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Curvature dimension bounds on the deltoid model

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RÉSUMÉ. — La courbe deltoide dans le plan est la frontière d'un domaine borné sur lequel il existe une famille de mesures de probabilité et des polynômes orthogonaux pour ces mesures qui sont aussi vecteurs propres d'opérateurs de diffusion. On peut donc considérer ces polynômes comme une extension des polynômes de Jacobi classiques. Ce domaine appartient à l'une des 11 familles de tels domaines bornés de $\mathbb{R}^2$. Nous étudions les inégalités de courbure-dimension associés à ces opérateurs, en en déduisons diverses bornes sur les polyômes associés, ainsi que des inégalités de Sobolev relatives aux formes de Dirichlet correspondantes.

ABSTRACT. — The deltoid curve in $\mathbb{R}^2$ is the boundary of a domain on which there exist probability measures and orthogonal polynomials for theses measures which are eigenvectors of diffusion operators. As such, those polynomials may be considered as a two dimensional extension of the classical Jacobi polynomials. This domain belongs to one of the 11 families of such bounded domains in $\mathbb{R}^2$. We study the curvature-dimension inequalities associated to these operators, and deduce various bounds on the associated polynomials, together with Sobolev inequalities related to the associated Dirichlet forms.

1. Introduction

The deltoid curve (also called Steiner's hypocycloid), see figure page 8, is one of the bounded domains in $\mathbb{R}^2$ on which there exist a probability

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measure μ and a symmetric diffusion process, the eigenvectors of it being orthogonal polynomials for μ. These orthogonal polynomials have been introduced in [17, 18] and appear in the classification of [19]. They appear in [4] as one of the eleven models in dimension two for which such polynomials exist. Moreover, it seems that is one of the most difficult models to analyse, since there does not exist many geometric interpretation for it. Beyond this, it is also interesting since it is deeply linked with the analysis of the A_2 root system and of the spectral analysis of $SU(3)$ matrices.

This deltoid model and the associated generators provide an interesting object to check various properties of diffusion operators, since one knows explicitly the eigenvalues, and has many informations on the eigenvectors. For example, they have satisfy recurrence formulas which allows for explicit computations, and in some cases generating functions, see [28].

Since the associated measures and operators depend on a real parameter $\lambda > 0$ (see equation (3.1)), one may try to understand how functional inequalities and curvature properties depend on this parameter λ, and hence on geometric properties of the model.

It turns out that for the specific cases $\lambda = 1$ and $\lambda = 4$, one may produce simple geometric interpretations : in the first case from the Euclidean Laplace operator through the symmetries of the triangular lattice, in the second case from the Casimir operator on $SU(3)$ acting on spectral measures. The $SU(3)$ model provides curvature-dimension inequalities for the generic model for $\lambda \geqslant 1$. It is not clear however that these inequalities are optimal. It turns out that they indeed are. Quite unexpectedly, the careful investigation of the $CD(\rho, n)$ inequality for this model does not produce better results than the direct consequence of the $SU(3)$ inequality. In comparison with the classical case of Jacobi polynomials, which are orthogonal with respect to the measure $C_{a,b}(1-x)^a(1+x)^b dx$ on $(-1, 1)$, this situation is similar to the symmetric case $a = b$, but differs from the dissymmetric case (see [1]).

It seems worth to point out at least two interesting features of the computations of curvature-dimension inequalities for this model. The first aspect concerns the existence of an optimal dimension in the inequality. When one looks for curvature-dimension inequalities on a compact Riemannian manifold with dimension n_0, but for a reversible measure which is not the Riemann measure, there is no optimal one. For any $n > n_0$, one may find some constant $\rho(n)$ such that a $CD(\rho(n), n)$ inequality holds. In general, $\rho(n)$ goes to $-\infty$ when $n \to n_0$. It is only for Laplace operators that one may expect some $CD(\rho, n_0)$ inequality. This is not the case here. For any $\lambda > 1$, there is a bound $n(\lambda) = 2\lambda$ such that no $CD(\rho, n)$ inequality may hold for

$n < n(\lambda)$. However, for this limiting value $n = 2\lambda$, the $CD(\frac{3}{4}(\lambda - 1), 2\lambda)$ holds. Of course, this phenomenon is due to the singularity of the density of the measure at the boundaries of it's support.

The second aspect concerns the use of appropriate coordinate systems. Since the underlying metric is a flat metric in two dimensions, the curvature-dimension inequality amounts to check for which values a and b one has an inequality of the form

$$-\nabla\nabla W \geqslant a\mathrm{Id} + b\nabla W \otimes \nabla W,$$

where W is the logarithm of the density measure with respect to the Riemann measure. It turns out that a proper choice of the coordinates leads to very simple formulas, which is not the case if the computation is made through the use of the naive usual coordinates in the Euclidean plane. This comes from the fact that we have indeed at disposal a polynomial structure, expressed through the choice of these coordinates, and the function W satisfies nice relations with respect to this, namely a "boundary equation" described in (3.3). This is a good indication that if one wants to study such inequalities for higher dimensional models, these "polynomial coordinates" should be used instead of the usual ones.

Then, the curvature-dimension inequalities provide through Sobolev inequalities (2.6) various uniform bounds on the orthogonal polynomials, which turn out, up to some change in the parameters, to be equivalent to the Sobolev inequality itself. They also provide bounds on kernels constructed from other spectral decompositions, that is for operators which do not necessary commute with our starting operator.

Orthogonal polynomials on the interior of the deltoid curve belong to the large family of Heckman-Opdam polynomials associated with root systems [12, 13], and in the even larger class of MacDonald's polynomials [21, 20, 22]. As such, they may serve as a guide for more general models of diffusions associated with orthogonal polynomials. One may find some extensive presentation of these pluri-dimensional families of orthogonal polynomials, for example in [9, 22].

The associated diffusion generator is associated with some reflection group in dimension 2. Operators associated with reflection groups in $\mathbb{R}^d$, known as Dunkl operators, are extensively studied in the literature, from the points of view of special functions related to Lie group analysis or Hecke algebras, as well as from the point of view of the associated heat equations, or in probability and statistics [8, 15, 14, 25, 24].

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Most of the language and notations related to diffusion operators, and in particular the links between curvature-dimension inequalities and Sobolev inequalities, together with the bounds one may deduce for eigenvectors, are borrowed from [3]. The fact that Sobolev inequalities are equivalent to bounds on the heat kernel goes back to [7, 27], and the relations between curvature dimension inequalities go back to [2] and are exposed in [3], among others. It is classical that one may deduce from them bounds on the spectral projectors. However, the fact that these bounds may in turn provide Sobolev inequalities, which is the content of Theorem 5.4, seems quite new, at least to our knowledge, although a similar result concerning logarithmic Sobolev inequalities is exposed in [3]. Notice that this recovers a Sobolev inequality with a weaker exponent, not to talk about the optimal constants, which are always out of reach with this kind of techniques.

Many of the properties concerning the spectral decomposition of the operator on the deltoid, recurrence formulas for the associated orthogonal polynomials, generating functions, etc., may be found in [28]. We shall not use the results of this paper here, apart the representations coming from symmetry groups in $\mathbb{R}^2$ and from $SU(3)$, that we recall briefly in Section 3.

The paper is organized as follows. In Section 2, we present the general curvature-dimension inequalities and the associated Sobolev inequalities. We show how this provides bounds on the eigenvectors. Section 3 is a short introduction to the model associated with the deltoid curve, where we explain the two geometric specific cases. Section 4 provides the associated curvature-dimension inequalities, first from the $SU(3)$-model, then from direct computation in subsection 4.2 with the adapted system of coordinates. Finally, in Section 5, we give the various bounds on polynomials and operators we are looking for.

