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Vertex algebroids à la Beilinson-Drinfeld

Fyodor Malikov[(1)]

To Vadik Schechtman on his 60th birthday

Résumé. — Ces notes informelles sont une introduction aux algébroïdes vertex en suivant les lignes suggérées par Beilinson et Drinfeld.

Abstract. — These informal notes are an introduction to vertex algebroids along the lines suggested by Beilinson and Drinfeld.

1. Introduction

The definition of a vertex algebra is bad enough, [7, 8, 13], but the definition of a vertex algebroid, an apparently simpler object as suggested in [9], is worse. The Borcherds identity [7], admittedly an infinite family of identities, can at least be written as a single formula, albeit depending on parameters. A vertex algebroid is a vector space with 3 partially defined operations satisfying a number of disparate identities that make some sense only if one discerns the Borcherds identity lurking behind.

To cite one problematic issue, the skew-symmetry, a fundamental property of a vertex algebra, usually is not part of the definition of a vertex algebra, but is indispensable when defining a vertex algebroid, where it appears in the form:

$$\xi_{(0)}\eta = -\eta_{(0)}\xi + \partial(\eta_{(1)}\xi).$$

It has always been clear that the prototype of the notion of a vertex algebroid is that of a Picard-Lie algebroid. The latter is a Lie A-algebroid that fits

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into an exact sequence

$$0 \longrightarrow A \longrightarrow \mathcal{L} \longrightarrow T_A \longrightarrow 0,$$

where A is a ring, T_A is the tangent Lie algebroid and arrows respect all the structures involved. The isomorphism classes of such sequences are in 1-1 correspondence with the De Rham cohomology group $\Omega^{2,cl}/d_{DR}\Omega^1_A$; 2-forms enter this classification as deformations of the Lie bracket restricted to T_A: $T_A \otimes T_A \longrightarrow T_A \oplus A$.

The vertex analogue, [9], is as follows:

$$0 \longrightarrow \Omega_A \longrightarrow \mathcal{V} \longrightarrow T_A \longrightarrow 0.$$

The corresponding vertex algebroids may or may not exist, but if they do, the isomorphism classes are a torsor over $\Omega^{3,cl}/d_{DR}\Omega^2_A$. 2-forms enter this classification as deformations of the $_{(0)}$-product restricted to T_A: $T_A \otimes T_A \longrightarrow T_A \oplus \Omega_A$; other products are treated separately.

While the former exact sequence brings one to the heart of the matter, the latter does not, not quite at least; for example, it does not contain the ring A, which is an integral part of the corresponding algebra of chiral differential operators.

A much more reasonable point of view was suggested by Beilinson and Drinfeld [5]. In the situation at hand this approach amounts to replacing all objects with their "jet" versions; the result is this:

$$0 \longrightarrow J_\infty A \longrightarrow \mathcal{L}^{ch} \longrightarrow J_\infty T_A \longrightarrow 0,$$

Defining the structure encoded by this exact sequence is straightforward and more or less parallel to the discussion of Picard-Lie algebroids, except that one has to work inside appropriate pseudo-tensor categories introduced in [5]; for example, $J_\infty A$ is a commutative$^!$ algebra (i.e., a commutative associative algebra with derivation), $J_\infty T_A$ is a Lie* (as opposed to ordinary Lie) algebra, etc. A chiral algebroid may or may not exist, but if it exists the isomorphism classes thereof are again, as in the Picard-Lie case, labeled by "the 2nd De Rham cohomology group," except that the latter has to be properly understood; "2-forms" enter this classification as deformations of the single Lie*-bracket. The results of [9] are then singled out by the requirement that the algebroid be $\mathbb{Z}_+$-graded. The clarity thus achieved by adopting the Beilinson-Drinfeld point of view is impressive.

This note is essentially an exposition of a tiny part of [5] that is needed to recover the results of [9]. The reader is advised having leafed through sect. 2, which is a quick reminder about algebras of twisted differential operators (TDO), to move directly to sect. 4, where algebras of chiral differential operators (CDO) are discussed, and return to sect. 3 only when needed. The Beilinson-Drinfeld theory is not only tremendously illuminating but is much more general than the conventional vertex algebra theory. We hope, however, that the relentless emphasis on the simplest case adopted here may serve the beginner well. Some other sources dealing with elementary aspects of [5] are [8], ch.19, 20, and [12]; an important example of a CDO is analyzed in [3].

A few points may be worth mentioning.

(i) In sect. 4.11 we show how a slight deviation from the graded case allows to obtain a family of CDOs labeled by the product of the De Rham cohomology groups $\Omega_A^{2,cl}/d_{DR}\Omega_A^1 \times \Omega_A^{3,cl}/d_{DR}\Omega_A^2$ thereby producing a cross between a TDO and a graded CDO. This is similar but different from from "twisted" chiral algebroids of [1, 2]. Similar inhomogeneities have somewhat surreptitiously crept into the works such as [10, 15]. I am grateful to A.Linshaw for pointing this out to me.

(ii) The construction of the universal enveloping algebra of a vertex algebroid has only appeared in a preprint version of [9]; the construction suggested here, sect. 4.7, is quite different.

(iii) We introduce, sect. 3.11, the notion of infinity-Lie* algebra, which seems essential for working with singular algebraic varieties, [11]. We hope to present the details elsewhere.

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It is a pleasure and honor to contribute to the celebrations of Vadik's birthday, and it is fitting that the subject of these notes owes its existence to Vadik.

2. TDO

2.1. Let A be a commutative unital $\mathbb{C}$-algebra, T_A the Lie algebra of derivations of A. The graded symmetric algebra $S_A^\bullet T_A$ is naturally a Poisson algebra. An algebra $\mathcal{D}_A^{tw}$ is called an algebra of twisted differential operators over A, TDO for brevity, if it carries a filtration $F_0(\mathcal{D}_A^{tw}) = A \subset \cdots \subset F_n(\mathcal{D}_A^{tw}) \subset \cdots$, $\bigcup_n F_n(\mathcal{D}_A^{tw}) = \mathcal{D}_A^{tw}$, s.t. the corresponding graded object is isomorphic to $S_A^\bullet T_A$ is a Poisson algebra.

In a word, a TDO is a quantization of $S_A^\bullet T_A$.

2.2. The key to classification of TDOs is the concept of a Picard-Lie A-algebroid. $\mathcal{L}$ is called a Lie A-algebroid if it is a Lie algebra, an A-module, and is equipped with anchor, i.e., a Lie algebra and an A-module map $\sigma : \mathcal{L} \to T_A$ s.t. the A-module structure map

$$A \otimes \mathcal{L} \longrightarrow \mathcal{L} \tag{2.1}$$

is an $\mathcal{L}$-module morphisms. Explicitly,

$$[\xi, a\tau] = \sigma(\xi)(a)\tau + a[\xi, \tau], \ a \in A, \ \xi, \tau \in \mathcal{L}. \tag{2.2}$$

A Picard-Lie A-algebroid is a Lie A-algebroid $\mathcal{L}$ s.t. the anchor fits in an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \mathcal{L} \xrightarrow{\sigma} T_A \longrightarrow 0, \tag{2.3}$$

where the arrows respect all the structures involved; in particular, A is regarded as an A-module and an abelian Lie algebra, and ι makes it an A-submodule and an abelian Lie ideal of $\mathcal{L}$.

Morphisms of Picard-Lie A-algebroids are defined in an obvious way to be morphisms of exact sequences (2.3) that preserve all the structure involved. Each such morphism is automatically an isomorphism and we obtain a groupoid $\mathcal{PL}_A$.

2.3. Classification of Picard-Lie A-algebroids that split as A-modules is as follows. We have a canonical such algebroid, $A \oplus T_A$ with bracket

$$[a + \xi, b + \tau] = \xi(b) - \tau(a) + [\xi, \tau].$$

Any other bracket must have the form

$$[\xi, \tau]_{new} = [\xi, \tau] + \beta(\xi, \tau), \ \beta(\xi, \tau) \in A.$$

The A-module structure axioms imply that $\beta(.,.)$ is A-bilinear, the Lie algebra axioms imply that, in fact, $\beta \in \Omega_A^{2,cl}$. Denote this Picard-Lie algebroid by $T_A(\beta)$. Clearly, any Picard-Lie A-algebroid is isomorphic to $T_A(\beta)$ for some β.

A morphism $T_A(\beta) \to T_A(\gamma)$ must have the form $\xi \to \xi + \alpha(\xi)$ for some $\alpha \in \Omega_A^1$. A quick computation will show that

$$Hom(T_A(\beta), T_A(\gamma)) = \{\alpha \in \Omega_A^1 \text{ s.t. } d\alpha = \beta - \gamma\}.$$

This can be rephrased as follows. Let $\Omega_A^{[1,2>}$ be a category with objects $\beta \in \Omega_A^{2,cl}$, morphisms $Hom(\beta, \gamma) = \{\alpha \in \Omega_A^1 \text{ s.t. } d\alpha = \beta - \gamma\}$. Then the assignment $(\gamma, T_A(\beta) \mapsto T_A(\beta+\gamma)$ defines a categorical action of $\Omega_A^{[1,2>}$ on $\mathcal{PL}_A$ which makes $\mathcal{PL}_A$ into an $\Omega_A^{[1,2>}$-torsor. The isomorphism classes of this catetgory are in 1-1 correspondence with the De Rham cohomology $\Omega_A^{2,cl}/d\Omega_A^1$, and the automorphism group of an object is $\Omega_A^{1,cl}$.

2.4. If X is a smooth algebraic variety, then the above considerations give the category of Picard-Lie algebroids over X, $\mathcal{PL}_X$, which is a torsor over $\Omega_X^{[1,2>}$ or, perhaps, a gerbe bound by the sheaf complex $\Omega_X^1 \to \Omega_X^{2,cl}$. This gerbe has a global section, the standard $\mathcal{O}_X \oplus \mathcal{T}_X$. The isomorphism of classes of such algebroids are in 1-1 correspondence with the cohomology group $H^1(X, \Omega_X^1, \to \Omega_X^{2,cl})$ (Ω_X^1 being placed in degree 0), and the automorphism group of an object is $H^0(X, \Omega_X^{1,cl})$.

2.5. The concept of the universal enveloping algebra of a Lie algebra has a Lie algebroid version, which reflects a partially defined multiplicative structure on $\mathcal{L}$.

Let $F(\mathcal{L})$ be a free unital associative $\mathbb{C}$-algebra generated be the Picard-Lie A-algebroid $\mathcal{L}$ regarded as a vector space over $\mathbb{C}$. We denote by $*$ its multiplication and by $\mathbf{1}$ its unit. Define the universal enveloping algebra $U_A(\mathcal{L})$ to be the quotient of $F(\mathcal{L})$ be the ideal generated by the elements $\xi * \tau - \tau * \xi - [\xi, \tau]$, $a * \xi - a\xi$, $\mathbf{1} - 1_A$, where 1_A is the unit of A.

It is rather clear that $U_A(\mathcal{L})$ is a TDO (sect. 2.1), and the assignemnt $\mathcal{L} \mapsto U_A(\mathcal{L})$ is an equivalence of categories if A is smooth, i.e., if $MaxSpec(A)$ is a smooth affine variety.

3. Beilinson-Drinfeld

3.1. Let $R = \mathbb{C}[\partial]$ be a polinomial ring regarded as a Hopf algebra with comultiplication $\Delta : R \to R \otimes R$, $\partial \mapsto \partial \otimes 1 + 1 \otimes \partial$, and counit $\epsilon : R \to \mathbb{C}$, $\partial \mapsto 0$.

We let $\mathcal{M}$ be the category of R-modules, and we choose to think of them as *right* R-modules. If I is a finite set, and $\{A_i\}$ is an I-family of objects, then the tensor product $\otimes_{i\in I} A_i$ is best understood as a system of "usual" products $A_{\sigma_1} \otimes A_{\sigma_2} \otimes \cdots$ defined for all orderings σ of I and obvious isomorphisms among them.

