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A rank formula for acylindrical splittings

Richard Weidmann[(1)]

Dedicated to Michel Boileau on the occasion of his 60th birthday

Résumé. — Une formule de rang pour les scindements acylindriques des groupes est demontrée. On en déduit que le genre de Heegaard d'une varieté graphée fermée est borné par une fonction linéaire en le rang du groupe fondamental.

Abstract. — We prove a rank formula for arbitrary acylindrical graphs of groups and deduce that the Heegaard genus of a closed graph manifold can be bounded by a linear function in the rank of its fundamental group.

Introduction

Grushko's Theorem states that the rank of groups is additive under free products, i.e. that rank $A * B = \text{rank } A + \text{rank } B$. The non-trivial claim is that rank $A * B \geqslant \text{rank } A + \text{rank } B$. In the case of amalgamated products $G = A *_C B$ with finite amalgam a similar lower bound for rank G can be given in terms of rank A, rank B and the order of C [25]. For arbitrary splittings this is no longer true.

It has first been observed by G. Rosenberger [16] that the naive rank formula rank $G \geqslant \text{rank } A + \text{rank } B - \text{rank } C$ does not hold for arbitrary amalgamated products $G = A *_C B$, in fact Rosenberger cites a class of Fuchsian groups as counterexamples. In [11] examples of Coxeter-groups where exhibited where rank $G_1 *_{\mathbb{Z}_2} G_2 = \text{rank } G_1 = \text{rank } G_2 = n$ with arbitrary n. In [23] examples of groups $G_n = A_n *_C B_n$ are constructed such that rank $A_n \geqslant n$, rank $B_n \geqslant n$ and rank $C = \text{rank } G_n = 2$.

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These examples clearly show that no non-trivial analogue of Grushko's Theorem holds for arbitrary splittings. If the splitting is k-acylindrical however, this situation changes. This has been shown in [24] for 1-acylindrical amalgamated products where it was also claimed that a similar result hold for arbitrary splittings. In this note we establish such a generalization to arbitrary k-acylindrical splittings. The combinatorics however turn out to be significantly more involved than those in [24]. The following is an immediate consequence of our main result, see Section 2 for more details.

Theorem 0.1. — *Let $\mathbb{A}$ be a k-acylindrical minimal graph of groups without trivial edge groups and finitely generated fundamental group. Then*

$$rank\,\pi_1(\mathbb{A}) \geqslant \frac{1}{2k+1}\left(\sum_{v\in VA}\text{rank }A_v - \sum_{e\in EA}\text{rank }A_e + e(A) + b(A) + 1 + 3k - \lfloor k/2\rfloor\right).$$

Here A is the graph underlying $\mathbb{A}$, $b(A)$ its Betti number, $e(A)$ the number of edge pairs. The vertex groups are the A_v and the edge groups the A_e. Moreover the rank of each edge group is counted twice as $A_e = A_{e^{-1}}$ for all $e \in EA$.

Answering a question of Waldhausen, Boileau and Zieschang [4] have shown that the rank of the fundamental group $r(M)$ of a closed 3-manifold M can be smaller than its Heegaard genus $g(M)$, this class of examples was extended in [27]. Since then it has been shown that the difference between $g(M)$ and $r(M)$ can be arbitrarily large, see [19] and [13]. It is however still unknown whether there exists a uniform positive lower bound of the quotient $r(M)/g(M)$. The following implies that such a bound exists for graph manifolds, it is an almost immediate consequence of Theorem 0.1.

Theorem 0.2. — *Let M be a closed orientable graph manifold. Then*

$$g(M) \leqslant 28 \cdot r(M).$$

Similar arguments show that the claim of Theorem 0.2 holds for arbitrary closed 3-manifolds provided is holds for closed hyperbolic 3-manifolds.

1. Acylindrical splittings

In this section we recall the definition of acylindrical splittings and establish some basic properties of such actions. We assume familiarity with the notation of Section 1 of [26].

Following Sela [20] a simplicial G-tree is called k-acylindrical if the pointwise stabilizer of any segment of length $k+1$ is trivial. A splitting is called k-acylindrical if the associated Bass-Serre tree is k-acylindrical. Note that a graph of groups $\mathbb{A}$ is k acylindrical iff there exists no vertices $v, v' \in VA$, elements $g_v \in A_v \backslash \{1\}$ and $g_{v'} \in A_{v'} \backslash \{1\}$ and a reduced $\mathbb{A}$-path $p = a_0, e_1, \ldots, e_{k+1}, a_{k+1}$ from v to v' such that $[g_v] = [p_v][g_{v'}][p_v^{-1}]$.

Acylindrical actions occur naturally in geometric group theory and low-dimensional topology, in particular the various JSJ-decompositions (see [9, 8, 15, 5, 6]) are usually 2-acylindrical or can be made 2-acylindrical. In [20] it is shown that the complexity of a reduced minimal k-acylindrical splitting of a group G can be bounded in terms of G and k, in [23] it is shown that this bound can be chosen to be $2k(\text{rank } G - 1) + 1$.

Recall that for any $r \in \mathbb{R}$ the largest integer $z \in \mathbb{Z}$ such that $z \leqslant r$ is denoted by $\lfloor r \rfloor$. We establish two simple lemmas.

LEMMA 1.1. — *Suppose that T is a k-acylindrical G-tree and that a vertex v of T is fixed by a non-trivial power g^n of some element $g \in G$. Then there exists a vertex w of T such that $d(v, w) \leqslant \lfloor k/2 \rfloor$ and that $gw = w$.*

Proof. Choose a vertex w of T such that $gw = w$ and that $d(v, w)$ is minimal. Such a vertex w clearly exists as g^n and therefore also g is elliptic. Note that the minimality assumption implies that $[v, w] \cup [w, gv]$ is a segment of length $2d(v, w)$. Clearly g^n fixes the segment $[v, w]$ and therefore $g^n = gg^n g^{-1}$ also fixes $g[w, v] = [gw, gv] = [w, gv]$. It follows that g^n fixes the segment $[v, w] \cup [w, gv]$. The acylindricity assumption therefore implies that $k \geqslant d(v, w) + d(w, gv) = 2d(v, w)$, the claim follows. □

LEMMA 1.2. — *Suppose that T is a k-acylindrical G-tree and that two vertices v and w are both fixed by non-trivial powers of g.*

Then there exists a vertex y fixed by g such that

$$d(v, y) + d(y, w) \leqslant k + [k/2]$$

and $d(v, y) + d(y, w) \leqslant k$ if G is torsion-free. In particular $d(v, w) \leqslant k + [k/2]$ and $d(v, w) \leqslant k$ if G is torsion-free.

Proof. — Suppose first that g fixes no vertex of the segment $[v, w]$. Choose vertices y and y' that are fixed under the action of g and that are in minimal distance of v and w, respectively. As the set of all points fixed by g is a subtree of T that is disjoint from $[v, w]$ it follows that $y = y'$. It follows from Lemma 1.1 that $d(y, v) \leqslant \lfloor k/2 \rfloor$ and $d(y, w) \leqslant \lfloor k/2 \rfloor$ which clearly implies the assertion.

We can therefore assume that g fixes a vertex of $[v, w]$. Thus it suffices to show the second claim of the lemma, i.e that $d(v, w) \leqslant k + \lfloor k/2 \rfloor$ and $d(v, w) \leqslant k$ if G is torsion-free.

Choose $n, m \in \mathbb{Z}$ such that $g^n v = v$, $g^m w = w$ where $g^n \neq 1$ and $g^m \neq 1$. We can assume that $d(v,w) \geqslant k$ as there is nothing to show otherwise. Choose vertices $x, y \in [v,w]$ such that $d(v,x) = d(w,y) = \lfloor k/2 \rfloor$. If follows as in the proof of Lemma 1.1 that g fixes x and y and therefore also $[x,y]$ point-wise. Thus g^n fixes the segment $[v,y]$ which implies that $d(v,y) \leqslant k$. It follows that $d(v,w) \leqslant d(v,y) + d(y,w) \leqslant k + \lfloor k/2 \rfloor$.

