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Bounds for invariant distances on pseudoconvex Levi corank one domains and applications

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Abstract. — Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex Levi corank one domain with defining function r, i.e., the Levi form $\partial\overline{\partial}r$ of the boundary ∂D has at least $(n-2)$ positive eigenvalues everywhere on ∂D. The main goal of this article is to obtain bounds for the Carathéodory, Kobayashi and the Bergman distance between a given pair of points $p, q \in D$ in terms of parameters that reflect the Levi geometry of ∂D and the distance of these points to the boundary. Applications include an understanding of Fridman's invariant for the Kobayashi metric on Levi corank one domains, a description of the balls in the Kobayashi metric on such domains that are centered at points close to the boundary in terms of Euclidean data and the boundary behaviour of Kobayashi isometries from such domains.

Résumé. — Soit D un domaine borné, lisse, de $\mathbb{C}^n$, de fonction définissante r. Nous supposons D de corang de Levi 1, c'est-à-dire tel que la forme de Levi $\partial\overline{\partial}r$ possède au moins $(n-2)$ valeurs propres strictement positives en tout point du bord ∂D de D. Le but principal de l'article est d'obtenir des estimées des distances de Caratheodory, Kobayashhi et Bergman, entre deux points quelconques de D, dépendant de la distance de ces points à

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la frontière ainsi que de paramètres reflétant la géométrie de Levi de ∂D. Comme applications, nous présentons certaines propriétés de l'invariant de Fridman pour la métrique de Kobayashi sur les domaines de corang de Levi 1, nous décrivons les boules pour la métrique de Kobayashi, centrées en des points proches de la frontière, en termes de données euclidiennes, et nous étudions le comportement au bord des isométries pour la métrique de Kobayashi sur de tels domaines.

1. Introduction

The efficacy and ubiquity of invariant metrics such as those of Bergman, Carathéodory and Kobayashi, whenever they are defined, is a fact that hardly needs to be justified. Each of these metrics has its own strengths and weaknesses. For example, while the Carathéodory and Kobayashi metrics are distance decreasing under holomorphic mappings, a property not enjoyed by the Bergman metric in general, it makes up by being Hermitian whereas the Carathéodory and Kobayashi metrics are only upper semicontinuous in general. For a bounded domain $D \subset \mathbb{C}^n$, a point $z \in D$ and a tangent vector $v \in \mathbb{C}^n$, let $B_D(z,v), C_D(z,v)$ and $K_D(z,v)$ be the infinitesimal Bergman, Carathéodory and Kobayashi metrics on D. To recall their definition, let $\Delta_r \subset \mathbb{C}$ be the open disc of radius $r > 0$ centered at the origin and this will be abbreviated as Δ when $r = 1$. Then

- $C_D(z,v) = \sup\big\{ |df(z)v| : f \in \mathcal{O}(D,\Delta)\big\}$,
- $K_D(z,v) = \inf\big\{1/r : \text{ there exists } f \in \mathcal{O}(\Delta_r, D) \text{ with } f(0) = z \text{ and } df(0) = v\big\}$, and
- $B_D(z,v) = b_D(z,v)/(K_D(z,\overline{z}))^{1/2}$, where

$$K_D(z,\overline{z}) = \sup\big\{ |f(z)|^2 : f \in \mathcal{O}(D), \|f\|_{L^2(D)} \leqslant 1\big\}$$

is the Bergman kernel and

$$b_D(z,v) = \sup\big\{ |df(z)v| : f \in \mathcal{O}(D), f(z) = 0 \text{ and } \|f\|_{L^2(D)} \leqslant 1\big\}.$$

While no exact formulae for these metrics are known in general, part of their utility derives from an understanding of how they behave when z is close to the boundary ∂D. This is a much studied question and we may refer to [1], [2], [18], [11], [12], [13], [26], [29], [30] and [40] which provide quantitative boundary estimates for these infinitesimal metrics on a wide variety of smoothly bounded pseudoconvex domains in $\mathbb{C}^n$. Now given a pair of distinct points $p, q \in D$, we may compute the lengths of all possible piecewise smooth curves in D that join p and q using these infinitesimal metrics. The infimum of all such lengths gives a distance function on D – these integrated versions are the Bergman, the inner Carathéodory and the Kobayashi distances on D and they will be denoted by $d^b_D(p,q), d^c_D(p,q)$ and $d^k_D(p,q)$ respectively for $p,q \in D$. Recall that the Carathéodory distance d^{Cara}_D for D is defined by setting

$$d^{Cara}_D(p,q) = \sup\big\{d^p_\Delta\big(f(p),f(q)\big) : f \in \mathcal{O}(D,\Delta)\big\},$$

where d^p_Δ denotes the Poincaré distance on the unit disc. It is well-known that d^c_D is always at least as big as d^{Cara}_D. In general, d^{Cara}_D need not coincide with the inner Carathéodory distance d^c_D. Moreover, it is known from the work of Reiffen ([49]) that the integrated distance of the infinitesimal metric $C_D(z,v)$, coincides with the inner distance associated to d^{Cara}_D which we denote by d^c_D. Recent work on the comparison of d^{Cara}_D and another invariant function, the Lempert function, may be found in [45]. Much less is known about the boundary behaviour of the integrated distances and our interest here lies in obtaining estimates of them. Partial answers to this question may be found in [1], [3] and [24], all of which deal with strongly pseudoconvex domains in $\mathbb{C}^n$. Optimal estimates of the boundary behaviour of invariant distances for domains with $C^{1+\epsilon}$-smooth boundary in dimension one, may be found in the recent work [46] where estimates for convexifiable domains are also dealt with. A more complete treatment for strongly pseudoconvex domains in $\mathbb{C}^n$ was given by Balogh–Bonk in [6] using the Carnot–Carathéodory metric that exists on the boundary of these domains. An analogue of these estimates was later obtained by Herbort in [31] on smoothly bounded weakly pseudoconvex domains of finite type in $\mathbb{C}^2$ using the bidiscs of Catlin (see [11]).

We are interested in supplementing the results of [6] and [31] by obtaining bounds for these distances on a Levi corank one domain in $\mathbb{C}^n$. A smoothly bounded pseudoconvex domain $D \subset \mathbb{C}^n$ of finite type with smooth defining function r (here the sign of r is chosen so that $D = \{r < 0\}$) is said to be a Levi corank one domain if the Levi form $\partial\overline{\partial} r$ of ∂D has at least $n-2$ positive eigenvalues everywhere on ∂D. An example of a Levi corank one domain is the egg domain

$$E_{2m} = \big\{z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \ldots + |z_{n-1}|^2 + |z_n|^2 < 1\big\} \tag{1.1}$$

for some integer $m \geqslant 1$. Let $2m$ be the least upper bound on the 1–type of the boundary points of D and let U be a tubular neighbourhood of ∂D such that for any $A \in U$ there is a unique orthogonal projection to ∂D which will be denoted by A^* such that $|A - A^*| = \operatorname{dist}(A, \partial D) = \delta_D(A)$. Furthermore, we may assume that the normal vector field, given at any $\zeta \in U$ by

$$\nu(\zeta) = \big(\partial r/\partial\overline{z}_1(\zeta), \partial r/\partial\overline{z}_2(\zeta), \ldots, \partial r/\partial\overline{z}_n(\zeta)\big)$$

has no zeros in U and is normal to the hypersurface $\Gamma = \{r(z) = r(\zeta)\}$. Fix $\zeta \in U$. After a permutation of coordinates if necessary, we may assume that $\partial r/\partial \overline{z}_n(\zeta) \neq 0$. Then note that the affine transform

$$\phi^{\zeta}(z) = \big(z_1 - \zeta_1, \ldots, z_{n-1} - \zeta_{n-1}, \langle \nu(\zeta), z - \zeta \rangle\big)$$

translates ζ to the origin and is invertible by virtue of the fact that $\partial r/\partial \overline{z}_n(\zeta) \neq 0$. Moreover, ϕ^{ζ} reduces the linear part of the Taylor expansion of $r^{\zeta} = r \circ (\phi^{\zeta})^{-1}$ about the origin, to

$$r^{\zeta}(z) = r(\zeta) + 2\Re z_n + \text{ terms of higher order.} \tag{1.2}$$

In particular, the origin lies on the hypersurface $\Gamma_{\zeta}^{r^{\zeta}}$, the zero set of $r^{\zeta}(z) - r(\zeta)$ and the normal to this hypersurface at the origin, is the unit vector along the $\Re z_n$-axis. In fact, by the continuity of $\partial r^{\zeta}/\partial z_n$, we get a small ball $B(0, R_0)$ (where as usual $B(p, r)$ is the ball around p with radius r), with the property that the vector field $\nu(z)$ has a non-zero component along (the constant vector field) $L_n = \partial/\partial z_n$ for all z in $B(0, R_0)$. Indeed, we may assume that $|\partial r^{\zeta}/\partial z_n(z)|$ is bounded below by any positive constant less than 1. By shrinking $B(0, R_0)$, if necessary, we can ensure that $|\partial r^{\zeta}/\partial z_n(z)| \geqslant 1/2$. We will perform such shrinking of neighbourhoods henceforth tacitly, taking care only that the number of times we have done this at the end of all, is finite. Furthermore, we may repeat the above procedure for any $\zeta \in U$. Since r and $(\phi^{\zeta})^{-1}$ are smooth (as functions of ζ), the family $\{\partial r^{\zeta}/\partial z_n(z)\}$ of functions parametrized by ζ, is equicontinuous. Moreover, the neighbourhood U is precompact and hence, we may choose the radius R_0 to be independent of ζ.

Continuing with some helpful background and terminology, we assign a weight of $1/2m$ to the variable z_1, $1/2$ to the variable z_α for $2 \leqslant \alpha \leqslant n-1$ and 1 to the variable z_n and for multi-indices $J = (j_1, j_2, \ldots, j_n)$ and $K = (k_1, k_2, \ldots k_n)$, the weight of the monomial $z^J \overline{z}^K = z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n} \overline{z}_1^{k_1} \overline{z}_2^{k_2} \ldots \overline{z}_n^{k_n}$, is defined to be

$$\mathrm{wt}(z^j \overline{z}^K) = (j_1 + k_1)/2m + (j_2 + k_2)/2 + \ldots + (j_{n-1} + k_{n-1})/2 + (j_n + k_n).$$

Analogous to the definition of degree, we say that a polynomial in $\mathbb{C}[z, \overline{z}]$ is of weight λ if the maximum of the weights of its monomials is λ; the weight of a polynomial mapping is the maximum of the weights of its components. Thus note that the weight of the map ϕ^{ζ} is 1 while its 'multiweight' is $(1/2m, 1/2, \ldots, 1/2, 1)$.

We now summarize the special normal form for Levi corank one domains (cf. [14]), details of which are discussed in the appendix. For each $\zeta \in U$, there is a radius $R > 0$ and an injective holomorphic mapping Φ^{ζ} :

$B(\zeta, R) \to \mathbb{C}^n$ such that the transformed defining function $\rho^\zeta = r^\zeta \circ (\Phi^\zeta)^{-1}$ becomes

$$\begin{aligned}\rho^\zeta(w) = r(\zeta) + 2\Re w_n + \sum_{l=2}^{2m} P_l(\zeta; w_1) + |w_2|^2 + \ldots + |w_{n-1}|^2 \\ + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leqslant m \\ j,k>0}} \Re\Big(\big(b^\alpha_{jk}(\zeta) w_1^j \overline{w}_1^k\big) w_\alpha\Big) + R(\zeta; w) \quad (1.3)\end{aligned}$$

where

$$P_l(\zeta; w_1) = \sum_{j+k=l} a^l_{jk}(\zeta) w_1^j \overline{w}_1^k$$

are real valued homogeneous polynomials of degree l without harmonic terms and the error function $R(\zeta, w) \to 0$ as $w \to 0$ faster than one of the monomials of weight 1. Further, the map Φ^ζ is actually a holomorphic polynomial automorphism of weight one of the form

$$\Phi^\zeta(z) = \Big(z_1 - \zeta_1, G_\zeta(\tilde{z} - \tilde{\zeta}) - Q_2(z_1 - \zeta_1), \langle \nu(\zeta), z - \zeta \rangle - Q_1('z - '\zeta)\Big) \quad (1.4)$$

where $G_\zeta \in GL_{n-2}(\mathbb{C}), \tilde{z} = (z_2, \ldots z_{n-1}), 'z = (z_1, z_2, \ldots, z_{n-1})$ and Q_2 is a vector valued polynomial whose α-th component is a polynomial of weight at most $1/2$ of the form

$$Q_2^\alpha(t) = \sum_{k=1}^{m} b_k^\alpha(\zeta) t^k$$

for $t \in \mathbb{C}$ and $2 \leqslant \alpha \leqslant n-1$. Finally, $Q_1('z - '\zeta)$ is a polynomial of weight at most 1 and is of the form $\hat{Q}_1\big(z_1 - \zeta_1, G_\zeta(\tilde{z} - \tilde{\zeta})\big)$ with $\hat{Q}_1$ of the form

$$\hat{Q}_1(t_1, t_2, \ldots, t_{n-1}) = \sum_{k=2}^{2m} a_{k0}(\zeta) t_1^k - \sum_{\alpha=2}^{n-1} \sum_{k=1}^{m} a_k^\alpha(\zeta) t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} c_\alpha(\zeta) t_\alpha^2.$$

Since G_ζ is just a linear map, $Q_1('z - '\zeta)$ also has the same form when considered as an element of the algebra of holomorphic polynomials $\mathbb{C}['z - '\zeta]$, when ζ is held fixed. The coefficients of all the polynomials, mentioned above, are smooth functions of ζ. By shrinking U, if needed, we can ensure that $R > 0$ is independent of ζ because these new coordinates depend smoothly on ζ. Further $Q_1(0, \ldots, 0) = 0$ and that the lowest degree of its monomials is at least two. On the other hand, while $Q_2(0) = (0, \ldots, 0)$, the lowest degree of the terms in Q_2^α is at least (and can be) one. In case, the polynomials Q_2^α and Q_1 are identically zero, it turns out that the arguments become even simpler and this will be evident from the sequel. Note that $\Phi^\zeta(\zeta) = 0$ and

$$\Phi^\zeta(\zeta_1, \ldots, \zeta_{n-1}, \zeta_n - \epsilon) = (0, \ldots, 0, -\epsilon\, \partial r/\partial \overline{z}_n(\zeta)).$$

Note that each Φ^ζ is a polynomial automorphism of $\mathbb{C}^n$ and it will be referred to as the canonical change of variables for a Levi corank one domain. Though it reduces the defining function to (1.3), it is not simple in the sense of being weighted homogeneous in the variables $z - \zeta$. This is evident in (1.2) where only the linear part of the expansion is reduced to its simplest form (up to a permutation) and later in the final transformation (1.4). Evidently, Φ^ζ is neither a 'decoupled polynomial mapping' nor a weighted homogeneous map in $z - \zeta$. This poses difficulties in imitating Herbort's calculations directly and we shall discuss these difficulties as we encounter them, namely in Section 2.6.

Recall that in general, the degree (or weight) of a holomorphic polynomial automorphism of $\mathbb{C}^n$ is not equal to that of its inverse. However, the inverse of Φ^ζ has the same form as Φ^ζ. The precise definition and properties of these maps is provided in the appendix. These maps belong to the algebraic group $\mathcal{E}_L$ consisting of weight preserving polynomial automorphisms whose first component has weight $1/2m$, the next $n-2$ components having weight $1/2$ each and the last component having weight 1; in particular, the weight of Φ^ζ is one.

To construct the distinguished polydiscs around ζ (more precisely, biholomorphic images of polydiscs), with notations as in [31] define for each $\delta > 0$, the special-radius

$$\tau(\zeta, \delta) = \min\Big\{ \Big(\delta/|P_l(\zeta, \cdot)|\Big)^{1/l}, \Big(\delta^{1/2}/B_{l'}(\zeta)\Big)^{1/l'} : 2 \leqslant l \leqslant 2m, 2 \leqslant l' \leqslant m \Big\}. \tag{1.5}$$

where

$$B_{l'}(\zeta) = \max\{|b^\alpha_{jk}(\zeta)| \ : \ j + k = l',\ 2 \leqslant \alpha \leqslant n-1\},\ 2 \leqslant l' \leqslant m.$$

Here, the norm of the homogeneous polynomials $P_l(\zeta, \cdot)$ of degree l, is taken according to the following convention: for a homogeneous polynomial

$$p(v) = \sum_{j+k=l} a_{j,k} v^j \bar{v}^k,$$

define $|p(\cdot)| = \max_{\theta \in \mathbb{R}} |p(e^{i\theta})|$. It was shown in [14] that the coefficients b^α_{jk}'s in the above definition of $\tau(\zeta, \delta)$ are insignificant and may be dropped out, so that

$$\tau(\zeta,\delta)=\min\Big\{\Big(\delta/|P_l(\zeta,\cdot)|\Big)^{1/l}\ :\ 2\leqslant l\leqslant 2m\Big\}.$$

Set

$$\tau_1(\zeta,\delta)=\tau(\zeta,\delta)=\tau,\tau_2(\zeta,\delta)=\ldots=\tau_{n-1}(\zeta,\delta)=\delta^{1/2},\tau_n(\zeta,\delta)=\delta$$

and define

$$R(\zeta,\delta)=\{z\in\mathbb{C}^n:|z_k|<\tau_k(\zeta,\delta),\ 1\leqslant k\leqslant n\}$$

which is a polydisc around the origin in $\mathbb{C}^n$ with polyradii $\tau_k(\zeta,\delta)$ along the z_k direction for $1\leqslant k\leqslant n$ and let

$$Q(\zeta,\delta)=(\Phi^{\zeta})^{-1}\big(R(\zeta,\delta)\big)$$

which is a distorted polydisc around ζ. It was shown in [14] that for all sufficiently small positive δ – say, for all δ in some interval $(0,\delta_e)$ with δ_e some number less than 1 – there is a uniform constant $C_0>1$ such that

(i) these 'polydiscs' satisfy the engulfing property, i.e., for all $\zeta\in U$ if $\eta\in Q(\zeta,\delta)$, then $Q(\eta,\delta)\subset Q(\zeta,C_0\delta)$ and

(ii) if $\eta\in Q(\zeta,\delta)$ then $\tau(\eta,\delta)\leqslant C_0\tau(\zeta,\delta)\leqslant C_0^2\tau(\eta,\delta)$.

For $A,B\in U$ let

$$M(A,B)=\{\delta>0:A\in Q(B,\delta)\}$$

which is a sub-interval of $[0,\infty)$ since the distorted polydiscs $Q(B,\delta)$ are monotone increasing as a function of δ. Then define

$$d'(A,B)=\inf M(A,B)$$

if $M(A,B)\neq\emptyset$ (otherwise we simply set it equal to $+\infty$) and let

$$d(A,B)=\min\{d'(A,B),|A-B|_{l^\infty}\}\tag{1.6}$$

which is an auxiliary pseudo-distance function on D. Here we work with the l^∞ norm of $A-B$ instead of the usual Euclidean distance for convenience. Now, let

$$\eta(A,B)=\log\left(1+\frac{d(A,B)}{\delta_D(A)}+\sum_{\alpha=2}^{n-2}\frac{\left|\hat{\Phi}^A(B)_\alpha\right|}{\sqrt{\delta_D(A)}}+\frac{\left|\hat{\Phi}^A(B)_1\right|}{\tau\big(A,\delta_D(A)\big)}\right)$$

where $\hat{\Phi}^A(B)$ differs from $\Phi^A(B)$ by a permutation of co-ordinates to ensure $\nu(A)_n \neq 0$ i.e., $\hat{\Phi}^A(B) = \Phi^A \circ P_A(B)$ where $P_A(z)$ is any permutation of the variables $z_1, \ldots, z_n$ such that $\partial(r \circ P_A^{-1})/\partial z_n \neq 0$. The existence of P_A follows from the assumption that $\nu(A) \neq 0$. If A, B are close enough, then we may certainly choose $P_A = P_B$. Finally, let

$$\varrho(A,B) = 1/2\Big(\eta(A,B) + \eta(B,A)\Big)$$

and as usual, for quantities S, T we will write $S \lesssim T$ to mean that there is a constant $C > 0$ such that $S \leqslant CT$, while $S \approx T$ means that $S \lesssim T$ and $T \lesssim S$ both hold.

It is known ([13]) that these infinitesimal metrics are uniformly comparable on a pseudoconvex Levi corank one domain D, i.e., $B_D(z,v) \approx C_D(z,v) \approx K_D(z,v)$ uniformly for all $(z,v) \in D \times \mathbb{C}^n$. Thus to get lower bounds for the integrated distances it suffices to understand d_D^c alone. We use the inequality $d_D^{Cara} \leqslant d_D^c$ to get sharper lower bounds on d_D^c and this is the only motivation for introducing d_D^{Cara}.

THEOREM 1.1. — *Let $D \subset \mathbb{C}^n$ be a smoothly bounded Levi corank one domain and U a tubular neighbourhood of ∂D as above. Then for all $A, B \in U \cap D$ we have that*

$$\varrho(A,B) - l \lesssim d_D^c(A,B) \leqslant d_D^k(A,B) \lesssim \varrho(A,B) + L$$

where l and L are some positive constants. In particular, the same bounds hold for $d_D^b(A,B)$ as well.

Obtaining a lower bound for d_D^c amounts to understanding the separation properties of bounded holomorphic functions on D. To do this, the boundary ∂D will be bumped outwards near a given $\zeta \in \partial D$ as in [13] to get a larger domain that contains D near ζ. The pluricomplex Green function for these large domains will then be used to construct weights for an appropriate $\overline{\partial}$-problem as in [31] and the solution thus obtained will be modified to get a holomorphic function with a suitable L^2 bound near ζ. Solving another $\overline{\partial}$-problem will extend this to a bounded holomorphic function on D with control on its separation properties. This function is then used to bound d_D^c from below. Theorem 1.1 has several consequences and we elaborate them in the following paragraphs.

In [21], Fridman defined an interesting non-negative continuous function on a given Kobayashi hyperbolic complex manifold of dimension n, say X that essentially determines the largest Kobayashi ball at a given point on X which is comparable to the unit ball $\mathbb{B}^n$. To be more precise, let $B_X(p,r)$

denote the Kobayashi ball around $p \in X$ of radius $r > 0$. The hyperbolicity of X ensures that the intrinsic topology on X is equivalent to the one induced by the Kobayashi metric. Thus for small $r > 0$, the ball $B_X(p, r)$ is contained in a coordinate chart around p and hence there is a biholomorphic imbedding $f : \mathbb{B}^n \to X$ with $B_X(p, r) \subset f(\mathbb{B}^n)$. Let $\mathcal{R}$ be the family of all $r > 0$ for which there is a biholomorphic imbedding $f : \mathbb{B}^n \to X$ with $B_X(p, r) \subset f(\mathbb{B}^n)$. Then $\mathcal{R}$ is evidently non-empty. Define

$$h_X(p, \mathbb{B}^n) = \inf_{r \in \mathcal{R}} \frac{1}{r}$$

which is a non-negative real valued function on X. Since the Kobayashi metric is biholomorphically invariant, the same holds for $h_X(p, \mathbb{B}^n)$ which shall henceforth be called Fridman's invariant. The same construction can be done using any invariant metric that induces the intrinsic topology on X and we may also work with homogeneous domains other than the unit ball. However we shall work with the Kobayashi metric exclusively. A useful property identified in [21] was that if $h_X(p_0, \mathbb{B}^n) = 0$ for some $p_0 \in X$, then $h_X(p, \mathbb{B}^n) = 0$ for all $p \in X$ and that X is biholomorphic to $\mathbb{B}^n$. Moreover, $p \mapsto h_X(p, \mathbb{B}^n)$ is continuous on X. The boundary behaviour of Fridman's invariant was studied in [44] for a variety of pseudoconvex domains and the following statement extends this to the class of Levi corank one domains.

Theorem 1.2. — *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain of finite type. Let $\{p^j\} \subset D$ be a sequence that converges to $p^0 \in \partial D$. Assume that the Levi form of ∂D has rank at least $n-2$ at p^0. Then*

$$h_D(p^j, \mathbb{B}^n) \to h_{D_\infty}(('0, -1), \mathbb{B}^n)$$

as $j \to \infty$ where D_∞ is a model domain defined by

$$D_\infty = \left\{ z \in \mathbb{C}^n : 2\Re z_n + P_{2m}(z_1, \overline{z}_1) + |z_2|^2 + \ldots + |z_{n-1}|^2 < 0 \right\}$$

and $P_{2m}(z_1, \overline{z}_1)$ is a subharmonic polynomial of degree at most $2m$ ($m \geqslant 1$) without harmonic terms, $2m$ being the 1-type of ∂D at p^0.

By scaling D along $\{p^j\}$, we obtain a sequence of domains D^j, each containing the base point $('0, -1)$, that converge to D_∞ as defined above. It should be noted that the polynomial $P_{2m}(z_1, \overline{z}_1)$ depends on how the sequence $\{p^j\}$ approaches p^0. This is simply restating the known fact that unlike the strongly pseudoconvex case, model domains near a weakly pseudoconvex point are not unique. Since Fridman's invariant is defined in terms of Kobayashi balls, the main technical step in this theorem is to show the convergence of the Kobayashi balls in D^j around $('0, -1)$ with a fixed radius

$R > 0$ to the corresponding Kobayashi ball on D_∞ with the same radius. Theorem 1.1 is used in this step.

Another consequence of scaling near a Levi corank one point combined with Theorem 1.1 is a description of the Kobayashi balls near such points in terms of parameters that reflect the Levi geometry of the boundary. This is well known in the strongly pseudoconvex case – indeed, the Kobayashi ball around a given point p near a strongly pseudoconvex boundary point is essentially an ellipsoid whose major and minor axis are of the order of $(\delta_D(p))^{1/2}$ and $\delta_D(p)$ respectively.

THEOREM 1.3. — *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex Levi corank one domain. Then for all $R > 0$, there are constants $C_1, C_2 > 0$ depending only on R and D such that*

$$Q(q, C_1\, \delta_D(q)) \subset B_D(q, R) \subset Q(q, C_2\, \delta_D(q))$$

for each $q \in D$ sufficiently close to ∂D.

Analogues of this for weakly pseudoconvex finite type domains in $\mathbb{C}^2$ and convex finite type domains in $\mathbb{C}^n$ were obtained in [44] by a direct scaling. These estimates are useful in establishing a generalized sub-mean value property for plurisubharmonic functions and defining suitable approach regions for boundary values of functions in H^p spaces at least on strongly pseudoconvex domains (see [38] for example).

The next and last class of applications of Theorem 1.1 deal with the problem of biholomorphic inequivalence of domains in $\mathbb{C}^n$. The paradigm underlying many of the results in this direction (see for example [9] and [47]) is that a pair of domains in $\mathbb{C}^n$ cannot have boundaries with different Levi geometry and yet be biholomorphic. For proper holomorphic mappings, it is known (see for example [19] and [17]) that the target domain cannot have a boundary with more complicated Levi degeneracies than the source domain. Fridman's invariant provides another approach to this problem with the advantage of quickly reducing it to the case of algebraic model domains. Here is an example to illustrate this point of view and we refer the reader to [5] for an alternative proof that works for proper holomorphic mappings as well.

THEOREM 1.4. — *Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded domains with $p^0 \in \partial D_1$ and $q^0 \in \partial D_2$. Assume that ∂D_1 is C^2-smooth strongly pseudoconvex near p^0 and that ∂D_2 is C^∞-smooth pseudoconvex and of finite type near q^0. Suppose further that the Levi form of ∂D_2 has rank exactly $n-2$ at q^0. Then there cannot exist a biholomorphism $f : D_1 \to D_2$ with $q^0 \in cl_f(p^0)$, the cluster set of p^0.*

When p^0 is also a Levi corank one point on ∂D_1, it is known ([55]) that a proper holomorphic mapping $f : D_1 \to D_2$ extends continuously to ∂D_1 near p^0. In fact, a similar result can be proved for isometries of these metrics as well. To set things in perspective, let D, G be bounded domains in $\mathbb{C}^n$ equipped with one of these invariant metrics. An isometry $f : D \to G$ is simply a distance preserving map. Note that no further assumptions such as smoothness or holomorphicity are being included as part of the definition of an isometry. A C^1-Kobayashi isometry is a C^1-diffeomorphism f from D onto G with $f^*(K_G) = K_D$, and C^1-Bergman and Carathéodory isometries are defined likewise. Of course, biholomorphisms are examples of isometries, but whether all isometries are necessarily holomorphic or conjugate holomorphic seems interesting to ask. Let us say that an isometry is *rigid* if it is either holomorphic or conjugate holomorphic. If isometries of the Bergman metric between a pair of strongly pseudoconvex domains in $\mathbb{C}^n$ are considered, a result in [27] shows that the isometry must be rigid. Recent work on the rigidity of local Bergman isometries may be found in [43]. Isometries of the Kobayashi metric between a strongly pseudoconvex domain and the ball are also shown to be rigid in [35] while a more recent result in [25] proves the rigidity of an isometry between a pair of strongly convex domains even in the non equidimensional case; the choice of either the Kobayashi or the Carathéodory metric is irrelevant here since the two coincide. However, this seems to be unknown for isometries of the Kobayashi or the Carathéodory metric between a pair of strongly pseudoconvex domains. On the other hand, the results of [42] and [6] show that isometries behave very much like holomorphic mappings. In particular, they exhibit essentially the same boundary behaviour as biholomorphisms. The following statements further justify this claim and extend some of the results in [42].

THEOREM 1.5. — *Let $f : D_1 \to D_2$ be a C^1-Kobayashi isometry between two bounded domains in $\mathbb{C}^n$. Let p^0 and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that ∂D_1 is C^∞-smooth pseudoconvex of finite type in a neighbourhood U of p^0 and that ∂D_2 is C^2-smooth strongly pseudoconvex in a neighbourhood V of q^0. Suppose further that the Levi form of ∂D_1 has rank at least $n-2$ near p^0 and that q^0 belongs to the cluster set of p^0 under f. Then f extends as a continuous mapping to a neighbourhood of p^0 in $\overline{D}_1$.*

The following result provides an explicit computation of $K_{E_{2m}}$, the Kobayashi metric of the egg domain E_{2m} introduced in (1.1) for $m \geqslant 1/2$ – notice that when m is not an integer, the boundary of E_{2m} is not smooth. This extends the computation done for such egg domains in $\mathbb{C}^2$ in [41] and [10]. It is straightforward to see that for any $\theta \in \mathbb{R}$ and $(p_1, \ldots, p_n) \in E_{2m}$,

$$(z_1, \ldots, z_n) \mapsto \left(e^{\iota\theta} \frac{\left(1 - |\hat{p}|^2\right)^{1/2m}}{\left(1 - \langle \hat{z}, \hat{p} \rangle\right)^{1/m}} z_1, \Psi(\hat{z}) \right)$$

is an automorphism of E_{2m}. Here $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product in $\mathbb{C}^{n-1}$, $z \in \mathbb{C}^n$ is written as $z = (z_1, \hat{z}), \hat{z} = (z_2, \ldots, z_n)$ and Ψ is an automorphism of $\mathbb{B}^{n-1}$ that takes $\hat{p}$ to the origin. More precisely,

$$\Psi(\hat{z}) = \frac{\left(1 - |\hat{p}|^2\right)^{1/2} \left(\hat{z} - \frac{\langle \hat{z}, \hat{p} \rangle}{|\hat{p}|^2} \hat{p} \right) - \left(1 - \frac{\langle \hat{z}, \hat{p} \rangle}{|\hat{p}|^2} \right) \hat{p}}{1 - \langle \hat{z}, \hat{p} \rangle}$$

for $\hat{p} \neq \hat{0}$. Since automorphisms are isometries for the Kobayashi metric, it is enough to compute the explicit formula for $K_{E_{2m}}$ at the point $(p, \hat{0}) \in E_{2m}$, for $0 < p < 1$, from which the general formula follows by composition with an appropriate automorphism of E_{2m} as described above.

Theorem 1.6. — *The Kobayashi metric for E_{2m} at the point $(p, \hat{0})$ for $0 < p < 1$, is given by*

$$K_{E_{2m}}\big((p, 0, \ldots, 0), (v_1, \ldots, v_n)\big) =$$

$$\begin{cases} \left(\frac{m^2 p^{2m-2} |v_1|^2}{(1-p^{2m})^2} + \frac{|v_2|^2}{1-p^{2m}} + \cdots + \frac{|v_n|^2}{1-p^{2m}} \right)^{1/2} & \text{for} \quad u \leqslant p, \\ \frac{m\alpha(1-t)|v_1|}{p(1-\alpha^2)(m(1-t)+t)} & \text{for} \quad u > p, \end{cases}$$

where

$$u = \left(\frac{m^2 |v_1|^2}{|v_2|^2 + \cdots + |v_n|^2} \right)^{1/2}, \tag{1.7}$$

$$t = \frac{2m^2 p^2}{u^2 + 2m(m-1)p^2 + u\big(u^2 + 4m(m-1)p^2\big)^{1/2}} \tag{1.8}$$

and α is the unique positive solution of

$$\alpha^{2m} - t\alpha^{2m-2} - (1-t)p^{2m} = 0.$$

Moreover, $K_{E_{2m}}$ is C^1-smooth on $E_{2m} \times (\mathbb{C}^n \setminus \{0\})$ for $m > 1/2$.

It should be mentioned that the Kobayashi metric at the origin, $K_{E_{2m}}\big((0, 0, \ldots, 0), v\big) = q_{E_{2m}}(v)$ where $q_{E_{2m}}$ denotes the *Minkowski functional* of E_{2m}. Moreover, the number α has an alternate definition: $\alpha =$

$q_{E_{2m}}\left(p(1-t)^{1/2m}, (t/(n-1))^{1/2}, \ldots, (t/(n-1))^{1/2}\right)$. Theorem 1.6 is useful in proving the isometric inequivalence of weakly spherical Levi corank one domains (the notion of weak sphericity is recalled from [7] and defined in the last section) and strongly pseudoconvex domains – see [42] for a related result in $\mathbb{C}^2$.

Theorem 1.7. — *Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded domains with $p^0 \in \partial D_1$ and $q^0 \in \partial D_2$. Assume that there are holomorphic coordinates in a neighbourhood U_1 around p^0 in which $U_1 \cap D_1$ is defined by*

$$\left\{z \in \mathbb{C}^n : 2\Re z_n + |z_1|^{2m} + |z_2|^2 + \ldots + |z_{n-1}|^2 + R(z, \overline{z}) < 0\right\}$$

where $m > 1$ is a positive integer and the error function $R(z, \overline{z}) \to 0$ faster than at least one of the monomials of weight one. Suppose further that ∂D_2 is C^2-smooth strongly pseudoconvex near q^0. Then there cannot exist a C^1-Kobayashi isometry $f : D_1 \to D_2$ with $q^0 \in cl_f(p^0)$.

To outline the proof of this statement, we scale D_1 along a sequence of points that converges to p^0 along the inner normal to D_1. The limit domain is exactly E_{2m}. In trying to adapt the scaling method for isometries, note that the normality of the scaled isometries needs the stability of the Kobayashi distance function on Levi corank one domains, i.e., the arguments used in analysing Fridman's invariant. This ensures the existence of the limit of scaled isometries, the limit being an isometry between E_{2m} and the unit ball. The final argument, which uses the explicit form of the Kobayashi metric on E_{2m} as given above, involves showing that this continuous isometry is in fact rigid which then leads to a contradiction.

Acknowledgements. — An independent and technically different proof of Theorem 1.1 has also been obtained by Gregor Herbort. We were informed of this after the completion of this manuscript and we would like to thank him for sharing a copy of his then unpublished manuscript and for a clarifying remark on an earlier version of this article that has been incorporated; his work has recently appeared in [32]. Thanks are also due to Peter Pflug for a timely comment that helped remove an ambiguity and to Nikolai Nikolov who pointed out the relevance of [45] and [46]. Last but not least, we thank the anonymous referee for a careful reading of this manuscript.

2. Proof of Theorem 1.1

We begin with some helpful generalities, notation and terminology. In general, if ρ is a smooth function defined on some open set V in $\mathbb{C}^n$, then there are 2 simple ways of constructing Hermitian forms associated to ρ.

We shall restrict to the case when the level sets of ρ are smooth real hypersurfaces, so let us assume that ρ has a nowhere-vanishing gradient in V. Consider the canonical $+ve$-semi definite Hermitian form associated to ρ by

$$\langle Y, W\rangle \to \Big(\sum_{j=1}^{n} \overline{\frac{\partial \rho}{\partial z_j}(z) Y_j}\Big)\Big(\sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k}(z) W_k\Big) = \sum_{j,k=1}^{n} \overline{\frac{\partial \rho}{\partial z_j}(z)} \frac{\partial \rho}{\partial z_k}(z) \overline{Y}_j W_k$$

for Y, W tangent vectors at z. The associated quadratic form is given by

$$C_\rho(z, Y) = \Big|\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) Y_j\Big|^2 \tag{2.1}$$

which is evidently positive semi-definite. Denote by $M_\rho(z)$, the matrix associated to this form with respect to (unless otherwise mentioned) the standard co-ordinates. Assume after a permutation of coordinates, that $\partial\rho/\partial z_n \neq 0$. Let $\zeta \in V$ and for $1 \leqslant j \leqslant n-1$ define the vectors

$$L_j(\zeta) = \Big(0, \ldots, 0, 1, 0, \ldots, 0, b_j^\zeta\Big)$$

where the jth entry in the above tuple is 1 and

$$b_j^\zeta = b_j(\zeta, \overline{\zeta}) = -\Big(\frac{\partial \rho}{\partial z_n}(\zeta)\Big)^{-1}\Big(\frac{\partial \rho}{\partial z_j}(\zeta)\Big)$$

This collection of $n-1$ vectors forms a basis for the complex tangent space at ζ to the hypersurface

$$\Gamma_\zeta^\rho = \Big\{z \in V : \rho(z) = \rho(\zeta)\Big\}$$

and is called the canonical basis for the complex tangent space denoted $H\Gamma_\zeta^\rho$, being independent of the choice of the defining function (up to a permutation to ensure $\partial\rho/\partial z_n \neq 0$). Next, observe that each L_j is an eigenvector for $M_\rho(z)$ with eigenvalue 0. Hence, $H\Gamma_\zeta^\rho$ is contained in the kernel of $M_\rho(\zeta)$ while the vector $\nu(\zeta)$ is an eigenvector of $M_\rho(\zeta)$ of eigenvalue $|\nu(\zeta)|^2$.

There is another way to construct a Hermitian form out of ρ whose associated quadratic form is

$$\mathcal{L}_\rho(z, Y) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \overline{z}_k} Y_j \overline{Y}_k \tag{2.2}$$

for any given $z \in V$ and $Y \in \mathbb{C}^n$. This is the complex Hessian and in general need not be positive or negative semi-definite. Restricted to $H\Gamma_\zeta^\rho$, this is the standard Levi form. When Γ_ζ^ρ is pseudoconvex, the restriction of $\mathcal{L}_\rho$ to $H\Gamma_\zeta^\rho$ is positive semi-definite. In particular, all this applies to $\rho = r$, the defining function of the given domain D as in the introductory section 1.

Henceforth, C will denote a positive constant that may vary from line to line. Positive constants will also be denoted by K, L or C_j for some integer j (for instance as in the last part of Lemma 2.1) and these may also vary as we move from one part of the text to another.

2.1. Basic properties of the pseudo-distance induced by the biholomorphically distorted polydiscs Q

Recall the function d defined in (1.6) and another notation from the introduction: for any $a \in D \cap U$, we denote by a^* the point closest to a in ∂D.