The symbol R^J stands for the tensor product $\otimes_J R$ of algebras; the iterated comultiplication gives an algebra morphism $R \to R^J$. Similarly, given a surjection $\pi : J \twoheadrightarrow I$ the repeated comultiplication defines a homomorphism $R^I \to R^J$. The former construction is a particular case of the latter one when I is a point; on the other hand, the latter construction is the tensor product of a number of former ones as follows: if we let $J_i = \pi^{-1}(i)$ and f_i be the map $R \to R^{J_i}$, then the map $R^I \to R^J$ is $\otimes_I f_i$.

For a finite set I and a collection of R-modules M_i, $i \in I$ and N, define

$$P_I^*(\{M_i\}, N) = Hom_{R^I}(\otimes_{i\in I} M_i, N \otimes_R R^I).$$

Elements of $P_I^*(\{M_i\}, N)$, often called **-operations*, can be composed as follows: for a surjection $\pi : J \twoheadrightarrow I$ define a map

$$P_I^*(\{M_i\}, N) \otimes (\otimes_I P_{J_i}^*(\{L_j\}, M_i)) \longrightarrow P_J^*(\{L_j\}, N), \text{ where } J_i = \pi^{-1}(i), \tag{3.1}$$

to send a collection of operations $\psi_i \in P_{J_i}^*(\{L_j\}, M_i)$, $i \in I$, and $\phi \in P_I^*(\{M_i\}, N)$ to the composite map:

$$\otimes_J L_j = \otimes_I \otimes_{J_i} L_j \xrightarrow{\otimes\psi_i} \otimes_I (M_i \otimes_R R^{J_i}) = (\otimes_I M_i) \otimes_{R^I} R^J$$

$$\xrightarrow{\phi} N \otimes_R R^I \otimes_{R^I} R^J = N \otimes_R R^J.$$

Denote this composition by $\phi(\psi_i) \in P_J^*(\{L_j\}, N)$.

An associativity property holds: if, in addition, there is a surjection $K \twoheadrightarrow J$ and operations $\chi_j \in P_{H_j}^*(\{A_k\}, L_j)$, $j \in J$, then $(\phi(\psi_i))(\chi) = \phi(\psi_i(\chi_j))$. This defines a pseudo-tensor category, to be denoted by $\mathcal{M}^*$.

We shall often encounter the situation when the I-family is constant, $M_i = M$, $J = I$ and π is a bijection. In this case, the composition $\phi(id_M, id_M, ...)$ also belongs to $P_I(\{M\}, N)$ and will be denoted $\pi\phi$. For example, if $\phi(a, b) = \sum_{i,j} \langle a, b\rangle_{ij} \partial_1^i \partial_2^j$ for some $\langle a, b\rangle_{ij} \in N$ and $\sigma = (1, 2)$ is the transposition, then

$$\sigma\phi(a, b) = \sum_{i,j} \langle b, a\rangle_{ij} \partial_2^i \partial_1^j.$$

This defines an action of the permutation group on each $P_I(\{M\}, N)$.

3.2. If a choice is made, then explicit formulas can be written down. If $I = \{1, 2, 3, ..., n\}$, then $N \otimes_R R^I$ can be identified with $N[\partial_1, \partial_2, ..., \partial_{n-1}]$, where ∂_i stands for $\underbrace{1 \otimes \cdots \otimes \partial}_{i} \otimes \cdots \otimes 1$. A binary operation $\mu \in P_{\{1,2\}}^*(\{M, M\}, N)$ can then be written as follows

$$\mu(a, b) = \sum_n a_{(n)} b \otimes \frac{\partial_1^n}{n!} \text{ for some } a_{(n)} b \in N.$$

One has for the transposition $\sigma = (1,2)$

$$\begin{aligned}\sigma\mu(a,b) &= \sum_n b_{(n)}a \otimes \frac{\partial_2^n}{n!} = \sum_n b_{(n)}a \otimes \frac{(\partial_1+\partial_2-\partial_1)^n}{n!}\\ &= \sum_{n\geqslant j}(-1)^j(b_{(n)}a)\frac{\partial^{n-j}}{(n-j)!}\otimes\frac{\partial_1^j}{j!}\end{aligned}$$

Similarly,

$$\mu(a,\mu(b,c)) = \sum_n \mu(a,b_{(n)}c)\otimes\frac{\partial_2^n}{n!} = \sum_{m,n} a_{(m)}(b_{(n)}c)\otimes\frac{\partial_1^m}{m!}\frac{\partial_2^n}{n!},$$

but

$$\begin{aligned}\mu(\mu(a,b),c) &= \sum_m \mu(a_{(m)}b,c)\otimes\frac{\partial_1^m}{m!} = \sum_{m,n}(a_{(m)}b)_{(n)}c\otimes\frac{\partial_1^m}{m!}\frac{\Delta(\partial)^n}{n!}\\ &= \sum_{m,n}(a_{(m)}b)_{(n)}c\otimes\frac{\partial_1^m}{m!}\frac{(\partial_1+\partial_2)^n}{n!}\end{aligned}$$

3.3. Along with $\mathcal{M}^*$ consider $\mathcal{V}ect$, the tensor category of vector spaces, hence a pseudo-tensor category where $P_I(\{V_i\},V) = Hom_{\mathbb{C}}(\otimes_I V_i, V)$. The assignment $\mathcal{M}^* \mapsto h(M) \stackrel{\text{def}}{=} M/\partial M$ defines a pseudo-tensor functor, called an *augmentation functor* in [5], 1.2.4, 1.2.9-11,

$$h : \mathcal{M}^* \longrightarrow \mathcal{V}ect,$$

as h defines, in an obvious manner, a map

$$h_I :\ P_I^*(\{M_i\},N) \longrightarrow Hom_{\mathbb{C}}(\otimes_I h(M_i), h(N)),$$

which is functorial in $\{M_i\}$ and N.

3.4. A pseudo-tensor category structure, i.e., a family of well-behaved spaces of "operations" $P_I^*(\{L_i\},N)$, is what is needed to define various algebraic structures. For example, a Lie* or associative* algebra is a pseudo-tensor functor

$$Lie \longrightarrow \mathcal{M}^* \text{ or } Ass \longrightarrow \mathcal{M}^*,$$

where *Lie* or *Ass* (resp.) is the corresponding operad (an operad being a pseudo-tensor category with a single object.) Explicitly, this means a choice of an R-module V and an operation $\mu(.,.) \in P_{\{1,2\}}^*(\{V,V\},V)$ that satisfies appropriate identities written by means of the above defined composition. For example, V is an associative* if $\mu(\mu(.,.),id)=\mu(id,\mu(.,.))$ as elements of $P^*_{\{1,2,3\}}(\{V,V,V\},V)$. Likewise, V is Lie* if

$$(1,2)\mu(.,.)=-\mu(.,.))$$

$$\text{and } \mu(\mu(.,.),id)+(1,2,3)\mu(\mu(.,.),id)+(1,2,3)^2\mu(\mu(.,.),id)=0.$$

It is easy to verify, using 3.2, that a Lie* algebra is an R-module with a family of multiplications $_{(n)}$ s.t. $a_{(n)}b=0$ if $n\gg 0$ and

$$a_{(n)}b=(-1)^{n+1}\sum_{j\geqslant 0}(b_{(n+j)}a)\frac{\partial^j}{j!}\text{: anti-symmetry} \tag{3.2}$$

$$a_{(n)}b_{(m)}c-b_{(m)}a_{(n)}c=\sum_{j\geqslant 0}\binom{n}{j}(a_{(j)}b)_{(n+m-j)}c\text{: Jacobi} \tag{3.3}$$

The last equality is known as the *Borcherds commutator formula*

It is convenient to denote by $a(\partial)$ the formal sum $\sum_n a_{(n)}\partial^n/n!$. We have

(i) the just written Jacobi identity is equivalent to

$$a(\partial_1)b(\partial_2)c-b(\partial_2)a(\partial_1)c=(a(\partial_1)b)(\partial_1+\partial_2)c.$$

(ii) the associativity condition $\mu(\mu(.,.),id)=\mu(id,\mu(.,.))$ is equivalent to

$$a(\partial_1)b(\partial_2)c=(a(\partial_1)b)(\partial_1+\partial_2)c.$$

This point of view has been introduced and developed by V.Kac and his collaborators, see [13] and references therein, especially [4], sect. 12.

3.5. Let L be a Lie* algebra with bracket $[.,.]\in P^*_{\{1,2\}}(\{L,L\},L)$. An R-module M is called an L-module if there is an operation $\mu\in P^*_{\{1,2\}}(\{L,M\},M)$ s.t.

$$\mu(.,\mu(.,.))-(1,2)\mu(.,\mu(.,.))=\mu([.,.],.).$$

The untiring reader will have no trouble verifying that in terms of $_{(n)}$-products this is nothing but an obvious version of (3.3).

The Chevalley complex is defined as follows. Denote by $C^n(L,M)$ the subspace of $P^*_{[n]}(\{L\},M)$, $[n]=\{1,2,...,n\}$, of skew-invariants of the symmetric group action. Set, mimicking the usual definition,

$$d : C^n(L, M) \longrightarrow C^{n+1}(L, M),\ d\phi(l_1, l_2, \dots, l_{n+1}) \\ = \sum_{1\leqslant i\leqslant n+1} (-1)^{i+1}\mu(l_i, \phi(l_1, ..., \widehat{l_i}, ..., l_{n+1})) \\ + \sum_{1\leqslant i<j\leqslant n+1} (-1)^{i+j}\phi([l_i, l_j], l_1, ..., \widehat{l_i}, ..., \widehat{l_j}, ..., l_{n+1}).$$

The last formula is somewhat symbolical and needs to be interpreted as follows: if $I = \{1, 2, ..., n\}$ and $\{x_i,\ i \in I\}$ is an I-family of elements of L, then we define $\phi(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ to be $\sigma\phi(x_1, x_2, \dots, x_n)$, where σ is a permutation such that $\sigma(j) = i_j$ and the action of the symmetric group on operations is the one defined in sect. 3.1; it is not simply the permutation of the variables. Essentially the familiar (from ordinary Lie theory) proof shows that $d^2 = 0$.

Various computations involving this complex, called there *reduced*, can be found in [4].

3.6. If L is a Lie* algebra and M an L-module, then $h(L)$ is an ordinary Lie algebra and $h(M)$, as well as M itself is an $h(L)$-module. This is true on general grounds, see sect. 3.3, but also easily follows from the explicit formulas of sect. 3.4.

3.7. In order to define a Poisson algebra object in $\mathcal{M}^*$ one needs, in addition to Lie*, another structure, associative commutative multiplication, and another constraint, the Leibniz rule. This is taken care of by another pseudo-tensor structure on $\mathcal{M}$, in fact, a genuine tensor category structure engendered by the fact that R is a Hopf algebra. Given $A, B \in \mathcal{M}$, let $A\otimes^! B$ be $A \otimes B$ acted upon by R via $\Delta : R \to R \otimes R$. The category $\mathcal{M}$ with this tensor structure will be denoted by $\mathcal{M}^!$.

