, x_1 \in \mathbb{R}^n$,

$$h_0(x_0)^1 - \theta h_1(x_1)^\theta = \exp \left\{ u(x_0, t) - \frac{1}{qt^{q-1}} |(\beta - 1)x_1|^q \right\} \leq \exp \left\{ \phi(\beta[1 - \theta]x_0 + \theta x_1]) \right\} = h((1 - \theta)x_0 + \theta x_1).$$

Thus, (2.6) holds for these $h_0, h_1, h$. Note that

$$\left( \int h_0(x)dx \right)^{1 - \theta} = \|e^{u(x, t)}\|_\beta, \quad \int h(x)dx = \frac{\|e^\phi\|_1}{\beta^n}.$$

By (1.2) and a slightly long calculation, we have

$$\int h_1(x)dx = \int e^{-C|x|^q} dx \left( C = \frac{\beta(\beta - 1)^{q-1}}{qt^{q-1}} \right) = \frac{\sigma_{n-1}}{q \, C^n} \Gamma(n/q) \quad (\sigma_{n-1} = \text{the surface area of the unit ball of } \mathbb{R}^n)$$

$$= \left[ \beta^{p-1} \frac{nL_p e^{p-1}(\beta - 1)}{p^p t} \right]^{-\frac{n}{p}}.$$

Thus, by (2.7), we conclude (1.5) for $\alpha = 1$, $\beta > 1$ and $t > 0$.

We prepare three lemmas for the next section.

**Lemma 2.1.** Assume that $\phi \in C^1(\mathbb{R}^n)$ and $D\phi$ is bounded on $\mathbb{R}^n$. Let $u \in C(\mathbb{R}^n \times [0, \infty))$ be the unique viscosity solution to the Cauchy problem (1.3) with (1.4). Then, we have

$$u(x, s) - \phi(x) \geq -\frac{s}{p} \left[ \max_{|z - x| \leq C_s} |D\phi(z)| \right]^p, \quad (x, s) \in \mathbb{R}^n \times (0, \infty),$$

where $C = (qL)^{\frac{1}{1 - q}}$ and $L$ is the constant of (2.2).

**Proof.** Fix $(x, s) \in \mathbb{R}^n \times (0, \infty)$ arbitrarily. Let $\hat{y} \in \mathbb{R}^n$ be a minimizer of the Hopf-Lax formula

$$u(x, s) = \inf_{y \in \mathbb{R}^n} \left[ \phi(x - y) + \frac{|y|^q}{qs^{q-1}} \right] = \phi(x - \hat{y}) + \frac{|\hat{y}|^q}{qs^{q-1}}.$$

Thus, by (2.7), we conclude (1.5) for $\alpha = 1$, $\beta > 1$ and $t > 0$.
Such a \( \hat{y} \) surely exists, since \( q > 1 \) and \( D\phi \) is bounded on \( \mathbb{R}^n \). Since \( u(x, s) \leq \phi(x) \) by (2.3), we have

\[
\frac{|\hat{y}|^q}{qs^{q-1}} \leq \phi(x) - \phi(x - \hat{y}) \leq L|\hat{y}|
\]

so that \( |\hat{y}| \leq Cs \). Note that, when \( |y| \leq Cs \), we have

\[
\phi(x - y) - \phi(x) = \int_0^1 \frac{d}{d\theta} \phi(x - \theta y) d\theta = \int_0^1 D\phi(x - \theta y) \cdot (-y) d\theta \\
\geq -|y| \max_{|z - x| \leq Cs} |D\phi(z)|.
\]

Thus,

\[
u(x, s) - \phi(x) = \inf_{|y| \leq Cs} \left[ \phi(x - y) - \phi(x) + \frac{|y|^q}{qs^{q-1}} \right] \\
\geq \inf_{|y| \leq Cs} \left[ -|y| \max_{|z - x| \leq Cs} |D\phi(z)| + \frac{|y|^q}{qs^{q-1}} \right] \\
\geq \inf_{y \in \mathbb{R}^n} \left[ -|y| \max_{|z - x| \leq Cs} |D\phi(z)| + \frac{|y|^q}{qs^{q-1}} \right] \\
= -\frac{s}{p} \left[ \max_{|z - x| \leq Cs} |D\phi(z)| \right]^p.
\]

\[\square\]

**Lemma 2.2.** — Let \( \{f_\epsilon\}_{0 < \epsilon < 1} \) be a family of nonnegative and measurable functions on \( \mathbb{R}^n \) such that \( f := \lim_{\epsilon \to 0^+} f_\epsilon \) exists a.e. on \( \mathbb{R}^n \). Assume that there exists a constant \( \delta \in (0, p) \) such that \( f_\epsilon, f \in L^{p-\delta}(\mathbb{R}^n) \) and

\[
\lim_{\epsilon \to 0^+} \int f_\epsilon(x)^{p-\delta} dx = \int f(x)^{p-\delta} dx. \tag{2.9}
\]

Then, we have (1.8).

