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Half-integral finite surgeries on knots in S^3

Eileen Li, Yi Ni[(1)]

Résumé. — Supposant qu'un nœud hyperbolique dans S^3 admet une chirurgie finie, Boyer et Zhang ont prouvé que la pente de la chirurgie doit être soit un entier, soit un demi-entier, et ils ont conjecturé que le dernier cas ne se produit pas. En utilisant les termes de correction dans l'homologie de Heegaard Floer, nous prouvons que si un noeud hyperbolique dans S^3 admet une chirurgie finie demi-entier, alors il doit avoir la même homologie de Floer des nœuds l'un des huit nœuds non-hyperboliques qui sont connus pour avoir ces chirurgies, et la variété résultante doit être l'une des dix formes de l'espace sphérique. Comme l'homologie de Floer des nœuds porte beaucoup d'informations sur le nœud, cela apporte une forte évidence à la conjecture de Boyer–Zhang.

Abstract. — Supposing that a hyperbolic knot in S^3 admits a finite surgery, Boyer and Zhang proved that the surgery slope must be either integral or half-integral, and they conjectured that the latter case does not happen. Using the correction terms in Heegaard Floer homology, we prove that if a hyperbolic knot in S^3 admits a half-integral finite surgery, then the knot must have the same knot Floer homology as one of the eight non-hyperbolic knots which are known to admit such surgeries, and the resulting manifold must be one of ten spherical space forms. As knot Floer homology carries a lot of information about the knot, this gives a strong evidence to Boyer–Zhang's conjecture.

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1. Introduction

Suppose that M is a 3–manifold with torus boundary, α is a slope on ∂M. Let $M(\alpha)$ be the Dehn filling along α. If M is hyperbolic, Thurston's Hyperbolic Dehn Surgery Theorem says that at most finitely many fillings are non-hyperbolic. These surgeries are called *exceptional surgeries*. The famous Cyclic Surgery Theorem [3] asserts that, if M is not Seifert fibered, α, β are slopes on ∂M such that both $M(\alpha)$ and $M(\beta)$ have cyclic fundamental groups, then $\Delta(\alpha, \beta)$, the distance between α, β, is at most 1. More generally, estimating the distance between any two exceptional surgery slopes is a central problem in Dehn surgery.

In [2], Boyer and Zhang proved that if M is hyperbolic, $M(\alpha)$ has finite fundamental group (or being a finite surgery) and $M(\beta)$ has cyclic fundamental group (or being a cyclic surgery), then $|\Delta(\alpha, \beta)| \leqslant 2$. In particular, if the $\frac{p}{q}$ surgery on a hyperbolic knot $K \subset S^3$, denoted $S^3_K(\frac{p}{q})$, has a finite fundamental group, then $|q| \leqslant 2$. In fact, Boyer and Zhang made the following conjecture.

Conjecture 1.1. — *Suppose that $K \subset S^3$ is a hyperbolic knot, and that $S^3_K(\frac{p}{q})$ has a finite fundamental group, then $\frac{p}{q}$ must be an integer.*

By Perelman's resolution of the Geometrization Conjecture [19, 20, 21], if a 3–manifold has a finite fundamental group, then it is necessarily a spherical space form. In order to prove Conjecture 1.1, we only need to rule out **T**- and **I**-type spherical space forms as results of half-integer surgeries, see Section 3 for more detail.

In this paper, all manifolds are oriented. If Y is an oriented manifold, then $-Y$ denotes the same manifold with the opposite orientation.

Let $\mathbb{T}$ be the exterior of the right hand trefoil, then $\mathbb{T}(\frac{p}{q})$ is the manifold obtained by $\frac{p}{q}$ surgery on the right hand trefoil. It is well-known that any **T**- or **I**-type manifold is homeomorphic to a $\pm\mathbb{T}(\frac{p}{q})$ (see Lemma 3.1).

Let $T(p, q)$ be the (p, q) torus knot, and let $[p_1, q_1; p_2, q_2]$ denote the (p_1, q_1) cable of $T(p_2, q_2)$.

Our main result is the following theorem.

Theorem 1.2. — *Suppose that $K \subset S^3$ is a hyperbolic knot, and that the $\frac{p}{2}$ surgery on K is a spherical space form for some odd integer $p > 0$, then K has the same knot Floer homology as either $T(5, 2)$ or a cable of a torus knot (which must be $T(3, 2)$ or $T(5, 2)$), and the resulting manifold is* *homeomorphic to the $\frac{p}{2}$ surgery on the corresponding torus or cable knot. More precisely, the possible cases are listed in the following table:*

knot type	*slope*	*resulting manifold*
$T(5,2)$	$17/2$	$-\mathbb{T}(17/2)$
$T(5,2)$	$23/2$	$\mathbb{T}(23/3)$
$[11,2;3,2]$	$43/2$	$\mathbb{T}(43/8)$
$[11,2;3,2]$	$45/2$	$\mathbb{T}(45/8)$
$[13,2;3,2]$	$51/2$	$\mathbb{T}(51/8)$
$[13,2;3,2]$	$53/2$	$\mathbb{T}(53/8)$
$[19,2;5,2]$	$77/2$	$-\mathbb{T}(77/12)$
$[21,2;5,2]$	$83/2$	$\mathbb{T}(83/13)$
$[17,3;3,2]$	$103/2$	$\mathbb{T}(103/18)$
$[19,3;3,2]$	$113/2$	$\mathbb{T}(113/18)$

Here the first column lists the knots with which K shares the same knot Floer homology.

Bleiler and Hodgson [1] have classified finite surgeries on all the iterated torus knots. The knots in the first column above are contained in their list.

The strategy of our proof is to compute the Heegaard Floer correction terms for the **T**- and **I**- type manifolds, then compare them with the correction terms of the half-integral surgeries on knots in S^3. If they match, then the knot Floer homology of the knots can be recovered from the correction terms.

Heegaard Floer homology has been successfully used in the study of finite surgery, see, for example, [16, 18, 6, 4]. The point here is that spherical space forms have the simplest possible Heegaard Floer homology, hence the information about the Heegaard Floer homology is completely contained in the correction terms. We will address this fact in more detail in Section 2.

Knot Floer homology tells us a lot about knots. For example, it detects the genus [17] and fiberedness [5, 10]. It is reasonable to expect that a knot with the same knot Floer homology as a knot in the above table must be the corresponding knot. Moreover, we propose the following conjecture, which would imply Conjecture 1.1.

CONJECTURE 1.3. — *Suppose $K \subset S^3$ has an L-space surgery. If all roots of its Alexander polynomial $\Delta_K(t)$ are unit roots, then K is an iterated torus knot.*

This paper is organized as follows. In Section 2, we recall the basic definition and properties of the correction terms. In Section 3, we discuss the strategy in our proof. In Section 4, we will show that half-integral finite surgeries do not exist when p is sufficiently large by concrete computations involving correction terms. Our approach in this section is inspired by Gu [7]. In Section 5, we discuss the computer search and the way to improve the estimates in Section 4.