2. Curvature dimension inequalities

We briefly recall in this Section the context of symmetric diffusion operators, following [3], in a specific context adapted to our setting. For a given probability space $(X, \mathcal{X}, \mu)$, we suppose given an algebra $\mathcal{A}$ of functions such that $\mathcal{A} \subset \cap_{1 \leqslant p < \infty} \mathcal{L}^p(\mu)$, $\mathcal{A}$ is dense in $\mathcal{L}^2(\mu)$, and which is stable under composition with smooth functions Φ. In our case, $\mathcal{A}$ may be chosen as the class of restrictions to the domain of smooth functions defined in a neighborhood of it, without any boundary condition. This particular choice for $\mathcal{A}$ is made possible thanks to a special property of the operator, which satisfies a "boundary equation", which is our model has the specific form (3.3). It is valid as soon one deals with operators having polynomial eigenvectors on a bounded domain, see [4]. In most of the cases, we may as well restrict our at-

tention to polynomials, although this would not be appropriate for Sobolev inequalities. A bilinear application $\Gamma : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ is given such that, $\forall f \in \mathcal{A}$, $\Gamma(f, f) \geqslant 0$, which satisfies $\Gamma(\Phi(f_1, \cdots, f_k), g) = \sum_i \partial_i \Phi \Gamma(f_i, g)$ for any smooth function Φ. A linear operator $\mathcal{L}$ is defined through

$$\int_X f \mathcal{L}(g) d\mu = - \int_X \Gamma(f, g) d\mu \qquad (2.1)$$

and we assume that $\mathcal{L}$ maps $\mathcal{A}$ into $\mathcal{A}$. In this context, all the properties of the model are described by Γ and μ, and the model is then entirely described by the triple (X, Γ, μ) (see [3]). We then extend $\mathcal{L}$ into a self adjoint operator and we suppose that $\mathcal{A}$ is dense in the domain of $\mathcal{L}$. The measure μ is often referred to the reversible measure for $\mathcal{L}$.

Then, for $f = (f_1, \cdots, f_k)$ and for any smooth function Φ

$$\mathcal{L}(\Phi(f)) = \sum_1^k \partial_i \Phi(f) \mathcal{L}(f_i) + \sum_{i,j=1}^k \partial_{ij}^2 \Phi(f) \Gamma(f_i, f_j). \qquad (2.2)$$

We have

$$\Gamma(f, g) = \frac{1}{2} \Big(\mathcal{L}(fg) - f\mathcal{L}(g) - g\mathcal{L}(f) \Big).$$

We moreover define the Γ_2 operator as

$$\Gamma_2(f, g) = \frac{1}{2} \Big(\mathcal{L}\Gamma(f, g) - \Gamma(f, \mathcal{L}g) - \Gamma(g, \mathcal{L}f) \Big). \qquad (2.3)$$

Then, for any parameters $\rho \in \mathbb{R}$ and $n \in [1, \infty]$, we say that $\mathcal{L}$ satisfies a curvature-dimension inequality $CD(\rho, n)$ if and only if

$$\forall f \in \mathcal{A}, \Gamma_2(f, f) \geqslant \rho \Gamma(f, f) + \frac{1}{n}(\mathcal{L}f)^2.$$

It is worth to observe that the $CD(\rho, n)$ inequality is local. For a general elliptic operator on a smooth manifold M with dimension n_0, one may always decompose $\mathcal{L}$ into $\Delta_g + \nabla \log V$, where Δ_g is the Laplace operator associated with the co-metric (g) and V is the density of μ with respect to the Riemann measure. In which case, the operator Γ_2 may be decomposed as

$$\Gamma_2(f, f) = |\nabla\nabla f|^2 + (\text{Ric}_g - \nabla\nabla \log V)(\nabla f, \nabla f), \qquad (2.4)$$

where Ric_g denotes the Ricci curvature computed for the Riemannian metric associated with g, $\nabla\nabla \log V$ is the Hessian of $\log V$, also computed in this metric, and $|\nabla\nabla f|^2$ is the Hilbert-Schmidt norm of the Hessian of f.

In this case, the $CD(\rho, n)$ inequality holds if and only if $n \geqslant n_0$ and, when V is not constant, when $n > n_0$ and

$$\mathrm{Ric}_g - \nabla\nabla \log V \geqslant \rho g + \frac{1}{n - n_0} \nabla \log V \otimes \nabla \log V. \qquad (2.5)$$

Of course, when $\mathcal{L} = \Delta_g$, this amounts to $n \geqslant n_0$ and $\mathrm{Ric}_g \geqslant \rho g$. In this case, there exists a best choice for both ρ and n, namely for n the dimension of the manifold and for ρ a lower bound on the Ricci tensor, that is the infimum over $\mathcal{M}$ of the lowest eigenvalue of this tensor.

In this paper, we are mainly mainly interested the case where $\mathcal{L} = \Delta + \nabla \log(V)$, where Δ is the Euclidean Laplace operator in some open set of $\mathbb{R}^{n_0}$. In which case the measure μ is $V dx$, and the $CD(\rho, n)$ inequality holds if and only if $n \geqslant n_0$ and

$$-\nabla\nabla \log(V) \geqslant \rho + \frac{1}{n - n_0} \nabla \log(V) \otimes \nabla \log(V).$$

In order for it to be satisfied, we may look at local inequalities $CD(\rho(x), n(x))$ and try to find such a pair $(\rho(x), n(x))$ for which $\rho(x)$ is bounded below and $n(x)$ bounded above. For a generic function V, there is no "best" local inequality in general. The $CD(\rho, n)$ inequality requires that the symmetric tensor $-\nabla\nabla \log V$ is bounded below by $\rho\mathrm{Id}$. If ρ_0 is the best real number such that $-\nabla\nabla \log V \geqslant \rho_0\mathrm{Id}$ (that is ρ_0 is the lowest eigenvalue at the point x of $-\nabla\nabla \log V$), then the inequality holds as soon as $\rho \geqslant \rho_0$ and $(\rho - \rho_0)(n - n_0) \geqslant |\nabla \log V|^2$.

But in our case, as already mentioned in the introduction, we are not in this situation. There is a lower bound on the admissible dimension, which is strictly bigger than n_0. To understand this phenomenon, one may analyse a bit further this $CD(\rho, n)$ inequality at a given point on the manifold.

It may happen that, at some point x, the eigenvector of $-\nabla\nabla \log V$ corresponding to some eigenvalue $\rho_1(x) > \rho_0(x)$ is parallel to $\nabla \log V$. Let then in this case, there is a best choice for both $\rho(x)$ and $n(x)$, which is

$$\begin{cases} \rho(x) = \rho_0(x), \\ n(x) = n_0 + \frac{1}{\rho_1(x) - \rho_0(x)} |\nabla \log V|^2. \end{cases}$$

In the model that we shall consider later, we shall see that this happens asymptotically on the boundary of the set we are working on, and the constants n and ρ computed at this boundary points are valid for all other points x.

When some $CD(\rho, n)$ inequality holds, with ρ and n constant, and whenever $\rho > 0$, and $2 < n < \infty$, then (X, Γ, μ) satisfies a tight Sobolev inequality. For $p = \frac{2n}{n-2}$, and for any $f \in \mathcal{A}$, we have

$$\left(\int_X f^p d\mu \right)^{2/p} \leqslant \int_X f^2 d\mu + \frac{4}{n(n-2)} \frac{n-1}{\rho} \int_X \Gamma(f, f) d\mu. \qquad (2.6)$$

More generally, an n- dimensional Sobolev inequality is an inequality of the form

$$\|f\|_{2n/(n-2)}^2 \leqslant A\|f\|_2^2 + C \int \Gamma(f, f) \, d\mu. \qquad (2.7)$$

When μ is a probability measure, we say that the inequality is tight when the constant A is 1, and provided some Sobolev inequality holds, tightness is equivalent to the fact that a Poincaré inequality occurs, which is automatic in our case since the spectrum is discrete (see [3]).