The 2 pseudo-tensor structures are related in that operations can sometimes be multiplied. Let us describe this product in the simplest possible case. Assume given $P^*_I(\{M_i\}, N_1)$, $P^*_J(\{L_j\}, N_1)$, where I and J are disjoint, and fix $i_0 \in I$, $j_0 \in J$. Denote by $I \vee J$ (or rather $I \vee_{i_0 j_0} J$) the union $I \sqcup J$ modulo the relation $i_0 = j_0$. There is a natural map

$$P^*_I(\{M_i\}, N_1) \otimes P^*_J(\{L_j\}, N_1) \\ \longrightarrow P^*_{I\vee J}(\{M_i, L_j, M_{i_0} \otimes^! L_{j_0}\}_{i\neq i_0, j\neq j_0}, N_1 \otimes^! N_2), \tag{3.4}$$

It is defined to be the following composition

$$
\begin{aligned}
&P_I^*(\{M_i\}, N_1) \otimes P_J^*(\{L_j\}, N_1)\\
&\quad = Hom_{R^I}(\otimes_I M_i, N_1 \otimes_R R^I) \otimes Hom_{R^J}(\otimes_I L_j, N_1 \otimes_R R^J)\\
&\quad \xrightarrow{\otimes} Hom_{R^{I\sqcup J}}(\otimes_I M_i \otimes \otimes_J L_j, (N_1 \otimes N_2) \otimes_{R^2} (R^I \otimes R^J))\\
&\xrightarrow{©} Hom_{R^{I\vee J}}(\otimes_{i\neq i_0} M_i \otimes \otimes_{j\neq j_0} L_j \otimes (M_{i_0} \otimes^! N_{j_0}), (N_1 \otimes N_2) \otimes_{R^2} (R^2 \otimes_R R^{I\vee J}))\\
&= Hom_{R^{I\vee J}}(\otimes_{i\neq i_0} M_i \otimes \otimes_{j\neq j_0} L_j \otimes (M_{i_0} \otimes^! N_{j_0}), (N_1 \otimes^! N_2) \otimes_R R^{I\vee J}))\\
&\qquad = P_{I\vee J}^*(\{M_i, L_j, M_{i_0} \otimes^! L_{j_0}\}_{i\neq i_0, j\neq j_0}, N_1 \otimes^! N_2).
\end{aligned}
$$

Of these arrows only the one marked by © needs an explanation. Consider the map

$$R^2 \otimes_R R^{I\vee J} \longrightarrow R^I \otimes R^J \tag{3.5}$$

defined on the generators to be the following two:

$$R^2 \longrightarrow R^I \otimes R^J \text{ and } R^{I\vee J} \longrightarrow R^I \otimes R^J.$$

The former is the tensor product of the iterated coproduct maps $R \to R^I$ and $R \to R^J$. The latter is defined to be $\partial_\alpha \mapsto \partial_\alpha$ if α is different from the equivalence class $\{i_0, j_0\}$ and $\partial_\alpha \mapsto \partial_{i_0} + \partial_{j_0}$ if α is the equivalence class $\{i_0, j_0\}$. The map (3.5) is an isomorphism as it simply is a coordinate system change in a polynomial ring. The arrow © is induced by its inverse.

Informally speaking, map (3.4) is essentially the conventional tensor product of 2 maps:

$$
\begin{aligned}
\otimes : &Hom_{R^I}(\otimes_I M_i, N_1 \otimes_R R^I) \otimes Hom_{R^J}(\otimes_I L_j, N_1 \otimes_R R^J)\\
&\longrightarrow Hom_{R^{I\sqcup J}}(\otimes_I M_i \otimes \otimes_J L_j, (N_1 \otimes N_2) \otimes_{R^2} (R^I \otimes R^J))
\end{aligned}
$$

except that the result must be reinterpreted. To indicate this denote by $\phi \otimes^! \psi$ the tensor product of 2 operations defined by (3.4).

The tensor product (3.4) is commutative, associative, and natural w.r.t. the composition (3.1); the reader can either figure out what this means on his own or read [5], 1.3.15. The structure so obtained is called compound pseudo-tensor category; if we want to emphasize this, we shall write $\mathcal{M}^{*!}$.

3.8. If index sets are ordered and operations are written in terms of $_{(n)}$-products, sect. 3.2, then the inherent symmetry of the definition is destroyed. For example, given $\phi \in P^*_{\{1,2\}}(\{M_1, M_2\}, N)$, with $\phi(a, b) = \sum_n (a_{(n)} b) \otimes \frac{\partial_1^n}{n!}$

and $id \in P^*_{\{1\}}(L,L)$ one easily computes $\phi\otimes^! id \in P^*_{\{1,2\}}(\{M_1, M_2\otimes^! L\}, N\otimes^! L)$ to be

$$\phi \otimes^! id(a, b \otimes^! c) = \sum_n (a_{(n)}b) \otimes^! c \otimes \frac{\partial_1^n}{n!}. \tag{3.6}$$

On the other hand, $id \otimes^! \phi \in P^*_{\{1,2\}}(\{L \otimes^! M_1, M_2\}, L \otimes^! N)$ is as follows

$$id \otimes^! \phi(c \otimes^! a, b) = \sum_{n \geqslant j} (-1)^{n-j} (c\frac{\partial^{n-j}}{(n-j)!}) \otimes^! a_{(n)}b) \otimes \frac{\partial_1^j}{j!}. \tag{3.7}$$

Indeed, if we consider $L \otimes M_1 \otimes M_2$ as an R^3-module, denoting ∂_x, ∂_y, ∂_z the copy of ∂ operating on L, M_1, M_2 resp., then the construction of map (3.4) gives the composition

$$c \otimes^! a \otimes b \mapsto \sum_n c \otimes a_{(n)}b \otimes \frac{\partial_y^n}{n!} \mapsto \sum_n c \otimes a_{(n)}b \otimes \frac{(\partial_1 - \partial_x)^n}{n!}$$
$$= \sum_{n \geqslant j} (-1)^{n-j} c \otimes a_{(n)}b \otimes \frac{\partial_1^j \partial_x^{n-j}}{j!(n-j)!} = \sum_{n \geqslant j} (-1)^{n-j} (c\frac{\partial^{n-j}}{(n-j)!}) \otimes^! a_{(n)}b \otimes \frac{\partial_1^j}{j!},$$

as desired. In this computation, the 2nd arrow uses the fact that $\Delta(\partial_1) = \partial_x + \partial_y$ and the last equality follows from the fact that $id \in P^*_{\{1\}}(L,L) = Hom_R(L, L \otimes_R R)$, which is identified with $Hom_R(L,L)$ by $c \otimes \partial^m \mapsto c\partial^m$.

3.9. A commutative$^!$ algebra is defined to be a commutative (associative unital) algebra in $\mathcal{M}^!$. In the present context, this is the same thing as the conventional commutative (associative unital) algebra with derivation. Modules over a commutative$^!$ algebra are defined (and described) similarly.

If $(L, [,.])$ is a Lie* algebra and (M_1, μ_1), (M_2, μ_2) are L-modules, $\mu_j \in P^*_{\{1,2\}}(\{L, M_j\}, M_j)$ being the action, $j = 1, 2$, then $M_1 \otimes^! M_2$ carries an L-module structure defined via the Leibniz rule. Namely, one defines $\mu = \mu_1 \otimes^! id_{M_2} + \mu_2 \otimes^! id_{M_1} \in P^*_{\{1,2\}}(\{L, M_1 \otimes^! M_2\}, M_1 \otimes^! M_2)$ and verifies, just as in the ordinary Lie algebra case, that this is a Lie* action.

If A is a commutative$^!$ algebra, then we say that L acts on A (or L acts on it by derivations) if A is an L-module s.t. the multiplication morphism

$$A \otimes^! A \longrightarrow A$$

is a morphism of L-modules.

In a similar vein, $\mathcal{L}$ is a Lie* A-algebroid if it is a Lie* algebra, an A-module, and it acts on A (by derivations) s.t.

(1) the action $\mu \in P^*_{\{1,2\}}(\{\mathcal{L}, A\}, A)$ is A-linear w.r.t. the $\mathcal{L}$-argument;

(2) the A-module morphism

$$A \otimes^! \mathcal{L} \longrightarrow \mathcal{L}$$

is an $\mathcal{L}$-module morphism, cf. (2.1).

A coisson algebra $\mathcal{P}$ is a Lie* algebra and a commutative$^!$ algebra s.t. the commutative$^!$-product map

$$\mathcal{P} \otimes^! \mathcal{P} \longrightarrow \mathcal{P}$$

is a Lie* algebra module morphism.

3.10. Let A be a conventional commutative associative unital algebra. Denote by $J_\infty A$ the universal commutative associative algebra with derivation generated by A. More formally, J_∞ is the left adjoint of the forgetful functor from the category of commutative algebras with derivation to the category of commutative algebras.

Lemma 3.1. — *If A is a Poisson algebra, then $J_\infty A$ is canonically a coisson algebra.*

Proof. — If $\{.,.\}$ is the Poisson bracket on A, then define $\{a\partial^l, b\partial^k\} = \{a, b\} \otimes \partial_1^l \partial_2^k$, then extend to all of $J_\infty A$ using the Leibniz property; this makes perfect sense thanks to the universal property of $J_\infty A$. The relation $\{a, (bc)\partial\} = \{a, (b\partial)c + b(c\partial)\}$ is almost tautological. $\square$

In hindsight, this simple assertion appears to be this theory's *raison d'être*.

To see an example, let A be a commutative algebra and consider the symmetric algebra $S^\bullet_A T_A$, which is canonically Poisson, sect. 2.1. It is graded, by assigning degree 1 to T_A, and so is the coisson algebra $J_\infty S^\bullet_A T_A$. Consider its degree 1 component, $J_\infty T_A$, which, by the way, can be equivalently described as the universal $J_\infty A$-module with derivation generated by T_A. The Lie* bracket on $J_\infty S^\bullet_A T_A$ restricts to $J_\infty T_A$ and makes it a Lie* algebra. Furthermore, $J_\infty S^\bullet_A T_A$ is a $J_\infty T_A$-module and $J_\infty A \subset J_\infty S^\bullet_A T_A$ is a submodule. Hence $J_\infty T_A$ acts on $J_\infty A$ be derivations. One easily verifies that, in fact, $J_\infty T_A$ is a Lie* $J_\infty A$-algebroid, sect. 3.9. Furthermore, it is not hard to prove that if a Lie* algebra L acts on $J_\infty A$ by derivations, then this action factors through a Lie* algebra morphism $L \to J_\infty T_A$.

A much more general discussion of tangent algebroids can be found in [5], 1.4.16.

3.11. The context of Lie* brackets makes it straightforward to suggest the definition of a Lie^*_∞ algebra, cf. [14], sect. 2. To begin with, let V be a graded vector space with homogeneous basis $\{x_i\}$. Denote by ΛV the graded symmetric algebra of this space, i.e., the associative algebra on generators $\{x_i\}$ and relations $x_i x_j = (-1)^{\deg x_i \cdot \deg x_j} x_j x_i$. Given a permutation $\sigma \in S_n$, define the sign $\epsilon(\sigma, \vec{x})$ s.t.

$$x_1 x_2 \ldots x_n = \epsilon(\sigma, \vec{x}) x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_n} \in S(V).$$

For the purposes of this section, we shall say that an R-module V is graded if $V = \oplus_{i\in\mathbb{Z}} V_i$ and $R(V_i) \subset V_i$. Similarly, if $\{V_i\}$ and W are graded R-modules, we shall say that an operation $\mu \in P^*_I(\{V_i\}, W)$ has degree k if

$$\mu(v_1, v_2 \ldots) \in V_N \otimes_R R^I, \text{ where } \sum_i \deg v_i + k = N.$$

Similarly, if V is a graded R-module, we shall say that an operation $\mu \in P^*_{[n]}(\{V\}, W)$ is *antisymmetric* if

$$\mu(v_{\sigma_1}, v_{\sigma_2} \ldots v_{\sigma_n}) = \mathrm{sgn}(\sigma)\epsilon(\sigma, \vec{x})\mu(v_1, v_2 \ldots v_n),$$

where the indices are used with the same reservations as in sect. 3.5 so that $\mu(v_{\sigma_1}, v_{\sigma_2} \ldots v_{\sigma_n})$ means $\sigma\mu(v_1, v_2 \ldots v_n)$ rather than a mere permutation of variables.

Definition. — A Lie^*_∞ algebra is a graded R-module L and a collection of antisymmetric $[n]$-operations $l_n \in P^*_{[n]}(\{L\}, L)$, $\deg l_n = 2 - n$, that for each $k = 1, 2 \ldots$ satisfy the following identity

$$\sum_{i+j=k+1} \sum_{\sigma} \mathrm{sgn}(\sigma)\epsilon(\sigma, \vec{x})(-1)^{i(j-1)} l_j(l_i(x_{\sigma_1} \ldots x_{\sigma_i}), x_{\sigma_{i+1}} \ldots x_{\sigma_n}) = 0, \tag{3.8}$$

where σ runs through the set of all $(i, n-i)$ unshuffles, i.e., $\sigma \in S_n$ s.t. $\sigma_1 < \sigma_2 < \cdots < \sigma_i$ and $\sigma_{i+1} < \sigma_{i+2} < \cdots < \sigma_n$.