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2. Preliminaries on Heegaard Floer homology and correction terms

Heegaard Floer homology was introduced by Ozsváth and Szabó [13]. Given a closed oriented 3–manifold Y and a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, one can define the Heegaard Floer homology groups $\widehat{HF}(Y,\mathfrak{s}), HF^+(Y,\mathfrak{s})$, $\dots$, which are invariants of $(Y,\mathfrak{s})$. When $\mathfrak{s}$ is torsion, there is an absolute $\mathfrak{Q}$–grading on $HF^+(Y,\mathfrak{s})$. When Y is a rational homology sphere, Ozsváth and Szabó [14] defined a *correction term* $d(Y,\mathfrak{s}) \in \mathfrak{Q}$, which is basically the shifting of $HF^+(Y,\mathfrak{s})$ relative to $HF^+(S^3)$ in the absolute grading.

The correction terms enjoy the following symmetries:

$$d(Y,\mathfrak{s}) = d(Y,J\mathfrak{s}), \quad d(-Y,\mathfrak{s}) = -d(Y,\mathfrak{s}), \tag{2.1}$$

where $J\colon \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y)$ is the conjugation.

Suppose that Y is an integral homology sphere, $K \subset Y$ is a knot. Let $Y_K(p/q)$ be the manifold obtained by $\frac{p}{q}$–surgery on K. Ozsváth and Szabó defined a natural identification $\sigma\colon \mathbb{Z}/p\mathbb{Z} \to \mathrm{Spin}^c(Y_K(p/q))$ [14, 18]. For simplicity, we often use an integer i to denote the Spin^c structure $\sigma([i])$, when $[i] \in \mathbb{Z}/p\mathbb{Z}$ is the congruence class of i modulo p.

A rational homology sphere Y is an *L-space* if $\mathrm{rank}\widehat{HF}(Y) = |H_1(Y)|$. Examples of L-spaces include spherical space forms. The information about the Heegaard Floer homology of an L-space is completely encoded in its correction terms.

Let $L(p,q)$ be the lens space obtained by $\frac{p}{q}$–surgery on the unknot. The correction terms for lens spaces can be computed inductively as in [14]:

$$\begin{aligned} d(S^3,0) &= 0, \\ d(L(p,q),i) &= -\frac{1}{4}+\frac{(2i+1-p-q)^2}{4pq}-d(L(q,r),j), \end{aligned} \tag{2.2}$$

where $0 \leqslant i < p+q$, r and j are the reductions modulo p of q and i, respectively.

For example, using the recursion formula (2.2), we can compute

$$\begin{aligned} d(L(p,1),i) &= -\frac{1}{4}+\frac{(2i-p)^2}{4p} \\ d(L(p,2),i) &= -\frac{1}{4}+\frac{(2i-p-1)^2}{8p}-d(L(2,1),j) \\ &= \frac{(2i-p-1)^2}{8p}-\frac{1+(-1)^i}{4} \qquad (2.3) \\ d(L(3,q),i) &= \begin{cases} (\frac{1}{2},-\frac{1}{6},-\frac{1}{6}), & q=1, i=0,1,2 \\ (\frac{1}{6},\frac{1}{6},-\frac{1}{2}), & q=2, i=0,1,2 \end{cases} \qquad (2.4) \\ d(L(5,q),i) &= \begin{cases} (1,\frac{1}{5},-\frac{1}{5},-\frac{1}{5},\frac{1}{5}), & q=1, i=0,1,2,3,4 \\ (\frac{2}{5},\frac{2}{5},-\frac{2}{5},0,-\frac{2}{5}), & q=2, i=0,1,2,3,4 \\ (\frac{2}{5},0,\frac{2}{5},-\frac{2}{5},-\frac{2}{5}), & q=3, i=0,1,2,3,4 \\ (-\frac{1}{5},\frac{1}{5},\frac{1}{5},-\frac{1}{5},-1), & q=4, i=0,1,2,3,4 \end{cases} \qquad (2.5) \end{aligned}$$

Given a null-homologous knot $K \subset Y$, Ozsváth–Szabó [15] and Rasmussen [22] defined the knot Floer homology. The basic philosophy is, if we know all the information about the knot Floer homology, then we can compute the Heegaard Floer homology of all the surgeries on K. In particular, if the $\frac{p}{q}$–surgery on $K \subset S^3$ is an L-space surgery, where $p,q>0$, then the correction terms of $S^3_K(p/q)$ can be computed from the Alexander polynomial $\Delta_K(T)$ of K as follows.

Suppose

$$\Delta_K(T)=\sum_i a_i t^i.$$

Define a sequence of integers

$$t_i=\sum_{j=1}^{\infty} j a_{i+j}, \quad i \geqslant 0.$$

then a_i can be recovered from t_i by

$$a_i = t_{i-1} - 2t_i + t_{i+1}, \quad \text{for } i > 0. \tag{2.6}$$

If K admits an L-space surgery, then one can prove [18, 22]

$$t_s \geqslant 0, \quad t_s \geqslant t_{s+1} \geqslant t_s - 1, \quad t_{g(K)} = 0. \tag{2.7}$$

Moreover, the following proposition holds.

Proposition 2.1. — *Suppose the $\frac{p}{q}$–surgery on $K \subset S^3$ is an L-space surgery, where $p, q > 0$. Then for any $0 \leqslant i \leqslant p-1$ we have*

$$d(S^3_K(p/q), i) = d(L(p,q), i) - 2\max\{t_{\lfloor \frac{i}{q} \rfloor}, t_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}.$$

This formula is contained in Ozsváth–Szabó [18] and Rasmussen [22]. A more general version of this formula can be found in Ni–Wu [12].

Lemma 2.2. — *Suppose i is an integer satisfying $0 \leqslant i < p+q$, then $J(\sigma([i]))$ is represented by $p+q-1-i$.*

Proof. — We only need to examine our result for surgeries on the unknot, as this is a homological statement.

As in the proof of [14, Proposition 4.8], there is a two-handle addition cobordism X from $-L(q,r)$ to $-L(p,q)$, where r is the reduction of p modulo q. Let $i \in \{0, 1, \dots, p+q-1\}$ be a number. Let r and j be the reduction of p and i modulo q. The proof of [14, Proposition 4.8] shows that there is a Spin^c structure $\mathfrak{s}_z(\psi_i)$ such that its restriction on $-L(p,q)$ is represented by i and its restriction on $-L(q,r)$ is represented by j. Moreover,

$$\langle c_1(\mathfrak{s}_z(\psi_i)), H\rangle = 2i + 1 - p - q$$

for a generator H of $H_2(X)$.