**Proof.** — Note that the inequality

\[
t^{\delta} \log t + \frac{1}{\delta e} \geq 0, \quad t \geq 0, \quad \delta > 0
\]

holds. Thus, applying the Fatou’s lemma to

\[
\int \left( f_\epsilon^p \log f_\epsilon + \frac{1}{\delta e} f_\epsilon^{p-\delta} \right) dx = \int f_\epsilon^{p-\delta} \left( f_\epsilon^\delta \log f_\epsilon + \frac{1}{\delta e} \right) dx
\]
we have
\[
\liminf_{\epsilon \to 0^+} \int \left( f_p^\epsilon \log f_\epsilon + \frac{1}{\delta e} f_p^{\epsilon \cdot \delta} \right) dx \geq \int f_p^{\cdot \delta} \left( f_\delta \log f + \frac{1}{\delta e} \right) dx.
\]
By our assumption, the left-hand side of this inequality is equal to
\[
\liminf_{\epsilon \to 0^+} \int f_p^\epsilon \log f_\epsilon dx + \frac{1}{\delta e} \lim_{\epsilon \to 0^+} \int f_p^{\epsilon \cdot \delta} dx.
\]
Therefore, we conclude (1.8) by (2.9). \(\square\)

**Lemma 2.3.** — For \(0 \leq f \in L^p(\mathbb{R}^n)\), let
\[
f_\epsilon(x) = \lambda(\epsilon x) f(x), \quad x \in \mathbb{R}^n, \ 0 < \epsilon < 1,
\]
where \(\lambda\) is a \(C(\mathbb{R}^n)\)–function such that \(\lambda(0) = 1\) and \(0 \leq \lambda \leq 1\) on \(\mathbb{R}^n\). Then, we have (1.8).

**Proof.** — We have
\[
\liminf_{\epsilon \to 0^+} \int f_p^\epsilon \log f_\epsilon dx
\]
\[
= \liminf_{\epsilon \to 0^+} \left[ \int \lambda(\epsilon \cdot) f_p^\epsilon \log \lambda(\epsilon \cdot) dx + \int_{\{ f \geq 1 \}} \lambda(\epsilon \cdot)^p f_p^\epsilon \log f dx \right.
\]
\[
+ \left. \int_{\{ f < 1 \}} \lambda(\epsilon \cdot)^p f_p^\epsilon \log f dx \right] \geq \liminf_{\epsilon \to 0^+} \int \lambda(\epsilon \cdot)^p f_p \log f dx
\]
\[
+ \liminf_{\epsilon \to 0^+} \int_{\{ f \geq 1 \}} \lambda(\epsilon \cdot)^p f_p \log f dx
\]
\[
+ \liminf_{\epsilon \to 0^+} \int_{\{ f < 1 \}} \lambda(\epsilon \cdot)^p f_p \log f dx
\]
\[
\equiv I + J + K.
\]

Since \(f \in L^p(\mathbb{R}^n)\), we have \(I = 0\) by Lebesgue’s dominated convergence theorem. By Fatou’s lemma, we have
\[
J \geq \int_{\{ f \geq 1 \}} f_p \log f dx.
\]
Since \(0 \leq \lambda \leq 1\) on \(\mathbb{R}^n\), we have
\[
K \geq \int_{\{ f < 1 \}} f_p \log f dx,
\]
\[
-125-
\]
so that
\[ I + J + K \geq \int f^p \log f \, dx. \]
Therefore, we conclude (1.8).