By definition, l_1 is simply a degree 1 linear map $L \longrightarrow L$, and (3.8) with $n = 1$ says that $l_1^2 = 0$; in other words, (L, l_1) is a complex.

Let us denote $l_2(.,.)$ by $[.,.]$. One has $[x_1, x_2] = -(-1)^{x_1 x_2}[x_2, x_1]$, and (3.8) with $n = 2$ reads, after an obvious re-arrangement,

$$l_1[x_1, x_2] = [l_1(x_1), x_2] + (-1)^{x_1}[x_1, l(x_2)].$$

We conclude that $[.,.]$ is an antisymmetric super-star-bracket of degree 0, and l_1 is its derivation. More explicitly, if we write $[x,y]=\sum_i x_{(i)}y\otimes\partial_1^i/i!$, then

$$l_1(x_{(i)}y)=l_1(x)_{(i)}y+(-1)^x x_{(i)}l_1(y),$$

hence l_1 is a derivation of all products $_{(i)}$.

The $n=3$ case of (3.8) involves terms such as $[[.,.],.]$, $l_3\circ l_1$, and $l_1\circ l_3$. The first one will give the "jacobiator," the last two will show that the super-Jacobi identity holds up to homotopy, l_3:

$$\begin{aligned}&[[x_1,x_2],x_3]+(-1)^{x_3(x_1+x_2)}[[x_3,x_1],x_2]+(-1)^{x_1(x_2+x_3)}[[x_2,x_3],x_1]=\\&\quad-\left(l_1l_3(x_1,x_2,x_3)+l_3(l_1(x_1),x_2,x_3)+(-1)^{x_1}l_3(x_1,l_1(x_2),x_3)\right.\\&\qquad\qquad\left.+(-1)^{x_1+x_2}l_3(x_1,x_2,l_1(x_3))\right)\end{aligned}$$

Writing $l_3(x_1,x_2,x_3)=\sum_{m,n}(x_1,x_2,x_3)_{mn}\otimes\partial_1^m\partial_2^n/m!n!$ and equating the terms in front of $\partial_1^m\partial_2^n$ in the last equality, we obtain, cf. (3.3),

$$a_{(n)}b_{(m)}c-(-1)^{ab}b_{(m)}a_{(n)}c-\sum_{j\geqslant 0}\binom{n}{j}(a_{(j)}b)_{(n+m-j)}c=$$
$$l_1((a,b,c)_{mn})+(l_1(a),b,c)_{mn}+(-1)^a(a,l_1(b),c)_{mn}+(-1)^{a+b}(a,b,l_1(c))_{mn},$$

where we took the liberty of using a,b,c in place of x_1,x_2,x_3 (resp.) so as to avoid being flooded by indices.

It is clear, of course, how the concept of a differential Lie* superalgebra is defined and how that of a Lie^*_∞ algebra generalizes it.

To push the analogy with the ordinary Lie_∞ algebras a little further, introduce, given a Lie^*_∞ algebra $(L,\{l_n\})$, the graded symmetric algebra with shifted grading $\Lambda L[1]$. Then each l_n can be extended to a degree 1 "coderivation" by mimicking the standard formula, [14, 16],

$$\hat{l}_n(x_1,\ldots,x_k)=\sum_\sigma \pm l_n(x_{\sigma_1},\ldots,x_{\sigma_n})x_{\sigma_{n+1}}\cdots x_{\sigma_k},$$

with summation extended to the set of all $(n,k-n)$-unshuffles. There are a few reasons to write "coderivation," one of them being that the target of $\hat{l}_n$ is neither $\Lambda L[1]$ nor even the tensor algebra $T(L[1])$, but the direct sum of spaces $L_{J\twoheadrightarrow I}$, which are defined for each surjection of finite sets $J\twoheadrightarrow I$ to be $L^{\otimes I}\otimes_{R^I}R^J$, cf. sect. 3.1. Nevertheless, such operations can be composed and one can verify that $(\sum_n\hat{l}_n)^2=0$; in fact, the proof in the ordinary

Lie_∞ algebra case, [16], goes through word for word thanks to its purely combinatorial nature. As they say, we are planning to return to this topic in future publications.

3.12. The discussion above is but a shadow of the genuine Beilinson-Drinfeld category [5], 2.2. Given a smooth algebraic curve X, their category is one of right D_X-modules with the pseudo-tensor structure defined by

$$P_I^*(\{\mathcal{M}_i\},\mathcal{N}) = \mathrm{Hom}_{D_{X^I}}(\boxtimes_I\mathcal{M}_i \longrightarrow \Delta_*\mathcal{N}),$$

where $\Delta : X \longrightarrow X^I$ is the diagonal embedding.

Seeking to spell out everything in the simplest possible case, let from now on X be $\mathbb{C}$, $X^I = \times_I X$, $\mathbb{C}[X^I]$ the corresponding polynomial ring D_{X^I} the corresponding algebra of globally defined differential operators; we let x be the coordinate on X, $\partial_x = \partial/\partial x$..

Given a surjection $\pi : J \twoheadrightarrow I$, there arise an embedding $X^I \hookrightarrow X^J$ and the corresponding algebra homomorphism $\mathbb{C}[X^J] \twoheadrightarrow \mathbb{C}[X^I]$, $x_j \mapsto x_{\pi(j)}$. Define $D_{X^I\to X^J} = \mathbb{C}[X^I]\otimes_{\mathbb{C}[X^J]} D_{X^J}$, which is operated on by D_{X^J} on the right – obviously, and by D_{X^I} on the left via $\partial_{x_i} \mapsto \sum_{j\in\pi^{-1}(i)}\partial_{x_j}$; this makes $D_{X^I\to X^J}$ into a $D_{X^I} - D_{X^J}$-bimodule. There are obvious isomorphisms:

$$\otimes_I D_X \xrightarrow{\sim} D_{X^I},\ \otimes_I D_{X\to X^{J_i}} \xrightarrow{\sim} D_{X^I\to X^J},$$

$$D_{X^I\to X^J}\otimes_{D_{X^J}} D_{X^J\to X^K} \xrightarrow{\sim} D_{X^I\to X^J};$$

J_i stands for $\pi^{-1}(i)$ in the 2nd isomorphism.

For a collection of right D_X-modules, $\mathcal{M}_i$, $i\in I$, $\mathcal{N}$, define

$$P_I^*(\{\mathcal{M}_i\},\mathcal{N}) = \mathrm{Hom}_{D_{X^I}}(\otimes_I\mathcal{M}_i \longrightarrow \mathcal{N}\otimes_{D_X} D_{X\to X^I}),$$

The composition is defined as follows: for a surjection $\pi : J \twoheadrightarrow I$, and a collection of operations $\psi_i \in P_{J_i}^*(\{\mathcal{L}_j\},\mathcal{M}_i)$, $i\in I$, $J_i = \pi^{-1}(i)$, and $\phi \in P_I^*(\{\mathcal{M}_i\},\mathcal{N})$, define $\phi(\psi_i) \in P_J^*(\{\mathcal{L}_j\},\mathcal{N})$ to be the composite map, cf. sect. 3.1:

$$\begin{aligned}\otimes_J\mathcal{L}_j = \otimes_I\otimes_{J_i}\mathcal{L}_j &\xrightarrow{\otimes\psi_i} \otimes_I(\mathcal{M}_i\otimes_{D_X} D_{X\to X^{J_i}}) = (\otimes_I\mathcal{M}_i)\otimes_{D_{X^I}} D_{X^I\to X^J}\\ &\xrightarrow{\phi} (\mathcal{N}\otimes_{D_X} D_{X\to X^I})\otimes_{D_{X^I}} D_{X^I\to X^J} \xrightarrow{\sim} \mathcal{N}\otimes_{D_X} D_{X\to X^J}.\end{aligned}$$

The associativity follows from the isomorphisms $D_{X^I\to X^J}\otimes_{D_{X^J}} D_{X^J\to X^K} \xrightarrow{\sim} D_{X^I\to X^J}$.

3.13. Denote by $\mathcal{M}^*_D$ the pseudo-tensor category just defined. The category of the right $\mathcal{D}_X$-modules also carries a tensor category structure, which gives us a compound pseudo-tensor category, $\mathcal{M}^{*!}_D$, cf. sect. 3.7, and so one can still talk about commutative associative, Lie, Poisson, etc. objects of $\mathcal{M}^{*!}_D$, which we will still be calling commutative$^!$, Lie, coisson, etc., algebras.

The obvious similarity between $\mathcal{M}^*_D$ and $\mathcal{M}^*$ is easily made into an assertion as follows. Given an R-module M, $M[x]$ is naturally a D_X-module if we stipulate $m\partial_x = m\partial$, $m \in M$. This defines a functor

$$\Phi : \mathcal{M}^{*!} \longrightarrow \mathcal{M}^{*!}_D,\ M \mapsto M[x],$$

which is clearly compound pseudo-tensor and faithful. In fact, it identifies $\mathcal{M}^{*!}$ with the *translation-invariant* subcategory of $\mathcal{M}^{*!}_D$, i.e., $\mathcal{L}$ is isomorphic to $\Phi(M)$ for $M \in \mathcal{M}^{*!}$ precisely when $\mathcal{L}$ is translation-invariant, and $\phi \in P^*_I(\{M_i[x]\}, N[x])$ belongs to $\Phi P^*_I(\{M_i\}, N)$ if and only if ϕ is translation-invariant. Therefore, an object of some type of $\mathcal{M}^{*!}$ is the same as a translation-invariant object of the same type in $\mathcal{M}^{*!}_D$.

3.14. We are exclusively interested in the translation invariant objects, but even then this more general point of view is helpful. The assignment $\mathcal{M}^*_D \ni \mathcal{L} \mapsto h(\mathcal{L}) \stackrel{\text{def}}{=} \mathcal{L}/\mathcal{L}\partial_x \in \mathcal{V}ect$ is still an augmentation functor, sect. 3.3, and if L is a Lie* algebra, then $h(L[x])$ is a Lie algebra, just as $h(L)$, sect. 3.6.

Likewise, since our discussion easily localizes, if L is a Lie* algebra, then $h(L[x, x^{-1}])$ is a Lie algebra. If we let $a_{[n]}$ denote the class of $a \otimes x^n$ in $h(L[x, x^{-1}])$, then it is immediate to derive from (3.3) a formula for the bracket:

$$[a_{[n]}, b_{[m]}] = \sum_{j\geqslant 0} \binom{n}{j} (a_{(j)}b)_{[n+m-j]}, \tag{3.9}$$

also called the Borcherds identity. Denote this Lie algebra Lie(L); cf. [13], pp.41-42, [8], 16.1.16.

3.15. Similarly, the concept of a chiral algebra, even in the translation-invariant setting, is most naturally introduced in the framework of D_X-modules. For an I-family $\{\mathcal{A}_i\} \subset \mathcal{M}_D$ denote by $\otimes_I \mathcal{A}_i[\cup\Delta_{\alpha\beta}] \in \mathcal{M}_D$ the localization at the union of the diagonals, $\Delta_{\alpha\beta}$ standing for $(x_\alpha - x_\beta)^{-1}$, $\alpha, \beta \in I$. Define

$$P^{ch}_I(\{\mathcal{A}_i\}, \mathcal{N}) = \mathrm{Hom}_{D_{X^I}}(\otimes_I \mathcal{A}_i[\cup\Delta_{\alpha\beta}], \mathcal{N} \otimes_{D_X} D_{X\to X^I}).$$

Elements of such sets are called *chiral* operations. They are composed in the same way as the *-operations of sect. 3.12, except that now one has to deal with the poles, and these are handled by expanding rational functions in appropriate domains. Let us examine the simplest and most important such composition; the pattern will then become clear.

