

ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

ERIC BEDFORD AND TANYA FIRSOVA

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Tome XXV, n° 1 (2016), p. 1-18.

http://afst.cedram.org/item?id=AFST_2016_6_25_1_1_0

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Geometric proof of the λ -Lemma

ERIC BEDFORD⁽¹⁾, TANYA FIRSOVA⁽²⁾

RÉSUMÉ. — Nous donnons une approche géométrique de la preuve de la λ -lemma. Nous soulignons, en particulier, le rôle que la pseudoconvexité joue dans la preuve.

ABSTRACT. — We give a geometric approach to the proof of the λ -lemma. In particular, we point out the role pseudoconvexity plays in the proof.

1. Introduction

A holomorphic motion in dimension one is a family of injections $f_\lambda : A \rightarrow \hat{\mathbb{C}}$ over a complex manifold $\Lambda \ni \lambda$. Holomorphic motions first appeared in [15, 14] where they were used to show that a generic rational map $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is structurally stable. This notion has since found numerous applications in holomorphic dynamics and Teichmüller Theory. Its usefulness comes from the fact that analyticity alone forces strong extendibility and regularity properties that are referred to as the λ -lemma. Let Δ be the unit disk in \mathbb{C} .

THEOREM 1.1. —

- **Extension λ -lemma** [14], [15] *Any holomorphic motion $f : \Delta \times A \rightarrow \hat{\mathbb{C}}$ extends to a holomorphic motion $\Delta \times \bar{A} \rightarrow \hat{\mathbb{C}}$.*
- **QC λ -lemma** [15] *The map $f(\lambda, a)$ is uniformly quasimetric in a .*

(*) Reçu le 12/02/2015, accepté le 30/03/2015

⁽¹⁾ IMS, Stony Brook University, Stony Brook NY 11794-3660, USA
ebedford@math.sunysb.edu

⁽²⁾ 138 Cardwell Hall, Manhattan, KS 66506, USA
tanyaf@math.ksu.edu

Article proposé par Vincent Guedj.

Note that when A has interior, $f(\lambda, a)$ is quasiconformal on the interior. For many applications it is important to know that a holomorphic motion can be extended to a holomorphic motion of the entire sphere. Bers & Royden [5] and Sullivan & Thurston [17] proved that there exists a universal $\delta > 0$ such that under the circumstances of the Extension λ -lemma, the restriction of f to the parameter disk Δ_δ of radius δ can be extended to a holomorphic motion $\Delta_\delta \times \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$. Ślodkowski [16] proved the strongest version asserting that δ is actually equal to 1:

λ -LEMMA [ŚLODKOWSKI]. — *Let $A \subset \hat{\mathbb{C}}$. Any holomorphic motion $f : \Delta \times A \rightarrow \hat{\mathbb{C}}$ extends to a holomorphic motion $\Delta \times \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$.*

Ślodkowski's proof builds on the work by Forstnerič [10] and Šnirel'man [18]. Astala and Martin [1] gave an exposition of Ślodkowski's proof from the point of view of 1-dimensional complex analysis. Chirka [7] gave an independent proof using solution to $\bar{\partial}$ -equation. (See [13] for a detailed exposition of Chirka's proof.) The purpose of this paper is to give a more geometric approach to the proof of the λ -lemma. We take Ślodkowski's approach and replace the major technical part in his proof (closedness, see [1, Theorem 4.1]) by a geometric pseudoconvexity argument.

The strongest λ -lemma fails when the dimension of the base manifold is greater than 1, even if the base is topologically contractible. This follows from the results of Earl-Kra [9] and Hubbard [12].

We give the necessary background on holomorphic motions, pseudoconvexity and Hilbert transform in Section 2. In Section 3, we show that the λ -lemma when A is finite implies the λ -lemma for arbitrary A . We set up the notations and terminology in Section 4. We state the filling theorem for the torus, and explain how it implies the finite λ -lemma in Section 5. In Section 6 we prove Hölder estimates for disks trapped inside pseudoconvex domains and construct such trapping pseudoconvex domains for “graphical tori”. We use these estimates to prove the filling theorem in Section 7.

1.1. Acknowledgments

We would like to thank Misha Lyubich, Yakov Eliashberg and the referee for fruitful discussions and useful suggestions.

2. Background

2.1. Holomorphic motion

Let Δ be a unit disk. Let $A \subset \hat{\mathbb{C}}$. A *holomorphic motion* of A is a map $f: \Delta \times A \rightarrow \hat{\mathbb{C}}$ such that

- (1) for fixed $a \in A$, the map $\lambda \mapsto f(\lambda, a)$ is holomorphic in Δ
- (2) for fixed $\lambda \in \Delta$, the map $a \mapsto f(\lambda, a) =: f_\lambda(a)$ is an injection and
- (3) the map f_0 is the identity on A .

2.2. Pseudoconvexity

Below we give definitions that are sufficient for our purposes.

A C^2 smooth function is (*strictly*) *plurisubharmonic* (written (strictly) psh) if its restriction to every complex line is strictly subharmonic. In coordinates $z = (z_1, \dots, z_n)$, $u(z)$ is strictly psh if the matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right)$ is positive definite.

A smoothly bounded domain $\Omega \subset \mathbb{C}^2$ is *strictly pseudoconvex* if there is a smooth, strictly psh function ρ in a neighborhood of $\bar{\Omega}$ such that $\{\Omega = \rho(z) < 0\}$.

LEMMA 2.1. — *Let $\Omega_s \subset \mathbb{C}^2$ be a family of pseudoconvex domains with defining functions ρ_s , $s \in [0, 1]$. We assume that the family ρ_s is continuous in s . Let $\phi_s: \Delta \mapsto \mathbb{C}^2$ be a continuous family of holomorphic non-constant functions that extend continuously to $\bar{\Delta}$. Set $D_s := \phi_s(\Delta)$. Suppose $\partial D_s \subset \partial \Omega_s$, $s \in [0, 1]$. And suppose $D_s \subset \Omega_s$, $s \in [0, 1]$. Then $D_1 \subset \Omega_1$.*

Proof. — Consider the restriction of the functions ρ_s to D_s . The functions $\rho_s \circ \phi_s: \Delta \mapsto \mathbb{R}$ are subharmonic functions, $\rho_1 \circ \phi_1$ is the limit of $\rho_s \circ \phi_s$. By the hypothesis of the lemma, $\rho_s \circ \phi_s \leq 0$ on Δ . Therefore, $\rho_1 \circ \phi_1 \leq 0$. If the maximum value 0 is attained in the interior point, $\rho_1 \circ \phi_1 \equiv 0$. It implies that $D_1 \subset \partial \Omega_1$, which is impossible. Therefore, $\rho_1 \circ \phi_1 < 0$ on Δ , and $D_1 \subset \Omega_1$. \square

Let $M \subset \mathbb{C}^2$ be a real two-dimensional manifold. We say that $p \in M$ is a *totally real* point if $T_p M \cap iT_p M = \{0\}$. M is a *totally real manifold* if all its points are totally real. If the manifold M is totally real, it is in fact homeomorphic to the torus (see [6] and [11]). Assume $M \subset \partial \Omega$, then one can define a characteristic field of directions on M .