If G is torsion-free then $g^{mn} \neq 1$, $g^{mn} v = v$ and $g^{mn} w = w$. Thus g^{mn} fixes the segment $[v,w]$. The acylindricity assumption then implies that $d(v,w) \leqslant k$. □

We will record a useful consequence of the above lemmas. In the following we assume familiarity with the theory of $\mathbb{A}$-graphs and folds as presented in [26]. Note that an $\mathbb{A}$-graph $\mathcal{B}$ (as introduced in [10]) encodes a morphism from an associated graph of groups $\mathbb{B}$ to $\mathbb{A}$ that is injective on vertex groups, see [1] for a discussion of morphisms.

If $\mathbb{A}$ is a graph of groups and if C is a subgraph of the underlying graph A then we call the graph of groups with underlying graph C whose edge groups, vertex groups and vertex morphisms coincide with those of $\mathbb{A}$ the subgraph of groups corresponding to C and denote it by $\mathbb{C}$. If the Bass-Serre tree of a graph of groups $\mathbb{A}$ is non-trivial, i.e. has no global fixed point, then it contains a unique minimal invariant subtree. In this case $\mathbb{A}$ contains a unique minimal subgraph of groups such that the inclusion map is π_1-injective. We call this subgraph of groups the *core* of $\mathbb{A}$ and write core $\mathbb{A}$.

If the Bass-Serre tree has global fixed points then the global fixed points form an invariant subtree. This subtree injects into A under the quotient map and the corresponding subgraph of groups consists of those vertices and edges for which the inclusion of the vertex, respectively edge, group is an isomorphism. We again call this subgraph of groups the *core* of $\mathbb{A}$ and write core $\mathbb{A}$.

In the second case we say that the core is a *simple core*. Note that A must be a tree if $\mathbb{A}$ has a simple core.

PROPOSITION 1.3. — *Let $\mathbb{A}$ be a k-acylindrical graph of groups and $\mathcal{B}$ be an $\mathbb{A}$-graph (with associated graph of groups $\mathbb{B}$). Suppose that $\mathcal{B}$ admits no move of type IA, IB, IIIA or IIIB and that all vertex groups of $\mathbb{B}$ are non-trivial. Let $\mathbb{B}' :=$ core $\mathbb{B}$. Then the following hold:*

1) $d_H(B, B') \leqslant k$ where d_H denotes the Hausdorff distance in B.

2) If B' is a simple core then $d_H(B, v) \leqslant k$ for any vertex $v \in VB'$.

3) If $\pi_1(\mathbb{B})$ is cyclic with simple core then $d_H(B, B') \leqslant \lfloor k/2 \rfloor$.

4) If $\pi_1(\mathbb{B})$ is cyclic with simple core then for any two vertices $v, w \in B$ there exists a tree $Y \subset B$ with at most $k + \lfloor k/2 \rfloor + 1$ vertices such that $v, w \in Y$ and $Y \cap B' \neq \emptyset$.

5) If $\pi_1(\mathbb{A})$ is torsion-free then Y can be chosen to have at most $k+1$ vertices.

6) If $d(v, w) > 2\lfloor k/2 \rfloor$ in (4) or (5) then $[v, w]$ contains an edge of B'.

Proof. — Let v be a vertex of B and $g \in B_v - 1$. By the definition of the core there exists a vertex $w \in B'$ and $g' \in G_w$ such that $[g] = [p][g'][p^{-1}]$ where $p = [1, e_1, 1, \ldots, 1, e_l, 1]$ is a reduced $\mathbb{B}$-path whose underlying path $q = e_1, \ldots, e_l$ in B is also reduced. To show (1) it clearly suffices to show that $l \leqslant k$.

As q is reduced and as $\mathcal{B}$ admits no move of type IA, IIIA, IB or IIIB it follows that the $\mathbb{A}$-path $\mu(p)$ is reduced. As

$$[g] = [\mu(g)] = [\mu(p)][\mu(g')][\mu(p)^{-1}] = [\mu(p)][g'][\mu(p)^{-1}]$$

it follows from the k-acylindricity of $\mathbb{A}$ that $l \leqslant k$.

The same argument shows (2). Using similar arguments (3) now follows from Lemma 1.1 and (4) and (5) follow from Lemma 1.2. (6) follows from the proof of Lemma 1.2. ☐

2. Formulation of the main result

Let $\mathbb{A}$ be a graph of groups with finitely generated fundamental group. We establish the notion of the *relative rank* of a vertex group A_v, which we denote by relrank A_v.

Let G be a group and $U \subset G$ a subgroup. We say that the corank of U in G is k if $k \in \mathbb{N}_0$ is minimal such that there exist elements $g_1, \ldots, g_k \in G$ with $G = \langle U, g_1, \ldots, g_k \rangle$. If no such k exists then we say that the corank is infinite. We denote the corank of U in G by corank (U, G). Let $v \in VA$ be a vertex. We denote by $E_v \subset EA$ the set of edges $e \in EA$ with $\alpha(e) = v$. We then define

$$\text{relrank } A_v := \min_{g: E_v \to A_v} \big(\text{corank } \big(\langle \bigcup_{e \in E_v} g(e)\alpha_e(A_e)g(e)^{-1} \rangle, A_v\big)\big).$$

Thus the relative rank of a vertex group A_v is the minimal number of elements needed to generated the vertex group in addition to appropriate conjugates of the adjacent edge groups. It is natural to allow conjugation of the edge groups as the boundary monomorphisms are only determined up to conjugation for a given isomorphism class of graphs of groups. It is not difficult to see that the relative rank of all vertex groups is finite as $\pi_1(\mathbb{A})$ is assumed to be finitely generated. We further define

$$\Sigma(\mathbb{A}) := \sum_{v \in VA} \text{relrank } A_v.$$

In the following we denote the Betti number of A by $b(A)$ and the number of edge pairs by $e(A)$. Our main result is the following:

Theorem 2.1. — *Let $\mathbb{A}$ be a k-acylindrical minimal graph of groups without trivial edge groups. Suppose that $\pi_1(\mathbb{A})$ is finitely generated. Then*

$$\mathit{rank}\ \pi_1(\mathbb{A}) \geqslant \frac{1}{2k+1}\left(\Sigma(\mathbb{A}) + e(A) + b(A) + 1 + 3k - \lfloor k/2 \rfloor\right).$$

If $\pi_1(\mathbb{A})$ is torsion-free then

$$\mathit{rank}\ \pi_1(\mathbb{A}) \geqslant \frac{1}{2k+1}\left(\Sigma(\mathbb{A}) + e(A) + b(A) + 1 + 3k + \eta\right).$$

where $\eta = 0$ if k is odd and $\eta = 1$ if k is even.

If all edge groups are finitely generated we clearly get the inequality

$$\text{relrank } A_v \geqslant \text{rank } A_v - \sum_{e \in E_v} \text{rank } A_e$$

and therefore

$$\Sigma(\mathbb{A}) \geqslant \sum_{v \in VA} \text{rank } A_v - \sum_{e \in EA} \text{rank } A_e.$$

Thus we obtain the following immediate consequence of Theorem 2.1. Recall that every edge-group is counted twice as every edge occurs with both orientations.

Corollary 2.2 (Theorem 0.1). — *Let $\mathbb{A}$ be a k-acylindrical minimal graph of groups with finitely generated non-trivial edge groups and finitely generated fundamental group. Then*

$\mathit{rank}\ \pi_1(\mathbb{A}) \geqslant$

$$\frac{1}{2k+1}\left(\sum_{v \in VA} \text{rank } A_v - \sum_{e \in EA} \text{rank } A_e + e(A) + b(A) + 1 + 3k - \lfloor k/2 \rfloor\right).$$

It is clear that the analogous statement can be made for torsion-free groups. The following two corollaries illustrate Corollary 2.2 by spelling out its content in the important case of 1-acylindrical amalgamated products and HNN-extensions. In the case of 1-acylindrical amalgamated products we recover the main theorem of [24].

Recall that a subgroups $U \subset G$ is called *malnormal* in G if $gUg^{-1} \cap U = 1$ for all $g \in G - U$. An amalgamated product $A *_C B$ is 1-acylindrical if and only if C is malnormal in A and B which is the same as saying that C is malnormal in $A *_C B$.