When a Sobolev inequality (2.7) holds, then the associated semigroup $P_t = \exp(t\mathcal{L})$ is ultracontractive, that is, for any $q \in [2, \infty]$

$$\|P_t f\|_q \leqslant \frac{C_1}{t^{\frac{n}{2}(\frac{1}{2} - \frac{1}{q})}} \|f\|_2, \ 0 < t \leqslant 1, \qquad (2.8)$$

with

$$C_1 = \left(\frac{Cn}{2} + 2A \right)^{n/2}.$$

This last constant C_1 is not sharp however. The bound is valid for $q = \infty$ and indeed, the result for any $q \in (2, \infty)$ is a consequence of the case $q = \infty$ through an interpolation argument. It turns out that his last ultra contractive bound (for some given q, but for any $t \in (0, 1)$) is in turn equivalent to the Sobolev inequality.

There is another equivalent version

$$\|P_t f\|_\infty \leqslant Ct^{-n/2}\|f\|_1, \ 0 < t \leqslant 1, \qquad (2.9)$$

for which one deduces immediately that the semigroup P_t has a density which is bounded above by $Ct^{-n/2}$.

When $1 \leqslant n \leqslant 2$, one may replace Sobolev inequalities by Nash inequalities, which play the same rle, see remark 5.6. However, the best constants that one may deduce from curvature-dimension inequalities for Nash inequalities are not known (see [6]).

As a consequence of ultracontractive bounds, whenever f is an eigenvector for $\mathcal{L}$ with eigenvalue $-\lambda$, and provided that $\int f^2 d\mu = 1$, one has

$$\|f\|_q \leqslant C_1 \inf_{t>0} \frac{e^{\lambda t}}{t^{\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}} = C_1 C_{n,q} \lambda^{\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}, \tag{2.10}$$

with $C_{n,q} = \inf_{s>0} e^s s^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}$, which follows immediately from the fact that $P_t f = e^{-\lambda t} f$. This applies in particular for $q = \infty$, and produces uniform bounds on the eigenvectors from the knowledge of their $\mathcal{L}^2(\mu)$ norms.

3. Diffusion processes on the interior of the deltoid curve

We describe first the operator associated with the deltoid curve associated with a family of orthogonal polynomials. Most of the details may be found in [28]. The deltoid curve is a degree 4 algebraic plane curve which may be parametrized as

$$x(t) = \frac{1}{3}(2\cos t + \cos 2t), \quad y(t) = \frac{1}{3}(2\sin t - \sin 2t)$$
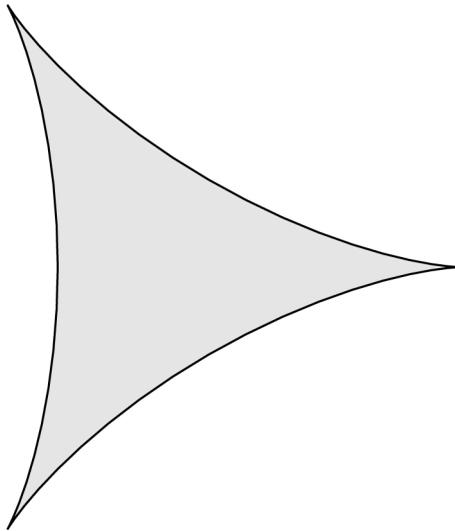

Figure 1. — The deltoid domain.

The unnatural factor $1/3$ in this description of the curve will produce more convenient formula in the sequel. The connected component of the complementary of the curve which contains 0 is a bounded open set, that

we refer to as the deltoid domain $\mathcal{D}$. It turns out that there exist on this domain a one parameter family $\mathcal{L}^{(\lambda)}$ of symmetric diffusion operator which may be diagonalized in a basis of orthogonal polynomials. It was introduced in [17, 18], and further studied in [28]. This is one of the 11 families of sets in dimension 2 carrying such diffusion operators, as described in [4].

In order to describe the operator, and thanks to the diffusion property (2.2), it is enough to describe $\Gamma(x, x)$, $\Gamma(x, y)$, $\Gamma(y, y)$, $\mathcal{L}^{(\lambda)}(x)$ and $\mathcal{L}^{(\lambda)}(y)$ (the Γ operator does not depend on λ here).

The symmetric matrix $\begin{pmatrix} \Gamma(x, x) & \Gamma(x, y) \\ \Gamma(y, x) & \Gamma(y, y) \end{pmatrix}$ is referred to in what follows as the metric associated with the operator, although properly speaking it is in fact a co-metric. It is indeed easier to use the complex structure of $\mathbb{R}^2 \simeq \mathbb{C}$, and use the complex variables $Z = x + iy$, $\bar{Z} = x - iy$, with

$$\mathcal{L}(Z) = \mathcal{L}(x) + i\mathcal{L}(y), \ \mathcal{L}(\bar{Z}) = \mathcal{L}(x) - i\mathcal{L}(y),$$

and

$$\Gamma(Z, Z) = \Gamma(x, x) - \Gamma(y, y) + 2i\Gamma(x, y), \ \Gamma(Z, \bar{Z}) = \Gamma(x, y) + \Gamma(y, y).$$

The formulas are much simpler with these variables, and $\mathcal{L}^{(\lambda)}$ is then described as

$$\begin{cases} \Gamma = \begin{pmatrix} \Gamma(Z, Z) = \bar{Z} - Z^2, & \Gamma(\bar{Z}, Z) = 1/2(1 - Z\bar{Z}) \\ \Gamma(\bar{Z}, Z) = 1/2(1 - Z\bar{Z}) & \Gamma(\bar{Z}, \bar{Z}) = Z - \bar{Z}^2 \end{pmatrix} \\ \mathcal{L}(Z) = -\lambda Z, \quad \mathcal{L}(\bar{Z}) = -\lambda\bar{Z}, \end{cases} \tag{3.1}$$

where $\lambda > 0$ is a real parameter.

The boundary of this domain turns out to the curve with equation

$$P(Z, \bar{Z}) = \det(\Gamma) = \Gamma(Z, \bar{Z})^2 - \Gamma(Z, Z)\Gamma(\bar{Z}, \bar{Z}) = 0, \tag{3.2}$$

and inside the domain, the associated metric is positive definite, so that it corresponds to some elliptic operator on it. The reversible measure associated with it has density $C_\alpha P(Z, \bar{Z})^\alpha$ with respect to the Lebsegue measure, where $\lambda = (6\alpha + 5)/2$, and is a probability measure exactly when $\lambda > 0$ (we refer to [28] for more details).

Indeed, this relation between the determinant of Γ, the measure and the equation of the boundary is not specific to this model. Following [4], any symmetric diffusion operator on a bounded open set $\Omega \subset \mathbb{R}^d$ with

piecewise $\mathcal{C}^1$ boundary, which may be diagonalized in a basis of orthogonal polynomials (for the reversible measure of $\mathcal{L}$) in the variables $(x_1, \cdots, x_d)$ shares the following properties

1. For any (i, j), $\Gamma(x_i, x_j)$ is a polynomial with degree at most 2.

2. The boundary $\partial \Omega$ is included in the algebraic set $\{\det(\Gamma) = 0\}$.

3. Whenever $P = \det(\Gamma)$ is an irreducible polynomial, then for any $i = 1, \cdots, d$, $\Gamma(x_i, \log P)$ is a degree 1 polynomial.

4. The reversible measures for $\mathcal{L}$ have a density P^α with respect to the Lebesgue measure.

There is a more general result when $\det(\Gamma)$ is not irreducible, and then the description of the various possible reversible measures is more complicated.

In our context, we use the complex variables $(Z, \bar{Z})$ in $\mathbb{R}^2$, which provide simpler forms. The deltoid domain is one of the sets Ω on which such structure exits. Here, the determinant of Γ is irreducible, and point 3., usually called the boundary equation, takes here a simple form

$$\Gamma(Z, \log P) = -3Z, \ \Gamma(\bar{Z}, \log P) = -3\bar{Z}, \tag{3.3}$$

and is easily checked from formulae (3.1).

There are two particular cases which are worth understanding, namely $\lambda = 1$ and $\lambda = 4$, corresponding to the parameters $\alpha = \pm 1/2$. We briefly present those two models, referring to [28] for more details, since we shall make a strong use of them in what follows.