Let $p \in M$. Let $H_p\partial\Omega := T_p\Omega \cap iT_p\Omega$ be the holomorphic tangent space. $\langle \xi_p \rangle := H_p\partial\Omega \cap T_pM$ is called the *characteristic direction*. We denote by $\chi(M, \Omega)$ the characteristic field of directions (see [8, Section 16.1]).

2.3. Hilbert transform

A function $u : \mathbb{S}^1 \rightarrow \mathbb{C}$ is *Hölder continuous with exponent α* if there is a constant A such that for all $x, y \in \mathbb{S}^1$:

$$|u(x) - u(y)| < A|x - y|^\alpha.$$

We will consider the space $C^{1,\alpha}(\mathbb{S}^1)$ of differentiable functions u with α -Hölder continuous derivative. The norm on the space $C^{1,\alpha}(\mathbb{S}^1)$ is defined by the formula:

$$\|u\|_{1,\alpha} := \sup_{x \in \mathbb{S}^1} |u(x)| + \sup_{x \in \mathbb{S}^1} |u'(x)| + \sup_{x \neq y \in \mathbb{S}^1} \frac{|u'(x) - u'(y)|}{|x - y|^\alpha}.$$

There exists a unique harmonic extension u_h of the function u to Δ . Let denote by u_h^* the harmonic conjugate of u_h , normalized by the condition $u_h(0) = 0$. The function u_h^* extends to $\mathbb{S}^1 = \partial\Delta$ as a Hölder continuous function with exponent α .

For a function $u \in C^{1,\alpha}(\mathbb{S}^1)$ we define its *Hilbert transform* Hu to be the boundary value of the harmonic conjugate function u_h^* . By definition, the function $u + iHu$ extends as a holomorphic function to the unit disk.

THEOREM 2.2. — *The Hilbert transform H is a bounded linear operator on $C^{1,\alpha}(\mathbb{S}^1)$ and $C^\alpha(\mathbb{S}^1)$.*

This Theorem makes it convenient for us to work with the spaces $C^{1,\alpha}(\mathbb{S}^1)$ and $C^\alpha(\mathbb{S}^1)$.

3. Finite λ -lemma

The first step in the proof of the λ -lemma is to reduce it to the λ -lemma for finitely many points, [15].

THEOREM 3.1. — **The Finite λ -lemma** *Assume $a_1, \dots, a_{n+1} \in \hat{\mathbb{C}}$, $a_i \neq a_j$ for $i \neq j$. Let $f : \Delta \times \{a_1, \dots, a_n\} \rightarrow \hat{\mathbb{C}}$ be a holomorphic motion. Then there exists a holomorphic motion $\tilde{f} : \Delta \times \{a_1, \dots, a_{n+1}\} \rightarrow \mathbb{C}$, so that \tilde{f} is an extension of f .*

Proof. — [Reduction of the λ -lemma to the finite λ -lemma (assuming the Extension- λ lemma):] We normalize the holomorphic motion f so that three points a_1, a_2, a_3 stay fixed. We can assume $a_1 = 0, a_2 = 1, a_3 = \infty$.

Let $\{a_n\}$ be a sequence of points that are dense in \bar{A} . Let $\{z_n\}$ be a sequence of points that are dense in $\mathbb{C} \setminus \bar{A}$. Let f_n be a holomorphic motion of $a_1, \dots, a_n, z_1, \dots, z_n$, such that

$$f_n(\lambda, a_i) = f(\lambda, a_i).$$

The existence of such holomorphic motion follows from the Finite λ -lemma.

For any fixed z_n , for $k \geq n$ and $n \geq 3$, maps f_k are defined at the point z_n , and functions $f_k(*, z_n) : \Delta \rightarrow \mathbb{C} \setminus \{0, 1\}$ form a normal family. So we can choose a convergent subsequence $f_k(*, z_n)$. Using the diagonal method, we get a holomorphic motion \tilde{f} , that is well defined for all a_n and z_n and coincides with f on a_i for all i . By the Extension λ -lemma, we extend it uniquely to the holomorphic motion of $\hat{\mathbb{C}}$. By construction, it coincides with the holomorphic motion f on the set A . \square

4. Notations and Terminology

We consider \mathbb{C}^2 with coordinates (λ, w) . The horizontal direction is parametrized by λ , the vertical by w . Throughout the paper we consider disks of the form

$$w = g(\lambda)$$

that will depend on two different parameters. We will use the following notations

$$\begin{aligned} g : \Delta \times \mathbb{S}^1 \times [0, t_0] &\rightarrow \mathbb{C}^2 \\ g_\xi^t(\lambda) := g^t(\lambda, \xi) &:= g_\xi(\lambda, t) := g(\lambda, \xi, t). \end{aligned}$$

4.1. Graphical Torus

Let $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$, $\pi(\lambda, w) = \lambda$, be the projection to the first coordinate.

We say that a torus $\Gamma \subset \{\partial\Delta\} \times \mathbb{C}$ is a **graphical torus** if for each $\lambda \in \partial\Delta$, $C_\lambda := \pi^{-1}(\lambda) \in \mathbb{C} \setminus \{0\}$ is a simple closed curve that has winding number 1 around 0.

