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Introduction

Over the last fifteen years, several long-standing open problems in 3-dimensional topology have been solved. The most celebrated achievement was the proof of Poincaré's and Thurston's geometrization conjectures by G. Perelman in 2003 [14], [15], [16]. The positive solution to these two conjectures ensures that each closed 3-manifold can be decomposed into a finite number of canonical "pieces" that admit one of Thurston's eight 3-dimensional geometric structures.

Among these structures, hyperbolic geometry is the richest. As such, it is still a source of a large variety of open problems, of which some have been solved just recently remarkable advances have been accomplished recently. For instance, we now know that every closed hyperbolic manifold admits a finite-sheeted cover that fibres over the circle. This fact, known as Thurston's virtually fibering conjecture, was shown to be true by D. Wise [18], [19], [20] under the assumption that the manifold is Haken, that is, contains an essential surface. The proof was then completed when I. Agol, building on work of J. Kahn and V. Markovic [11], proved Waldhausen's virtually Haken conjecture, saying that every closed hyperbolic 3-manifold admits a finite-sheeted cover which is Haken [2].

As for the structure of complete (non closed) hyperbolic 3-manifolds with finitely generated fundamental group, the tameness theorem, originally conjectured by A. Marden in the 1970s, was finally established independently by I. Agol [1] and D. Calegari–D. Gabai [9]. It states that they are homeomorphic to the interior of compact 3-manifolds.

In many cases, the proofs of these results, that greatly improved our understanding of the structure of 3-dimensional manifolds, relied on new tools and ideas borrowed from other fields. For instance, Perelman's result was established building on work of R. Hamilton about the Ricci flow and relies heavily on analytical methods. Agol and Wise's main ingredient to prove the virtually fibering conjecture is the use of cubulations, a notion that stems from geometric group theory, and, more precisely, CAT(0) spaces and groups.

The interplay between low-dimensional topology and other fields of mathematics has been fruitful not only because it had led to proofs of open problems, but also because it has provided new perspectives on the subject and with them the quest for further insight.

It is worth mentioning at least some of the questions that arise from the interaction of 3-dimensional topology with other subjects.

- How special are the fundamental groups of 3-manifolds?

  It is well-known that any finitely presented group can be realised as the fundamental group of a 4-manifold. On the other hand, 3-manifold groups seem to be somehow special, but to what extent? Is it possible to characterise them? For instance, are there Poincaré duality groups in dimension 3 that are not 3-manifold groups? Is their large scale geometry sufficient to classify them? In this vein, one of the paramount open problems in the field, is Cannon's conjecture that a Gromov-hyperbolic group having boundary homeomorphic to the 2-sphere admits a quotient by a finite group that contains a finite index Kleinian subgroup.

- Given a knot or link, how is the geometry of its exterior reflected in its quantum and combinatorial invariants?

  For any knot, Kashaev [12] conjectures that the hyperbolic volume of its complement is determined by the values of its Jones polynomial at roots of unity. Thus far, most connections between these different realms remain very elusive. For instance, is it possible to provide geometric characterisations of alternating knots?

- Given a 3-manifold, what are the relations between its topological properties, its contact geometry structure, and its Heegaard-Floer homology?

  In this setting, one natural question is to provide a topological characterisation of L-spaces, that is manifolds that have the same Heegaard-Floer homology of a lens space. S. Boyer, C. McA. Gordon, and L. Watson conjectured that an irreducile rational homology sphere is an L-space if and only if its fundamental group is not left-orderable and that happens if and only if the manifold does not admit a taut foliation [7]. This is known to be true for Seifert fibred manifolds.

In 2013, on the occasion of Michel Boileau's 60th birthday, a conference was organised in Toulouse from the 24th to the 28th of June. Under the auspices of CIMI (as part of the thematic month on Topology, Symplectic, and Contact Geometry in Toulouse), IMT, Université P. Sabatier, CNRS, GDR "Tresses", GEAR, Région Midi-Pyrénées, and ANR (projects ETTT, TCGD, and GDS).

Michel Boileau's research interests cover a number of subjects related to 3-dimensional topology: from knot theory to the study of singularities, from properties of Heegaard splittings and 3-manifold groups to geometric structures. He notably contributed to the geometrization program: He wrote a complete and self-contained proof of the orbifold theorem in collaboration with B. Leeb and J. Porti [5] and of the geometrization conjecture in collaboration with L. Bessières, G. Besson, S. Maillot, and J. Porti [3]. The conference provided the opportunity to review the state of the art on all these subjects and discuss the future challenges.

This special volume collects eleven papers by some of the plenary lecturers of the conference and their coauthors. The contents of the papers span a wide range of diverse topics.

