PETER B. SHALEN

Homology and volume of hyperbolic 3-orbifolds, and enumeration of arithmetic groups

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Peter B. Shalen[1]

Résumé. — Selon un théorème de Borel, les volumes d'orbifolds hyperboliques arithmétiques de dimension 3 constituent un ensemble discret. Ce théorème soulève le problème d'énumérer les orbifolds hyperboliques arithmétiques de dimension 3 dont le volume est majoré par une constante donnée. Une étape cruciale dans ce programme est de majorer le rang d'un certain 2-groupe abélien élémentaire associé à un tel orbifold O. Ce rang est majoré par la dimension de $H_1(O; \mathbb{Z}_2)$. Étant donné une variété hyperbolique M dont le volume est majoré par une constante convenable, des résultats que j'ai établis en collaboration avec Marc Culler et d'autres auteurs donnent des bornes supérieures utiles pour la dimension de $H_1(M; \mathbb{Z}_2)$. Dans cet article je décrirai mes progrès sur le problème d'étendre les résultats de ce genre au cadre des orbifolds.

Abstract. — Borel's theorem that volumes of arithmetic hyperbolic 3-orbifolds form a discrete set raises the problem of enumerating those arithmetic hyperbolic 3-orbifolds whose volume is subject to a given upper bound. A key step is bounding the rank of a certain elementary abelian 2-group associated with such an orbifold O. This rank is bounded above by the dimension of $H_1(O; \mathbb{Z}_2)$. Joint work of mine with Marc Culler and others gives good bounxds for the dimension of $H_1(M; \mathbb{Z}_2)$, where M is a hyperbolic 3-manifold whose volume has a suitable upper bound. I will report on progress on the problem of extending results of this kind to orbifolds.

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1. Introduction

It is a great honor to help celebrate Michel Boileau's 60th birthday.

Nobody has contributed more than Michel to our understanding of 3-dimensional orbifolds. It is therefore a pleasant coincidence that I have found something to say about 3-orbifolds in time for this birthday volume.

The motivation for studying arithmetic orbifolds originally came from number theory. Although I cannot yet call myself a number theorist, I have devoted Sections 2 and 3 of this paper to explaining the number-theoretic background of the problem to the best of my understanding. In Section 3 I state Borel's result that there are only a finite number of arithmetic 3-orbifolds of at most a given volume V, raise the question of enumerating these for a given V, and explain why the problem of getting good bounds on the dimension of $H_1(O; \mathbb{Z}_2)$ for orbifolds O of volume at most V is relevant to the problem. In Section 4 I review some joint work with Marc Culler and others that gives good bounds on the dimension of $H_1(M; \mathbb{Z}_2)$ for manifolds M of at most a given volume. In Section 5 I report on some progress on the corresponding problem for orbifolds, and give a few hints on the method of proof of the tentative results that I have obtained so far.

Bon anniversaire, Michel. Tous mes voeux de bonheur pour les 60 années à venir!

2. Quaternion algebras and arithmetic groups

2.1. If K is a field of characteristic 0, a *quaternion algebra* over K is defined to be an associative algebra over K which is a 4-dimensional K-vector space, and has a basis $\{1, i, j, k\}$ such that

$$i^2 = \alpha, \ j^2 = \beta, \ ij = k \text{ and } ji = -k$$

for some non-zero elements $\alpha, b \in K$. It turns out that every quaternion algebra either is a division algebra (the interesting case) or is isomorphic to the algebra $\mathcal{M}_2(K)$ of 2×2 matrices over K. The basic example of a division quaternion algebra is of course the algebra $\mathbb{H}$ of Hamiltonian quaternions over $K = \mathbb{R}$.

If B is a quaternion algebra over K, then K is identified with $K \cdot 1 \subset B$, which is the center of B.

2.2. In the case where K is a number field (i.e. a finite extension of $\mathbb{Q}$), a division quaternion algebra over K just misses being a field because of the failure of commutativity. It should be thought of as a non-commutative

analogue of a number field, and is a natural object of study from the number-theoretical point of view.