Corollary 2.3. *[24]. — Let $G = A *_C B$ with $C \neq 1$ malnormal in A and B. Then*

$$\mathit{rank}\ G \geqslant \frac{1}{3}(\mathit{rank}\ A + \mathit{rank}\ B - 2\mathit{rank}\ C + 5).$$

In the case of a HNN-extension $\langle H, f | fU_1f^{-1} = U_2\rangle$ the splitting is 1-acylindrical if the associated subgroups U_1 and U_2 are malnormal and conjugacy separated in H. Recall that two subgroups V_1, V_2 of G are conjugacy separated if $gV_1g^{-1} \cap V_2 = 1$ for all $g \in G$.

Corollary 2.4. — *Let $G = \langle H, f | fU_1f^{-1} = U_2\rangle$ where U_1 and U_2 are non-trivial, conjugacy separated and malnormal in H. Then*

$$\mathit{rank}\ G \geqslant \frac{1}{3}(\mathit{rank}\ H - 2\mathit{rank}\ U_1 + 6).$$

3. The proof of the rank formula

One way of proving Grushko's theorem, i.e. to show the non-trivial inequality rank $A*B \geqslant$ rank A+rank B, is to start with a wedge of rank $(A*B)$ circles and fold it onto the graph of groups corresponding to the free product. One then merely needs to observe that for the sequence of graphs of groups one obtains the complexity that is the sum of the ranks of the vertex groups and the Betti number of the underlying graph does not increase. The proof of the main theorem follows a similar strategy however the complexity is less obvious.

The complexity is defined via the free decomposition and free complexity of a graph of groups: For a graph of groups $\mathbb{A}$ we denote the maximal subgraphs of groups that have no trivial edge groups the free factors, the free rank of $\mathbb{A}$ is the Betti number of the graph obtained from A by collapsing all edges with non-trivial edge group. If $\mathbb{A}$ has r free factors and free rank n then we call the pair (r, n) the free complexity of $\mathbb{A}$ which we denote by

$c_f(\mathbb{A})$, see also Section 1.2 of [26]. The free factors will play the role of the vertex groups in the complexity in the proof of Grushko's theorem.

Let $\mathbb{A}$ be a graph of groups and $\mathbb{A}' :=$ core $\mathbb{A}$. We say that a subgraph of groups $\mathbb{D}$ of $\mathbb{A}$ carries $\pi_1(\mathbb{A})$ if the injection of $\pi_1(\mathbb{D})$ into $\pi_1(\mathbb{A})$ is bijective. We denote the set of subgraphs of groups of $\mathbb{A}$ that carry $\pi_1(\mathbb{A})$ by $\mathcal{C}(\mathbb{A})$. The following is trivial:

1) If $\mathbb{A}'$ is a simple core then $\mathbb{D} \in \mathcal{C}(\mathbb{A})$ if and only if $D \cap A' \neq \emptyset$.

2) If $\mathbb{A}'$ is not a simple core then $\mathbb{D} \in \mathcal{C}(\mathbb{A})$ if and only if $A' \subset D$.

We define

$$\tilde{c}_1(\mathbb{D}) := \Sigma(\mathbb{D}) + e(D) + b(D)$$

and

$$c_1(\mathbb{A}) := \min_{\mathbb{D} \in \mathcal{C}(\mathbb{A})} \tilde{c}_1(\mathbb{D}).$$

Lemma 3.1. — *Let $\mathbb{A}$ be a graph of groups with $\pi_1(\mathbb{A}) \neq 1$. Suppose that $\mathbb{A}$ has a simple core $\mathbb{A}'$ and that A' does not consist of a single vertex.*

Then $c_1(\mathbb{A}) = 1$. Moreover $\tilde{c}_1(\mathbb{D}) = 1$ for some $\mathbb{D} \in \mathcal{C}(\mathbb{A})$ if and only if one of the following holds:

1) D consists of a single edge of A'.

2) $\pi_1(\mathbb{A})$ is cyclic and D consists of a single vertex of A'.

Proof. — Choosing $\mathbb{D}$ to be the subgraph of groups whose underlying graph is an arbitrary edge of A' yields $\tilde{c}_1(\mathbb{D}) = 1$ as both vertex groups of $\mathbb{D}$ have trivial relative rank. This shows that $c_1(\mathbb{A}) = 1$ as by assumption $\pi_1(\mathbb{A}) \neq 1$ and therefore $c_1(\mathbb{D}) \geqslant 1$ for all $\mathbb{D} \in \mathcal{C}(\mathbb{A})$.

It is further clear that $\tilde{c}_1(\mathbb{D}) = 1$ implies that D contains at most one edge. If it contains no edge then the single vertex group must be a cyclic vertex group of the core which implies that $\pi_1(\mathbb{A})$ is cyclic. If D consists of one edge that is not contained in A' then the vertex group of the vertex lying in A' must be of relative rank at least 1 which implies that $\tilde{c}_1(\mathbb{D}) \geqslant 2$. The claim follows. □

This implies the following useful fact:

Corollary 3.2. — *Suppose that $\mathbb{A}$ has a simple core $\mathbb{A}'$. Let v be a vertex of A'. Then there exists $\mathbb{D} \in \mathcal{C}(\mathbb{A})$ such that $v \in VD$ and that $c_1(\mathbb{A}) = \tilde{c}_1(\mathbb{D})$.*

The following lemma shows that c_1 behaves well when moves (folds) are applied. It is one of the keys to the proof of the rank formula. Recall that $c_f(\mathbb{A})$ denotes the free complexity of $\mathbb{A}$.

LEMMA 3.3. — *Let $\mathbb{A}$ be a graph of groups, $\mathcal{B}$ an $\mathbb{A}$-graph and $\bar{\mathcal{B}}$ an $\mathbb{A}$-graph obtained from $\mathcal{B}$ by a move. Suppose that $c_f(\mathbb{B}) = (1, 0)$. Then*

$$c_1(\bar{\mathbb{B}}) \leqslant c_1(\mathbb{B}).$$

Proof. — Let $p : B \to \bar{B}$ be the map induced by the elementary move. Note that $\mathbb{D} \in \mathcal{C}(\mathbb{B})$ implies that the subgraph of groups $\bar{\mathbb{D}}$ of $\bar{\mathbb{B}}$ with underlying graph $\bar{D} := p(D)$ lies in $\mathcal{C}(\bar{\mathbb{B}})$ as it carries the fundamental group of $\bar{\mathbb{B}}$. In order to show that $c_1(\bar{\mathbb{B}}) \leqslant c_1(\mathbb{B})$ it therefore suffices to show that $\tilde{c}_1(\bar{\mathbb{D}}) \leqslant \tilde{c}_1(\mathbb{D})$ for all $\mathbb{D} \in \mathcal{C}(\mathbb{B})$ where $\bar{\mathbb{D}}$ is defined as before. It is easily verified that we only need to consider the case where all edges involved in the move lie in D as otherwise $\bar{\mathbb{D}} = \mathbb{D}$ and therefore $\tilde{c}_1(\bar{\mathbb{D}}) = \tilde{c}_1(\mathbb{D})$.

Recall the following facts: If the move is of type IA, IB, IIIA or IIIB then $e(\bar{D}) = e(D) - 1$ and if it is of type IIA or IIB then $e(\bar{D}) = e(D)$. If the move is of type IIIA or IIIB then $b(\bar{D}) = b(D) - 1$ and in the remaining cases $b(\bar{D}) = b(D)$. In order to prove the lemma it therefore suffices to establish the following three statements:

1) If the move is of type IA or IB then $\Sigma(\bar{\mathbb{D}}) \leqslant \Sigma(\mathbb{D}) + 1$.

2) If the move is of type IIA or IIB then $\Sigma(\bar{\mathbb{D}}) \leqslant \Sigma(\mathbb{D})$.

3) If the move is of type IIIA or IIIB then $\Sigma(\bar{\mathbb{D}}) \leqslant \Sigma(\mathbb{D}) + 2$.