LEMMA 2.1. — *The function d satisfies*

(i) For all $a, b \in U$ we have $d(a,b) = 0$ if and only if $a = b$,

(ii) There exists $C > 0$ such that for all $a, b \in U$,

$$d(a,b) \leqslant C d(b,a)$$

(iii) There exist constants R_0 and $L > 0$ such that for all $a, b \in U$ with $|a-b| < R_0$, one has

$$d(a,b) \geqslant 1/2L\ d'(a,b)$$

(iv) There exists $C > 0$ such that for all $a, b, c \in U$, we have

$$d(a,b) \leqslant C\big(d(a,c) + d(b,c)\big)$$

(v) For a suitable constant $C > 0$, we have $d(a,a^*) \leqslant C\delta_D(a)$ for all $a \in U$ and

(vi) There exist constants $C_1, C_2 > 0$ such that for all $a, b \in U$ with $d'(a,b) < 1$, we have

$$C_1 \left|a-b\right|^{2m} \leqslant d'(a,b) \leqslant C_2 \left|a-b\right|.$$

Proof. — The proofs of parts (i)-(v) follow exactly as in [31]. To establish (vi), observe that $a \in Q(b,\delta)$ if and only if $|\Phi^b(a)_1| < \tau(b,\delta)$, $|\Phi^b(a)_\alpha| < \sqrt{\delta}$ for all $2 \leqslant \alpha \leqslant n-1$ and $|\Phi^b(a)_n| < \delta$. As a consequence,

$$|G_b(\tilde{a} - \tilde{b}) - Q_2(a_1 - b_1)| < \sqrt{\delta}$$

which implies that

$$|G_b(\tilde{a} - \tilde{b})| < \sqrt{\delta} + Q_2(a_1 - b_1).$$

But we already know that

$$|a_1 - b_1| < \tau(b, \delta) \lesssim \delta^{1/2m}.$$

Moreover, since G_b^{-1} is uniformly bounded below in norm in a neighbourhood of b, we see that

$$|\tilde{a} - \tilde{b}| \lesssim \sqrt{\delta} + \tau(b, \delta) \lesssim \delta^{1/2m},$$

provided $\delta < 1$. It follows that $|'a - 'b| \lesssim \delta^{1/2m}$ and consequently that $|a_n - b_n| \lesssim \delta^{1/2m}$. To summarize, we conclude that $|a_1 - b_1|$, $|\tilde{a} - \tilde{b}|$, $|'a - 'b|$ and $|a_n - b_n|$ are all less than $\delta^{1/2m}$ times a constant. Hence, $d(a, b) \gtrsim |a - b|^{2m}$. Now to prove the upper inequality of (vi), write

$$\begin{aligned} \left|\Phi^b(a)\right|^2 = \left|a_1 - b_1\right|^2 + \left|G_\zeta(\tilde{a} - \tilde{b}) - Q_2(a_1 - b_1)\right|^2 \\ + \left|(b_n^\zeta)^{-1}\big(a_n - b_n\big) - Q_1('a - 'b)\right|^2. \end{aligned}$$

Note that the right hand side above is at most $C_b|a - b|^2$, where C_b is the maximum of the absolute values of G_ζ, $(b_n^\zeta)^{-1}$ and the coefficients of the polynomials $Q_1('a - 'b)$, $Q_2^\alpha(a_1 - b_1)$ for $2 \leqslant \alpha \leqslant n - 1$, all of which are smooth functions of b. Hence, C_b is bounded above by a positive constant (say, C_2) that depends only on the domain D. This proves the lemma. □

Observe that sufficiently small balls in this pseudo-distance d are the analytic polydiscs $Q(\cdot, \delta)$. In particular, the topology generated by d coincides with the Euclidean topology. However as part (vi) of Lemma 2.1 shows, the pseudodistance d is not bi-Lipschitz equivalent to the Euclidean distance. Infact a more precise description of the local behaviour of d in terms of the various components of the canonical transforms $\Phi^b(a)$ which are explicitly known, is laid down in lemma 2.11.

2.2. Local domains of comparison and plurisubharmonic weights

The task now is to compare the pseudo-distance d induced by the distorted polydiscs which captures certain key aspects of the CR-geometry of the boundary of our Levi corank one domain D, to the distance induced by an algebra of uniformly bounded holomorphic functions, d_D^{Cara}. This will be a starter towards our ultimate goal of obtaining better bounds on d_D^c. To the end of obtaining a lower bound on d_D^{Cara}, we first construct for every given pair of points $A, B \in D \cap U$ which are at least ϵ-apart in the pseudo-distance d, a function $f \in H^\infty(D)$ separating the given pair by an amount depending only on ϵ. The plan for this construction is to set up a $\bar{\partial}$-problem whose solution will give a smooth L^2-function with the required separation properties. Modification of this solution to ensure holomorphicity is a simple routine matter. But extracting information about the values of this L^2-holomorphic function near the boundary, so as to determine whether it also lies in $H^\infty(D)$, and thereby obtain a function which will bound d_D^{Cara} from below, seems difficult as explained by Catlin in his introduction to [11]. A strategy followed therein to overcome this difficulty, is to bump the domain D_ζ, pushing the boundary as far as possible subject to certain constraints: $\text{dist}(\zeta, \partial D_t^\zeta) < t$ for one and among others, that ∂D_t^ζ be pseudoconvex; here D_t^ζ with $t \in \mathbb{R}_+$ denotes the bumped domain. The afore-mentioned $\bar{\partial}$-problem is solved first on D_t^ζ to obtain a solution $\tilde{f} \in H^2(D_t^\zeta)$. The bumping technique as well as the $\bar{\partial}$-problem are sophisticated enough, so as to achieve good control of the L^2-norm of $\tilde{f}$ in terms of the distances of the giver pair of points – that $\tilde{f}$ separates – to the boundary along different directions. In particular, ∂D_ζ lies well within the bumped domain D_t^ζ; well enough, to extract information about the values of $\tilde{f}$ near ∂D_ζ via an appropriate application of the Cauchy integral formula. This leads to the sought after function in $H^\infty(D)$ with the desired separation properties as well.

The technical tools required for this bumping procedure are a family $\{\lambda_\delta\}$ of uniformly bounded smooth plurisubharmonic functions with growing Hessians near ζ, which exist owing to the pseudoconvex finite type character of the boundary. The condition that the Levi form of ∂D_ζ have at most one zero eigenvalue, furnishes from [13] further control on the blow-up rate of the derivatives of $\{\lambda_\delta\}$. The bumping functions $E_{\zeta,t}$ is manufactured by suitably summing up these λ_δ's. The defining function ρ^ζ of the bumped domain is taken to be $\rho^{\zeta,t} = \rho^\zeta + E_{\zeta,t}$. Now we would like these functions to be as simple as possible – simplicity given the constraints and needs of the problem at hand. In our present context, this means tangible estimates of $E_{\zeta,t}$. Indeed, to bring out what tangible estimates mean, we introduce the one parameter family of decoupled algebraic functions

$$\hat{J}_{\zeta,t}(w) = J_{\zeta,t}(w) - t,$$

where

$$J_{\zeta,t}(w) = \Big(t^2 + |w_n|^2 + \sum_{j=2}^{2m} |P_j(\zeta,\cdot)|^2 |w_1|^{2j} + |w_2|^4 + \ldots + |w_{n-1}|^4\Big)^{1/2}. \quad (2.3)$$

Now, all relevant analytic behaviour of $E_{\zeta,t}$ can be estimated in terms of $J_{\zeta,t}$ as stated more precisely, in the lemma 2.2 below. It also turns out that

$J_{\zeta,t}(w)$ is useful in estimating the L^2 norms of holomorphic functions on D_t^ζ. Above all, these functions are tangible in the sense that they are defined directly in terms of the defining function – all one needs to know is the size of the derivatives of $\rho^\zeta(w)$ of order up to $2m$ in the variable w. At this point, we would like to remark that $\hat{J}_{\zeta,t}(w)$ is a (inhomogeneous) pseudo-norm in the sense that $\hat{J}_{\zeta,t}(w-z)$ is a pseudo-distance. Observe that $\hat{J}_{\zeta,t}(w-z)$ is symmetric and satisfies the triangle inequality up to a positive constant. Indeed, let $P(|w|)$ denote the expression

$$|w_n|^2 + \sum_{j=1}^{2m} |P_j(\zeta,\cdot)|^2 |w_1|^{2j} + |w_2|^4 + \ldots + |w_{n-1}|^4.$$

It is well known that

$$|A+B|^{2j} \leqslant 2^{2j-1}(|A|^{2j} + |B|^{2j})$$

for $A, B \in \mathbb{R}$ and $j \in \mathbb{N}$. Applying the above inequality to each term in $P(|w+z|)$, it follows that $P(|w+z|) \lesssim P(|w|) + P(|z|)$. This gives the pseudo-triangle inequality for $\hat{J}_{\zeta,t}(w-z)$. The simplicity of the bumpings D_t^ζ of interest here, can be kept track of, by observing their difference from the trivial bumpings $D^{t,\zeta} := \{w \in \mathbb{C}^n \ : \ \rho_{t,\zeta}(w) := \rho^\zeta(w) - t < 0\}$. It turns out that this difference can be measured via the anisotropic pseudo-norm $\hat{J}_{\zeta,t}$ as in (2.21) below. Thus, these bumpings D_t^ζ are tractable and useful. Suffice it to say that they have been a useful intermediate tool in yielding optimal estimates for our infinitesimal invariant metrics as in [11] and [13] and their integrated distances in dimension two as in [31].

We begin our study of the bumping procedure applied to a segment of a tubular neighbourhood $U_{\zeta,t}$ of ∂D_ζ, near $\zeta = 0$, defined by

$$U_{\zeta,t} = \{w : |\rho^\zeta(w)| < s|J_{\zeta,t}(w)|\} \cap B(0,R), \tag{2.4}$$

where s, R are small positive constants. So, we carry out the bumping process in the region where the variation in ρ^ζ is small in the pseudo-norm $\hat{J}_{\zeta,t}$ – we may also rewrite the defining inequality for $U_{\zeta,t}$ as $|\rho_{st,\zeta}(w)| < s\hat{J}_{\zeta,t}(w)$ when $\rho^\zeta(w) > 0$ i.e., when w lies outside our domain D_ζ; since D_t^ζ can be compared with $D^{t,\zeta}$ via $\hat{J}_{\zeta,t}$ we may include $D^{t,\zeta}$ into our domain of study. In this domain $U_{\zeta,t} \cup D^{t,\zeta}$, it is possible to perform well-controlled bumping; in particular, the afore-mentioned special control available on the λ_δ's contributes to attaining sharp lower bounds on the Hessian of $E_{\zeta,t}$. We shall now just put down the summary of these well-known matters more precisely, for our case of interest, from Proposition 2.2 and Theorem 2.4 of [13].

LEMMA 2.2. — *For sufficiently small* $R_1 < R_0$, $s > 0$ *and each* $\zeta \in \partial D \cap B(0, R_1)$, *there exists on* $\{\rho^\zeta < t\} \cup U_{\zeta,t}$, *a smooth negative real valued plurisubharmonic function* $E_{\zeta,t}$ *with the following properties:*

(i) $-C_3 J_{\zeta,t} \leqslant E_{\zeta,t} \leqslant -1/C_3 J_{\zeta,t}$

(ii) The complex Hessian of $E_{\zeta,t}$ *satisfies*

$$\mathcal{L}_{E_{\zeta,t}}(w,Y) \approx J_{\zeta,t}(w)\Big(\Big|\frac{Y_1}{\tau(\zeta, J_{\zeta,t}(w))}\Big|^2 + \sum_{k=2}^{n-1} \Big|\frac{Y_k}{\sqrt{J_{\zeta,t}(w)}}\Big|^2 + \Big|\frac{Y_n}{J_{\zeta,t}(w)}\Big|^2 \Big)$$

for all $w \in \{\rho^\zeta < t\} \cup U_{\zeta,t}$ *and* $Y \in \mathbb{C}^n$.

(iii) The first directional derivative of $E_{\zeta,t}$ *satisfies*

$$|\langle \partial E_{\zeta,t}(w), Y\rangle|^2 \leqslant C_4 J_{\zeta,t}(w) \mathcal{L}_{E_{\zeta,t}}(w,Y)$$

(iv) Let $\rho^{\zeta,t} = \rho^\zeta + \epsilon_0 E_{\zeta,t}$. *Then for small enough* $\epsilon_0 > 0$ *the domain*

$$D_t^\zeta = \Big\{ w \in \{\rho^\zeta < t\} \cup U_{\zeta,t} \ : \ \rho^{\zeta,t}(w) < 0 \Big\}$$

is pseudoconvex.

The positive constants C_3, C_4 and s here are and independent of the parameters t and ζ. It is clear that $D_t^\zeta \supset D_\zeta$. Part (iii) above compares the complex Hessian of $E_{\zeta,t}(w)$ with the canonical positive semi-definite Hermitian form $|\langle \partial E_{\zeta,t}(w), Y\rangle|^2$ associated to the smooth function $E_{\zeta,t}$. It ensures that $\mathcal{L}_{E_{\zeta,t}}$ satisfies a good enough lower bound for the bumpings to retain pseudoconvexity while pushing out ∂D_ζ as far as possible. Part (ii) provides a more concrete estimate on the complex Hessian of $E_{\zeta,t}(w)$. Together with part (iii), this aids in checking that the bumpings are optimal. Let us just examine what happens at ζ, the origin:

$$\langle \partial \rho^{\zeta,t}(w), Y\rangle = \langle \partial \rho^\zeta(w), Y\rangle + \epsilon_0 \langle \partial E_{\zeta,t}(w), Y\rangle \tag{2.5}$$

for $Y \in \mathbb{C}^n$. Setting $w = 0$ and restricting Y to the complex tangent space to D^ζ at the origin, i.e., $\{z \in \mathbb{C}^n : z_n = 0\}$, we see that

$$\langle \partial \rho^{\zeta,t}(0), Y\rangle = \epsilon_0 \langle \partial E_{\zeta,t}(0), Y\rangle.$$

But we already know from parts (iii) and (ii) that

$$\begin{aligned} |\langle \partial \rho^{\zeta,t}(0), Y\rangle|^2 &\leqslant C_4 \epsilon_0^2 \big(J_{\zeta,t}(0)\big)^2 \left(\left|\frac{Y_1}{\tau\big(\zeta, J_{\zeta,t}(0)\big)}\right|^2 + \sum_{\alpha=2}^{n-1} \left|\frac{Y_\alpha}{\sqrt{J_{\zeta,t}(0)}}\right|^2 \right) \\ &\lesssim \epsilon_0^2 |'Y|^2. \end{aligned} \tag{2.6}$$

since $\tau\big(\zeta, J_{\zeta,t}(0)\big) \gtrsim \sqrt{J_{\zeta,t}(0)}$ and $J_{\zeta,t}(0) = t$. Now, consider

$$\mathcal{L}_{\rho^{\zeta,t}}(0,Y) = \mathcal{L}_{\rho^{\zeta}}(0,Y) + \epsilon_0 \mathcal{L}_{E_{\zeta,t}}(0,Y).$$

Recall that $\mathcal{L}_{\rho^{\zeta}}(0,Y)$ is positive semi-definite due to pseudoconvexity of D_ζ at the origin. Hence,

$$\begin{aligned}\mathcal{L}_{\rho^{\zeta,t}}(0,Y) &\gtrsim \epsilon_0 J_{\zeta,t}(0) \left(\left| \frac{Y_1}{\tau\big(\zeta, J_{\zeta,t}(0)\big)} \right|^2 + \sum_{\alpha=2}^{n-1} \left| \frac{Y_\alpha}{\sqrt{J_{\zeta,t}(0)}} \right|^2 \right) \\ &= \epsilon_0 t \Big(\big| \frac{Y_1}{\tau(\zeta,t)} \big|^2 + \frac{1}{t} \sum_{\alpha=2}^{n-1} \big| Y_\alpha \big|^2 \Big). \end{aligned} \tag{2.7}$$

Using the fact that $\tau(\zeta,t) \lesssim t^{1/2m}$, we get that

$$\mathcal{L}_{\rho^{\zeta,t}}(0,Y) \gtrsim \epsilon_0 t^{1-1/m} |'Y|^2.$$

Observe that (2.6) and (2.7) together imply that the bumped domains touch the domain D_ζ minimally in the complex tangential directions and approximately to the first order along the normal direction at the origin. Furthermore, rewriting (2.5) together with Lemma 2.2(iii) yields the following estimates about the variation of the normal vector fields with respect to the bumping procedure:

$$\begin{aligned} &\big| \langle \partial \rho^{\zeta,t}(w), Y \rangle - \langle \partial \rho^{\zeta}(w), Y \rangle \big|^2 = \epsilon_0^2 \big| \langle \partial E_{\zeta,t}(w), Y \rangle \big|^2 \\ &\quad \leqslant C_4 \epsilon_0^2 \big(J_{\zeta,t}(w) \big)^2 \left(\left| \frac{Y_1}{\tau\big(\zeta, J_{\zeta,t}(w)\big)} \right|^2 + \sum_{\alpha=2}^{n-1} \left| \frac{Y_\alpha}{\sqrt{J_{\zeta,t}(w)}} \right|^2 + \left| \frac{Y_n}{J_{\zeta,t}(w)} \right|^2 \right) \\ &\quad = C_4 \epsilon_0^2 \left(\left| \frac{J_{\zeta,t}(w)}{\tau\big(\zeta, J_{\zeta,t}(w)\big)} \right|^2 |Y_1|^2 + \sum_{\alpha=2}^{n-1} \big| \sqrt{J_{\zeta,t}(w)} Y_\alpha \big|^2 + \big| Y_n \big|^2 \right) \\ &\quad \lesssim C_4 \epsilon_0^2 \Big(J_{\zeta,t}(w) \big| 'Y \big|^2 + |Y_n|^2 \Big). \end{aligned}$$

Here we use the fact that $\tau\big(\zeta, J_{\zeta,t}(w)\big) \gtrsim J_{\zeta,t}(w)^{1/2}$ and that $J_{\zeta,t}(w) \ll 1$ for w near the origin. The above analysis shows that the bumping process pushes the boundary ∂D_t^ζ in the 'normal' direction Y_n relatively more than in the complex tangential directions.

As in [13] and [31], the domains D_t^ζ, serve as local pseudoconvex domains of comparison containing certain special polydiscs $P(w,t,\theta)$ of optimal size:

$$\begin{aligned} P(w,t,\theta) = \Delta\Big(w_1, \tau\big(\zeta, \theta J_{\zeta,t}(w)\big) \Big) \times \Delta\big(w_2, \sqrt{\theta J_{\zeta,t}(w)} \big) \times \ldots \\ \times \Delta\big(w_{n-1}, \sqrt{\theta J_{\zeta,t}(w)} \big) \times \Delta\big(w_n, \theta J_{\zeta,t}(w) \big) \end{aligned}$$

In an earlier notation, we note that $P(w,t,\theta)$ is just the translate of the polydisc $R(w,\theta J_{\zeta,t}(w))$ centered at the origin, to the point w. The polyradii of $P(w,t,\theta)$ are so chosen as to have Lemma 2.3. This lemma is contained in Proposition 2.5 of [13] and its proof therein. This lemma allows us to fit such optimally sized polydiscs about any given point $w \in D_\zeta$ near ζ, within the intersection of the bumped domain D_ζ^t and $U_{\zeta,t}$. The reason for interpolating a *polydisc* herein, is that they are the simplest domains for which the Cauchy integral formula is valid. In particular, we may obtain estimates on the value at a point z of any function holomorphic in it, in terms of the distance of z to the boundary along various (coordinate) directions. Let us state this more precisely and in general terms, as this elementary estimate will play a key role (for instance at (2.34)). Let Ω be a domain in $\mathbb{C}^n$ and $h \in \mathcal{O}(\Omega)$. Then for each $z \in \Omega$ and any polydisc $P = \Delta^1 \times \ldots \times \Delta^n$ around z contained in Ω, it follows from the Cauchy integral formula that

$$|h(z)| \leqslant C \frac{\| h \|_{L^2}}{\delta_{\Delta^1}(z_1) \times \ldots \times \delta_{\Delta^n}(z_n)} \tag{2.8}$$

for some universal constant C that depends only on the dimension n. Here, each Δ^j denotes a disc in the $\mathbb{C}_{z_j}$-plane around z_j. In particular we have (2.8) for large enough polydiscs which reach out to the boundary along as many of the coordinate directions as possible while being contained within Ω. Recall that the L^2-norm of a function is bounded above by its L^∞-norm (up to a constant) on any compact measure space. Here holomorphicity enables a reverse estimate – of course we cannot claim that an L^2 holomorphic function is bounded but we get a control on its rate of blow-up as $z \to \partial P$, in terms of the direction of approach, from (2.8). Now, let us get back to our polydiscs which geometrically estimate the region gained by our bumpings.

LEMMA 2.3. — *There exist numbers $M_0, r_0, \theta > 0$ such that*

(a) *For all $\zeta \in \partial D$, any $t > 0$ and $w \in D^\zeta \cap B(0, r_0)$ one has*

$$D_t^\zeta \supset P(w,t,\theta).$$

(b) *Next, the variation of the defining function ρ^ζ on the polydisc $P(w,t,\theta)$ is described by the statement*

$$|\rho^\zeta(x)| \leqslant M_0 \theta J_{\zeta,t}(w).$$

for all $x \in P(w,t,\theta)$.

Subjecting the defining functions $\rho^{\zeta,t}$ of our bumped domains to the Diederich – Fornaess modification for producing a strongly plurisubharmonic exhaustion function, yields a family of uniformly bounded smooth functions $\psi^w_{\zeta,t}$ on D^ζ_t, whose Hessians satisfy sharp lower bounds on the polydiscs $P(w,t,\theta)$. This owes to a finer control on the derivatives of λ_δ available for Levi corank one domains, as compared to a less sharp control known for general smooth pseudoconvex domains of finite type. The reader is referred to [12] for the general but less refined bumping constructs which can also be used to derive a lower bound on the infinitesimal Kobayashi metric. It turns out that $\psi^w_{\zeta,t}$ is a plurisubharmonic barrier function for $\zeta \in \partial D_\zeta$ of algebraic growth (see [12], for a proof). In particular, ∂D_ζ near ζ is regular in the sense of Sukhov (cf. [55]) and B-regular in the sense of Sibony (see [51] and [52]). This, in turn, implies that $\overline{D}_\zeta$ has a Stein neighbourhood basis. This fact will be used in the sequel. But before all, let us put down the construction of $\psi^w_{\zeta,t}$, which will be useful for constructing the weight functions for our $\bar\partial$-problem.

Lemma 2.4. — *After shrinking θ, given $w \in D^\zeta_t \cap \{\rho^\zeta < 0\}$ there exists on D^ζ_t a plurisubharmonic function $\psi^w_{\zeta,t} < 0$ such that*

(i) $\psi^w_{\zeta,t} \geqslant -1$ on $P(w,t,\theta)$

(ii) For any $Y \in \mathbb{C}^n$ and $x \in P(w,t,\theta)$, we have that

$$\mathcal{L}_{\psi^w_{\zeta,t}}(x,Y) \geqslant C_5\Big(\frac{|Y_1|^2}{\tau(\zeta,J_{\zeta,t}(w))^2} + \sum_{k=2}^{n-1}\frac{|Y_k|^2}{J_{\zeta,t}(w)} + \frac{|Y_n|^2}{\big(J_{\zeta,t}(w)\big)^2}\Big)$$

where C_5 is a positive constant.

Proof. — Following the Diederich – Fornaess technique ([20]) as mentioned above, set

$$\psi(x) = -\Big(-\rho^{\zeta,t}(x)e^{-K|x|^2}\Big)^{\eta_0}$$

which is strongly plurisubharmonic on D^ζ_t. Following [31], we define

$$\psi^w_{\zeta,t}(x) = \psi(x)\big/J_{\zeta,t}(w)^{\eta_0}.$$

First note that the pseudoconvexity of $\Phi^\zeta\big(\partial D \cap B(\zeta,R_0)\big)$ gives

$$\mathcal{L}_{\rho^\zeta}(x,Y) \geqslant -K_1|\rho^\zeta(x)||Y|^2 - K_1|Y||\langle\partial\rho^\zeta(x),Y\rangle| \tag{2.9}$$

whenever $x \in \Phi^\zeta\big(B(\zeta, R_0)\big)$, for some constant $K_1 > 0$. Furthermore, a standard calculation of the complex Hessian of ψ (see [48] for details) yields

$$\begin{aligned}
&\mathcal{L}_\psi(x, Y)\\
&= \eta_0|\psi(x)|\Bigg(\frac{1-\eta_0}{2}\frac{|\langle\partial\rho^{\zeta,t}(x), Y\rangle|^2}{\rho^{\zeta,t}(x)^2} + \Big|\sqrt{\frac{1-\eta_0}{2}}\frac{|\langle\partial\rho^{\zeta,t}(x), Y\rangle|}{\rho^{\zeta,t}(x)} + \frac{\sqrt{2}K\eta_0}{\sqrt{1-\eta_0}}\overline{\langle x, Y\rangle}\Big|^2\\
&\qquad + K\Big(|Y|^2 - K\eta_0\big(1 + \frac{2\eta_0}{1-\eta_0}\big)|\langle x, Y\rangle|^2\Big) + \frac{\mathcal{L}_{\rho^{\zeta,t}}(x, Y)}{\rho^{\zeta,t}(x)}\Bigg)\\
&\geqslant \eta_0|\psi(x)|\left(\frac{1-\eta_0}{2}\frac{|\langle\partial\rho^{\zeta,t}(x), Y\rangle|^2}{\rho^{\zeta,t}(x)^2} + K\Big(|Y|^2 - K\eta_0\big(1 + \frac{2\eta_0}{1-\eta_0}\big)R_*^2|Y|^2\Big)\right.\\
&\qquad \left. + \frac{\mathcal{L}_{\rho^{\zeta,t}}(x, Y)}{\rho^{\zeta,t}(x)}\right).
\end{aligned}$$

Here we have used the estimate $|\langle x, Y\rangle|^2 \leqslant R_*^2|Y|^2$ for some positive constant R_*. Choosing K and η_0 suitably, we can ensure that the following inequality holds throughout D_t^ζ:

$$\mathcal{L}_\psi(x, Y) \geqslant \eta_0|\psi(x)|\Big(\frac{1-\eta_0}{2}\frac{|\langle\partial\rho^{\zeta,t}(x), Y\rangle|^2}{\rho^{\zeta,t}(x)^2} + \frac{1}{2}K|Y|^2 + \frac{\mathcal{L}_{\rho^{\zeta,t}}(x, Y)}{\rho^{\zeta,t}(x)}\Big). \tag{2.10}$$

To verify (i), let $x \in P(w, t, \theta)$. Writing this analytically, translates to the following string of inequalities:

$$\begin{aligned}
&|x_1 - w_1| < \tau\big(\zeta, \theta J_{\zeta,t}(w)\big),\\
&|x_\alpha - w_\alpha| < \sqrt{\theta J_{\zeta,t}(w)} \quad \text{for all } 2 \leqslant \alpha \leqslant n-1, \text{ and}\\
&|x_n - w_n| < \theta J_{\zeta,t}(w).
\end{aligned} \tag{2.11}$$

Recall that $J_{\zeta,t}(w) = \sqrt{t^2 + P(|w|)}$, where $P(|w|)$ is the inhomogeneous pseudo-norm given by:

$$P(|w|) = |w_n|^2 + \sum_{j=2}^{2m}|P_j(\zeta, \cdot)|^2|w_1|^{2j} + |w_2|^4 + \ldots + |w_{n-1}|^4.$$

Write the difference $J_{\zeta,t}(x) - J_{\zeta,t}(w)$ as

$$J_{\zeta,t}(x) - J_{\zeta,t}(w) = \frac{\big(J_{\zeta,t}(x)\big)^2 - \big(J_{\zeta,t}(w)\big)^2}{J_{\zeta,t}(x) + J_{\zeta,t}(w)}. \tag{2.12}$$

The aim now is to obtain an analogue of the estimate (4.3) in [31]. This estimate is essentially an assertion about the uniform comparability

$J_{\zeta,t}(x) \approx J_{\zeta,t}(y)$. Furthermore, such an estimate will lead to the engulfing property for the polydiscs $P(w,t,\theta)$ analogous to that for the Catlin – Cho polydiscs $Q(\cdot,\cdot)$. To this end, we begin by applying the triangle inequality to the numerator which is $\big|P(|w|) - P(|x|)\big|$, on the right side of (2.12). This gives

$$\begin{aligned}
&\Big||x_n|^2 - |w_n|^2 + \sum_{\alpha=2}^{n-1}\big(|x_\alpha|^4 - |w_\alpha|^4\big) + \sum_{j=2}^{2m}|P_j(\zeta,\cdot)|^2\big(|x_1|^{2j} - |w_1|^{2j}\big)\Big| \\
&\leqslant \big||x_n|^2 - |w_n|^2\big| + \sum_{\alpha=2}^{n-1}\big||x_\alpha|^4 - |w_\alpha|^4\big| + \sum_{j=2}^{2m}|P_j(\zeta,\cdot)|^2\big||x_1|^{2j} - |w_1|^{2j}\big|.
\end{aligned} \tag{2.13}$$

Note that the first term on the right here namely, $\big||x_n|^2 - |w_n|^2\big|$, is bounded above up to a constant by $\theta J_{\zeta,t}(w)$. Indeed, $\big||x_n|^2 - |w_n|^2\big| \lesssim |x_n - w_n| \lesssim \theta J_{\zeta,t}(w)$ by the last inequality in (2.11). Next, to likewise – estimate the last summand of (2.13), we use the following well-known inequality:

$$|x_1|^{2j} \leqslant 2^{2j-1}(|x_1 - w_1|^{2j} + |w_1|^{2j}),$$

which implies that

$$\Big||x_1|^{2j} - |w_1|^{2j}\Big| \leqslant 2^{2j-1}|x_1 - w_1|^{2j} + (2^{2j-1} - 1)|w_1|^{2j}$$

for each $j \leqslant 2m$. Hence,

$$|P_j(\zeta,\cdot)|^2\big||x_1|^{2j} - |w_1|^{2j}\big| \leqslant 2^{2j-1}|P_j(\zeta,\cdot)|^2|x_1 - w_1|^{2j} + (2^{2j-1} - 1)|P_j(\zeta,\cdot)|^2|w_1|^{2j}. \tag{2.14}$$

It follows from the definition of $\tau\big(\zeta, \theta J_{\zeta,t}(w)\big)$ that

$$|P_j(\zeta,\cdot)|^2 \leqslant \frac{\theta^2(J_{\zeta,t}(w))^2}{\Big(\tau\big(\zeta,\theta J_{\zeta,t}(w)\big)\Big)^{2j}}. \tag{2.15}$$

But we know from (2.11) that $|x_1 - w_1|^{2j} \leqslant \Big(\tau\big(\zeta,\theta J_{\zeta,t}(w)\big)\Big)^{2j}$. Therefore,

$$2^{2j-1}|P_j(\zeta,\cdot)|^2|x_1 - w_1|^{2j} \leqslant 2^{2j-1}\theta^2\big(J_{\zeta,t}(w)\big)^2 \lesssim \theta J_{\zeta,t}(w). \tag{2.16}$$

Finally, since $\tau\big(\zeta,\theta J_{\zeta,t}(w)\big) \gtrsim \big(\theta J_{\zeta,t}(w)\big)^{1/2}$, we may rewrite (2.15) as

$$|P_j(\zeta,\cdot)|^2 \lesssim \frac{\theta^2(J_{\zeta,t}(w))^2}{\theta^j(J_{\zeta,t}(w))^j}.$$

We may estimate the second summand on the right of (2.14) as follows: use part (b) of lemma 2.3 as well as the fact that w is close to ζ (which is the origin), to get

$$|w| \approx |\rho_\zeta(w)| \leqslant M_0 \theta J_{\zeta,t}(w).$$

Therefore, $|w_1|^{2j} \leqslant |w|^{2j} \lesssim \big(\theta J_{\zeta,t}(w)\big)^{2j}$ so that

$$(2^{2j-1}-1)|P_j(\zeta,\cdot)|^2|w_1|^{2j} \lesssim \frac{\theta^2\big(J_{\zeta,t}(w)\big)^2}{\theta^j\big(J_{\zeta,t}(w)\big)^j}\big(\theta J_{\zeta,t}(w)\big)^{2j} = \big(\theta J_{\zeta,t}(w)\big)^{j+2}$$

Now since $w \in D_t^\zeta$, we have $\rho^\zeta(w) + \epsilon_0 E_{\zeta,t}(w) < 0$ i.e.,

$$\epsilon_0 E_{\zeta,t}(w) < -\rho^\zeta(w) = |\rho^\zeta(w)|.$$

Note that the constants hidden in the various $\lesssim, \gtrsim$-bounds here and in what follows can be taken to be independent of ζ, θ and t unless otherwise spelled out. Therefore, by part (i) of lemma 2.2, we obtain that $|J_{\zeta,t}(w)| \lesssim |\rho^\zeta(w)|$ and subsequently by lemma 2.3(b) that

$$\big(\theta J_{\zeta,t}(w)\big)^{j+2} \lesssim \big(\theta|\rho^\zeta(w)|\big)^{j+2} \lesssim \theta^{j+3}|J_{\zeta,t}(w)| < \theta J_{\zeta,t}(w)$$

since we may assume $\theta < 1$. In all therefore, we obtain for the second summand at (2.14) that

$$(2^{2j-1}-1)|P_j(\zeta,\cdot)|^2|w_1|^{2j} \lesssim \theta J_{\zeta,t}(w),$$

just as we did for the first summand at (2.14) in the estimation (2.16). It is now clear that we will similarly also have that the 'mid-summands' in the right of the inequality (2.13) namely, $\big||x_\alpha|^4 - |w_\alpha|^4\big|$, must be bounded above by $\theta J_{\zeta,t}(w)$ up to some positive constant. Getting back to our aim at (2.12), we put together the various foregoing estimates to conclude that the numerator on the right of (2.12) is bounded above in modulus by $\theta J_{\zeta,t}(w)$ up to some positive constant, as well. For the denominator therein, note that it is bounded below by $2t$. The above analysis yields

$$|J_{\zeta,t}(x) - J_{\zeta,t}(w)| \leqslant (C_6'/2t)\,\theta J_{\zeta,t}(w) \tag{2.17}$$

We may choose some $t > 0$ such that the foregoing inequalities here are valid; we may shrink the value of t in what follows if necessary, but only finitely many times. In particular, fixing such a value of t now and letting $C_6 = C_6'/2t$, we rewrite (2.17) as the pair of bounds

$$J_{\zeta,t}(x) \leqslant (1+C_6\theta)J_{\zeta,t}(w) \;\text{ and }\; J_{\zeta,t}(x) \geqslant (1-C_6\theta)J_{\zeta,t}(w)$$

which describe the variation of $J_{\zeta,t}(\cdot)$ on $P(w,t,\theta)$. We have thus established (4.3) of [31] in our setting. Here, θ is chosen small enough so as to ensure $1 - C_6\theta > 0$. We remark in passing that (2.17) may also be rewritten as

$$|\hat{J}_{\zeta,t}(x) - \hat{J}_{\zeta,t}(w)| \leqslant (C_6'/2t)\ \theta\hat{J}_{\zeta,t}(w) + C_6'/2, \tag{2.18}$$

where as we know C_6' is a constant depending on the given domain D but with no more particular dependence on w, ζ, t, θ. Of course the above inequality is just a rephrased form of the triangle inequality for the pseudo-norm $\hat{J}_{\zeta,t}$ up to constants; so (2.17), (2.18) are expected. However, we need to go through their derivation to keep track of what the constants depend on.