Fix $\phi \in \mathrm{Hom}_{D_{X^2}}(\mathcal{L} \otimes \mathcal{L}[\Delta], \mathcal{L})$. The composition $\phi(\phi, id)$ is defined as follows:

$$\begin{aligned}
&\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}[\Delta_{12}, \Delta_{13}, \Delta_{23}] \xrightarrow{\sim} (\mathcal{L} \otimes \mathcal{L}[\Delta_{12}]) \otimes \mathcal{L}[\Delta_{13}, \Delta_{23}] \\
&\xrightarrow{\phi\otimes id} (\mathcal{L} \otimes_{D_X} D_{X\to X^2}) \otimes \mathcal{L}[\Delta_{13}, \Delta_{23}] \xrightarrow{\sim} (\mathcal{L} \otimes \mathcal{L}[\Delta]) \otimes_{D_{X^2}} D_{X^2\to X^3} \\
&\qquad \xrightarrow{\phi\otimes id} (\mathcal{L} \otimes_{D_X} D_{X\to X^2}) \otimes_{D_{X^2}} D_{X^2\to X^3} \xrightarrow{\sim} \mathcal{L} \otimes_{D_X} D_{X\to X^3}.
\end{aligned}$$

Here the isomorphism

$$(\mathcal{L} \otimes_{D_X} D_{X\to X^2}) \otimes \mathcal{L}[\Delta_{13}, \Delta_{23}] \xrightarrow{\sim} (\mathcal{L} \otimes \mathcal{L}[\Delta]) \otimes_{D_{X^2}} D_{X^2\to X^3},$$

the only not so evident step, is made as follows: if we let t be the coordinate on the diagonal $X \hookrightarrow X^2$, then $t - x_1$ and $t - x_2$ act nilpotently on $\mathcal{L} \otimes_{D_X} D_{X\to X^2}$ and we use the geometric series to replace

$$\Delta_{13} = \frac{1}{x_1 - x_3} \text{ with } -\sum_{n=0}^{\infty} \frac{(x_1 - t)^n}{(x_3 - t)^{n+1}},$$

$$\Delta_{23} = \frac{1}{x_2 - x_3} \text{ with } -\sum_{n=0}^{\infty} \frac{(x_2 - t)^n}{(x_3 - t)^{n+1}}. \tag{3.10}$$

This gives the category of right D_X-modules another pseudo-tensor structure, to be denoted $\mathcal{M}_D^{ch}$.

3.16. It is often useful to use an isomorphism of right D_{X^2}-modules

$$\mathcal{L} \otimes_{D_X} D_{X\to X^2} \xrightarrow{\sim} \Omega_X^1 \otimes \mathcal{L}[\Delta]/\Omega_X^1 \otimes \mathcal{L},\ l \otimes 1 \mapsto \frac{dx \otimes l}{x_1 - x_2} \bmod \Omega_X^1 \otimes \mathcal{L},$$

which is a manifestation of the Kashiwara lemma, [6], 7.1. Notice that from this point of view, the composite map

$$\mathcal{L} \otimes_{D_X} D_{X\to X^2} \twoheadrightarrow \mathcal{L} \otimes_{D_X} D_{X\to X^2}/(\mathcal{L} \otimes_{D_X} D_{X\to X^2})\partial_1 \xrightarrow{\sim} \mathcal{L}$$

is defined by the residue

$$\Omega_X^1 \otimes \mathcal{L}[\Delta] \longrightarrow \mathcal{L},\ \omega \otimes l \mapsto \left(\int_{x_1:|x_1-x_2|=r} \omega\right) l.$$

3.17. A Lie^{ch} algebra (on X) is a Lie object in $\mathcal{M}_D^{ch}$; explicitly, it is a right D_X-module $\mathcal{L}$ with chiral bracket $[.,.]^{ch} \in P_2^{ch}(\{\mathcal{L},\mathcal{L}\},\mathcal{L})$ that is anticommutative and satisfies the Jacobi identity. The simplest example is Ω_X^1 with the canonical right D_X-module structure (given by the negative Lie derivative) and the chiral Lie bracket

$$\Omega_X^1 \otimes \Omega_X^1[\Delta] \twoheadrightarrow \Omega_X^1 \otimes \Omega_X^1[\Delta]/\Omega_X^1 \otimes \Omega_X^1 \xrightarrow{\sim} \Omega_X^1 \otimes_{D_X} D_{X\to X^2},$$

where the rightmost isomorphism has just been discussed, sect. 3.16.

The chiral algebra is a Lie^{ch} algebra $\mathcal{L}$ with a unit, i.e., a morphism $\iota : \Omega_X^1 \longrightarrow \mathcal{L}$ s.t. the composition $[\iota(.),.]$ coincides with the map

$$\Omega_X^1 \otimes \mathcal{L}[\Delta] \twoheadrightarrow \Omega_X^1 \otimes \mathcal{L}[\Delta]/\Omega_X^1 \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{L} \otimes_{D_X} D_{X\to X^2},$$

The obvious map $\otimes_I \mathcal{A}_i \longrightarrow \otimes_I \mathcal{A}_i[\cup\Delta_{\alpha\beta}]$ defines, by restriction, a map

$$P_I^{ch}(\{\mathcal{A}_i\},\mathcal{N}) \longrightarrow P_I^{*}(\{\mathcal{A}_i\},\mathcal{N}),$$

hence a *forgetful* functor $\mathcal{M}_D^{ch} \longrightarrow \mathcal{M}_D^{*}$. It follows that each Lie^{ch} algebra can be regarded as a Lie* algebra. Further composing with $h : \mathcal{M}_D^* \to \mathcal{V}ect$, sect. 3.14, will attach an ordinary Lie algebra $h(\mathcal{L})$ to each chiral algebra $\mathcal{L}$.

A chiral algebra is called *commutative* if the corresponding Lie* algebra is abelian, i.e., the corresponding Lie* bracket is 0. In the translation-invariant setting, a commutative chiral algebra is the same thing as an ordinary unital commutative associative algebra with derivation; we shall have more to say on this in sect. 3.20.

The definition of a chiral algebra module should be evident; any chiral algebra module is automatically a module over the corresponding Lie* algebra. If $\mathcal{L}$ is a chiral algebra and $\mathcal{M}$ an $\mathcal{L}$-module, then $h(\mathcal{L})$ is a Lie algebra, and both $\mathcal{M}$ and $h(\mathcal{M})$ are $h(\mathcal{L})$-modules. If the structure involved is translation invariant, in particular, $\mathcal{L} = L[x]$, $\mathcal{M} = M[x]$, then the fiber M is also an $h(\mathcal{L})$-module, as well as Lie$(\mathcal{L})$-module, see sect. 3.14.

3.18. $\mathcal{M}$, a module over a chiral algebra $\mathcal{L}$, is called *central* if it is trivial over the corresponding Lie* algebra $h(\mathcal{L})$, [5], 3.3.7.

In view of what is said at the end of sect. 3.17 it may sound as a surprise that a module over a commutative chiral algebra $\mathcal{L} = L[x]$ is not the same thing as a module over L regarded as a commutative associative algebra with derivation. However, if the module in question is central, then the two notions coincide; we shall explain this in sect. 3.20 and show an example in sect. 4.4.

3.19. An explicit description of a chiral algebra usually arises in the following situation. Let $\mathcal{V}$ be a translation-invariant *left* D_X-module, which amounts to having $\mathcal{V} = V[x]$, V being a *left* R-module. Let $\mathcal{V}^r \stackrel{\text{def}}{=} \mathcal{V} \otimes_{\mathbb{C}[x]} \Omega^1_X$ be the corresponding *right* D_X-module; we shall sometimes write simply $V[x]dx$ for $\mathcal{V}^r$.

Notice canonical isomorphisms of right D_{X^2}-modules

$$\begin{aligned}\mathcal{V}^r \otimes_{D_X} D_{X\to X^2} &\longrightarrow \mathbb{C}[x] \otimes V[y][(x-y)^{-1}]dx \wedge dy/\mathbb{C}[x] \otimes V[y]dx \wedge dy\\ &\longleftarrow V[x] \otimes \mathbb{C}[y][(x-y)^{-1}]dx \wedge dy/V[x] \otimes \mathbb{C}[y]dx \wedge dy; \quad (3.11)\end{aligned}$$

the first is discussed in sect. 3.16, the second is the result of a formal Taylor series expansion

$$v(x) \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n v(y)(x-y)^n,$$

which is essentially Grothendieck's definition of a connection.

In this setting, the translation-invariant chiral bracket $[.,.] \in P_2^{ch}(\{\mathcal{V}^r, \mathcal{V}^r\}, \mathcal{V}^r)$ is conveniently encoded by a map, usually referred to as an OPE:

$$V \otimes V \longrightarrow V((x-y)),\ a \otimes b \mapsto \sum_{n\in\mathbb{Z}} a_{(n)} b \otimes (x-y)^{-n-1}. \qquad (3.12)$$

Given an OPE, one recovers the chiral bracket

$$(V[x]dx \otimes V[y]dy)[(x-y)^{-1}]$$

$$\longrightarrow \mathbb{C}[x] \otimes V[y][(x-y)^{-1}]dx \wedge dy/\mathbb{C}[x] \otimes V[y]dx \wedge dy$$

by defining

$$\frac{1}{(x-y)^N}(a \otimes f(x)) \otimes (b \otimes g(y)) \mapsto \sum_{n\in\mathbb{Z}} a_{(n)} b \frac{f(x)g(y)}{(x-y)^{N+n+1}} dx \wedge dy \text{ mod reg},$$

"mod reg." meaning, of course, "modulo $\mathbb{C}[x] \otimes V[y]dx \wedge dy$." In fact, this sets up a 1-1 correspondence between binary chiral operations and OPEs, [8], 19.2.11, or [5], 3.5.10.

In this vein, the Jacobi identity can also be made explicit. The diagonal in X^3 being of codimension 2, $\mathcal{V}^r \otimes_{D_X} D_{X\to X^3}$ does not allow a description as simple as (3.11), and one relies instead on iterations of (3.11). Writing $\mathcal{V}^r \otimes_{D_X} D_{X\to X^3}$ as $(\mathcal{V}^r \otimes_{D_X} D_{X\to X^2}) \otimes_{D_{X^2}} D_{X^2\to X^3}$, which requires a choice of an embedding $X^2 \to X^3$, such as $(u,v) \mapsto (u,v,u)$, one obtains identifications, such as

$$\mathcal{V}^r \otimes_{D_X} D_{X\to X^3} \longrightarrow \mathbb{C}[x]\otimes(\mathbb{C}[y]\otimes V[z][(y-z)^{-1}] \text{ mod reg. })[(x-z)^{-1}] \text{ mod reg.};$$

we omit differentials, $dx \wedge dy \wedge dz$, for typographical reasons.

Write $a(x-y)b$ for OPE (3.12). Various compositions that enter the Jacobi identity involve expressions such as

$$\begin{aligned} &[a[,b,c]] = (a(x-z)(b(y-z)c \text{ mod reg.}) \text{ mod reg. })dx\wedge dy \wedge dz,\\ &[[a,b]c] = ((a(x-y)b \text{ mod reg.})(y-z)c \text{ mod reg. })\, dx\wedge dy\wedge dz, \text{ etc..}\end{aligned}$$

The Jacobi identity,

$$[a,[b,c]] - (1,2)[a,[b,c]] - [[a,b],c] = 0,$$

implies, as above, that for any $F(x,y,z) = (x-y)^r(x-z)^s(y-z)^t$, $r,s,t\in\mathbb{Z}$,

$$\begin{aligned} &\oint_{x:|x-z|=R} dx \oint_{y:|y-z|=r} dy F(x,y,z) a(x-z)b(y-z)c\, dz\\ &\qquad - \oint_{y:|y-z|=R} dy \oint_{x:|x-z|=r} dx F(x,y,z) b(y-z)a(x-z)c\, dz\\ &- \oint_{y:|x-z|=R} dy \oint_{x:|x-y|=r} dx F(x,y,z)(a(x-y)b)(y-z)c\, dz = 0, \end{aligned} \quad (3.13)$$

where $R > r$. Let us explain this.