2.3. The analogy becomes complicated from the very beginning, because there is no *canonical* analogue for division quaternion algebras of the ring of integers in a number field. The useful non-canonical analogue is given by the notion of a maximal order. An *order* in a finite-dimensional associative $\mathbb{Q}$-algebra A is a subring of A which, when regarded as an additive subgroup, is a free abelian group on some $\mathbb{Q}$-basis of A. If E is a number field, the ring of integers of E is the unique order which is maximal (with respect to inclusion). In the case where A is a quaternion algebra, a maximal order exists, but in general it is not unique.

2.4. Another basic object that arises in studying number field, and does not quite have a canonical analogue in the context of a quaternion algebra over a number field, is the group of units of the ring of integers. If E is a number field, it is an easy exercise to show that the group R^* of units of the ring of integers R in E cannot be properly contained with finite index in any other subgroup of E^*. Thus R^* is maximal within its *commensurability class* in E^*. (Two subgroups A and B of a given group are said to be *commensurable* if $A \cap B$ has finite index in both A and B.) In the case of a quaternion algebra B over a number field K, even if we fix a maximal order $\mathfrak{O}$ for B, the group of units $\mathfrak{O}^*$ is not in general maximal in its commensurability class in B^*. For example, the normalizer $N_{B^*}(\mathfrak{O}^*)$ contains $\mathfrak{O}^*$ with a finite index which may well be greater than 1.

The group B^* has center K^*. I will denote by $\Delta_\mathfrak{O}$ and $\Gamma_\mathfrak{O}$ the respective images of $\mathfrak{O}^*$ and $N_{B^*}(\mathfrak{O}^*)$ in the quotient group B^*/K^*; the groups $\Delta_\mathfrak{O}$ and $\Gamma_\mathfrak{O}$ contain almost the same information as $\mathfrak{O}^*$ and $N_{B^*}(\mathfrak{O}^*)$, and turn out to be more convenient to work with. Up to isomorphism, the subgroups of B^*/K^* commensurable with $\Delta_\mathfrak{O}$ (including $\Gamma_\mathfrak{O}$) are "arithmetic lattices" in a sense that I will now explain.

2.5. If K is a number field, there are a finite number of (necessarily injective) homomorphisms from K to $\mathbb{C}$. Some of these may have images contained in $\mathbb{R}$; these are called *real places* of K. Those homomorphisms (if any) whose images are not contained in $\mathbb{R}$ occur in conjugate pairs; each such pair is called a *complex place* of K. The degree of K over $\mathbb{Q}$ is $r_1 + 2r_2$, where r_1 and r_2 denote the numbers of real and complex places, respectively.

2.6. Now suppose that B is a quaternion algebra over a number field K, and that $\mathcal{P}$ is a real or complex place of K, which we use to identify

K with a subfield of $\mathbb{R}$ or $\mathbb{C}$ respectively. Then $B \otimes \mathbb{R}$ or, respectively, $B \otimes \mathbb{C}$, is a quaternion algebra over $\mathbb{R}$ or $\mathbb{C}$. Any quaternion algebra over $\mathbb{R}$ is isomorphic to $\mathbb{H}$ or to $\mathcal{M}_2(\mathbb{R})$, and any quaternion algebra over $\mathbb{C}$ is isomorphic to $\mathcal{M}_2(\mathbb{C})$. Hence $\mathcal{P}$ defines an injection $I_{\mathcal{P}}$ from B to $\mathbb{H}$, $\mathcal{M}_2(\mathbb{R})$, or $\mathcal{M}_2(\mathbb{C})$. When $\mathcal{P}$ is a complex place, $i_{\mathcal{P}}$ is defined only up to complex conjugation. In the case where $\mathcal{P}$ is a real place and $B \otimes \mathbb{R}$ is isomorphic to $\mathbb{H}$, so that $i_{\mathcal{P}} \subset \mathbb{H}$, the quaternion algebra B is said to *ramify at $\mathcal{P}$*.