Moving on further, it follows from Lemma 2.2(i) and Lemma 2.3(b) that

$$\begin{aligned}
|\rho^{\zeta,t}(x)| &\geqslant \epsilon_0|E_{\zeta,t}(x)| - |\rho^{\zeta}(x)| \\
&\geqslant \epsilon/C_3\ J_{\zeta,t}(x) - M_0\theta J_{\zeta,t}(w) \\
&\geqslant \epsilon_0/C_3(1 - C_6\theta)J_{\zeta,t}(w) - M_0\theta J_{\zeta,t}(w) \\
&\geqslant \epsilon_0/2C_3\ J_{\zeta,t}(w)
\end{aligned}$$

and

$$\begin{aligned}
|\rho^{\zeta,t}(x)| &\leqslant \epsilon_0|E_{\zeta,t}(x)| + |\rho^{\zeta}(x)| \\
&\leqslant \epsilon_0 C_3\ J_{\zeta,t}(x) + M_0\theta J_{\zeta,t}(w) \\
&\leqslant C_3\epsilon_0(1 + C_6\theta)J_{\zeta,w}(w) + M_0\theta J_{\zeta,w}(w) \\
&\lesssim J_{\zeta,t}(w)
\end{aligned}$$

provided θ and ϵ_0 are sufficiently small. Putting the above two observations together, we see that

$$(\epsilon_0/2C_3)J_{\zeta,t}(w) \leqslant |\rho^{\zeta,t}(x)| \leqslant J_{\zeta,t}(w). \tag{2.19}$$

Furthermore,

$$|E_{\zeta,t}(x)| \leqslant C_3 J_{\zeta,t}(x) \leqslant C_3(1 + C_6\theta)J_{\zeta,t}(w) \leqslant C_7|\rho^{\zeta,t}(x)|, \tag{2.20}$$

which implies that

$$|\rho^{\zeta,t}(x)| \approx |E_{\zeta,t}(x)| \approx J_{\zeta,t}(w).$$

for all $x \in P(w,t,\theta)$. At this point we would like to note that since $\rho^{\zeta,t}(x) \approx \operatorname{dist}(x, \partial D_t^{\zeta})$, it follows that on $P(w,t,\theta)$

$$|\operatorname{dist}(x, \partial D_t^{\zeta}) - \operatorname{dist}(x, \partial D^{t,\zeta})| \approx \hat{J}_{\zeta,t}(w) \tag{2.21}$$

where $\hat{J}_{\zeta,t}(x) = J_{\zeta,t}(x) - t$, and $D^{t,\zeta} = \big\{w \in \mathbb{C}^n : \rho_{t,\zeta}(w) := \rho^\zeta(w) - t < 0\big\}$. Thus the anisotropic pseudo-norm $\hat{J}_{\zeta,t}$ measures the difference between the trivial bumpings given by $\{\rho_{t,\zeta}\}$ and the bumpings $\{\rho^{\zeta,t}\}$ engineered taking into account the CR geometry of ∂D_ζ. Getting back from this note to the lemma at hand, it follows from (2.19) that $\psi(x) \geqslant -|\rho^{\zeta,t}(x)|^{\eta_0} \geqslant -\big(J_{\zeta,t}(w)\big)^{\eta_0}$ for all $x \in P(w,t,\theta)$. This gives

$$\psi^w_{\zeta,t}(x) = \frac{\psi(x)}{\big(J_{\zeta,t}(w)\big)^{\eta_0}} \geqslant -1.$$

To establish (ii), let us examine the complex Hessian of $\rho^{\zeta,t}$:

$$\begin{aligned}\mathcal{L}_{\rho^{\zeta,t}}(x,Y) &= \mathcal{L}_{\rho^\zeta}(x,Y) + \epsilon_0 \mathcal{L}_{E_{\zeta,t}}(x,Y)\\ &\geqslant -K_1\big(|\rho^\zeta(x)||Y_1|^2 + |\langle \partial\rho^\zeta(x), Y\rangle||Y|\big) + \epsilon_0\mathcal{L}_{E_{\zeta,t}}(x,Y) \quad \text{by (2.9)}\\ &\geqslant -K_1\big(|\rho^{\zeta,t}(x)| + \epsilon_0|E_{\zeta,t}(x)|\big)|Y|^2 - K_1|\langle\partial\rho^{\zeta,t}(x),Y\rangle||Y|\\ &\quad - K_1\epsilon_0|\langle\partial E_{\zeta,t}(x),Y\rangle||Y| + \epsilon_0\mathcal{L}_{E_{\zeta,t}}(x,Y) \end{aligned}\tag{2.22}$$

for $x \in P(w,t,\theta)$. Note that, by (2.20), the first term $-K_1\big(|\rho^{\zeta,t}(x)| + \epsilon_0|E_{\zeta,t}(x)|\big)|Y|^2$ in (2.22) is at least

$$-K_1\big(|\rho^{\zeta,t}(x)| + \epsilon_0 C_7|\rho^{\zeta,t}(x)|\big)|Y|^2. \tag{2.23}$$

Now, consider the third term in (2.22):

$$-\epsilon_0 K_1|\langle\partial E_{\zeta,t}(x),Y\rangle||Y| \geqslant -\epsilon_0 K_1\sqrt{C_3}\sqrt{J_{\zeta,t}(x)}\sqrt{\mathcal{L}_{E_{\zeta,t}}(x,Y)}|Y| \tag{2.24}$$

by Lemma 2.2(iii). The part (ii) of the same lemma gives

$$\begin{aligned}\sqrt{\mathcal{L}_{E_{\zeta,t}}(x,Y)} &\lesssim \frac{\sqrt{J_{\zeta,t}(x)}}{\sqrt{C_3}}\left(\frac{|Y_1|^2}{\big(\tau(\zeta,J_{\zeta,t}(x))\big)^2} + \sum_{\alpha=2}^{n-1}\frac{|Y_\alpha|^2}{J_{\zeta,t}(x)} + \frac{|Y_n|^2}{\big(J_{\zeta,t}(x)\big)^2}\right)^{1/2}\\ &= \frac{1}{\sqrt{C_3}}\left(\frac{J_{\zeta,t}(x)}{\tau\big(\zeta,J_{\zeta,t}(x)\big)^2}|Y_1|^2 + \sum_{\alpha=2}^{n-1}|Y_\alpha|^2 + \frac{|Y_n|^2}{J_{\zeta,t}(x)}\right)^{1/2}. \end{aligned}\tag{2.25}$$

Using the fact that $\tau\big(\zeta,J_{\zeta,t}(x)\big) \gtrsim \big(J_{\zeta,t}(x)\big)^{1/2}$ in (2.25), we get

$$\sqrt{J_{\zeta,t}(x)}\sqrt{C_3}\sqrt{\mathcal{L}_{E_{\zeta,t}}(x,Y)} \lesssim \Big(J_{\zeta,t}(x)|Y_1|^2 + J_{\zeta,t}(x)\sum_{\alpha=2}^{n-1}|Y_\alpha|^2 + |Y_n|^2\Big)^{1/2}.$$

Since $J_{\zeta,t}(x)$ is small, it follows that

$$-\epsilon_0 K_1 |\langle \partial E_{\zeta,t}(x), Y\rangle||Y| \gtrsim -\epsilon_0 K_1 |Y|^2 \gtrsim -\epsilon_0 K_1 |\rho^{\zeta,t}(x)||Y|^2. \tag{2.26}$$

Here we used that $|\rho^{\zeta,t}(x)|$ is bounded away from zero on the polydiscs $P(w,t,\theta) \subset D_t^{\zeta}$. To estimate the second term in (2.22), we use the following version of the Cauchy-Schwarz inequality:

$$xy \leqslant \epsilon/2\, x^2 + 1/2\epsilon\, y^2$$

where x, y and ϵ are positive numbers. Applying this inequality for $x = |Y|$, $y = |\langle \partial\rho^{\zeta,t}(x), Y\rangle|$ and $\epsilon = |\rho^{\zeta,t}(x)|$, we see that

$$-K_1|\langle \partial\rho^{\zeta,t}(x), Y\rangle||Y| \geqslant -K_1/2\, |\rho^{\zeta,t}(x)||Y|^2 - K_1/2\, |\rho^{\zeta,t}(x)|\, |\langle \partial\rho^{\zeta,t}(x), Y\rangle|^2. \tag{2.27}$$

The inequalities (2.23), (2.27) and (2.26) together with (2.22) imply that

$$\frac{\mathcal{L}_{\rho^{\zeta,t}}(x,Y)}{|\rho^{\zeta,t}(x)|} \geqslant -K_2|Y|^2 - \frac{1}{K_2}\frac{|\langle \partial\rho^{\zeta,t}(x), Y\rangle|^2}{|\rho^{\zeta,t}(x)|^2} + \frac{\epsilon_0}{2}\frac{\mathcal{L}_{E_{\zeta,t}}(x,y)}{|\rho^{\zeta,t}(x)|}.$$

Using the above inequality and (2.10), and adjusting constants as in [31], we finally get

$$\mathcal{L}_{\psi}(x,Y) \geqslant \frac{\eta_0\epsilon_0}{2}|\psi(x)|\frac{\mathcal{L}_{E_{\zeta,t}}(x,Y)}{|\rho^{\zeta,t}(x)|} \geqslant C_8|\psi(x)|\frac{\mathcal{L}_{E_{\zeta,t}}(x,Y)}{J_{\zeta,t}(w)}. \tag{2.28}$$

This estimate in conjunction with Lemma 2.2(ii) yields the required estimate for the complex Hessian of ψ and consequently, for $\psi^w_{\zeta,t}$ as well. Indeed,

$$|\psi(x)| = |\rho^{\zeta,t}(x)|^{\eta_0} e^{-K\eta_0|x|^2}.$$

Consequently, $|\psi(x)| \gtrsim |J_{\zeta,t}(w)|^{\eta_0}$ by using (2.19). Therefore, we get from (2.28) that

$$\begin{aligned}
\mathcal{L}_{\psi^w_{\zeta,t}}(x,Y) &= \frac{\mathcal{L}_{\psi}(x,Y)}{\big(J_{\zeta,t}(w)\big)^{\eta_0}} \\
&\geqslant C_8 \frac{|\psi(x)|}{\big(J_{\zeta,t}(w)\big)^{\eta_0}}\frac{\mathcal{L}_{E_{\zeta,t}}(x,Y)}{J_{\zeta,t}(w)} \\
&\geqslant C_8 \frac{\mathcal{L}_{E_{\zeta,t}}(x,Y)}{J_{\zeta,t}(w)} \\
&\geqslant C_{10}\frac{J_{\zeta,t}(x)}{J_{\zeta,t}(w)}\left(\frac{|Y_1|^2}{\tau\big(\zeta, J_{\zeta,t}(w)\big)^2} + \sum_{k=2}^{n-1}\frac{|Y_k|^2}{J_{\zeta,t}(w)} + \frac{|Y_n|^2}{\big(J_{\zeta,t}(w)\big)^2}\right) \\
&\geqslant C_{10}(1 - C_6\theta)\left(\frac{|Y_1|^2}{\tau\big(\zeta, J_{\zeta,t}(w)\big)^2} + \sum_{k=2}^{n-1}\frac{|Y_k|^2}{J_{\zeta,t}(w)} + \frac{|Y_n|^2}{\big(J_{\zeta,t}(w)\big)^2}\right).
\end{aligned}$$

This proves the lemma. □

2.3. Separation properties of the pluri-complex Green function

The weight functions for setting up the appropriate $\overline{\partial}$-problem will be formulated in terms of the pluri-complex Green function which was introduced by Klimek [36]. For any domain Ω, these are given by

$$\mathcal{G}_\Omega(z,w) = \sup\left\{u(z) \ : \ u \in PSH_w(\Omega)\right\}.$$

Here for $w \in \Omega$, $PSH_w(\Omega)$ denotes the family of all plurisubharmonic functions that are negative on Ω and which have the property that the function $u(z) - \log|z-w|$ is bounded from above near w. The Green function itself is again a member of $PSH_w(\Omega)$. Any fixed sub-level set of the pluri-complex Green function $\mathcal{G}_\Omega(\cdot,w)$ is contained in the corresponding sublevel set of any member of $PSH_w(\Omega)$. The following lemma provides a link between the separation properties of the sub-level sets of the pluri-complex Green function associated with the bumped domain D_t^ζ and the polydiscs $P(w,t,\theta)$. To this end, note that the polydiscs $P(w,t,\theta)$ are balls in the metric defined by

$$\left|\Delta_\zeta^{\theta J_{\zeta,t}(w)}(z-w)\right|_{l^\infty}$$
$$\approx \hat{V}_w(z) := \left(\frac{|z_1-w_1|^2}{\tau(\zeta,\theta J_{\zeta,t}(w))^2} + \sum_{\alpha=2}^{n-1}\frac{|z_\alpha - w_\alpha|^2}{\theta J_{\zeta,t}(w)} + \frac{|z_n-w_n|^2}{\theta J_{\zeta,t}(w)^2}\right)^{1/2}.$$

The pluri-complex Green function with a pole $w \in D_\zeta$ for our bumped domain, can then be controlled by a suitable blend of the logarithm of this metric $\hat{V}_w$ near w, and the negative plurisubharmonic exhaustion for D_t^ζ as in the previous section. Recall that the $\psi_{\zeta,t}^w$ are uniformly bounded in modulus on polydiscs $P(w,t,\theta)$ which as noted are the sub-level sets of $\hat{V}_w(z)$.

Lemma 2.5. —

(a) There is a bound $M_1 \gg 1$ (depending on θ) such that given $\sigma \in (0,1)$ one has for all $w \in D_t^\zeta \cap D_\zeta$ that

$$P(w,t,\sigma\theta) \supset \left\{\mathcal{G}_{D_t^\zeta}(\cdot,w) < \log\sigma - M_1\right\}.$$

(b) Further, one can find σ_0 in such a way that for any two points $w,w' \in D_t^\zeta$ but with $w' \notin P(w,t,\theta)$ one has

$$P(w',t,\sigma_0\theta) \cap P(w,t,\sigma_0\theta) = \emptyset.$$

In particular, we have

$$\left\{\mathcal{G}_{D_t^\zeta}(\cdot,w') < \log\sigma_0 - M_1\right\} \cap \left\{\mathcal{G}_{D_t^\zeta}(\cdot,w) < \log\sigma_0 - M_1\right\} = \emptyset.$$

for all $w,w' \in D_\zeta$.

Proof. — Let $\xi : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function with $\xi(s) = s$ for all $s \leqslant 1/4$ and with $\xi(s) = 3/4$ if $s \geqslant 7/8$. Thereafter, let $V_w(z)$ denote the smooth function $\hat{V}_w(z)^2$. Also choose a convex increasing function χ on $\mathbb{R}$ satisfying $\chi(s) = -7/4$ for $s \leqslant -2$ and $\chi(s) = s$ if $s \geqslant -3/2$. Then, for a large enough M' (depending on θ), the function

$$\phi_w = 1/2 \ \log \xi \circ V_w + M' \chi \circ \psi^w_{\zeta,t}$$

becomes plurisubharmonic on D^ζ_t and hence a candidate for the supremum that defines $\mathcal{G}_{D^\zeta_t}(\cdot, w)$. We have

$$\phi_w \geqslant 1/2 \ \log \xi \circ V_w - 7/4 \ M'.$$

If now $z \in D^\zeta_t$ is a point for which $\mathcal{G}_{D^\zeta_t}(z, w) < \log \sigma - 7/4 \ M'$, then

$$\log \xi \circ V_w(z) \leqslant 2 \log \sigma,$$

which implies $V_w(z) \leqslant \sigma^2$, provided $\sigma < 1/2$. But then this ensures that $z \in P(w, t, \sigma\theta)$ for V_w dominates the square of the metric that defines the polydisc P. So we may let $M_1 = 7/4 \ M'$, to obtain the assertion of part (a) of the lemma.

For part (b) of the lemma, we argue by contradiction. So, assume that (b) fails to hold for any choice of σ_0. This assumption means that there exists, corresponding to every choice of $\sigma_0 > 0$, a pair of points $w, w' \in D^\zeta_t$ with $w' \notin P(w, t, \theta)$ and such that $P(w', t, \sigma_0\theta) \cap P(w, t, \sigma_0\theta) \neq \emptyset$. Now, firstly the condition that $w' \notin P(w, t, \theta)$ means that at least one out of the following list of inequalities must hold.

(1) $|w_1 - w'_1| > \sigma_0 \tau\big(\zeta, \theta J_{\zeta,t}(w)\big)$ or,

(α) $|w_\alpha - w'_\alpha| > \sigma_0 \sqrt{\theta J_{\zeta,t}(w)}$ for $2 \leqslant \alpha \leqslant n-1$ or,

(n) $|w_n - w'_n| > \sigma_0 \theta J_{\zeta,t}(w)$.

Next, by our assumption, there is point x (dependent on σ_0) lying in both the polydiscs $P(w, t, \sigma_0\theta)$ and $P(w', t, \sigma_0\theta)$ – where the value of σ_0 will be chosen appropriately in a moment, to obtain a contradiction. To this end, first note what happens to the above list of inequalities as we vary the value of σ_0 in a small interval $(0, \delta)$, say: at least one of the inequalities must hold for most values of σ_0. Indeed, let us just suppose that the first of the inequalities in the above list holds, for infinitely many values of σ_0 which decrease to 0. Next, assume that $x = x(\sigma_0) \in P(w', t, \sigma_0\theta) \cap P(w, t, \sigma_0\theta)$. Then a combined application of the facts that $J_{\zeta,t}(x) \approx J_{\zeta,t}(w) \approx J_{\zeta,t}(w')$ and that $\tau(w, \delta) \approx \tau(w', \delta)$ gives

$$|w_1 - w'_1| \leqslant |w_1 - x_1| + |w'_1 - x_1| \leqslant \sigma_0 C_{11} \tau(\zeta, \theta J_{\zeta,t}(w))$$

with some constant C_{11} which does not depend on σ_0, w, w' and θ. Recalling a basic estimate for $\tau(\cdot,\cdot)$, we conclude – in the event that (1) holds infinitely often – that for a positive constant $\tilde{C}_{11}$ independent of σ_0 we must have

$$\tau\big(\zeta, \theta J_{\zeta,t}(w)\big) \leqslant \tilde{C}_{11}\big(\sigma_0 \theta J_{\zeta,t}(w)\big)^{1/2m} \tag{2.29}$$

holding for infintely many values of σ_0 which decrease to 0; noting now, that the left side here is independent of σ_0 while the right side $\to 0$ as $\sigma_0 \to 0$, we obtain the desired contradiction. It is now also clear that similar arguments will take care of the remaining cases – namely, the cases where one of the other inequalities out of the list above, holds for most (small) values of σ_0 – as well. □

2.4. Localization lemmas

LEMMA 2.6. — *There exists $L > 0$ with the following property*

(a) If t is sufficiently small and $w' \in P(w,t,\theta/2) \cap D_\zeta$ and $f \in H^\infty\big(P(w,t,\theta)\big)$, then there exists $\tilde{f} \in H^2(D_t^\zeta)$ such that $\tilde{f}(w) = f(w)$, $\tilde{f}(w') = f(w')$ and

$$|\tilde{f}|_{L^2(D_t^\zeta)} \leqslant L\; J_{\zeta,t}(w)\Big(\sqrt{J_{\zeta,t}(w)}\Big)^{n-2}\tau\big(\zeta, J_{\zeta,t}(w)\big)|f|_{L^\infty}.$$

(b) If $w' \notin P(w,t,\theta/2) \cap D_\zeta$ and $f \in H^\infty\big(P(w,t,\theta)\big)$, then there exists $\tilde{f} \in H^2(D_t^\zeta)$ such that $\tilde{f}(w) = f(w)$, $\tilde{f}(w') = 0$ and

$$|\tilde{f}|_{L^2(D_t^\zeta)} \leqslant L\; J_{\zeta,t}(w)\Big(\sqrt{J_{\zeta,t}(w)}\Big)^{n-2}\tau\big(\zeta, J_{\zeta,t}(w)\big)|f|_{L^\infty}.$$

Proof. — Let f be a function from $H^\infty\big(P(w,t,\theta)\big)$. For part (a), we choose a non-negative cut-off function $\xi \in C^\infty(\mathbb{R})$ such that $\xi(s) = 1$ for $s \leqslant 1/3$ and $\xi(s) = 0$, if $s \geqslant 7/8$. Now, define

$$v = \overline{\partial}\Big[\xi\Big(\frac{|z_1 - w_1|^2}{\tau(\zeta, \theta J_{\zeta,t}(w))^2}\Big)\xi\Big(\frac{|z_2 - w_2|^2}{\theta^2 J_{\zeta,t}(w)}\Big) \dots \xi\Big(\frac{|z_{n-1} - w_{n-1}|^2}{\theta^2 J_{\zeta,t}(w)}\Big)\xi\Big(\frac{|z_n - w_n|^2}{\theta^2\big(J_{\zeta,t}(w)\big)^2}\Big)\, f\Big] \tag{2.30}$$

which is a smooth $\overline{\partial}$-closed $(0,1)$-form on D_t^ζ with support

$$\operatorname{supp}(v) \subset P(w,t,\theta) \setminus P(w,t,\theta/\sqrt{3}).$$

By Lemma 2.5 we have

$$\mathcal{G}_{D_t^\zeta}(\cdot, w) \geqslant \log(1/\sqrt{3}) - M_1$$

on supp(v) and then since w' lies in $P(w, t, \theta/2)$, we similarly have

$$\mathcal{G}_{D_t^\zeta}(\cdot, w') \geqslant \log(1/\sqrt{3} - 1/2) \; - \; \log(1 + 2C_{12}) - M_1$$

on supp(v), because $P\big((w', t, \frac{1/\sqrt{3}-1/2}{1+2C_{12}}\theta)\big) \subset P(w, t, \theta/\sqrt{3})$. The plurisubharmonic function

$$\Phi = \psi^w_{\zeta,t} + 4\mathcal{G}_{D_t^\zeta}(\cdot, w) + 4\mathcal{G}_{D_t^\zeta}(\cdot, w')$$

is bounded from below on supp(v) by some constant $-T < 0$. From theorem 5 of [31] we obtain a solution $u \in C^\infty(D_t^\zeta)$ to the equation $\overline{\partial} u = v$ such that

$$\int_{D_t^\zeta} |u|^2 e^{-\Phi} d^{2n}z \leqslant 2 \int_{D_t^\zeta} |v|^2_{\partial\overline{\partial}\psi^w_{\zeta,t}} e^{-\Phi} d^{2n}z \tag{2.31}$$

where $d^{2n}z$ denotes the standard Lesbegue measure on $\mathbb{C}^n$ identified with $\mathbb{R}^{2n}$. Now, by Lemma 2.4, it follows that $|v|^2_{\partial\overline{\partial}\psi^w_{\zeta,t}} \leqslant L_1 |f|^2_{L^\infty}$ for some unimportant constant L_1. Indeed, note first by the holomorphicity of f that we have

$$v(z_1, \ldots, z_n) = f(z) \cdot \overline{\partial}\big(W_1 W_2 \ldots W_n\big)$$

where

$$W_j = \xi\Big(\Big|\frac{z_j - w_j}{\tau_j(\zeta, \theta J_{\zeta,t}(w))}\Big|^2\Big).$$

Let

$$\hat{W}_j = \Pi^n_{k=1, k\neq j} W_k.$$

Then observe that $|\hat{W}_j| \leqslant 1$. Choose some constant C with

$$\Big|\frac{d\xi}{ds}(s)\Big|^2 \leqslant C$$

for $s \in [1/3, 7/8]$. Note next that the $(0,1)$-form v can be written as

$$v = f \sum_{j=1}^{n} \hat{W}_j \frac{\partial W_j}{\partial \overline{z}_j} d\overline{z}_j.$$

Next, note that

$$\Big|\frac{\partial W_n}{\partial \overline{z}_n}\Big|^2 \leqslant C \Big|\frac{z_n - w_n}{\big(\theta J_{\zeta,t}(w)\big)^2}\Big|^2 \lesssim \frac{1}{|\theta J_{\zeta,t}(w)|^2}$$

where the second inequality follows from $|z_n - w_n| \leqslant \big(\theta J_{\zeta,t}(w)\big)^2$ since $z \in P(w,t,\theta)$ which in turn comes from the fact $\mathrm{supp}(v) \subset P(w,t,\theta)$. Similarly, we have for each $2 \leqslant \alpha \leqslant n-1$ that

$$\Big|\frac{\partial W_\alpha}{\partial \bar{z}_\alpha}\Big|^2 \lesssim \Big|\frac{z_\alpha - v_\alpha}{\big(\sqrt{\theta J_{\zeta,t}(w)}\big)^2}\Big|^2 \lesssim \frac{1}{\theta J_{\zeta,t}(w)}$$

and finally

$$\Big|\frac{\partial W_1}{\partial \bar{z}_1}\Big|^2 \leqslant C\Big|\frac{z_n - w_n}{\big(\tau\big(w,\theta J_{\zeta,t}(w)\big)\big)^2}\Big|^2 \lesssim \frac{\tau\big(w,\theta J_{\zeta,t}(w)\big)^2}{\tau\big(w,\theta J_{\zeta,t}(w)\big)^4} = \frac{1}{\tau\big(w,\theta J_{\zeta,t}(w)\big)^2}$$

as well, so that an application of part (ii) of Lemma 2.4 to the form $|v|^2_{\partial\bar{\partial}\psi^w_{\zeta,t}}$ (which will read as an upper bound) evaluated at the pair

$$(x,Y) = \Big(z, f(z)\big(\hat{W}_1(z),\ldots,\hat{W}_n(z)\big)\Big)$$

actually cancels the scaling factors in the definition of the form $|v|^2_{\partial\bar{\partial}\psi^w_{\zeta,t}}$ and yields our afore-mentioned claim that $|v|^2_{\partial\bar{\partial}\psi^w_{\zeta,t}} \lesssim |f|^2_{L^\infty}$. This subsequently extends the estimate (2.31) as

$$\begin{aligned}\int_{D_t^\zeta} |u|^2 d^{2n}z &\leqslant \int_{D_t^\zeta} |v|^2 e^{-\Phi} d^{2n}z \\ &\leqslant 2L_1 e^T \mathrm{Vol}(\mathrm{supp}\ v)\ |f|^2_{L^\infty} \\ &\leqslant 2L_1 e^T \theta^{2n}\ J_{\zeta,t}(w)^2\Big(\sqrt{J_{\zeta,t}(w)}\Big)^{2(n-2)}\tau\big(\zeta, J_{\zeta,t}(w)\big)^2|f|^2_{L^\infty} \\ &\leqslant 2L_1 e^T J_{\zeta,t}(w)^2\Big(\sqrt{J_{\zeta,t}(w)}\Big)^{2(n-2)}\tau\big(\zeta, J_{\zeta,t}(w)\big)^2|f|^2_{L^\infty}.\end{aligned}$$

since θ is a small constant which may be assumed to be < 1. The function

$$\tilde{f}(z) = \Big[\xi\big(\tfrac{|z_1-w_1|^2}{\tau(\zeta,\theta J_{\zeta,t}(w))^2}\big)\xi\big(\tfrac{|z_2-w_2|^2}{\theta^2 J_{\zeta,t}(w)}\big)\cdots$$

$$\xi\big(\frac{|z_{n-1}-w_{n-1}|^2}{\theta^2 J_{\zeta,t}(w)}\big)\xi\big(\frac{|z_n-w_n|^2}{\theta^2\big(J_{\zeta,t}(w)\big)^2}\big)\Big]\cdot f(z)\ -\ u(z)$$

now becomes holomorphic and has the desired properties, as $u(w) = u(w') = 0$ while the L^2-estimate for $\tilde{f}$ follows immediately from that for the function u.

Next we do part (b) in a manner similar to (a): Firstly, if $w' \notin P(w,t,\theta/2)$ then with a number $\sigma_0 > 0$ (independent of w,w',t) we have

$$P(w,t,\sigma_0\theta/2) \cap P(w',t,\sigma_0\theta/2) = \phi.$$

Let us denote by $f_1(z)$ the product

$$\xi\Big(\frac{|z_1-w_1|^2}{(\sigma_0\theta/2)^2\tau(\zeta,J_{\zeta,t}(w))^2}\Big)\xi\Big(\frac{|z_2-w_2|^2}{(\sigma_0\theta/2)^2 J_{\zeta,t}(w)}\Big) \ldots\xi\Big(\frac{|z_{n-1}-w_{n-1}|^2}{(\sigma_0\theta/2)^2 J_{\zeta,t}(w)}\Big)\xi\Big(\frac{|z_n-w_n|^2}{(\sigma_0\theta/2)^2\big(J_{\zeta,t}(w)\big)^2}\Big)$$

and solve $\overline{\partial}u=v$ with $v=\overline{\partial}(f_1 f)$. Now,

$$\operatorname{supp}(v)\subset P(w,t,\sigma_0\theta/2)\setminus P\big(w,t,(\sigma_0\theta/2\sqrt{3})\big).$$

Then, just as in part (a), we see that

$$\mathcal{G}_{D_t^\zeta}(\cdot,w)\geqslant\log\big(\sigma_0/(2\sqrt{3})\big)-M_1$$

on supp(v). Furthermore, supp(v) is disjoint from $P(w',t,\sigma_0\theta/2)$ and so

$$\mathcal{G}_{D_t^\zeta}(\cdot,w')\geqslant\log\big(\sigma_0/(2\sqrt{3})\big)-M_1-\log(1+2C_{12}).$$

Proceeding exactly as in part (a) from here on, we arrive at the statement in (b) of our Lemma. □

Lemma 2.7. — *Let $x\in D$ and denote by ζ the point closest to x in the boundary ∂D. Let y be a point such that $w_y=\Phi^\zeta(y)\in D_t^\zeta$. Let $w_x=\Phi^\zeta(x)$. Then, given $f\in H^\infty\big(P(w_x,t,\theta)\big)$ it is possible to find a function $\hat{f}\in H^\infty(D)$ such that $\hat{f}(x)=f(w_x)$ and $|\hat{f}|_{L^\infty}\leqslant L^*|f|_{L^\infty}$, where L^* is some positive constant (independent of f). Further, we may also arrange for $\hat{f}$ to satisfy $\hat{f}(y)=f(w_y)$ in case $w_y\in P(w_x,t,\theta/2)$ and $\hat{f}(y)=0$ if $w_y\notin P(w_x,t,\theta/2)$.*

Proof. — We apply Lemma 2.6 to the pair of points $w=w_x$, $w'=w_y$ and the function f. This yields a function $\tilde{f}\in H^2(D_t^\zeta)$ with $\tilde{f}(w_x)=f(w_x)$ and $\tilde{f}(y)=f(w_y)$ in case $w_y\in P(w_x,t,\theta/2)$ and $\tilde{f}(y)=0$, if $w_y\notin P(w_x,t,\theta/2)$. It satisfies the L^2-estimate

$$|\tilde{f}|_{L^2\left(D_t^\zeta\right)}\leqslant LJ_{\zeta,t}(w_x)\Big(\sqrt{J_{\zeta,t}(w)}\Big)^{n-2}\tau(\zeta,J_{\zeta,t}(w_x))|f|_{L^\infty}. \tag{2.32}$$

Let $\lambda\geqslant 0$ be a smooth function on $\mathbb{R}$ such that $\lambda(x)=1$ for $x\leqslant(3/4)^2$ and $\lambda(x)=0$ for $x\geqslant(7/8)^2$. We note that there exists δ_0 (independent of ζ,t,x,y and f) such that

$$\hat{v}=\begin{cases}\overline{\partial}\Big(\lambda\big(\frac{|\Phi^\zeta(z)|^2}{r_0^2}\big)\cdot\tilde{f}\circ\Phi^\zeta\Big) & ;\text{ on } (\Phi^\zeta)^{-1}(D_t^\zeta)\cap\{r<\delta_0\}\\ 0 & ;\text{ on } \{r<\delta_0\}\cap\big(\{|\Phi^\zeta|\geqslant 7r_0/8\}\cup\{|\Phi^\zeta|\leqslant 3r_0/4\}\big)\end{cases}$$

defines a smooth $\overline{\partial}$-closed $(0,1)$-form on $\{r<\delta_0\}$; here r_0 is the radius from Lemma 2.3. This follows from the fact that

$$(\Phi^\zeta)^{-1}(D_t^\zeta)\cap\Big(\mathbb{B}(\zeta,7/8r_0)\backslash\mathbb{B}(\zeta,3/4r_0)\Big)\supset\{r<\delta_0\}\cap\Big(\mathbb{B}(\zeta,7/8r_0)\backslash\mathbb{B}(\zeta,3/4r_0)\Big).$$

Consulting now the discussion on the existence of a Stein-neighbourhood basis in [52] and as noted prior to Lemma 2.4, we ascertain for ourselves the possibility of being able to choose a Stein neighbourhood Ω of $\overline{D}$ such that

$$\overline{D}\subset\{r<\delta_2\}\subset\Omega\subset\{r<\delta_0\}.$$

On Ω, using results from [28] – where we work with the elementary weight $4\log|\cdot-x|+4\log|\cdot-y|$ – we can solve the equation $\overline{\partial}\hat{u}=\hat{v}$ with a smooth function $\hat{u}$ such that $\hat{u}(x)=\hat{u}(y)=0$ and

$$|\hat{u}|_{L^2(\Omega)}\leqslant C_{13}|\tilde{f}\circ\Phi^\zeta|_{L^2(\{r<\delta_0\})}\leqslant C_{14}|\tilde{f}|_{L^2(D_t^\zeta)} \tag{2.33}$$

for some positive constants C_{13},C_{14}. Then certainly the function

$$\hat{f}(z)=\lambda(\frac{|\Phi^\zeta(z)|^2}{r_0^2})\cdot\tilde{f}\circ\Phi^\zeta(z)-\hat{u}(z)$$

is holomorphic on D. We need to estimate the L^∞-norm of $\hat{f}$.

Let $z\in D$ and suppose first that $w_z=\Phi^\zeta(z)\in\mathbb{B}(0,r_0)\cap\{\rho^\zeta<0\}$. Then we know by part (a) of Lemma 2.3 that

$$P(w_z,t,\theta)\subset D_t^\zeta.$$

Using (2.8) this gives

$$\begin{aligned}|\hat{f}(z)|&=|\hat{f}\circ(\Phi^\zeta)^{-1}(w_z)|\\&\leqslant\frac{|\hat{f}\circ\Phi^\zeta|_{L^2(D_t^\zeta)}}{\pi\tau(\zeta,J_{\zeta,t}(w_z))\big(\sqrt{J_{\zeta,t}(w_z)}\big)^{n-2}J_{\zeta,t}(w_z)}\\&\leqslant\frac{|\hat{f}|_{L^2(D_t^\zeta)}+|u\circ(\Phi^\zeta)^{-1}|_{L^2(D_t^\zeta)}}{\pi\tau(\zeta,J_{\zeta,t}(w_z))\big(\sqrt{J_{\zeta,t}(w_z)}\big)^{n-2}J_{\zeta,t}(w_z)}\\&\leqslant C_{15}\frac{|\hat{f}|_{L^2(D_t^\zeta)}}{\pi\tau(\zeta,J_{\zeta,t}(w_z))\big(\sqrt{J_{\zeta,t}(w_z)}\big)^{n-2}J_{\zeta,t}(w_z)}\\&\leqslant C_{15}L|f|_{L^\infty}.\end{aligned} \tag{2.34}$$

The last estimate here comes from (2.32). Now assume that $w_z\in\{\rho^\zeta<0\}\cap\{|\Phi^\zeta|\geqslant 0.9r_0\}$. Then we see that $|\hat{f}(z)|=|\hat{u}(z)|$. But $\hat{u}$ is defined on

$\{r < \delta_2\}$ and holomorphic on $\{r < \delta_2\} \cap \{\Phi^\zeta| \geqslant 0.9r_0\}$. After possibly shrinking $\delta_2 > 0$ we find using the mean value inequality that

$$|\hat{u}(z)| \leqslant \delta_2^{-n} |\hat{u}|_{L^2(\{r<\delta_2\})} \leqslant L^* |f|_{L^\infty}$$

because of (2.32) and (2.33), proving the desired L^∞-estimate for $\hat{f}$. □

Next, as in [31] again, we have the following separation of points Lemma and given the foregoing lemmas, the proof is verbatim as in [31] and this time we shall not repeat it.

Lemma 2.8. — *There is a uniform constant $c_0 > 0$ such that for any $a, b \in D \cap U$ with $b \notin Q_{2\delta_D(a)}(\zeta)$ where $\zeta = a^*$, one has*

$$d_D^{Cara}(a, b) \geqslant c_0.$$

2.5. Estimation of the inner Caratheodory distance from below

Let U be a tubular neighbourhood of ∂D as before, U' a relatively compact neighbourhood of ∂D inside U. We intend to estimate $d_D^c(A, B)$ for two points $A, B \in U'$. We split the procedure into two cases

$$d'(B, A) > 4C_e \delta_D(A) \tag{2.35}$$

$$d'(B, A) \leqslant 4C_e \delta_D(A) \tag{2.36}$$

Before we begin, we put down 2 elementary inequalities which will be of recurrent use in the sequel. The first, is the following simple version of the Cauchy – Schwarz inequality

$$|z_1 + \ldots + z_N|^2 \leqslant N\big(|z_1|^2 + \ldots |z_N|^2\big)$$

where $N \in \mathbb{N}$ for any set of complex numbers $\{z_j\}_{j=1}^N$. Second, is the following logarithmic inequality

$$\log(1 + rx) \geqslant r \log(1 + x)$$

where x and r are positive reals with $r < 1$.

We now begin with case (2.35).

Lemma 2.9. — *Assume that (2.35) holds. Then with some universal constant $C_* > 0$ we have*

$$d_D^c(A, B) \geqslant C_* \log\Big(1 + \frac{d(B, A)}{\delta_D(A)}\Big).$$

Proof. — Let $A, B \in D \cap U'$ be points such that $d'(B,A) \geqslant 4C_e\delta_D(A)$. Choose a smooth path $\gamma : [0,1] \to D$ from A to B satisfying

$$2d_D^c(A,B) \geqslant L_D^c(\gamma)$$

where $L_D^c(\gamma)$ refers to the length of γ in the inner Caratheodory metric.

We shall split again into two cases:

If $\gamma([0,1]) \not\subset U \cap D$, then we can find an exit time, i.e., a number $t_1' \in (0,1)$ with $\gamma([0,t_1')) \subset D \cap U$ and $\gamma(t_1') \in \partial U \cap D$. Now we apply Cho's lower bound on the differential metric as in [13] and find

$$\begin{aligned} L_D^c(\gamma) &\geqslant L_D^c(\gamma_{|_{[0,t_1']}}) \\ &\geqslant C_1 \log \frac{\delta_D(\partial U)}{\delta_D(A)}. \end{aligned} \tag{2.37}$$

Indeed, the lower bound from [13] reads

$$C_D(z,X) \gtrsim \frac{|\langle L_1(z), X\rangle|}{\tau\big(z,\delta_D(z)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\langle L_\alpha(z), X\rangle|}{\sqrt{\delta_D(z)}} + \frac{|X_n|}{\delta_D(z)}$$

– recall that $L_n \equiv 1$; here we shall let z vary in a small neighbourhood of $A^* = \pi(A)$, assumed to be the origin after a translation, on which such an estimate is guaranteed by [13]. Further we may also assume after a rotation that $\nu\big(A^*\big) = L_n = ('0,1)$; at this point we may also want to note that the hypothesis on $d'(A,B)$, of the case under consideration remains intact, since these transformations preserve d' in the sense that they transform the d' associated with the initial domain D into the d' of the transformed domain. In particular, we have for z in a small ball $\mathbb{B}_\delta$ and $X \in \mathbb{C}^n$ the estimate

$$C_D(z,X) \gtrsim \frac{|X_n|}{\delta_D(z)}.$$

which contains in it the rate of blow-up of the Caratheodory metric along the normal direction, since $\nu(\pi(z))$ must have a non-zero component along L_n. To unravel this information precisely from the above inequality and in a more useful form for our purposes, we need to restrict ourselves to the cone

$$\mathcal{C}_\alpha^z = \big\{X \in T_{\pi(z)}(\mathbb{C}^n) \;:\; \big|\langle \nu\big(\pi(z)\big), X\rangle\big| > \alpha\big|\nu\big(\pi(z)\big)\big|\big\}$$

where $\alpha \in (0,1]$. Let

$$\mathcal{C} = \bigcup_{z \in D} \{z\} \times \mathcal{C}_\alpha^z$$

and consider the function defined on $\mathcal{C}$ by

$$R(z,X) = \Big| \frac{|X_n|}{|\langle \nu\big(\pi(z)\big), X\rangle|} - 1 \Big|$$

which is zero for all those z whose $\pi(z)$ is the origin, i.e., the z_n-axis. Now note that $R(z,X)$ is continuous on $\mathcal{C}$ as $|\langle \nu\big(\pi(z)\big), X\rangle|$ is bounded away from 0 and also that $R(A^*,X) = 0$. Therefore, given any ϵ (1/2, say) there is a δ_0 (which we may take to be $< \delta$) such that

$$|R(z,X) - R(A^*,X)| < \epsilon$$

for all $z \in \mathbb{B}_{\delta_0}$ which is to say, we have

$$\Big| \frac{X_n}{\langle \nu\big(\pi(z)\big), X\rangle} \Big| > 1 - \epsilon = 1/2.$$

or equivalently that

$$|X_n| > 1/2 \, \big|\langle \nu\big(\pi(z)\big), X\rangle\big|. \tag{2.38}$$

Now getting to our setting, since the curve $\gamma(t)$ moves away from the boundary during the interval $I = [0, t_1')$ meaning, $\text{dist}\big((\gamma(t_1'), \partial D)\big) = \text{dist}(\partial U, \partial D)$ $= \delta_D(U)$ is greater than $\gamma(0) = A \in U$, we must have that the 'normal component' of the curve, namely $\langle \dot\gamma(t), \nu\big(\pi(\gamma(t))\big)\rangle$, must be non-zero (bigger than some $\alpha > 0$) for some non-trivial stretch of time, i.e., for a sub-interval of I of non-zero length – call this sub-interval I again – so that we may apply the fore-going considerations, in particular (2.38) to pass to a further sub-interval of I of non-zero length, if necessary, where we have

$$|\dot\gamma_n(t)| \geqslant 1/2 \, |\langle \dot\gamma(t), \nu\big(\pi(\gamma(t))\big)\rangle|.$$

Indeed as mentioned above, the existence of such an interval follows just from continuity and the fact that $\nu\big(\pi(\gamma(t))\big)$ at $t = 0$ is just $('0, 1)$, so that

$$\frac{|\dot\gamma_n(0)|}{|\langle \dot\gamma(0), \nu\big(\pi(\gamma(0))\big)\rangle|} = 1$$

and we may take this sub-interval, which we shall denote again by I, to be of the form $[0, t_2')$. We then have on this sub-interval that

$$\begin{aligned} C_D(\gamma(t), \dot\gamma(t)) &\gtrsim \frac{|\dot\gamma_n(t)|}{\delta_D(\gamma(t))} \\ &\gtrsim \frac{|\langle \dot\gamma(t), \nu\big(\pi(\gamma(t))\big)\rangle|}{\delta_D(\gamma(t))} \\ &\gtrsim \frac{|\langle \dot\gamma(t), \nu\big(\pi(\gamma(t))\big)\rangle|}{|r(\gamma(t))|}. \end{aligned}$$

We have elaborately presented the steps that lead to this lower bound because of its re-occurrence later in a more complicated setting. Integrating the final estimate in the above with respect to t, leads to (2.37) which subsequently yields,

$$\begin{aligned} L_D^c(\gamma) &\geqslant C_1 \log \frac{\delta_D(\partial U)}{\delta_D(A)} \\ &\geqslant \frac{1}{2} C_1 \log \Big(1 + \frac{\delta_D(\partial U)}{\delta_D(A)}\Big) \\ &\geqslant \frac{1}{2} C_1 \log \Big(1 + \frac{\delta_D(\partial U)}{\mathrm{diam}(D)} \frac{d(B,A)}{\delta_D(A)}\Big) \\ &\geqslant C_2 \log \Big(1 + \frac{d(B,A)}{\delta_D(A)}\Big) \end{aligned}$$

where

$$C_2 = \frac{1}{2} C_1 \frac{\delta_D(\partial U)}{\mathrm{diam}(D)}.$$

The second inequality in the foregoing string of inequalities, can be ensured by choosing U_0 such that $\delta_D(A) \ll \delta_D(\partial U)$, while the third one follows because $\delta_D(\partial U) \ll \mathrm{diam}(D)$. This finishes case (i).