Thus, the vertical slices $\{C_\lambda : \lambda \in \mathbb{S}^1\}$ give a foliation of Γ . We wish to construct a transverse foliation of Γ . We will consider holomorphic functions $g_\xi : \Delta \rightarrow \mathbb{C}$, which extend continuously to $\bar{\Delta}$ and such that the boundary $\gamma_\xi := g_\xi(\partial\Delta) \subset \Gamma$. We will construct a family of holomorphic disks such

that $\{\gamma_\xi : \xi \in \mathbb{S}^1\}$ form another foliation of the torus Γ that is transverse to the original foliation.

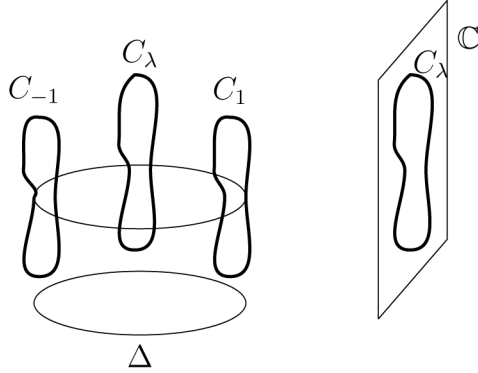


Figure 1. — Torus Γ .

4.2. Family of Graphical Tori

Let $\{C_\lambda^t : t > 0, \lambda \in \partial\Delta\}$ be smooth curves, such that

- (1) C_λ^t have winding number 1 around 0;
- (2) for fixed λ , C_λ^t form a smooth foliation of $\mathbb{C} \setminus \{0\}$;
- (3) there exists $\epsilon > 0$, so that $C_\lambda^t = \{|w|^2 = t\}$ for $t < \epsilon$.

Let

$$\Gamma^t = \{(\lambda, w) : \lambda \in \partial\Delta, w \in C_\lambda^t\}.$$

We set $\Gamma^0 = \{(\lambda, 0) : \lambda \in \partial\Delta\}$. We refer to Γ^t , $t \geq 0$ as smooth family of graphical tori, though for $t = 0$ it degenerates to a circle Γ^0 . The superscript t will be applied to indicate the dependence on the torus Γ^t .

4.3. Holomorphic Transverse Foliation of a Graphical Torus

Let Γ be a graphical torus. Let $g : \Delta \rightarrow \mathbb{C}$ be a holomorphic function that extends continuously to the closure $\bar{\Delta}$. We say that the function $g : \bar{\Delta} \rightarrow \mathbb{C}$ defines a **holomorphic disk** $D := \{(\lambda, g(\lambda)) : \lambda \in \Delta\} \subset \mathbb{C}^2$ with a **trace** $\gamma := \partial D$.

We will construct foliations of graphical tori by traces of holomorphic disks. To do this, we will require additional properties:

We say that a function $g : \bar{\Delta} \times \mathbb{S}^1 \rightarrow \mathbb{C}$ defines a **holomorphic transverse foliation** of a graphical torus Γ if

- (1) g is continuous.
- (2) for each $\xi \in \mathbb{S}^1$, we let $\{\gamma_\xi := g(\lambda, \xi) : \lambda \in \partial\Delta\}$. The curves γ_ξ are simple, pairwise disjoint and define a foliation of Γ .
- (3) Let $g_\xi(\lambda) := g(\lambda, \xi)$, $g_\xi : \Delta \rightarrow \mathbb{C}$ is holomorphic, $g_\xi \in C^{1,\alpha}(\bar{\Delta})$
- (4) $g_\xi(\lambda) \neq 0$, for all $\xi \in \mathbb{S}^1$, $\lambda \in \Delta$
- (5) $g_\xi(\lambda) \neq g_\eta(\lambda)$, for every $\lambda \in \Delta$ and distinct $\xi, \eta \in \mathbb{S}^1$.

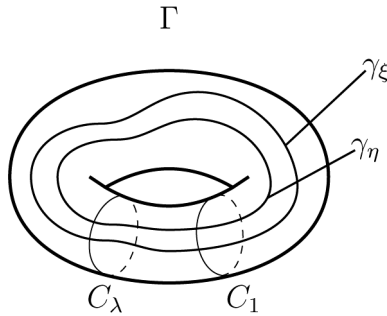


Figure 2. — Holomorphic Transverse Foliation of the Torus Γ .

We will also consider holomorphic transverse foliations of a smooth family graphical tori $\{\Gamma^t\}$. This refers to a smooth family of foliations of graphical tori Γ^t with the additional assumption that the disks from Γ^{t_1} are disjoint from the disks from Γ^{t_2} if $t_1 \neq t_2$.

In fact the leaves in all of our foliations will be closed, and thus they are also fibrations by curves.

5. Holomorphic transverse foliations and the Finite λ -lemma

FILLING THEOREM. — *Let Γ be a graphical torus, then there exist a function $g : \bar{\Delta} \times \mathbb{S}^1 \rightarrow \mathbb{C}$ that defines a holomorphic transverse foliation of Γ . Moreover, the foliation is unique in the following strong sense: if there is a function $h : \bar{\Delta} \rightarrow \mathbb{C}$ that defines a holomorphic disk with trace in Γ , and if $h(\lambda) \neq 0$ for $\lambda \in \Delta$, then there exists $\xi \in \mathbb{S}^1$ so that $h = g_\xi$.*

We need the following slightly stronger statement to deduce the Finite λ -lemma:

FILLING THEOREM'. — *Let Γ^t , $t \in [0, \infty)$ be a family of graphical tori. There exists a function $g : \bar{\Delta} \times \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{C}$ that defines a holomorphic transverse foliation of the family Γ^t . And the foliation is unique in the above mentioned strong sense.*

The reduction of the Finite λ -lemma to Filling Theorem' can be found in [16].

Reduction of the Finite λ -lemma to Filling Theorem'. — Let f be a holomorphic motion of the points a_1, \dots, a_n . We need to extend the motion f to one more point a_{n+1} . To achieve that we construct a holomorphic motion of all of \mathbb{C} and pick the leaf that passes through the point a_{n+1} .

We normalize the motion so that $a_1 = 0$, $f(\lambda, 0) = 0$ for all $\lambda \in \Delta$. Let $\lambda = re^{i\theta}$. For each $r \in [0, 1)$, $e^{i\theta} \in \mathbb{S}^1$ the derivative $\frac{\partial f}{\partial r}(\lambda, a_i)$ defines a vector $v_\theta(r, a_i)$ in \mathbb{C} . We can extend it to a smooth family of vector fields $v_\theta(r, \cdot)$ on \mathbb{C} . By integrating the vector field for $r \in [0, 1)$ and taking the union of solutions over $\xi \in \mathbb{S}^1$, we get a smooth motion $g : \Delta \times \mathbb{C} \rightarrow \mathbb{C}$ such that $g(\lambda, a_i) = f(\lambda, a_i)$.