Denote by $Jac \in P_2^{ch}(\{\mathcal{L},\mathcal{L}\},\mathcal{L})$ the left hand side of the Jacobi identity; it is a map

$$Jac : (V[x]\otimes V[y]\otimes V[z])[(x-y)^{-1},(x-z)^{-1},(y-z)^{-1}]dx\wedge dy\wedge dz$$

$$\longrightarrow V[t]dt \otimes_{D_X} D_{X\to X^3}.$$

Written down it gives the left hand side of (3.13) without the $\int$ signs but with the function $F(x,y,z)$ expanded in powers of appropriate variabes, $(x-z)$ and $(y-z)$ for the 1st and 3rd term, $(x-y)$ and $(y-z)$ for the 2nd one, in domains prescribed by (3.10). For example, in the case of the 1st integral, one has

$$F(x,y,z) = (x-y)^r(x-z)^s(y-z)^t = (x-z)^{s+r}(y-z)^t \sum_{j=0}^{\infty}(-1)^j \binom{r}{j}\left(\frac{y-z}{x-z}\right)^j.$$

Notice that the choice of the domain coincides with the one determined by the contour of integration.

Treating the arising 3 expressions requires an effort as they belong to 3 different realizations of the same space, $V[t]dt \otimes_{D_X} D_{X\to X^3}$. However, part of this computation is easy: the composition

$$(V[x]\otimes V[y]\otimes V[z])[(x-y)^{-1},(x-z)^{-1},(y-z)^{-1}]dx\wedge dy\wedge dz \xrightarrow{Jac} V[t]dt\otimes_{D_X} D_{X\to X^3}$$
$$\twoheadrightarrow V[t]dt = V[t]dt \otimes_{D_X} D_{X\to X^3}/(V[t]dt \otimes_{D_X} D_{X\to X^3})\mathbb{C}[\partial_x,\partial_y]$$

is defined simply by taking the residues, just as in sect. 3.16, hence it equals the left hand side of (3.13).

Formula (3.13) is the Borcherds identity [7] in the form suggested in [13], 4.8. Therefore, a translation invariant chiral algebra on $\mathbb{C}$ defines a vertex algebra. A passage in the opposite direction is carefully explained in [8], Ch.15.

3.20. The case $F(x,y,z) = (x-z)^m(y-z)^n$ of (3.13) reproduces the Borcherds commutator formula (3.3)

$$a_{(n)}b_{(m)}c - b_{(m)}a_{(n)}c = \sum_{j\geqslant 0}\binom{n}{j}(a_{(j)}b)_{(n+m-j)}c. \tag{3.14}$$

The case $F(x,y,z) = (x-y)^{-1}(y-z)^{-1}$ becomes the celebrated normal ordering formula

$$(a_{(-1)}b)_{(-1)}c = \sum_{j=0}^{\infty} b_{(-j-2)}a_{(j)}c + a_{(-1)}b_{(-1)}c + \sum_{j=0}^{\infty} a_{(-j-2)}b_{(j)}c \tag{3.15}$$

In fact, these particular cases suffice to reproduce the entire (3.13), [13], 4.8.

One sees at once that in the language where a chiral algebra is a vector space V with a family of multiplications, ${}_{(n)}$, $n\in\mathbb{Z}$, "V is commutative" (see sect. 3.17) means the "nth product is 0 if $n\geqslant 0$." Borcherds commutator formula (3.14) implies then that $(V,{}_{(-1)})$ is a commutative algebra; in fact, $[a_{(m)}, b_{(n)}] = 0$ for all m,n. Further, (3.15) shows that the product ${}_{(-1)}$ is associative, and so $(V_{(-1)})$ is an associative, commutative algebra with derivation. The passage in the opposite direction is explained in [13, 8].

Similarly, the conceptual definition of a chiral algebra module, reviewed in sect. 3.17, boils down to a vector space M with multiplications

$${}^{M}_{(n)} : V\otimes M \longrightarrow M,\ a\otimes m \mapsto a^{M}_{(n)}m$$

so that

$$a^M_{(n)}b^M_{(m)}c - b^M_{(m)}a^M_{(n)}m = \sum_{j\geqslant 0}\binom{n}{j}(a_{(j)}b)^M_{(n+m-j)}m, \tag{3.16}$$

$$\begin{aligned}(a_{(-1)}b)^M_{(-1)}c &= \sum_{j=0}^{\infty} b^M_{(-j-2)}a^M_{(j)}c + a^M_{(-1)}b^M_{(-1)}c \\ &\quad + \sum_{j=0}^{\infty} a^M_{(-j-2)}b^M_{(j)}m; \end{aligned} \tag{3.17}$$

we are deliberately omitting some of the obvious axioms.

One sees clearly how the concept of a module over a commutative chiral algebra is different from one over a commutative associative algebra with derivation: the associativity condition $(ab)m = a(bm)$ in the latter is replaced by the more cumbersome (3.17) in the former. If, however, M is central, sect. 3.17, which means that $a^M_{(n)}m = 0$, $n \geqslant 0$, then the "correction terms" in (3.17) vanish, and the two concepts become equivalent.

3.21. We have seen, sect. 3.17, that there is a forgetful functor that makes a chiral algebra into a Lie* algebra. This functor admits the left adjoint called the *chiral enveloping* algebra. Let us sketch its construction, cf. [8], 16.1.11, [5], 3.7.1. (We work in the translation-invariant setting, this goes without saying.)

Given a Lie* algebra L, consider the Lie algebra $\mathrm{Lie}(L)$, sect. 3.14. Formula (3.9) implies that $\mathrm{Lie}(L)_+$ defined to be spanned by $a_{[n]}$, $a \in L$, $n \geqslant 0$, is a Lie subalgebra. Define $U^{ch}L$ to be $U(\mathrm{Lie}(L))/U(\mathrm{Lie}(L)_+)$. Here $U(.)$ is the ordinary universal enveloping of a Lie algebra.

It is easy to see that the map $L \longrightarrow \mathrm{Lie}(L)$, $a \mapsto a_{[-1]}$, is injective, and so is the composition

$$L \longrightarrow \mathrm{Lie}(L) \hookrightarrow U(\mathrm{Lie}(L)) \twoheadrightarrow U(\mathrm{Lie}(L))/U(\mathrm{Lie}(L)_+)$$

The Reconstruction Theorem, [8], 2.3.11 or [13], 4.5, implies that $U^{ch}L$ carries a chiral algebra structure defined, in terms of ${}_{(n)}$-products, by a slightly tautological formula

$$(\overline{a_{[-1]}})_{(n)}v = a_{[n]} \cdot v;$$

here $\overline{a_{[-1]}}$ is the image of $a_{[-1]}$ under the above composition, and $\cdot$ on the right means the action of $\mathrm{Lie}(L)$ on $U(\mathrm{Lie}(L))/U(\mathrm{Lie}(L)_+)$.

4. CDO

4.1. We shall work exclusively in the translation-invariant situation, although much of what we are about to say does not require this assumption, and so we shall typically deal with fibers of the actual objects, cf. sect. 3.13, 3.20. Thus, for example, the phrase " a chiral (Lie*, etc.) algebra V" means the fiber of a translation-invariant chiral (Lie*, etc.) algebra $V[x]$, and a chiral (Lie*, etc.) algebra morphism $f: \ V \longrightarrow W$ means $f \otimes id: \ V[x] \longrightarrow W[x]$.

4.2. Let A be a commutative associative unital algebra. A chiral algebra $\mathcal{D}^{ch}_A$ is called an algebra of *chiral differential operators* over A if it carries a filtration $F_{-1}\mathcal{D}^{ch}_A = \{0\} \subset F_0\mathcal{D}^{ch}_A \subset F_1\mathcal{D}^{ch}_A \subset \cdots$, $\cup_n F_n\mathcal{D}^{ch}_A = \mathcal{D}^{ch}_A$, s.t. the graded object

$$\mathrm{gr}\mathcal{D}^{ch}_A = \bigoplus_{n=0}^{\infty} F_n\mathcal{D}^{ch}_A / F_{n-1}\mathcal{D}^{ch}_A$$

is a coisson algebra, sect. 3.9, which is isomorphic, as a coisson algebra, to $J_\infty S^\bullet_A T_A$, sect 3.10, 3.13.

By definition, $F_0\mathcal{D}^{ch}_A = J_\infty A$ and $F_1\mathcal{D}^{ch}_A$ fits in the short exact sequence

$$0 \longrightarrow J_\infty A \longrightarrow F_1\mathcal{D}^{ch}_A \longrightarrow J_\infty T_A \longrightarrow 0, \tag{4.1}$$

loc. cit. Notice that both $F_1\mathcal{D}^{ch}_A$ and $J_\infty T_A$ are Lie* algebras and chiral $J_\infty A$-modules, but while $J_\infty T_A$ is a Lie* $J_\infty A$-algebroid, see sect 3.10, $F_1\mathcal{D}^{ch}_A$ is not. This has to do with the fact that $J_\infty A$ being a commutative algebra with derivation is both a commutative$^!$ algebra, sect. 3.9, and a commutative chiral algebra, sect. 3.17; in its former capacity it operates on $J_\infty T_A$, but it acts on $F_1\mathcal{D}^{ch}_A$ only as a chiral algebra, sect. 3.18.

This prompts the following definition.

4.3. A chiral algebroid (A-algebroid)[1] is a short exact sequence

$$0 \longrightarrow J_\infty A \stackrel{\iota}{\longrightarrow} \mathcal{L}^{ch}_A \stackrel{\sigma}{\longrightarrow} J_\infty T_A \longrightarrow 0, \tag{4.2}$$

where $\mathcal{L}^{ch}_A$ is a Lie* algebra and a chiral module over $J_\infty A$, and the arrows respect all the structures. Here is what this amounts to.

Denote by $\mu \in P^{ch}_{\{1,2\}}(\{J_\infty A, \mathcal{L}^{ch}_A\}, \mathcal{L}^{ch}_A)$ the chiral $J_\infty A$-module structure on $\mathcal{L}^{ch}_A$ and by $[.,.] \in P^*_{\{1,2\}}(\{\mathcal{L}^{ch}_A, \mathcal{L}^{ch}_A\}, \mathcal{L}^{ch}_A)$ the Lie* algebra bracket.

---

(1) we should have said " a translation-invariant chiral algebroid on $\mathbb{C}$ in the case of a jet-scheme"

There arises $\mu^* \in P^*_{\{1,2\}}(\{J_\infty A, \mathcal{L}^{ch}_A\}, \mathcal{L}^{ch}_A)$, the Lie* algebra action corresponding to μ, sect. 3.17. We demand the following.

(i) $\mathcal{L}^{ch}_A \stackrel{\sigma}{\longrightarrow} J_\infty T_A$ is a Lie* algebra morphism.

(ii) Point (i) implies that $J_\infty A \subset \mathcal{L}^{ch}_A$ is a Lie* ideal and, therefore, the Lie* algebra $J_\infty T_A$ operates on $J_\infty A$. We require that this action be equal to the canonical action of $J_\infty T_A$ on $J_\infty A$, sect. 3.10.

(iii) $J_\infty A \longrightarrow \mathcal{L}^{ch}_A$ is simultaneously a chiral $J_\infty A$-module morphism and a Lie* algebra morphism; furthermore, $\mu^*(.,.) = [\iota, .]$, which means that the two Lie* actions of $J_\infty A$ on $\mathcal{L}^{ch}_A$, one defined by μ^* another by the embedding ι, coincide.

(iv) Points (i) and (iii) imply that the Lie* action of $J_\infty A$ on the quotient $\mathcal{L}^{ch}_A/J_\infty A$ is trivial; in other words, $\mathcal{L}^{ch}_A/J_\infty A$ is a *central* chiral $J_\infty A$-module, sect. 3.18. Therefore $\mathcal{L}^{ch}_A/J_\infty A$ is a module over $J_\infty A$ regarded as an ordinary associative commutative algebra, *loc. cit.*. We demand that the induced by (4.2) vector space isomorphism $\mathcal{L}^{ch}_A/J_\infty A \stackrel{\sim}{\longrightarrow} J_\infty T_A$ be a $J_\infty A$-module morphism.

(v) The chiral action μ is $\mathcal{L}^{ch}_A$-linear, cf. (2.1).