3. Proof of inequality (1.1)

In this section, we provide a complete proof of inequality (1.1) for all \( f \in W^{1,p}(\mathbb{R}^n) \) and \( p > 1 \). First, we show (1.1) for \( f \in W^{1,p}(\mathbb{R}^n) \) satisfying (1.7). We put \( \phi := \log f \). Then, \( \phi \) fulfills
\[ \phi \in C^1(\mathbb{R}^n), \quad \phi \leq 0 \text{ in } \mathbb{R}^n, \quad e^\phi \in L^1(\mathbb{R}^n), \quad \text{and} \quad D\phi \text{ is bounded on } \mathbb{R}^n. \] (3.1)

Further, note that (3.1) implies (2.1). Thus, if \( f \in W^{1,p}(\mathbb{R}^n) \) fulfills (1.7), Cauchy problem (1.3) with (1.4) for \( \phi := \log f \) admits the unique viscosity solution \( u \in C(\mathbb{R}^n \times [0, \infty)) \).

**Lemma 3.1.** Let \( p > 1 \) and \( k > 0 \). Assume that \( f \in W^{1,p}(\mathbb{R}^n) \) fulfills (1.7). Let \( u \in C(\mathbb{R}^n \times [0, \infty)) \) be the unique viscosity solution of Cauchy problem (1.3) with (1.4) for \( \phi := \log f \). We define the function \( F(\cdot) \) on \([0, \infty[, s \geq 0\) by
\[ F(s) = \int e^{(ks+1)\phi(x)} \, dx, \quad s \geq 0. \]
If \( \Ent(e^\phi) > -\infty \), then we have
\[ \liminf_{s \to 0^+} \frac{F(s) - F(0)}{s} \geq -\frac{1}{p} \int e^{\phi(x)} |D\phi(x)|^p \, dx + k \int \phi(x)e^{\phi(x)} \, dx. \] (3.2)

**Proof.** 1. Since \( \phi \leq 0 \) in \( \mathbb{R}^n \), we have, by (2.8),
\[ e^{(ks+1)\phi(x)} \leq e^{(ks+1)\phi(x)} \leq e^{\phi(x)} \in L^1(\mathbb{R}^n), \quad s \geq 0. \] (3.3)
Thus, \( F \) is well–defined. Furthermore, note that
\[ 0 \leq -\int \phi(x)e^{\phi(x)} \, dx < \infty, \] (3.4)
since \( \Ent(e^\phi) > -\infty \). Thus, by (2.5), (3.3) and (3.4), we have, for \((x, s) \in \mathbb{R}^n \times (0, \infty)\),
\[ 0 \leq (ks + 1)u(x,s)\] (ks + 1)(|\phi(x)| + Ms)e^{\phi(x)} \in L^1(\mathbb{R}^n).
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2. We show that

$$F(s) - F(0) \geq -\frac{s}{p} (ks + 1) \int e^{(ks+1)\phi(x)} \left[ \max_{|z-x| \leq C_s} |D\phi(z)|^p \right] dx$$

$$+ \int \int_{0}^{s} k\phi(x)e^{(k\theta+1)\phi(x)} d\theta dx \tag{3.5}$$

(note that all terms in (3.5) are well–defined by the arguments above). In order to show (3.5), we see that

$$F(s) - F(0) = \int \left[ e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)} \right] dx + \int \left[ e^{(ks+1)\phi(x)} - e^{\phi(x)} \right] dx =: I + J.$$ Using the inequalities $u(x, s) \leq \phi(x)$ and

$$|e^b - e^a| = \left| \int_a^b e^t dt \right| \leq \max\{e^a, e^b\}|b - a|, \quad a, b \in \mathbb{R},$$

we have

$$0 \leq -e^{(ks+1)u(x,s)} + e^{(ks+1)\phi(x)} = \left| e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)} \right| \leq (ks + 1) \max\{e^{(ks+1)u(x,s)}, e^{(ks+1)\phi(x)}\} |u(x, s) - \phi(x)| \leq (ks + 1) e^{(ks+1)\phi(x)} |\phi(x) - u(x, s)|,$$

so that, by Lemma 2.1,

$$e^{(ks+1)u(x,s)} - e^{(ks+1)\phi(x)} \geq (ks + 1) e^{(ks+1)\phi(x)} [u(x, s) - \phi(x)] \geq -\frac{s}{p} (ks + 1) e^{(ks+1)\phi(x)} \left[ \max_{|z-x| \leq C_s} |D\phi(z)|^p \right].$$

This implies that

$$I \geq -\frac{s}{p} (ks + 1) \int e^{(ks+1)\phi(x)} \left[ \max_{|z-x| \leq C_s} |D\phi(z)|^p \right] dx.$$

On the other hand, we have

$$J = \int \left[ e^{(ks+1)\phi(x)} - e^{\phi(x)} \right] dx = \int \int_{0}^{s} \frac{d}{d\theta} e^{(k\theta+1)\phi(x)} d\theta dx = \int \int_{0}^{s} k\phi(x)e^{(k\theta+1)\phi(x)} d\theta dx.$$

Thus, we have obtained (3.5). Then, by Lebesgue’s dominated convergence theorem, we conclude (3.2). \qed
Proposition 3.2. — Let $p > 1$. Then, inequality (1.1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.7).