Now we choose $i_1, i_2 \in \{0, 1, \dots, p+q-1\}$ such that $i_1 + i_2 = p+q-1$, and let j_1, j_2 be the reductions of i_1, i_2 modulo q. We have

$$\langle c_1(\mathfrak{s}_z(\psi_{i_1})), H\rangle = -\langle c_1(\mathfrak{s}_z(\psi_{i_2})), H\rangle. \tag{2.8}$$

We claim that

$$H^2(X) \cong H_2(X, \partial X) \cong \mathbb{Z}.$$

In fact, let $Y_1 = Y_3 = -L(q,r)$, $Y_2 = -L(p,q)$. We have exact sequences:

$$0 \to H_2(X, Y_i) \to H_2(X, \partial X) \to H_1(Y_{i+1}) \to H_1(X, Y_i), \quad i = 1, 2,$$

Since X is a 2–handle cobordism,

$$H_2(X,Y_1) \cong H_2(X,Y_2) \cong \mathbb{Z}, \quad H_1(X,Y_1) \cong H_1(X,Y_2) \cong 0.$$

So the above exact sequences become

$$0 \to \mathbb{Z} \to H_2(X,\partial X) \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

$$0 \to \mathbb{Z} \to H_2(X,\partial X) \to \mathbb{Z}/q\mathbb{Z} \to 0.$$

As $\gcd(p,q)=1$, it is easy to see $H_2(X,\partial X) \cong \mathbb{Z}$. This finishes the proof of the claim.

It follows from the claim and (2.8) that $\psi_{i_2} = J\psi_{i_1}$. Hence i_2 represents $J(\sigma([i_1]))$. □

Corollary 2.3. — *If p is odd, then $J\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ has a unique fixed point:*

$$C(p,q) = \begin{cases} \dfrac{p+q-1}{2}, & \text{if } q \text{ is even,} \\[2ex] \dfrac{q-1}{2}, & \text{if } q \text{ is odd.} \end{cases}$$

3. The strategy of our proof

By the Geometrization theorem [19, 20, 21], if a 3–manifold has a finite fundamental group, then the manifold must be a spherical space form. Besides S^3, there are five types of spherical space forms: **C, D, T, O, I**. The **C**-type manifolds are the lens spaces with cyclic fundamental groups; the **D**-type manifolds are Seifert fibered spaces over the orbifold $S^2(2,2,n)$ with dihedral type fundamental groups; the **T**-type manifolds are Seifert fibered spaces over the orbifold $S^2(2,3,3)$ with tetrahedral type fundamental groups; the **O**-type manifolds are Seifert fibered spaces over the orbifold $S^2(2,3,4)$ with octahedral type fundamental groups; the **I**-type manifolds are Seifert fibered spaces over the orbifold $S^2(2,3,5)$ with icosahedral type fundamental groups.

It follows from the Cyclic Surgery Theorem [3] that **C**-type manifolds cannot be obtained by half-integral surgeries on hyperbolic knots. The **D**- and **O**-type manifolds have even order H_1, so they cannot be obtained from half-integral surgery. We only need to consider **T**- and **I**-type manifolds.

Lemma 3.1. — *Any* **T**-*type manifold is homeomorphic to* $\pm\mathbb{T}(\frac{6n\pm3}{n})$ *for some positive integer n with* $\gcd(n,3)=1$. *Any* **I**-*type manifold is homeomorphic to* $\pm\mathbb{T}(\frac{6n\pm5}{n})$ *for some positive integer n with* $\gcd(n,5)=1$.

Proof. — Suppose Y is a **T**-type manifold, then it is Seifert fibered over the base orbifold $S^2(2,3,3)$. Removing the neighborhood of a multiplicity 3 singular fiber, we get a Seifert fibered space over the orbifold $D^2(2,3)$. The classification of Seifert fibered spaces tells us that there is only one such manifold up to orientation reversal, which is the trefoil complement $\mathbb{T}$. So Y or $-Y$ can be obtained by Dehn filling on $\mathbb{T}$. The same argument works for **I**-type manifolds.

Now we consider the problem when we get Seifert fibered spaces with base orbifold $S^2(2,3,3)$ and $S^2(2,3,5)$ by Dehn filling on $\mathbb{T}$. The regular fiber on $\partial\mathbb{T}$ has slope 6, so $\frac{p}{q}$–filling will create a multiplicity $\Delta(\frac{p}{q},6)$ fiber. To get a Seifert fibered space with base orbifold $S^2(2,3,3)$ or $S^2(2,3,5)$, we need to have $\Delta(\frac{p}{q},6)=3$ or 5. So $\frac{p}{q}=\frac{6n\pm3}{n}$ or $\frac{6n\pm5}{n}$ for some $n>0$. □

Let $p,q>0$ be coprime integers. Using Proposition 2.1, we get

$$d(\mathbb{T}(p/q),i)=d(L(p,q),i)-2\chi_{[0,q)}(i), \tag{3.1}$$

where

$$\chi_{[0,q)}(i)=\begin{cases}1, & \text{when } 0\leqslant i<q,\\ 0, & \text{when } q\leqslant i<p.\end{cases}$$

Suppose $S^3_K(p/2)$ is a spherical manifold, then by (2.3) and Proposition 2.1

$$\begin{aligned}d(S^3_K(p/2),i) &= d(L(p,2),i)-2\max\{t_{\lfloor\frac{i}{2}\rfloor},t_{\lfloor\frac{p+1-i}{2}\rfloor}\}\\ &= -\frac{1+(-1)^i}{4}+\frac{(2i-p-1)^2}{8p}-2t_{\min\{\lfloor\frac{i}{2}\rfloor,\lfloor\frac{p+1-i}{2}\rfloor\}}.\end{aligned}$$

If $S^3_K(p/2)\cong\varepsilon\mathbb{T}(p/q)$ for $\varepsilon\in\{-1,1\}$, where "$\cong$" stands for orientation preserving homeomorphism, then the two sets

$$\{d(S^3_K(p/2),i)|\, i\in\mathbb{Z}/p\mathbb{Z}\},\quad \{d(\varepsilon\mathbb{T}(p/q),i)|\, i\in\mathbb{Z}/p\mathbb{Z}\}$$

are necessarily equal. However, the two parametrizations of Spin^c may not be equal: they could differ by an affine isomorphism of $\mathbb{Z}/p\mathbb{Z}$. More precisely, there exists an affine isomorphism $\phi\colon\ \mathbb{Z}/p\mathbb{Z}\to\mathbb{Z}/p\mathbb{Z}$, such that

$$d(S^3_K(p/2),i)=d(\varepsilon\mathbb{T}(p/q),\phi(i)).$$

This map ϕ commutes with J, so it follows from Corollary 2.3 that $\phi(C(p,2)) = C(p,q)$. For any integer a, define $\phi_a\colon\ \mathbb{Z}/p\mathbb{Z}\to\mathbb{Z}/p\mathbb{Z}$ by

$$\phi_a(i)\equiv a\big(i-C(p,2)\big)+C(p,q). \tag{3.2}$$

By (2.1) and Lemma 2.2, $d(\mathbb{T}(p/q), \phi_a(i)) = d(\mathbb{T}(p/q), \phi_{p-a}(i))$. So we may assume

$$d(S^3_K(p/2), i) = \varepsilon d(\mathbb{T}(p/q), \phi_a(i)), \quad \text{for any } i \in \mathbb{Z}/p\mathbb{Z},$$

and for some a satisfying

$$0 < a < \frac{p}{2}, \quad \gcd(p, a) = 1. \tag{3.3}$$

Let

$$\delta^{\varepsilon}_a(i) = d(L(p,2), i) - \varepsilon d(\mathbb{T}(p/q), \phi_a(i)). \tag{3.4}$$

By Proposition 2.1, we should have

$$\delta^{\varepsilon}_a(i) = 2t_{\min\{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{p+1-i}{2} \rfloor\}} \tag{3.5}$$

if $S^3_K(p/2) \cong \varepsilon\mathbb{T}(p/q)$ and ϕ_a (or ϕ_{p-a}) identifies their Spin^c structures.