We give the argument for moves of type IA, IIA and IIIA, the argument for moves of type IB, IIB and IIIB is analogous.

Suppose first that the move is of type IA. Let $f_1, f_2 \in ED$ with $x = \alpha(f_1) = \alpha(f_2)$ be the edges that are identified by the move. Let $f = p(f_1) = p(f_2) \in E\bar{D}$ be their image in $\bar{D}$.

In this case $D_x = \bar{D}_{p(x)}$. The two edges f_1 and f_2 adjacent to x are replaced by a single edge f adjacent to $p(x)$ with edge group $D_f = \langle D_{f_1}, D_{f_2} \rangle$. Now the two conjugates of D_{f_1} and D_{f_2} that occur in the realization of the minimal corank in the definition of the relative rank of D_x can clearly be replaced by two conjugates of D_f and therefore by one conjugate of D_f and one more (conjugating) element. Thus relrank $\bar{D}_{p(x)} \leqslant$ relrank $D_x + 1$.

The vertices $y = \omega(f_1)$ and $z = \omega(f_2)$ are replaced by the vertex $p(y) = p(z)$ with vertex group $\langle D_y, D_z \rangle$ whose adjacent edge groups coincide with those of y and z except that D_{f_1} and D_{f_2} are replaced with D_f. As in the definition of the relative ranks of D_y and D_z the conjugacy factors of D_{f_1} and D_{f_2} can be chosen to be trivial it follows that relrank $\bar{D}_{p(y)} \leqslant$ relrank D_y + relrank D_z.

As the relative ranks of the remaining vertex groups are trivially preserved it follows that $\Sigma(\bar{\mathbb{D}}) \leqslant \Sigma(\mathbb{D}) + 1$.

Suppose now that the move is of type IIA. Let $f \in ED$ be the edge involved, $x := \alpha(f)$ and $y := \omega(f)$. Thus we have $D_x = \bar{D}_{p(x)}$. As one adjacent edge groups increases it is obvious that relrank $\bar{D}_{p(x)} \leqslant$ relrank D_x.

The vertex group D_y increases but only by as much as the edge group of f, thus relrank $\bar{D}_{p(y)} \leqslant$ relrank D_y. It follows that $\Sigma(\bar{\mathbb{D}}) \leqslant \Sigma(\mathbb{D})$.

Suppose now that the move is of type IIIA. As in the case of a move of type IA it follows that relrank $\bar{D}_{p(x)} \leqslant$ relrank $D_x + 1$ and it remains to show that relrank $\bar{D}_{p(y)} \leqslant$ relrank $D_y + 1$. Clearly $\bar{D}_{p(y)} = \langle D_y, h\rangle$ for some $h \in \bar{D}_{p(y)}$. The analysis of the fold further shows that h can be chosen such that $\omega_{f_2}(D_{f_2}) \subset h\omega_f(\bar{D}_f)h^{-1}$. It now follows similar to the case of a move of type IA that relrank $\bar{D}_{p(y)} \leqslant$ relrank $D_y + 1$. □

We can now define the needed complexity which we denote by c_2 and by $\bar{c}_2$ in the torsion-free case.

Suppose that $\mathbb{A}$ has free complexity (r, n) with free factors $\mathbb{A}_i$ for $1 \leqslant i \leqslant r$. Suppose further that q of the free factors have cyclic fundamental group (and therefore a simple core). After reordering we can assume that the free factors with cyclic fundamental group are the factors $\mathbb{A}_1, \ldots, \mathbb{A}_q$.

For any $k \in \mathbb{Z}$ we define $\eta_k := \frac{1+(-1)^k}{2}$, thus $\eta_k = 0$ if k is odd and $\eta_k = 1$ if k is even. Recall that this is the η occurring in the statement of the main result.

We assume that $\pi_1(\mathbb{A})$ is not cyclic which implies that the free complexity $\mathbb{A}$ cannot be of type $(0, 0)$, $(0, 1)$ or of type $(1, 0)$ with $q = r = 1$. The definitions of c_2 and $\bar{c}_2$ are as follows:

If $r = q$ then $n + r \geqslant 2$ as we assume that $\pi_1(\mathbb{A})$ is not cyclic and we put

$$c_2(\mathbb{A}) = 1 + k + \lfloor k/2 \rfloor + (2k + 1)(n + r - 2)$$

and

$$\bar{c}_2(\mathbb{A}) = 1 + k + \eta_k + (2k + 1)(n + r - 2).$$

If $r > q$, i.e. if there exist free factors with non-cyclic fundamental group, then

$$c_2(\mathbb{A}) = \bar{c}_2(\mathbb{A}) = \sum_{i=q+1}^{r} c_1(\mathbb{A}_i) + 2k(r-q-1) + (2k+1)(n+q).$$

The main task will be to prove the following Proposition which almost immediately implies Theorem 2.1:

Proposition 3.4. — *Let G be a finitely generated group and $\mathbb{A}$ be a non-trivial, minimal k-acylindrical graph of groups with $\pi(\mathbb{A}) \cong G$ and without trivial edge groups. Then there exists a sequence $\mathcal{B}_1, \ldots, \mathcal{B}_m$ of $\mathbb{A}$-graphs with associated graphs of groups $\mathbb{B}_1, \ldots, \mathbb{B}_m$ such that the following hold:*

1) $\phi : \pi_1(\mathbb{B}_i) \to \pi_1(\mathbb{A})$ is surjective for $1 \leqslant i \leqslant m$.

2) For $1 \leqslant i \leqslant m-1$ $\mathcal{B}_{i+1}$ is obtained from $\mathcal{B}_i$ by a single move.

3) The free complexity of $\mathbb{B}_1$ is $(0, rank\ G)$, thus $\pi_1(\mathbb{B}_1) \cong F_{rank\ G}$.

4) $\mathbb{B}_m$ has no trivial edge groups and the map $[.] : B_m \to A$ is bijective.

5) $c_2(\mathbb{B}_i) \geqslant c_2(\mathbb{B}_{i+1})$ for $1 \leqslant i \leqslant m-1$.

6) If G is torsion-free then $\bar{c}_2(\mathbb{B}_i) \geqslant \bar{c}_2(\mathbb{B}_{i+1})$ for $1 \leqslant i \leqslant m-1$

The main step is the following lemma. It implies that the complexity of an $\mathbb{A}$-graph (with k-acylindrical $\mathbb{A}$) does not increase if certain moves are applied. The difficult cases are those moves that change the free complexity. This can happen in three different ways:

1) A move of type IIIA or IIIB occurs in such a way that it adds a (possibly trivial) element to a trivial vertex group. If this element is non-trivial then a new free factor with cyclic fundamental group and consisting of a single vertex emerges while the remaining free factors remain unaffected. In this case the number of free factors increases by at most one and the free rank decreases by one.

2) There exists a subgraph of groups of free complexity $(1,1)$ whose image under the move has free complexity $(1,0)$. We say that an HNN-move occurs. The following situations are possible:
   (a) A move of type IA or IB identifies vertices that belong to the same free factors and at least one of the edge involved has a trivial edge group.

(b) A move of type IIA or IIB is applied to an edge e with trivial edge group and $\alpha(e)$ and $\omega(e)$ belong to the same free factor.

(c) A move of type IIIA or IIIB is applied, at least one edge group involved is trivial and the vertex group to which an element is added is non-trivial.

3) There exists a subgraph of groups of free complexity $(2, 0)$ whose image under the move has free complexity $(1, 0)$. We say that an amalgamation move occurs. The following situations are possible:

(a) A move of type IA or IB identifies vertices that belong to distinct free factors.

(b) A move of type IIA is applied to an edge e with trivial edge group and $\alpha(e)$ and $\omega(e)$ belong to distinct free factors.

Lemma 3.5. — *Let $\mathbb{A}$ be a minimal k-acylindrical graph of groups and $\mathcal{B}$ be an $\mathbb{A}$-graph that admits no fold of type IA, IB, IIIA or IIIB that restricts to a free factor of $\mathbb{B}$.*

Suppose that $\mathcal{B}'$ is an $\mathbb{A}$-graph such that one of the following holds:

1) $\mathcal{B}'$ is obtained from $\mathcal{B}$ by a move of type IA, IB, IIIA or IIIB.