Let C_0^t be a smooth family of simple curves that foliate $\mathbb{C} \setminus \{0\}$. We choose the foliation so that different a_i belong to different curves C_0^t . Take $r < 1$. Let $\mathbb{S}_r = \{\lambda : |\lambda| = r\}$. Let $C_\lambda^t = g(\lambda, C_0^t)$ for $\lambda \in \mathbb{S}_r$.

By Filling Theorem', there exists a holomorphic motion with the prescribed traces $\Gamma_r^t = \{(\lambda, C_\lambda^t) : \lambda \in \mathbb{S}_r\}$. By the uniqueness, it coincides with f on points a_1, \dots, a_n . By taking the limit as $r \rightarrow 1$, we obtain a holomorphic motion of \mathbb{C} that coincides with f on a_1, \dots, a_n . \square

6. Trapping holomorphic disks inside pseudoconvex domains

The aim of the section is to prove a priori estimates for the derivative of a disk with the trace in a graphical torus (Corollary 6.8), which is the heart of our proof of the λ -lemma.

6.1. Estimates for holomorphic disks trapped inside strictly pseudoconvex domains

The next theorem is from [4], [3]. We do not use the result of the theorem. We provide the proof to shed light on the technique we use and put the results in a general context.

THEOREM 6.1. — *[4], [3] Let Ω be a strictly pseudoconvex domain, and let M be a totally real 2-dimensional manifold, $M \subset \partial\Omega$. Let $g : \Delta \rightarrow \Omega$ be*

an injective holomorphic function that extends as a C^1 smooth function to the closure $\bar{\Delta}$. Set $D = g(\Delta)$. Assume that $\gamma := \partial D \subset M$. Then there is a constant $\alpha = \alpha(M, \Omega)$, so that the angle $\angle(T_p\gamma, \xi_p) > \alpha$ is uniformly large, independently of D .

LEMMA 6.2. — Under hypothesis of Theorem 6.1, for every point $p \in \gamma$, $T_p\gamma$ is transverse to the characteristic field of directions $\chi(M, \Omega)$.

Proof. — Let ρ be a strictly psh function such that $\Omega = \{\rho < 0\}$. The function $\rho \circ g : \Delta \rightarrow \mathbb{R}$ is subharmonic. Let $p \in \partial\Delta$. By the Hopf Lemma, the radial derivative $\frac{\partial(\rho \circ g)}{\partial r}(p) > 0$. Let ξ_p be a vector that defines the characteristic direction in a point p . The normal vector to the disk $g(\Delta)$ in a point p is $iT_p\gamma$. It does not belong to the tangent plane to $\partial\Omega$, so $iT_p\gamma$ is transverse to $i\xi_p$. Therefore, $T_p\gamma$ is transverse to ξ_p . \square

Let n_p be the unit outward normal vector to the hypersurface $\partial\Omega$. The vectors $(\xi_p, i\xi_p, n_p, in_p)$ form an orthonormal basis in $\mathbb{C}^2 \approx \mathbb{R}^4$ with respect to Euclidean inner product (\cdot, \cdot) . The vectors in_p and ξ_p form an orthonormal basis for T_pM . Given α , we define a conical neighborhood of ξ_p :

$$K_\alpha = \{v \in T_pM : (v, \xi_p) > \alpha(v, in_p)\} \subset T_pM.$$

LEMMA 6.3. — Let Ω be a strictly pseudoconvex domain, and let $M \subset \partial\Omega$ be totally real. There exist $\alpha > 0$, and a continuous family of strictly pseudoconvex domains Ω_ϵ such that $M \subset \partial\Omega_\epsilon$, and the characteristic fields of directions $\chi(M, \Omega_\epsilon)$ fill the cone-fields K_α .

Proof. — The manifold M separates $\partial\Omega$ into two parts $(\partial\Omega)_1, (\partial\Omega)_2$. Let h be a smooth function such that

- (1) $h|_M = 0$;
- (2) $h|_{(\partial\Omega)_1} > 0, h|_{(\partial\Omega)_2} < 0$;
- (3) $\frac{\partial h}{\partial(i\xi_p)} > 0$, for each $p \in M$.

Let us denote by \vec{n} the normal field to the hypersurfaces $\rho = \text{const}$. Since we can identify $T_p\mathbb{C}^2$ with \mathbb{C}^2 , we can treat the normal vector field n as a function defined in a neighborhood of $\partial\Omega$. We use the same letter n for this function. Let $\rho_\epsilon(z) = \rho(z + \epsilon h\vec{n})$, $\Omega_\epsilon = \{\rho_\epsilon < 0\}$. Then there exists δ , so that for $|\epsilon| < \delta$, ρ_ϵ are plurisubharmonic. Therefore, Ω_ϵ are strictly pseudoconvex, and characteristic fields of directions to Ω_ϵ fill the cone field K_α . \square

Proof of Theorem 6.1. — Let $D \subset \Omega$, $\partial D \subset \partial\Omega$. Then by Lemma 6.3, there exists a continuous family of strictly pseudoconvex domains Ω_ϵ , $|\epsilon| < \delta$ so that their characteristic fields of directions fill C_α , for some $\alpha > 0$. By Lemma 2.1, $D \subset \Omega_\epsilon$ for $|\epsilon| < \delta$. Therefore, an angle estimate follows. \square

6.2. Pseudoconvex domains for Graphical Tori

We wish to obtain the angle estimates for graphical tori. Let η_p be a vector that is tangent to the curve C_λ in a point p . We want to think of η_p as a characteristic direction. However, a priori a graphical torus Γ does not belong to a pseudoconvex domain. It belongs to a Levi flat domain $\{|\lambda| = 1\} \times \mathbb{C}$. Our strategy is to curve this Levi flat domain to obtain a family of pseudoconvex domains whose boundaries contain the torus Γ and so that characteristic directions span a wedge around η_p .