The injective homomorphism $i_{\mathcal{P}}|B^*$ maps B^* into $\mathrm{GL}_2(\mathbb{C})$ (if $\mathcal{P}$ is complex), $\mathrm{GL}_2(\mathbb{R})$ (if $\mathcal{P}$ is real and B does not ramify at $\mathcal{P}$) or $\mathbb{H}^*$ (if $\mathcal{P}$ is real and B ramiies at $\mathcal{P}$)). Hence, in these respective cases, $i_{\mathcal{P}}|B^*$ defines a homomorphism from B^*/K^* to $\mathrm{PGL}_2(\mathbb{C})$, $\mathrm{PGL}_2(\mathbb{R})$, or $\mathbb{H}^*/\mathbb{R}^*$, which is also readily seen to be injective.

If a, c, and b denote, respectively, the number of real places of K at which B does not ramify, the number of real places of K at which B ramifies, and the number of complex places of K, the construction that I have just described gives a injections from B^*/K^* to $\mathrm{PGL}_2(\mathbb{R})$, c injections from B^*/K^* to $\mathbb{H}^*/\mathbb{R}^*$, and b injections from B^*/K^* to $\mathrm{PGL}_2(\mathbb{C})$. These in turn define a diagonal injection from B^*/K^* to $\mathrm{PGL}_2(\mathbb{R})^a \times (\mathbb{H}^*/\mathbb{R}^*)^c \times \mathrm{PGL}_2(\mathbb{C})^b$. It is a fairly elementary matter to show that the image of $\Delta_{\mathfrak{O}}$ under this diagonal injection is a discrete group; in fact, this can be deduced almost formally from the fact that the various field homomorphisms from K to $\mathbb{C}$ are $\mathbb{C}$-linearly independent, which is a basic fact from Galois theory.

Since $\mathbb{H}^*/\mathbb{R}^*$ is compact, it is not hard to show that the product projection from $\mathrm{PGL}_2(\mathbb{R})^a \times (\mathbb{H}^*/\mathbb{R}^*)^c \times \mathrm{PGL}_2(\mathbb{C})^b$ to $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$ maps any discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^a \times (\mathbb{H}^*/\mathbb{R}^*)^c \times \mathrm{PGL}_2(\mathbb{C})^b$ onto a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$. In particular, this projection maps the image of $\Delta_{\mathfrak{O}}$ in $\mathrm{PGL}_2(\mathbb{R})^a \times (\mathbb{H}^*/\mathbb{R}^*)^c \times \mathrm{PGL}_2(\mathbb{C})^b$ onto a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$. The natural map from $\Delta_{\mathfrak{O}}$ on to this discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$ is injective provided that $a + b > 0$. In this case, one identifies $\Delta_{\mathfrak{O}}$ with a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$. It is a deep number-theoretical fact that $\Delta_{\mathfrak{O}}$, regarded as a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$, has finite covolume, i.e. is a lattice. Note that any subgroup of B^*/K^* which is commensurable with $\Delta_{\mathfrak{O}}$, such as the subgroup $\Gamma_{\mathfrak{O}}$ defined in subsection 2.4, is also identified with a lattice in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$.

2.7. Suppose that a and b are non-negative integers, not both 0. An *arithmetic lattice* in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$ is a lattice Γ in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$ such that for some number field having exactly b complex places, some quaternion algebra B over K which fails to ramify at exactly a places

of K, and some maximal order $\mathfrak{O}$ of B, the lattice Γ is commensurable with $\Delta_{\mathfrak{O}}$. The lattice $\Gamma_{\mathfrak{O}}$ in in $\mathrm{PGL}_2(\mathbb{R})^a \times \mathrm{PGL}_2(\mathbb{C})^b$ is an example.

In the case where $a = 0$ and $b = 1$, i.e. the case where K has exactly one complex case and B ramifies at all real places of K, the arithmetic lattices associated to B lie in $\mathrm{PGL}_2(\mathbb{C})$. The quotients of hyperbolic 3-space by such lattices are orientable hyperbolic 3-orbifolds of finite volume, called *arithmetic 3-orbifolds*. In particular, torsion-free arithmetic lattices in $\mathrm{PGL}_2(\mathbb{C})$ define *arithmetic 3-manifolds*.

3. Borel's theorem and the enumeration problem

A theorem of Borel's [9] asserts that for any positive real number V, and for given a and b, there are at most finitely many arithmetic lattices of covolume at most V. Determining all of these for given values of V, a and b is algorithmically possible thanks to work by Chinburg and Friedman [11], but appears to be impractical except for very small values of V, up to about $V = 0.41$. (The smallest covolume of a lattice in $PGL(2, \mathbb{C})$ is about 0.39.)