2) $\mathcal{B}$ admits no move of type IA, IB, IIIA or IIIB and $\mathcal{B}'$ is obtained from $\mathcal{B}$ by a move of type IIA or IIB.

Then $c_2(\mathbb{B}') \leqslant c_2(\mathbb{B})$ and $\bar{c}_2(\mathbb{B}') \leqslant \bar{c}_2(\mathbb{B})$.

Proof. — As before we only give the argument for moves of type IA, IIA and IIIA, the same argument also applies for moves of type IB, IIB and IIIB.

Case 1: $\mathcal{B}'$ is obtained from $\mathcal{B}$ by a move of type IA: If the free complexity does not change then the move must involve at least one edge with trivial edge group as the move is assumed to not restrict to a free factor. It is then easily verified that all free factors are being preserved by the move, thus the claim is immediate. We therefore only need to check the cases that the move is an HNN-move or an amalgamation move.

Subcase 1A: Suppose that the move is an HNN-move. Thus there exists a free factor $\mathbb{B}_i$ of $\mathbb{B}$ and vertices $y \neq z \in VB_i$ that are identified under the fold. Moreover at least one of the edge groups involved is trivial. We may assume that $\mathbb{B}_i'$ is the free factor of $\mathbb{B}'$ that contains the image of $\mathbb{B}_i$ under the move.

We first argue in the case that $\mathbb{B}_i$ does not have a simple core. Choose $\mathbb{D} \in \mathcal{C}(\mathbb{B}_i)$ such that $\tilde{c}_1(\mathbb{D}) = c_1(\mathbb{B}_i)$. Recall that core $\mathbb{B}_i \subset \mathbb{D}$. It follows from Proposition 1.3 that $d(y, \text{core } \mathbb{B}_i) \leqslant k$ and $d(z, \text{core } \mathbb{B}_i) \leqslant k$. Choose $x_1, x_2 \in VD$ such that $d(y, x_1) = d(y, \mathbb{D}) \leqslant k$ and $d(z, x_2) = d(z, \mathbb{D}) \leqslant k$. Put

$$\bar{D} := D \cup [x_1, y] \cup [x_2, z]$$

and let $\bar{\mathbb{D}}$ be the corresponding subgraph of groups of $\mathbb{B}_i$. As at most $2k$ vertex groups with trivial relative rank are added when going from $\mathbb{D}$ to $\bar{\mathbb{D}}$ it follows that

$$\tilde{c}_1(\bar{\mathbb{D}}) \leqslant \tilde{c}_1(\mathbb{D}) + 2k.$$

Let $D' := p(\bar{D}) \subset \mathbb{B}_i'$, thus $\mathbb{D}'$ is the image of $\bar{\mathbb{D}}$ under the move. Note that the sum of the relative ranks of the vertex groups does not change as the vertex group of the vertex $p(y) = p(z)$ has trivial relative rank. It follows that $\tilde{c}_1(\mathbb{D}') \leqslant \tilde{c}_1(\bar{\mathbb{D}})+1$ as the Betti number increases by one. It follows that

$$\tilde{c}_1(\mathbb{D}') \leqslant \tilde{c}_1(\mathbb{D}) + 2k + 1 = c_1(\mathbb{B}_i) + 2k + 1.$$

As $\mathbb{D}' \in \mathcal{C}(\mathbb{B}_i')$ it follows that $c_1(\mathbb{B}_i') \leqslant \tilde{c}_1(\mathbb{D}') \leqslant c_1(\mathbb{B}_i) + 2k + 1$. As the free rank of $\mathbb{B}'$ is one less than the free rank of $\mathbb{B}$ and as all other free factors are unchanged it follows that $\bar{c}_2(\mathbb{B}') = c_2(\mathbb{B}') \leqslant c_2(\mathbb{B}) = \bar{c}_2(\mathbb{B})$.

Next we consider the case that $\mathbb{B}_i$ has simple core with non-cyclic fundamental group. Choose $\mathbb{D} \in \mathcal{C}(\mathbb{B}_i)$ as before. Choose $x \in VD$ such that v is also contained in core $\mathbb{B}_i$. By Lemma 1.3(2) we know that $d(x, y), d(x, y) \leqslant k$, thus we can argue as in the first case with $x = x_1 = x_2$.

If $\mathbb{B}_i$ has simple core with cyclic fundamental group then Proposition 1.3 implies that there exists a tree $\bar{D} \subset B_i$ containing y and z such that $\bar{D}$ contains at most $k + \lfloor k/2 \rfloor$ edges (k if G is torsion-free) and that $\bar{\mathbb{D}} \in \mathcal{C}(\mathbb{B}_i)$. Note that all edge groups of $\bar{\mathbb{D}}$ have trivial relative rank if $\bar{D}$ contains more than $2\lfloor k/2 \rfloor$ edges as in this case $\bar{D}$ contains an edge of core $\mathbb{B}_i$. Otherwise one vertex group can be of relative rank 1. Define D' and $\mathbb{D}'$ as before. It follows that $\tilde{c}_1(\mathbb{D}') \leqslant k+\lfloor k/2 \rfloor+1$ and that $\tilde{c}_1(\mathbb{D}') \leqslant 2\lfloor k/2 \rfloor+2 = k+1+\eta_k$ if $\pi_1(\mathbb{A})$ is torsion free. It follows that $c_2(\mathbb{B}') \leqslant c_2(\mathbb{B})$ and that $\bar{c}_2(\mathbb{B}') \leqslant \bar{c}_2(\mathbb{B})$.

Subcase 1B: Suppose that the move is an amalgamation-move. The argument is very similar to the HNN-case. There exist free factors $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ of $\mathbb{B}$ and vertices $y \in VB_{i_1}$ and $z \in VB_{i_2}$ that are identified by the move. Let $\mathbb{B}_i'$ be the free factor of $\mathbb{B}'$ that contains the image of $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$.

Suppose first that not both $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ have cyclic fundamental group. For $j = 1, 2$ we can choose $\mathbb{D}_j \in \mathcal{C}(\mathbb{B}_{i_j})$ such that $\tilde{c}_1(\mathcal{D}_j) = c_1(\mathbb{B}_{i_j})$ and that $d(y, D_1) \leqslant k$ and $d(z, D_2) \leqslant k$. Thus we can choose $x_1 \in VD_1$ and $x_2 \in VD_2$ such that $d(y, x_1) = d(y, D_1) \leqslant k$ and $d(z, x_2) = d(z, D_2) \leqslant k$.

Put $D_1' = D_1 \cup [x_1, y]$ and $D_2' = D_2 \cup [x_2, z]$ and let $\mathbb{D}_1'$ and $\mathbb{D}_2'$ be the corresponding subgraphs of groups of $\mathbb{B}$. Clearly

$$\tilde{c}_1(\mathbb{D}_j') \leqslant \tilde{c}_1(\mathbb{D}_j) + k = c_1(\mathbb{B}_{i_j}) + k$$

for $j = 1, 2$ as at most k edges and vertices with trivial relative rank are added.

Note that $\mathbb{B}_i'$ has underlying graph $B_i' = p(B_{i_1} \cup B_{i_2})$. Let further $D' = p(D_1' \cup D_2')$ and $\mathbb{D}'$ the corresponding graph of groups. It is clear that $\mathbb{D}' \in \mathcal{C}(\mathbb{B}_i')$. As in the proof of Lemma 3.3 we see that

$$\begin{aligned} c_1(\mathbb{B}_i') \leqslant \tilde{c}_1(\mathbb{D}') \leqslant \tilde{c}_1(\mathbb{D}_1') + \tilde{c}_1(\mathbb{D}_2') &\leqslant c_1(\mathbb{B}_{i_1}) + k + c_1(\mathbb{B}_{i_2}) + k \\ &= c_1(\mathbb{B}_{i_1}) + c_1(\mathbb{B}_{i_2}) + 2k. \end{aligned}$$

The remaining free factors are left unchanged and the free rank is preserved.