In the first case $\lambda = 1$, one sees that this operator is nothing else that the image of the Euclidean Laplace operator on $\mathbb{R}^2$ acting on the functions which are invariant under the symmetries around the lines of a regular triangular lattice. To see this, one considers the three unit third roots of identity in $\mathbb{C}$, say $(e_1, e_2, e_3) = (1, j, \bar{j})$. Then, consider the functions Z and $z_k : \mathbb{C} \mapsto \mathbb{C}$ which are defined as

$$z_k(z) = e^{i\Re(z\bar{e}_k)}, \ Z = \frac{1}{3}(z_1 + z_2 + z_3) \tag{3.4}$$

They satisfy $|z_k| = 1$ and $z_1 z_2 z_3 = 1$.

It is easily seen that, for the Euclidean Laplace operator on $\mathbb{R}^2$, Z and $\bar{Z}$ satisfy the relations (3.1) with $\lambda = 1$. Moreover, the function $Z : \mathbb{C} \mapsto \mathbb{C}$ is a diffeomorphism between the interior $\mathcal{T}$ of the triangle T and the deltoid

domain $\mathcal{D}$, where T is one of the equilateral triangles containing the two edges 0 and $4\pi/3$. The functions which are invariant under the symmetries of the triangular lattice generated by this triangle T are exactly functions of Z. Therefore, the image of $\mathcal{L}^{(1)}$ through $Z^{-1} : \mathcal{D} \mapsto \mathcal{T}$ is nothing else that the Laplace operator on $\mathcal{T}$. We may as well look at the image of the operator $\mathcal{L}^{(\lambda)}$ and it is then simply

$$\mathcal{L}^{(\lambda)}(f) = \Delta(f) + (\alpha + 1/2)\nabla \log W \nabla f = \Delta(f) + \frac{\lambda - 1}{3}\nabla \log W \nabla f \quad (3.5)$$

where Δ is the usual Laplace operator in $\mathbb{R}^2$ and the function W is defined form the functions z_j described in equation (3.4) as

$$W = -(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2. \quad (3.6)$$

One should be aware here that thanks to the properties of the functions z_j, $-(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2$ is a real valued function taking values in $(0, \infty)$ (and vanishes only at the boundaries of $\mathcal{T}$).

This representation provides a way of computing $CD(\rho, n)$ inequalities for $\mathcal{L}^{(\lambda)}$, following the description of Section 2.

The second description follows from the Casimir operator on $SU(3)$. This latter group is a semi-simple compact Lie group, and as such as a canonical Laplace (Casimir) operator which commutes (both from left and right) to the group action [16, 10]. The form of the Casimir operator acting on central functions (that is functions which depend only on the spectrum of the matrix) is well known and given by Weyl's formula. Nevertheless, for the comfort of the reader, we provide here a completely naive approach which allows for extension to larger classes of operators acting on sets of matrices. In any such semi simple compact Lie group G, one considers it's Lie algebra $\mathcal{G}$, naturally endowed with a Lie algebra structure $\mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$, $(X, Y) \mapsto [X, Y]$. The Lie algebra structure provides on $\mathcal{G}$ a natural quadratic form K (the Killing form) as follows : for any element $X \in \mathcal{G}$, one considers $ad(X) : \mathcal{G} \mapsto \mathcal{G}$, $Y \mapsto [X, Y]$, and $K(X, Y) = -\text{trace}\left(ad(X)ad(Y)\right)$. It turns out that this quadratic form is positive definite exactly when the group is compact and semi-simple. If one considers, for this Killing form, any orthonormal basis (X_i) in the Lie algebra, the quantity $\sum_i X_i^2$, computed in the enveloping algebra, does not depend on the choice of the basis, and commutes with any element on the Lie algebra itself (this means that this commutation property depends only on the Lie-algebra structure and not on the way that the elements of this Lie algebra are effectively represented as linear operators).

Now, to any $X \in \mathcal{G}$ is associated a first order operator D_X on G defined as follows

$$D_X(f)(g) = \partial_t f(ge^{tX}) \mid_{t=0}. \tag{3.7}$$

The application $X \mapsto D_X$ is a representation of the Lie algebra into the linear space of vector fields ($[D_X, D_Y] = D_{[X,Y]}$), and any identity in the Lie algebra (or on it's enveloping algebra) translates to an identity on those differential operators. We work here with the right action $g \mapsto ge^{tX}$ but we could as well work with the left action $g \mapsto e^{tX}g$. For any orthonormal basis X_i of $\mathcal{G}$ for the Killing form K, one defines the Casimir operator

$$\mathcal{L} = \sum_i D_{X_i}^2. \tag{3.8}$$

It does not depend of the choice of the basis and commutes with the group action, that is $[\mathcal{L}, D_X] = 0$ for any $X \in \mathcal{G}$.

This Killing form provides an Euclidean quadratic form in the tangent plane at identity in G (the Lie algebra $\mathcal{G}$), which may be translated to the tangent plane at any point $g \in G$ through the group action, and endows G with a natural Riemanian structure. It turns out that the Casimir operator $\mathcal{L}$ is also the Laplace operator for this structure. For the group $SU(d)$ that we are interested in, one wants to precisely describe the action of this Casimir operator on the entries of the matrix $g = (z_{ij})$ in $SU(d)$. That is, writing the entries $z_{pq} = x_{pq} + iy_{pq}$, we consider x_{pq} and y_{pq} as functions $G \mapsto \mathbb{R}$, and, for any i, j, k, l, we want to compute

$$\mathcal{L}^{SU(d)}(x_{ij}), \; \mathcal{L}^{SU(d)}(y_{ij}), \; \Gamma^{SU(d)}(x_{ij}, x_{k,l}), \; \Gamma^{SU(d)}(x_{ij}, y_{k,l}), \; \Gamma^{SU(d)}(y_{ij}, x_{k,l}),$$

where $\Gamma^{SU(d)}$ is the square field operator associated with $\mathcal{L}^{SU(d)}$. In order to get simpler formulae, it is once again better to work with the complex valued functions z_{pq}, writing for such a function $z = x + iy$,

$$\mathcal{L}^{SU(d)}(z) = \mathcal{L}^{SU(d)}(x) + iL^{SU(d)}(y),$$
$$\Gamma^{SU(d)}(z, z) = \Gamma(x, x) - \Gamma(y, y) + 2i\Gamma(x, y),$$
$$\Gamma^{SU(d)}(z, \bar{z}) = \Gamma(x, x) + \Gamma(y, y).$$

If one denotes by $E_{p,q}$ the matrix with entries $(E_{p,q})_{i,j} = \delta_{ip}\delta_{jq}$, a base of the Lie algebra of $SU(d)$ is given by

$$R_{k,l} = (E_{k,l} - E_{l,k})_{k<l}$$

$$S_{k,l} = i(E_{k,l} + E_{l,k})_{k<l}$$

$$D_{1,l} = i(E_{1,1} - E_{l,l})_{1<l}.$$

In order to describe the Casimir operator in a compact form, it is better to work with $D_{k,l} = i(E_{k,k} - E_{l,l})_{k<l}$, and one may write in this way (up to a factor 2 that will play no rle in the future)

$$\mathcal{L}^{SU(d)} = \sum_{k<l} (D^2_{R_{k,l}} + D^2_{S_{k,l}} + \frac{2}{n} D^2_{D_{k,l}}). \tag{3.9}$$

The reader should pay attention here that this operator defined by (3.9) is not the classical Casimir operator on $SU(d)$, since the norm of $R_{k,l}$ for the Killing form is actually $2\sqrt{3}$, and therefore $\mathcal{L}^{SU(d)}$ defined by (3.9) is 12 times the usual one. This change of normalization in $\mathcal{L}^{SU(d)}$ plays no rle in what follows, except for the computations of the Ricci curvature in the next Section 4.1.