Proof. — By (1.7), we put $\phi(x) = p \log f(x)$. When $\text{Ent}(f^p) = -\infty$, (1.1) is trivial. So, we may assume that $\text{Ent}(e^\phi) = \text{Ent}(f^p) > -\infty$.

For any $k > 0$, we consider the functions $F$ of Lemma 3.1 and

$$B(s) = \left( \frac{nL_p e^{p-1}k}{p^p} \right)^{\frac{nk}{p}} (ks + 1)^{-\frac{n(k+1)}{p}}, \quad s \geq 0.$$ 

Note that (1.5) with $\alpha = 1$ and $\beta = ks + 1$ can be rewritten as

$$F(s) \leq F(0)^{ks + 1} B(s).$$ 

Since $B(0) = 1$, we have

$$\liminf_{s \to 0^+} \frac{F(s) - F(0)}{s} \leq F(0) \liminf_{s \to 0^+} \frac{F(0)^{ks} B(s) - B(0)}{s}.$$ 

Note that

$$\liminf_{s \to 0^+} \frac{F(0)^{ks} B(s) - B(0)}{s} = \frac{d}{ds} \left[ F(0)^{ks} B(s) \right] \bigg|_{s=0} = k \log \left( \int e^{\phi(x)} dx \right) + \frac{nk}{p} \log \left( \frac{nL_p k}{p^p e} \right).$$ 

Therefore, by Lemma 3.1, we obtain

$$\left[ -\frac{1}{p} \int e^{\phi(x)} |D\phi(x)|^p dx + \int \phi(x) e^{\phi(x)} dx \right] \leq \int e^{\phi(x)} dx \left[ k \log \left( \int e^{\phi(x)} dx \right) + \frac{nk}{p} \log \left( \frac{nL_p k}{p^p e} \right) \right],$$ 

so that

$$k \text{Ent}(e^\phi) \leq \frac{1}{p} \int e^{\phi(x)} |D\phi(x)|^p dx + \int e^{\phi(x)} dx \frac{nk}{p} \log \left( \frac{nL_p k}{p^p e} \right).$$ 

Since $e^{\phi(x)} = f(x)^p$ and $e^{\phi(x)} |D\phi(x)|^p = p^p |Df(x)|^p$ in $\mathbb{R}^n$, we have obtained

$$\text{Ent}(f^p) \leq \frac{p^{p-1}}{k} \int |Df(x)|^p dx + \frac{n}{p} \int f(x)^p dx \log \left( \frac{nL_p k}{p^p e} \right).$$ 

Minimizing the right-hand side with respect to $k > 0$ over $(0, \infty)$, we obtain (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (1.7).
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Now, we state the theorem of this paper.

**Theorem 3.3.** — Let $p > 1$. Inequality (1.1) holds true for all $f \in W^{1,p}(\mathbb{R}^n)$.

**Proof.** — We divide the proof of Theorem 3.3 into six steps as follows:

(i) We show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying

$$f \in C^1(\mathbb{R}^n), \quad 0 < f \text{ in } \mathbb{R}^n, \text{ and } D(\log f) \text{ is bounded on } \mathbb{R}^n. \quad (3.6)$$

(ii) We show (1.1) for $0 \leq f \in C^1_0(\mathbb{R}^n)$, where $C^1_0(\mathbb{R}^n)$ is the set of all $C^1(\mathbb{R}^n)$–functions with compact supports in $\mathbb{R}^n$.

(iii) We show (1.1) for $0 \leq f \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$.

(iv) We show (1.1) for $0 \leq f \in W^{1,p}(\mathbb{R}^n) \cap L^{p-\delta}(\mathbb{R}^n)$ with some $\delta \in (0, p-1)$.

(v) We show (1.1) for $0 \leq f \in W^{1,p}(\mathbb{R}^n)$.