If neither $\mathbb{B}_{i_1}$ nor $\mathbb{B}_{i_2}$ have cyclic fundamental group it follows that $\bar{c}_2(\mathbb{B}') = c_2(\mathbb{B}') \leqslant c_2(\mathbb{B}) = \bar{c}_2(\mathbb{B})$ as in the definition of $c_2 = \bar{c}_2$ the sum increases by at most $2k$, r decreases by one and n and q are unchanged.

If $\mathbb{B}_{i_2}$ has cyclic fundamental group and $\mathbb{B}_{i_1}$ does not, the argument shows that $c_1(\mathbb{B}_i') \leqslant c_1(\mathbb{B}_{i_1}) + c_1(\mathbb{B}_{i_2}) + 2k \leqslant c_1(\mathbb{B}_{i_1}) + 2k + 1$. It follows that $\bar{c}_2(\mathbb{B}') = c_2(\mathbb{B}') \leqslant c_2(\mathbb{B}) = \bar{c}_2(\mathbb{B})$ as the sum increases by at most $2k + 1$, r and q decrease by one and n remains unchanged.

Suppose now that both $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ have cyclic fundamental group. In this case it follows from Proposition 1.3(3) that we can choose D_1, D_2, $x_1 \in VD_1$ and $x_2 \in VD_2$ such that $d(y, x_1) = d(y, D_1) \leqslant \lfloor k/2 \rfloor$ and $d(z, x_2) = d(z, D_2) \leqslant \lfloor k/2 \rfloor$. Defining D_1', D_2' and D' as before and applying the same arguments we get

$$c_1(\mathbb{B}_i') \leqslant c_1(\mathbb{B}_{i_1}) + c_1(\mathbb{B}_{i_2}) + 2\lfloor k/2 \rfloor \leqslant 2 + 2\lfloor k/2 \rfloor = 1 + k + \eta_k \leqslant 1 + k + \lfloor k/2 \rfloor.$$

It is easily verified that this implies that $c_2(\mathbb{B}') \leqslant c_2(\mathbb{B})$ and that $\bar{c}_2(\mathbb{B}') \leqslant \bar{c}_2(\mathbb{B})$.

Case 2: $\mathcal{B}'$ is obtained from $\mathcal{B}$ by a move of type IIA and $\mathcal{B}$ admits no move of type IA, IB, IIIA or IIIB: Thus there exists an edge $e \in EB \backslash EB_i$ with $B_e = 1$ such that the move adds a non-trivial element of $B_{\alpha(e)}$ to the edge group. Put $x := \alpha(e)$ and $y := \omega(e)$.

Note first that if the move does not affect the free complexity then it restricts to a subgraph of groups of free complexity $(1, 0)$ and therefore does not change the free complexity by Lemma 3.3. Thus we may assume that the move is an HNN-move or an amalgamation move.

Subcase 2A: Suppose first that the move is an HNN-move, i.e. that $x, y \in V\mathbb{B}_i$ for some free factor $\mathbb{B}_i$ of $\mathbb{B}$. Let $\mathbb{B}_i'$ be the free factor of $\mathbb{B}'$ that contains the image of $\mathbb{B}_i$ (and of e) under the move.

Suppose first that $\mathbb{B}_i$ does not have a cyclic fundamental group. Choose $\mathbb{D} \in \mathcal{C}(\mathbb{B}_i)$ such that $\tilde{c}_1(\mathbb{D}) = c_1(\mathbb{B}_i)$ and choose $x_1, x_2 \in VD$ such $d(x, x_1) = d(x, D)$ and $d(y, x_2) = d(y, D)$. Note that $d(y, x_2) \leqslant k$ by Proposition 1.3 and that $d(x, x_1) \leqslant k-1$ as the move at x is possible. Put $\bar{D} = D \cup [x_1, x] \cup [x_2, y]$. Clearly $\bar{\mathbb{D}}$ is obtained from $\mathbb{D}$ by adding at most $d(x, D) + d(y, D) \leqslant 2k-1$ edges and vertices with trivial relative rank. It follows that

$$\tilde{c}_1(\bar{\mathbb{D}}) \leqslant \tilde{c}_1(\mathbb{D}) + d(x, D) + d(y, D) = c_1(\mathbb{B}_i) + d(x, D) + d(y, D).$$

Let further $\hat{D}$ be the graph obtained from $\bar{D}$ by adding the edge e. Clearly $\tilde{c}_1(\hat{\mathbb{D}}) = \tilde{c}_1(\bar{\mathbb{D}}) + 2$ as the relative rank of all vertex groups is preserved, one edge is added to $\bar{D}$ and $b(\hat{D}) = b(\bar{D}) + 1$. As $\tilde{c}_1(\mathbb{D}) = c_1(\mathbb{B}_i)$ follows that

$$\tilde{c}_1(\hat{\mathbb{D}}) \leqslant \tilde{c}_1(\mathbb{D}) + d(x, D) + d(y, D) = c_1(\mathbb{B}_i) + d(x, D) + d(y, D) + 2.$$

Let $D' = p(\hat{D})$ and $\mathbb{D}'$ the corresponding subgraph groups of $\mathbb{B}'$. The proof of Lemma 3.3 implies that

$$\tilde{c}_1(\hat{\mathbb{D}}) \geqslant \tilde{c}_1(\mathbb{D}') \geqslant c_1(\mathbb{B}_i')$$

where the second inequality holds as $\mathbb{D}' \in \mathcal{C}(\mathbb{B}_i')$. Thus

$$c_1(\mathbb{B}_i') \leqslant c_1(\mathbb{B}_i) + d(x, D) + d(y, D) + 2 \leqslant c_1(\mathbb{B}_i) + 2k + 1$$

which proves the claim.

If $\mathbb{B}_i$ has cyclic fundamental group then precisely the same arguments yield the assertion together with the various claims of Proposition 1.3.

Subcase 2B: Suppose now that the move is an amalgamation move. Thus there exist free factors $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ such that $x = \alpha(e) \in VB_{i_1}$ and $y = \omega(e) \in VB_{i_2}$. Denote the free factor of $\mathbb{B}'$ that contains the images of $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ by $\mathbb{B}_i'$.

Suppose first that not both $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ have cyclic fundamental group. Choose $\mathbb{D}_1$ and $\mathbb{D}_2$ as before. Choose $x_1 \in \mathbb{D}_1$ and $x_2 \in \mathbb{D}_2$ such that $d(x, x_1) = d(x, \mathbb{D}_1)$ and $d(y, x_2) = d(y, \mathbb{D}_2)$. As in subcase 2A we see that $d(y, x_2) \leqslant k$ and that $d(x, x_1) \leqslant k-1$. Define $\bar{D} := D_1 \cup D_2 \cup [x, x_1] \cup [y, x_2] \cup e$. It follows as before that

$$\tilde{c}_1(\bar{\mathbb{D}}) = \tilde{c}_1(\mathbb{D}_1) + \tilde{c}_1(\mathbb{D}_2) + d(x, x_1) + d(y, x_2) + 1 \leqslant \tilde{c}_1(\mathbb{D}_1) + \tilde{c}_1(\mathbb{D}_2) + 2k$$

which implies the claim.

If both $\mathbb{B}_{i_1}$ and $\mathbb{B}_{i_2}$ have cyclic fundamental group then the same argument together with the discussion in case 1B proves the claim.

Case 3: $\mathcal{B}'$ is obtained from $\mathcal{B}$ by a move of type IIIA. Thus there are edges f_1 and f_2 with $x = \alpha(f_1) = \alpha(f_2)$ and $y = \omega(f_1) = \omega(f_2)$ and the move identifies f_1 and f_2 and adds an element to B_y. We may assume that at least one edge group is trivial as otherwise the move restricts to a free factor.

Suppose first that B_y is trivial. Then the Betti number decreases by one and a new free factor with cyclic fundamental group emerges. In this case the complexities c_2 and $\bar{c}_2$ are clearly preserved.