Borel's proof of finiteness shows that in order to enumerate all the arithmetic lattices of covolume at most V, it suffices to enumerate those that have the form $\Gamma_{\mathfrak{O}}$, where $\mathfrak{O}$ is a maximal order in a quaternion algebra B. (See Subsections 2.4 and 2.6.) On the other hand, the arithmetic lattices to which Borel's argument applies the most directly are those of the form $\Gamma_{\mathfrak{O}}^1$, where $\Gamma_{\mathfrak{O}}^1 \leqslant \Delta_{\mathfrak{O}}$ denotes the image in B^*/K^* of the subgroup $\mathfrak{O}^1$ of $\mathfrak{O}^*$ consisting of all elements of $\mathfrak{O}$ whose *reduced norm* is equal to 1. We may define the reduced norm of an element of B by identifying the algebra $B \otimes L$, where L is the algebraic closure of K, with $\mathcal{M}_2(L)$; the reduced norm of $x \in B$ is then the determinant of $x \otimes 1 \in \mathcal{M}_2(L)$.

Borel gives a purely number-theoretical formula for the covolume of $\Gamma_{\mathfrak{O}}^1$, regarded as a lattice in $\mathrm{PGL}_2(\mathbb{C})$. Using this formula, one can pass from an upper bound on the covolume of $\Gamma_{\mathfrak{O}}^1$ to upper bounds on such quantities as the root discriminant of the field K (defined to be $|d_K|^{1/n_K}$, where d_K and n_K denote respectively the discriminant and degree of K), and on number-theoretic data that determine the quaternion algebra K. This makes it fairly practical to enumerate the lattices of the form $\Gamma_{\mathfrak{O}}^1$ having at most a given covolume.

In order to enumerate lattices of the form $\Gamma_{\mathfrak{O}}$ whose covolume is bounded above by a given V, one shows that $\Gamma_{\mathfrak{O}}^1$ is a finite-index normal subgroup of $\Gamma_{\mathfrak{O}}$, and that $\Gamma_{\mathfrak{O}}/\Gamma_{\mathfrak{O}}^1$ is an elementary abelian 2-group, i.e. a direct product of groups of order 2. If r denotes the rank of $\Gamma_{\mathfrak{O}}/\Gamma_{\mathfrak{O}}^1$, then $|\Gamma_{\mathfrak{O}}/\Gamma_{\mathfrak{O}}^1| = 2^r$, so that $\mathrm{covol}\,\Gamma_{\mathfrak{O}}^1 = 2^r\,\mathrm{covol}\,\Gamma_{\mathfrak{O}} \leqslant 2^r V$. Hence if one has a good bound

on r one can enumerate the possibilities for $\Gamma_{\mathfrak{D}}^1$; as $\Gamma_{\mathfrak{D}}$ contains $\Gamma_{\mathfrak{D}}^1$ with index 2^r, one can then enumerate the possibilities for $\Gamma_{\mathfrak{D}}$ as well. This makes finding a good bound on r the biggest difficulty in the enumeration problem. Chinburg and Friedman found a purely number-theoretical way to give a bound, but it quickly becomes impractical as V increases beyond 0.41.

The elementary abelian 2-group $\Gamma_{\mathfrak{D}}/\Gamma_{\mathfrak{D}}^1$ may be regarded as a $\mathbb{Z}_2$-vector space of dimension r. Hence r is bounded above by $\dim H_1(\Gamma_{\mathfrak{D}}, \mathbb{Z}/2\mathbb{Z})$. This suggests a topological approach to the most difficult step in the enumeration problem: find an upper bound for $\dim H_1(\Gamma_{\mathfrak{D}}, \mathbb{Z}/2\mathbb{Z})$ in terms of an upper bound for covol M.