One may compute then the associated vector fields following formula (3.7), and we get

$$D_{R_{pq}} = \sum_k z_{kq}\partial_{kp} - z_{kp}\partial_{kq} + \bar{z}_{kq}\bar{\partial}_{kp} - \bar{z}_{kp}\bar{\partial}_{kq}$$

$$D_{S_{pq}} = i\left[\sum_k z_{kq}\partial_{kp} + z_{kp}\partial_{kq} - \bar{z}_{kq}\bar{\partial}_{kp} - \bar{z}_{kp}\bar{\partial}_{kq} \right].$$

$$D_{D_{pq}} = i\left[\sum_k z_{kp}\partial_{kp} - z_{kq}\partial_{kq} - \bar{z}_{kp}\bar{\partial}_{kp} + \bar{z}_{kq}\bar{\partial}_{kq} \right].$$

With these relations, one may directly compute the action of $\mathcal{L}^{SU(d)}$ on the entries of the matrix, and we get

$$\begin{cases} \mathcal{L}^{SU(d)}(z_{pq}) = \frac{-2(d^2-1)}{d} z_{pq} \\ \mathcal{L}^{SU(d)}(\bar{z}_{pq}) = \frac{-2(d^2-1)}{d} \bar{z}_{pq} \\ \Gamma^{SU(d)}(z_{kl}, z_{rq}) = -2z_{kq}z_{rl} + \frac{2}{d}z_{kl}z_{rq}, \\ \Gamma^{SU(d)}(z_{kl}, \bar{z}_{rq}) = 2(\delta_{kr}\delta_{lq} - \frac{1}{d}z_{kl}\bar{z}_{rq}). \end{cases} \tag{3.10}$$

Now, let us consider the special case $d = 3$ and consider the function $Z : G \mapsto \mathbb{C}$ defined by $Z(g) = \frac{1}{3}\text{trace}(g) = \frac{1}{3}(z_{11} + z_{22} + z_{33})$. Any function on G depending only on the spectrum of g is a function of $(Z, \bar{Z})$, since, denoting by $\lambda_1, \lambda_2, \lambda_3$ the eigenvalues of g, with $|\lambda_i| = 1$, $\lambda_1\lambda_2\lambda_3 = 1$, such a spectral function is a function of $(3Z = \lambda_1 + \lambda_2 + \lambda_3, \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$, but thanks to the properties of the eigenvalues,

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 3\bar{Z}.$$

Then, the characteristic polynomial of g, that is $P(X) = \det(X - g)$, may be written as

$$P(X) = X^3 - 3ZX^2 + 3\bar{Z}X - 1.$$

It turns out that formulae (3.10) produce an easy way to compute the action of $\mathcal{L}^{SU(d)}$ on the entries of the characteristic polynomial. For this, we use the following formulae, valid for any square matrix M with entries (m_{ij})

$$\partial_{m_{ij}} \log \det(M) = M_{ji}^{-1}, \;\; \partial_{m_{ij}m_{kl}}^2 \log \det(M) = -M_{jk}^{-1}M_{li}^{-1}. \qquad (3.11)$$

Using the change of variable formula (2.2), when applying equation (3.11) to $X\mathrm{Id} - M$ together with formulae (3.10), one gets, on $SU(d)$ and with $P(X) = \det(X\mathrm{Id} - M)$,

$$\begin{aligned}
\Gamma^{SU(d)}(\log P(X), \;\; \log P(Y)) &= (d - \frac{XP'(X)}{P(X)})(d - \frac{YP'(Y)}{P(Y)}) \\
&\quad -d\Big(d + \frac{X+Y}{X-Y}\big(\frac{YP'(Y)}{P(Y)} \\
&\quad -\frac{XP'(X)}{P(X)}\big) - \frac{XY}{X-Y}\big(\frac{P'(Y)}{P(Y)} - \frac{P'(X)}{P(X)}\big)\Big) \\
&= XY\Big(\frac{P'(X)}{P(X)}\frac{P'(Y)}{P(Y)} + \frac{d}{X-Y}\big(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\big)\Big).
\end{aligned}$$

which in turn gives

$$\Gamma(P(X), P(Y)) = XY\Big(P'(X)P'(Y) + d\frac{P'(X)P(Y) - P'(Y)P(X)}{X-Y}\Big), \qquad (3.12)$$

and

$$\mathcal{L}^{SU(d)}(P) = (1 - d^2)XP' + (1 + d)X^2P''. \qquad (3.13)$$

One may now compare this for $d = 3$ with the formula given by (3.1) for $\lambda = 4$, to observe that indeed, with $\mathcal{L} = \frac{3}{4}\mathcal{L}^{SU(3)}$, they give the same result.

In the end, we see that $\mathcal{L}^{(4)}$ is nothing else than $\frac{3}{4}\mathcal{L}^{SU(3)}$ when acting on functions of $Z = T/3$, where T is the trace of the matrix.

4. Curvature dimension for the deltoid model

4.1. Curvature-dimension inequalities for $SU(3)$

It is well known that the Casimir operator of any compact semi simple Lie group has a constant Ricci curvature (see for example [11], prop. 3.17, [5], thm 7.93, [23], 4.2.2, or [26]). Moreover, the constant is universal and $\mathrm{Ric}(f,f) = \frac{1}{4}\Gamma(f,f)$ (for the standard Casimir operator on the group). Here, with our modified definition (3.9), we get $\mathrm{Ric}(f,f) = 3\Gamma(f,f)$. Remembering that the dimension of $SU(3)$ is 8, we get

PROPOSITION 4.1. — *The Casimir operator* $\mathcal{L}^{SU(3)}$ *defined by equation* (3.9) *satisfies the optimal* $CD(3,8)$ *inequality.*

From the definition of the Γ_2 operator and of the curvature-dimension inequality, it is immediate that if if $\mathcal{L}$ satisfies $CD(\rho,n)$, then $c\mathcal{L}$ satisfies $CD(c\rho,n)$. We therefore see that $\frac{3}{4}\mathcal{L}^{SU(3)}$ satisfies a $CD(\frac{9}{4},8)$ inequality.

Then, $\mathcal{L}^{(4)}$ being the image of $\frac{3}{4}\mathcal{L}^{SU(3)}$ through the map $g \mapsto Z = \frac{1}{3}(z_{11} + z_{22} + z_{33})$, applying the $CD(\rho,n)$ inequality on $SU(3)$ on function of $(Z,\bar{Z})$ provided directly the

COROLLARY 4.2. — $\mathcal{L}^{(4)}$ *satisfies the* $CD(\frac{9}{4},8)$ *inequality.*

4.2. Curvature dimension for the general deltoid model

We may now come back to the general deltoid model. Let us write, in the triangle representation, the decomposition (3.5)

$$\mathcal{L}^{(\lambda)} = \Delta + \frac{\lambda - 1}{3}\nabla \log W,$$

where $W^{(1-\lambda)/3}$ is the density the invariant measure of $\mathcal{L}^{(\lambda)}$ with respect to the Lebesgue measure, and W is given in formula (3.6).

We also know from the general formulation of the $CD(\rho,n)$ inequality (2.5) that the $CD(\frac{9}{4},8)$ for $\lambda = 4$ translates into the following

$$-\nabla\nabla \log W \geqslant \frac{9}{4}\mathrm{Id} + \frac{1}{6}\nabla \log W \otimes \nabla \log W. \qquad (4.1)$$

For $\lambda > 1$, multiplying the previous inequality by $\frac{\lambda-1}{3}$, this in turns gives for the general case provides

COROLLARY 4.3. — *For any $\lambda \geqslant 1$, $\mathcal{L}^{(\lambda)}$ satisfies a $CD(\frac{3(\lambda-1)}{4}, 2\lambda)$ inequality.*

Observe that indeed the limiting case $\lambda = 1$ corresponds to the Laplace operator on $\mathbb{R}^2$ which satisfies a $CD(0,2)$ inequality.