The other sub-case is (ii): $\gamma([0,1]) \subset U \cap D$.
Since $d'(B,A) > 4C_e \delta_D(A)$ we have

$$4C_e \delta_D(A) < \inf M(B,A)$$

and therefore

$$B \notin Q_{4C_e\delta_D(A)}(A) \supset Q_{2C_e\delta_D(A)}(A^*).$$

Thus we can choose a number $t_1 \in (0,1)$ such that

$$\gamma(t_1) \in \partial Q_{2C_e\delta_D(A)}(A^*).$$

This shows that the following set is not empty

$$\begin{aligned} S^{A,B} := \big\{ j \in \mathbb{Z}^+ : \exists\, 0 = t_0 < t_1 < \ldots < t_j < 1 \\ \text{such that } \gamma(t_\nu) \in \partial Q_{2C_e\delta_D(\gamma(t_{\nu-1}))}\big(\gamma(t_{\nu-1})^*\big),\ 1 \leqslant \nu \leqslant j \big\}. \end{aligned}$$

Since $\gamma([0,1]) \subset D \cap U$, the boundary point $\gamma(t)^*$ is well-defined for any $t \in [0,1]$. As in [31] again, $S^{A,B}$ can be ascertained to be a finite set and consequently we may define the number

$$m := \max S^{A,B}$$

and choose numbers $0 = t_0 < t_1 \ldots < t_m < 1$ such that

$$\gamma(t_\nu) \in \partial Q_{2C_e \delta_D(\gamma(t_{\nu-1}))}\big(\gamma(t_{\nu-1})^*\big),\ 1 \leqslant \nu \leqslant m.$$

Further following [31], we get $2d_D^c(A,B) \geqslant c_0 m$ and subsequently follow the steps therein to estimate m from below, which uses the pseudo-distance property of d and leads eventually to the estimate

$$d(B,A) \leqslant C_*^{m+2}\delta_D(A)$$

where C_* is a constant bigger than 1 (in fact bigger than 6). This gives

$$C_*^{2m} \geqslant \frac{d(B,A)}{\delta_D(A)}.$$

From this it follows that

$$(1+C_*)^{2m} > 1 + C_*^{2m} \geqslant 1 + \frac{d(B,A)}{\delta_D(A)}$$

which subsequently leads to

$$2m\log(1+C_*) > \log\left(1 + \frac{d(B,A)}{\delta_D(A)}\right)$$

which gives

$$m \gtrsim \log\left(1 + \frac{d(B,A)}{\delta_D(A)}\right)$$

But then recalling that

$$c_0 \cdot m \leqslant 2d_D^c(A,B)$$

we finally see that

$$d_D^c(A,B) \gtrsim \log\left(1 + \frac{d(B,A)}{\delta_D(A)}\right)$$

□

Now we turn to the other possibility:

Lemma 2.10. — *Assume that (2.36) holds. Then with some universal constant $C_{21} > 0$ we have*

$$d_D^{Cara}(A,B) \geqslant C_{21}\ \log\Big(1\ +\ |\Phi^A(B)_n|^2/\delta_D(A)^2 + \sum_{\alpha=2}^{n-1}|\Phi^A(B)_\alpha|^2/\delta_D(A)\ +\ |\Phi^A(B)_1|^2/\tau(A^*,\delta_D(A))^2\Big)$$

Proof. — Suppose that A, B are points in $D \cap U'$ which satisfy (2.36). Then clearly

$$B \in Q_{4C_e\delta_D(A)}(A)$$

and

$$\begin{aligned}\Phi^A(B) \in \Delta\big(0, \tau(A, 4C_e\delta_D(A))\big) \times \Delta\big(0, \sqrt{4C_e\delta_D(A)}\big) \times \dots \\ \times \Delta\big(0, \sqrt{4C_e\delta_D(A)}\big) \times \Delta\big(0, 4C_e\delta_D(A)\big) \subset P(w_A, t, \theta)\end{aligned}$$

with a number $\theta > 0$ independent of A, B and $t := 4C_e\delta_D(A)$. Here we put $w_A = \Phi^{A^*}(A)$. According to Lemma 2.7 applied to the point $x = A$ and $y = B$, there exists for any function $f \in H^\infty\big(P(w, t, \theta)\big)$ having norm equal to one, a function $\hat{f} \in H^\infty(D)$ with $|\hat{f}|_{L^\infty} \leqslant L^*$ for some constant L^* such that

$$\begin{aligned}\hat{f}(A) &= f(w_A) \\ \hat{f}(B) &= f(w_B)\end{aligned}$$

where $w_B = \Phi^{A^*}(B)$. This implies

$$d_D^{Cara}(A, B) \geqslant d^P\Big(\frac{1}{L^*} f(w_A), \frac{1}{L^*} f(w_B)\Big).$$

We now make our choice of the function f namely, put

$$f(v) = f_n(v_1, v_2, \dots, v_n) = \frac{1}{C'\ \delta_D(A)}\Big(\big(\Phi^A \circ (\Phi^{A^*})^{-1}\big)(v_1, v_2, \dots, v_n)\Big)_n \tag{2.39}$$

where C' is a constant chosen so that $|f|_{L^\infty} \leqslant 1$ and is independent of A, B – to see that this can indeed be done, notice that

$$\Phi^A \circ (\Phi^{A^*})^{-1}(v) = \big(\Phi^A - \Phi^{A^*}\big) \circ (\Phi^{A^*})^{-1}(v) + v$$

and hence

$$\begin{aligned}C'|f(v)| &\leqslant \frac{\big|\big(\Phi^A - \Phi^{A^*}\big) \circ (\Phi^{A^*})^{-1}(v)\big| + |v_n|}{\delta_D(A)} \\ &\leqslant C + \frac{|v_n|}{\delta_D(A)}.\end{aligned}$$

But then on $P(w_A, t, \theta)$ we have

$$|v_n| \leqslant |v_n - (w_A)_n| + |(w_A)_n| \leqslant \theta J_{A^*,t}(w_A) + t \leqslant C''t \leqslant C'''\delta_D(A).$$

Certainly $f(w_A)=0$. Together with a basic estimate concerning the Poincaré distance d^p_Δ on the unit disc – estimate (6.6) in [31] – we get

$$d_D^{Cara}(A,B) \geqslant d^p_\Delta\Big(0, \frac{1}{L^*} f(w_B)\Big) \geqslant \frac{1}{2} \log\left(1 + \frac{|\Phi^A(B)_n|^2}{(L^*C')^2(\delta(A))^2}\right)$$

with some suitable constant $C' > 0$. Similarly next, choosing the function f to be

$$f(v) = f_\alpha(v) = \frac{1}{C'\sqrt{\delta(A)}}\big(\Phi^A \circ (\Phi^{A^*})^{-1}\big)(v)_\alpha,$$

for all $2 \leqslant \alpha \leqslant n-1$ – which also has L^∞-norm not bigger than 1, viewed as a function on $P(w_A,t,\theta)$ – we also obtain (since again $f(w_A)=0$) that

$$d_D^{Cara}(A,B) \geqslant d^p_\Delta\big(0, \frac{1}{L^*} f(w_B)\big) \geqslant \frac{1}{2} \log\left(1 + \frac{|\Phi^A(B)_\alpha|^2}{(L^*C')^2(\sqrt{\delta(A)})^2}\right)$$

and similarly again, choosing

$$f(v) = f_1(v) = \frac{1}{C'\tau\big(A,\delta_D(A)\big)}\big(\Phi^A \circ (\Phi^{A^*})^{-1}\big)(v)_1$$

with C' a suitable constant adjusted so that $f_1 \in L^\infty\big(P(w_A,t,\theta)\big)$, we get

$$d_D^{Cara}(A,B) \geqslant d^p_\Delta\big(0, \frac{1}{L^*} f(w_B)\big) \geqslant \frac{1}{2} \log\left(1 + \frac{|\Phi^A(B)_1|^2}{(L^*C')^2\tau\big(A,\delta_D(A)\big)^2}\right)$$

To summarize then, we have for each $1 \leqslant j \leqslant n$, that

$$d_D^{Cara}(A,B) \gtrsim \log\Big(1 + c\big|\frac{\Phi^A(B)_j}{\tau_j(A)}\big|^2\Big)$$

for some constant $c<1$. Adding together these inequalities over the index j gives the estimate asserted in the Lemma. Let us note here for later purposes that the foregoing inequalities may also be rewritten as

$$d_D^{Cara}(A,B) \gtrsim \max_{1\leqslant j\leqslant n} \log\left(1 + \big|\frac{\Phi^A(B)_j}{\tau_j(A)}\big|^2\right) \tag{2.40}$$

as $c<1$. □

We now proceed to demonstrate that the estimates of the last two lemmas fit together well to yield Theorem 1.1. For reasons of symmetry it is enough to verify that

$$d^c_D(A,B) \gtrsim \eta(A,B).$$

Before we begin, let us just record one useful fact which is the analogue of lemma 3.2 of [31] and follows by the very same line of proof therein.

Lemma 2.11. — *If $a, b \in D \cap U$ are points with $|a-b| < R_0$, we have*

$$\max\Big\{|\big(\Phi^a(b)\big)_n|, |\big(\Phi^a(b)\big)_2|^2, \ldots, |\big(\Phi^a(b)\big)_{n-1}|^2, \max_{2\leqslant l\leqslant 2m} |P_l(a)||\big(\Phi_a(b)\big)_1|^l\Big\}$$
$$\leqslant d'(b,a)$$
$$\leqslant 2\max\Big\{|\big(\Phi^a(b)\big)_n|, |\big(\Phi^a(b)\big)_2|^2, \ldots, |\big(\Phi^a(b)\big)_{n-1}|^2, \max_{2\leqslant l\leqslant 2m} |P_l(a)||\big(\Phi_a(b)\big)_1|^l\Big\}$$

Now suppose (2.35) holds. Then we claim that for some $c_1 > 0$ we have

$$d(B,A)/\delta_D(A) \geqslant C_{12}\Big(|\Phi^A(B)_n|/\delta_D(A) + \sum_{\alpha=2}^{n-1} |\Phi^A(B)_\alpha|/\big(\delta_D(A)\big)^{1/2} + |\Phi^A(B)_1|/\tau(A,\delta_D(A))\Big) \quad (2.41)$$

For the proof of this, let $0 < \epsilon < 2d'(B,A)$ be a number such that $B \in Q_\epsilon(A)$. Then

$$\Phi^A(B) \in \Delta\big(0, \tau(A,\epsilon)\big) \times \Delta(0,\sqrt{\epsilon}) \times \ldots \times \Delta(0,\sqrt{\epsilon}) \times \Delta(0,\epsilon).$$

In particular, $|\Phi^A(B)_n| \leqslant \epsilon \leqslant 2d'(B,A)$. But then we also know $|\Phi^A(B)| \leqslant c_1'|A-B|$, which implies

$$|\Phi^A(B)_n| \leqslant \min\big\{2d'(B,A), c_1'|A-B|\big\} \leqslant c_2' d(B,A).$$

with some constant $c_2' > 1$. In particular,

$$\frac{|\Phi^A(B)_n|}{\delta_D(A)} \leqslant c_2' \frac{d(B,A)}{\delta_D(A)}$$

Next we estimate $|\Phi^A(B)_1|/\tau\big(A,\delta_D(A)\big)$. Since the function $t \to t/\tau(A,t)$ is increasing and $\epsilon \geqslant d'(B,A) > 4C_e\delta_D(A)$, we get

$$\frac{|\Phi^A(B)_1|}{\tau\big(A,\delta_D(A)\big)} \leqslant \frac{\tau(A,\epsilon)}{\tau\big(A,\delta_D(A)\big)} \leqslant c_3' \frac{\epsilon}{\delta_D(A)} \leqslant 2c_3' \frac{d'(B,A)}{\delta_D(A)}.$$

Moreover since $|\Phi^A(B)_1| \leqslant c_1'|A-B|$ and $\tau\big(A,\delta_D(A)\big) \geqslant c_4'\delta_D(A)$ we get

$$\frac{|\Phi^A(B)_1|}{\tau\big(A,\delta_D(A)\big)} \leqslant c_5' \frac{|A-B|}{\delta_D(A)}$$

and subsequently that

$$\frac{|\Phi^A(B)_1|}{\tau\big(A, \delta_D(A)\big)} \leqslant c_6' \frac{d(B,A)}{\delta_D(A)}.$$

Also

$$\frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} \leqslant \frac{\sqrt{\epsilon}}{\sqrt{\delta_D(A)}} \lesssim \frac{\epsilon}{\delta_D(A)} \leqslant \frac{2d'(B,A)}{\delta_D(A)} < \frac{2Ld(B,A)}{\delta_D(A)}.$$

This completes the verification of the claim (2.41) and then Lemma 2.9 proves

$$d_D^c(A,B) \geqslant C_* \rho(A,B)$$

for those points A, B that satisfy (2.35).

Next we move on to the case when A, B satisfy (2.36). In this case we claim that for some positive constant $c_2 > 0$ we have

$$\frac{d(B,A)}{\delta_D(A)} \leqslant c_2 \left(\frac{|\Phi^A(B)_n|}{\delta_D(A)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} + \frac{|\Phi^A(B)_1|}{\tau\big(A, \delta_D(A)\big)} \right) \tag{2.42}$$

First, we note that we have $d'(B,A) \leqslant \epsilon$ where we now let

$$\epsilon = 2 \ \max \Big\{ |\big(\Phi^A(B)\big)_n|, |\big(\Phi^A(B)\big)_2|^2, \\ \dots, |\big(\Phi^A(B)\big)_{n-1}|^2, \max_{2 \leqslant l \leqslant 2m} |P_l(A)| |\big(\Phi_A(B)\big)_1|^l \Big\}$$

We split into the various possible cases for the value of ϵ and deal with them one by one. First, suppose that $\epsilon = 2|\Phi^A(B)_n|$. Then we get

$$\frac{d(B,A)}{\delta_D(A)} \leqslant \frac{d'(B,A)}{\delta_D(A)} \leqslant \frac{\epsilon}{\delta_D(A)} = 2 \frac{|\Phi^A(B)_n|}{\delta_D(A)}.$$

Next we look at what happens when ϵ happens to be $2 \ \max \{|P_l(A)||\Phi^A(B)_1|^l\}$. In this case note that

$$\begin{aligned} \frac{d(B,A)}{\delta_D(A)} &\leqslant \frac{\epsilon}{\delta_D(A)} \\ &= 2 \frac{\max \{|P_l(A)||\Phi^A(B)_1|^l\}}{\delta_D(A)} \\ &\lesssim \ \max \left(\frac{|\Phi^A(B)_1|}{\tau\big(A, \delta_D(A)\big)} \right)^l \\ &\lesssim \frac{|\Phi^A(B)_1|}{\tau\big(A, \delta_D(A)\big)} \end{aligned}$$

provided we assure ourselves that $|\Phi^A(B)_1|/\tau\big(A, \delta_D(A)\big) \leqslant C$ for some constant C. To see this, choose any sequence $\eta_j \to d'(B,A)$ from above. Then by definition of $d'(B,A)$ we have $B \in Q_{\eta_j}(A)$ for all j. This gives $|\Phi^A(B)_1| \leqslant \tau(A, \eta_j)$. Letting $j \to \infty$ we find $|\Phi^A(B)_1| \leqslant \tau\big(A, d'(B,A)\big)$. Then since we are in the case (2.36) i.e., $d'(B,A) \leqslant 4C_e \delta_D(A)$, we obtain

$$\tau\big(A, d'(B,A)\big) \leqslant C' \tau\big(A, \delta_D(A)\big)$$

Therefore, $|\Phi^A(B)_1| \leqslant C'\tau\big(A, \delta_D(A)\big)$, finishing this case.
If it happens that $\epsilon = 2|\Phi^A(B)_\alpha|^2$ for some $2 \leqslant \alpha \leqslant n-1$, then similar arguments with $\tau_1(A) = \tau_1\big(A, \delta(A)\big)$ replaced by $\tau_\alpha(A) = \sqrt{\delta(A)}$ gives

$$\frac{d(B,A)}{\delta_D(A)} \leqslant 2\frac{|\Phi^A(B)_\alpha|^2}{\delta_D(A)} \lesssim \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}}$$

Summarizing the results of the various cases depending on the value of ϵ, we thus get that for some $1 \leqslant j \leqslant n$ the inequality

$$\frac{d(B,A)}{\delta(A)} \lesssim \frac{|\Phi^A(B)_j|}{\tau_j(A)} \tag{2.43}$$

must hold. This will be used in the sequel.

Putting together what we inferred for each of the possible values that ϵ may take, we may now also assert that (2.42) holds, from which it in-turn follows from Lemma 2.10 that

$$d_D^c(A,B) \geqslant d_D^{Cara}(A,B) \geqslant C \log\Big(1 + \big(d(B,A)/\delta_D(A)\big)^2\Big). \tag{2.44}$$

Now we get to the end result, the lower bound as stated in Theorem 1.1; but we wish to first summarise for convenience, a couple of results from our variety of estimates encountered in course of our dealings of the two cases (2.35) and (2.36) which will be useful in the sequel – to be precise, parts (i) and (ii) of part (a) of the following proposition, come from the discussion of Lemmas 2.9 and 2.10 respectively. We then conclude by showing how the desired end, re-written in part (b) of this proposition, follows from its previous parts.

Proposition 2.12. —

(a) *For points $A, B \in D \cap U'$, depending on whether they are far or near, as measured by the pseudo-distance d', we have two cases and correspondingly various estimates as in the first two statements below:*

(i) $d'(B,A) > 4C_e\delta_D(A)$. *Then for some constants* $K_{11}, K_{12} > 0$ *we have*

$$d^c_D(A,B) \geqslant K_{11}\ \log\Big(1\ +\ \big(d(B,A)/\delta_D(A)\big)\Big)$$

and

$$d(B,A)/\delta_D(A) \geqslant K_{12}\Big(|\Phi^A(B)_n|/\delta_D(A) + \sum_{\alpha=2}^{n-1}|\Phi^A(B)_\alpha|/\big(\delta_D(A)\big)^{1/2}\ +\ |\Phi^A(B)_1|/\tau(A,\delta_D(A))\Big)$$

(ii) $d'(B,A) \leqslant 4C_e\delta_D(A)$. *Then for some constants* $K_{21}, K_{22}, K_{23} > 0$ *we have*

$$d^{Cara}_D(A,B) \geqslant K_{21}\ \log\Big(1\ +\ |\Phi^A(B)_n|^2/\delta_D(A)^2 + \sum_{\alpha=2}^{n-1}|\Phi^A(B)_\alpha|^2/\delta_D(A)\ +\ |\Phi^A(B)_1|^2/\tau(A,\delta_D(A))^2\Big)$$

and

$$d(B,A)/\delta_D(A) \leqslant K_{22}\Big(|\Phi^A(B)_n|/\delta_D(A) + \sum_{\alpha=2}^{n-1}|\Phi^A(B)_\alpha|/\big(\delta_D(A)\big)^{1/2}\ +\ |\Phi^A(B)_1|/\tau\big(A,\delta_D(A)\big)\Big)$$

from which it was seen to follow in the foregoing lemmas, that

$$d^c_D(A,B) \geqslant d^{Cara}_D(A,B) \geqslant K_{23}\log\Big(1\ +\ \big(d(B,A)/\delta_D(A)\big)^2\Big)$$

(b) Finally, we have the lower bound valid for all $A, B \in D\cap U$ *and some constant* $K > 0$*:*

$$d^c_D(A,B) \geqslant K\log\Big(1+\frac{|\Phi^A(B)_n|}{\delta_D(A)}+\sum_{\alpha=2}^{n-2}\frac{|\Phi^A(B)_\alpha|}{\big(\delta_D(A)\big)^{1/2}}+\frac{|\Phi^A(B)_1|}{\tau\big(A,\delta_D(A)\big)}\Big)-\frac{K}{n}\ \log 2$$

obtained by combining the first two inequalities in (ii) and (i) of part(a).

Proof. — As noted earlier, it remains only to combine the two cases to get the final inequality as stated in (b). We may assume that the constant $K_{12} < 1$ in the second inequality in (i). Then

$$K_{12}\log\big(1+Q\big) \leqslant \log(1+K_{12}Q) \leqslant \log\big(1+d(B,A)/\delta_D(A)\big)$$

where Q is the quantity

$$|\Phi^A(B)_n|/\delta_D(A) + \sum_{\alpha=2}^{n-1} |\Phi^A(B)_\alpha|/\big(\delta_D(A)\big)^{1/2} \ + \ |\Phi^A(B)_1|/\tau(A, \delta_D(A)).$$

We therefore have

$$d_D^c(A, B) \geqslant K_{11} \log\big(1 + d(B, A)/\delta_D(A)\big) \geqslant K_{11} K_{12} \log(1 + Q),$$

when in the first case of (a) of the proposition. In particular

$$d_D^c(A, B) \gtrsim \log(1 + Q) - \frac{1}{n} \log 2 \tag{2.45}$$

To deal with case (ii), let us denote by E the expression

$$|\Phi^A(B)_n|^2/\delta_D(A)^2 + \sum_{\alpha=2}^{n-1} |\Phi^A(B)_\alpha|^2/\delta_D(A) \ + \ |\Phi^A(B)_1|^2/\tau(A, \delta_D(A))^2.$$

Next we use the inequality

$$|z_1 + \ldots + z_N|^2 \leqslant N\big(|z_1|^2 + \ldots |z_N|^2\big)$$

with $N = n$, to convert the inequality in (ii) of the proposition and express it in terms of E to get

$$\begin{aligned} d_D^{Cara}(A, B) \geqslant K_{21} \log(1 + E) &\geqslant K_{21} \log(1 + Q^2/n) \\ &\geqslant \frac{K_{21}}{n} \log(1 + Q^2) \\ &\geqslant \frac{K_{21}}{n} \log \frac{1}{2}(1 + Q)^2 \\ &= \frac{2K_{21}}{n} \log(1 + Q) - \frac{K_{21}}{n} \log 2 \end{aligned} \tag{2.46}$$

Combining (2.45) and (2.46) gives the final inequality of the proposition. □

Finally, we can also get from part (b) of the last Proposition that

$$d_D^c(A, B) \gtrsim \log\left(1 + \frac{|d(B, A)|}{\delta_D(A)} + \sum_{\alpha=2}^{n-2} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} + \frac{|\Phi^A(B)_1|}{\tau\big((A, \delta_D(A)\big)}\right) - l. \tag{2.47}$$

for some positive constant l.

To see this, first let us finish the easy case namely, when we are in the case (a)(i) of the last Proposition. Then, recall from our arguments for the inequality (2.41) that we had

$$\frac{d(B,A)}{\delta_D(A)} \geqslant c\frac{|\Phi^A(B)_j|}{\tau_j(A)}$$

for all $1 \leqslant j \leqslant n$ and for some constant $c < 1$. Now from the first inequality in (a)(i) of Proposition 2.12 we have

$$\begin{aligned} d^c_D(A,B) &\geqslant \log\left(1 + \frac{1}{2}\frac{d(B,A)}{\delta_D(A)} + \frac{1}{2}\frac{d(B,A)}{\delta_D(A)}\right) \\ &\geqslant \log\left(1 + \frac{1}{2}\frac{d(B,A)}{\delta_D(A)} + \frac{c}{2}\sum_{j=1}^{n-1}\frac{|\Phi^A(B)_j|}{\tau_j(A)}\right) \\ &\gtrsim \log\left(1 + \frac{d(B,A)}{\delta_D(A)} + \sum_{j=1}^{n-1}\frac{|\Phi^A(B)_j|}{\tau_j(A)}\right) \end{aligned}$$

as required.

The other case to deal with is when we are in the situation of (a)(ii) Proposition 2.12. Again to finish off the easy sub-case first, suppose that the maximum on the right hand inequality in Lemma 2.11 happens to be $|\Phi^A(B)_n|$; we then have the following chain of inequalities giving the claim:

$$\begin{aligned} &d^c_D(A,B) \\ &\geqslant K\log\left(1 + \frac{|\Phi^A(B)_n|}{\delta_D(A)} + \sum_{\alpha=2}^{n-2}\frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} + \frac{|\Phi^A(B)_1|}{\tau\big((A,\delta_D(A)\big)}\right) - K\log n/2 \\ &\geqslant K\log\left(1 + \frac{1}{2}\frac{|d'(B,A)|}{\delta_D(A)} + \sum_{\alpha=2}^{n-2}\frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} + \frac{|\Phi^A(B)_1|}{\tau\big((A,\delta_D(A)\big)}\right) - K\log n/2 \\ &\geqslant \frac{K}{2}\log\left(1 + \frac{|d'(B,A)|}{\delta_D(A)} + \sum_{\alpha=2}^{n-2}\frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} + \frac{|\Phi^A(B)_1|}{\tau\big((A,\delta_D(A)\big)}\right) - K\log n/2 \\ &\geqslant \frac{K}{2}\log\left(1 + \frac{|d(B,A)|}{\delta_D(A)} + \sum_{\alpha=2}^{n-2}\frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta_D(A)}} + \frac{|\Phi^A(B)_1|}{\tau\big((A,\delta_D(A)\big)}\right) - K\log n/2. \end{aligned} \tag{2.48}$$

The remaining possibilities are when the maximum on the right inequality in the Lemma 2.11 is attained by $|\Phi^A(B)_k|^2$ for some $2 \leqslant k \leqslant n-1$ or by $\max_{2\leqslant l\leqslant 2m} |P_l(A)||\Phi^A(B)_k|^l$ with $k=1$. In this case, we first appeal to the inequality (2.43) and put it in the from

$$\frac{1}{2}\frac{|\Phi^A(B)_k|}{\tau_k(A)} \gtrsim \frac{1}{2}\frac{d(B,A)}{\delta_D(A)} \tag{2.49}$$

Next recall from (2.40) that for all $1 \leqslant j \leqslant n$ we have that

$$e^{d_D^{Cara}(A,B)} \gtrsim 1 + \left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|^2$$

Summing over the first $n-1$ indices gives

$$\begin{aligned} e^{d_D^{Cara}(A,B)} &\gtrsim (n-1) + \sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|^2 \\ &\geqslant (n-1) + \frac{1}{2}\sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j|}{\tau_j(A)}\right|^2 + \frac{1}{2}\frac{|\Phi^A(B)|}{\tau_k(A)} \end{aligned}$$

which can be re-written using (2.49) as

$$\begin{aligned} d_D^{Cara}(A,B) &\geqslant \log\left(1 + \frac{1}{2(n-1)}\sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|^2 + \frac{1}{2(n-1)}\frac{|\Phi^A(B)_k|}{\tau_k(A)}\right) - l \\ &\geqslant \log\left(1 + \frac{1}{2(n-1)}\sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|^2 + \frac{c}{(n-1)}\frac{d(B,A)}{\delta_D(A)}\right) - l \\ &\geqslant \log\left(1 + \frac{c}{(n-1)}\frac{d(B,A)}{\delta_D(A)} + \frac{1}{2(n-1)^2}\left|\sum_{j=1}^{n-1}\frac{\Phi^A(B)_j}{\tau_j(A)}\right|^2\right) - l \\ &\gtrsim \log\left(1 + \sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|^2 + \frac{d(B,A)}{\delta_D(A)}\right) - l \\ &\gtrsim \log\left(\frac{1}{n}\Big(1 + \sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|\Big)^2 + \frac{d(B,A)}{\delta_D(A)}\right) - l \\ &\geqslant \log\left(\frac{1}{n}\Big(1 + \sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right|\Big) + \frac{1}{n}\frac{d(B,A)}{\delta_D(A)}\right) - l \\ &= \log\left(1 + \sum_{j=1}^{n-1}\left|\frac{\Phi^A(B)_j}{\tau_j(A)}\right| + \frac{d(B,A)}{\delta_D(A)}\right) - \log n - l. \end{aligned} \tag{2.50}$$

finishing the proof of the inequality claimed at (2.47).

2.6. Upper bound on the invariant distances

In this section we shall establish the upper bound on the Kobayashi distance $d_D^k(A, B)$ between two points $A, B \in D \cap U$. We have already mentioned the availability of optimal upper bounds on the infinitesimal Kobayashi metric for Levi corank one domains, in the introductory section. We shall rephrase it here in a form useful for the methods of the present sub-section. For now, we begin the proof of the upper bound in theorem 1.1 right away, by first getting an upper bound on the Kobayashi distance between two points which are on the same inner normal to ∂D at an arbitrary boundary point and close to it. This can be obtained following [31] with almost no changes.

Lemma 2.13. — *Suppose that $\tilde{A}, \tilde{B}$ are points in $D \cap U$ such that $\tilde{A}^* = \tilde{B}^*$, then*

$$d_D^k(\tilde{A}, \tilde{B}) \leqslant \frac{1}{2} \log \Big(1 + \tilde{C} \frac{|\delta(\tilde{A}) - \delta(\tilde{B})|}{\min\{\delta(\tilde{A}), \delta(\tilde{B})\}}\Big)$$

for some positive constant $\tilde{C}$.

Here we have denoted the boundary distance by $\delta_D(\cdot)$ more simply by $\delta(\cdot)$ and we shall continue doing so for the rest of this section. With this lemma as a start, we now proceed to get an upper bound on the Kobayashi distance between the points A and B, in terms of the distances of A, B to the boundary and the distance between them measured in the pseudo-distance d', as expressed in Theorem 1.1. So suppose that $A, B \in D \cap U$; if $|A - B| \geqslant R$ then the claim follows from proposition 2.5 of [24]. So we have only to deal with the case when $A, B \in D \cap U$ with $|A - B| < R$ in which case $d(A, B) \approx d'(A, B)$. Now we shall split-up again into two cases depending upon whether A, B are near or far, when their distance is measured by the pseudo-distance d'; to quantify the definition of nearness here, we first choose positive constants C_1, L, η with the following properties:

(a): $d(x, y) \leqslant C_1\big(d(x, z_1) + d(z_1, z_2) + d(z_2, y)\big)$ holds for all $x, y, z_1, z_2 \in U$ and

(b): $d(x, y) \geqslant d'(x, y)/L$ whenever $d'(x, y)$ is finite.

If we choose a thin enough tubular neighbourhood $U_0 \subset U$ of ∂D and the constant η small enough, we can achieve for any pair of points $A, B \in U_0$ that, the points $A - \eta d'(A, B)\nu_{A^*}$ and $B - \eta d'(B, A)\nu_{B^*}$, still lie in U. Define $M = 3LC_1/\eta$ and shrink U to ensure $\delta(z) < \delta_e/2M$ for all $z \in D \cap U$ – this in turn ensures that the special analytic polydiscs $Q(A, 2M\delta_D(A))$ are well defined – here δ_e is the number introduced in section 1. Recall that δ_e was taken to be less than 1 and therefore the same will hold for all boundary distances $\delta(\cdot)$. We shall continue doing such adjustments of neighbourhoods and constants tacitly; the sloppiness caused due to overloaded notation is taken care of by keeping track of what entities such changes depend on.

We now proceed with the two cases mentioned above. The first one, is laid down precisely in

Lemma 2.14. — *Suppose that the points $A, B \in D \cap U$ satisfy $d'(A,B) \leqslant M\ max\{\delta(A), \delta(B)\}$. Then*

$$d_D^k(A,B) \leqslant C_1 \log\left(1 + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}} + \frac{d(A,B)}{\delta(A)}\right)$$
$$+ \log\left(1 + \frac{|\Phi^B(A)_1|}{\tau\big(B,\delta(B)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^B(A)_\alpha|}{\sqrt{\delta(B)}} + \frac{d(B,A)}{\delta(B)}\right) + C_2.$$

for some positive constants C_1, C_2 depending on the domain D but free from any particular dependence on A, B.

Proof. — We begin with the observation that the upper bound on the infinitesimal Kobayashi metric from Theorem 1 of [13] – using the fact from [14] that $\eta(z,\delta) \approx \tau(z,\delta)$ where $\eta(\cdot,\cdot)$ (only for now in the notations of [13], [14]) is the quantity occurring in bound as stated in [13] – can be put in the form:

$$K_D(z,X) \leqslant C\Big(\frac{|\langle L_1(z), X\rangle|}{\tau\big(z,\delta(z)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\langle L_\alpha(z), X\rangle|}{\sqrt{\delta(z)}} + \frac{|X_n|}{\delta(z)}\Big). \tag{2.51}$$

Here $z \in D \cap U$; of course this estimate as it stands, is valid only on a small ball about $\pi(z)$ and when we are in the normalization of the foregoing sections, in particular $\partial r/\partial z_n \neq 0$. However, to rephrase the above estimate to ensure its validity on the entire tubular neighbourhood $D \cap U$, only requires z to be replaced by $\sigma(z)$ for an appropriate permutation σ (depending on which portion of the tubular neighbourhood z is situated). As we shall be interested only in size estimates of various quantities here and since permutations are isometries – so, in particular the distance to the boundary and the quantity τ remain preserved i.e., $\tau(z,\delta(z)) = \tau\big(\sigma(z), \delta_{\sigma(D)}(\sigma(z))\big)$ – we shall as always, suppress these permutations in the calculations. Actually, we can even overlook these issues completely, in view of our standing assumption for the rest of this section that $\operatorname{dist}(A,B) < R$. Next, note that

$$\begin{aligned}\Big||\langle L_j(z),X\rangle| - |\langle L_j(\pi(z)),X\rangle|\Big| &\leqslant \Big|\langle L_j(z),X\rangle - \langle L_j(\pi(z)),X\rangle\Big| \\ &= \Big|\frac{\partial r/\partial z_j}{\partial r/\partial z_n}(z) - \frac{\partial r/\partial z_j}{\partial r/\partial z_n}\big(\pi(z)\big)\Big||X_n| \\ &\lesssim |z-\pi(z)||X_n| \text{ by the mean value inequality} \\ &= \delta(z)|X_n|. \end{aligned} \tag{2.52}$$

We thus get, for some constant $C_2 > 0$ and for each $1 \leqslant j \leqslant n-1$, that

$$\begin{aligned}\frac{|\langle L_j(z),X\rangle|}{\tau_j(z,\delta(z))} &\lesssim \frac{|\langle L_j(\pi(z)),X\rangle|}{\tau_j(z,\delta(z))} + \frac{\delta(z)}{\tau_j(z,\delta(z))} \\ &< \frac{|\langle L_j(\pi(z)),X\rangle|}{\tau_j(z,\delta(z))} + C_2\end{aligned}$$

using which we rewrite (2.51) for clarity in the sequel as:

$$K_D(z,X) \leqslant C\Big(\frac{|\langle L_1(\pi(z)),X\rangle|}{\tau\big(z,\delta(z)\big)} + \sum_{\alpha=2}^{n-1}\frac{|\langle L_\alpha(\pi(z)),X\rangle|}{\sqrt{\delta(z)}} + \frac{|X_n|}{\delta(z)}\Big) + \text{some constant} \tag{2.53}$$

Now assume that $\delta(A) \geqslant \delta(B)$. Let B' be the point in $D \cap U$ such that $\delta(B') = \delta(A)$ and $(B')^* = B^*$. So in particular, B, B' lie on the same normal to ∂D at B^*. Lemma 2.13 prompts an estimation of $d_D^k(A,B)$ in terms of the distances $d_D(A,B')$ and $d_D(B',B)$, because the latter can be estimated by that lemma; while the former distance $d_D(A,B')$ is bounded above by the Kobayashi-length of any curve joining A, B'. More precisely, if c is any one such curve, we will have the estimation as in the following string of inequalities:

$$\begin{aligned} d_D^k(A,B) &\leqslant d_D^k(A,B') + d_D^k(B',B) \\ &\leqslant L_D^{Kob}(c) + d_D^k(B,B') \\ &\leqslant L_D^{Kob}(c) + C' \log\Big(1 + \tilde{C}\frac{|\delta(B')-\delta(B)|}{\min\{\delta(B'),\delta(B)\}}\Big) \\ &= L_D^{Kob}(c) + C' \log\Big(1 + \tilde{C}\frac{|\delta(B')-\delta(B)|}{\delta(B)}\Big) \\ &\leqslant L_D^{Kob}(c) + C' \log\Big(1 + \tilde{C}\frac{d'(A,B)}{\delta(B)}\Big). \end{aligned} \tag{2.54}$$

The last inequality follows from our hypotheses of lemma 2.14, so that $|\delta(A)-\delta(B)| \leqslant d'(A,B)$ and subsequently since $|\delta(B')-\delta(A)| = 0$, we get $|\delta(B')-\delta(B)| \leqslant d'(A,B)$.

We have now to construct a suitable path c connecting A and B' in $D \cap U$, trying to keep its Kobayashi-length as small as possible while also taking care that it be estimable in tangible terms. First, let $\Psi_A = (\Phi^A)^{-1}$ and $\gamma(t) = (\Psi^A)\big(t\Phi^A(B)\big)$ denote the pull-back of the straight line joining $\Phi^A(A) = 0$ and the point $\Phi^A(B)$. Then $\gamma(0) = A$, $\gamma(1) = B$; but it is not clear that γ is contained within our domain D. However, γ lies trapped within the analytic polydisc $Q\big(A, 2d'(A,B)\big)$. To see this indeed, we must go back to the definitions of the special analytic polydisc $Q(\cdot,\cdot)$, the pseudo-distance d' and the curve γ, all of which are defined in terms of the simplifying change of variables as in (1.4). Recall that

$$Q\big(A, 2d'(A,B)\big) = \Psi_A\Big(R\big(A, 2d'(A,B)\big)\Big) \tag{2.55}$$

where we recall $R(A, 2d'(A,B))$ is the polydisc centered at the origin with polyradius $(\tau_1, \ldots, \tau_n)$ with $\tau_j = \tau_j(A, 2d'(A,B))$ as introduced in section 1. To verify that $\gamma(t) = \Psi_A(t\Phi^A(B))$ lies in the sets at (2.55) is tantamount to checking the following list of n-inequalities:

(1) $|t\Phi^A(B)_1| < \tau\big(A, 2d'(A,B)\big)$,

(α) $|t\Phi^A(B)_\alpha| < \sqrt{2d'(A,B)}$, for $2 \leqslant \alpha \leqslant n-1$,

(n) $|t\Phi^A(B)_n| < 2d'(A,B)$.

We confine to display the checking of the first:

$$|t\Phi^A(B)_1| \leqslant |\Phi^A(B)_1| \leqslant \tau\big(A, d'(A,B)\big) < \tau\big(A, 2d'(A,B)\big),$$

just by the definition of $d'(A,B)$. Next, by our hypotheses of our the lemma we are currently dealing with, $d'(A,B) \leqslant M\delta(A)$; so, by the monotonically increasing nature of $\tau(A,\delta)$ with respect to δ and thereby of the distorted polydiscs $Q(A,\cdot)$, we get in all that

$$\gamma \subset Q_{2d'(B,A)}(A) \subset Q_{2M\delta(A)}(A).$$

We now define our path c in terms of γ as follows. Let $\gamma^*(t)$ denote $\pi_{\partial D}(\gamma(t)) = \big(\gamma(t)\big)^*$ for short; let $c(t) = \gamma^*(t) - \delta(A)\nu_{\gamma^*(t)}$. Then $\delta(c(t)) = \delta(A)$ for all t. Note that

$$c(1) = \gamma^*(1) - \delta(A)\nu_{\gamma^*(1)} = B^* - \delta(A)\nu_{B^*} = B',$$

so c is a curve in $D \cap U$ connecting A and B' which maintains a constant distance from ∂D.

Next, to extend the chain of inequalities in (2.54) and reach the required upper bound, we need to estimate the length of c in the Kobayashi metric.