The first step towards the study and classification of manifolds is to provide ways to construct them and to encode them, possibly via a set of combinatorial data. It is then necessary to establish when different constructions result in homeomorphic manifolds. It is well-known that every closed 3-manifold can be obtained by gluing together along their boundaries two handlebodies of the same genus. A decomposition of a 3-manifolds into two handlebodies is called a *Heegaard spliting*. It is not difficult to see that any manifold admits different Heegaard splittings. A classical result of Reidemeister and Singer says that any two Heegaard splittings of a 3-manifold can be obtained from one another via a finite sequence of stabilisations and destabilisations. In his contribution to this volume, **L. Siebenmann** gives a self-contained proof of the Reidemeister-Singer theorem. His proof relies on the operation of "linear bisection".

Any Heegaard splitting of a manifold also provides a natural presentation of its fundamental group with a number of generators equal to the genus of the handlebodies of the splitting. It is thus clear that the minimum genus of a Heegaard splitting of a manifold, that is the *Heegaard genus* of the manifold, gives an upper bound on the rank of its fundamental group. However such an upper bound is not sharp in general. Boileau and Zieschang in [6] exhibited the first example, a Seifert fibered manifold, for which the bound is not sharp. Hyperbolic manifolds with the same property were constructed by T. Li in [13]. Although the discrepancy between Heegaard genus and rank can be arbitrary large as a difference, it is still not known whether it can even be arbitrarly large as a ratio. In his paper, **R. Weidmann** shows that for graph manifolds this ratio is always bounded above by a constant. This result is obtained as a consequence of a formula for the rank of acylindrical graphs of groups in terms of the ranks of their vertex and edge groups.

The main result in the article by **I. Agol and M. Freedman** shows that Heegaard splittings appear in a natural way even when 3-manifolds are seen as submanifolds of simply connected 4-dimensional manifolds. Indeed, under weak assumptions, the authors show that a smooth embedding of a

3-manifold into $\mathbf{S}^3 \times \mathbb{R}$ or $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be isotoped so that all generic levels are Heegaard surfaces. Understanding whether a given 3-manifold admits a smooth embedding into a given 4-manifold and, if it does, determining whether such an embedding is somehow unique, are still largely open questions that include, as particular cases, the smooth 4-dimensional Poincaré conjecture and the Schoenflies problem. The contents of Agol and Freedman's paper, which includes a condition for a manifold to stably embed in 4-space, can be seen as possible approaches towards a solution of these open problems.

If Heegaard splittings provide one natural topological presentation of 3-manifolds, triangulations are possibly the most combinatorial way to encode them. Since triangulations come equipped with a notion of complexity (i.e. the number of tetrahedra), they can be used to write algorithms that enumerate manifolds systematically. Triangulations presenting special features may also imply that the manifolds that admit them enjoy special properties. With this viewpoint in mind, **C. Hodgson, H. Rubinstein, H. Segerman, and S. Tillmann** consider *essential* and *strongly essential triangulations* of 3-manifolds that are either closed or compact with non empty boundary. A triangulation with a single vertex of a manifold is essential if no edge loop is null-homotopic, and it is strongly essential if, in addition, no two edge loops are homotopic keeping the vertex fixed. The authors provide ways to construct triangulations of this type, on one hand, and algorithms to decide whether a given triangulation is (strongly) essential on the other hand.

P. Shalen's contribution to the volume is also somehow concerned with the problem of enumerating 3-manifolds and orbifolds, namely arithmetic ones. A result of Borel ensures that the set of volumes of arithmetic orbifolds is discrete. It should thus be possible to enumerate all arithmetic orbifolds whose volume is bounded by a given constant. It turns out that the enumeration requires the ability to bound the rank of certain elementary abelian 2-groups that are quotients of the orbifold fundamental group. The author shows how previous work of his in collaboration with M. Culler allows one to give bounds on the $\mathbb{Z}/2$-homology rank of manifolds assuming certain volume bounds. Clearly such bounds are also bounds for the rank of every quotient of the orbifold group that is an elementary abelian 2-group.

Yet another way of presenting 3-manifolds is via Dehn surgery on links in the 3-sphere. According to Thurston's hyperbolic Dehn surgery theorem, manifolds resulting from Dehn surgery on hyperbolic knots have the nice property to be hyperbolic except for finitely many of them. There is a vast literature dedicated to the study of these exceptional, i.e. non hyperbolic, surgeries. A celebrated result is Culler, Gordon, Lucke, and Shalen's cyclic surgery theorem [10] which gives conditions that must be fulfilled by surgery slopes in order to give rise to a manifold with cyclic fundamental group. In a similar vein, a conjecture of Boyer and Zhang [8] states that if a p/q-surgery on a hyperbolic knot in the 3-sphere yields a manifold with finite fundamental group, then $q = 0, 1$. It is known that if p/q is a slope giving rise to a manifold with finite fundamental group, then $|q| \leqslant 2$ and the equality case imposes further restrictions on the manifold.