THEOREM 6.4. — *Let Γ be a graphical torus. Assume that $g : \Delta \rightarrow \mathbb{C}$ defines a holomorphic disk D with the trace $\gamma \subset \Gamma$, $g(\lambda) \neq 0$. Then there exists a constant $\alpha = \alpha(\Gamma) > 0$ (independent of D) so that the angle $\angle(\eta_p, T_p\gamma)$ is bounded below by α independently of D .*

We need Lemmas 6.5, 6.6 and 6.7 to prove Theorem 6.4.

Consider a family of the graphical tori Γ^t , $\Gamma^1 = \Gamma$. Let $F : \mathbb{S}^1 \times \mathbb{C} \rightarrow \mathbb{R}$ be a defining function, $F^{-1}(t) = \Gamma^t$. Let us extend F to a smooth function $F : \bar{\Delta} \times \mathbb{C} \rightarrow \mathbb{R}$, so that $F(\lambda, w) = |w|^2$ for all $\lambda \in \bar{\Delta}$, $|w| \leq \epsilon$. We can also satisfy the condition $F'_w \neq 0$.

LEMMA 6.5. — *There exists a function $\phi : \bar{\Delta} \times \mathbb{C} \rightarrow \mathbb{R} \geq 0$, so that ϕ is smooth, $\Delta_w \phi > 0$, and restriction of ϕ to $\mathbb{S}^1 \times \mathbb{C}$ defines a foliation of $\mathbb{S}^1 \times \mathbb{C}$ by Γ^t . We also require that for $|\lambda| = 1$ $\phi_\lambda^{-1}(1) = C_\lambda$.*

Proof. — Let $F(\lambda, w)$ be the extension defined earlier. Let $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing convex function, $\rho(0) = 0$, $\rho(1) = 1$. Then $\phi = \rho \circ F$ is also an extension of a defining function of the foliation as well.

$$\Delta_w(\rho \circ F) = \frac{1}{4}\rho''|F_w|^2 + \frac{1}{4}\rho'\Delta_w F \tag{6.1}$$

Since $F'_w(\lambda, w) \neq 0$, when $w \neq 0$, so that $\Delta_w(\rho \circ F) > 0$ away from a neighborhood of $w = 0$. In a neighborhood of 0, $\Delta_w F = 4$. By taking $\rho'(0) > 0$, one can insure that $\Delta(\rho \circ F) > 0$.

Let us set $\phi = \rho \circ F$, then $\phi_\lambda^{-1} = C_\lambda$. \square

LEMMA 6.6. — *There exists a function $\psi : \bar{\Delta} \times \mathbb{C} \rightarrow \mathbb{R} \cup \{-\infty\}$, so that ψ is smooth, $\Delta_w \psi < 0$, and restriction of ψ to $\mathbb{S}^1 \times \mathbb{C}$ defines a foliation of $\mathbb{S}^1 \times \mathbb{C}$ by Γ^t . We require that $\psi(\lambda, 0) = -\infty$ for all $\lambda \in \bar{\Delta}$. We also require that for $|\lambda| = 1$, $\psi_\lambda^{-1}(t) = C_\lambda$.*

Proof. — Consider a function $\psi = c\rho \circ \ln F$, where ρ is increasing, concave function, $\rho(-\infty) = -\infty$.

$$\Delta_w(\rho \circ \ln F) = \frac{1}{4}\rho'' \frac{|F_w|^2}{F^2} + \frac{1}{4}\rho' \Delta_w(\ln F)$$

Since $F'_w \neq 0$ when $w \neq 0$, we can make $\Delta_w(\rho \circ \ln F) < 0$. In a neighborhood of $w = 0$, $\Delta_w(\ln F) = 0$, therefore $\Delta_w(\rho \circ \ln F) < 0$. By choosing a constant c , we can ensure that $\psi_\lambda^{-1}(1) = C_\lambda$. \square

Let $T\Gamma$ be the tangent space of the graphical torus Γ . Let $K_\alpha \subset T\Gamma$ be the cone field:

$$K_\alpha := \{(p, v) : v \in T_p T, (v, \eta_p) > \alpha(v, \frac{\partial}{\partial \theta})\}.$$

$$K_\alpha^\circ := \{(p, v) \in K_\alpha : v \neq c\eta_p, c \in \mathbb{R}\}$$

LEMMA 6.7. — *For a graphical torus Γ , there exist a family of pseudoconvex domains Ω_ϵ , $\epsilon \in [-\delta, 0) \cup (0, \delta]$ and $\alpha > 0$, so that $\Gamma \subset \partial\Omega_\epsilon$ and characteristic directions $\chi(T, \Omega_\epsilon)$ fill K_α° .*

Proof. — Take

$$\omega_\epsilon := \frac{1}{\epsilon}(|\lambda|^2 - 1) + \phi,$$

where ϕ is a function constructed in Lemma 6.5.

$$\text{Hess } \omega_\epsilon = \begin{pmatrix} \frac{1}{\epsilon} + \frac{\partial^2 \phi}{\partial \lambda \partial \bar{\lambda}} & \frac{\partial^2 \phi}{\partial w \partial \bar{\lambda}} \\ \frac{\partial^2 \phi}{\partial \bar{w} \partial \lambda} & \Delta_w \phi \end{pmatrix}$$

For small enough ϵ , the Hessian is positive definite, so the function ω_ϵ is strictly plurisubharmonic. The domains

$$\Omega_\epsilon = \{(\lambda, w) : \omega_\epsilon(\lambda, w) < 1\}.$$

are strictly pseudoconvex for small ϵ .

Let D be a holomorphic disk with the trace in Γ . The domains Ω_ϵ converge to $|\lambda| < 1$. Therefore, by Lemma 2.1, the disk D is trapped in Ω_ϵ for all small enough ϵ .