To this end, we first write out the estimate for the size of $\dot{c}(t)$, measured in the infinitesimal Kobayashi metric provided by (2.53):

$$\begin{aligned} K_D\big(c(t),\dot{c}(t)\big) &\lesssim \frac{|\langle L_1\big(c^*(t)\big),\dot{c}(t)\rangle|}{\tau\big(c(t),\delta\big(c(t)\big)} + \sum_{\alpha=2}^{n-1}\frac{|\langle L_\alpha\big(c^*(t)\big),\dot{c}(t)\rangle|}{\sqrt{\delta\big(c(t)\big)}} + \frac{|\dot{c}(t)_n|}{\delta\big(c(t)\big)} + C_3 \\ &= \frac{|\langle L_1\big(\gamma^*(t)\big),\dot{c}(t)\rangle|}{\tau\big(c(t),\delta\big(c(t)\big)} + \sum_{\alpha=2}^{n-1}\frac{|\langle L_\alpha\big(\gamma^*(t)\big),\dot{c}(t)\rangle|}{\sqrt{\delta\big(c(t)\big)}} + \frac{|\dot{c}(t)_n|}{\delta\big(c(t)\big)} + C_3 \end{aligned} \tag{2.56}$$

for some constant $C_3 > 0$; here we have used the fact that $\gamma^*(t) = c^*(t)$ for all $t \in [0,1]$. Since the curve c is defined in terms of the curve γ, let us replace $\dot{c}(t)$ by $\dot{\gamma}(t)$ in the above inequality. For this, if we write $N(x)$ for $\nu\big(\pi_{\partial D}(x)\big)$, we have

$$\begin{aligned} \dot{c}(t) &= D\big(\pi_{\partial D}\big)_{|_{\gamma(t)}}(\dot{\gamma}(t)) - \delta(A)\cdot\frac{d}{dt}\nu\big(\pi_{\partial D}(\gamma(t))\big) \\ &= \Big(D(\pi_{\partial D})(\gamma(t)) - \delta(A)D(N)(\gamma(t))\Big)\dot{\gamma}(t) \\ &= \dot{\gamma}^*(t) - \delta(A)DN\big(\gamma(t)\big)\big(\dot{\gamma}(t)\big) \end{aligned} \tag{2.57}$$

since

$$\dot{\gamma}^*(t) = D\big(\pi_{\partial D}\big)(\gamma(t))\cdot\dot{\gamma}(t). \tag{2.58}$$

Now we upper bound (2.56), by the length of $\dot{\gamma}$ as measured by the metric appearing on the right hand side of (2.53). To do this, we analyze and estimate each summand in (2.56). First consider

$$\frac{|\langle L_1\big(\gamma^*(t)\big),\dot{c}(t)\rangle|}{\tau\big(c(t),\delta\big(c(t)\big)}$$

and the numerator herein, which by (2.57) may be estimated as

$$\begin{aligned} \big|\langle L_1\big(\gamma^*(t)\big),\dot{c}(t)\rangle\big| &= \Big|\Big\langle L_1\big(\gamma^*(t)\big),\ \dot{\gamma}^*(t) - \delta(A)DN(\gamma(t))\big(\dot{\gamma}(t)\big)\Big\rangle\Big| \\ &= \big|\langle L_1\big(\gamma^*(t)\big),\dot{\gamma}^*(t)\rangle\big| + \delta(A)\Big|\Big\langle L_1\big(\gamma^*(t)\big),DN(\gamma(t))\big(\dot{\gamma}(t)\big)\Big\rangle\Big| \\ &\leqslant \big|\langle L_1\big(\gamma^*(t)\big),\dot{\gamma}^*(t)\rangle\big| + \delta(A)\big|L_1\big(\gamma^*(t)\big)\big|\Big|DN(\gamma(t))\Big||\dot{\gamma}(t)| \\ &\lesssim \big|\langle L_1\big(\gamma^*(t)\big),\dot{\gamma}^*(t)\rangle\big| + \delta(A)|\dot{\gamma}(t)|. \end{aligned} \tag{2.59}$$

Now we may use the techniques in the proof of lemma (2.2) of [6], to compare the components along the complex tangential direction L_1, of the tangent vector of the curve γ and that of its projection onto the boundary. More precisely,

$$\Big|\Big\langle L_1\big(\gamma^*(t)\big), \dot\gamma^*(t)\Big\rangle \; - \; \Big\langle L_1\big(\gamma^*(t)\big), \dot\gamma(t)\Big\rangle\Big| \leqslant C\delta\big(\gamma(t)\big)|\dot\gamma(t)|.$$

for some constant $C > 0$ independent of γ. Indeed, first recall the geometrically obvious relation between the curve γ and its orthogonal projection onto the boundary γ^*, namely:

$$\gamma^*(t) - \gamma(t) = -r\big(\gamma(t)\big)\nu\big(\gamma^*(t)\big). \tag{2.60}$$

A differentiation then yields the desired connection between the tangent vectors,

$$\begin{aligned}\dot\gamma^*(t) - \dot\gamma(t) &= -\langle \nu\big(\gamma(t)\big), \dot\gamma(t)\rangle \nu\big(\gamma^*(t)\big) - r\big(\gamma(t)\big)DN(\gamma(t))\big(\dot\gamma(t)\big)\\ &= -r\big(\gamma(t)\big)DN(\gamma(t))\big(\dot\gamma(t)\big).\end{aligned} \tag{2.61}$$

In particular then, we get for any $j = 1, \dots, n$, that

$$\begin{aligned}\big|\langle L_j\big(\dot\gamma^*(t)\big), \dot\gamma^*(t)\rangle - \langle L_j\big(\dot\gamma^*(t)\big), \dot\gamma(t)\rangle\big| &= -r\big(\gamma(t)\big)\Big|DN(\gamma(t))\Big||\dot\gamma(t)|\\ &\lesssim \delta(\gamma(t))|\dot\gamma(t)|.\end{aligned}$$

Consequently, (2.59) can be rewritten now as:

$$\big|\langle L_1\big(\gamma^*(t)\big), \dot c(t)\rangle\big| \lesssim \big|\langle L_1\big(\gamma^*(t)\big), \dot\gamma(t)\rangle\big| + C_1\delta\big(\gamma(t)\big)\big|\dot\gamma(t)\big| + \delta(A)|\dot\gamma(t)|.$$

Let us rewrite this as an estimate for the first term in (2.56).

$$\begin{aligned}\frac{|\langle L_1\big(\gamma^*(t)\big), \dot c(t)\rangle|}{\tau\big(c(t), \delta(c(t))\big)} &\lesssim \frac{|\langle L_1\big(\gamma^*(t)\big), \dot\gamma(t)\rangle|}{\tau\big(c(t), \delta(c(t))\big)}\\ &\quad + \frac{\delta\big(\gamma(t)\big)}{\tau\big(c(t), \delta(c(t))\big)}|\dot\gamma(t)| + \frac{\delta(A)}{\tau\big(c(t), \delta(c(t))\big)}|\dot\gamma(t)|\end{aligned} \tag{2.62}$$

Consider the ratio of distances appearing in the second term on the right namely,

$$\frac{\delta\big(\gamma(t)\big)}{\tau\big(c(t), \delta(c(t))\big)} = \frac{\delta\big(\gamma(t)\big)}{\tau\big(c(t), \delta(A)\big)}.$$

Now, from the fact that the curve γ lies trapped within the analytic polydisc $Q\big(A, 2M\delta(A)\big)$, by hypothesis of lemma 2.14, we may estimate the distance of $\gamma(t)$ to the boundary ∂D as follows. First, for all points z which lie in the polydisc $R(A, \delta)$ about the origin, we have

$$\big|\Psi_A(z) - \Psi_A(0)\big| \leqslant C|z|$$

where the Lipschitz constant C is independent of A, owing to the fact that the polynomial maps $\{\Psi_A(\cdot)\}$, are Lipschitz uniformly in $A \in D \cap U$. Of course the special radii of the polydisc R is of no relevance to the last inequality but use will be made of the fact the constant C here, may also be taken independent $\delta \in (0, \delta_e)$. Put differently, if a point w lies in $Q(A, \cdot)$, then it must satisfy

$$|w - A| \leqslant C|\Phi^A(w)|. \tag{2.63}$$

Let us apply this observation to the situation at hand: since $\gamma(t) \in Q(A, 2M\delta(A))$, we have

$$\begin{aligned}|\gamma(t) - A| \leqslant C|\Phi^A(\gamma(t))| = |t\Phi^A(B)| \leqslant C|\Phi^A(B)| &\lesssim \max_{1\leqslant j\leqslant n}\{\tau_j(A, \delta(A))\}\\ &\lesssim \tau_1\big(A, \delta(A)\big)\end{aligned}$$

where all constants at the various $\lesssim$-bounds here are independent of γ i.e., of the points A, B. Therefore,

$$\delta\big(\gamma(t)\big) \leqslant \operatorname{dist}\big(\gamma(t), A\big) + \delta\big(A\big) \leqslant C\tau\big(A, \delta(A)\big)$$

for some constant C independent of A, B. Thus we get that the ratio

$$\frac{\delta(\gamma(t))}{\tau\big(c(t), \delta(c(t))\big)} \lesssim \frac{\delta(\gamma(t))}{\tau\big(A, \delta(A)\big)} \tag{2.64}$$

is bounded above by a constant independent of A and B. In these calculations, we use the fact that $\tau\big(c(t), \delta(c(t))\big) = \tau\big(c(t), \delta(A)\big) \approx \tau\big(A, \delta(A)\big)$. This fact follows from the uniform comparability of these distinguished radii at different points within a distorted polydisc. In addition, it also uses the fact that the path $c(t)$ remains at a fixed small distance from the boundary of D (small enough to define the polydiscs and to validate the application of the comparability of the radii, by covering the path by finitely many such polydiscs). Also, the lower bound on the special radius $\tau\big(A, \delta(A)\big) \gtrsim \sqrt{\delta(A)} \geqslant \delta(A)$ is used repeatedly in this paper.

Recall that we were seeking to obtain some control on the terms occurring in the right hand side of (2.62). The ratio on the left of (2.64) is a factor appearing in the inequality (2.62); we may now express that inequality as

$$\frac{|\langle L_1\big(\gamma^*(t)\big), \dot{c}(t)\rangle|}{\tau\big(c(t), \delta(c(t))\big)} \lesssim \frac{|\langle L_1\big(\gamma^*(t)\big), \dot{\gamma}(t)\rangle|}{\tau\big(A, \delta(A)\big)} + C|\dot{\gamma}(t)|. \tag{2.65}$$

What we have just obtained is a bound on the first term in (2.56). The next $n-2$ summands in (2.56) can be similarly dealt with. For the last term in (2.56), we proceed as follows. First recall (2.57) and write out the equality which it contains for the last component, namely,

$$\dot{c}(t)_n = \dot{\gamma}^*(t)_n - \delta(A)\Big(DN(\gamma(t))\big(\dot{\gamma}(t)\big)\Big)_n$$

Therefore,

$$\frac{|\dot{c}(t)_n|}{\delta(c(t))} \leqslant \frac{|\dot{\gamma}^*(t)_n|}{\delta(c(t))} + \frac{\delta(A)}{\delta(c(t))}\Big|DN(\gamma(t))\Big||\dot{\gamma}(t)| \leqslant \frac{|\dot{\gamma}^*(t)_n|}{\delta(A)} + C|\dot{\gamma}(t)| \quad (2.66)$$

where C is again some constant independent of A, B since $DN(z) = D(\nu \circ \pi)(z)$ is Lipchitz uniformly in $z \in D \cap U$.

Let us now pause to sum up the discussion so far, of the estimate (2.56), from the estimates (2.65), through its analogues for the components of $\dot{c}(t)$ along L_α for $\alpha = 2, \ldots, n-1$ and (2.66), as:

$$K_D\big(c(t), \dot{c}(t)\big) \lesssim \frac{|\langle L_1\big(\gamma^*(t)\big), \dot{\gamma}(t)\rangle|}{\tau\big(A, \delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\langle L_\alpha\big(\gamma^*(t)\big), \dot{\gamma}(t)\rangle|}{\sqrt{\delta(A)}} + \frac{|\dot{\gamma}^*(t)_n|}{\delta(A)} + C|\dot{\gamma}(t)| + \text{some constant}$$

which more simply may be recorded for now as

$$K_D\big(c(t), \dot{c}(t)\big) \lesssim \frac{|\langle L_1\big(\gamma^*(t)\big), \dot{\gamma}(t)\rangle|}{\tau\big(A, \delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\langle L_\alpha\big(\gamma^*(t)\big), \dot{\gamma}(t)\rangle|}{\sqrt{\delta(A)}} + \frac{|\dot{\gamma}^*(t)_n|}{\delta(A)} + \text{ some constant.} \quad (2.67)$$

Here we have used the fact that $|\dot{\gamma}(t)|$ can be bounded above by a uniform constant independent of the curve γ, A, B – to verify this, let us write down:

$$\dot{\gamma}(t) = D\Psi_A(t\Phi^A(B))\big(\Phi^A(B)\big).$$

So, $|\dot{\gamma}(t)| \leqslant \big|D\Psi_A(t\Phi^A(B))\big||\Phi^A(B)|$ which is bounded above on the pre-compact neighbourhood U, as Φ^A, Ψ_A are polynomial automorphisms with coefficients varying smoothly in the variable A which comes from the thin neighbourhood U.

Now, recall that the goal of lemma 2.14, was reduced at (2.54) to obtaining an upper-bound for $L_D^{Kob}(c)$; this will now of course be obtained by integrating the upper bound (2.67) above. To obtain a concrete upper bound for $d_D^k(A, B)$, we must now estimate the summands in (2.67), involving the coordinates of the points A, B or some explicit function of them alone, as far as possible. We now turn towards this end. We begin by noticing for example that

$$\begin{aligned}\Big|\langle L_1\big(\gamma^*(t)\big),\dot\gamma(t)\rangle\Big| &= \Big|\dot\gamma(t)_1 - \big(\frac{\partial r/\partial z_1}{\partial r/\partial z_n}\big)\big(\gamma^*(t)\big)\ \dot\gamma(t)_n\Big| \\ &\lesssim |\dot\gamma(t)_1| + |\dot\gamma(t)_n|\end{aligned}$$

and subsequently rewrite this out for convenience as

$$\frac{|\langle L_1\big(\gamma^*(t)\big),\dot\gamma(t)\rangle|}{\tau\big(A,\delta(A)\big)} \lesssim \frac{|\dot\gamma(t)_1|}{\tau\big(A,\delta(A)\big)} + \frac{|\dot\gamma(t)_n|}{\tau\big(A,\delta(A)\big)}.$$

Further, recalling that $1/\tau\big(A,\delta(A)\big) \lesssim 1/\delta(A)$, we obtain

$$\frac{|\langle L_1\big(\gamma^*(t)\big),\dot\gamma(t)\rangle|}{\tau\big(A,\delta(A)\big)} \lesssim \frac{|\dot\gamma(t)_1|}{\tau\big(A,\delta(A)\big)} + \frac{|\dot\gamma(t)_n|}{\delta(A)}. \tag{2.68}$$

Similar calculations give for each $2 \leqslant \alpha \leqslant n-1$, that

$$\frac{|\langle L_\alpha\big(\gamma^*(t)\big),\dot\gamma(t)\rangle|}{\sqrt{\delta(A)}} \lesssim \frac{|\dot\gamma(t)_\alpha|}{\sqrt{\delta(A)}} + \frac{|\dot\gamma(t)_n|}{\delta(A)}. \tag{2.69}$$

Finally, it remains to replace $\dot\gamma^*(t)_n$ by the n-th component (*alone*) of $\dot\gamma(t)$ in (2.67). To do this, let us go back to (2.61) and read that equation for the n-th component. Apply the triangle inequality therein and the fact that $|r(z)| \approx \delta(z)$ for all $z \in D \cap U$, to get

$$\frac{|\dot\gamma^*(t)_n|}{\delta(A)} \lesssim \frac{|\dot\gamma(t)_n|}{\delta(A)} + \frac{\delta(\gamma(t))}{\delta(A)}\Big|\Big(DN(\gamma(t))\big(\dot\gamma(t)\big)\Big)_n\Big|. \tag{2.70}$$

At this point, we realize that we require a sharper estimate on $\delta(\gamma(t))$ than that obtained at (2.64). The best grasp on the distance $\delta(\gamma(t))$ is attained by observing the key feature of the canonical transform Φ^A which defines the curve γ – it is that transform (unique among such maps in the group $\mathcal{E}_L$, up to a unitary linear map) which casts the weighted homogeneous Taylor expansion of the defining function of our Levi corank one domain, in the new coordinates about the point A, as in (1.3). Indeed, such a Taylor expansion of $r(\gamma(t)) = \rho^A(t\zeta)$ where $\zeta = \Phi^A(B)$, reads

$$\begin{aligned}\rho^A(t\zeta) &= r(A) + 2\Re(t\zeta_n) + \sum_{l=2}^{2m}\sum_{j+k=l} a^l_{jk}(A)t^{j+k}\zeta_1^j\overline{\zeta}_1^k \\ &\quad + \sum_{\alpha=2}^{n-1}\sum_{\substack{j+k\leqslant m\\ j,k>0}} \Re\Big(\big(b^\alpha_{jk}(A)t^{j+k+1}\zeta_1^j\overline{\zeta}_1^k\big)\zeta_\alpha\Big) + \sum_{\alpha=2}^{n-1} t^2|\zeta_\alpha|^2 + R(A;t\zeta).\end{aligned}$$

This gives the following upper bound on the boundary distance on γ:

$$\delta(\gamma(t)) \approx |r(\gamma(t))| = |\rho^A(t\zeta)| \lesssim \delta(A) + 2|\zeta_n| + \sum_{l=2}^{2m} \sum_{j+k=l} |a^l_{jk}(A)||\zeta_1|^{j+k} + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leqslant m \\ j,k>0}} |b^\alpha_{jk}(A)|\zeta_1|^{j+k}|\zeta_\alpha| + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2 + \big|R(A;t\zeta)\big|$$

where we recall that the remainder function satisfies

$$R\Big(A; \big(\tau(A,\delta), \sqrt{\delta}, \dots, \sqrt{\delta}, \delta\big)\Big)/\delta \to 0 \text{ as } \delta \to 0.$$

In particular, $|R(A;t\zeta)| \leqslant C\delta_D(A)$ for some positive constant independent of A, B. Subsequently, therefore we may put the foregoing estimate in the following simplified form, where the right side is only a 'weight one' polynomial in the components of $\Phi^A(B)$.

$$\delta(\gamma(t)) \lesssim \delta(A) + |\Phi^A(B)_n| + \sum_{l=2}^{2m} |\Phi^A(B)_1|^l \Big(\sum_{j+k=l} |a^l_{jk}(A)|\Big) + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leqslant m \\ j,k>0}} |b^\alpha_{jk}(A)||\Phi^A(B)_1|^{j+k}|\Phi^A(B)_\alpha| + \sum_{\alpha=2}^{n-1} |\Phi^A(B)_\alpha|^2. \quad (2.71)$$

Now just by the definition of $\tau(A,\delta(A))$ we have

$$|b^\alpha_{j,k}(A)| \leqslant \sqrt{\delta(A)}\tau(A,\delta(A))^{-(j+k)}$$

Combined with the fact that $\Phi^A(B) \in R(A, 2M\delta(A))$, this gives the following estimate on the coupled monomials in the $|\Phi^A(B)_j|$'s occurring in (2.71):

$$\sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leqslant m \\ j,k>0}} |b^\alpha_{jk}(A)||\Phi^A(B)_1|^{j+k}|\Phi^A(B)_\alpha| \leqslant \hat{C}\delta(A)^{1/2}\tau(A,\delta(A))^{-(j+k)}\tau(A, 2M\delta_D(A))^{j+k}\sqrt{\delta(A)} \leqslant C\delta(A),$$

where $\hat{C}$ is a universal constant and C a constant which depends on the domain D but not on the points A, B.

A similar treatment with all other monomials in the components of $\Phi^A(B)$ appearing on the right of the inequality (2.71) verifies that they are all bounded above by $\delta(A)$ up to some constant. Altogether this yields that the factor $\delta(\gamma(t))/\delta(A)$ is bounded above by a constant independent for the points A, B; combined with the earlier observation that the same is true of $|DN(\gamma(t))\big(\dot\gamma(t)\big)|$ occurring in the second term on the right side of (2.70) as well, we may now conclude that

$$\frac{|\dot\gamma^*(t)_n|}{\delta(A)} \leqslant \frac{|\dot\gamma(t)_n|}{\delta(A)} + \text{ some constant.}$$

Hence the calculations so far, have lead to reducing (2.67) as

$$K_D\big(c(t), \dot c(t)\big) \lesssim \Big(\frac{|\dot\gamma_1(t)|}{\tau\big(A, \delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\dot\gamma_\alpha(t)|}{\sqrt{\delta(A)}} + \frac{|\dot\gamma(t)_n|}{\delta(A)}\Big) + \text{some constant.} \tag{2.72}$$

This subsequently entails the estimation of the components of $\dot\gamma(t)$. We now proceed towards this. Denoting $\Phi^A(B)$ by ζ as above, let us begin with the expression for $\dot\gamma$ given by:

$$\dot\gamma(t) = D\Psi_{A|_{t\zeta}}(\zeta).$$

The expression for the inverse map Ψ_A and its derivative have been put down in the appendix, section 9, using which we may write down $\dot\gamma(t)$ more explicitly in the form

$$\dot\gamma(t) = \Big(\zeta_1, H_A(\tilde\zeta) + H_A\big(\zeta_1 \frac{\partial Q_2}{\partial z_1}(t\zeta_1)\big), b_n^A\big(\zeta_n + \sum_{j=1}^{n-1} \zeta_j \frac{\partial \tilde Q_1}{\partial z_j}(t'\zeta)\big)\Big)$$

where $H_A = G_A^{-1}$ and

$$\tilde Q_1('z) = (b_n^A)^{-1}\Big(\langle \tilde b^A, H_A\big(\tilde z + Q_2(z_1)\big)\rangle + b_1^A z_1\Big) + Q_1\Big(z_1, H_A\big(\tilde z + Q_2(z_1)\big)\Big).$$

where Q_1 and Q_2 are the same polynomials that occur in the expression for Φ^ζ as in (1.4). To carry out the afore-mentioned estimation at (2.72), we first note that it now reads up to an additive constant, as follows

$$\begin{aligned} K_D\big(c(t), \dot c(t)\big) \lesssim & \frac{|b_n^A|\Big|\zeta_n + \sum_{j=1}^{n-1} \zeta_j \frac{\partial \tilde Q_1}{\partial z_j}(t'\zeta)\Big|}{\delta(A)} \\ & + \left|\frac{H_A(\tilde\zeta) + H_A\big(\zeta_1\big(\frac{\partial Q_2}{\partial z_1}(t\zeta_1)\big)\big)}{\sqrt{\delta(A)}}\right| + \frac{|\zeta_1|}{\tau\big(A, \delta(A)\big)}. \end{aligned} \tag{2.73}$$

Now we estimate the numerator in the first summand, which contains in it the following sum [1]

$$\sum_{j=1}^{n-1} \zeta_j \frac{\partial Q_1}{\partial z_j}(t\,'\zeta).$$

This is of course a polynomial of the same form as Q_1 in ζ (i.e., up to multiplication by some factors which are either powers of t or some integers), since it is obtained by applying the differential operator $z_1\partial/\partial z_1 + \ldots + z_{n-1}\partial/\partial z_{n-1}$ which is of weight zero, to the polynomial $Q_1('z)$ and evaluated at $t\,'\zeta$. We shall confine ourselves to presenting a few samples of the estimations required here as they are all based on the same idea. The key point on which the calculations here rely on, is the fact from [14] and [56] that, within the special analytic polydiscs $Q(A,\cdot)$, variation of not only the defining function but also its derivatives obtained by applying differential operators of weight no less than -1, are controlled by the 'radii' of such polydiscs. We now illustrate how this works. Let us recall the expression for Q_1, in the notations of [56] for convenience:

$$Q_1('z) = \sum_{k=2}^{2m} d_k z_1^k + \sum_{\alpha=2}^{n-1}\sum_{k=1}^{m} d_{\alpha,k} z_1^k \big(z_\alpha + P_2^\alpha(z_1)\big) + \sum_{\alpha=2}^{n-1} c_\alpha \big(z_\alpha + P_2^\alpha(z_1)\big)^2 \tag{2.74}$$

where we recall that

$$P_2^\alpha(z_1) = \sum_{l=1}^{m} e_l^\alpha z_1^l.$$

Then we will have

$$\begin{aligned}\sum_{j=1}^{n-1} \zeta_j \frac{\partial \tilde{Q}_1}{\partial z_j}(t'\zeta) = &\sum_{k=2}^{2m} k d_k t^{k-1}\zeta_1^k + \sum_{\alpha=2}^{n-1}\sum_{k=2}^{m}(k+1) d_{\alpha k} t^k \zeta_1^k \zeta_\alpha \\ &+ \sum_{\alpha=2}^{n-1}\sum_{k=1}^{m}\sum_{l=1}^{m}(l+k) d_{\alpha,k} e_l^\alpha t^{l+k-1}\zeta_1^{l+k} \\ &+ \sum_{\alpha=2}^{n-1} 2c_\alpha^2 t \zeta_\alpha^2 + 2\sum_{\alpha=2}^{n-1}\sum_{l=1}^{m} c_\alpha e_l^\alpha t^l (l+1)\zeta_1^l \zeta_\alpha \\ &+ \sum_{\alpha=2}^{n-1} 2\Big(\sum_{l=1}^{m} e_l^\alpha t^l \zeta_1^l\Big)\Big(\sum_{l=1}^{m} e_l^\alpha t^{l-1} l \zeta_1^l\Big)\end{aligned}$$

Now using the estimates on the various coefficients from Lemma (3.4) of [56] – for instance again, the coefficient $(l+k)d_{\alpha,k}e_\alpha^l t^{l+k-1}$ occurring in the

(1) There is no typo here; we wish to focus only on Q_1 rather than $\tilde{Q}_1$.

second summand in the above, which we shall denote by $\mathbf{c}_{\alpha,l,k,t}$, can be estimated as

$$
\begin{aligned}
|\mathbf{c}_{\alpha,l,k,t}| &\lesssim (l+k)\big\{\delta(A)\tau_1\big(A,\delta(A)\big)^{-k}\tau_\alpha\big(A,\delta(A)\big)^{-1}\big\} \\
&\qquad\qquad \big\{\delta(A)\tau_1\big(A,\delta(A)\big)^{-l}\tau_\alpha\big(A,\delta(A)\big)^{-1}\big\}t^{l+k-1} \\
&\lesssim (\delta(A))^2\tau_1\big(A,\delta(A)\big)^{-(k+l)}\tau_\alpha\big(A,\delta(A)\big)^{-2} \\
&= \delta(A)\tau_1\big(A,\delta(A)\big)^{-(k+l)}
\end{aligned}
$$

since $\tau_\alpha\big(A,\delta(A)\big) = \sqrt{\delta(A)}$ and $t < 1$. Consequently, the corresponding monomial can be estimated as

$$
\begin{aligned}
|\mathbf{c}_{\alpha,l,k,t}\zeta_1^{l+k}| = |\mathbf{c}_{\alpha,l,k,t}||\Phi^A(B)_1|^{l+k} &\lesssim \delta(A)\tau_1\big(A,\delta(A)\big)^{-(k+l)}\tau_1\big(A,2M\delta_D(A)\big)^{l+k} \\
&\lesssim \delta(A).
\end{aligned}
$$

Subsequent similar computations yield

$$
\Big|\sum_{j=1}^{n-1}\zeta_j\frac{\partial Q_1}{\partial z_j}(t'\zeta)\Big| \lesssim \delta(A),
$$

where the constants hidden in the $\lesssim$-inequalities are all independent of our chosen pair of points A, B. Thereafter a likewise treatment for the remaining terms in $\sum_{j=1}^{n-1}\zeta_j\partial\tilde{Q}_1/\partial z_j(t'\zeta)$, result finally in the same upper bound as well for the numerator of the first term in (2.73), which means that the first term in (2.73) above is bounded above up to a constant, by

$$
\frac{|\Phi^A(B)_n|}{\delta(A)} + \text{some constant}.
$$

Next, to say a few words about the second term at (2.73), we first observe that it is bounded above by

$$
|H_A|\Big(\frac{|\tilde{\zeta}|}{\sqrt{\delta(A)}} + \frac{\big|\zeta_1\big(\frac{\partial Q_2}{\partial z_1}(t\zeta_1)\big)\big|}{\sqrt{\delta(A)}}\Big)
$$

and then note that

$$\begin{aligned}
\Big|\zeta_1 \frac{\partial Q_2}{\partial z_1}(t\zeta_1)\Big| &= \Big|\sum_{k=1}^{m} k b_k^{\alpha} t^{k-1}\zeta_1^k\Big| \\
&\leqslant \sum_{k=1}^{m} k|b_k^{\alpha}||\Phi^A(B)_1|^k \\
&\lesssim \sum_{k=1}^{m} \delta(A)\tau_1\big(A,\delta(A)\big)^{-k}\tau_\alpha(A,\delta(A))^{-1}\tau_1\big(A,2M\delta(A)\big)^k, \\
&\qquad \text{(by Lemma 3.4 of [56])} \\
&\lesssim \sqrt{\delta(A)}\,.
\end{aligned}$$

Thus the upshot is that the second summand in (2.73) is bounded above, up to a multiplicative constant, by

$$\sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}} + \text{ some constant.}$$

In all (2.73) transforms to the more concrete upper bound

$$K_D\big(c(t),\dot c(t)\big) \lesssim \frac{|\Phi^A(B)_n|}{\delta(A)} + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}} + C$$

for some positive constant C independent of A, B. We have thus reached the explicit upper bound sought for at (2.56). Thus (2.54) now becomes

$$\begin{aligned}
d_D^k(A,B) \lesssim{}& \frac{|\Phi^A(B)_n|}{\delta(A)} + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}} + C \\
&+ C' \log\Big(1 + \tilde C \frac{d'(B,A)}{\delta(B)}\Big) \\
\lesssim{}& \frac{|\Phi^A(B)_n|}{\delta(A)} + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}} \\
&+ \log\Big(1 + \frac{d'(B,A)}{\delta(B)}\Big) + L,
\end{aligned}$$

for some positive constant L depending on the domain D but free of any particular dependence on A, B. We may rewrite this now as

$$d_D^k(A,B) \lesssim \log\left(1 + \frac{|\Phi^A(B)_n|}{\delta(A)} + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}}\right) + \log\left(1 + \frac{d'(B,A)}{\delta(B)}\right) + L \tag{2.75}$$

Indeed, notice that by the hypothesis of lemma 2.14 we have: $B \in Q_{2M\delta(A)}(A)$, using which we get that

$$\frac{|\Phi^A(B)_n|}{\delta(A)} + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}}$$

is uniformly bounded above; we then get (2.75) by simply using the fact that the function $(\log(1+x))/x$ is bounded below by a positive constant when x varies over a compact interval of positive reals. Next, note that A, B are close enough for $d'(A,B) \approx d(A,B)$ to hold. Since $|\Phi^B(A)_n| \lesssim d(A,B)$, we may write the last inequality as

$$d_D^k(A,B) \lesssim \log\left(1 + \frac{d(A,B)}{\delta(A)} + \frac{|\Phi^A(B)_1|}{\tau\big(A,\delta(A)\big)} + \sum_{\alpha=2}^{n-1} \frac{|\Phi^A(B)_\alpha|}{\sqrt{\delta(A)}}\right) + \log\left(1 + \frac{d(B,A)}{\delta(B)}\right) + L.$$

Now this is the bound when $\delta(A) \geqslant \delta(B)$, an assumption made just after (2.53). Notice that the second logarithmic term here involves $\delta(B)$, whereas the preceding term involves $\delta(A)$ and Φ^A. So to drop the condition 'if $\delta(A) \geqslant \delta(B)$', it is not artificial to restate it in a more symmetric fashion, as in the statement of the lemma (2.14), to complete its proof herewith. □

Finally, the case complementary to the one dealt by lemma 2.14 namely, $d(A,B) \geqslant M \max\{\delta(A), \delta(B)\}$, can be reduced essentially to the first, as in section 7 of [31] following the line of arguments therein and the foregoing estimate may be obtained in that case as well.

3. Fridman's invariant function on Levi corank one domains

The purpose of this section is to prove Theorem 1.2. But before that, we gather some interesting properties of Fridman's invariant function $h_D(\cdot, \mathbb{B}^n)$ that were proved in [21].

Proposition 3.1. — *Let Ω be a Kobayashi hyperbolic manifold of complex dimension n. Then*

- *if there is a $p^0 \in \Omega$ such that $h_\Omega(p^0, \mathbb{B}^n) = 0$, then $h_\Omega(\cdot, \mathbb{B}^n) \equiv 0$ and Ω is biholomorphically equivalent to $\mathbb{B}^n$.*
- *$p \mapsto h_\Omega(p, \mathbb{B}^n)$ is continuous on Ω.*

To put things in perspective, we state the following result on the boundary behaviour of Fridman's invariant for strongly pseudoconvex domains.

Theorem 3.2. — *Let $D \subset \mathbb{B}^n$ be a bounded domain, $p^0 \in \partial D$ and let $\{p^j\} \subset D$ be a sequence that converges to p^0. If D is C^2-smooth strongly pseudoconvex equipped with the Kobayashi metric, then $h_D(p^j, \mathbb{B}^n) \to 0$ as $j \to \infty$.*

The reader is referred to [44] for a proof. It should be noted that, for D a Levi corank one domain, as in Theorem 1.2, the limit $h_{D_\infty}\big(('0,-1), \mathbb{B}^n\big)$ can be strictly positive, unlike the strongly pseudoconvex case, and, in general, depends on the nature of approach $p^j \to p^0 \in \partial D$. Recall that, here, and in the sequel, for any $z \in \mathbb{C}^n$, $z = ('z, z_n)$ and $'z$ will denote $(z_1, \ldots, z_{n-1})$.

Before going further, let us briefly recall the scaling technique (cf. [56]) for a smoothly bounded pseudoconvex domain $D \subset \mathbb{C}^n$ of finite type when the Levi form of ∂D has rank at least $(n-2)$ at $p^0 \in \partial D$. Assume that D is given by a smooth defining function r, p^0 is the origin and that $\frac{\partial r}{\partial \bar z}(p^0) = ('0, 1)$. Consider a sequence $p^j \in D$ that converges to the origin and denote by ζ^j, the point on ∂D chosen so that $\zeta^j = p^j + ('0, \epsilon_j)$ for some $\epsilon_j > 0$. Also, $\epsilon_j \approx \delta_D(p^j)$.

Let Φ^{ζ^j} be the polynomial automorphisms of $\mathbb{C}^n$ corresponding to $\zeta^j \in \partial D$ as described in (1.4). It can be checked from the explicit form of Φ^{ζ^j} that $\Phi^{\zeta^j}(\zeta^j) = ('0, 0)$ and

$$\Phi^{\zeta^j}(p^j) = \big('0, -\epsilon_j / d_0(\zeta^j)\big),$$

where $d_0(\zeta^j) = \Big(\partial r / \partial \overline{z}_n(\zeta^j)\Big)^{-1} \to 1$ as $j \to \infty$. Define a dilation of coordinates by

$$\Delta^{\epsilon_j}_{\zeta^j}(z_1, z_2, \ldots, z_n) = \Big(\frac{z_1}{\tau(\zeta^j, \epsilon_j)}, \frac{z_2}{\epsilon_j^{1/2}}, \ldots, \frac{z_{n-1}}{\epsilon_j^{1/2}}, \frac{z_n}{\epsilon_j}\Big),$$

where $\tau(\zeta^j, \epsilon_j)$ are as defined in (1.5). Note that $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(p^j) = \big('0, -1/d_0(\zeta^j)\big)$. For brevity, we write $\big('0, -1/d_0(\zeta^j)\big) = z^j$ and $('0, -1) = z^0$. It was shown in [56] that the scaled domains $D^j = \Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}(D)$ converge in the Hausdorff sense to

$$D_\infty = \left\{ z \in \mathbb{C}^n : 2\Re z_n + P_{2m}(z_1, \overline{z}_1) + |z_2|^2 + \ldots + |z_{n-1}|^2 < 0 \right\}$$

where $P_{2m}(z_1, \overline{z}_1)$ is a subharmonic polynomial of degree at most $2m$ ($m \geqslant 1$) without harmonic terms, $2m$ being the 1-type of ∂D at p^0. Recall that the polynomial $P_{2m}(z_1, \overline{z}_1)$ depends on how the sequence p^j approaches p^0 and hence, the domain D_∞ is canonically linked to the given domain D and the sequence $p^j \to p^0$. More can be said about the polynomial $P_{2m}(z_1, \overline{z}_1)$ if p^j approaches p^0 along the inner normal to ∂D at p^0. Furthermore, observe that D_∞ is complete hyperbolic (each point on ∂D_∞, including the point at infinity, is a local holomorphic peak point – cf. Lemma 1 of [8]) and hence D_∞ is taut.

It is natural to investigate the stability of the Kobayashi metric at the infinitesimal level first. The following lemma can be proved using the same ideas as in Lemma 5.2 of [44]. The only requirement is to establish the normality of a scaled family of holomorphic mappings which follows from Theorem 3.11 of [56].

Lemma 3.3. — *For $(z, v) \in D_\infty \times \mathbb{C}^n, \lim_{j\to\infty} K_{D^j}(z, v) = K_{D_\infty}(z, v)$. Moreover, the convergence is uniform on compact sets of $D_\infty \times \mathbb{C}^n$.*

Proof of Theorem 1.2. — There are two cases to be examined:

(i) $\liminf_{j\to\infty} h_D(p^j, \mathbb{B}^n) = 0$, or

(ii) $\liminf_{j\to\infty} h_D(p^j, \mathbb{B}^n) > c$ for some positive constant c.

In the first case, arguments similar to the ones employed in Theorem 5.1(i) of [44] together with Lemma 3.3 show that the limit domain D_∞ is biholomorphic to $\mathbb{B}^n$. On the other hand, this will not be true in the second case.

To analyse case (ii), it will be useful to consider the stability of the Kobayashi balls on the scaled domains D^j around $z^j \in D^j$ with a fixed radius $R > 0$. The proof of this is accomplished in two steps. In the first part, we show that the sets $B_{D^j}(z^j, R)$ do not accumulate at the point at infinity in ∂D_∞. The proof of this statement relies on Theorem 1.1 and unravelling the definition of the 'normalizing maps' Φ^{ζ^j} and the dilations $\Delta_{\zeta^j}^{\epsilon_j}$. The second part is to show that the sets $B_{D^j}(z^j, R)$ do not cluster at any finite boundary point of ∂D_∞.

Lemma 3.4. — *For each $R > 0$ fixed, $B_{D^j}(z^j, R)$ is uniformly compactly contained in D_∞ for all j large.*

Proof. — Since the scaling maps $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}$ are biholomorphisms and therefore Kobayashi isometries, it immediately follows that

$$B_{D^j}(z^j, R) = \Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}\big(B_D(p^j, R)\big).$$

We assert that $B_{D^j}(z^j, R)$ cannot accumulate at the point of infinity in ∂D_∞. To establish this, assume that $q \in B_D(p^j, R)$ and consider the lower bound on the Kobayashi distance given by Proposition 2.12:

$$C_* \log\left(1 + \left(\frac{d(p^j, q)}{\delta_D(p^j)}\right)^2\right) \leqslant d^k_D(p^j, q) \leqslant R$$

which yields that

$$d(p^j, q) \leqslant \left(\exp(R/C_*) - 1\right)^{1/2} \delta_D(p^j) < \exp(R/C_*)\delta_D(p^j)$$

where C_* is a positive constant (uniform in j). Then the definition of the pseudodistance d quickly leads to the following two possibilities:

- either $|p^j - q|_{l^\infty} < \exp(R/C_*)\delta_D(p^j)$ or
- for each j, there exists $\delta_j \in \big(0, \exp(R/C_*)\delta_D(p^j)\big)$ such that $p^j \in Q(q, \delta_j)$.