In their paper **E. Li and Y. Ni** prove that if a $p/2$-surgery on K contradicts Boyer and Zhang's conjecture then the knot Floer homology of K is prescribed, p can only assume a finite number of explicit values, and the resulting manifolds can be determined.

As remarked above, exteriors of hyperbolic knots represent a well-behaved and much studied class of 3-manifolds. Nonetheless, several natural questions about this class remain unanswered. One of these is a conjecture of Reid and Walsh [17] : Is it true that a commensurability class of cusped hyperbolic manifolds can contain at most three knot exteriors? It was shown by **M. Boileau, S. Boyer, R. Cebanu, and G. Walsh** [4] that the only possible counterexamples to the conjecture must be knots with *hidden symmetries*. So far, only three knots with hidden symmetries are known, and conjecturally they are the only ones.

In order to corroborate this latter conjecture, the same four authors consider covers of reflection orbifolds. Indeed, if a knot exterior covers a reflection orbifold, then it has hidden symmetries. A detailed analysis of covers of small reflection orbifolds is carried out in their paper for this volume, leading to a verification that no counterexample to the conjecture arises in this way. This implies in particular that the conjecture is satisfied by any knot having the property that every closed essential surface in its exterior contains an accidental parabolic.

Properties of specific knots in higher dimensions are the subject of **S. Friedl and P. Orson**'s paper. They give a new proof of a result of Zeeman about k-twist spun knots stating that the exteriors of such knots fibre over the cirle. A consequence of this fact is that any knot of the form $K\sharp - K$, that is which decomposes as the connected sum of a knot and of its opposite, is doubly slice.

As mentioned before, every closed 3-manifold has a canonical decomposition into pieces that admit a geometric structure, modelled on one of Thurston's eight geometries of dimension 3. In dimension 2, the situation is simpler and each closed surface admits one among three possible geometries: Euclidean, spherical, and hyperbolic. Moreover, in dimension 2, every surface admits a *projective structure*, that is every surface is the quotient of some open subset of the projective plane. One might wonder whether a similar result could hold in dimension 3, notably if all closed 3-manifolds could be modelled on one single geometry. Of course, keeping in mind what happens in dimension 2, one could ask whether every 3-manifold admits a (real) projective structure. In their paper, **D. Cooper and W. Goldman** give the first known example of a connected 3-manifold that cannot admit a real projective structure; it is the connected sum of two copies of real projective 3-space. Although this fact can be deduced from Y. Benoist classification of manifolds admitting a real projective structure with abelian holonomy, the authors furnish here a direct proof.

F. Guéritaud's result in his contribution to this volume, can also be seen as a study of geometric structures but in dimension 2. The author is concerned with understanding the effect of infinitesimal deformations of some hyperbolic metric on a given surface perhaps with singularities or cusps, and more specifically how the lengths of geodesics vary under the deformation. The case considered in the paper is that of a torus with one puncture. The author determines which deformations either increase or shrink the lengths of all geodesics.

Left-orderable groups enjoy a variety of nice properties. It is still quite unclear how left-orderability of a 3-manifold group is reflected in the topological properties of the manifold and vice-versa. In their paper, **D. Calegari and D. Rolfsen** provide a class of examples of left-orderable groups by showing that if G is the group of orientation-preserving PL-homeomorphisms of a connected n-manifold M fixing point-wise a codimenson-1 closed and non-empty submanifold K (e.g. its boundary) then G is left-orderable (in fact locally indicable). If all the above hypotheses are fulfilled except that K is now supposed to have codimension 2, then the authors prove that G satisfies a weaker property, namely it is circularly orderable. The authors concentrate their attention on the special case in which M is the n-dimensional cube and K its boundary. These groups are of special interest because every right-angled Artin groups embeds in one of them. For these groups they observe that their cyclic subgroups are at most exponentially distorted. Analogous results are obtained for the groups of C^1-diffeomorphisms, while it is observed that the case of homeomorphisms is still open. Note that in the case of 1-dimensional cubes the group of PL-homeomorphism is biorderable, while the group of homeomorphisms cannot be so, since it contains every left-orderable group. Finally, as a corollary, it is shown that the group of C^1-diffeomorphisms of a hyperbolic knot which are orientation-preserving and isotopic to the identity is left-orderable.

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