It is not clear however that this inequality is sharp. Indeed, going from the CD inequality on $SU(3)$ to the same CD inequality for $\mathcal{L}^{(4)}$, we may as well have lost information. In general, as we already mentioned, on a smooth compact manifold (with no boundaries) with dimension n_0, there is no optimal $CD(\rho, n)$ inequality, and for any $n > n_0$ one may find some $\rho(n_0)$ such that the operator $\Delta + \nabla \log V$ satisfies a $CD(\rho, n)$ inequality. Moreover, this does not tell us anything about the case where $\lambda < 0$.

We shall show the following

PROPOSITION 4.4. —

1. For $\lambda < 1$, the operator $\mathcal{L}^{(\lambda)}$ does not satisfy any $CD(\rho, \infty)$ inequality.

2. For $\lambda > 1$, the operator satisfy no $CD(\rho, n)$ inequality for any $n < 2\lambda$. Moreover, the best constant ρ in the $CD(\rho, 2\lambda)$ inequality is $\rho = \frac{3(\lambda-1)}{4}$.

Proof. — To perform the computations, one may use the triangle model, which is the natural one since in the associated coordinates, the metric is the usual Euclidean one. It turns out that this leads to painful computations. In the deltoid model, one may use the coordinates $(Z, \bar{Z})$, which, thanks to the fact that they provide a polynomial system, satisfy the boundary equation (3.3). This identity simplify the computations in a considerable way.

In this system of coordinates, the operator $\mathcal{L}^{(\lambda)}$ may be decomposed as

$$\mathcal{L}^{(\lambda)} = \mathcal{L}^{(1)} + \frac{\lambda - 1}{3} \nabla \log P,$$

where P is defined in equation (3.2), where $\mathcal{L}^{(1)}$ corresponds to the Euclidean Laplace operator, and following (2.5), the $CD(\rho, n)$ inequality amounts to show that

$$-\frac{\lambda - 1}{3} \nabla \nabla \log P \geqslant \rho \Gamma + \frac{1}{n - 2} \frac{(\lambda - 1)^2}{9} \nabla \log P \otimes \nabla \log P.$$

For $\lambda < 1$, everything boils down to check that the tensor $\nabla\nabla \log P$ is not bounded below by any $\rho\Gamma$ for any $\rho \in \mathbb{R}$, and for $\lambda > 1$, to show that an inequality of the form

$$-\nabla\nabla \log P \geqslant c_1\Gamma + c_2\nabla \log P \otimes \nabla \log P, \qquad (4.2)$$

may not occur when $c_2 > 1/6$ and that, when $c_2 = 1/6$, the best constant c_1 is $c_1 = 9/4$.

Our aim is to compute the Hessian of the function $\log P$.

Following [3], the Hessian of f, applied to dh, dk, that is in a local system of coordinates $\nabla\nabla^{ij}(f)\partial_i h\partial_j k$, may be defined as

$$H[f](h,k) = \frac{1}{2}\big(\Gamma(h,\Gamma(f,k)) + \Gamma(k,\Gamma(f,h)) - \Gamma(f,\Gamma(k,h))\big). \qquad (4.3)$$

We want to apply this with $f = \log P$ and $h, k = Z, \bar{Z}$. For this, one may use the boundary equation (3.3).

From this, we deduce that, for any function $G(Z, \overline{Z})$, $\Gamma(\log P, G) = -3D(G)$, where D is the Euler operator $Z\partial_Z + \overline{Z}\partial_{\overline{Z}}$

Let us write

$$H^{11} = H[\log P](Z, Z),\ \ H^{12} = H[\log P](Z, \overline{Z}),\ \ H^{22} = H[\log P](\overline{Z}, \overline{Z}).$$

From the previous remarks, we get

$$H^{11} = -3\Gamma(Z, Z) + \frac{3}{2}D(\Gamma(Z, Z)).$$

$$H^{12} = -3\Gamma(Z, \overline{Z}) + \frac{3}{2}D(\Gamma(Z, \overline{Z})),$$

and

$$H^{22} = -3\Gamma(\overline{Z}, \overline{Z}) + \frac{3}{2}D(\Gamma(Z, Z)).$$

In other words, with the obvious notations, $H = -3\Gamma + \frac{3}{2}D\Gamma$

In the same way, the tensor $\nabla \log P \otimes \nabla \log P$ may be computed in this system of coordinates as $9M$, where

$$M = \begin{pmatrix} Z^2 & Z\overline{Z} \\ Z\overline{Z} & \overline{Z}^2 \end{pmatrix}$$

.

For a tensor (R^{ij}) in complex coordinates, to represent a non negative real tensor amounts to ask that

$$R^{12} \geqslant 0 \text{ and } (R^{12})^2 \geqslant R^{11}R^{22}.$$

This shows immediately that $\nabla\nabla \log P$ is not bounded above (at a point where $Z\bar{Z} = 1$)

In the end, the inequality

$$-\nabla\nabla \log P \geqslant c_1\Gamma + c_2\nabla \log P \otimes \nabla \log P \qquad (4.4)$$

amounts to

$$(3 - c_1)\Gamma - \frac{3}{2}D\Gamma - 9c_2 M \geqslant 0,$$

where Γ denotes the matrix

$$\begin{pmatrix} \Gamma(Z,Z) & \Gamma(Z,\overline{Z}) \\ \Gamma(Z,\overline{Z}) & \Gamma(\overline{Z},\overline{Z}) \end{pmatrix}.$$

$R^{12} \geqslant 0$ reads $3 - c_1 + (c_1 - 18c_2)Z\overline{Z} \geqslant 0$, and for this to be true on the deltoid domain $\mathcal{D}$ amounts to ask

$$c_1 \leqslant 3, \quad c_2 \leqslant 1/6,$$

since $Z\overline{Z}$ varies from 0 to 1 on $\mathcal{D}$.

The second one writes, with $\rho^2 = Z\overline{Z}$

$$\left(3/2 - c_1/2 + (c_1/2 - 9c_2)\rho^2\right)^2 \geqslant$$

$$(3/2 - c_1)^2\rho^2 + (c_1 - 9c_2)^2\rho^4 + (Z^3 + \overline{Z}^3)(c_1 - 9c_2)(3/2 - c_1). \qquad (4.5)$$

Writing everything in polar coordinates $Z = \rho e^{i\theta}$, this writes as

$$P_2(\rho^2) \geqslant 2\rho^3 \cos(3\theta)(c_1 - 9c_2)(3/2 - c_1),$$

where P_2 is a degree 2 polynomial.

Observe that this requires to be true for any $\rho \in [0, 1]$ when $\cos(3\theta) = 1$ (which corresponds to the cusps of the deltoid curve).

But, with the explicit computation of P_2, one gets

$$P_2(\rho^2) - 2\rho^3(c_1 - 9c_2)(3/2 - c_1) =$$

$$\frac{1}{4}(1 - \rho)(3 - c_1 + c_2\rho)\big(3 - c_1 + \rho(3 - 2c_1) + 3\rho^2(c_1 - 12c_2)\big).$$

For the maximal value $c_2 = 1/6$,

$$P_2(\rho^2) - 2\rho^3(c_1 - 9c_2)(3/2 - c_1) = (1 - \rho)^2(3 - c_1 + 3\rho(2 - c_1)),$$

and we get the bound $c_1 \leqslant 9/4$ (for $\rho = 1$) . Observe that this bound is attained at the cusps.

For these values $c_2 = 1/6$ and $c_1 = 9/4$, equation (4.5) writes

$$4(1 - \rho^2)^2 \geqslant \rho^2 + \rho^4 - 2\rho^3 \cos(3\theta),$$

while the condition $\Gamma(Z, \overline{Z})^2 \geqslant \Gamma(Z, Z)\Gamma(\overline{Z}, \overline{Z})$, which characterizes the points in $\bar{\mathcal{D}}$, writes

$$\frac{1}{4}(1 - \rho^2)^2 \geqslant \rho^2 + \rho^4 - 2\rho^3 \cos(3\theta),$$

so the the inequality is satisfied everywhere in $\mathcal{D}$. Observe that the critical points for the curvature-dimension inequality for the critical values are attained at the cusps. □

Remark 4.5. — Observe that the values $c_1 = 9/4$ and $c_2 = 1/6$ in inequality (4.4) are exactly the bounds obtained in equation (4.1). Moreover, we know that even with $c_2 = 0$ (corresponding to a $CD(\rho, \infty)$ inequality), if we look for the optimal value for c_1, it is clear from this method that the best constant c_1 is $c_{max} < 3$, so that whatever the constant $c_2 \in [0, 1/6]$, the optimal value for c_1 lies in the interval $[9/4, 3)$. (The optimal constant $c_1(c_2)$ may be explicitly computed but has no real interest.)