Now, Proposition 3.5 of [56] tells us that if $p^j \in Q(q, \delta_j)$ then $q \in Q(p^j, C\delta_j)$ for some uniform constant $C > 0$. Hence, the second statement above can be rephrased in the following manner – for every j there exists $\delta_j \in \big(0, \exp(R/C_*)\delta_D(p^j)\big)$ such that $q \in Q(p^j, C\delta_j)$ or equivalently that

$$q \in \big(\Phi^{p^j}\big)^{-1}\Big(\Delta(0, \tau(p^j, C\delta_j)) \times \Delta(0, \sqrt{C\delta_j}) \times \ldots \times \Delta(0, \sqrt{C\delta_j})) \times \Delta(0, C\delta_j)\Big)$$

In other words, each Kobayashi ball $B_D(p^j, R)$ is contained in the union $G^j_1 \cup G^j_2$ where

$$G^j_1 = \big\{z \in \mathbb{C}^n : |z - p^j|_{l^\infty} < \exp(R/C_*)\delta_D(p^j)\big\} \text{ and}$$
$$G^j_2 = \big(\Phi^{p^j}\big)^{-1}\Big(\Delta(0, \tau(p^j, C\delta_j)) \times \Delta(0, \sqrt{C\delta_j}) \times \ldots \times \Delta(0, \sqrt{C\delta_j}) \times \Delta(0, C\delta_j)\Big).$$

The idea is to verify that the sets $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(G^j_1)$ and $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(G^j_2)$ are uniformly bounded. For this, consider

$$\begin{aligned}\Phi^{\zeta^j}(G^j_1) &= \Phi^{\zeta^j}\Big(\big\{z \in \mathbb{C}^n : |z - p^j|_{l^\infty} < \exp(R/C_*)\delta_D(p^j)\big\}\Big)\\ &= \big\{w \in \mathbb{C}^n : |\big(\Phi^{\zeta^j}\big)^{-1}(w) - p^j|_{l^\infty} < \exp(R/C_*)\delta_D(p^j)\big\}.\end{aligned}$$

Now, write

$$w - \Phi^{\zeta^j}(p^j) = \Phi^{\zeta^j}\Big(\big(\Phi^{\zeta^j}\big)^{-1}(w)\Big) - \Phi^{\zeta^j}(p^j)$$

and note that the derivatives $\{D\Phi^{\zeta^j}\}$ are uniformly bounded in the operator norm by L, say. Therefore, for $w \in \Phi^{\zeta^j}(G_1^j)$, we have that

$$\big|w - \Phi^{\zeta^j}(p^j)\big|_{l^\infty} \leqslant L\big|\big(\Phi^{\zeta^j}\big)^{-1}(w) - p^j\big|_{l^\infty} < L\exp(R/C_*)\delta_D(p^j),$$

and consequently that,

$$\Phi^{\zeta^j}(G_1^j) \subset \big\{w \in \mathbb{C}^n : \big|w - \Phi^{\zeta^j}(p^j)\big|_{l^\infty} < L\exp(R/C_*)\delta_D(p^j)\big\}.$$

Since $\Phi^{\zeta^j}(p^j) = \big('0, -\epsilon_j/d_0(\zeta^j)\big)$, the above inclusion can be rewritten as

$$\Phi^{\zeta^j}(G_1^j) \subset \Bigg\{w : |w_k| < L\exp(R/C_*)\delta_D(p^j) \\ \text{for } 1 \leqslant k \leqslant n-1, \left|w_n + \frac{\epsilon_j}{d_0(\zeta^j)}\right| < L\exp(R/C_*)\delta_D(p^j)\Bigg\}.$$

Hence

$$\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}(G_1^j) \subset \Bigg\{w : |w_1| < \frac{L\exp(R/C_*)\delta_D(p^j)}{\tau(\zeta^j, \epsilon_j)}, \\ |w_\alpha| < \frac{L\exp(R/C_*)\delta_D(p^j)}{{\epsilon_j}^{1/2}} \text{ for } 2 \leqslant \alpha \leqslant n-1, \\ \left|w_n + \frac{1}{d_0(\zeta^j)}\right| < \frac{L\exp(R/C_*)\delta_D(p^j)}{\epsilon_j}\Bigg\}. \tag{3.1}$$

If $w = (w_1, \ldots, w_n)$ belongs to the set described by (3.1) above, then

$$\begin{cases} |w_1| < \frac{L\exp(R/C_*)\delta_D(p^j)}{\tau(\zeta^j,\epsilon_j)} \lesssim \frac{L\exp(R/C_*)\epsilon_j}{\tau(\zeta^j,\epsilon_j)}, \\ |w_\alpha| < \frac{L\exp(R/C_*)\delta_D(p^j)}{{\epsilon_j}^{1/2}} \lesssim L\exp(R/C_*){\epsilon_j}^{1/2} \text{ for } 2 \leqslant \alpha \leqslant n-1, \\ |w_n| < \frac{L\exp(R/C_*)\delta_D(p^j)}{\epsilon_j} + \frac{1}{d_0(\zeta^j)} \lesssim L\exp(R/C_*) + 1 \end{cases} \tag{3.2}$$

since $\epsilon_j \approx \delta_D(p^j)$ and $d_0(\zeta^j) \approx 1$. Furthermore, to examine the sets $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}(G_2^j)$, first note that these sets are the images of the unit polydisc in $\mathbb{C}^n$ under the maps $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j} \circ \big(\Phi^{p^j}\big)^{-1} \circ \big(\Delta_{p^j}^{C\delta_j}\big)^{-1}$. Let K be a positive constant such that $|\det D\big(\Phi^{\zeta^j} \circ \big(\Phi^{p^j}\big)^{-1}\big)| < K$ for all j large. It follows that each set $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}(G_2^j)$ is contained in a polydisc centered at $\big('0, -1/d_0(\zeta^j)\big)$, given by

$$\Delta\left(0, K^2\frac{\tau(p^j, C\delta_j)}{\tau(\zeta^j, \epsilon_j)}\right) \times \Delta\left(0, K^2\left(\frac{C\delta_j}{\epsilon_j}\right)^{1/2}\right) \times \ldots$$
$$\ldots \times \Delta\left(0, K^2\left(\frac{C\delta_j}{\epsilon_j}\right)^{1/2}\right) \times \Delta\left(-\frac{1}{d_0(\zeta^j)}, K^2\frac{C\delta_j}{\epsilon_j}\right). \quad (3.3)$$

Observe that, if w belongs to the polydisc as defined above, then

$$\begin{cases} |w_1| < K^2\frac{\tau(p^j, C\delta_j)}{\tau(\zeta^j, \epsilon_j)}, \\ |w_\alpha| < K^2\left(\frac{C\delta_j}{\epsilon_j}\right)^{1/2} \lesssim K^2\left(\frac{\delta_D(p^j)}{\epsilon_j}\right)^{1/2} \lesssim K^2 \text{ for } 2 \leqslant \alpha \leqslant n-1, \\ |w_n| < K^2\frac{C\delta_j}{\epsilon_j} + \frac{1}{d_0(\zeta^j)} \lesssim K^2\frac{\delta_D(p^j)}{\epsilon_j} + 1 \lesssim K^2 + 1. \end{cases} \quad (3.4)$$

It follows from [14] that $\epsilon_j^{1/2} \lesssim \tau(\zeta^j, \epsilon_j)$ and $\tau(p^j, C\delta_j) \lesssim \tau(p^j, \epsilon_j) \approx \tau(\zeta^j, \epsilon_j)$. As a consequence, we see that if w belongs to either of the sets (3.2) or (3.4), then $|w|$ is uniformly bounded. Hence, by virtue of the inclusions (3.1) and (3.3), it is immediate that the sets $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(G^j_1)$ and $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(G^j_2)$ are uniformly bounded. This in turn implies that the sets $B_{D^j}(z^j, R)$ cannot cluster at the point at infinity in ∂D_∞.

It remains to show that the sets $B_{D^j}(z^j, R)$ do not cluster at any finite point of ∂D_∞. Suppose that there is a sequence of points $q^j \in B_{D^j}(z^j, R)$ such that $q^j \to q^0 \in \partial D_\infty$ where q^0 is any finite boundary point. Recall Theorem 2.3 of [5] which provides a neighbourhood U of any given boundary point of the limit domain D_∞, such that on the portion of each scaled domain D^j intercepted by U, we have a uniform rate of blow up of the infinitesimal Kobayashi metric of D^j. More precisely here, there is a neighbourhood U of q^0 in $\mathbb{C}^n$, a positive constant C and $J \in \mathbb{N}$, such that

$$K_{D^j}(z, v) \geqslant C\Big(\frac{|\big(D^j\Phi^z(z)(v)\big)_1|}{\tau\big(z, \delta_{D^j}(z)\big)} + \sum_{\alpha=2}^{n-1}\frac{|\big(D^j\Phi^z(z)(v)\big)_\alpha|}{(\delta_{D^j}(z))^{1/2}} + \frac{|\big(D^j\Phi^z(z)(v)\big)_n|}{\delta_{D^j}(z)}\Big) \quad (3.5)$$

for all $z \in U \cap D^j$ with $j \geqslant J$ and tangent vector $v \in \mathbb{C}^n$. Here, the notation ${}^j\Phi^z(\cdot)$ is the special boundary chart (as described by (1.4)) corresponding to z, when z is viewed as a point in the scaled domain D^j. Evidently, the last component of $D\Phi^\zeta(z)(v)$ is given by

$$\big(D\Phi^\zeta(z)(v)\big)_n = \langle \nu(\zeta), v\rangle - \sum_{j=1}^n \frac{\partial Q_1}{\partial z_j}('z - '\zeta)v_j,$$

from which we get

$$\big(D^j\Phi^z(z)(v)\big)_n = \langle \nu(z), v\rangle.$$

Consider a neighbourhood $\tilde{U}$ of z^0 disjoint from U and which is compactly contained in D_∞. Then $z^j \in \tilde{U}$ for all j large. Let γ^j be any piecewise C^1-path connecting $z^j = \gamma^j(0)$ and $q^j = \gamma^j(1)$. Let t_j be the last of the timings of entry of γ^j into U. That is, $t_j \in (0,1)$ is such that the sub-curve of γ^j defined by

$$\sigma^j(t) = \gamma^j(t) \ \text{ for } t \in (t_j, 1]$$

is contained entirely in $U \cap D^j$. We particularly note then that the uniform lower bound (3.5) holds with (z,v) replaced by $(\sigma^j(t), \dot{\sigma}^j(t))$ for all j large. Note also that σ^j is contained in an ϵ-neighbourhood of ∂D^j for some fixed uniform $\epsilon > 0$ and for all j large. It thus follows from (3.5) that

$$\begin{aligned}\int_0^1 K_{D^j}\big(\gamma^j(t), \dot{\gamma}^j(t)\big)dt &\geqslant \int_{t_j}^1 K_{D^j}\big(\sigma^j(t), \dot{\sigma}^j(t)\big)dt \\ &\gtrsim \int_{t_j}^1 \frac{\Big|\Big(D^j\Phi^{(\sigma^j(t))}\big(\sigma^j(t)\big)\big(\dot{\sigma}^j(t)\big)\Big)_n\Big|}{\delta_{D^j}\big(\sigma^j(t)\big)}dt \\ &= \int_{t_j}^1 \frac{\langle \nu\left(\sigma^j(t)\right), \dot{\sigma}^j(t)\rangle}{\delta_{D^j}\big(\sigma^j(t)\big)}dt.\end{aligned}$$

The fact that the last integrand is positive follows from the considerations as in the arguments following (2.37). To be more precise, we focus on a stretch of time where the curve σ^j has a non-zero component along the normal to ∂D^j at $\pi_j\big(\sigma^j(t)\big)$ (as explained in Section 2). It can be checked that this stretch of time can be taken to be a non-zero constant, uniform in j, since the domains D^j converge to D_∞. Furthermore, it can be checked that the last integrand is up to a factor of 2 at least

$$\frac{\Re\langle \nu\left(\sigma^j(t)\right), \dot{\sigma}^j(t)\rangle}{2\,\delta_{D^j}\big(\sigma^j(t)\big)} = \frac{1}{2\,\delta_{D^j}\big(\sigma^j(t)\big)}\,\frac{d}{dt}\delta_{D^j}\big(\sigma^j(t)\big) = \frac{1}{2}\frac{d}{dt}\log\delta_{D^j}\big(\sigma^j(t)\big)$$

so that

$$\int_0^1 K_{D^j}\big(\gamma^j(t), \dot{\gamma}^j(t)\big)dt \gtrsim \frac{1}{2}\log\frac{1}{\delta_{D^j}(q^j)} - C$$

for a uniform positive constant C. Finally, taking the infimum over all such γ^j, we see that

$$d^k_{D^j}(q^j, z^j) \gtrsim \frac{1}{2}\log\frac{1}{\delta_{D^j}(q^j)} - C.$$

Note that the left hand side here is bounded above by R while the right hand side becomes unbounded as $q^j \to q^0 \in \partial D_\infty$. This contradiction completes the proof of the lemma. □

Once we are able to control the behaviour of Kobayashi balls $B_{D^j}(z^j, R)$ as $j \to \infty$, we intend to use the following comparison estimate due to K.T. Kim and D. Ma ([37], [35]) to conclude the stability of the integrated Kobayashi distance under scaling.

Lemma 3.5. — *Let D be a Kobayashi hyperbolic domain in $\mathbf{C}^n$ with a subdomain $D' \subset D$. Let $p, q \in D'$, $d^k_D(p,q) = a$ and $b > a$. If D' satisfies the condition $B_D(q,b) \subset D'$, then the following two inequalities hold:*

$$d^k_{D'}(p,q) \leqslant \frac{1}{\tanh(b-a)} d^k_D(p,q),$$
$$K_{D'}(p,v) \leqslant \frac{1}{\tanh(b-a)} K_D(p,v).$$

This statement compares the Kobayashi distance on the subdomain D' against its ambient domain D. Recall that the estimate $d^k_D \leqslant d^k_{D'}$ is always true.

The proofs of Lemmas 5.7, 5.8 and 5.9 of [44] go through verbatim in our setting, thereby, yielding the following two propositions – which are stated here without proof.

Proposition 3.6. — $\lim_{j\to\infty} d^k_{D^j}(z^j, \cdot) = d^k_{D_\infty}(z^0, \cdot)$ *and* $\lim_{j\to\infty} d^k_{D^j}(z^0, \cdot) = d^k_{D_\infty}(z^0, \cdot)$*. Moreover, the convergence is uniform on compact sets of D_∞.*

Proposition 3.7. — *Fix $R > 0$, then the sequence of domains $B_{D^j}(z^j, R)$ converges in the Hausdorff sense to $B_{D_\infty}(z^0, R)$. Moreover, for any $\epsilon > 0$*

- $B_{D_\infty}(z^0, R) \subset B_{D^j}(z^j, R+\epsilon)$*, and*
- $B_{D^j}(z^j, R-\epsilon) \subset B_{D_\infty}(z^0, R)$

for all j large.

Proof of Theorem 1.2(ii). — By the biholomorphic invariance of the function h, it follows that $h_D(p^j, \mathbb{B}^n) = h_{D^j}(z^j, \mathbb{B}^n)$ and therefore, it suffices to show that $h_{D^j}(z^j, \mathbb{B}^n) \to h_{D_\infty}(z^0, \mathbb{B}^n)$. To verify this, let $1/R$ be a positive number that almost realizes $h_{D_\infty}(z^0, \mathbb{B}^n)$, i.e., $1/R < h_{D_\infty}(z^0, \mathbb{B}^n) + \epsilon$ for some $\epsilon > 0$ fixed. Then there exists a biholomorphic imbedding $F : \mathbb{B}^n \to D_\infty$ satisfying $F(0) = z^0$ and $B_{D_\infty}(z^0, R) \subset F(\mathbb{B}^n)$. Pick $\delta > 0$ such that $B_{D_\infty}(z^0, R-\epsilon) \subset F(B(0,1-\delta))$. Since $F(B(0,1-\delta))$ is relatively compact in D_∞ and $D^j \to D_\infty$, it follows that $F(B(0,1-\delta))$ is compactly contained in D^j for all large j. Now, by Proposition 3.7, we see that

$$B_{D^j}(z^j, R-2\epsilon) \subset B_{D_\infty}(z^0, R-\epsilon),$$

and consequently that

$$B_{D^j}(z^j, R-2\epsilon) \subset F(B(0,1-\delta)) \subset D^j,$$

which, in turn, implies that

$$h_{D^j}(z^j, \mathbb{B}^n) \leqslant \frac{1}{R-2\epsilon}$$

for all j large. Therefore, by the choice of R, it follows that

$$\limsup_{j\to\infty} h_{D^j}(z^j, \mathbb{B}^n) \leqslant h_{D_\infty}(z^0, \mathbb{B}^n). \tag{3.6}$$

The goal now is to show that

$$h_{D_\infty}(z^0, \mathbb{B}^n) \leqslant \liminf_{j\to\infty} h_{D^j}(z^j, \mathbb{B}^n).$$

Firstly, observe that $\liminf_{j\to\infty} h_{D^j}(z^j, \mathbb{B}^n)$ is finite (cf. inequality (3.6) above) and there is a subsequence $h_{D^{j_k}}(z^{j_k}, \mathbb{B}^n)$ of $h_{D^j}(z^j, \mathbb{B}^n)$ that converges to $\liminf_{j\to\infty} h_{D^j}(z^j, \mathbb{B}^n)$. Let R_k be a sequence of positive numbers and $F^{j_k} : \mathbb{B}^n \to D^{j_k}$ be a sequence of biholomorphic imbeddings with the property that $F^{j_k}(0) = z^{j_k}$, $B_{D^{j_k}}(z^{j_k}, R_k) \subset F^{j_k}(\mathbb{B}^n)$ and $1/R_k \leqslant h_{D^{j_k}}(z^{j_k}, \mathbb{B}^n) + \epsilon$. Recall the scalings $\Delta^{\epsilon_{j_k}}_{\zeta^{j_k}} \circ \Phi^{\zeta^{j_k}}$ associated with the sequence p^{j_k} and consider the mappings

$$\theta^{j_k} := \left(\Delta^{\epsilon_{j_k}}_{\zeta^{j_k}} \circ \Phi^{\zeta^{j_k}}\right)^{-1} \circ F^{j_k} : \mathbb{B}^n \to D,$$

and note that $\theta^{j_k}(0) = p^{j_k} \to p^0 \in \partial D$ as $k \to \infty$. We claim that F^{j_k} admits a convergent subsequence. Indeed, applying Theorem 3.11 from [56] to θ^{j_k} assures us that the family $\Delta^{\epsilon_{j_k}}_{\zeta^{j_k}} \circ \Phi^{\zeta^{j_k}} \circ \theta^{j_k} = F^{j_k}$ is normal and the uniform limit F is a holomorphic mapping from $\mathbb{B}^n$ into D_∞. Moreover, $F(0) = \lim_{k\to\infty} F^{j_k}(0) = z^0$. Furthermore, by the first part of the proof and choice of the numbers R_k, it follows that

$$1/R_k \leqslant h_{D^{j_k}}(z^{j_k}, \mathbb{B}^n) + \epsilon \leqslant h_{D_\infty}(z^0, \mathbb{B}^n) + 2\epsilon$$

for all k large. On the other hand, the largest possible radii admissible in the definition of Fridman's invariant function $h_D(p^{j_k}, \mathbb{B}^n)$ (which equals

$h_{D^{j_k}}(z^{j_k}, \mathbb{B}^n))$ is at most $1/c$. Hence, we may assume that the sequence R_k converges to $R_0 > 0$.

We show that $F : \mathbb{B}^n \to D_\infty$ is an imbedding and that $F(\mathbb{B}^n)$ contains the Kobayashi ball $B_{D_\infty}(z^0, R_0 - 2\epsilon)$. To this end, it is straightforward to check that

$$B_{D^{j_k}}(z^{j_k}, R_0 - \epsilon) \subset B_{D^{j_k}}(z^{j_k}, R_k) \subset F^{j_k}(\mathbb{B}^n)$$

for all k large. But by Proposition 3.7,

$$B_{D_\infty}(z^0, R_0 - 2\epsilon) \subset B_{D^{j_k}}(z^{j_k}, R_0 - \epsilon)$$

and consequently, that $B_{D_\infty}(z^0, R_0 - 2\epsilon) \subset F^{j_k}(\mathbb{B}^n)$ for all large k. It follows that $B_{D_\infty}(z^0, R_0 - 2\epsilon) \subset F(\mathbb{B}^n)$, which, in turn, implies that F is non-constant.

To establish the injectivity of F, consider any point $a \in \mathbb{B}^n$. Each mapping $F^{j_k}(\cdot) - F^{j_k}(a)$ never vanishes in $\mathbb{B}^n \setminus \{a\}$ because of the injectivity of F^{j_k} in $\mathbb{B}^n$. Applying Hurwitz's theorem to the sequence $F^{j_k}(\cdot) - F^{j_k}(a) \in \mathcal{O}(\mathbb{B}^n \setminus \{a\}, \mathbb{C}^n)$, we have that $F(z) \neq F(a)$ for all $z \in \mathbb{B}^n \setminus \{a\}$. Since a is any arbitrary point of $\mathbb{B}^n$, this exactly means that F is injective.

To conclude, observe that the above analysis shows that $R_0 - 2\epsilon$ is a candidate for the infimum that defines $h_{D_\infty}(z^0, \mathbb{B}^n)$, and hence $h_{D_\infty}(z^0, \mathbb{B}^n) \leqslant 1/(R_0 - 2\epsilon)$. This last observation, in turn, implies that

$$h_{D_\infty}(z^0, \mathbb{B}^n) \leqslant \liminf_{j \to \infty} h_{D^j}(z^j, \mathbb{B}^n)$$

as desired.

4. A quantitative description of Kobayashi balls in terms of Euclidean parameters - Proof of Theorem 1.3

The fact that the topology induced by the Kobayashi distance and the pseudo-distance d' coincide (indeed, with the Euclidean topology) means that

$$Q\left(p, C_1(p, R)\delta_D(p)\right) \subset B_D(p, R) \subset Q\left(p, C_2(p, R)\delta_D(p)\right) \tag{4.1}$$

holds true for each $p \in D \cap U$ and for some positive constants $C_1(p, R)$, $C_2(p, R)$ (which depend on the point p, the radius R and the domain D). The purpose of Theorem 1.3 is to show that these constants can be chosen independent of the point p; such a uniform 'ball-box' estimate is expected to naturally follow from theorem 1.1. Indeed it does; but working it out rigorously is not trivial. The proof is based on the fact that the integrated Kobayashi distance is stable under scaling (cf. Propositions 3.6 and 3.7).

Proof of Theorem 1.3. — For $p \in D$, let us first prove that $Q\left(p, C_1 \delta_D(p)\right) \subset B_D(p, R)$ for some uniform constant C_1. Suppose that this is not the case. Then there are points $p^j \in D$, $p^0 \in \partial D$, $p^j \to p^0$ and a sequence of positive numbers $C_j \to 0$ with the property that for each j, the 'polydisc' $Q\left(p^j, C_j \delta_D(p^j)\right)$ is not entirely contained in the Kobayashi ball $B_D(p^j, R)$.

Applying a biholomorphic change of coordinates, if needed, we may assume that p^0 is the origin and the domain D near the origin is defined by

$$\Big\{ z \in \mathbb{C}^n : 2\Re z_n + \sum_{l=2}^{2m} P_l(z_1) + |z_2|^2 + \ldots + |z_{n-1}|^2 \\ + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leqslant m \\ j,k>0}} \Re\Big(\big(b_{jk}^{\alpha} w_1^j \overline{w}_1^k \big) w_\alpha \Big) + \text{ terms of higher weight} < 0 \Big\}$$

as in (1.3). Denote by ζ^j, the point on ∂D closest to p^j chosen such that $\zeta^j = p^j + ('0, \epsilon_j)$. Then $\epsilon_j \approx \delta_D(p^j)$ by construction. Furthermore, pick points $q^j \in Q\left(p^j, C_j \delta_D(p^j)\right)$ that lie on the boundary of the Kobayashi ball $B_D(p^j, R)$. The idea is to scale D with respect to the sequence $p^j \to p^0$, and analyse the sets $Q\left(p^j, C_j \delta_D(p^j)\right)$ under the scalings $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}$. Recall from the proof of Lemma 3.4 that the images $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}\left(Q\left(p^j, C_j \delta_D(p^j)\right)\right)$ are contained in polydiscs centered at $\left('0, -1/d_0(\zeta^j)\right)$, given by

$$\Delta\left(0, K^2 \frac{\tau\left(p^j, C_j \delta_D(p^j)\right)}{\tau(\zeta^j, \epsilon_j)}\right) \times \Delta\left(0, K^2 \left(\frac{C_j \delta_D(p^j)}{\epsilon_j}\right)^{1/2}\right) \times \ldots \\ \ldots \times \Delta\left(0, K^2 \left(\frac{C_j \delta_D(p^j)}{\epsilon_j}\right)^{1/2}\right) \times \Delta\left(\frac{-1}{d_0(\zeta^j)}, \frac{K^2 C_j \delta_D(p^j)}{\epsilon_j}\right), \tag{4.2}$$

where $K > 0$ is independent of j. Among other things, S. Cho in [12] proved that $\tau(\zeta^j, \epsilon_j) \approx \tau(p^j, \epsilon_j)$. But we know that $\delta_D(p^j) \approx \epsilon_j$ so that $\tau\left(p^j, C_j \delta_D(p^j)\right) \lesssim C_j \tau(\zeta^j, \epsilon_j)$. Also, $d_0(\zeta^j) \approx 1$ and $C_j \to 0$. These estimates show that the sets described in (4.2) are uniformly bounded, and consequently that, $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}\left(Q\left(p^j, C_j \delta_D(p^j)\right)\right)$ are also uniformly bounded. In particular, the sequence $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}(q^j)$ is bounded and since $C_j \to 0$, we have $\Delta_{\zeta^j}^{\epsilon_j} \circ \Phi^{\zeta^j}(q^j) \to ('0, -1)$.

Observe that $d^k_D(p^j,q^j) = R$ by construction. Since the scalings $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j} : D \to D^j$ are isometries in the Kobayashi metric on D and D^j, it follows that

$$d^k_{D^j}\left(\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(p^j), \Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j)\right) = d^k_D(p^j,q^j) = R$$

or, equivalently that

$$d^k_{D^j}\left(('0,-1/d_0(\zeta^j)), \Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j)\right) = R.$$

But we know that $\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j) \to ('0,-1)$. Hence, it follows from Proposition 3.6 that $d^k_{D_\infty}(('0,-1),('0,-1)) = R$ which is not possible. This contradiction validates that there is a constant C_1 (uniform in p) such that $Q\big(p, C_1\delta_D(p)\big)$ is contained in the Kobayashi ball $B_D(p,R)$.

To verify the inclusion $B_D(p,R) \subset Q(p, C_2\ \delta_D(p))$ for some uniform constant C_2, we suppose, on the contrary, that this does not hold true. Evidently, in view of (4.1), there are points $p^j \in D$, $p^0 \in \partial D$, $p^j \to p^0$ and a sequence of positive numbers $C_j \to +\infty$ such that – for each j, $Q\left(p^j, C_j\delta_D(p^j)\right)$ does not contain the Kobayashi ball $B_D(p^j,R)$. As before, pick points $\zeta^j \in \partial D$ closest to p^j and define ϵ_j, D^j, Φ^{ζ^j}, $\Delta^{\epsilon_j}_{\zeta^j}$ and D_∞ analogously. Furthermore, choose q^j in the complement of the closure of $Q\left(p^j, C_j\delta_D(p^j)\right)$ such that $q^j \in B_D(p^j,R)$, so that, as before,

$$d^k_{D^j}\left(\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(p^j), \Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j)\right) = d_D(p^j,q^j) < R,$$

and consequently, that

$$d^k_{D^j}\left(('0,-1/d_0(\zeta^j)), \Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j)\right) < R,$$

which implies that

$$\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j) \in B_{D^j}(z^j,R) \subset B_{D_\infty}(z^0, R+\epsilon)$$

for all j large. Here the last inclusion follows from Proposition 3.7 and as before, $('0,-1)$ and $('0,-1/d_0(\zeta^j))$ are written as z^0 and z^j respectively for brevity. The above observation can be restated as

$$d^k_{D_\infty}\left(\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j), z^0\right) < R+\epsilon. \tag{4.3}$$

However, we claim that

$$\left|\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j) - z^0\right| \to +\infty,$$

which violates (4.3). Therefore, the theorem is completely proven once the claim is established. To prove the claim, recall that $\Phi^{\zeta^j}(p^j) = \big('0, -\epsilon_j/d_0(\zeta^j)\big)$ and $d_0(\zeta^j) \approx 1$. Therefore, we see that $p^j \in Q(\zeta^j, \epsilon_j)$. Next, Proposition 3.5 of [56] quickly leads to the following statement: $Q(\zeta^j, \epsilon_j) \subset Q(p^j, C\epsilon_j)$ for some uniform positive constant C. Moreover, $\delta_D(p^j) \approx \epsilon_j$ so that $Q(\zeta^j, CC_j\delta_D(p^j)) \subset Q\left(p^j, C_j\delta_D(p^j)\right)$, where the constant C is independent of j. Hence, q^j lies in the complement of $Q(\zeta^j, CC_j\delta_D(p^j))$, by construction, and therefore, the first component

$$\left|\left(\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j)\right)_1\right| \geqslant \frac{\tau\big(\zeta^j, CC_j\delta_D(p^j)\big)}{\tau(\zeta^j, \epsilon_j)} \gtrsim C_j \to +\infty.$$

As a consequence, $\left|\Delta^{\epsilon_j}_{\zeta^j} \circ \Phi^{\zeta^j}(q^j) - ('0, -1)\right| \to +\infty$, and hence the claim.

5. Proof of Theorem 1.4

Suppose there exists a biholomorphism f from D_1 onto D_2 with the property that q^0 belongs to the cluster set of f at p^0. To begin with, we assert that f extends as a continuous mapping to p^0. This requires the fact that p^0 and q^0 are both *plurisubharmonic barrier points* (cf. [55], [17]). For the strongly pseudoconvex case, this is well known due to Fornaess and Sibony ([23]), and for smooth pseudoconvex finite type point q^0, the above statement was proved in [12], [55].

Assume that D_1 and D_2 are given by a smooth defining functions r_1 and r_2 respectively, both $p^0 = 0$ and $q^0 = 0$ and that $\frac{\partial r_1}{\partial \bar z}(p^0) = ('0, 1) = \frac{\partial r_2}{\partial \bar z}(q^0)$. Let q^j be a sequence of points in D_2 converging to q^0 along the inner normal to the origin, i.e., $q^j = ('0, -\delta_j)$, each $\delta_j > 0$ and $\delta_j \searrow 0$. Since $f : D_1 \to D_2$ is a biholomorphism and $0 \in cl_f(0)$, there exists a sequence $p^j \in D_1$ with $p^j \to 0$ such that $f(p^j) = q^j$. Now, scale D_1 with respect to $\{p^j\}$ and D_2 with respect to $\{q^j\}$.

To scale D_1, recall that by [47], for each ξ near $p^0 \in \partial D_1$, there is a unique automorphism h^ξ of $\mathbb{C}^n$ with $h^\xi(\xi) = 0$ such that the domain $h^\xi(D_1)$ is given by

$$\left\{z \in \mathbf{C}^n : 2\Re\big(z_n + K^\xi(z)\big) + H^\xi(z) + \alpha^\xi(z) < 0\right\}$$

where $K^\xi(z) = \sum_{i,j=1}^{n} a_{ij}(\xi) z_i z_j, H^\xi(z) = \sum_{i,j=1}^{n} b_{ij}(\xi) z_i \bar z_j$ and $\alpha^\xi(z) = o(|z|^2)$ with $K^\xi('z, 0) \equiv 0$ and $H^\xi('z, 0) \equiv |'z|^2$. The automorphisms h^ξ converge to the identity uniformly on compact subsets of $\mathbb{C}^n$ as $\xi \to p^0$. For $\xi =$

$(\xi_1, \xi_2, \ldots, \xi_n) \in D_1$ as above, consider the point $\tilde{\xi} = (\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n + \epsilon)$ where $\epsilon > 0$ is chosen to ensure that $\tilde{\xi} \in \partial D_1$. Then the actual form of h^{ξ} shows that $h^{\tilde{\xi}}(\xi) = ('0, -\epsilon)$.

In order to apply Pinchuk's scalings to the sequence $p^j \to p^0 \in \partial D_1$, choose $\xi^j \in \partial D_1$ such that if $p^j = ('p^j, p^j_n)$, then $\xi^j = ('p^j, p^j_n + \epsilon_j) \in \partial D_1$ for some $\epsilon_j > 0$. Then $\epsilon_j \approx \delta_{D_1}(p^j)$ by construction. Now, define the dilations

$$T^j(z_1, z_2, \ldots, z_n) = \left({\epsilon_j}^{-\frac{1}{2}} z_1, \ldots, {\epsilon_j}^{-\frac{1}{2}} z_{n-1}, {\epsilon_j}^{-1} z_n \right)$$

and the dilated domains $D_1^j = T^j \circ h^{\xi^j}(D_1)$. It was shown in [47] that D_1^j converge to

$$D_{1,\infty} = \left\{ z \in \mathbb{C}^n : 2\Re z_n + |z_1|^2 + |z_2|^2 + \ldots + |z_{n-1}|^2 < 0 \right\}$$

which is the unbounded realization of the unit ball in $\mathbb{C}^n$.

For clarity and completeness, we briefly describe the scalings for the domain D_2, which are simpler this time as the sequence q^j approaches q^0 normally. To start with, consider $\Delta^j : \mathbb{C}^n \to \mathbb{C}^n$, a sequence of dilations, defined by

$$\Delta^j(w_1, w_2, \ldots, w_{n-1}, w_n) = \left(\delta_j^{-\frac{1}{2m}} w_1, \delta_j^{-\frac{1}{2}} w_2, \ldots, \delta_j^{-\frac{1}{2}} w_{n-1}, \delta_j^{-1} w_n \right).$$

Note that $\Delta^j('0, -\delta_j) = ('0, -1)$ for all j and the domains $D_1^j = \Delta^j(D_1)$ converge in the Hausdorff sense to

$$D_{2,\infty} = \left\{ w \in \mathbb{C}^n : 2\Re w_n + Q_{2m}(w_1, \overline{w}_1) + |w_2|^2 + \ldots + |w_{n-1}|^2 < 0 \right\},$$

where Q_{2m} is the homogeneous polynomial of degree $2m$ that coincides with the polynomial of same degree in the homogeneous Taylor expansion of the defining function for ∂D_2 near the origin.

By the biholomorphic invariance of the function h, it follows that $h_{D_1}(p^j, \mathbb{B}^n) = h_{D_2}(q^j, \mathbb{B}^n)$. Then Theorem 1.2 assures us that the right hand side above converges to $h_{D_{2,\infty}}\big(('0, -1), \mathbb{B}^n\big)$. Furthermore, since ∂D_1 is strongly pseudoconvex near p^0, by Theorem 3.2, we see that the left hand side above $h_{D_1}(p^j, \mathbb{B}^n) \to 0$ as $j \to \infty$. It follows that $h_{D_{2,\infty}}\big(('0, -1), \mathbb{B}^n\big) = 0$. As a result, $D_{2,\infty}$ is biholomorphic to $\mathbb{B}^n$ by virtue of Proposition 3.1. Therefore, the problem has been quickly reduced to investigation for the special case of model domains, namely, $D_{2,\infty}$ and $\mathbb{B}^n$, which are algebraic.

By composing with a suitable Cayley transform, if necessary, we may assume that there is a biholomorphism F from $D_{1,\infty}$ (which is biholomorphic to $\mathbb{B}^n$) onto $D_{2,\infty}$ with the additional property that the cluster set of F^{-1} at some point $('0, \iota a) \in \partial D_{2,\infty}$ (for $a \in \mathbb{R}$) contains a finite point of $\partial D_{1,\infty}$. Then Theorem 2.1 of [16] tells us that F^{-1} extends holomorphically past the boundary to a neighbourhood of $('0, \iota a)$. It turns out that F^{-1} extends biholomorphically across some point $('0, \iota a^0) \in \partial D_{2,\infty}$. To prove this claim, it suffices to show that the Jacobian of F^{-1} does not vanish identically on the complex plane

$$L = \big\{('0, \iota a) : a \in \mathbb{R}\big\} \subset \partial D_{2,\infty}.$$

If the claim were false, then the Jacobian of F^{-1} vanishes on the entire w_n-axis, which intersects the domain $D_{2,\infty}$. However, F^{-1} is injective on $D_{2,\infty}$, and consequently, has nowhere vanishing Jacobian determinant on $D_{2,\infty}$. This contradiction proves the claim.

Furthermore, it is evident that the translations in the imaginary w_n-direction leave $D_{2,\infty}$ invariant. Therefore, we may assume that $('0, \iota a^0)$ is the origin and that F preserves the origin. Now recall that the Levi form is preserved under local biholomorphisms around a boundary point, thereby yielding the strong pseudoconvexity of $\partial D_{2,\infty}$. In particular, $Q_{2m}(w_1, \overline{w}_1) = |w_1|^2$ which gives the strong pseudoconvexity of $q^0 \in \partial D_2$. This contradicts the assumption that the Levi form of ∂D_2 has rank exactly $n-2$ at q^0. Hence the result.

6. Continuous extension of isometries – Proof of Theorem 1.5

The proof of Theorem 1.5 will be accomplished in several steps. The first step is to analyse the behaviour of the Kobayashi metric on a smoothly bounded pseudoconvex Levi corank one domain.

Proposition 6.1. — *Let D be a bounded domain in $\mathbb{C}^n$. Assume that ∂D is smooth pseudoconvex and of finite type near a point $p^0 \in \partial D$. Suppose further that the Levi form of ∂D has rank at least $n-2$ near p^0. Then for any $\epsilon > 0$, there exist positive numbers $r_2 < r_1 < \epsilon$, C and C' (where r_1, r_2, C and C' depend on A) such that*

$$d_D^k(A, B) \geqslant \frac{C'}{2} \log \frac{1}{\delta_D(B)} - C, \qquad A \in D \setminus B(p^0, r_1), B \in B(p^0, r_2) \cap D.$$

Proof. — By Theorem 3.10 of [56], there exists a neighbourhood U of p^0 in $\mathbb{C}^n$ such that

$$K_D(z, v) \approx \frac{(D\Phi^z(z)v)_1}{\tau\left(z, \delta_D(z)\right)} + \frac{(D\Phi^z(z)v)_2}{\left(\delta_D(z)\right)^{1/2}} + \cdots + \frac{(D\Phi^z(z)v)_{n-1}}{\left(\delta_D(z)\right)^{1/2}} + \frac{(D\Phi^z(z)v)_n}{\delta_D(z)} \tag{6.1}$$

for all $z \in U \cap D$ and v a tangent vector at z. The neighbourhood U is so chosen to avoid the point A. Let γ be a piecewise C^1-smooth curve in D joining A and B, i.e., $\gamma(0) = A$ and $\gamma(0) = B$. As we travel along γ starting from A, there is a last point α on the trace γ with $\alpha \in \partial U \cap D$. Let $\gamma(t_0) = \alpha$ and denote by σ, the subcurve of γ with end-points α and B. Observe that the trace σ is contained in a δ-neighbourhood of ∂D for some fixed $\delta > 0$. Here we choose $\delta > 0$ in such a way that the δ-neighbourhood of ∂D does not contain the point A. Evidently,

$$\int_0^1 K_D\left(\gamma(t), \dot{\gamma}(t)\right) dt \geqslant \int_{t_0}^1 K_D\left(\sigma(t), \dot{\sigma}(t)\right) dt.$$

From this point, proceeding exactly as in the proof of Lemma 3.4, it follows using (6.1) that

$$\int_0^1 K_D\left(\gamma(t), \dot{\gamma}(t)\right) dt \geqslant \frac{C'}{2} \log \frac{1}{\delta_D(B)} - C$$

for some positive constants C and C'. Now, taking the infimum over all such paths γ yields

$$d_D^k(A, B) \geqslant \frac{C'}{2} \log \frac{1}{\delta_D(B)} - C$$

as desired. □

Proposition 6.2. — *Let $f : D_1 \to D_2$ be a C^1-Kobayashi isometry between two bounded domains in $\mathbb{C}^n$. Let p^0 and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that ∂D_1 is C^∞-smooth pseudoconvex of finite type in a neighbourhood U of p^0 and the Levi form of ∂D_1 has rank at least $n-2$ near p^0. Suppose that ∂D_2 is C^∞-smooth pseudoconvex finite type in a neighbourhood V of q^0. Then there exist smaller neighbourhoods $U_1 \subset U$, $V_2 \subset V$ of p^0 and q^0 respectively with the following property: if z is an arbitrary point of $U_1 \cap D_1$ such that $f(z) \in V_2 \cap D_2$, then*

$$|df(z)v| \leqslant C \frac{|v|}{\left(\delta_{D_1}(z)\right)^\nu}$$

where $\nu \in (0,1)$ and $C > 0$ are constants independent of $z \in U_1 \cap D_1$.

Proof. — Since ∂D_2 is smooth pseudoconvex of finite type near $q^0 \in \partial D_2$, it follows from [12] that there exists a neighbourhood $V_2 \subset V$ of q^0 and constants $C_1 > 0$ and $\eta \in (0,1)$ such that

$$K_{D_2}(w, u) \geqslant C_1 \frac{|u|}{\left(\delta_{D_2}(w)\right)^\eta}$$

for all $w \in V_2 \cap D_2$ and tangent vectors u. Furthermore, since f is a C^1-Kobayashi isometry from D_1 onto D_2, it follows that

$$C_1 \frac{|df(z)v|}{(\delta_{D_2}(f(z)))^\eta} \leqslant K_{D_2}(f(z), df(z)v) = K_{D_1}(z,v) \leqslant \frac{|v|}{\delta_{D_1}(z)},$$

and hence,

$$|df(z)v| \lesssim \frac{(\delta_{D_2}(f(z)))^\eta \, |v|}{\delta_{D_1}(z)} \tag{6.2}$$

for $z \in D_1$ such that $f(z) \in V_2 \cap D_2$ and tangent vectors v.

Fix $A \in D_1$ and let $U_1 \subset U$ be a neighbourhood of p^0 such that

$$d^k_{D_1}(z, A) \geqslant \frac{C_2}{2} \log \frac{1}{\delta_{D_1}(z)} - C_3$$

for $z \in U_1 \cap D_1$ and some uniform positive constants C_2, C_3. This follows from Proposition 6.1 above. Without loss of generality, we may assume that the constant C_2 is at most $1 - \eta$. Moreover,

$$d^k_{D_1}(z, A) = d^k_{D_2}(f(z), f(A)) \leqslant \frac{1}{2} \log \frac{1}{\delta_{D_2}(f(z))} + C_4,$$

and hence

$$\delta_{D_2}(f(z)) \leqslant C_5 \left(\delta_{D_1}(z)\right)^{C_2} \tag{6.3}$$

for all z in $U_1 \cap D_1$ and uniform constants C_4, C_5.