Remarks. — (1) Point (v) is self-explanatory: $\mu \in P^{ch}_{\{1,2\}}(\{J_\infty A, \mathcal{L}^{ch}_A\}, \mathcal{L}^{ch}_A)$ is called $\mathcal{L}^{ch}_A$-linear if the composition

$$[., \mu(.,.)] \in Hom_{D_{X^3}}((\mathcal{L}^{ch}_A[x] \otimes J_\infty A[y] \otimes \mathcal{L}^{ch}_A[z])[(y-z)^{-1}], \mathcal{L}^{ch}_A[t] \otimes_{D_X} D_{X\to X^3})$$

equals the sum of the compositions $\mu([.,.],.) + \mu(.,[.,.])$, where $[.,.] \in P^*_{\{1,2\}}(\{\mathcal{L}^{ch}_A, \mathcal{L}^{ch}_A\}, \mathcal{L}^{ch}_A)$ is the Lie*-bracket. In terms of $_{(n)}$-products this amounts to the fact that the commutator formula, cf. sect. 3.20,

$$a_{(n)}b_{(m)}c - b_{(m)}a_{(n)}c = \sum_{j\geqslant 0} \binom{n}{j} (a_{(j)}b)_{(n+m-j)}c, \tag{4.3}$$

whose validity for $m, n \geqslant 0$ is the consequence of $\mathcal{L}^{ch}_A$ being a Lie* algebra, is also valid for $m < 0$ if $b \in J_\infty A$. This is an analogue of (2.2).

(2) Item (iv) is the only point where this definition is conceptually different from the one in sect. 2.2 and it is ultimately responsible for there being an obstruction to the existence of a chiral algebroid.

4.4. A well-known example arises when $A = \mathbb{C}[x_1, ..., x_N]$. Introduce », a Lie algebra with generators x_{ij}, ∂_{mn}, $1 \in \mathbb{C}$ and relations $[\partial_{mn}, x_{ij}] = \delta_{mi}\delta_{n,-j}$. There is a subalgebra, $»_-$, defined to be the linear span of x_{ij}, ∂_{mn}, $j > 0$,

$m \geqslant 0$. The induced representation $\mathrm{Ind}_{\mathfrak{»}_-}^{\mathfrak{»}}\mathbb{C}$, which is naturally identified with $\mathbb{C}[x_{ij}, \partial_{mn};\ j \leqslant 0, n < 0]$, is well known to carry a vertex algebra structure; it is often referred to as a "β-γ-system. Explicit formulas can be found in [17]. For example, one has

$$\begin{aligned}&(\partial_{i,-1})_{(0)}(x_{j,0}) = \delta_{ij},\\ &(\partial_{i,-1})_{(n+1)}(x_{j,0}) = (\partial_{i,-1})_{(n)}(\partial_{j,-1}) = (x_{i,0})_{(n)}(x_{j,0}) = 0 \text{ if } n \geqslant 0.\end{aligned} \tag{4.4}$$

Fix $\mathbb{C}[x_1, ..., x_n] \hookrightarrow \mathbb{C}[x_{ij}, \partial_{mn};\ j \leqslant 0, n < 0]$, $x_i \mapsto x_{i0}$. For any étale localization $\mathbb{C}[x_1, ..., x_N] \subset A$, the space

$$A[x_{ij}, \partial_{mn};\ j, n < 0] \stackrel{\text{def}}{=} A \otimes_{\mathbb{C}[x_1,\ldots,x_N]} \mathbb{C}[x_{ij}, \partial_{mn};\ j \leqslant 0, n < 0]$$

inherits a vertex algebra structure from $\mathbb{C}[x_{ij}, \partial_{mn};\ j \leqslant 0, n < 0]$.

The increasing filtration $\{F_r A[x_{ij}, \partial_{mn};\ j, n < 0]\}$, $r \geqslant 0$, is defined by counting the letters ∂_{mn}, $n < 0$. The graded object is identified with $J_\infty S_A^\bullet T_A$, and so $A[x_{ij}, \partial_{mn};\ j, n < 0]\}$ is a CDO, sect. 4.2.

The space $F_1 A[x_{ij}, \partial_{mn};\ j, n < 0]$ is a chiral algebroid. Exact sequence (4.2) in this case becomes

$$0 \longrightarrow A[x_{ij};\ j < 0] \longrightarrow F_1 A[x_{ij}, \partial_{mn};\ j, n < 0]$$

$$\longrightarrow \oplus_m \oplus_{n<0} A[x_{ij};\ j < 0]\partial_{mn} \longrightarrow 0.$$

It is easy to see exactly how $F_1 A[x_{ij}, \partial_{mn};\ j, n < 0]$ fails to be a central chiral $J_\infty A$-module and $J_\infty T_A = F_1 A[x_{ij}, \partial_{mn};\ j, n < 0]/J_\infty A$ does not: suppressing extraneous indices we derive using (4.4, 3.17)

$$((x_0)^2)_{(-1)}\partial_{-1} = x_0^2\partial_{-1} - 2x_{-1}.$$

4.5. Classification of chiral algebroids is delightfully similar to that of Picard-Lie algebroids, sect. 2.3.

To begin with, assume that the tangent Lie algebroid T_A is a free A-module with basis $\{\xi_i\}$. Then it is easy to see that the chiral module structure on $\mathcal{L}_A^{ch}$ cannot be deformed. Indeed, we have $\mathcal{L}_A^{ch} = J_\infty A \oplus \{\sum_i f_{i,(-1)}\partial^{j_i}\xi_i\}$, with coefficients $f_{i,(-1)}$, $f_i \in J_\infty A$, determined uniquely; this uses requirement (iv) of the definition in sect. 4.3. Elements $f_{(n)}\xi$, $n \geqslant 0$, are then completely determined: these elements form the Lie* action of $J_\infty A$ on $\mathcal{L}_A^{ch}$, which by (iii) is the same as minus the adjoint action of $\mathcal{L}_A^{ch}$

restricted to $J_\infty A$, which by (ii) is the pull-back via $\mathcal{L}_A^{ch} \longrightarrow J_\infty T_A$ of the canonical action of $J_\infty T_A$ on $J_\infty A$. Along with the designation $f_{(-n-1)} = 1/n!(\partial^n f)_{(-1)}$, which is imposed by the definition of a chiral module, this determines $f_{(n)}$ with $n<0$. Indeed, the expression of the type $f_{(-1)}(g_{(-1)}\xi)$ can be computed by reading (3.17) backwards:

$$f_{(-1)}(g_{(-1)}\xi) = (fg)_{(-1)}\xi - \sum_{j=0}^{\infty} g_{(-2-j)}(f_{(j)}\xi) - \sum_{j=0}^{\infty} f_{(-2-j)}(g_{(j)}\xi).$$

Therefore, the room for maneuver is only provided by the Lie* bracket on $\mathcal{L}_A^{ch}$. If $[.,.]$ is one such bracket, then any bracket is

$$[\xi.,.\tau]_\alpha = [\xi,\tau] + \alpha(\xi,\tau) \text{ for some } \alpha \in P^*_{\{1,2\}}(\{J_\infty T_A, J_\infty T_A\}, J_\infty A).$$

It easily follows from (4.3) that α must be $J_\infty A$-linear. The antisymmetry of a Lie* bracket implies that α must be antisymmetric. The Jacobi identity,

$$[\xi_1[\xi_2,\xi_3]_\alpha]_\alpha - [\xi_2[\xi_1,\xi_3]_\alpha]_\alpha - [[\xi_1,\xi_2]_\alpha,\xi_3]_\alpha = 0,$$

implies

$$[\xi_1,\alpha(\xi_2,\xi_3)] - [\xi_2,\alpha(\xi_1,\xi_3)] + [\xi_3,\alpha(\xi_1,\xi_2)] - \\ \alpha(\sigma[\xi_1,\xi_2],\xi_3) + \alpha(\sigma[\xi_1,\xi_3],\xi_2) - \alpha(\sigma[\xi_2,\xi_3],\xi_1) = 0$$

Since the Lie* bracket $[.,.]$ restricted to $J_\infty A$ is the pull-back of the canonical action of $J_\infty T_A$ on $J_\infty A$ (item (ii) of the definition in sect. 4.2), this means that α is a closed 2-cochain of $J_\infty T_A$ with coefficients in $J_\infty A$, which satisfies an extra condition of being $J_\infty A$-linear, see the definition of the Chevalley complex in sect. 3.5.

More generally, define $C^n_{J_\infty A}(J_\infty T_A, J_\infty A) \subset C^n(J_\infty T_A, J_\infty A)$ to be the subspace of $J_\infty A$-linear operations. It is easy to see that $C^\bullet_{J_\infty A}(J_\infty T_A, J_\infty A) \subset C^\bullet(J_\infty T_A, J_\infty A)$ is a subcomplex, and as such it is called the *Chevalley–De Rham complex*; this definition makes sense for any Lie* algebroid, [5], 1.4.14.

To conclude, given a chiral algebroid $\mathcal{L}_A^{ch}$ and $\alpha \in C^{2,cl}_{J_\infty A}(J_\infty T_A, J_\infty A)$ we have defined another chiral algebroid, to be denoted $\mathcal{L}_A^{ch}(\alpha)$; furthermore, any chiral algebroid is isomorphic to $\mathcal{L}_A^{ch}(\alpha)$ for some α.

The description of morphisms is also similar to sect. 2.3. By definition, each morphism must have the form

$$\xi \mapsto \xi + \beta(\xi),\ \beta \in C^1_{J_\infty A}(J_\infty T_A, J_\infty A) = \mathrm{Hom}_{J_\infty A}(J_\infty T_A, J_\infty A).$$

A quick computation, no different from the ordinary case, will show

$$\mathrm{Hom}_{Ch-Alg}(\mathcal{L}_A^{ch}(\alpha_1), \mathcal{L}_A^{ch}(\alpha_2)) = \{\beta \in C^1_{J_\infty A}(J_\infty T_A, J_\infty A) \text{ s.t. } d\beta = \alpha_1 - \alpha_2\}.$$

This can be rephrased as follows. Let $C^{[1,2>}(J_\infty T_A)$ be the category with objects $C^{2,cl}_{J_\infty A}(J_\infty T_A, J_\infty A)$ and morphisms $\mathrm{Hom}(\alpha_1, \alpha_2) = \{\beta \in C^1_{J_\infty A}(J_\infty T_A, J_\infty A) \text{ s.t. } d\beta = \alpha_1 - \alpha_2\}$. Then the category of chiral A-algebroids, if non-empty, is a $C^{[1,2>}(J_\infty T_A)$-torsor. It is non-empty if T_A has a finite abelian basis; this follows from sect. 4.4.

4.6. These considerations can be localized in an obvious manner. For any smooth X, one obtains a tangent Lie* algebroid T_X^{ch} and a gerbe of chiral algebroids over $J_\infty X$, bound by the complex $C^1_{J_\infty X}(T_X^{ch}, \mathcal{O}_{J_\infty X}) \to C^{2,cl}_{J_\infty X}(T_X^{ch}, \mathcal{O}_{J_\infty X})$. This gerbe is locally non-empty, as follows from sect. 4.4. The calculation of its characteristic class, in this and much greater generality, can be found in [5], 3.9.22. We shall review below (sect. 4.9) the case of a graded chiral agebroid.

4.7. The chiral enveloping algebra $U^{ch}(\mathcal{L}_A^{ch})$ attached to $\mathcal{L}_A^{ch}$ if the latter is regarded as a Lie* algebra, sect. 3.21, does not "know" about the chiral structure that $\mathcal{L}_A$ carries. This leads to the existence of a canonical ideal as follows. Consider two elements $a_{(-n)}\xi \in \mathcal{L}_A^{ch}$ and $a_{[-n]}\xi \in \mathcal{L}_A \cdot U^{ch}(\mathcal{L}_A^{ch})$, $n > 0$, where $a \in J_\infty A \subset U^{ch}(\mathcal{L}_A^{ch})$ and $\xi \in \mathcal{L}_A^{ch} \subset U^{ch}(\mathcal{L}_A^{ch})$. Since both these products, $_{(-n)}$, reflecting the chiral $J_\infty A$-module structure of $\mathcal{L}_A^{ch}$, and $_{[n]}$, reflecting the chiral algebra structure of $U^{ch}(\mathcal{L}_A^{ch})$, sect. 3.21, satisfy the same Borcherds commutator formula, cf. (3.9) and (4.3), their difference satisfies

$$\tau_{[m]}(a_{(-n)}\xi - a_{[-n]})\xi \in \mathcal{L}_A^{ch} \cdot U^{ch}(\mathcal{L}_A^{ch})) = 0 \text{ if } m \geqslant 0.$$

This is a familiar *singular vector* condition. Denote by J the maximal chiral ideal of $U^{ch}(\mathcal{L}_A^{ch})$ generated by all such elements along with the difference $1_A - 1_{U^{ch}}$, where $1_A \in J_\infty A$ and $1_{U^{ch}} \in U^{ch}(\mathcal{L}_A^{ch})$ are units. It is practically obvious that $D^{ch}_{\mathcal{L}_A^{ch}}$ defined to be the quotient $U^{ch}(\mathcal{L}_A^{ch})/J$ is a CDO over A, sect. 4.2, at least if the tangent algebroid T_A is a (locally) free A-module. In fact, the assignment $\mathcal{L}_A \mapsto D^{ch}_{\mathcal{L}_A^{ch}}$ is an equivalence of categories.