4. Homology and volume: the torsion-free case

In the case of a torsion-free lattice Γ, not necessarily arithmetic, joint work of mine with Marc Culler and others [2], [13], [14], gives good bounds on the dimension of $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ in the presence of a suitable bound on the volume of Γ. The results are stated in terms of hyperbolic 3-manifolds: if Γ is a torsion-free lattice in $PGL(2, \mathbb{C})$ then $M = H^3/\Gamma$ is an orientable hyperbolic 3-manifold, the volume of M is the covolume of Γ, and we have $H_1(M, \mathbb{Z}/2\mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. These results should be seen as belonging to the realm of quantitative Mostow rigidity, because they relate hyperbolic volume—which is a topological invariant in view of Mostow rigidity—to the rank of a homology group, which is a classical and well-understood topological invariant. (I should mention that there are elementary results that give linear bounds on the rank of $\pi_1(M)$ in terms of vol M, but these results, and their homological consequences, are quantitatively very weak.)

Given an orientable hyperbolic 3-manifold M, let us set $d = \dim H_1(M, \mathbb{Z}/2\mathbb{Z})$ and let v denote the volume of M. It was shown in [2], [13], and [14], respectively, that

- if $v \leqslant 1.22$ then $d \leqslant 3$;
- if $v \leqslant 3.08$ then $d \leqslant 5$; and
- if $v \leqslant 3.44$ then $d \leqslant 7$.

These results are deep, and the proofs represent many years of work. Some of the ingredients are the "$\log(2k - 1)$ theorem" [12], [5], [2], a result on displacements of points in H^3 under elements of a free Kleinian group, which is based on a study of a Banach-Tarski decomposition of the Patterson-Sullivan measure, and which, in its final form, requires the Marden Conjecture, proved in [1] and [10]; a topological study of the nerve of a

covering of H^3 by hyperbolic cylinders [5], [14]; the homological group theory methods of [18] and the criteria for freeness of subgroups of 3-manifold groups developed in [15], [6], and [5]; the results about volumes of Haken manifolds proved in [4], and involving both Perelman's work on the Ricci flow [7] and the work of Kojima and Miyamoto [16], [17] on volumes of hyperbolic manifolds with totally geodesic boundary; and deep topological results [3], [13] on desingularization of immersed π_1-incompressible surfaces in 3-manifolds. The results of [3] and [13] use towers of two-sheeted coverings as in Shapiro and Whitehead's proof of Dehn's Lemma [19]; this is why $\mathbb{Z}_2$ coefficients are needed for the results of [13] and [14] relating volume to homology.

It is a pleasant coincidence that the results of [13] and [14] give upper bounds specifically for the rank of $H_1(M, \mathbb{Z}_2)$ from upper bounds for $\mathrm{vol}\, M$, as this is very similar to what is needed for enumeration of arithmetic lattices. However, they cannot be applied directly to maximal arithmetic lattices, because the latter typically have torsion. When Γ has torsion, $O = H^3/\Gamma$ is an orientable hyperbolic 3-orbifold, the volume of O is the covolume of Γ, and we have $H_1(O, \mathbb{Z}/2\mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. In the next section I will describe work in progress concerned with finding results qualitatively similar to the ones given in [2], [13], and [14], which apply to the orbifold case, and which I hope will be of practical use in enumerating arithmetic lattices with covolume subject to certain upper bounds.

5. Some results on groups with torsion

I will denote hyperbolic 3-space by H^3. Recall that if Γ is a lattice in $\mathrm{PGL}_2(\mathbb{C})$, possibly with torsion, then the orbit space $M = H^3/\Gamma$, equipped with the quotient topology, is a 3-manifold. The quotient map $q : H^3 \to H^3/\Gamma$ maps the set of all fixed points of non-trivial elements of Γ onto a subset $\mathcal{G}$ of M which is topologically a graph. The orbifold $O = H^3/\Gamma$ is described by specifying the manifold M, the graph $\mathcal{G} \subset M$, and a labeling of each point $x \in \mathcal{G}$ by a finite group, which is the stabilizer in Γ of an arbitrary point of $q^{-1}(x)$. (Up to conjugacy this stabilizer is independent of the choice of a point of $q^{-1}(x)$.) If C is a component of the complement in $\mathcal{G}$ of the set of nodes of $\mathcal{G}$, all points of C are labeled with the same finite group. I will refer to M as the *underlying manifold* of O, and to $\mathcal{G}$ as its *singular set*.