Proof of Theorem 1.2. — We will compute the correction terms of the **T**- and **I**-type manifolds using (3.1). For all a satisfying (3.3), we compute the sequence $\delta^{\varepsilon}_a(i)$. Then we check if this sequence satisfies (3.5) for any $\{t_s\}$ as in (2.7).

By Proposition 4.1, the equality (3.5) does not hold when p is sufficiently large. For small p, the strategy in the last paragraph can be realized by direct computer calculatioins. See Section 5 for more details.

As a calculation result, we found all the possible p/q, which are exactly $7/2$ and the numbers given in the table in Theorem 1.2. We also get the corresponding correction terms, from which we can recover the Alexander polynomials using (3.5) and (2.6). By [16, Theorem 1.2], from the Alexander polynomials we can get the knot Floer homology of the corresponding knots, which should be the knot Floer homology of either $T_{3,2}$ (when $p/q = 7/2$), or $T_{5,2}$, or their cable knots as in the table in Theorem 1.2. By Ghiggini [5], if the knot Floer homology is the same as that of $T_{3,2}$, then the knot must be $T_{3,2}$. So we are left with the knots corresponding to the knots in the table in Theorem 1.2. □

4. The case when p is large

In this section, we will assume that $S^3_K(p/2) \cong \varepsilon\mathbb{T}(p/q)$, and

$$p = 6q + \zeta r, \quad r \in \{3, 5\}, \quad \varepsilon, \zeta \in \{1, -1\}.$$

We will prove that this does not happen when p is sufficiently large. More precisely, we will show:

Proposition 4.1. — *If $p \geqslant 192r(36r+2)^2$, then $S^3_K(p/2) \not\cong \varepsilon\mathbb{T}(p/q)$, where $p = 6q + \zeta r$, $r \in \{3,5\}$.*

Let $s \in \{0,1,\dots,r-1\}$ be the reduction of q modulo r. For any integer n, let $\theta(n) \in \{0,1\}$ be the reduction of n modulo 2, and let $\bar\theta(n) = 1-\theta(n)$. More generally, let $[n]_p$ be the integer in $[0,p)$ such that $n \equiv [n]_p \pmod p$.

The equation (3.2) becomes

$$\phi_a(i) \equiv a(i - \frac{p+1}{2}) + \frac{\bar\theta(q)p+q-1}{2} \pmod p.$$

Using (2.3), (3.1) and (3.4), we get

$$\begin{aligned}\delta^\varepsilon_a(i) &= d(L(p,2),i) - \varepsilon d(\mathbb{T}(p/q),\phi_a(i)) \\ &= \frac{(2i-p-1)^2}{8p} - \frac{\bar\theta(i)}{2} - \varepsilon d(L(p,q),\phi_a(i)) + 2\varepsilon\chi_{[0,q)}(\phi_a(i)) \quad (4.1)\end{aligned}$$

Lemma 4.2. — *Assume that $S^3_K(p/2) \cong \varepsilon\mathbb{T}(p/q)$. Let $m \in \{0,1,2,3\}$ satisfy that*

$$0 \leqslant a - mq + \frac{\bar\theta(q)\zeta r + q - 1}{2} < q,$$

then

$$\left|a - \frac{mp}{6}\right| < \sqrt{\frac{4rp}{3}}.$$

Proof. — Since $S^3_K(p/2) \cong \varepsilon\mathbb{T}(p/q)$, there exists an integer a satisfying (3.3) such that (3.5) holds. It follows from (2.7) and (3.5) that

$$\delta^\varepsilon_a(\frac{p+3}{2}) - \delta^\varepsilon_a(\frac{p+1}{2}) = 0 \text{ or } 2. \quad (4.2)$$

Using (4.1), we get

$$\begin{aligned}&\delta^\varepsilon_a(\frac{p+3}{2}) - \delta^\varepsilon_a(\frac{p+1}{2}) = \frac{1}{2p} - \frac{\bar\theta(\frac{p+3}{2})}{2} - \varepsilon d(L(p,q), a \\ &+\frac{\bar\theta(q)p+q-1}{2}) + 2\varepsilon\chi_{[0,q)}([a + \frac{\bar\theta(q)p+q-1}{2}]_p) \\ &+\frac{\bar\theta(\frac{p+1}{2})}{2} + \varepsilon d(L(p,q), \frac{\bar\theta(q)p+q-1}{2}) - 2\varepsilon\chi_{[0,q)}(\frac{\bar\theta(q)p+q-1}{2}). \quad (4.3)\end{aligned}$$

Let

$$\begin{aligned}C_0 = \varepsilon\zeta&\left(-d(L(r,s), a - mq + \frac{\bar\theta(q)\zeta r+q-1}{2}) + d(L(r,s), \frac{\bar\theta(q)\zeta r+q-1}{2})\right) \\ &+\frac{1}{2p} + \frac{1}{2} - \theta(\frac{p+1}{2}) + 2\varepsilon\left(\chi_{[0,q)}([a + \frac{\bar\theta(q)p+q-1}{2}]_p) - \chi_{[0,q)}(\frac{\bar\theta(q)p+q-1}{2})\right).\end{aligned}$$

When $\zeta = 1$, by the recursive formula (2.2), the right hand side of (4.3) becomes

$$\begin{aligned}
&\varepsilon\left(\frac{-(2a-\theta(q)p)^2+(\theta(q)p)^2}{4pq}+d(L(q,r),a-mq+\frac{\bar\theta(q)r+q-1}{2})\right.\\
&\left.-d(L(q,r),\frac{\bar\theta(q)r+q-1}{2})\right)+\frac{1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})\\
&+2\varepsilon\left(\chi_{[0,q)}([a+\frac{\bar\theta(q)p+q-1}{2}]_p)-\chi_{[0,q)}(\frac{\bar\theta(q)p+q-1}{2})\right)\\
=\;&\varepsilon\left(-\frac{a^2}{pq}+\frac{\theta(q)a}{q}+\frac{(2a-2mq-\theta(q)r)^2-(\theta(q)r)^2}{4qr}\right)+C_0\\
=\;&\varepsilon\left(\frac{6}{pr}(a-\frac{mp}{6})^2-\frac{m^2-6m\theta(q)}{6}\right)+C_0.
\end{aligned}$$