Suppose now that B_y is non-trivial. Let $\mathbb{B}_i$ be the free factor containing y and $\mathbb{B}'_i$ the free factor of $\mathbb{B}'$ containing the image of $\mathbb{B}_i$ under the move. Choose $\mathbb{D} \in \mathcal{C}(\mathbb{B}_i)$. We may choose $\mathbb{D}$ such that $d(D, y) \leqslant k$ and that $d(D, y) \leqslant \lfloor k/2 \rfloor$ if $\mathbb{B}_i$ has cyclic fundamental group. Choose $z \in VD$ such that $d(y, z) = D(y, D)$ and put $\bar{D} = D \cup [y, z]$. Clearly $\tilde{c}_1(\bar{\mathbb{D}}) \leqslant \tilde{c}_1(\mathbb{D}) + k$ and $\tilde{c}_1(\bar{\mathbb{D}}) \leqslant \tilde{c}_1(\mathbb{D}) + \lfloor k/2 \rfloor = 1 + \lfloor k/2 \rfloor$ if $\mathbb{B}_i$ has cyclic fundamental group. Let now $D' = p(\bar{D})$. Now $\mathbb{D}'$ emerges from $\bar{\mathbb{D}}$ by adding one generator to a vertex group. Thus $\tilde{c}_1(\mathbb{D}') \leqslant \tilde{c}_1(\bar{\mathbb{D}}) + 1 \leqslant \tilde{c}_1(\mathbb{D}) + k + 1$ and $\tilde{c}_1(\mathbb{D}') \leqslant \tilde{c}_1(\bar{\mathbb{D}}) + 1 \leqslant \lfloor k/2 \rfloor + 2$ if $\mathbb{B}_i$ has cyclic fundamental group. As $\mathbb{D}') \in \mathcal{C}(\mathbb{B}'_i)$ it follows that $c_1(\mathbb{B}'_i) \leqslant c_1(\mathbb{B}_i) + k + 1$ and that $c_1(\mathbb{B}'_i) \leqslant \lfloor k/2 \rfloor + 2$ if $\mathbb{B}_i$ has cyclic fundamental group. It follows that $c_2(\mathbb{B}') \leqslant c_2(\mathbb{B})$ and that $\bar{c}_2(\mathbb{B}') \leqslant \bar{c}_2(\mathbb{B})$. $\square$

Proof of Proposition 3.4. The existence of an $\mathbb{A}$-graph $\mathcal{B}_1$ satisfying (1) and (3) has been established in [10] (see also [26]), it is simply the S-wedge corresponding to a minimal generating set S of $\pi_1(\mathbb{A})$, see also Remark 1.3 of [26]. It now suffices to establish the existence of $\mathbb{A}$-graphs $\mathcal{B}_2, \ldots, \mathcal{B}_m$ such that (2), (4), (5) and (6) are satisfied. Indeed (1) will be automatically satisfied as the surjectivity of ϕ is preserved under moves.

Having constructed $\mathcal{B}_i$ we construct $\mathcal{B}_{i+1}$ by applying a move of type IA, IB, IIIA, IIIB that restricts to a free factor of $\mathbb{B}_i$ if possible. In this case the other free factors and the free rank are unchanged and therefore $c_2(\mathbb{B}) = c_2(\mathbb{B}')$ and $\bar{c}_2(\mathbb{B}) = \bar{c}_2(\mathbb{B}')$ if $\pi_1(\mathbb{A})$ is torsion free by Lemma 3.3.

If no such move is possible we construct $\mathcal{B}_{i+1}$ by applying an arbitrary move of type IA, IB, IIIA, IIIB. In this case the claim follows from Lemma 3.5.

If no move of type IA, IB, IIIA, IIIB is possible we construct $\mathcal{B}_{i+1}$ by applying a move of type IIA or IIB, again the claim follows from Lemma 3.5.

Note that in any such sequence the number of moves of type IA, IB, IIIA and IIIB is bounded from above by the number of edges of B_1. It remains to argue that we can find such a sequence such that (4) holds. This however follows from the Proposition in [2] where it is shown that edges of B_i that get mapped to the same edge in A get identified by finitely many folds and that any element of an edge group A_e can be added to any edge group of $\mathbb{B}_i$ with finitely many folds. The first implies that we can find moves that make $p : B_i \to A$ injective, the second guarantees that we can find moves such that the resulting $\mathbb{A}$-graph has no trivial edge groups. $\square$

Note that that under the hypothesis of Proposition 3.4 the group $\pi_1(\mathbb{A})$ is non-cyclic as the splitting is non-trivial and as all edge groups are assumed non-trivial.

The last ingredient in the proof of Theorem 2.1 is the following simple observation:

Lemma 3.6. — *Let $\mathbb{A}$ be a minimal graph of groups with finitely generated fundamental group and without trivial edge groups. Let $\mathcal{B}$ be an $\mathbb{A}$-graph such that the associated morphism $[.] : B \to A$ is bijective, that $\mathbb{B}$ has no trivial edge groups and that $\phi : \pi_1(\mathbb{B}, u_0) \to \pi_1(\mathbb{A}, [v_0])$ is surjective. Then*

$$c_2(\mathbb{A}) \leqslant c_2(\mathbb{B}) \text{ and } \bar{c}_2(\mathbb{A}) \leqslant \bar{c}_2(\mathbb{B}).$$

Proof. — As both $\mathbb{A}$ and $\mathbb{B}$ have only non-trivial edge groups it follows that both graphs of groups have free complexity $(1, 0)$. As $\mathbb{A}$ is minimal and ϕ is surjective it follows that $\mathbb{B}$ is also minimal. Thus $\mathcal{C}(\mathbb{A}) = \{A\}$ and $\mathcal{C}(\mathbb{B}) = \{B\}$. It follows that $c_2(\mathbb{A}) = \bar{c}_2(\mathbb{A}) = c_1(\mathbb{A}) = \tilde{c}_1(\mathbb{A})$ and that $c_2(\mathbb{B}) = \bar{c}_2(\mathbb{B}) = c_1(\mathbb{B}) = \tilde{c}_1(\mathbb{B})$. In order to prove the lemma it therefore suffices to show that relrank $B_v \geqslant$ relrank $A_{[v]}$ for all $v \in VB$.

Let $v \in VB$ and put $k :=$ relrank B_v. Thus there exist $g_1, \ldots, g_k \in B_v$ and $g_e \in B_v$ for all $e \in E_v$ such that

$$B_v = \langle g_1, \ldots, g_k, \bigcup_{e \in E_v} g_e \alpha_e(B_e) g_e^{-1} \rangle = \langle g_1, \ldots, g_k, \bigcup_{e \in E_v} g_e e_\alpha \alpha_{[e]}(B_e) e_\alpha^{-1} g_e^{-1} \rangle.$$

To conclude the proof of the lemma it clearly suffices to show that

$$A_{[v]} = \langle g_1, \ldots, g_k, \bigcup_{e \in E_v} g_e e_\alpha \alpha_{[e]}(A_{[e]}) e_\alpha^{-1} g_e^{-1} \rangle.$$

Suppose that $g \in A_{[v]}$. It now follows as in the proof of the proposition on page 455 of [2] that there exist a finite sequence of folds of type IIA or IIB (only those are possible) that can be applied to $\mathcal{B}$ that add g to B_v. Denote the resulting $\mathbb{A}$-graph by $\mathcal{B}'$ and denote the natural map from B to

B' by p. We may assume that the labels of the edges are unchanged. By the proof of Lemma 3.3 it follows that

$$g \in \langle g_1, \ldots, g_k, \bigcup_{e \in E_v} g_e \alpha_{p(e)}(A_{p(e)}) g_e^{-1} \rangle =$$

$$\langle g_1, \ldots, g_k, \bigcup_{e \in E_v} g_e e_\alpha \alpha_{[e]}(A_{p(e)}) e_\alpha^{-1} g_e^{-1} \rangle \subset \langle g_1, \ldots, g_k, \bigcup_{e \in E_v} g_e e_\alpha \alpha_{[e]}(A_{[e]}) e_\alpha^{-1} g_e^{-1} \rangle,$$

the lemma is proven. □

We now have all the tools necessary to give the proof of the rank formula.