For small ϵ , the function

$$\sigma_\epsilon(\lambda, w) := \frac{1}{\epsilon}(|\lambda|^2 - 1) - \psi$$

is strictly plurisubharmonic. By the same reasoning, the disks are trapped in

$$\Sigma_\epsilon = \{(\lambda, w) : \sigma_\epsilon < -1\}$$

when ϵ is sufficiently small. \square

Proof of Theorem 6.4. — By Lemma 6.2, the tangent $T_p\gamma$ is transverse to characteristic directions. Therefore, the angle estimate follows. \square

COROLLARY 6.8. — *Let $g : \Delta \rightarrow \mathbb{C}$ define a holomorphic disk with the trace in Γ , $g(\lambda) \neq 0$ for $\lambda \in \Delta$. Assume that $g \in C^1(\bar{\Delta})$. Then there exists C depending only on Γ such that $|g'(\lambda)| < C$ for all $\lambda \in \bar{\Delta}$. The derivative estimate stays valid for graphical tori that are small perturbations of Γ .*

Proof. — It is enough to estimate $g'(\lambda)$ for $|\lambda| = 1$. Then $\lambda = e^{i\theta}$, so $|g'_\lambda| = |g'_\theta|$. Let u, v, θ, r be an orthonormal system of coordinates in a neighborhood of Γ . We assume that $u|_\Gamma$ is a coordinate along C_λ and v, u are coordinates in $\lambda = \text{const}$ plane. Then $g'_\theta = u'_\theta$ and the angle estimate implies that $|u'_\theta|$ is uniformly bounded from below. \square

7. Proof of the Filling Theorem

The proof is by continuity method. At many points we follow the treatment of [1]. For each $\lambda \in \mathbb{S}^1$ we can foliate interior of $C_\lambda^t \setminus \{0\}$ by simple smooth curves C_λ^s , $s \in (0, t)$ so that

- (1) $C_\lambda^s = \{|z| = s\}$ for $s \leq \epsilon$;
- (2) C_λ^s depend smoothly on λ .

Let

$$\begin{aligned} \Gamma^t &= \{(\lambda, w) : \lambda \in \mathbb{S}^1, w \in C_\lambda^t\} \\ \Gamma^0 &= \{(\lambda, 0) : \lambda \in \mathbb{S}^1\} \end{aligned}$$

Γ^t by definition is a smooth family of graphical tori, $\Gamma^1 = \Gamma$. For $t \leq \epsilon$, the tori Γ^t are foliated by the vertical leaves $w = \text{const}$. We will prove that the set S of parameters t such that Γ^t is foliated is open and closed in $[0, 1]$, so $S = [0, 1]$, and the torus Γ is foliated. Moreover, we will prove that the foliation is unique in the strong sense.

Let $F : \mathbb{S}^1 \times \mathbb{C} \rightarrow \mathbb{R}$ be a defining function of the foliations C_λ^t . For each fixed λ ,

$$C_\lambda^t = \{(\lambda, w) : F(\lambda, w) = t\}.$$

The function F depends smoothly on λ . We assume that $F'_w(\lambda, w) \neq 0$ for $w \neq 0$, $\lambda \in \mathbb{S}^1$.

LEMMA 7.1. — *Assume that the winding number of a curve $\{\gamma(\lambda) : \lambda \in \mathbb{S}_\lambda^1\}$ around 0 is equal to zero. Then the winding number of the curve $\{F'_w(\lambda, \gamma(\lambda)) : \lambda \in \mathbb{S}^1\}$ around 0 is equal to zero.*

Proof. — There is a homotopy of the curve γ , $G : \gamma \times [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ so that $G(\gamma \times \{0\}) = \gamma$, $G(\gamma \times \{1\}) = \text{const}$. The winding number of the curves $\{F'_w(\lambda, \gamma^t(\lambda)) : \lambda \in \mathbb{S}^1\}$ around 0 is well defined, so it stays constant. Hence, it is equal to zero. \square

7.1. Regularity

THEOREM 7.2. — *Let Γ be a graphical torus. Let $g : \bar{\Delta} \rightarrow \mathbb{C}$ be a function that defines a holomorphic disk with the trace $g(\partial\Delta) \in \Gamma$. Assume $g' \in L^\infty(\Delta)$, $g \neq 0 \forall \lambda \in \Delta$. Then $g \in C^{1,\alpha}(\bar{\Delta})$, $0 < \alpha < 1$.*

Proof. — We include Γ into a family of graphical tori Γ^t with $\Gamma^1 = \Gamma$. Let $F : \mathbb{S}^1 \times \mathbb{C} \rightarrow \mathbb{R}$ be a defining function for Γ^t , $F^{-1}(t) = \Gamma^t$. Since the trace of g is in Γ we have equation:

$$F(\lambda, g(\lambda)) = 1. \tag{7.1}$$

Let $\lambda = e^{i\theta}$. Since $g' \in L^\infty(\Delta)$, the bounded radial limits exist almost everywhere. The function g extends to be C^α on the closed disk, and the partial derivative g_θ exist a.e. We differentiate equation (7.1) a.e. with respect to θ and obtain:

$$\lambda i F_\lambda(\lambda, g(\lambda)) - \text{Im}(F_w(\lambda, g(\lambda))g'(\lambda)\lambda) = 0. \tag{7.2}$$

The winding number of $\{g(\lambda) : \lambda \in \mathbb{S}^1\}$ around 0 is zero, and by Lemma 7.1, the winding number of $\{F_w(\lambda, g(\lambda)) : \lambda \in \mathbb{S}^1\}$ around 0 is zero as well. Thus we can take the logarithm and obtain

$$F_w(\lambda, g(\lambda)) = e^{a(\lambda)+ib(\lambda)}.$$

The left hand-side is α -Hölder continuous, so $b(\lambda)$ is α -Hölder continuous function, and so is its Hilbert transform $Hb(\lambda)$. Thus equation (7.2) becomes

$$\text{Im} \left(\lambda e^{Hb(\lambda)-ib(\lambda)} g'(\lambda) \right) = e^{-a(\lambda)} F'_\lambda(\lambda, g(\lambda)) \lambda$$

for almost every θ . Since the right hand side is C^α , so is the left hand side. Further the left hand side is the imaginary part of an analytic function so the function $\lambda e^{Hb(\lambda)-ib(\lambda)}g'(\lambda)$ itself is C^α . Therefore, $g' \in C^\alpha(\bar{\Delta})$. \square

7.2. Openness

In [2], the stability of foliation by holomorphic disks is proved if one starts from the standard torus.