When $\zeta = -1$, the right hand side of (4.3) becomes

$$\begin{aligned}
&\varepsilon\left(\frac{-(2a-\theta(q)p)^2+(\theta(q)p)^2}{4pq}+d(L(q,q-r),a-mq+\frac{-\bar\theta(q)r+q-1}{2})\right.\\
&\left.-d(L(q,q-r),\frac{-\bar\theta(q)r+q-1}{2})\right)+\frac{1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})\\
&+2\varepsilon\left(\chi_{[0,q)}([a+\frac{\bar\theta(q)p+q-1}{2}]_p)-\chi_{[0,q)}(\frac{\bar\theta(q)p+q-1}{2})\right)\\
=&\varepsilon\left(-\frac{a^2}{pq}+\frac{\theta(q)a}{q}+\frac{(2a-2mq+\theta(q)r-q)^2-(\theta(q)r-q)^2}{4q(q-r)}\right.\\
&\left.-d(L(q-r,r),a-mq+\frac{-\bar\theta(q)r+q-1}{2})+d(L(q-r,r),\frac{-\bar\theta(q)r+q-1}{2})\right)\\
&+\frac{1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})+2\varepsilon\left(\chi_{[0,q)}([a+\frac{\bar\theta(q)p+q-1}{2}]_p)-\chi_{[0,q)}(\frac{\bar\theta(q)p+q-1}{2})\right)\\
=&\varepsilon\left(-\frac{a^2}{pq}+\frac{\theta(q)a}{q}+\frac{(a-mq+\theta(q)r-q)(a-mq)}{q(q-r)}-\frac{(a-mq-\bar\theta(q)r)(a-mq)}{(q-r)r}\right)\\
&+C_0\\
=&\varepsilon\left(-\frac{a^2}{pq}+\frac{\theta(q)a}{q}-\frac{(a-mq+\theta(q)r)(a-mq)}{qr}\right)+C_0\\
=&\varepsilon\left(-\frac{6}{pr}(a-\frac{mp}{6})^2-\frac{m^2-6m\theta(q)}{6}\right)+C_0.
\end{aligned}$$

Using (2.4), (2.5), we have

$$|C_0|\leqslant\frac{6}{5}+\frac{1}{2p}+\frac{1}{2}+2=\frac{37}{10}+\frac{1}{2p}<\frac{9}{2}.$$

Moreover, $|\frac{m^2-6m\theta(q)}{6}| \leqslant \frac{3}{2}$. It follows from (4.2) that

$$\left|\frac{6}{pr}(a-\frac{mp}{6})^2\right| \leqslant 2+\frac{3}{2}+\frac{9}{2}=8,$$

so our conclusion holds. □

Lemma 4.3. — *Let k be an integer satisfying*

$$0 \leqslant k < \frac{1}{48}\frac{p-13r+6}{\sqrt{3rp}} - \frac{1}{6}. \tag{4.4}$$

Let

$$i_k = \frac{\bar\theta(q)p+q-1}{2} + a(6k) - kmp, \quad j_k = \frac{\bar\theta(q)\zeta r+q-1}{2} + a(6k) - kmp.$$

Then

$$\delta_a^\varepsilon(\frac{p+1}{2}+6k+1) - \delta_a^\varepsilon(\frac{p+1}{2}+6k) = Ak+B+C_k,$$

where

$$\begin{aligned} A &= \varepsilon\zeta\cdot\frac{2(6a-mp)^2}{pr}+\frac{6}{p}, \\ B &= \varepsilon\left(\frac{6\zeta}{pr}(a-\frac{mp}{6})^2-\frac{m^2-6m\theta(q)}{6}\right), \\ C_k &= \varepsilon\zeta\big(-d(L(r,s),j_k+a-mq)+d(L(r,s),j_k)\big) \\ &\quad +\frac{1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})+2\varepsilon\chi_{[0,q)}([i_k+a]_p)-2\varepsilon\chi_{[0,q)}([i_k]_p). \end{aligned}$$

Proof. — By (4.4), we have

$$(6k+1)\sqrt{\frac{4rp}{3}} < \frac{p-13r+6}{12} \leqslant \frac{q-2r+1}{2}. \tag{4.5}$$

It follows from (3.3), (4.5) and Lemma 4.2 that

$$0 \leqslant i_k < i_k+a < p+q, \qquad 0 \leqslant j_k, j_k+a-mq < q. \tag{4.6}$$

For example,

$$\begin{aligned} j_k+a-mq &= j_k+a-m\frac{p-\zeta r}{6} \\ &= \frac{\bar\theta(q)\zeta r+q-1}{2}+(6k+1)(a-\frac{mp}{6})+\frac{m\zeta r}{6} \\ &< \frac{r+q-1}{2}+\frac{q-2r+1}{2}+\frac{r}{2} \\ &= q. \end{aligned}$$

Similar argument shows other inequalities.

Using (4.1), we can compute

$$\begin{aligned}
&\delta_a^\varepsilon(\frac{p+1}{2}+6k+1)-\delta_a^\varepsilon(\frac{p+1}{2}+6k)\\
= \quad & \frac{(6k+1)^2}{2p}-\frac{\bar{\theta}(\frac{p+1}{2}+6k+1)}{2}-\varepsilon d(L(p,q),i_k+a)+2\varepsilon\chi_{[0,q)}([i_k+a]_p)\\
&-\frac{(6k)^2}{2p}+\frac{\bar{\theta}(\frac{p+1}{2}+6k)}{2}+\varepsilon d(L(p,q),i_k)-2\varepsilon\chi_{[0,q)}([i_k]_p). \qquad (4.7)
\end{aligned}$$

When $\zeta=1$, using (4.6) and the recursion formula (2.2), the right hand side of (4.7) becomes

$$\begin{aligned}
&\varepsilon\left(\frac{-(2i_k+2a+1-p-q)^2+(2i_k+1-p-q)^2}{4pq}+d(L(q,r),j_k+a-mq)-d(L(q,r),j_k)\right)\\
&+\frac{12k+1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})+2\varepsilon\chi_{[0,q)}([i_k+a]_p)-2\varepsilon\chi_{[0,q)}([i_k]_p)\\
= \ &\varepsilon\left(\frac{-a(a(12k+1)-2kmp-\theta(q)p)}{pq}+\frac{(2j_k+2a-2mq+1-q-r)^2-(2j_k+1-q-r)^2}{4qr}\right)\\
&+\frac{6k}{p}+C_k\\
= \ &\varepsilon\left(\frac{-a(a(12k+1)-2kmp-\theta(q)p)}{pq}+\frac{(a-mq)((12k+1)a-2kmp-mq-\theta(q)r)}{qr}\right)\\
&+\frac{6k}{p}+C_k\\
= \ &Ak+B+C_k.
\end{aligned}$$