(vi) We show (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$.

Here, in (iv) and (v), $f \geq 0$ means that $f \geq 0$ a.e. in $\mathbb{R}^n$. In (iv), we consider a constant $\delta \in (0, p-1)$, although we considered the case $\delta \in (0, p)$ in Lemma 2.2.

(i) Let $f \in W^{1,p}(\mathbb{R}^n)$ be a function satisfying (3.6). We denote by $L_0$ the Lipschitz constant of $\log f$. Note that there exists a constant $M > 0$ such that $\log f(x) \leq M$ on $\mathbb{R}^n$. If not, we find a sequence $\{x_j\}$ of $\mathbb{R}^n$ such that $\log f(x_j) \geq j + 1$ for each $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ arbitrarily. Since $\log f$ is Lipschitz continuous on $\mathbb{R}^n$, we have

$$\log f(x_j) - \log f(x) \leq L_0|x - x_j| \leq 1, \quad |x - x_j| \leq \frac{1}{L_0}, \quad j \in \mathbb{N},$$

so that $j \leq \log f(x)$ on $\{|x - x_j| \leq 1/L_0\}$. Thus,

$$\infty > \int f(x)^p dx = \int e^{p \log f(x)} dx \geq \int_{\{|x - x_j| \leq 1/L_0\}} e^{p \log f(x)} dx \geq e^{p j} \omega_n \left(\frac{1}{L_0}\right)^n,$$

where $\omega_n$ is the volume of the unit ball of $\mathbb{R}^n$. Since $j \in \mathbb{N}$ is arbitrary, this is a contradiction. Hence, there exists a constant $M > 0$ such that $\log f(x) \leq M$ on $\mathbb{R}^n$. Set

$$f_M(x) = f(x)e^{-M} = e^{\log f(x) - M}, \quad x \in \mathbb{R}^n.$$
It is easy to see that $f_M \in W^{1,p}(\mathbb{R}^n)$ fulfills (1.7). Thus, we have, by Proposition 3.2,

$$\text{Ent}(f^p_M) \leq \frac{n}{p} \int f_M(x)^p \log \left( L_p \frac{\int |Df_M(x)|^p \, dx}{\int f_M(x)^p \, dx} \right).$$

Since $\text{Ent}(f^p_M) = e^{-pM} \text{Ent}(f^p)$, we have shown (1.1) for $f \in W^{1,p}(\mathbb{R}^n)$ satisfying (3.6).

(ii) Let $0 \leq f \in C^1_0(\mathbb{R}^n)$. We set

$$f_{\epsilon}(x) = \left[ f(x)^p + \epsilon e^{-\langle x \rangle} \right]^{1/p}, \quad x \in \mathbb{R}^n, \quad 0 < \epsilon < 1,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then, $0 < f_{\epsilon} \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. Since $f$ has a compact support in $\mathbb{R}^n$, $D(\log f_{\epsilon})$ is bounded on $\mathbb{R}^n$. Thus, $f_{\epsilon}$ belongs to $W^{1,p}(\mathbb{R}^n)$ and fulfills (3.6). By (i), we see that $f_{\epsilon}$ satisfies

$$\int f_{\epsilon}^p \, dx \log \int f_{\epsilon}^p \, dx + \frac{n}{p} \int f_{\epsilon}^p \, dx \log \left( L_p \frac{\int |Df_{\epsilon}|^p \, dx}{\int f_{\epsilon}^p \, dx} \right),$$

$$\geq \int f_{\epsilon}^p \log f_{\epsilon}^p \, dx.$$  

Let $\delta \in (0, p - 1)$. Using the inequality

$$(a + b)^\kappa \leq a^\kappa + b^\kappa \quad a, b \geq 0, \quad 0 < \kappa < 1,$$

we have

$$|f_{\epsilon}(x)|^{p-\delta} \leq f(x)^{p-\delta} + e^{-\frac{\delta}{p} \langle x \rangle}.$$

Thus, $f_{\epsilon}, f \in L^{p-\delta}(\mathbb{R}^n)$. By Lemma 2.2, we see that (1.8) holds for this $\{f_{\epsilon}\}$ and $f$. Since $f_{\epsilon}, f \in W^{1,p}(\mathbb{R}^n)$ fulfill

$$\lim_{\epsilon \to 0^+} \int f_{\epsilon}(x)^p \, dx = \int f(x)^p \, dx, \quad \lim_{\epsilon \to 0^+} \int |Df_{\epsilon}(x)|^p \, dx = \int |Df(x)|^p \, dx,$$

we have shown (1.1) for $0 \leq f \in C^1_0(\mathbb{R}^n)$ by letting $\epsilon$ to $0+$ in (3.7).