THEOREM 7.3. — *Let Γ^t be a family of graphical tori, $t \in [0, \infty)$. Assume that a function $g^{t_0} : \bar{\Delta} \times \mathbb{S}^1 \rightarrow \mathbb{C}$ defines a holomorphic transverse foliation of a graphical torus Γ^{t_0} . Then there exists δ and a function $\tilde{g} : \bar{\Delta} \times \mathbb{S}^1 \times (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{C}$ that defines a transverse holomorphic foliation of Γ^t for $|t - t_0| < \delta$.*

Proof. — Hilbert transform

$$H : C^{1,\alpha}(\mathbb{S}^1) \rightarrow C^{1,\alpha}(\mathbb{S}^1)$$

is a bounded linear operator. We change the standard normalization $Hu(0) = 0$ to $Hu(1) = 0$. We denote by $C_{\mathbb{R}}^{1,\alpha}(\mathbb{S}^1) \subset C^{1,\alpha}(\mathbb{S}^1)$ be the subspace of real-valued functions. The curve $\{g_\xi^{t_0}(\lambda), \lambda \in \mathbb{S}^1\}$ has winding number 0 around zero, since $g_\xi^{t_0}(\lambda) \neq 0$ for $\lambda \in \Delta$. Therefore, by Lemma 7.1, the curve $\{F_w(\lambda, g_\xi^{t_0}(\lambda)) : \lambda \in \mathbb{S}^1\}$ has winding number 0 around 0:

$$F_w(\lambda, g_\xi^{t_0}(\lambda)) = e^{a_\xi(\lambda)+ib_\xi(\lambda)},$$

where $\alpha_\xi(\lambda)$, $b_\xi(\lambda)$ are Hölder continuous with exponent α . Thus, $Hb_\xi(\lambda)$ is Hölder continuous as well.

$X_\xi(\lambda) := e^{Hb_\xi(\lambda)-ib_\xi(\lambda)}$ is a holomorphic function on Δ and is proportional to the normal vector to C_λ^t in points $(\lambda, g_\xi^t(\lambda))$.

Functions of the form $(u(\lambda) + iHu(\lambda))X_\xi(\lambda)$ give all holomorphic functions that are Hölder continuous up to the boundary with the condition that $(u_\xi(1) + iHu_\xi(1))X_\xi(1)$ is proportional to the normal vector to C_1^t in a point $g_\xi^t(1)$. There exists an ϵ such that for each point $\eta \in C_1^t$, $|t - t_0| < \epsilon$, there is only one normal vector that intersects C_1^t in a point η .

The space $C^0(\mathbb{S}^1, C^{1,\alpha}(\mathbb{S}^1))$ is a Banach space with the norm

$$\|u\| = \sup_{\xi \in \mathbb{S}^1, \lambda \in \mathbb{S}^1} |u_\xi(\lambda)| + \sup_{\xi \in \mathbb{S}^1, \lambda \in \mathbb{S}^1} |u'_\xi(\lambda)| + \sup_{\xi \in \mathbb{S}^1, \lambda_1 \neq \lambda_2 \in \mathbb{S}^1} \frac{|u'_\xi(\lambda_1) - u'_\xi(\lambda_2)|}{|\lambda_1 - \lambda_2|^\alpha}.$$

Consider an operator

$$\mathcal{F} : \mathbb{R}t \times C^0(\mathbb{S}_\xi^1, C_{\mathbb{R}}^{1,\alpha}(\mathbb{S}_\lambda^1)) \rightarrow C^0(\mathbb{S}_\xi^1, C_{\mathbb{R}}^{1,\alpha}(\mathbb{S}_\lambda^1)) :$$

where \mathcal{F} is a function of two variable (t, u_ξ) . We consider function $u_\xi(\lambda)$ as an element of $C^0(\mathbb{S}_\xi^1, C^{1,\alpha}(\mathbb{S}_\lambda^1))$.

$$\mathcal{F}(t, u) : \mathbb{S}^1 \ni (t, \xi) \rightarrow F(\lambda, g_\xi(\lambda) + (u_\xi + iHu_\xi)X_\xi(\lambda)) - t \in C^{1,\alpha}(\mathbb{S}_\lambda^1)$$

For $0 < \alpha < 1$, H is a bounded linear operator, so \mathcal{F} is a continuous mapping of Banach spaces. Further, when \mathcal{F} is considered as a map from $\mathbb{R} \times \mathbb{S}_\xi^1$ to $C^{0,\alpha}(\mathbb{S}_\lambda^1)$, it is differentiable, and we compute the differential of \mathcal{F} at $u_\xi = 0$ in the direction δu_ξ :

$$D\mathcal{F}(t, 0; \delta u_\xi) = e^{\alpha_\xi(\lambda) - Hb_\xi(\lambda)} \delta u_\xi(\lambda).$$

Since $\mathcal{F}(t, 0; \delta u_\xi)$ is an invertible linear operator, we can define u_ξ^t as the unique element of $C^0(\mathbb{S}_\xi^1, C^{1,\alpha}(\mathbb{S}_\lambda^1))$ satisfying $\mathcal{F}(t, u_\xi^t) = 0$. And the function $\tilde{g}(\lambda, \xi, t) = g_\xi^{t_0}(\lambda) + u_\xi^t(\lambda)$ defines a holomorphic transverse foliation and is of class $C^{1,\alpha}$ on $\bar{\Delta}$. By continuity, for $\xi \neq \eta$, $g_\xi^t(\lambda) \neq g_\eta^t(\lambda)$ for $\lambda \in \Delta$. \square

This also gives us the openness for one disk.

THEOREM 7.4. — *Let Γ^t , $t \in I$ be a family of graphical tori. Let $g^{t_0} : \bar{\Delta} \rightarrow \mathbb{C}$ be a function that defines a holomorphic disk with the trace in the torus Γ^{t_0} . Assume that $g^{t_0} \in C^{1,\alpha}(\bar{\Delta})$, $g^{t_0}(\lambda) \neq 0$ for $\lambda \in \Delta$. Then there exists δ and a continuous function $g : \bar{\Delta} \times (t_0 - \delta, t_0 + \delta) \rightarrow \mathbb{C}$ such that $g^t(\lambda) := g(\lambda, t)$ defines a holomorphic disk with the trace in Γ^t and $g^t \in C^{1,\alpha}(\bar{\Delta})$, $g^t(\lambda) \neq 0$ for $\lambda \in \Delta$.*

7.3. Closedness

THEOREM 7.5. — *Let Γ^t , $t \in [0, \infty)$ be a family of graphical tori. Suppose that there exists $g : \bar{\Delta} \times \mathbb{S}^1 \times [0, t_0) \rightarrow \mathbb{C}$ that defines a holomorphic transverse foliation of Γ^t . Then g can be extended to $g : \bar{\Delta} \times \mathbb{S}^1 \times [0, t_0] \rightarrow \mathbb{C}$ that defines a holomorphic transverse foliation.*

Proof. — By Corollary 6.8, there exists C that depends only on Γ^{t_0} so that $|(g_\xi^t)'| < C$ for $t < t_0$ close to t_0 . Since the space of bounded holomorphic functions on Δ is compact, we can pass to the limit. Let $g_\xi^{t_0}$ be the limits, $|(g_\xi^{t_0})'| \leq C$. By Regularity Theorem 7.2, $g_\xi^{t_0} \in C^{1,\alpha}(\bar{\Delta})$. \square

This also give us closedness for a family of disks.