Proof of Theorem 2.1. — Choose a sequence of $\mathbb{A}$-graphs $\mathcal{B}_1, \ldots, \mathcal{B}_m$ as in Proposition 3.4. It follows in particular that $c_2(\mathbb{B}_1) \geqslant c_2(\mathbb{B}_m)$. As $\mathbb{B}_m$ has no trivial edge group and therefore free complexity $(1, 0)$ it follows from Lemma 3.6 that $c_2(\mathbb{B}_m) \geqslant c_2(\mathbb{A})$. Thus $c_2(\mathbb{B}_1) \geqslant c_2(\mathbb{A})$. As $\mathbb{A}$ contains no trivial edge groups it follows that $\mathbb{A}$ has free complexity $(1, 0)$. As $\mathbb{A}$ is minimal we further have that core $\mathbb{A} = \mathbb{A}$. This implies that $c_2(\mathbb{A}) = c_1(\mathbb{A}) = \tilde{c}_1(\mathbb{A}) = \Sigma(\mathbb{A}) + e(A) + b(A)$. It follows that

$$c_2(\mathbb{B}_1) \geqslant \Sigma(\mathbb{A}) + e(A) + b(A).$$

The conclusion follows as $\mathbb{B}_1$ has free complexity $(0, \text{rank } \pi_1(\mathbb{A}))$ and therefore $c_2(\mathbb{B}_1) = 1 + k + [k/2] + (\text{rank } \pi_1(\mathbb{A}) - 2)(2k + 1)$.

If $\pi_1(\mathbb{A})$ is torsion-free the same argument shows that

$$\bar{c}_2(\mathbb{B}_1) \geqslant \bar{c}_2(\mathbb{A}) = c_2(\mathbb{A}) = \Sigma(\mathbb{A}) + e(A) + b(A)$$

and the conclusion follows as $\bar{c}_2(\mathbb{B}_1) = 1 + k + (\text{rank } \pi_1(\mathbb{A}) - 2)(2k + 1) + \eta_k$. □

4. Rank versus genus

This section is dedicated to the proof of the following theorem, recall that we denote the Heegaard genus of a 3-manifold M by $g(M)$ and the rank of its fundamental group by $r(M)$. We assume that the reader is familiar with Heegaard splittings and amalgamation of Heegaard splittings, for our purposes [17] is a good source.

THEOREM 4.1. *[Theorem 0.2]. — Let M be a closed orientable graph manifold. Then*

$$g(M) \leqslant 28 \cdot r(M).$$

Proof. — Let M be a closed orientable graph manifold. Let A be the graph underlying the JSJ-decomposition, for any $v \in VA$ we denote the Seifert piece corresponding to v by M_v. Let $\mathbb{A}$ be the graph of groups corresponding to the JSJ, in particular $\pi_1(M) = \pi_1(\mathbb{A})$ and $A_v = \pi_1(M_v)$ for all $v \in VA$.

In the following we denote the orientable circle bundle over the Möbius band by Q. Q is a Seifert manifold with two distinct Seifert fibrations. Moreover the subgroup of the fundamental group of Q corresponding to the boundary is of index 2 and therefore normal. This makes $\pi_1(\mathbb{A})$ behave differently if M has pieces anthropomorphic to Q. If no Seifert piece is anthropomorphic to Q then the graph of groups $\mathbb{A}$ is 2-acylindrical and unless M consists of two Seifert pieces both homeomorphic to Q it is 3-acylindrical; this follows from the discussion in Section 3 of [3]. If M consists of two pieces homeomorphic to Q then the action on the Bass-Serre tree is dihedral with infinite kernel, in particular it is not k-acylindrical for any k. As in the later case $g(m) \leqslant 4$ and $r(M) \geqslant 2$ the claim holds. Thus we may assume that $\mathbb{A}$ is 3-acylindrical.

We will use the following facts:

1) For any $v \in VA$ we have relrank $(A_v) \geqslant$ rank $A_v - |E_v| - 1$. This holds as for function $g : E_v \to A_v$ the subgroup $\langle \bigcup_{e \in E_v} g(e)\alpha_e(A_e)g(e)^{-1} \rangle$ is generated by $|E_v| + 1$ elements, namely the element corresponding to the fiber and one more element from each subgroup $g(e)\alpha_e(A_e)g(e)^{-1}$.

2) $g(M_v) \leqslant r(M_v) =$ rank A_v for any $v \in VA$. This holds as any vertical Heegaard splitting is of genus $r(M_v)$, see [17] for Seifert manifolds with orientable base space and [14] for Seifert manifolds with non-orientable base space.

3) $g(M) \leqslant \sum_{v \in VA} g(M_v) - e(A) + b(A)$, this follows as the splitting obtained from amalgamating minimal genus Heegaard splittings of the Seifert pieces is of genus $\sum_{v \in VA} g(M_v) - e(A) + b(A)$.

It now follows from Theorem 2.1 that

$$r(M) = \text{rank } \pi_1(M) = \text{rank } \pi_1(\mathbb{A}) \geqslant \frac{1}{7}\left(\Sigma(\mathbb{A} + e(A) + b(A) + 10\right) =$$

$$= \frac{1}{7}\left(\sum_{v \in VA} \text{relrank } A_v + \frac{1}{2}\sum_{v \in V} |E_v| + b(A) + 10\right)$$

where the last equality holds as any (oriented) edge occurs in precisely one set E_v and as $e(A)$ counts edge pairs. With fact (1) we conclude that

$$r(M) > \frac{1}{7}\left(\sum_{v\in VA}\max(0,\text{rank } A_v - |E_v| - 1) + \frac{1}{2}\sum_{v\in VA}|E_v| + b(A)\right) =$$

$$= \frac{1}{7}\left(\sum_{v\in VA}\left(\max(0,\text{rank } A_v - |E_v| - 1) + \frac{1}{2}|E_v|\right) + b(A)\right).$$

We claim that $\max(0,\text{rank } A_v - |E_v| - 1) + \frac{1}{2}|E_v| \geqslant \frac{1}{4}\text{rank } A_v$ for all $v \in VA$.

If $\text{rank } A_v - |E_v| - 1 < 0$ then $|E_v| > \text{rank } A_v - 1 > \frac{1}{2}\text{rank } A_v$ as $\text{rank } A_v \geqslant 2$ and therefore

$$\max(0,\text{rank } A_v - |E_v| - 1) + \frac{1}{2}|E_v| = 0 + \frac{1}{2}|E_v| \geqslant \frac{1}{4}\text{rank } A_v.$$

If $\text{rank } A_v - |E_v| - 1 \geqslant 0$ then $\frac{1}{2}|E_v| \leqslant \frac{1}{2}\text{rank } A_v - \frac{1}{2}$. It follows that

$$\max(0,\text{rank } A_v - |E_v| - 1) + \frac{1}{2}|E_v| = \text{rank } A_v - \frac{1}{2}|E_v| - 1 \geqslant$$

$$\geqslant \text{rank } A_v - 1 - \left(\frac{1}{2}\text{rank } A_v - \frac{1}{2}\right) = \frac{1}{2}(\text{rank } A_v - 1) \geqslant \frac{1}{4}\text{rank } A_v.$$

Thus the claim is proven. Thus

$$r(M) \geqslant \frac{1}{7}\left(\sum_{v\in VA}\frac{1}{4}\text{rank } A_v + b(A)\right) = \frac{1}{28}\sum_{v\in VA}\text{rank } A_v + \frac{1}{7}b(A).$$

Using fact (2) and (3) this implies that

$$r(M) \geqslant \frac{1}{28}\sum_{v\in VA}g(M_v) + \frac{1}{7}b(A) > \frac{1}{28}\left(\sum_{v\in VA}g(M_v) + b(A) - e(A)\right) \geqslant g(M).$$

This concludes the proof. □

The bound given in the above proof is probably far from being sharp and with more care it is probably possible to get a significantly smaller constant than 28. However getting to significantly better constant would require an analysis of the folding sequence used in the proof of Theorem 2.1 which would certainly be extremely technical.

If the manifold is not a graph manifold but has non-trivial JSJ then the argument given above will show that the quotient $g(M)/r(M)$ is uniformly bounded provided it is bounded for the pieces. Thus it is bounded if it is bounded for hyperbolic pieces.

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