(iii) Let $0 \leq f \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. Let $\rho$ be a $C^1_0(\mathbb{R}^n)$–function with $\rho(0) = 1$ and $0 \leq \rho \leq 1$ on $\mathbb{R}^n$. We set

$$f_{\epsilon}(x) = \rho(\epsilon x) f(x), \quad x \in \mathbb{R}^n, \quad 0 < \epsilon < 1.$$
Then, \(0 \leq f_\epsilon \in C^1_0(\mathbb{R}^n)\). Thus, by (ii), we see that (3.7) holds for this function \(f_\epsilon\). Since \(f_\epsilon\) and \(f\) satisfy (3.8), we conclude (1.1) for \(0 \leq f \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)\) by using Lemma 2.3 and letting \(\epsilon\) to 0+ in (3.7).

(iv) Let \(0 \leq f \in W^{1,p}(\mathbb{R}^n) \cap L^{p-\delta}(\mathbb{R}^n)\) with some \(\delta \in (0, p-1)\). Let \(\eta \in C_0^\infty(\mathbb{R}^n)\) be a nonnegative function such that \(\int \eta(x)dx = 1\). For a sufficiently small \(\epsilon > 0\), we define \(f_\epsilon\) by

\[
f_\epsilon(x) = \frac{1}{\epsilon^n} \int f(y) \eta \left(\frac{x-y}{\epsilon}\right) dy, \quad x \in \mathbb{R}^n.
\]

Then, \(0 \leq f_\epsilon \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) \cap L^{p-\delta}(\mathbb{R}^n)\) with some \(\delta \in (0, p-1)\), since \(p-\delta > 1\). Thus, by (iii), we see that (3.7) holds for this function \(f_\epsilon\).

Next, since \(f_\epsilon \to f\) in \(L^{p-\delta}(\mathbb{R}^n)\), we find a sequence \(\{\epsilon_j\} \subset (0, 1)\) such that \(\epsilon_j \to 0\) as \(j \to \infty\) and \(f_{\epsilon_j} \to f\) a.e. on \(\mathbb{R}^n\). Thus, by Lemma 2.2, we have

\[
\liminf_{j \to \infty} \int f_{\epsilon_j}(x)^p \log f_{\epsilon_j}(x) dx \geq \int f(x)^p \log f(x) dx.
\]

Since (3.8) is fulfilled, we have shown (1.1) for \(0 \leq f \in W^{1,p}(\mathbb{R}^n) \cap L^{p-\delta}(\mathbb{R}^n)\) with some \(\delta \in (0, p-1)\) by using Lemma 2.2 and letting \(\epsilon\) to 0+ in (3.7).

(v) Let \(0 \leq f \in W^{1,p}(\mathbb{R}^n)\). Set

\[
f_\epsilon(x) = \rho(\epsilon x)f(x), \quad x \in \mathbb{R}^n, \quad 0 < \epsilon < 1.
\]

Here, \(\rho\) is a \(C^1_0(\mathbb{R}^n)\)–function with \(\rho(0) = 1\) and \(0 \leq \rho \leq 1\) on \(\mathbb{R}^n\). Then, it is easy to see that \(0 \leq f_\epsilon \in W^{1,p}(\mathbb{R}^n) \cap L^{p-\delta}(\mathbb{R}^n)\) for all \(\delta \in (0, p-1)\). Thus, by (iv), (3.7) holds for this function \(f_\epsilon\). By the same arguments as those of (iii), we conclude (1.1) for \(0 \leq f \in W^{1,p}(\mathbb{R}^n)\).

(vi) We show (1.1) for \(f \in W^{1,p}(\mathbb{R}^n)\). Note that if \(f \in W^{1,p}(\mathbb{R}^n)\) then \(|f| \in W^{1,p}(\mathbb{R}^n)\). Hence, by (v) and the fact that \(|D|f|| \leq |Df|\) a.e. in \(\mathbb{R}^n\), we conclude (1.1) for \(f \in W^{1,p}(\mathbb{R}^n)\). \(\square\)
Bibliography