THEOREM 7.6. — *Let Γ^t , $t \in [0, \infty)$ be a family of graphical tori. Assume that $g : \bar{\Delta} \times [0, t_0) \rightarrow \mathbb{C}$ is a continuous function such that $g^t(\lambda) := g(\lambda, t)$ defines a holomorphic disk with the trace in Γ^t , $g^t \in C^{1,\alpha}(\bar{\Delta})$, $g^t(\lambda) \neq 0$ for $\lambda \in \Delta$. Then g can be extended to a continuous $g : \bar{\Delta} \times [0, t_0] \rightarrow \mathbb{C}$ such that g^{t_0} defines a holomorphic disk with the trace in Γ^{t_0} , $g^{t_0}(\lambda) \neq 0$ for $\lambda \in \Delta$.*

7.4. Uniqueness

Let $\Gamma^c = \{(\lambda, w) : |w| = c, |\lambda| = 1\}$ be standard tori. Let $g : \Delta \rightarrow \mathbb{C}$ be a function that defines a holomorphic disk with the trace of $g(\partial\Delta) \in \Gamma^c$. By the Minimum Modulus Theorem, \min of g is attained on the boundary. Maximum modulus is attained on the boundary as well. So $|g(\lambda)| = \text{const}$. Therefore, $g(\lambda) = \text{const}$.

THEOREM 7.7. — *Let Γ be a graphical torus. Let $g, h : \bar{\Delta} \rightarrow \mathbb{C}$ be functions that define holomorphic disks with traces in Γ , $g(\lambda) \neq 0$, $h(\lambda) \neq 0$ for $\lambda \in \Delta$. Assume that $g(1) = h(1)$. Then there exists ϵ such that $|g(\lambda) - h(\lambda)| < \epsilon$ for $\lambda \in \mathbb{S}^1$ implies $g(\lambda) \equiv h(\lambda)$.*

Note that the same ϵ works for tori close to Γ .

Proof. —

Let $a(\lambda, s) = F(\lambda, g(\lambda) + s(h(\lambda) - g(\lambda)))$, $a(\lambda, 0) = a(\lambda, 1) = t$. Then

$$\int_0^1 a_s ds = 0 = \text{Re}(h(\lambda) - g(\lambda)) \int_0^1 F_w(g(\lambda) + s(h(\lambda) - g(\lambda))) ds. \quad (7.3)$$

The winding number of the curve $\{g(\lambda) : \lambda \in \mathbb{S}^1\}$ around 0 is equal to zero. Hence, by Lemma 7.1, the winding number of

$$\{F_w(\lambda, g(\lambda)) : \lambda \in \mathbb{S}^1\}$$

around 0 is equal to zero.

Therefore, for small enough ϵ , the winding number of the curve

$$\left\{ \int_0^1 F_w(g(\lambda) + s(h(\lambda) - g(\lambda))) ds : \lambda \in \mathbb{S}^1 \right\}$$

around 0 is equal to zero, so

$$\int_0^1 F_w(g(\lambda) + s(h(\lambda) - g(\lambda))) ds = e^{a(\lambda) + ib(\lambda)}.$$

The function $b(\lambda)$ is a bounded Hölder continuous function.

By equation (7.3), $\arg(g(\lambda) - h(\lambda)) = \frac{\pi}{2} - b(\lambda)$, where $b(\lambda)$ is a bounded function. It contradicts the fact that $g(1) - h(1) = 0$. \square

7.5. Global Uniqueness

THEOREM 7.8. — *Let Γ be a graphical torus. Let $g^1, h^1 : \bar{\Delta} \rightarrow \mathbb{C}$ be functions that define holomorphic disks with traces in Γ , $g^1, h^1 \in C^{1,\alpha}(\bar{\Delta})$. Assume that $g^1(1) = h^1(1)$. Then $g^1(\lambda) = h^1(\lambda)$.*

Proof. — We include torus Γ into a family of graphical tori Γ^t , $t \in [0, 1]$, $\Gamma^1 = \Gamma$. By Theorems 7.4, 7.6, there exist functions $g, h : \bar{\Delta} \times [0, 1] \rightarrow \mathbb{C}$ such that $g(\lambda, 1) = g^1$, $h(\lambda, 1) = h^1$ and $g^t(\lambda) := g(\lambda, t)$ define holomorphic disks with the traces in tori Γ^t . There exists ϵ such that for $t < \epsilon$, $\Gamma^t = \{(\lambda, w) : |w| = t, \lambda \in \mathbb{S}_\lambda^1\}$, $t \in [0, 1]$ are standard tori with uniqueness of solutions. For $t < \epsilon$, $g^t \equiv h^t$. Let $t_0 = \sup\{t : h^t \equiv g^t\}$. If $t_0 \neq 1$, then by applying Theorem 7.7, we get a contradiction. \square

At this point we have proved the Filling theorem. For Filling Theorem' the only statement remains is to show that disks for Γ^{t_1} are disjoint from disks for Γ^{t_2} when $t_1 \neq t_2$. Suppose D^{t_j} is a disk with boundary in Γ^{t_j} . If $D^{t_1} \cap D^{t_2} \neq \emptyset$, then since the traces are in Γ^{t_1} and Γ^{t_2} we will have $D_{\xi_1}^{t_1} \cap D_{\xi_2}^{t_2} \neq \emptyset$ for all $\xi_1, \xi_2 \in \mathbb{S}^1$. By Filling Theorem, the disks $D_{\xi_1}^{t_1} \cap D_{\xi_2}^{t_2} = \emptyset$ for $\xi_1 \neq \xi_2$. However, there is a continuous family of disks D_ξ^t , $t \in [t_1, t_2]$, which is a contradiction.

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