The singular set of the orbifold $O = H^3/\Gamma$ is a link in the underlying manifold of O, i.e. has no nodes, if and only if every finite subgroup of Γ is cyclic.

Here is the result that I have been able to prove so far relating mod 2 homology to hyperbolic volume for orbifolds:

THEOREM 5.1. — *Let O be an orientable hyperbolic 3-orbifold. Suppose that the singular set of O is a link and that $\pi_1(O)$ contains no triangle groups. If O has volume at most 1.72, then*

$$\dim H_1(\Omega; \mathbb{Z}_2) \leqslant 7 + 5\left(\left\lfloor \frac{10}{3} \operatorname{vol}(\Omega) \right\rfloor + \left\lfloor \frac{5}{3} \operatorname{vol}(\Omega) \right\rfloor\right).$$

In particular, $\dim H_1(O, \mathbb{Z}/2\mathbb{Z}) \leqslant 42$.

(The bound of 42 appears to be the best one I can obtain so far. I should mention that although the manuscript is already more than 70 pages long, it is not yet complete. The bound has fluctuated a bit during the course of the writing, and may be slightly different when the paper is finished.)

The assumption that $\pi_1(O)$ contains no triangle groups is a harmless one from the point of view of applications to maximal arithmetic lattices, because if an arithmetic lattice Γ in $PGL(2, \mathbb{C})$ contains a triangle group, then the triangle group is itself isomorphic to an arithmetic lattice in $PGL(2, \mathbb{R})$, and the field that defines Γ is a degree-2 extension of the field that defines the triangle group as an arithmetic lattice. The arithmetic lattices in Γ in $PGL(2, \mathbb{C})$ that are isomorphic to triangle groups are finite in number and are classified; using these facts and the arguments involving discriminants and Borel's volume formula that I have described above, it is possible (in a practical sense) to list all arithmetic lattices in Γ in $PGL(2, \mathbb{C})$ that do contain triangle groups.

The assumption that the singular set of O is a link—or equivalently that the finite subgroups of the corresponding lattice are cyclic—is a natural one from the topological point of view; however, it is too restrictive for applications to maximal arithmetic lattices, because maximal arithmetic lattices typically contain dihedral groups (or, at the very least, copies of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which are dihedral groups of order 4). Thus the projected application will depend on relaxing this assumption. At present I am attempting to remove this hypothesis, and to improve the bound of 42 in the conclusion.

Theorem 5.1 follows formally from two propositions:

PROPOSITION 5.2. — *Let $O = H^3/\Gamma$ be an orientable hyperbolic 3-orbifold, and let M denote the underlying manifold of O. Suppose that the singular set of O is a link, and that the underlying manifold of O has no connected summand homeomorphic to $S^2 \times S^1$ or a (nontrivial) lens space. If O has*

volume at most 3.44, *then*

$$\dim H_1(|\Omega|; \mathbb{Z}/2\mathbb{Z}) \leqslant 3 + 5 \left\lfloor \frac{5}{3} \operatorname{vol}(\Omega) \right\rfloor.$$

In particular, $\dim H_1(|\Omega|; \mathbb{Z}/2\mathbb{Z}) \leqslant 28.$ $\dim H_1(M, \mathbb{Z}/2\mathbb{Z}) \leqslant 15.$

PROPOSITION 5.3. — *Let* $O = H^3/\Gamma$ *be an orientable hyperbolic* 3-*orbifold. Suppose that the singular set of* O *is a link. Then* O *has a two-sheeted orbifold cover* O' *such that the underlying manifold* M' *of* O' *satisfies* $2\dim H_1(M', \mathbb{Z}/2\mathbb{Z}) \geqslant \dim H_1(O, \mathbb{Z}/2\mathbb{Z}) - 1.$

The proof of Proposition 5.3 is an elementary application of Smith Theory.

In the special case where M is hyperbolic, Proposition 5.2 can be deduced from the result that I quoted above from [14]. When M is not hyperbolic, if $\dim H_1(M, \mathbb{Z}/2\mathbb{Z}) \geqslant 4$, there always exists an essential sphere or torus in M by Perelman's geometrization theorem [8]. Such a sphere or torus gives rise to an incompressible suborbifold of O. The results of [4], which are stated for manifolds but are easily adapted to orbifolds, give lower volumes for the volume of O in terms of data involving incompressible suborbifolds of O. These estimates are used in the proof of Proposition 5.3. The details are rather involved.