When $\zeta=-1$, the right hand side of (4.7) becomes

$$\begin{aligned}
&\varepsilon\left(\frac{-(2i_k+2a+1-p-q)^2+(2i_k+1-p-q)^2}{4pq}+d(L(q,q-r),j_k+a-mq)-d(L(q,q-r),j_k)\right)\\
&+\frac{12k+1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})+2\varepsilon\chi_{[0,q)}([i_k+a]_p)-2\varepsilon\chi_{[0,q)}([i_k]_p)\\
= \ &\varepsilon\Big(\frac{-a(a(12k+1)-2kmp-\theta(q)p)}{pq}+\frac{(2j_k+2a-2mq+1-2q+r)^2-(2j_k+1-2q+r)^2}{4q(q-r)}\\
&-d(L(q-r,r),j_k+a-mq)+d(L(q-r,r),j_k)\Big)\\
&+\frac{12k+1}{2p}+\frac{1}{2}-\theta(\frac{p+1}{2})+2\varepsilon\chi_{[0,q)}([i_k+a]_p)-2\varepsilon\chi_{[0,q)}([i_k]_p)\\
= \ &\varepsilon\Big(\frac{-a(a(12k+1)-2kmp-\theta(q)p)}{pq}+\frac{(a-mq)(-\bar{\theta}(q)r+12ka-2kmp+a-mq-q+r)}{q(q-r)}\\
&-\frac{(2j_k+2a-2mq+1-q)^2-(2j_k+1-q)^2}{4(q-r)r}\Big)+\frac{6k}{p}+C_k\\
= \ &\varepsilon\left(\frac{-a(a(12k+1)-2kmp-\theta(q)p)}{pq}-\frac{(a-mq)((12k+1)a-2kmp-mq+\theta(q)r)}{qr}\right)\\
&+\frac{6k}{p}+C_k\\
= \ &Ak+B+C_k.
\end{aligned}$$

□

Proof of Proposition 4.1.. — If $S^3_K(p/2) \cong \varepsilon\mathbb{T}(p/q)$, then (3.5) holds, so

$$\delta^\varepsilon_a(\frac{p+1}{2}+6k+1)-\delta^\varepsilon_a(\frac{p+1}{2}+6k)=0 \text{ or } 2 \tag{4.8}$$

for all k satisfying (4.4). If $p \geqslant 192r(36r+2)^2$, then

$$(6\cdot 6r+1)\cdot 8\sqrt{3r} \leqslant \sqrt{p}-1 < \frac{p-13r+6}{\sqrt{p}},$$

hence $k=6r$ satisfies (4.4).

Let A, B, C_k be as in Lemma 4.3. If $A \neq 0$, then $Ak+B+C$ is equal to 0 or 2 for at most two values of k for any given C. Given $p, q, a, \varepsilon, \zeta$, as k varies, C_k can take at most $3r$ values. It follows that $Ak+B+C_k$ can not be 0 or 2 for $k=0,1,\dots,6r$. As a consequence, if $p \geqslant 192r(36r+2)^2$, then (4.8) does not hold.

The remaining case we need to consider is that $A=0$. In this case

$$r=3, \quad \varepsilon\zeta=-1, \quad 6a-mp=\pm 3. \tag{4.9}$$

So

$$\begin{aligned} B+C_k \;=\;& d(L(3,s),j_k+a-mq)-d(L(3,s),j_k)+\frac{1}{2}-\frac{\varepsilon m^2}{6} \\ & -\theta(\frac{p+1}{2})+\varepsilon\big(m\theta(q)+2\chi_{[0,q)}([i_k+a]_p)-2\chi_{[0,q)}([i_k]_p)\big) \end{aligned} \tag{4.10}$$

Note that the second row in the above expression is always an integer. Using (2.4), the value of $d(L(3,s),j_k+a-mq)-d(L(3,s),j_k)$ is 0 or $\pm\frac{2}{3}$, so $B+C_k$ is an integer only if $m=1$ or 3.

If $m=3$, then $a=\frac{p-1}{2}\equiv 1 \pmod 3$. Since

$$j_k=\frac{3\zeta\bar{\theta}(q)+q-1}{2}+a(6k)-kmp \equiv 1-s \pmod 3,$$

it follows from (2.4) that

$$d(L(3,s),j_k+a-mq)-d(L(3,s),j_k)=d(L(3,s),2-s)-d(L(3,s),1-s)=\pm\frac{2}{3},$$

so $B+C_k$ is not an integer.

If $m = 1$, $a = \frac{p\pm 3}{6} \in \{q, q+\zeta\}$. If $a = q$, $B + C_k$ is not an integer. So we must have $a = q + \zeta$. Consider the first row on the right hand side of (4.10) and let $k = 0$, we get

$$d(L(3,s), 1-s+\zeta) - d(L(3,s), 1-s) + \frac{1}{2} - \frac{\varepsilon}{6}.$$

Since this number is an integer, using (2.4), we get

$$s = \begin{cases} 2, & \text{if } \varepsilon = 1, \\ 1, & \text{if } \varepsilon = -1. \end{cases}$$

We consider $\delta_a^\varepsilon(6) - \delta_a^\varepsilon(7)$, which is 0 by (3.5).

If $\varepsilon = 1$, it follows from (4.9) that $\zeta = -1$. Since $a = q + \zeta = q - 1$, we can compute

$$\phi_a(6) = (q-1)(6 - \frac{p+1}{2}) + \frac{\bar{\theta}(q)p + q - 1}{2} \equiv 6q - 6 \pmod{p}.$$

So we have

$$\begin{aligned} & \delta_a^{+1}(6) - \delta_a^{+1}(7) \\ = \; & \frac{(11-p)^2}{8p} - \frac{1}{2} - d(L(p,q), 6q-6) - \frac{(13-p)^2}{8p} + d(L(p,q), q-4) - 2 \\ = \; & \frac{p-12}{2p} - \frac{1}{2} - \frac{(5q-8)^2 - (-5q-4)^2}{4pq} + d(L(q,q-3), q-6) \\ & -d(L(q,q-3), q-4) - 2 \\ = \; & -\frac{6}{p} + \frac{30q-12}{pq} + \frac{(-8)^2 - (-4)^2}{4q(q-3)} \\ & -d(L(q-3,3), q-6) + d(L(q-3,3), q-4) - 2 \\ = \; & -\frac{6}{p} + \frac{30q-12}{pq} + \frac{12}{q(q-3)} - \frac{(q-11)^2 - (q-7)^2}{12(q-3)} \\ & +d(L(3,2), 2) - d(L(3,2), 1) - 2 \\ = \; & -2, \end{aligned}$$

a contradiction.