5. Sobolev inequalities and bounds on the eigenvectors

As described in Section 2, from the curvature dimension inequality, we may obtain bounds on the supremum of the associated eigenvectors. More precisely, whenever a $CD(\rho, n)$ inequality holds with $\rho > 0$ and $n < \infty$, there exists a constant C such that for any eigenvector P satisfying $\mathcal{L}(P) = -\mu P$, then

$$\|P\|_\infty \leqslant C\mu^{n/4}\|P\|_2,$$

where the $\mathcal{L}^2$ norm is computed with respect to the invariant measure of the operator $\mathcal{L}$.

Turning to the case of the operator $\mathcal{L}^{(\lambda)}$ on the deltoid, we recall from [28] that the associated eigenvectors which are polynomials with total degree n have eigenvalues $\mu_{p,q} = (\lambda - 1)(p + q) + p^2 + q^2 + pq$, with $p + q = n$. More precisely, of any $n \geqslant 1$ such that $p + q = n$, when $p \neq q$, there is a dimension 2 associated eigenspace. In complex variables, for such value $\mu_{p,q}$, there is a unique degree n polynomial $P_{p,q}(Z, \bar{Z})$ with highest degree term $Z^p \bar{Z}^q$ and another one which is $\bar{P}_{p,q}(Z, \bar{Z}) = P_{p,q}(\bar{Z}, Z) = P_{q,p}(Z, \bar{Z})$ eigenvector (the polynomial $P_{p,q}$ having real coefficients). For $p = q$ however, the associated eigenspace is one dimensional. The real forms are $S_{p,q} = \frac{1}{2}(P_{p,q} + P_{q,p})$ and $A_{p,q} = \frac{-i}{2}(P_{p,q} - P_{q,p})$, which form a real basis for this eigenspace.

When $\lambda > 1$, for $\mathcal{L}^{(\lambda)}$, for any $\mu_{p,q}$ and of any polynomial P in the associated eigenspace, one gets from the $CD(\frac{3(\lambda-1)}{4}, 2\lambda)$ inequality

$$\|P\|_\infty \leqslant C(\lambda) \mu^{\lambda/2}. \tag{5.1}$$

Looking at the constants, this does not produce any estimates for $\lambda = 1$ or $0 < \lambda < 1$. However, for $\lambda = 1$, one may consider the following. The operator $\mathcal{L}^{(1)}$ is nothing else than the usual Laplace operator acting on functions $f(Z)$, where the function Z is given in (3.4). As functions of (x, y) in the real plane, those functions are periodic in x with period 4π and in y with period $4\pi/\sqrt{3}$. As such, the associated semigroup $P_t^{(1)}$ is an image of the product semigroup of the associated 1 dimensional torus (that is the semigroup on the real line acting on periodic functions). More precisely, when considering a function on the deltoid as a function of $(x, y) \in \mathbb{R}^2$,

$$P_t^{(1)}((x, y, dx', dy') = P_t^{S^1(4\pi)}(x, dx') P_t^{S^1(4\pi/\sqrt{3})}(y, dy'),$$

where $P_t^{S^1(\tau)}(x, dx')$ is the semigroup of the torus with radius τ, that is the semigroup of the one-dimensional Brownian motion acing on τ-periodic functions. Since both semigroups have a density which is bounded above by $C/\sqrt{t}$ for some constant C and for $0 < t \leqslant 1$, it turns out that $P_t^{(1)}$ has a density which is bounded above by C'/t. This is enough to get the bound on the associated eigenvectors. In the end, we get

PROPOSITION 5.1. — *For any $\lambda \geqslant 1$, there exists a constant $C(\lambda)$ depending on λ only, such that for any polynomial P eigenvector of $\mathcal{L}^{(\lambda)}$ with eigenvalue $\mu \neq 0$, one has*

$$\|P\|_\infty \leqslant C(\lambda) \mu^{\lambda/2} \|f\|_2. \tag{5.2}$$

Remark 5.2. — Looking at the constants, whenever $\lambda \to 1$, the constant $C(\lambda)$ in (5.2) goes to ∞, and there is an unexpected discontinuity in the

constants. Indeed, our computations are not the best possible. One may sharpen them with the help of spectral gaps, that is the knowledge of the lowest non 0 eigenvalue, which here is λ. More precisely, one may reinforce the constants in the ultracontractive bounds under a $CD(\rho, n)$ inequality and the knowledge of this lowest eigenvalue (see remark page 313 in [6]). But the argument in this modified estimate produces a Sobolev inequality with some dimensional parameter $m > n$ (where n is the dimension appearing in the $CD(\rho, n)$ inequality and m the dimensional parameter in the Sobolev inequality, and if we force m to be close to n, and the constant in the inequality is not improved. There is therefore a balance in the optimal bound on P between the value of μ (for μ large one wants m to be the lowest possible), and for $\lambda \to 1$ (when $\lambda \to 1$, one wants $C(\lambda)$ not too big). We could such have produced a better bound. But indeed, the remark in [3] as it stands is not really valid for $\rho = 0$ which corresponds in our case to $\lambda = 1$. One would have to sharpen this estimate, both for the case $\rho = 0$ and for the value of m. It is indeed true that one may obtain a n-dimensional Sobolev inequality (under it's entropic form) under a estimate on the lowest eigenvalue and some $CD(\rho, n)$ inequality, even for $\rho < 0$, but the argument in [6] is clearly not sufficient for that and requires further analysis.

From the point of view of the invariant measure $\mu^{(\lambda)}$ of $\mathcal{L}^{(\lambda)}$, what is relevant is the decomposition of $\mathcal{L}^2(\mu^{(\lambda)})$ into spaces of orthogonal polynomials. More precisely, when denoting $\mathcal{P}_k$ the space of polynomials with total degree less than or equal to k, one considers the subspace $\mathcal{H}_k$ of $\mathcal{P}_k$ which is orthogonal to $\mathcal{P}_k$, such that one has the orthogonal decomposition

$$\mathcal{L}^2(\mu^{(\lambda)}) = \oplus_{k=1}^{\infty} \mathcal{H}_k,$$

where $\mathcal{H}_0$ is the space of constant functions.

One has

PROPOSITION 5.3. — *There exists a constant $C_1(\lambda)$ such that, for any $k \geqslant 1$ and any $P \in \mathcal{H}_k$*

$$\|P\|_\infty \leqslant C_1(\lambda) k^{\lambda + 1/2} \|P\|_2. \tag{5.3}$$

Proof. — One may decompose $\mathcal{H}_k$ into the eigenspaces associated to $\mathcal{L}^{(\lambda)}$. There are $r_k = [k/2] + 1$ such eigenspaces, and all the eigenvalues belong to the interval $[k(\lambda + k - 5/4), k(\lambda + k - 1)]$, or, when $k \geqslant 1$, in the interval $[3/4k^2, \lambda k^2]$.