Fix neighbourhoods $U_1 \subset U$, $V_2 \subset V$ of p^0 and q^0 respectively as above. If z is any point of $U_1 \cap D_1$ such that $f(z) \in V_2 \cap D_2$, then it follows from inequalities (6.2) and (6.3) that

$$|df(z)v| \lesssim \frac{|v|}{(\delta_{D_1}(z))^\nu}.$$

where $\nu = 1 - C_2 - \eta$. □

Proof of Theorem 1.5. — The proof involves two steps. The first step is to show that f extends to $D_1 \cup \{p^0\}$ as a continuous mapping. Once this claim is established, the second step is to verify that f is continuous on a neighbourhood of p^0 in $\overline{D}_1$.

By hypothesis, there exists a sequence $p^j \in D_1$ with $p^j \to p^0$ and $f(p^j) \to q^0$. Assume, on the contrary, that there exists a sequence s^j in D_1 with $s^j \to p^0 \in \partial D_1$ such that the sequence $f(s^j)$ does not converge to q^0. Consider polygonal paths γ^j in D_1 joining p^j and s^j defined as follows: for each j, choose p^{j0}, s^{j0} on ∂D_1 closest to p^j and s^j respectively. Set

$p^{j'} = p^j - |p^j - s^j|\nu(p^{j0})$ and $s^{j'} = s^j - |p^j - s^j|\nu(s^{j0})$ where $\nu(z)$ denotes the outward unit normal to ∂D_1 at $z \in \partial D_1$. Define $\gamma^j = \gamma_1^j \cup \gamma_2^j \cup \gamma_3^j$ as the union of three segments, where

- γ_1^j is the straight line path joining p^j and $p^{j'}$ along the inner normal to ∂D_1 at the point p^{j0},
- γ_2^j is the straight line joining $p^{j'}$ and $s^{j'}$, (in case, the straight line segment joining $p^{j'}$ and $s^{j'}$ does not lie entirely in D_1, take γ_2^j to be any curve in D_1 at a constant distance from ∂D_1 and joining $p^{j'}$ and $s^{j'}$),
- γ_3^j is the straight line segment joining $s^{j'}$ and s^j along the inward normal to ∂D_1 at the point s^{j0}.

Evidently, the composition $f \circ \gamma^j$ yields a continuous path in D_2 joining $f(p^j)$ and $f(s^j)$. Let $U_1 \subset U$ and $V_2 \subset V$ be neighbourhoods of p^0 and q^0 respectively as given by Proposition 6.2. Fix a smaller neighbourhood $V' \subset V_2$ of q^0 with V' relatively compact in V_2. For each j, pick points $u^j \in \partial V' \cap D_2$ on the trace$(f \circ \gamma^j)$. Let $t^j \in D_1$ be such that $f(t^j) = u^j$. Note that the points t^j lie on trace(γ^j) and hence $t^j \to p^0$ by construction. Moreover, $f(t^j) = u^j \to u^0 \in V_2 \cap \partial D_2$, where u^0 is different from q^0. Denote by σ^j, the sub-curve of γ^j with end-points p^j and t^j. Now, trace$(f \circ \sigma^j)$ is contained in $V_2 \cap D_2$, and hence, it follows from Proposition 6.2 that there are constants $\nu \in (0,1)$ and $C > 0$ such that

$$|df(z)v| \leqslant C \frac{|v|}{\left(\delta_{D_1}(z)\right)^{\nu}}$$

for all points z on trace(σ^j). Integrating along the path σ^j, we obtain that

$$\left|f(t^j) - f(p^j)\right| \lesssim \left|t^j - p^j\right|^{1-\nu},$$

a contradiction. Hence, f extends continuously to $D_1 \cup \{p^0\}$.

Now, we use the strong pseudoconvexity of ∂D_2 at q^0 and the continuity of f at p^0 to get neighbourhoods $U' \subset U$ and $V'' \subset V$ of p^0 and q^0 respectively, satisfying the following property: $V'' \cap \partial D_2$ is C^2-smooth strongly pseudoconvex and $f(U' \cap D_1) \subset V'' \cap D_2$. By the lower semi-continuity of rank, there is a neighbourhood $U'' \subset U'$ of p^0, U'' compactly contained in U' such that $U'' \cap \partial D_1$ is of Levi-rank at least $n-2$. Furthermore, we may assume $U'' \cap \partial D_1$ to be finite type and pseudoconvex. Let $a^0 \in U'' \cap \partial D_1$ and a^j be a sequence of points in D_1 with $a^j \to a^0$. The goal now is to show that f extends continuously to the point a^0. There are two cases to be considered. After passing to a subsequence, if needed,

(i) $f(a^j) \to b^0 \in V'' \cap \partial D_2$,

(ii) $f(a^j) \to b^1 \in V'' \cap D_2$ as $j \to \infty$.

We investigate case (ii) first. Fix $a' \in U' \cap D_1$ and observe that the quantity $d^k_{V \cap D_2}\big(f(a^j), f(a')\big)$ is uniformly bounded (say by R) because of the completeness of $V \cap D_2$. Therefore, for all j large

$$d^k_{D_1}(a^j, a') = d^k_{D_2}\big(f(a^j), f(a')\big) \leqslant d^k_{V \cap D_2}\big(f(a^j), f(a')\big) < R,$$

which implies that $a' \in B_{D_1}(a^j, R)$. This contradicts the completeness of D_1. Hence, the sequence $f(a^j) \to b^0 \in V'' \cap \partial D_2$ and consequently b^0 belongs to the cluster set of a^0 under f. From this point, proceeding exactly as in the first part of the proof yields that f is continuous at the point a^0. Since $a^0 \in U'' \cap \partial D_1$ was arbitrary, it follows that f extends as a continuous map on $U'' \cap \partial D_1$ and Theorem 1.5 is completely proven.

Next is the corrected version of one of the earlier works of the second author (Theorem 1.2 of [42]), which follows as a corollary of Theorem 1.5.

Theorem 6.3. — *Let $f : D_1 \to D_2$ be a C^1-Kobayashi isometry between two bounded domains in $\mathbf{C}^2$. Let p^0 and q^0 be points on ∂D_1 and ∂D_2 respectively. Assume that ∂D_1 is C^∞-smooth weakly pseudoconvex of finite type near p^0 and that ∂D_2 is C^2-smooth strongly pseudoconvex in a neighbourhood U_2 of q^0. Suppose that q^0 belongs to the cluster set of p^0 under f. Then f extends as a continuous mapping to a neighbourhood of p^0 in $\overline{D}_1$.*

7. Kobayashi metric of a complex ellipsoid in $\mathbb{C}^n$ – Proof of Theorem 1.6

Let D be a domain in $\mathbb{C}^n$, $z \in D$ and $v \in \mathbb{C}^n$ a tangent vector at the point z. Recall that a mapping $\phi \in \mathcal{O}(\Delta, D)$ is said to be a complex geodesic for (z, v) if $\phi(0) = z$ and $K_D(z, v)\phi'(0) = v$. Such mappings are also sometimes referred to as Kobayashi extremals in the literature. A complex ellipsoid is a domain of the form

$$E(2m_1, \ldots, 2m_n) = \Big\{z \in \mathbb{C}^n : |z_1|^{2m_1} + \ldots + |z_n|^{2m_n} < 1\Big\}$$

where $m_j > 0$ for each $j = 1, \ldots, n$. It is well known that complex ellipsoids are convex if and only if $m_j \geqslant 1/2$ for $j = 1, \ldots, n$. Moreover, they are taut domains (i.e., $\mathcal{O}(\Delta, E(2m_1, \ldots, 2m_n))$ is a normal family) and hence, there always exist complex geodesics through a given point $z \in E(2m_1, \ldots, 2m_n)$ and any tangent vector at the point z. The primary goal of this section is to describe the Kobayashi metric on the complex ellipsoids for the case $m_1 \geqslant$

$1/2$ and $m_2 = \cdots = m_n = 1$ – notice that these are exactly the domains E_{2m} introduced in (1.1). To understand the Kobayashi metric on E_{2m}, we use the characterisation of all complex geodesics $\phi : \Delta \to E(2m_1, \ldots, 2m_n)$, each $m_j \geqslant 1/2$, due to Jarnicki, Pflug and Zeinstra ([34]). Observe that it suffices to consider only those complex geodesics $\phi = (\phi_1, \ldots, \phi_n) : \Delta \to \mathbb{C}^n$ for which

$$\phi_j \text{ is not identically zero for any } j = 1, \ldots, n. \tag{7.1}$$

After a suitable permutation of variables, we may assume that for some $0 \leqslant s \leqslant n$,

$$\begin{cases} 0 \notin \phi_j(\Delta) \text{ for } j = 1, \ldots, s \text{ and} \\ 0 \in \phi_j(\Delta) \text{ for } j = s+1, \ldots, n. \end{cases} \tag{7.2}$$

The main result of ([34]) that is needed is:

Theorem 7.1. — *A non-constant mapping $\phi = (\phi_1, \ldots, \phi_n) : \Delta \to \mathbb{C}^n$ with (7.1) and (7.2) is a complex geodesic in $E(2m_1, \ldots, 2m_n)$ if and only if ϕ is of the form*

$$\phi_j(\lambda) = \begin{cases} a_j \left(\frac{1-\bar{\alpha}_j\lambda}{1-\bar{\alpha}_0\lambda}\right)^{1/m_j} & \textit{for} \quad j = 1, \ldots, s, \\ a_j \left(\frac{\lambda-\alpha_j}{1-\bar{\alpha}_j\lambda}\right)\left(\frac{1-\bar{\alpha}_j\lambda}{1-\bar{\alpha}_0\lambda}\right)^{1/m_j} & \textit{for} \quad j = s+1, \ldots, n, \end{cases}$$

where

$$a_j \in \mathbb{C} \setminus \{0\} \textit{ for } j = 1, \ldots, n,$$
$$\alpha_1, \ldots, \alpha_s \in \bar{\Delta}, \alpha_0, \alpha_{s+1}, \ldots, \alpha_n \in \Delta,$$
$$\alpha_0 = \sum_{j=1}^{n} |a_j|^{2m_j} \alpha_j,$$
$$1 + |\alpha_0|^2 = \sum_{j=1}^{n} |a_j|^{2m_j}(1 + |\alpha_j|^2),$$
$$\textit{the case } s = 0, \alpha_0 = \alpha_1 = \cdots = \alpha_n \textit{ is excluded and,}$$
$$\textit{the branches of powers are such that } 1^{1/m_j} = 1, j = 1, \ldots, n.$$

Proof of Theorem 1.6. — We proceed by induction on the index n. First, note that the case $n = 2$ is Theorem 2 of [10]. Next, assume that the result holds for all integers between 2 and $n - 1$. To prove the inductive step, fix $(p, 0, \ldots, 0) \in E_{2m}$ and $(v_1, \ldots, v_n) \in \mathbb{C}^n$. The main objective is to find an effective formula for $\tau = K_{E_{2m}}\left((p,0,\dots,0),(v_1,\dots,v_n)\right)$. It is well known (see, for example, Proposition 2.2.1 in [33]) that if $p=0$, then

$$\tau = K_{E_{2m}}\left((0,0,\dots,0),(v_1,\dots,v_n)\right) = q_{E_{2m}}(v_1,\dots,v_n) \tag{7.3}$$

where $q_{E_{2m}}$ denotes the *Minkowski functional* of E_{2m}. Furthermore, $q_{E_{2m}}(v_1,\dots,v_n)$ is the only positive solution of the equation

$$\frac{|v_1|^{2m}}{\left(q_{E_{2m}}(v_1,\dots,v_n)\right)^{2m}} + \frac{|v_2|^2}{\left(q_{E_{2m}}(v_1,\dots,v_n)\right)^2} + \cdots + \frac{|v_n|^2}{\left(q_{E_{2m}}(v_1,\dots,v_n)\right)^2} = 1. \tag{7.4}$$

For $0<p<1$, if $\hat{v} = (v_2,\dots,v_n) = \hat{0}$, then

$$\tau = K_{E_{2m}}\left((p,0,\dots,0),(v_1,0,\dots,0)\right) = K_\Delta(p,v_1) = \frac{|v_1|}{1-p^2}.$$

Hence, we may assume that $\hat{v} \neq \hat{0}$ in the sequel.

Let $\phi : \Delta \to E_{2m}$ be a complex geodesic with $\phi(0) = (p,\hat{0})$ and $\tau\phi'(0) = (v_1,\dots,v_n)$. Evidently, ϕ_1 is not identically zero and $\phi_2,\dots,\phi_n$ cannot be identically zero simultaneously. Suppose that $\phi_4 = \cdots = \phi_n \equiv 0$, then the mapping $\tilde{\phi} = (\phi_1,\phi_2,\phi_3) : \Delta \to E(2m,2,2) \subset \mathbb{C}^3$ is a complex geodesic through the point $(p,0,0)$ in $E(2m,2,2)$ and consequently,

$$\tau = K_{E_{2m}}\left((p,0,\dots,0),(v_1,\dots,v_n)\right) = K_{E(2m,2,2)}\left((p,0,0),(v_1,v_2,v_3)\right).$$

The right hand side above is known explicitly by induction hypothesis, thereby, yielding an explicit formula for τ. The above analysis shows that we may assume that ϕ_j is not identically zero for any $j=1,\dots,n$. This assumption does not restrict the generality, since mappings with zero-components are exactly *lower dimensional* complex geodesics, as observed above. Then, ϕ satisfies (7.1) and (7.2) (with $s=0$ or $s=1$).

Consider the case $s=1$ first. Applying Theorem 7.1 gives that

$$\phi_1(\lambda) = a_1\left(\frac{1-\bar{\alpha}_1\lambda}{1-\bar{\alpha}_0\lambda}\right)^{1/m},$$
$$\phi_j(\lambda) = a_j\left(\frac{\lambda-\alpha_j}{1-\bar{\alpha}_j\lambda}\right)\left(\frac{1-\bar{\alpha}_j\lambda}{1-\bar{\alpha}_0\lambda}\right) \text{ for } j=2,\dots,n,$$

where a_j, α_j are as stated in Theorem 7.1. It follows that

$$\phi(0) = (a_1, -a_2\alpha_2, \dots, -a_n\alpha_n) \text{ and}$$
$$\phi_1'(0) = a_1(\bar{\alpha}_0 - \bar{\alpha}_1)/m,$$
$$\phi_j'(0) = a_j\left((1-|\alpha_j|^2) - \alpha_j(\bar{\alpha}_0 - \bar{\alpha}_j)\right) \text{ for } j=2,\dots,n.$$

But we know that $\phi(0) = (p, \hat{0})$ and $\tau\phi'(0) = (v_1, \ldots, v_n)$. Therefore,

$$a_1 = p, \ \alpha_2 = \cdots = \alpha_n = 0, \ \text{and}$$
$$\tau p(\bar{\alpha}_0 - \bar{\alpha}_1)/m = v_1, \tau a_2 = v_2, \ldots, \tau a_n = v_n.$$

These conditions, in turn, imply that

$$\alpha_0 = p^{2m}\alpha_1, \ \text{and}$$
$$1 + |\alpha_0|^2 = p^{2m}\left(1 + |\alpha_1|^2\right) + |a_2|^2 + \ldots + |a_n|^2,$$

and consequently,

$$\tau p\bar{\alpha}_1(p^{2m} - 1)/m = v_1, \ \text{and} \tag{7.5}$$
$$1 + p^{4m}|\alpha_1|^2 = p^{2m}\left(1 + |\alpha_1|^2\right) + |v_2|^2/\tau^2 + \ldots + |v_n|^2/\tau^2. \tag{7.6}$$

Eliminating τ from the above two equations, we get

$$\frac{p^2\left(|v_2|^2 + \ldots + |v_n|^2\right)}{m^2|v_1|^2} = \frac{1 - |\alpha_1|^2 p^{2m}}{|\alpha_1|^2\left(1 - p^{2m}\right)}.$$

Observe that the right hand side above is at least 1 and hence for

$$\frac{m^2|v_1|^2}{|v_2|^2 + \ldots + |v_n|^2} \leqslant p^2,$$

solving equations (7.5) and (7.6) for τ, it follows that

$$\tau = \left(\frac{m^2p^{2m-2}|v_1|^2}{(1 - p^{2m})^2} + \frac{|v_2|^2}{1 - p^{2m}} + \ldots + \frac{|v_n|^2}{1 - p^{2m}}\right)^{1/2}. \tag{7.7}$$

To summarize, the condition $s = 1$ is equivalent to requiring that $\frac{m^2|v_1|^2}{|v_2|^2+\cdots+|v_n|^2} \leqslant p^2$ and in this case, τ is defined by (7.7).

For the second case, when $s = 0$, let u and t be parameters as defined by equations (1.7) and (1.8) respectively. Observe that the condition $s = 0$ is equivalent to requiring that $u > p$. Moreover, the parameter t is a solution of

$$(m-1)^2p^2t^2 - \left(u^2 + 2m(m-1)p^2\right)t + m^2p^2 = 0, \tag{7.8}$$

and hence, satisfies $0 < t < 1$. As before, Theorem 7.1 applied once again gives

$$\phi_1(\lambda) = a_1\left(\frac{\lambda - \alpha_1}{1 - \bar{\alpha}_1\lambda}\right)\left(\frac{1 - \bar{\alpha}_1\lambda}{1 - \bar{\alpha}_0\lambda}\right)^{1/m},$$
$$\phi_j(\lambda) = a_j\left(\frac{\lambda - \alpha_j}{1 - \bar{\alpha}_j\lambda}\right)\left(\frac{1 - \bar{\alpha}_j\lambda}{1 - \bar{\alpha}_0\lambda}\right) \ \text{for } j = 2, \ldots, n,$$

where a_j, α_j satisfy the conditions listed in Theorem 7.1. It is immediate that

$$\begin{aligned}\phi(0) &= (-a_1\alpha_1, \ldots, -a_n\alpha_n) \text{ and}\\ \phi_1'(0) &= a_1\left((1-|\alpha_1|^2) - \frac{\alpha_1}{m}(\bar\alpha_0 - \bar\alpha_1)\right),\\ \phi_j'(0) &= a_j\left((1-|\alpha_j|^2) - \alpha_j(\bar\alpha_0 - \bar\alpha_j)\right) \text{ for } j = 2, \ldots, n.\end{aligned}$$

It follows that

$$\begin{aligned}-a_1\alpha_1 = p,\ \alpha_2 = \ldots = \alpha_n = 0, \text{ and}\\ \tau a_1\left(1-|\alpha_1|^2 - \frac{\alpha_1}{m}(\bar\alpha_0 - \bar\alpha_1)\right) = v_1, \tau a_2 = v_2, \ldots, \tau a_n = v_n.\end{aligned}$$

As a consequence,

$$\begin{aligned}\alpha_0 &= |a_1|^{2m}\alpha_1, \text{ and}\\ 1 + |\alpha_0|^2 &= |a_1|^{2m}\left(1+|\alpha_1|^2\right) + |a_2|^2 + \ldots + |a_n|^2,\end{aligned}$$

so that

$$\frac{|v_1|}{\tau} = \frac{p}{|\alpha_1|}\left(1 - |\alpha_1|^2 - \frac{|\alpha_1|^2}{m}\Big(\frac{p^{2m}}{|\alpha_1|^{2m}} - 1\Big)\right), \text{ and} \tag{7.9}$$

$$1 + \frac{p^{4m}}{|\alpha_1|^{4m-2}} - \frac{p^{2m}}{|\alpha_1|^{2m}}\left(1+|\alpha_1|^2\right) = \frac{|v_2|^2}{\tau^2} + \ldots + \frac{|v_n|^2}{\tau^2}. \tag{7.10}$$

Writing $|\alpha_1| = \alpha$, the goal now is to solve the equations (7.9) and (7.10) for α. To achieve this, first eliminate τ from the above two equations, so that

$$\frac{\left((m-1)\alpha^{2m} - m\alpha^{2m-2} + p^{2m}\right)^2}{\alpha^{4m-2} - p^{2m}\alpha^{2m} - p^{2m}\alpha^{2m-2} + p^{4m}} = \frac{m^2|v_1|^2}{p^2\left(|v_2|^2 + \ldots + |v_n|^2\right)}.$$

But the right hand side above is exactly the quotient u^2/p^2 and hence,

$$\begin{aligned}p^2\left((m-1)\alpha^{2m} - m\alpha^{2m-2} + p^{2m}\right)^2 =\\ u^2\left(\alpha^{4m-2} - p^{2m}\alpha^{2m} - p^{2m}\alpha^{2m-2} + p^{4m}\right).\end{aligned}$$

The above equality can be rewritten as (refer Example 8.4.7 of [33])

$$\alpha^{2m} - t\alpha^{2m-2} - (1-t)p^{2m} = 0, \text{ or} \tag{7.11}$$

$$(m-1)^2\frac{p^2}{u^2}\alpha^{2m} - \frac{m^2p^2}{tu^2}\alpha^{2m-2} + \frac{\left(u^2-p^2\right)}{(1-t)u^2}p^{2m} = 0.$$

Further observe that the equation (7.11) is equivalent to

$$\alpha = q_{E(2m,2)}\left(p(1-t)^{\frac{1}{2m}}, t^{\frac{1}{2}}\right), \tag{7.12}$$

where $q_{E(2m,2)}$ denotes the Minkowski functional of $E(2m,2) = \{z \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\}$, and therefore satisfies

$$\frac{|v_1|^{2m}}{\left(q_{E(2m,2)}(v_1,v_2)\right)^{2m}} + \frac{|v_2|^2}{\left(q_{E(2m,2)}(v_1,v_2)\right)^2} = 1.$$

It follows from the formulation (7.12) that the equation (7.11) has a unique solution α in the open interval $(0,1)$. Once we know α rather explicitly, it is easy to compute τ – Indeed, substituting (7.11) into the expression (7.9) yields

$$\tau = \frac{m\alpha^{2m-1}|v_1|}{p\,(m(1-t)+t)\,(p^{2m}-\alpha^{2m-2})}$$

which, in turn, equals

$$\tau = \frac{m\alpha(1-t)|v_1|}{p(1-\alpha^2)\,(m(1-t)+t)} \text{ whenever } u > p. \tag{7.13}$$

The equations (7.7) and (7.13) together give a comprehensive formula for the infinitesimal Kobayashi metric of the ellipsoid E_{2m}.

We assume that $m > 1/2$ for the rest of this section. To complete the proof, it remains to establish smoothness of $K_{E_{2m}}$. To this end, we first show that away from the zero section of the tangent bundle of the domain E_{2m}, both expressions (7.7) and (7.13) are C^1-smooth in each of the variables $p, v_1, \ldots, v_n$. While it is straightforward to infer smoothness from (7.7), to verify the claim for (7.13), the following observation will be needed: The *Kobayashi indicatrix* of the complex ellipsoid $E(2m,2)$ at the origin, i.e., $\{(v_1,v_2) \in \mathbb{C}^2 : K_{E(2m,2)}((0,0),(v_1,v_2)) = 1\}$ is given by the equation $|v_1|^{2m} + |v_2|^2 = 1$. The indicatrix is evidently C^1 since $m > 1/2$. As a consequence, the Kobayashi metric $K_{E(2m,2)}((0,0),(v_1,v_2))$ must be a C^1-function of the variables v_1 and v_2. Equivalently, the Minkowski functional $q_{E(2m,2)}$ of the domain $E(2m,2)$ (which equals $K_{E(2m,2)}((0,0),(v_1,v_2))$) is C^1. It follows from (7.12) that α is C^1-smooth with respect to the parameter t, which in turn, varies smoothly as a function of $p, v_1, \ldots, v_n$. This proves the claim.

Recall that every point of E_{2m} is in the orbit of the point $(p,\hat{0})$ for $0 \leqslant p < 1$. Moreover, since the action of the automorphism group of E_{2m} is real analytic, to conclude that $K_{E_{2m}}$ is C^1, it suffices to show that the Kobayashi metric is C^1-smooth at the point $v = (v_1,\hat{v}) \neq (0,\hat{0})$ in the tangent space $T_{(p,\hat{0})}E_{2m} = \mathbb{C}^n$ with $u = p$. We will show that for each

$j = 1, \ldots, n$,

$$\lim_{p \geqslant u \to p} \frac{\partial K_{E_{2m}}}{\partial |v_j|} \left((p, \hat{0}), (v_1, \hat{v})\right) = \lim_{p < u \to p} \frac{\partial K_{E_{2m}}}{\partial |v_j|} \left((p, \hat{0}), (v_1, \hat{v})\right), \text{ and} \quad (7.14)$$

$$\lim_{p \to u^+} \frac{\partial K_{E_{2m}}}{\partial p} \left((p, \hat{0}), (v_1, \hat{v})\right) = \lim_{p \to u^-} \frac{\partial K_{E_{2m}}}{\partial p} \left((p, \hat{0}), (v_1, \hat{v})\right). \quad (7.15)$$

When $u \leqslant p$, we have

$$K_{E_{2m}} \left((p, \hat{0}), (v_1, \hat{v})\right) = \left(\frac{m^2 p^{2m-2} |v_1|^2}{(1 - p^{2m})^2} + \frac{|v_2|^2}{1 - p^{2m}} + \ldots + \frac{|v_n|^2}{1 - p^{2m}} \right)^{1/2}$$

from which it follows that

$$\lim_{p \geqslant u \to p} \frac{\partial K_{E_{2m}}}{\partial |v_1|} \left((p, \hat{0}), (v_1, \hat{v})\right) = \frac{m p^{2m-1}}{1 - p^{2m}}, \text{ and} \quad (7.16)$$

$$\lim_{p \geqslant u \to p} \frac{\partial K_{E_{2m}}}{\partial |v_2|} \left((p, \hat{0}), (v_1, \hat{v})\right) = \left(\frac{m^2 |v_1|^2 - p^2 |v_3|^2 - \ldots - p^2 |v_n|^2}{m^2 |v_1|^2} \right)^{1/2}. \quad (7.17)$$

The computation in the second case, $0 < p < u$, will involve several steps. To begin with, it is straightforward to check that

$$\lim_{p < u \to p} t = 1 \quad \text{and} \quad \lim_{p < u \to p} \alpha = 1. \quad (7.18)$$

Next, differentiating (7.8) with respect to $|v_1|$ gives

$$\frac{\partial t}{\partial |v_1|} = - \frac{2u^2 t}{|v_1| \left(2(m-1)p^2 \left(m(1-t) + t\right) + u^2\right)},$$

which implies that

$$\lim_{p < u \to p} \frac{\partial t}{\partial |v_1|} = \frac{2m}{(1 - 2m)p} \left(|v_2|^2 + \ldots + |v_n|^2\right)^{-1/2}. \quad (7.19)$$

Differentiating (7.11) with respect to $|v_1|$ quickly leads to

$$2 \left(m \alpha^{2m-1} - (m-1) t \alpha^{2m-3}\right) \frac{\partial \alpha}{\partial |v_1|} = \left(\alpha^{2m-2} - p^{2m}\right) \frac{\partial t}{\partial |v_1|}. \quad (7.20)$$

Taking the limit as u tends to p from above in (7.20), and using (7.19), we get

$$\lim_{p < u \to p} \frac{\partial \alpha}{\partial |v_1|} = \frac{m \left(1 - p^{2m}\right)}{(1 - 2m)p} \left(|v_2|^2 + \ldots + |v_n|^2\right)^{-1/2}. \quad (7.21)$$

Also, observe that the equations (7.11) and (7.8) are equivalent to

$$\frac{1-t}{1-\alpha^2} = \frac{\alpha^{2m-2}}{\alpha^{2m-2}-p^{2m}} \text{ and } u^2 = \frac{p^2\left(m(1-t)+t\right)^2}{t}$$

respectively, so that (7.13) can be rewritten as

$$K_{E_{2m}}\left((p,\hat{0}),(v_1,\hat{v})\right) = \frac{\alpha^{2m-1}}{\alpha^{2m-2}-p^{2m}}\left(\frac{|v_2|^2}{t}+\ldots+\frac{|v_n|^2}{t}\right)^{1/2}, \quad (7.22)$$

which upon differentiation turns out to be

$$\frac{\partial K_{E_{2m}}}{\partial|v_1|}\left((p,\hat{0}),(v_1,\hat{v})\right) = \left(\frac{(2m-1)\alpha^{2m-2}}{t^{1/2}\left(\alpha^{2m-2}-p^{2m}\right)}\frac{\partial\alpha}{\partial|v_1|} - \frac{\alpha^{2m-1}}{2t^{3/2}\left(\alpha^{2m-2}-p^{2m}\right)}\frac{\partial t}{\partial|v_1|} - \frac{2(m-1)\alpha^{4m-4}}{t^{1/2}(\alpha^{2m-2}-p^{2m})^2}\frac{\partial\alpha}{\partial|v_1|}\right)\left(|v_2|^2+\ldots+|v_n|^2\right)^{1/2}.$$

So that

$$\lim_{p<u\to p}\frac{\partial K_{E_{2m}}}{\partial|v_1|}\left((p,\hat{0}),(v_1,\hat{v})\right) = \frac{mp^{2m-1}}{1-p^{2m}} \quad (7.23)$$

owing to (7.19), (7.21) and (7.18). The expressions (7.16) and (7.23) together verify that (7.14) holds for $j=1$. Furthermore, a similar computation yields that

$$\lim_{p<u\to p}\frac{\partial t}{\partial|v_2|} = \frac{2p}{(2m-1)m^2|v_1|^2}\left(m^2|v_1|^2-p^2|v_3|^2-\ldots-p^2|v_n|^2\right)^{1/2}, \text{ and}$$

$$\lim_{p<u\to p}\frac{\partial\alpha}{\partial|v_2|} = \frac{p\left(1-p^{2m}\right)}{(2m-1)m^2|v_1|^2}\left(m^2|v_1|^2-p^2|v_3|^2-\ldots-p^2|v_n|^2\right)^{1/2}.$$

Consequently, we find that, in agreement with the first case (cf. (7.17))

$$\lim_{p<u\to p}\frac{\partial K_{E_{2m}}}{\partial|v_2|}\left((p,\hat{0}),(v_1,\hat{v})\right) = \left(\frac{m^2|v_1|^2-p^2|v_3|^2-\ldots-p^2|v_n|^2}{m^2|v_1|^2}\right)^{1/2}.$$

Finally, an argument similar to the one used above shows that

$$\lim_{p\geqslant u\to p}\frac{\partial K_{E_{2m}}}{\partial|v_j|}\left((p,\hat{0}),(v_1,\hat{v})\right) = \lim_{p<u\to p}\frac{\partial K_{E_{2m}}}{\partial|v_j|}\left((p,\hat{0}),(v_1,\hat{v})\right)$$

for each $j=3,\ldots,n$.

In the third case, $0 = p < u$, the Kobayashi indicatrix of E_{2m} at the origin, defined as

$$\{v \in \mathbb{C}^n : K_{E_{2m}}\left((0,\ldots,0),(v_1,\ldots,v_n)\right) = 1\},$$

is given by the equation $|v_1|^{2m} + |v_2|^2 + \ldots + |v_n|^2 = 1$. Since $m > 1/2$, the indicatrix is C^1 which, in turn, implies that the Kobayashi metric $K_{E_{2m}}\left((0,\ldots,0),(v_1,\ldots,v_n)\right)$ is also a C^1 function of the variable v.

To verify that (7.15) holds, we evaluate the left hand side first. Note that from (7.7), we obtain

$$\lim_{p\to u^+} K_{E_{2m}}\left((p,\hat{0}),(v_1,\hat{v})\right) = \frac{m|v_1|}{u\left(1-|u|^{2m}\right)}.$$

On differentiating (7.7) with respect to p, we have that

$$\begin{aligned}&2K_{E_{2m}}\left((p,\hat{0}),(v_1,\hat{v})\right)\frac{\partial K_{E_{2m}}}{\partial p}\left((p,\hat{0}),(v_1,\hat{v})\right)\\&=\frac{2mp^{2m-1}}{(1-p^{2m})^2}\left(|v_2|^2+\ldots+|v_n|^2\right)+\frac{2m^2(m-1)p^{2m-3}}{\left(1-p^{2m}\right)^2}|v_1|^2+\frac{4m^3p^{4m-3}}{\left(1-p^{2m}\right)^3}|v_1|^2.\end{aligned}$$

Letting $p \to u^+$, and using the above observation, we get

$$\lim_{p\to u^+}\frac{\partial K_{E_{2m}}}{\partial p}\left((p,\hat{0}),(v_1,\hat{v})\right) = \frac{mu^{2m-2}\left(u^{2m}+2m-1\right)}{\left(1-u^{2m}\right)^2}|v_1|.$$

Working with the right hand side of (7.15), observe that

$$\lim_{p\to u^-} t = 1 \text{ and } \lim_{p\to u^-}\alpha = 1.$$

It follows from the definition of the parameter t that

$$\frac{\partial t}{\partial p} = \frac{2p\left(m(1-t)+t\right)^2}{2p^2(m-1)\left(m(1-t)+t\right)+u^2},$$

which implies that

$$\lim_{p\to u^-}\frac{\partial t}{\partial p} = \frac{2}{(2m-1)u}. \tag{7.24}$$

To compute $\partial\alpha/\partial p$, we differentiate both sides of (7.11). After simplification, we get

$$\left(2m\alpha^{2m-1} - 2t(m-1)\alpha^{2m-3}\right)\frac{\partial\alpha}{\partial p} = \left(\alpha^{2m-2}-p^{2m}\right)\frac{\partial t}{\partial p} + 2m(1-t)p^{2m-1},$$

so that

$$\lim_{p\to u^-}\frac{\partial\alpha}{\partial p}=\frac{1-u^{2m}}{(2m-1)u}. \tag{7.25}$$

Using (7.13) to write the derivative of $K_{E_{2m}}$ with respect to p, it follows, by virtue of (7.24) and (7.25), that

$$\lim_{p\to u^-}\frac{\partial K_{E_{2m}}}{\partial p}\left((p,\hat{0}),(v_1,\hat{v})\right)=\frac{mu^{2m-2}\left(u^{2m}+2m-1\right)}{\left(1-u^{2m}\right)^2}|v_1|,$$

as required.

To conclude that $K_{E_{2m}}$ is C^1, it remains to verify that $K_{E_{2m}}\big((p,\hat{0}),(v_1,\hat{v})\big)$ is C^1 at $p=0$ with respect to the variable p. Firstly, recall from (7.7) that when $u\leqslant p$,

$$K_{E_{2m}}\left((p,\hat{0}),(v_1,\hat{v})\right)=\left(\frac{m^2p^{2m-2}|v_1|^2}{(1-p^{2m})^2}+\frac{|v_2|^2}{1-p^{2m}}+\ldots+\frac{|v_n|^2}{1-p^{2m}}\right)^{1/2},$$

so that

$$2\left(\frac{m^2p^{2m-2}|v_1|^2}{(1-p^{2m})^2}+\frac{|v_2|^2}{1-p^{2m}}+\ldots+\frac{|v_n|^2}{1-p^{2m}}\right)^{1/2}\frac{\partial K_{E_{2m}}}{\partial p}\left((p,\hat{0}),(v_1,\hat{v})\right)$$
$$=\frac{2mp^{2m-1}}{(1-p^{2m})^2}\left(|v_2|^2+\ldots+|v_n|^2\right)+\frac{2m^2(m-1)p^{2m-3}}{\left(1-p^{2m}\right)^2}|v_1|^2+\frac{4m^3p^{4m-3}}{\left(1-p^{2m}\right)^3}|v_1|^2,$$

from which it can be derived that

$$\lim_{u\leqslant p\to 0}\frac{\partial K_{E_{2m}}}{\partial p}\left((p,\hat{0}),(v_1,\hat{v})\right)=0. \tag{7.26}$$

Next, we need to rewrite (7.13) (or, equivalently, (7.22)) so that the alternate formulation of $K_{E_{2m}}\left((p,\hat{0}),(v_1,\hat{v})\right)$ for $u>p$ is well-defined for $p=0$. To this end, formally define

$$\tilde{t}=\frac{t^{1/2}|v_1|}{p\left(|v_2|^2+\ldots+|v_n|^2\right)^{1/2}} \tag{7.27}$$

and

$$\tilde{\alpha}=\frac{p}{\alpha|v_1|}, \tag{7.28}$$

so that the expression (7.22) for the Kobayashi metric takes the form

$$K_{E_{2m}}\left((p,\hat{0}),(v_1,\hat{v})\right)=\frac{1}{\tilde{\alpha}\tilde{t}\left(1-p^2\tilde{\alpha}^{2m-2}|v_1|^{2m-2}\right)} \tag{7.29}$$

whenever $u > p$. Note that, substituting for t in (1.8) gives $\tilde{t}$ as

$$\tilde{t}^2 = \frac{2|v_1|^2}{|v_1|^2 + 2\left(1 - \frac{1}{m}\right)p^2\left(|v_2|^2 + \ldots + |v_n|^2\right) + |v_1|\Big(|v_1|^2 + 4\left(1 - \frac{1}{m}\right)p^2\left(|v_2|^2 + \ldots + |v_n|^2\right)\Big)^{1/2}}. \tag{7.30}$$

Next, write t and α in terms of $\tilde{t}$ and $\tilde{\alpha}$ using (7.27) and (7.28) and plug in the resulting expressions into the equation (7.12) in place of t and α respectively. This will show that $\tilde{\alpha}$ is uniquely determined by the following equation:

$$\left(1 - \frac{\tilde{t}^2 p^2\left(|v_2|^2 + \ldots + |v_n|^2\right)}{|v_1|^2}\right)|v_1|^{2m}\tilde{\alpha}^{2m} + \tilde{t}^2\left(|v_2|^2 + \ldots + |v_n|^2\right)\tilde{\alpha}^2 = 1. \tag{7.31}$$

Henceforth, we use equations (7.30) and (7.31) to define $\tilde{t}$ and $\tilde{\alpha}$ respectively. In particular, when $u > p = 0$ (i.e., $|v_1| > 0$), it follows from (7.30) that $\tilde{t} = 1$, and consequently, (7.31) and (7.29) reduce to

$$|v_1|^{2m}\tilde{\alpha}^{2m} + |v_2|^2\tilde{\alpha}^2 + \ldots + |v_n|^2\tilde{\alpha}^2 = 1, \tag{7.32}$$

and

$$K_{E_{2m}}\left((0, 0, \ldots, 0), (v_1, v_2, \ldots, v_n)\right) = \frac{1}{\tilde{\alpha}}$$

respectively for $|v_1| > 0$. Observe that the above formulae are in agreement with the expression for the Kobayashi metric at the origin given in terms of the Minkowski functional of E_{2m} (cf. equations (7.3) and (7.4)). Moreover, (7.29) gives a formula for the Kobayashi metric when $u > p$, that is equivalent to (7.22). Furthermore, (7.29) has an advantage over (7.22) that it defines $K_{E_{2m}}$ at $p = 0$.

We now use (7.29) to show that $K_{E_{2m}}\big((p, \hat{0}), (v_1, \hat{v})\big)$ is C^1-smooth at $p = 0$ with respect to p. To start with, differentiating (7.30) and (7.31) with respect to p and letting p tend to 0 gives

$$\lim_{p \to 0}\frac{\partial \tilde{t}}{\partial p} = 0 \;\text{ and }\; \lim_{p \to 0}\frac{\partial \tilde{\alpha}}{\partial p} = 0,$$

respectively, for $|v_1| > 0$. Moreover, differentiating (7.29) with respect to p, we see that

$$\frac{\partial K_{E_{2m}}}{\partial p}\left((p, \hat{0}), (v_1, \hat{v})\right) = \frac{\tilde{t}\left((2m-1)p^2|v_1|^{2m-2}\tilde{\alpha}^{2m-2} - 1\right)\frac{\partial \tilde{\alpha}}{\partial p} + \tilde{\alpha}\left(p^2|v_1|^{2m-2}\tilde{\alpha}^{2m-2} - 1\right)\frac{\partial \tilde{t}}{\partial p} + 2p\tilde{t}|v_1|^{2m-2}\tilde{\alpha}^{2m-1}}{\tilde{\alpha}^2\tilde{t}^2\left(1 - p^2|v_1|^{2m-2}\tilde{\alpha}^{2m-2}\right)^2},$$

and hence

$$\lim_{u>p\to 0} \frac{\partial K_{E_{2m}}}{\partial p}\big((p,\hat{0}),(v_1,\hat{v})\big) = 0$$

whenever $|v_1| > 0$. Furthermore, it can be checked that

$$\lim_{u>p\to 0} \frac{\partial K_{E_{2m}}}{\partial p}\big((p,\hat{0}),(v_1,\hat{v})\big) = 0 \ \text{ for } \ |v_1| \to 0.$$

This finishes the proof of the theorem.