It appears that if one removes the hypothesis that the singular set is a link, one can prove a result similar to Proposition 5.2, but with a larger bound in the conclusion. On the other hand, I do not yet know how to prove a result qualitatively similar to Proposition 5.3 without the hypothesis that the singular set is a link.

Bibliography

[1] AGOL (I.). — Tameness of hyperbolic 3-manifolds. arXiv:math.GT/0405568.

[2] AGOL (I.), CULLER (M.), and SHALEN (P. B.). — Dehn surgery, homology and hyperbolic volume. Algebr. Geom. Topol., 6, p. 2297-2312 (2006).

[3] AGOL (I.), CULLER (M.), and SHALEN (P. B.). — Singular surfaces, mod 2 homology, and hyperbolic volume. I. Trans. Amer. Math. Soc., 362(7), p. 3463-3498 (2010).

[4] AGOL (I.), STORM (P. A.), and THURSTON (W. P.). — Lower bounds on volumes of hyperbolic Haken 3-manifolds. J. Amer. Math. Soc., 20(4), p. 1053-1077 (electronic) (2007). With an appendix by Nathan Dunfield.

[5] ANDERSON (J. W.), CANARY (R. D.), CULLER (M.), and SHALEN (P. B.). — Free Kleinian groups and volumes of hyperbolic 3-manifolds. J. Differential Geom., 43(4), p. 738-782 (1996).

[6] BAUMSLAG (G.) and SHALEN (P. B.). — Groups whose three-generator subgroups are free. Bull. Austral. Math. Soc., 40(2), p. 163-174 (1989).

[7] BESSIÈRES (L.), BESSON (G.), MAILLOT (S.), BOILEAU (M.), and PORTI (J.). Geometrisation of 3-manifolds, volume 13 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich (2010).

[8] BESSIÈRES (L.), BESSON (G.), MAILLOT (S.), BOILEAU (M.), and PORTI (J.). Geometrisation of 3-manifolds, volume 13 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich (2010).

[9] BOREL (A.). — Commensurability classes and volumes of hyperbolic 3-manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 8(1), p. 1-33 (1981).

[10] CALEGARI (D.) and GABAI (D.). — Shrinkwrapping and the taming of hyperbolic 3-manifolds. J. Amer. Math. Soc., 19(2), p. 385-446 (electronic) (2006).

[11] CHINBURG (T.) and FRIEDMAN (E.). — The smallest arithmetic hyperbolic three-orbifold. Invent. Math., 86(3), p. 507-527 (1986).

[12] CULLER (M.) and SHALEN (P. B.). — Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds. J. Amer. Math. Soc., 5(2), p. 231-288 (1992).

[13] CULLER (M.) and SHALEN (P. B.). — Singular surfaces, mod 2 homology, and hyperbolic volume, II. Topology Appl., 158(1), p. 118-131 (2011).

[14] CULLER (M.) and SHALEN (P. B.). — 4-free groups and hyperbolic geometry. J. Topol., 5(1), p. 81-136 (2012).

[15] JACO (W. H.) and SHALEN (P. B.). — Seifert fibered spaces in 3-manifolds. Mem. Amer. Math. Soc., 21(220):viii+192 (1979).

[16] KOJIMA (S.) and MIYAMOTO (Y.). — The smallest hyperbolic 3-manifolds with totally geodesic boundary. J. Differential Geom., 34(1), p. 175-192 (1991).

[17] MIYAMOTO (Y.). — Volumes of hyperbolic manifolds with geodesic boundary. Topology, 33(4), p. 613-629 (1994).

[18] SHALEN (P. B.) and WAGREICH (P.). — Growth rates, Zp-homology, and volumes of hyperbolic 3-manifolds. Trans. Amer. Math. Soc., 331(2), p. 895-917 (1992).

[19] SHAPIRO (A.) and WHITEHEAD (J. H. C.). — A proof and extension of Dehn's lemma. Bull. Amer. Math. Soc., 64, p. 174-178 (1958).