Writing $P \in \mathcal{H}_k$ as $P = \sum_{i=1}^{r_k} a_i P_i$ where P_i is an eigenvector with $\|P_i\|_2 = 1$ and $\|P\|^2 = \sum_1^{r_k} a_i^2$, one has from (5.2) and the bound on the

eigenvalues in $\mathcal{H}_k$

$$\|P\|_\infty \quad \leqslant \sum_1^{r_k} |a_i| \|P_i\|_\infty \leqslant C(\lambda) \lambda^{(1+\lambda/2)} \sum_1^{r_k} |a_i| k^\lambda$$

$$\leqslant C(\lambda) \lambda^{(1+\lambda/2)} k^\lambda r_k^{1/2} \sqrt{\sum_1^{r_k} a_i^2},$$

from which the bound follows immediately. $\qquad\qquad\square$

One may wonder how far inequality (5.3) is from the Sobolev inequality we started from. Observe first that, for the heat kernel $\mathcal{P}_t^{(\lambda)}$, for any function $P \in \mathcal{H}_k$, one has

$$\|\mathcal{P}_t^{(\lambda)} P\|_2 \leqslant \exp(-3/4tk^2) \|P\|_2,$$

since all eigenvalues of $\mathcal{P}_t$ on $\mathcal{H}_k$ are bounded below by $\exp(-3/4tk^2)$.

Therefore, we have, for any $P \in \mathcal{H}_k$

$$\|\mathcal{P}_t^{(\lambda)} P\|_\infty \leqslant C_1(\lambda) \exp(-3/4tk^2) k^{\lambda+1/2} \|P\|_2.$$

Observe that this relies only on the bound (5.2) together with the knowledge of the eigenvalues.

THEOREM 5.4. — *Let P_t be a symmetric Markov semigroup with reversible probability measure μ and generator $\mathcal{L}$. Assume that $\mathcal{L}$ satisfies a Poincaré inequality and that one has a decomposition into orthogonal spaces $\mathcal{L}^2(\mu) = \oplus_k \mathcal{H}_k$, where $\mathcal{H}_k$ is a linear space, with the property that, for some real number $a > 0$ and for any $f \in \mathcal{H}_k$,*

$$\|P_t f\|_\infty \leqslant C k^p e^{-atk^2} \|f\|_2.$$

Then, $\mathcal{L}$ satisfies a tight Sobolev inequality with dimension $m = 2p + 1$.

Proof. — Following [6], and from the existence of a Poincaré inequality, it is enough to prove that, for $t \in (0,1]$ and for some constant C, $\|P_t f\|_\infty \leqslant Ct^{-m/4} \|f\|_2$. We may restrict to the case where $\|f\|_2 = 1$. For $f \in \mathcal{L}^2(\mu)$, let us write $f = \sum_k f_k$, where $f_k \in \mathcal{H}_k$ and $\sum_k \|f_k\|_2^2 = 1$.

$$\|P_t f\|_\infty \leqslant \sum_k \|P_t f_k\|_\infty \leqslant \sum_k k^p e^{-atk^2} \|f_k\|_2 \leqslant \left(\sum_k k^{2p} e^{-2atk^2} \right)^{1/2}.$$

One may compare the sum $\sum_k k^{2p}e^{-2atk^2}$ with $\int_0^\infty x^{2p}\exp(-2atx^2)\,dx$, where the function $x^{2p}\exp(-2atx^2)$ is increasing on $(0,\sqrt{p/(2at)}$ and decreasing on $(\sqrt{p/(2at)},\infty)$, and we see that, for $0 < t \leqslant 1$,

$$\sum_k k^{2p}e^{-2atk^2} \leqslant C(a,p)t^{-(p+1)/2}, \ 0 < t \leqslant 1.$$

Therefore, following the results exposed in Section 2, we get a Sobolev inequality with dimension $m = 2(p+1)$. The existence of a Poincaré inequality (that is of a strictly positive first non zero eigenvalue for $-\mathcal{L}$) insures that we may get a tight Sobolev inequality (2.7). This gives the result. $\qquad\square$

Looking at the values for the deltoid model, we see that the estimate provides a Sobolev inequality with dimension $m = 2\lambda + 3$, whereas we started from a Sobolev inequality with dimension 2λ. One may wonder if this lost in dimension (from n to $n+3$) is due to too crude estimates on both the eigenvalues and the summation in the series, or from the fact that the spaces $\mathcal{H}_k$ are k dimensional. Indeed, even in the case or one dimensional Jacobi operators, where the eigenspaces are one dimensional, where the eigenvalues for the associated operator are $k(k+c)$ for polynomials with degree k, one would pass with the same method from a Sobolev inequality with dimension n to a Sobolev inequality with dimension $n+1$. This is in big contrast with the case of logarithmic Sobolev inequalities, where estimates on the $\mathcal{L}^p$ bounds on the eigenvectors are indeed equivalent to logarithmic Sobolev inequalities (see [3]).

Finally, we directly get from this a criterium for a symmetric operator constructed from orthogonal polynomial would have a bounded density.

PROPOSITION 5.5. — *Let K be a symmetric operator in $\mathcal{L}^2(\mu^{(\lambda)})$ which maps $\mathcal{H}_k$ into $\mathcal{H}_k$ and is such that, for any $P \in \mathcal{H}_k$, $\|K(P)\|_2 \leqslant \nu_k\|P\|_2$. If $A = \sum_k \nu_k^2 k^{2\lambda+1} < \infty$, then K^2 may be represented by a bounded kernel,*

$$K^2(f)(x) = \int f(y)k(x,y)\,d\mu(y),$$

where $|k| \leqslant A$.

Proof. — Arguing as in the proof of Theorem (5.4), we may write $f = \sum_k f_k$ with $f_k \in \mathcal{H}_k$. Then,

$$\begin{aligned}\|K(f)\|_\infty &\leqslant \sum_k \|K(f_k)\|_\infty \leqslant \sum_k k^{\lambda/1/2}\|K(f)\|_2\\ &\leqslant \sum_k \nu_k k^{\lambda/1/2}\|f_k\|_2 \leqslant (\sum_k \nu_k k^{\lambda/1/2})^{1/2}\|f\|_2.\end{aligned}$$

Therefore, the operator K is bounded from $\mathcal{L}^2$ into $\mathcal{L}^\infty$ with norm $A^{1/2}$. By symmetry and duality, the same is true from $\mathcal{L}^1$ into $\mathcal{L}^2$, and by composition, K^2 is bounded from $\mathcal{L}^1$ into $\mathcal{L}^\infty$ with norm A. It therefore may be represented by a kernel k bounded by A. $\qquad\square$

Remark 5.6. — The method presented here says nothing about the case where $0 < \lambda < 1$. Indeed, in this case, one may expect to have a two-dimensional behavior for the heat kernel, that is $\|P_1\|_{2,\infty} \leqslant Ct^{-1/2}$, $0 < t \leqslant 1$. In this context, it is better to replace Sobolev inequalities by Nash inequalities, that is inequalities of the form

$$\|f\|_2^2 \leqslant \|f\|_1^{2\theta}\left(\|f\|_2^2 + C\int \Gamma(f,f)d\mu\right)^{1-\theta},$$

where $\theta = \frac{2}{n+2}$ is a dimensional parameter. When $n > 2$, this is equivalent to a Sobolev inequality with dimension n, but for $n \in (1,2)$, this is still equivalent to a bound $\|P_t\|_{2,\infty} \leqslant C't^{-n/4}$ (see [3]). As mentioned, we may expect when $0 < \lambda < 1$ some Nash inequality with dimensional parameter $n = 2$. We cannot expect any smaller value for n since, applied to any function compactly supported in the interior of $\mathcal{D}$, this would contradict the classical two dimensional Nash inequality in an open domain of $\mathbb{R}^2$. However, the singularity of the measure at the cusps of the deltoid make things a bit hard to analyze. Following the method developed in [3], pages 370-371, we are able to prove Nash inequalities with dimension $n = 5/2$, with however a constant $C(\lambda)$ which goes to infinity when $\lambda \to 0$. This is not satisfactory for many points of view. First, because of the bad dimension, and for the lack of continuity in these inequalities when $\lambda \to 1$. Secondly, when $\lambda \to 0$, the measure converges to the uniform measure on the three cusps $\frac{1}{3}(\delta_{1/3} + \delta_{j/3} + \delta_{\bar{j}/3})$, and the associated polynomials converge to the corresponding polynomials on three points). It is therefore a challenging question to produce for small λ some functional inequality which recaptures this particular structure in the limit.

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