8. Proof of Theorem 1.7

Suppose that there is a C^1-Kobayashi isometry f from D_1 onto D_2 with $q^0 \in cl_f(p^0)$, the cluster set of p^0. Firstly, from the explicit form of the defining function for $U_1 \cap \partial D_1$, it is clear that ∂D_1 near p^0 is smooth pseudoconvex and of finite type with Levi rank exactly $(n-2)$.

Assume that both $p^0 = 0$ and $q^0 = 0$ and choose a sequence $p^j = ('0, -\delta_j) \in U_1 \cap D_1$ on the inner normal approaching the origin. By Theorem 1.5, it readily follows that f extends continuously up to p^0. As a consequence, $q^j = f(p^j)$ converges to q^0 which is a strongly pseudoconvex point in ∂D_2. The idea is to apply the scaling technique to (D_1, D_2, f) as in the proof of Theorem 1.4. To scale D_1, we only consider dilations

$$\Delta^j(z_1, z_2, \ldots, z_{n-1}, z_n) = \left(\delta_j^{-\frac{1}{2m}} z_1, \delta_j^{-\frac{1}{2}} z_2, \ldots, \delta_j^{-\frac{1}{2}} z_{n-1}, \delta_j^{-1} z_n\right).$$

Note that $\Delta^j('0, -\delta_j) = ('0, -1)$ for all j and the domains $D_1^j = \Delta^j(D_1)$ converge in the Hausdorff sense to

$$D_{1,\infty} = \big\{z \in \mathbb{C}^n : 2\Re z_n + |z_1|^{2m} + |z_2|^2 + \ldots + |z_{n-1}|^2 < 0\big\},$$

which is biholomorphic to E_{2m}.

While for D_2, we use the composition $T^j \circ h^{\xi^j}$ as in Section 5 - we include a brief exposition here for completeness: Consider points $\xi^j \in \partial D_2$ defined by $\xi^j = q^j + ('0, \epsilon_j)$, for some $\epsilon_j > 0$. As before, h^{ξ^j} are the 'centering maps' (cf. [47]) corresponding to $\xi^j \in \partial D_2$, and T^j are the dilations

$$T^j(w_1, w_2, \ldots, w_n) = \left({\epsilon_j}^{-\frac{1}{2}} w_1, \ldots, {\epsilon_j}^{-\frac{1}{2}} w_{n-1}, {\epsilon_j}^{-1} w_n\right).$$

It was shown in [47] that the dilated domains $D_2^j = T^j \circ h^{\xi^j}(D_2)$ converge to

$$D_{2,\infty} = \big\{w \in \mathbb{C}^n : 2\Re w_n + |w_1|^2 + |w_2|^2 + \ldots + |w_{n-1}|^2 < 0\big\}$$

which is the unbounded representation of the unit ball in $\mathbb{C}^n$. Among other things, the following claim was verified in [53]: For $w \in D_{2,\infty}$,

$$d^k_{D^j_2}(w,\cdot) \to d^k_{D_{2,\infty}}(w,\cdot) \tag{8.1}$$

uniformly on compact sets of $D_{2,\infty}$. As a consequence, the sequence of Kobayashi metric balls $B_{D^j_2}(\cdot, R) \to B_{D_{2,\infty}}(\cdot, R)$, and, for large j,

$$\begin{aligned} B_{D_{2,\infty}}(\cdot, R) \subset B_{D^j_2}(\cdot, R+\epsilon)\,, \text{ and} \\ B_{D^j_2}(\cdot, R-\epsilon) \subset B_{D_{2,\infty}}(\cdot, R)\,. \end{aligned} \tag{8.2}$$

The scaled maps $f^j = T^j \circ h^{\xi^j} \circ f \circ (\Delta^j)^{-1} : D^j_1 \to D^j_2$ are isometries in the Kobayashi metric on D^j_1 and D^j_2 and note that $f^j('0,-1) = ('0,-1)$ for all j. Exhaust $D_{1,\infty}$ by an increasing union $\{K_\nu\}$ of relatively compact domains, each containing $('0,-1)$. Fix a pair K_1 compactly contained in K_2 say, and let $\omega(K_1)$ be a neighbourhood of K_1 such that $\omega(K_1) \subset K_2$. Since the domains D^j_1 converge to $D_{1,\infty}$, it follows that $\omega(K_1) \subset K_2$ is relatively compact in D^j_1 for all j large. Now, to establish that f^j admits a convergent subsequence, it will suffice to show that f^j restricted to $\omega(K_1)$ is uniformly bounded and equicontinuous. For each $z \in K_2$, note that for all j large,

$$d^k_{D^j_2}\left(f^j(z), ('0,-1)\right) = d^k_{D^j_1}(z, ('0,-1)) \leqslant d^k_{D_{1,\infty}}(z, ('0,-1)) + \epsilon$$

where the last inequality follows from Proposition 3.6. Observe that the right hand side above is bounded above by a uniform positive constant, say $\tilde{R} > 0$. Therefore, by (8.2), it follows that

$$f^j(K_2) \in B_{D^j_2}\left(('0,-1), \tilde{R}\right) \subset B_{D_{2,\infty}}\left(('0,-1), \tilde{R}+\epsilon\right), \tag{8.3}$$

which exactly means that $\{f^j(K_2)\}$ is uniformly bounded.

The following observation will be needed to deduce the equicontinuity of f^j restricted to $\omega(K_1)$. For each $z \in \omega(K_1)$, there is a small ball $B(z,r)$ around z with radius $r > 0$, which is compactly contained in $\omega(K_1)$. For $R' \gg 2\tilde{R}$, we intend to apply Lemma 3.5 to the domain D^j_2 with the Kobayashi ball $B_{D^j_2}(('0,-1), R')$ as the subdomain D'. Let $\tilde{z} \in B(z,r)$, then, for $f^j(\tilde{z}), f^j(z)$ as p and q respectively, and $R'/2$ as b, it can be checked that all the conditions of Lemma 3.5 are satisfied. So that

$$d^k_{B_{D_2^j}(('0,-1),R')}\left(f^j(z),f^j(\tilde z)\right) \leqslant \frac{d^k_{D_2^j}\left(f^j(z),f^j(\tilde z)\right)}{\tanh\left(\frac{R'}{2}-d^k_{D_2^j}\left(f^j(z),f^j(\tilde z)\right)\right)} \leqslant \frac{d^k_{D_2^j}\left(f^j(z),f^j(\tilde z)\right)}{\tanh\left(\frac{R'}{2}-2\tilde R\right)} \tag{8.4}$$

where the last inequality above is a simple consequence of triangle inequality and (8.3).

Furthermore, it turns out that for any small neighbourhood W of $q^0 \in \partial D_2$ and for all large j, $B_{D_{2,\infty}}(('0,-1),R'+\epsilon) \subset T^j \circ h^{\xi^j}(W\cap D_2)$. On the other hand, let $R>1$ be such that

$$\begin{aligned} h^{\xi^j}(W\cap D_2) &\subset \{w\in\mathbb{C}^n : |w_1|^2+\ldots+|w_{n-1}|^2+|w_n+R|^2<R^2\} \\ &\subset \{w\in\mathbb{C}^n : 2R(\Re w_n)+|w_1|^2+\ldots+|w_{n-1}|^2<0\} = D_0 \approx \mathbb{B}^n. \end{aligned}$$

Observe that T^j leaves the domain D_0 invariant for each j, therefore, we may conclude that

$$B_{D_{2,\infty}}(('0,-1),R'+\epsilon) \subset T^j\circ h^{\xi^j}(W\cap D_2) \subset D_0 \approx \mathbb{B}^n.$$

Now, using the explicit form of the Kobayashi metric on $\mathbb{B}^n$, it follows that

$$|f^j(z)-f^j(\tilde z)| \lesssim d^k_{D_0}\left(f^j(z),f^j(\tilde z)\right) \leqslant d^k_{B_{D_{2,\infty}}(('0,-1),R'+\epsilon)}\left(f^j(z),f^j(\tilde z)\right). \tag{8.5}$$

While from (8.2), we see that

$$d^k_{B_{D_{2,\infty}}(('0,-1),R'+\epsilon)}\left(f^j(z),f^j(\tilde z)\right) \leqslant d^k_{B_{D_2^j}(('0,-1),R')}\left(f^j(z),f^j(\tilde z)\right). \tag{8.6}$$

Finally, we deduce, using the expression for the Kobayashi metric (which equals the Poincaré metric) on $B(z,r)$, that

$$d^k_{D_2^j}\left(f^j(z),f^j(\tilde z)\right) = d^k_{D_1^j}(z,\tilde z) \leqslant d^k_{B(z,r)}(z,\tilde z) \lesssim |z-\tilde z|. \tag{8.7}$$

so that

$$|f^j(z)-f^j(\tilde z)| \lesssim |z-\tilde z|$$

by virtue of (8.5), (8.6), (8.4) and (8.7). Hence, there is a well-defined continuous limit of some subsequence of f^j. Denote this limit by $\tilde f : D_{1,\infty} \to \overline{D}_{2,\infty}$.

The proof now divides into two parts. The first part is to show that $\tilde{f}$ is an isometry between $D_{1,\infty}$ and $D_{2,\infty}$ and the second step is to show that $\tilde{f}$ is either holomorphic or conjugate holomorphic.

I. $\tilde{f}$ is an isometry

If $\tilde{f}$ were known to be holomorphic, then the maximum principle would imply that $\tilde{f} : D_{1,\infty} \to D_{2,\infty}$. However, $\tilde{f}$ is known to be just continuous. To overcome this difficulty, consider $\Omega_1 \subset D_{1,\infty}$, the set of all points $z \in D_{1,\infty}$ such that $\tilde{f}(z) \in D_{2,\infty}$. Note that $\tilde{f}\left(('0,-1)\right) = ('0,-1) \in D_{2,\infty}$ and hence Ω_1 is non-empty. Also, since $\tilde{f}$ is continuous, it follows that Ω_1 is open in $D_{1,\infty}$.

Assertion. — $d^k_{D_{1,\infty}}(p,q) = d^k_{D_{2,\infty}}\left(\tilde{f}(p), \tilde{f}(q)\right)$ for all $p, q \in \Omega_1$.

Grant this for now. Then, for $s \in \partial\Omega_1 \cap D_{1,\infty}$, let $s^j \in \Omega_1$ such that $s^j \to s$. Assuming that the assertion holds true, we have that

$$d^k_{D_{1,\infty}}\left(s^j, ('0,-1)\right) = d^k_{D_{2,\infty}}\left(\tilde{f}(s^j), ('0,-1)\right) \tag{8.8}$$

for all j. Since $s \in \partial\Omega_1 \cap D_{1,\infty}$, $\tilde{f}(s^j)$ converges to a point of $\partial D_{2,\infty}$. Furthermore, $D_{2,\infty}$ is complete in the Kobayashi metric, and hence, the right hand side in (8.8) is unbounded. However, the left hand side remains bounded again because of completeness of $D_{1,\infty}$. This contradiction shows that $\Omega_1 = D_{1,\infty}$. In other words, $\tilde{f} : D_{1,\infty} \to D_{2,\infty}$ and $d^k_{D_{1,\infty}}(p,q) = d^k_{D_{2,\infty}}\big(\tilde{f}(p), \tilde{f}(q)\big)$ for all $p, q \in D_{1,\infty}$.

To verify the assertion, recall that $d^k_{D_1^j}(p,q) = d^k_{D_2^j}\big(f^j(p), f^j(q)\big)$ for all j. The statement $d^k_{D_1^j}(p,q) \to d^k_{D_{1,\infty}}(p,q)$ follows from the proof of Proposition 3.6. Hence, it remains to show that the right hand side above converges to $d^k_{D_{2,\infty}}\left(\tilde{f}(p), \tilde{f}(q)\right)$. To achieve this, note that

$$\left|d^k_{D_2^j}\big(f^j(p), f^j(q)\big) - d^k_{D_2^j}\big(\tilde{f}(p), \tilde{f}(q)\big)\right| \leqslant d^k_{D_2^j}\big(f^j(p), \tilde{f}(p)\big) + d^k_{D_2^j}\big(\tilde{f}(q), f^j(q)\big)$$

by the triangle inequality. Since $f^j(p) \to \tilde{f}(p)$ and $D_2^j \to D_{2,\infty}$, it follows that there is a small ball $B\big(\tilde{f}(p), r\big)$ around $\tilde{f}(p)$ which contains $f^j(p)$ and which is contained in D_2^j for all large j, where $r > 0$ is independent of j. Thus

$$d^k_{D_2^j}\big(f^j(p), \tilde{f}(p)\big) \lesssim \left|f^j(p) - \tilde{f}(p)\right|.$$

Similarly, it can be checked that $d^k_{D_2^j}\big(\tilde{f}(q), f^j(q)\big)$ is arbitrarily small. Furthermore, it follows from (8.1) that $d^k_{D_2^j}\big(\tilde{f}(p), \tilde{f}(q)\big) \to d^k_{D_{2,\infty}}\big(\tilde{f}(p), \tilde{f}(q)\big)$. Hence the assertion.

Next, we claim that $\tilde{f}$ is surjective. To see this, consider any point $t^0 \in \partial\big(\tilde{f}(D_{1,\infty})\big) \cap D_{2,\infty}$ and let $t^j \in \tilde{f}(D_{1,\infty})$ satisfying $t^j \to t^0$. Pick $s^j \in D_{1,\infty}$ such that $\tilde{f}(s^j) = t^j$. Then

$$d^k_{D_{1,\infty}}\big(('0,-1), s^j\big) = d^k_{D_{2,\infty}}\big(\tilde{f}(('0,-1)), \tilde{f}(s^j)\big).$$

There are two cases to be considered depending on whether $s^j \to s \in \partial D_{1,\infty}$ or $s^j \to s^0 \in D_{1,\infty}$ as $j \to \infty$. In case $s^j \to s$, observe that the right hand side above remains bounded because of the completeness of $D_{2,\infty}$. But left hand side is unbounded since $D_{1,\infty}$ is complete in the Kobayashi metric. This contradiction shows that $s^j \to s^0 \in D_{1,\infty}$, which, in turn, implies that $\tilde{f}(s^0) = t^0$. Now, consider the isometries $(f^j)^{-1} : D_2^j \to D_1^j$. Exactly the same arguments as above show that some subsequence of $(f^j)^{-1}$ converges uniformly on compact sets of $D_{2,\infty}$ to $\tilde{g} : D_{2,\infty} \to D_{1,\infty}$. It follows that $\tilde{f} \circ \tilde{g} \equiv id_{D_{2,\infty}}$. In particular, $\tilde{f}$ is surjective and $\tilde{f}$ is a isometry between $D_{1,\infty}$ and $D_{2,\infty}$ in the Kobayashi metric.

II. $\tilde{f}$ is holomorphic or conjugate holomorphic

The proof of the fact that $\tilde{f}$ is a biholomorphic mapping follows exactly as in [54]. To outline the key ingredients, the first step is to show that $\tilde{f}$ is differentiable everywhere, which implies that the Kobayashi metric $K_{D_{1,\infty}}$ is Riemannian. By Theorem 1.6, we know that $K_{E_{2m}}$ or equivalently that $K_{D_{1,\infty}}$ is C^1-smooth. Recall that $D_{1,\infty} \approx E_{2m}$ and $D_{2,\infty} \approx \mathbb{B}^n$. So that $\tilde{f}$ after composing with appropriate Cayley transforms, leads to a continuous isometry $\tilde{F}$ between two C^1-smooth Riemannian manifolds $\big(E_{2m}, K_{_{2m}}\big)$ and $\big(\mathbb{B}^n, K_{\mathbb{B}^n}\big)$. Applying the Myers-Steenrod theorem repeatedly to $\tilde{F}$, yields the desired result.

Once we know that $\tilde{F} : E_{2m} \to \mathbb{B}^n$ is holomorphic, which, additionally, may be assumed to preserve the origin, it follows that $2m = 2$. Indeed, $\tilde{F}$ is a biholomorphism between two circular domains, E_{2m} and $\mathbb{B}^n$ such that $\tilde{F}(0) = 0$ and is, hence, linear. In particular $2m = 2$. Said differently, there are holomorphic coordinates at p^0 in which a tiny neighbourhood around p^0 can be written as

$$\big\{z \in \mathbb{C}^n : 2\Re z_n + |z_1|^2 + |z_2|^2 + \ldots + |z_{n-1}|^2 + \text{higher order terms} < 0\big\},$$

which violates the assumption that the Levi rank at $p^0 \in \partial D_1$ is exactly $n-2$. Alternatively, one can use Theorem 1.4 or results from [5] to arrive at a contradiction. Hence the theorem. □

9. Appendix

The group of all polynomial automorphisms of $\mathbb{C}^n$ is usually denoted by $GA_n(\mathbb{C})$ and two special subgroups of $GA_n(\mathbb{C})$ are the affine subgroup

$$Af_n(\mathbb{C}) = \{F \in GA_n(\mathbb{C}) \ : \ \deg(F) \leqslant 1\}$$

and the triangular subgroup

$$BA_n(\mathbb{C}) =$$

$$\{F \in GA_n(\mathbb{C}) \ : \ F_j = a_j z_j + H_j \text{ where } a_j \in \mathbb{C}^* \text{ and } H_j \in \mathbb{C}[z_1, \ldots, z_{j-1}]\}$$

whose members are also called *elementary automorphisms*.

The Jung-van der Kulk theorem says that every polynomial automorphism in dimension $n = 2$ can be obtained as a finite composition of affine and elementary automorphisms; using this fact it can be derived (cf. [22]) that the degree of the inverse Φ^{-1} of a polynomial automorphism of $\mathbb{C}^2$ is the same as the degree of Φ. However, these facts are known to be false in higher dimensions.

The weight of a polynomial automorphism – or more generally a polynomial endomorphism of $\mathbb{C}^n$ – is by definition the maximum of the weights of its components.

Let us assign a weight of $1/2m$ to the variable z_1 where $m \in \mathbb{N}$, $1/2$ to the variables z_α for all $2 \leqslant \alpha \leqslant n-1$ and 1 to the variable z_n as in the introduction and consider the collection $\mathcal{E}_L = \mathcal{E}_{(1/2m,1/2,1)}$, of all weight preserving polynomial automorphisms of the form

$$F(z) = \big(a_1 z_1 + b_1, A(''z) + P_2(z_1), a_n z_n + b_n + P_n('z)\big) \tag{9.1}$$

where $''z = (z_2, \ldots, z_{n-1})$, $'z = (z_1, \ldots, z_{n-1})$, A is an invertible affine transform on $\mathbb{C}^{n-2}$, P_2 is a vector valued polynomial all of whose components are polynomials of weight at-most $1/2$ and P_n is a polynomial of weight at-most 1 while $b_1, b_n \in \mathbb{C}$ and $a_1, b_n \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. One may also note that if $m > 1$, then any (component wise) weight preserving polynomial automorphism has necessarily got to be of the form (9.1). In general, it is not true that the degree (resp. weight) of the inverse Φ^{-1} of a polynomial automorphism Φ, is same as the degree (resp. weight) of Φ. However, since $\mathcal{E}_L$ consists of 'elementary like' polynomial automorphisms that preserve the weight of each component, we have that the weights of the components of the inverse of F is same as those of F, for each $F \in \mathcal{E}_L$. Indeed, note that the inverse of F is given by

$$F^{-1}(z) = \Big(a_1^{-1}(z_1 - b_1), A^{-1}\big(''z - P_2\big(a_1^{-1}(z_1 - b_1)\big)\big), a_n^{-1}\big(z_n - b_n - P_n'('z)\big)\Big)$$

where $P_n'('z) = P_n\Big(a_1^{-1}(z_1 - b_1), A^{-1}\big('z - P_2(a_1^{-1}(z_1 - b_1))\big)\Big)$. It is easy to see that F^{-1} lies in $\mathcal{E}_L$. Next note that if we pick another $G \in \mathcal{E}_L$ given by

$$G(z) = \big(c_1 z_1 + d_1, B(''z) + Q_2(z_1), c_n z_n + d_n + Q_n('z)\big)$$

say, then

$$\begin{aligned} G(F(z)) = \Big(&c_1 a_1 z_1 + c_1 b_1 + d_1, BA(''z) + BP_2(z_1) + Q_2(a_1 z_1 + b_1), \\ &c_n a_n z_n + c_n b_n + d_n + Q_n\big(c_1 z_1 + d_1, A(''z) + P_2(z_1)\big)\Big) \end{aligned}$$

which is again in $\mathcal{E}_{(1/2m,1/2,1)}$ completing the verification that $\mathcal{E}_L$ is a group. In fact noting that $\mathcal{E}_L$ is in *bijection* with a product of finitely many copies of $\mathbb{C}^M \setminus \{0\}$, $\mathbb{C}^N$ for some $M, N \in \mathbb{N}$ and $GL_{n-2}(\mathbb{C})$ we see that $\mathcal{E}_L$ is a *non-singular* affine algebraic variety; next noting that the composition of maps in $\mathcal{E}_L$ when viewed as an operation on the various coefficients here, is a polynomial operation on these coefficients and likewise for taking inverses as well, we conclude that $\mathcal{E}$ is a complex algebraic group. Finally, we note that the topology on $\mathcal{E}_L$ obtained from its identification as mentioned above, is the same as the topology of uniform convergence on compacts i.e., the topology of uniform convergence on compacts on $\mathcal{E}_L$ is the same as the topology of 'convergence of the coefficients of the polynomial maps'.

Recall the special reduction procedure for the Taylor expansion of any given smooth defining function for a piece of Levi corank one hypersurface. Let Σ be a smooth pseudoconvex real-hypersurface in $\mathbb{C}^n$ of finite type with the property that the Levi-rank is at-least $n-2$ at each of its points. We assume that the origin lies in Σ and that the D' Angelo 1-type of the points of Σ is bounded above by some integer $2m$. Let r be a smooth defining function for Σ with $\partial r/\partial z_n(z) \neq 0$ for all z in a small neighborhood U in $\mathbb{C}^n$ of Σ such that the vector fields

$$L_n = \partial/\partial z_n, \ L_j = \partial/\partial z_j + b_j(z,\bar{z})\partial/\partial z_n,$$

where $b_j = \big(\partial r/\partial z_n\big)^{-1}\partial r/\partial z_j$, form a basis of $\mathbb{C}T^{(1,0)}(U)$ and satisfy $L_j r \equiv 0$ for $1 \leqslant j \leqslant n-1$ and for each $z \in U$, all eigenvalues of $\partial\bar{\partial} r(z)(L_i, \bar{L}_j)_{2 \leqslant i,j \leqslant n-1}$ are positive.

The main objective of this reduction procedure is to obtain a certain normal form near $\zeta \in U$ in which there are no harmonic monomials of weight less than one, when weights are taken with respect to the inverses of the multitype at $\zeta \in U$. Recall that Levi corank one hypersurfaces are h-extendible/semiregular i.e., their Catlin multitype and D' Angelo multitype agree at every point.

We digress a little here, to recall and introduce the term 'weakly spherical' used in the introduction. First, we recall the notion of weak sphericity of Barletta – Bedford from [7]: a smooth pseudoconvex hypersurface $M \subset \mathbb{C}^2$ of finite type $2m$ at $p \in M$ can after a change of coordinates centered at $p = 0$, be defined by a function of the form

$$2\Re z_2 + P_{2m}(z_1, \bar{z}_1) + \Im z_2 \sum_{l=1}^{k} Q_l(z_1) + \sigma_{2m+1}(z_1) + \sigma_2(\Im z_2) + (\Im z_2)\sigma_{m+1}(z_1)$$

Here $P_{2m}(z_1, \bar{z}_1)$ is a non-zero homogeneous subharmonic polynomial of degree $2m$ without harmonic terms, the Q_l's are homogeneous polynomials of degree l and the σ_j's vanish to order j in z_1 or $\Im z_2$. To put it succinctly, the lowest weight component in the weighted homogeneous expansion of the defining function with respect to the weight $(1, 1/2m)$ given by the type, is of weight one and of the form $2\Re z_2 + P_{2m}(z_1, \bar{z}_1)$. Now according to [7], M is weakly spherical at p if $P_{2m}(z_1, \bar{z}_1) = |z_1|^{2m} = |z_1^m|^2$ – note that a 'squared norm of a polynomial' of weight $1/2$ in the single variable z_1, is necessarily of the form $c|z_1|^{2m}$ for some positive constant c. We may extend this notion to higher dimensions for h-extendible/semiregular hypersurfaces as follows: call a smooth pseudoconvex hypersurface $M \subset \mathbb{C}^n$ which is h-extendible at a point $p \in M$, to be *weakly spherical* at p if there is a change of coordinates that maps p to the origin, in which the lowest weight component of the weighted homogeneous expansion of the defining function of M about $p \in M$ which we may assume to be the origin, performed with respect to the weights given by the inverse of the multitype $(m_n, \ldots, m_1)$ of M at $p = 0$, is of the form

$$2\Re z_n + |P_1('z)|^2 + \ldots + |P_{n-1}('z)|^2.$$

This when expanded is of the form

$$2\Re z_n + c_1|z_1|^{m_n} + c_2|z_2|^{m_{n-1}} + \ldots + c_{n-1}|z_{n-1}|^{m_2} + \text{mixed terms} \quad (9.2)$$

where the phrase 'mixed terms' denotes a sum of weight 1 monomials annihilated by at-least one of the natural quotient maps $\mathbb{C}['z, '\overline{z}] \to \mathbb{C}['z, '\overline{z}]/(z_j \overline{z}_k)$ for $1 \leqslant j, k \leqslant n-1$, $j \neq k$. It was shown in [5] that if a h-extendible hypersurface admits a Lipschitz CR mapping into a strongly pseudoconvex hypersurface, then the source must be weakly spherical i.e., the fibres of the mapping must consist only of weakly spherical points. Stated differently, the local model for the source of our CR-mapping must be the pull-back of a piece of the sphere $\partial\mathbb{B}^n$ via a proper weighted homogeneous polynomial endomorphism of $\mathbb{C}^n$.

Now, the mixed terms in (9.2) involving the z_α's for $2 \leqslant \alpha \leqslant n-1$, must be of the form $z_\alpha \overline{z}_\beta$ where $2 \leqslant \beta \leqslant n-1$ with $\beta \neq \alpha$, since $\text{wt}(z_\alpha) = 1/2$. An application of the spectral theorem, removes the occurrence of such terms with $\alpha \neq \beta$. The remaining mixed terms must be of the form $z_\alpha \overline{z}_1^j$ where $2 \leqslant \alpha \leqslant n-1$ and $1 \leqslant j \leqslant m$ and constitutes the polynomial

$$\sum_{\alpha=2}^{n-1} \sum_{j=1}^{m} \Re(b_j^\alpha \overline{z}_1^j z_\alpha),$$

say. Then an application of the change of variables given by

$$\begin{aligned} w_1 &= z_1, \ w_n = z_n \\ w_\alpha &= z_\alpha - P(z_1) \ \text{ for } 2 \leqslant \alpha \leqslant n-1 \end{aligned}$$

where $P(z_1) = \sum_{j=1}^{m} b_j^\alpha z_1^j$, removes the occurrence of such terms as well and the transformed defining function when expanded about $p = 0$ reads

$$2\Re z_n + |z_1|^{2m} + |z_2|^2 + \ldots + |z_{n-1}|^2 + R(z, \overline{z})$$

with the error function $R(z, \overline{z}) \to 0$ faster than atleast one of the monomials of weight 1. The domain D_1 in the hypothesis of theorem 1.7 has its defining function about a boundary point p in the above form; we shall say that p is *weakly* spherical if the integer $m > 1$.

Getting back from this digression about weakly spherical Levi corank one hypersurfaces, to the afore-mentioned reduction procedure for arbitrary Levi corank one hypersurfaces, we recall that it can be split up into five simpler steps – for $\zeta \in U$, the map $\Phi^\zeta = \phi_5 \circ \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ where each ϕ_j is described below.

The first step is to normalize the linear part of the Taylor series as in (1.2). Recall that this was done via the affine map ϕ_1 given by

$$\begin{aligned} \phi_1(z_1, \ldots, z_n) &= \Big(z_1 - \zeta_1, \ldots, z_{n-1} - \zeta_{n-1}, \big(z_n - \zeta_n - \sum_{j=1}^{n-1} b_j^\zeta (z_j - \zeta_j)\big)(b_n^\zeta)^{-1}\Big) \\ &= \Big(z_1 - \zeta_1, \ldots, z_{n-1} - \zeta_{n-1}, \langle \nu(\zeta), z - \zeta \rangle\Big) \end{aligned}$$

where the coefficients $b_n^\zeta = \big(\partial r / \partial z_n(\zeta)\big)^{-1}$ and b_j^ζ are clearly smooth functions of ζ on U. Therefore, ϕ_1 translates ζ to the origin and

$$r(\phi_1^{-1}(z)) = r(\zeta) + 2\Re z_n + \text{terms of higher order}.$$

where the constant term disappears when $\zeta \in \Sigma$.

Now, since the Levi form restricted to the subspace

$$L_* = \mathrm{span}_{\mathbb{C}^n}\langle L_2, \ldots, L_{n-1}\rangle$$

of $T^{(1,0)}_\zeta(\partial\Omega)$ is positive definite, we may diagonalize it via a unitary transform ϕ_2 and a dilation ϕ_3 will then ensure that the Hermitian – quadratic part involving only $z_2, z_3, \ldots, z_{n-2}$ in the Taylor expansion of r is $|z_2|^2 + |z_3|^2 + \ldots + |z_{n-2}|^2$. The entries of the matrix that represents the composite of the last two linear transformations are smooth functions of ζ and in the new coordinates still denoted by $z_1, \ldots z_n$, the defining function is in the form

$$\begin{aligned} r(z) = r(\zeta) + 2\Re z_n + \sum_{\alpha=2}^{n-1}\sum_{j=1}^{m} 2\Re\big((a_j^\alpha z_1^j + b_j^\alpha \bar{z}_1^j) z_\alpha\big) + 2\Re\sum_{\alpha=2}^{n-1} c_\alpha z_\alpha^2 \\ + \sum_{2\leqslant j+k\leqslant 2m} a_{j,k} z_1^j \bar{z}_1^k + \sum_{\alpha=2}^{n-1}|z_\alpha|^2 + \sum_{\alpha=2}^{n-1}\sum_{\substack{j+k\leqslant m\\ j,k>0}} 2\Re\big(b_{j,k}^\alpha z_1^j \bar{z}_1^k z_\alpha\big) \\ + O(|z_n||z| + |z_*|^2|z| + |z_*||z_1|^{m+1} + |z_1|^{2m+1}) \quad (9.3) \end{aligned}$$

This still does not reduce the quadratic component of the Taylor series as far as we can; more can be done: the pluriharmonic terms of weights up to 1 here i.e., z_α^2 as also $z_1^k, \bar{z}_1^k, z_1^k z_\alpha, \bar{z}_1^k \bar{z}_\alpha$ can all be removed by absorbing them into the normal variable z_n by the following standard change of coordinates ϕ_4 given by

$$\begin{aligned} z_j &= t_j \ \ (1 \leqslant j \leqslant n-1), \\ z_n &= t_n - \hat{Q}_1(t_1, \ldots, t_{n-1}) \end{aligned}$$

where

$$\hat{Q}_1(t_1, \ldots, t_{n-1}) = \sum_{k=2}^{2m} a_{k0} t_1^k - \sum_{\alpha=2}^{n-1}\sum_{k=1}^{m} a_k^\alpha t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} c_\alpha t_\alpha^2$$

with coefficients that are smooth functions of ζ. Even with this, we may still have quadratic (degree 2) terms involving z_n remaining in the expansion. We do not care to remove them, as they are of weight > 1.

The final step removes all other harmonic monomials of weight up to one remaining in (9.3), rewritten in the t-coordinates, which are of the form $\bar{t}_1^j t_\alpha$ by applying the transform ϕ_5 given by

$$\begin{aligned} t_1 &= w_1, \ t_n = w_n, \\ t_\alpha &= w_\alpha - Q_2^\alpha(w_1) \ \ (2 \leqslant \alpha \leqslant n-1) \end{aligned}$$

where

$$Q_2^\alpha(w_1) = \sum_{k=1}^{m} \overline{b_k^\alpha} w_1^k$$

with coefficients smooth in ζ, as before (since all these coefficients are simply the derivatives of some order of the smooth defining function r evaluated at ζ).

We then have that the composite Φ^ζ of these various simplifying maps is as given in (1.4) and the normal form for the Taylor expansion is as given in (1.3). Further, Φ^ζ belongs to the group $\mathcal{E}_L$ for each fixed $\zeta \in U$. Let Q_ζ denote the biholomorphically distorted polydisc $Q(\zeta, \epsilon(\zeta))$, which is also the ball of radius $\epsilon(\zeta)$ about ζ in the pseudo-distance d defined at (1.6). Then in particular we now have that, the collection $\{Q_\zeta, \Phi^\zeta\}$, where ζ varies over points in the *one sided* tubular neighbourhood U^-, the pseudocovex side of Σ (in our particular setting it is $U \cap D$) forms an atlas of special charts, giving U^- the structure of a $\mathcal{E}_L$-manifold i.e., the associated transition maps lie in the complex Lie group $\mathcal{E}_L$, the group of weight preserving, elementary-like, weight one polynomial automorphisms described above. Alternately, we may cover the tubular neighbourhood U with the atlas of charts $\{Q(\zeta, \delta_e), \Phi^\zeta\}$ which gives U itself the structure of an $\mathcal{E}_L$ manifold. It is the former atlas which is of interest to us.

We need to compute the inverse $\Psi_\zeta = (\Phi^\zeta)^{-1}$ in the last subsection of Section 2. Let

$$(w_1, \ldots, w_n) = \Phi^\zeta(z) = \Big(z_1 - \zeta_1, G_\zeta(\tilde{z} - \tilde{\zeta}) - Q_2(z_1 - \zeta_1), \langle \nu(\zeta), z - \zeta\rangle - Q_1('z - '\zeta)\Big) \tag{9.4}$$

where $Q_2 : \mathbb{C} \to \mathbb{C}^{n-2}$ is a polynomial map whose components are the polynomials Q_2^α as above. Now, we find out the components of Ψ_ζ, the first component of which is

$$z_1 = w_1 + \zeta_1 \tag{9.5}$$

Next, $\tilde{w} = G_\zeta(\tilde{z} - \tilde{\zeta}) - Q_2(z_1 - \zeta_1)$ i.e., $G_\zeta(\tilde{z} - \tilde{\zeta}) = \tilde{w} - Q_2(w_1)$ so that

$$\tilde{z} = H_\zeta\big(\tilde{w} + Q_2(w_1)\big) + \tilde{\zeta} \tag{9.6}$$

and finally

$$\begin{aligned} w_n &= \langle \nu(\zeta), z - \zeta\rangle - Q_1('z - '\zeta) \\ &= \partial r/\partial z_n(\zeta)(z_n - \zeta_n) + \sum_{j=1}^{n-1} \partial r/\partial z_j(\zeta)(z_j - \zeta_j) - Q_1('z - '\zeta) \end{aligned}$$

Now note that $\tilde{z}-\tilde{\zeta} = H_\zeta\big(\tilde{w}+Q_2(w_1)\big)$. So $Q_1(z_1-\zeta_1, \tilde{z}-\tilde{\zeta}) = Q_1\big(w_1, H_\zeta(\tilde{w}+ Q_2(w_1))\big)$ and subsequently,

$$\begin{aligned} w_n &= \partial r/\partial z_n(\zeta)\Big(z_n - \zeta_n - \sum_{\alpha=2}^{n-2} b_j^\zeta(z_j-\zeta_j) - b_1^\zeta(z_1-\zeta_1)\Big) \\ &\qquad\qquad\qquad\qquad - Q_1\big(w_1, H_\zeta(\tilde{w}+Q_2(w_1))\big) \\ &= (b_n^\zeta)^{-1}\Big(z_n-\zeta_n - \langle\tilde{b}^\zeta, \tilde{z}-\tilde{\zeta}\rangle - b_1^\zeta w_1\Big) - Q_1\big(w_1, H_\zeta(\tilde{w}+Q_2(w_1))\big) \end{aligned}$$

and subsequently,

$$b_n^\zeta w_n = \Big(z_n-\zeta_n - \langle\tilde{b}^\zeta, \tilde{z}-\tilde{\zeta}\rangle - b_1^\zeta w_1\Big) - b_n^\zeta Q_1\big(w_1, H_\zeta(\tilde{w}+Q_2(w_1))\big)$$

giving finally that the last component of $z = \Psi_\zeta(w)$ is of the form

$$z_n = b_n^\zeta w_n + \zeta_n + \langle\tilde{b}^\zeta, \tilde{z}-\tilde{\zeta}\rangle + b_1^\zeta w_1 + b_n^\zeta Q_1\big(w_1, H_\zeta(\tilde{w}+Q_2(w_1))\big). \quad (9.7)$$

which we shall also write more shortly as

$$z_n = b_n^\zeta w_n + b_n^\zeta \tilde{Q}_1('w) + \zeta_n$$

where for some slight convenience in the section where it is used, we take $\tilde{Q}_1$ to be of the form

$$\tilde{Q}_1('w) = (b_n^\zeta)^{-1}\Big(\langle\tilde{b}^\zeta, H_\zeta\big(\tilde{w}+Q_2(w_1)\big)\rangle + b_1^\zeta w_1\Big) + Q_1\Big(w_1, H_\zeta\big(\tilde{w}+Q_2(w_1)\big)\Big).$$

with Q_1 and Q_2 are the same very polynomials occurring in the expression for Φ^ζ as in (9.4). Now, altogether, equations (9.5), (9.6) and (9.7) give the expressions for the various components that constitute the mapping $\Psi_\zeta(w)$. A straightforward computation shows that the derivative of the map $\Phi^\zeta(z)$ in standard co-ordinates, is represented by the matrix

$$D\Phi^\zeta(z) = \begin{pmatrix} 1 & 0 \ \cdots & 0 & 0 \\ -\frac{\partial Q_2^2}{\partial z_1}(z_1-\zeta_1) & & & 0 \\ \vdots & G_\zeta & & \vdots \\ -\frac{\partial Q_2^{n-1}}{\partial z_1}(z_1-\zeta_1) & & & 0 \\ \frac{\partial r}{\partial z_1}(\zeta) - \frac{\partial Q_1}{\partial z_1}('z-'\zeta) & \cdots & \frac{\partial r}{\partial z_{n-1}}(\zeta) - \frac{\partial Q_1}{\partial z_1}('z-'\zeta) & \frac{\partial r}{\partial z_n}(\zeta) \end{pmatrix} \quad (9.8)$$

so that in particular, the mapping occurring in the definition of the M-metric namely,

$$D\Phi^z(z)(X) = \Big(X_1, G_z(\tilde{X}), \sum_{j=1}^{n} \partial r/\partial z_j(z) X_j\Big) = \Big(X_1, G_z(\tilde{X}), \langle \nu(z), X\rangle\Big)$$

which of course is linear for each fixed z but also is weighted homogeneous in the variables z_1 through z_{n-1}, with respect to the weights that we have assigned to the variables $z_1, \ldots, z_n$, as well. The above expression is needed in section 3. We notice in passing that $D\Phi^\zeta$ for each fixed $\zeta \in U$ belongs to the linear Lie group $\mathcal{E}L_n = \mathcal{E}_L \cap GL_n(\mathbb{C})$ and gives the tangent bundle of the $\mathcal{E}_L$-manifold mentioned above, the structure of a fibre bundle with structure group $\mathcal{E}L_n$.

Next, we record the derivative of the inverse map Ψ_ζ needed in the last sub-section of section 2.

$$D\Psi_\zeta(w) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \langle (H_\zeta)_{R_2}, \frac{\partial Q_2}{\partial w_1}(w)\rangle & & & & 0 \\ \vdots & & H_\zeta & & \vdots \\ \langle (H_\zeta)_{R_{n-1}}, \frac{\partial Q_2}{\partial w_1}(w)\rangle & & & & 0 \\ b_n^\zeta \partial\tilde{Q}_1/\partial w_1('w) & b_n^\zeta \partial\tilde{Q}_1/\partial w_2('w) & \ldots & b_n^\zeta \partial\tilde{Q}_1/\partial w_{n-1}('w) & b_n^\zeta \end{pmatrix} \tag{9.9}$$

where we index the rows and columns of the square matrix $H_\zeta = G_\zeta^{-1}$ of order $n-2$ by the integers $2, \ldots, n-1$ and denote its rows herein by $(H_\zeta)_{R_\alpha}$.

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