If $\varepsilon = -1$, then $\zeta = 1$. Since $a = q + \zeta = q + 1$, we get

$$\phi_a(6) = (q+1)(6 - \frac{p+1}{2}) + \frac{\bar{\theta}(q)p + q - 1}{2} \equiv 2 \pmod{p}.$$

So we have

$$\begin{aligned}
& \delta_a^{-1}(6)-\delta_a^{-1}(7) \\
= & \frac{(11-p)^2}{8p}-\frac{1}{2}+d(L(p,q),2)-2-\frac{(13-p)^2}{8p}-d(L(p,q),q+3) \\
= & \frac{p-12}{2p}-\frac{1}{2}+\frac{(5-p-q)^2-(2q+7-p-q)^2}{4pq} \\
& -d(L(q,3),2)+d(L(q,3),3)-2 \\
= & -\frac{6}{p}+\frac{(q+1)(p-6)}{pq}-\frac{(5-q-3)^2-(7-q-3)^2}{12q} \\
& +d(L(3,1),2)-d(L(3,1),0)-2 \\
= & -2,
\end{aligned}$$

a contradiction. □

5. Improved estimates and the computer search

Given Proposition 4.1, in order to prove Theorem 1.2, we only need to check finitely many p (for $p \leqslant 3.18 \times 10^7$ by Proposition 4.1) and show that the only possibilities are the ones given in Theorem 1.2. The calculation can always be carried out by a computer search.

As the reader may find, the bound $192r(36r+2)^2$ in Proposition 4.1 can be greatly decreased by carefully improving our estimates. The first author [9] has carried out a case-by-case analysis, which shows that p cannot be greater than $6,000$. Our computer search is based on this more practical bound, rather than Proposition 4.1.

We wrote a *Mathematica* program [11] to carry out the calculation, which is done on our desktop computer within 10 minutes. During the calculation, all numbers involved are integers or rationals, for which *Mathematica* always gives exact results. So our computer calculation is reliable.

In the rest of this section, we will illustrate in the following lemma how we can lower the bound in the case when $r=3$, $\varepsilon=\zeta=1$.

Lemma 5.1. — *If $p \geqslant 951$, then $S^3_K(p/2) \ncong \mathbb{T}(p/q)$, where $p=6q+3$.*

Proof. — We use a similar strategy as in the proof of Proposition 4.1. Assume that $S^3_K(p/2) \cong \mathbb{T}(p/q)$, then (3.5) holds for some a satisfying (3.3) and a sequence of integers $\{t_s\}$ satisfying (2.7).

Step 1. If $q > 2$, then $q \equiv 2 \pmod 6$.

Using (4.1), we can compute

$$\delta_a(3q+2) = \begin{cases} 0, & \text{if } q \equiv 1 \pmod 6, \\ 0, & \text{if } q \equiv 2 \pmod 6, \\ -1, & \text{if } q \equiv 4 \pmod 6, \\ 1, & \text{if } q \equiv 5 \pmod 6. \end{cases}$$

Since $\delta_a(i)$ should be even, the only possible cases are $q \equiv 1$ or $2 \pmod 6$, and $\delta_a(3q+2) = 0$.

If $q \equiv 1 \pmod 6$, $\phi_a(3q+3) = \frac{q-1}{2} + a$. As in Section 4, suppose

$$\frac{q+1}{2} + (m-1)q \leqslant a \leqslant \frac{q-1}{2} + mq,$$

where $0 \leqslant m \leqslant 3$. The computation in Lemma 4.2 shows

$$\begin{aligned} \delta_a(3q+3) &= \frac{2}{6q+3}\left(a - \frac{m(2q+1)}{2}\right)^2 - \frac{m^2-6m}{6} \\ &\quad -d(L(3,1), a-m) + d(L(3,1),0) + \frac{1}{12q+6} - \frac{1}{2} \\ &\quad +2\left(\chi_{[0,q)}(a + \frac{q-1}{2}) - 1\right) \\ &= \frac{4a^2+1}{12q+6} - \frac{2am}{3} + \frac{m^2(2q+1)}{6} - \frac{m^2-6m}{6} \\ &\quad -d(L(3,1), a-m) + 2\left(\chi_{[0,q)}(a + \frac{q-1}{2}) - 1\right). \end{aligned}$$

We claim that $\delta_a(3q+3)$ is never an integer. In fact, the denominators of all the terms except $\frac{4a^2+1}{12q+6}$ in $\delta_a(3q+3)$ are divisors of 6. Since 18 divides $12q+6$, we have

$$(12q+6) \cdot \delta_a(3q+3) \equiv 4a^2 + 1 \pmod 3,$$

which is impossible if $\delta_a(3q+3)$ is an integer. This finishes the proof of this claim. As a result, $q \equiv 2 \pmod 6$.

Step 2. When $q \geqslant 158$, we must have $\delta_a(3q+3) = 0$.

As in Section 4, suppose

$$\frac{q-2}{2} + (m-1)q \leqslant a \leqslant \frac{q-4}{2} + mq,$$

where $0 \leqslant m \leqslant 3$.

By (4.1) and (2.2), we have

$$\begin{aligned}\delta_a(i) &= \frac{(i-3q-2)^2}{12q+6}-\frac{\bar{\theta}(i)}{2}+2\chi_{[0,q)}(\phi_a(i))-\frac{(2\phi_a(i)-7q-2)^2}{4q(6q+3)} \\ &\quad +\frac{(2[\phi_a(i)]_b-q-2)^2}{12q}-d(L(3,2),[\phi_a(i)]_b). \qquad (5.1)\end{aligned}$$

In particular,

$$\begin{aligned}\delta_a(3q+3) &= \frac{2}{6q+3}\left(a-\frac{m(2q+1)}{2}\right)^2-\frac{m^2}{6}-d(L(3,2),a-2m+2) \\ &\quad +\frac{1}{12q+6}+2\chi_{[0,q)}([a+\frac{p+q-1}{2}]_p) \qquad (5.2)\end{aligned}$$

By (3.5) and (2.7), we must have $\delta_a(3q+3)=0$ or 2. This allows us to compute the possible values of a.

Let

$$\begin{aligned}D &= (12q+6)\Bigg(\delta_a(3q+3)+\frac{m^2}{6}+d(L(3,2),a+2-2m) \\ &\quad -\frac{1}{12q+6}-2\chi_{[0,q)}([a+\frac{p+q-1}{2}]_p)\Bigg), \qquad (5.3)\end{aligned}$$

it follows from (5.2) that

$$a=\frac{m(2q+1)\pm\sqrt{D}}{2}. \qquad (5.4)$$

So

$$\left|a-\frac{mp}{6}\right|=\frac{\sqrt{D}}{2}<\frac{1}{2}\sqrt{22(2q+1)}\leqslant\frac{q-1}{2}, \qquad (5.5)$$

when $q\geqslant 47$.

Since D is a perfect square, $D\equiv 0$ or $1 \pmod 3$. From (5.3) we get

$$2m^2+2\cdot 6d(L(3,2),a+2-2m)-1\equiv 0 \text{ or } 1 \pmod 3.$$

Using (2.4), we conclude that

$$d(L(3,2),a+2-2m)=\frac{1}{6}, \qquad \text{if } m=0,3.$$

It follows from (5.5) that $[a+\frac{p+q-1}{2}]_p\in[0,q)$ if and only if $m=3$. So

(5.3) becomes

$$D = \begin{cases} (2q+1)(6\delta_a(3q+3)+1)-1, & \text{if } m=0; \\ (2q+1)(6\delta_a(3q+3)-2)-1, & \text{if } m=1,\ a\equiv 2 \pmod 3; \\ (2q+1)(6\delta_a(3q+3)+2)-1, & \text{if } m=1,\ a\not\equiv 2 \pmod 3; \\ (2q+1)(6\delta_a(3q+3)+1)-1, & \text{if } m=2,\ a\equiv 1 \pmod 3; \\ (2q+1)(6\delta_a(3q+3)+5)-1, & \text{if } m=2,\ a\not\equiv 1 \pmod 3; \\ (2q+1)(6\delta_a(3q+3)-2)-1, & \text{if } m=3. \end{cases} \quad (5.6)$$

Since $\delta_a(3q+3)=0$ or 2, it follows from (5.4) that

$$|a-mq| \leqslant \frac{\sqrt{D}}{2}+\frac{m}{2} < \frac{1}{2}\sqrt{17(2q+1)}+\frac{3}{2} < \frac{1}{2}(\frac{q}{2}-2), \quad (5.7)$$

when $q \geqslant 158$.

If $m=0,1$, from (5.7) we see that

$$q < \frac{7q+2}{2}+2a < 6q+3$$

and

$$0 \leqslant \frac{q+2}{2}+2(a-mq) < q.$$

Using (5.1), (5.4) and (5.6) we can compute

$$\begin{aligned} & \delta_a(3q+4) \\ = \ & \frac{8}{6q+3}\left(a-\frac{m(2q+1)}{2}\right)^2-\frac{2}{3}m^2-\frac{1}{2}+\frac{4}{12q+6}-d(L(3,2),2a-m+2) \\ = \ & \begin{cases} 4\delta_a(3q+3)+\frac{1}{6}-d(L(3,2),2a+2), & \text{if } m=0, \\ 4\delta_a(3q+3)-\frac{5}{2}-d(L(3,2),2a+1), & \text{if } m=1,\ a\equiv 2 \pmod 3), \\ 4\delta_a(3q+3)+\frac{1}{6}-d(L(3,2),2a+1), & \text{if } m=1,\ a\not\equiv 2 \pmod 3). \end{cases} \quad (5.8) \end{aligned}$$

If $m=2,3$, from (5.7) we see that

$$q < \frac{7q+2}{2}+2a-(6q+3) < 6q+3,$$

and

$$0 \leqslant \frac{q+2}{2}+2(a-mq)-3 < q.$$

Using (5.1), (5.4) and (5.6) we can compute

$$\delta_a(3q+4) = \frac{8}{6q+3}\left(a - \frac{m(2q+1)}{2}\right)^2 - \frac{2}{3}m^2 + 4m$$
$$-\frac{13}{2} + \frac{4}{12q+6} - d(L(3,2), 2a-m+2)$$
$$= \begin{cases} 4\delta_a(3q+3) - \frac{1}{2} - d(L(3,2), 2a), & \text{if } m=2, a \equiv 1 \pmod 3; \\ 4\delta_a(3q+3) + \frac{13}{6} - d(L(3,2), 2a), & \text{if } m=2, a \not\equiv 1 \pmod 3; \\ 4\delta_a(3q+3) - \frac{11}{6} - d(L(3,2), 2a-1), & \text{if } m=3. \end{cases} \quad (5.9)$$

By (3.5), we have $\delta_a(3q+4) = \delta_a(3q+3) = 0$ or 2. If $\delta_a(3q+3) = 2$, (5.8) and (5.9) do not hold. So we must have $\delta_a(3q+3) = 0$.

Step 3. When $q \geqslant 158$, the equality (3.5) does hold for any a satisfying (3.3) and any sequence $\{t_s\}$ satisfying (2.7).

Since $\delta_a(3q+4) = \delta_a(3q+3) = 0$, it follows from (5.8) and (5.9) that (m, a) falls into one of three cases:

$$m=0, a \not\equiv 0 \pmod 3; \quad m=1, a \not\equiv 2 \pmod 3; \quad m=2, a \equiv 1 \pmod 3.$$

In these cases, $D = 2q$ when $m = 0, 2$ and $D = 4q+1$ when $m = 1$. It follows from (5.4) that

$$|a - mq| \leqslant \sqrt{\frac{q}{2}} + 1 < \frac{1}{6}(\frac{q}{2} - 5) \quad (5.10)$$

when $m = 0, 2$ and $q \geqslant 112$, and

$$|a - mq| \leqslant \sqrt{q + \frac{1}{4}} + \frac{1}{2} < \frac{1}{3}(\frac{q}{2} - 2) \quad (5.11)$$

when $m = 1$ and $q \geqslant 50$.

If $m = 0$, it follows from (5.10) that

$$q \leqslant \frac{7q+2}{2} + 5a < \frac{7q+2}{2} + 6a < 6q+3$$

and

$$0 \leqslant \frac{q+2}{2} + 5a < \frac{q+2}{2} + 6a < q.$$

Using (5.1) we can compute

$$\begin{aligned}\delta_a(3q+7) &= \frac{100}{12q+6}a^2 + \frac{25}{12q+6} - d(L(3,2), 2+5a) \\ &= 4,\end{aligned}$$

and

$$\begin{aligned}\delta_a(3q+8) &= \frac{24}{2q+1}a^2 - \frac{1}{2} + \frac{36}{12q+6} - d(L(3,2),2) \\ &= 6.\end{aligned}$$

This is impossible, since we should have $\delta_a(3q+7) = \delta_a(3q+8)$ by (3.5).

If $m = 1$, it follows from (5.11) that

$$0 \leqslant \frac{7q+2}{2} + 3a - (6q+3) < q.$$

Using (5.1) we can compute

$$\begin{aligned}\delta_a(3q+5) &= \frac{6}{2q+1}\left(a - \frac{2q+1}{2}\right)^2 + \frac{9}{12q+6} - \frac{3}{2} - d(L(3,2),2) + 2 \\ &= 4.\end{aligned}$$

Since $\delta_a(3q+4) = 0$, we can get a contradiction with (3.5) and (2.7).

If $m = 2$, it follows from (5.10) that

$$q \leqslant \frac{7q+2}{2} + 5a - 2(6q+3) < \frac{7q+2}{2} + 6a - 2(6q+3) < 6q+3$$

and

$$\frac{q+2}{2} + 5(a-2q) - 6, \frac{q+2}{2} + 6(a-2q) - 6 \in [0,q).$$

Using (5.1) we can compute

$$\begin{aligned}\delta_a(3q+7) &= \frac{100}{12q+6}\big(a-(2q+1)\big)^2 - \frac{2}{3} + \frac{25}{12q+6} - d(L(3,2),2a) \\ &= 4,\end{aligned}$$

and

$$\begin{aligned}\delta_a(3q+8) &= \frac{144}{12q+6}\big(a-(2q+1)\big)^2 - \frac{1}{2} + \frac{36}{12q+6} - d(L(3,2),2) \\ &= 6.\end{aligned}$$

Again, this is impossible since $\delta_a(3q+7) = \delta_a(3q+8)$. $\square$

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