All of this is, of course, parallel to sect. 2.5.

Example. — If $\mathcal{L}_A^{ch}$ is $F_1 A[x_{ij}, \partial_{mn};\ j, n < 0]$ introduced in sect. 4.4, then $D^{ch}_{\mathcal{L}_A^{ch}} = A[x_{ij}, \partial_{mn};\ j, n < 0]\}$.

4.8. We shall say that a chiral algebra $\mathcal{V}$ is $\mathbb{Z}$-*graded* if $\mathcal{V} = \oplus_{n\in\mathbb{Z}} \mathcal{V}_n$ s.t. $\mathcal{V}_{n(j)}\mathcal{V}_m \subset \mathcal{V}_{m+n-j-1}$ and $\partial(\mathcal{V}_n) \subset \mathcal{V}_{n+1}$. A similar definition also applies to coisson algebras, sect. 3.9. Here is the origin of this concept.

Let L be a Lie* algebra. We say that L acts on a chiral algebra $\mathcal{V}$ if $\mathcal{V}$ is an L-module such that the chiral bracket $\mu \in P^{ch}_{\{1,2\}}(\{\mathcal{V},\mathcal{V}\},\mathcal{V})$ is L-linear, cf. sect. 4.3 (v) and Remark (1).

Let $\mathcal{V}ec$ be a free $R = \mathbb{C}[\partial]$-module on 1 generator l. Make it into a Lie* algebra by defining a Lie* bracket so that

$$l \otimes l \mapsto -l\partial \otimes 1 + 2l \otimes \partial_1.$$

This is equivalent to saying that $l_{(0)}l = -l\partial$, $l_{(1)}l = 2l$. Call an action of $\mathcal{V}ec$ on $\mathcal{V}$ *nice* if $l^{\mathcal{V}}_{(-1)}v = -v\partial$. For example, the adjoint action of L on itself is nice.

One readily verifies that $\mathcal{V}$ is $\mathbb{Z}$-graded iff $\mathcal{V}$ carries a nice action of $\mathcal{V}ec$ such that the operator $l^{\mathcal{V}}_{(1)}$ is diagonalizable. The equivalence is established by stipulating $\mathcal{V}_n = \{v \text{ s.t. } l^{\mathcal{V}}_{(1)}v = nv\}$.

Notice the (easy to verify) isomorphism $h(\mathcal{V}ec[x]) \xrightarrow{\sim} T_{\mathbb{C}[x]}$ defined by $l \otimes x^n \mapsto -x^n\partial/\partial x$, sect. 3.14, and so the grading operator has the meaning of $-x\partial/\partial x$.

4.9. Now it should be clear what a $\mathbb{Z}$-graded chiral algebroid is; we call it $\mathbb{Z}_+$-graded if $(\mathcal{L}^{ch}_A)_n = \{0\}$ provided $n < 0$. Classification of $\mathbb{Z}_+$-graded chiral algebroids is simpler and more explicit, [9]. We continue under the assumption that T_A is a free A-module with a finite *abelian* basis $\{\tau_i\}$. Denote by $\{\omega_i\} \subset \Omega_A$ the dual basis: $\omega_i(\tau_j) = \delta_{ij}$. In this case there is always an $\mathcal{L}^{ch}_A$ determined by the requirements $\tau_{i(n)}\tau_j = 0$ if $n \geqslant 0$, see sect. 4.4.

Notice that the quasiclassical object $J_\infty A \oplus J_\infty T_A$ is naturally $\mathbb{Z}_+$-graded: place $A \subset J_\infty A$ in degree 0, $T_A \subset J_\infty T_A$ in degree 1, and use the fact that ∂ has degree 1. We seek, therefore, a classification of those $\mathbb{Z}_+$-graded chiral algebroids whose grading induces the indicated one on the quasiclassical object.

Having split $\mathcal{L}^{ch}_A$ into the direct sum $J_\infty A \oplus J_\infty T_A$ as in sect. 4.5, we obtain that a variation of the Lie* bracket is an operation $\alpha(.,.) \in P^{*}_{\{1,2\}}(\{J_\infty T_A, J_\infty T_A\}, J_\infty A)$, which is $J_\infty A$-bilinear. Since $J_\infty T_A$ is a free $\mathbb{C}[\partial]$ module, it is determined by its values on $T_A \subset J_\infty T_A$:

$$\alpha(\xi,\eta) = \sum_{n=0}^{\infty} \alpha_n(\xi,\eta) \otimes \frac{\partial_1^n}{n!},\ \xi,\eta \in T_A.$$

The grading condition demands that at most 2 components may be nonzero:

$$\alpha(\xi,\eta) = \alpha_0(\xi,\eta) \otimes 1 + \alpha_1(\xi,\eta) \otimes \partial_1,$$

where $\alpha_0(\xi,\eta) \in \Omega_A = (J_\infty A)_1$, $\alpha_1(\xi,\eta) \in A = (J_\infty A)_0$. Furthermore, varying the splitting $J_\infty T_A \hookrightarrow (\mathcal{L}_A^{ch})_1$ by sending $\tau_i \mapsto \tau_i - 1/2\sum_j \alpha_1(\tau_i,\tau_j)\omega_j$ ensures that α_1 is 0.

Component α_0, as it stands, is an antisymmetric A-bilinear map from T_A to Ω_A, hence $\alpha_0 \in \Omega_A^2 \otimes_A \Omega_A$. The relation $\xi_{(0)}(\eta_{(1)}\gamma) = (\xi_{(0)}\eta)_{(1)}\gamma + \eta_{(1)}(\xi_{(0)}\gamma)$, which is (3.3) with $n=0$, $m=1$, shows that in fact α_0 is totally antisymmetric and so belongs to Ω_A^3. Finally, the relation $\xi_{(0)}(\eta_{(0)}\gamma) = (\xi_{(0)}\eta)_{(0)}\gamma + \eta_{(0)}(\xi_{(0)}\gamma)$, which is (3.3) with $n=m=0$, shows that α_0 is, moreover, a closed 3-form.

Similarly, a change of splitting $\xi \mapsto \xi + \beta(\xi)$ preserves the grading precisely when $\beta \in \Omega_A \otimes_A \Omega_A$ and the normalization we chose ($\tau_{i(1)}\tau_j = 0$) requires that $\beta \in \Omega_A^2$. The effect of this on α_1 is $\alpha_1 \mapsto \alpha_1 - d_{DR}\beta$.

More formally, the meaning of these computations is as follows. The truncated Chevalley–De Rham complex $C^1_{J_\infty A}(J_\infty T_A, J_\infty A) \longrightarrow C^{2,cl}_{J_\infty A}(J_\infty T_A, J_\infty A)$, introduced in sect. 4.5, is graded, and its degree 0 component, $C^1_{J_\infty A}(J_\infty T_A, J_\infty A)[0] \longrightarrow C^{2,cl}_{J_\infty A}(J_\infty T_A, J_\infty A)[0]$, describes the category of $\mathbb{Z}_+$-graded chiral algebroids. Described above is a map of the truncated De Rham complex $\Omega_A^2 \longrightarrow \Omega_A^{3,cl}$ to $C^1_{J_\infty A}(J_\infty T_A, J_\infty A)[0] \longrightarrow C^{2,cl}_{J_\infty A}(J_\infty T_A, J_\infty A)[0]$; e.g., this map sends

$$\begin{aligned}
\Omega_A^{3,cl} \ni \alpha_0 \quad &\mapsto \quad \alpha \in C^{2,cl}_{J_\infty A}(J_\infty T_A, J_\infty A)[0] \text{ s.t. } \alpha(\xi,\eta) = \alpha_0(\xi,\eta,.)\otimes 1,\\
\Omega_A^2 \ni \beta_0 \quad &\mapsto \quad \beta \in C^1_{J_\infty A}(J_\infty T_A, J_\infty A)[0] \text{ s.t. } \alpha(\xi) = \alpha_0(\xi,.)\otimes 1.
\end{aligned}$$

Analogously to $C^{[1,2>}(J_\infty T_A)$, sect. 4.5, introduce $\Omega_A^{[2,3>}$, the category with objects $\Omega_A^{3,cl}$ and morphisms $\mathrm{Hom}(\alpha_1,\alpha_2) = \{\beta \in \Omega_A^2 \text{ s.t. } \alpha_1 - \alpha_2 = d_{DR}\beta\}$. The map of complexes just defined gives a functor $\Omega_A^{[2,3>} \longrightarrow C^{[1,2>}(J_\infty T_A)[0]$. The point is: *this functor is an equivalence of categories.*

To summarize: *if A is such that T_A is a free A-module with a finite abelian basis, then the category of chiral A-algebroids is a $\Omega_A^{[2,3>}$-torsor.*

4.10. These considerations can be localized so as to obtain, over any smooth X, a gerbe of $\mathbb{Z}_+$-graded CDOs bound by the complex $\Omega_X^2 \longrightarrow \Omega_X^{3,cl}$; this gerbe is locally non-empty. Its characteristic class is $ch_2(T_X)$. The details of this computation can be found in [9]; cf. [5], 3.9.23.

4.11. One can slightly relax the $\mathbb{Z}_+$-graded condition by demanding that a CDO be filtered, i.e., that

$$[T_A, T_A] \subset T_A \otimes 1 \oplus \Omega_A \otimes 1 \oplus A \otimes 1 \oplus A \otimes \partial_1,$$

here the summand $A \otimes 1$ is the one that was prohibited in sect. 4.9. In other words, we allow variations of the form

$$[\xi, \eta]_{\alpha,\beta} = [\xi, \eta] + \alpha(\xi, \eta) + \beta(\xi, \eta),\ \alpha(\xi, \eta) \in \Omega_A,\ \beta(\xi, \eta) \in A.$$

Just as before, one obtains $\alpha(.,.) \in \Omega_A^{3,cl}$, $\beta(.,.) \in \Omega^{2,cl}$, and (provided T_A has an abelian basis) the category of filtered CDOs is an $\Omega_A^{[2,3>} \times \Omega_A^{[1,2>}$-torsor, thereby getting a cross between the Picard-Lie (sect. 2.2 and graded chiral algebroid. This is similar to but different from the concept of a *twisted* CDO introduced (and used) in [1, 2].

Bibliography

[1] Arakawa (T.), Chebotarov (D.), Malikov (F.). — 'Algebras of twisted chiral differential operators and affine localization of g-modules", Selecta Math. New Series, Vol. 17 Issue 1, p. 1-46 (2011).

[2] Arakawa (T.), Malikov (F.). — "A chiral Borel-Weil-Bott theorem", Adv. in Math. Vol. 229, Issue 5, p. 2908-2949 (2012).

[3] Arkhipov (S.), Gaitsgory (D.). — Differential operators on the loop group via chiral algebras,Int. Math. Res. Not. no. 4, p. 165-210 (2002).

[4] Bakalov (B.), Kac (V.G.), Voronov (A.A.). — Cohomology of conformal algebras. Comm. Math. Phys., 200 (1999), pp. 561-598.

[5] Beilinson (A.), Drinfeld (V.). — Chiral algebras. American Mathematical Society Colloquium Publications, 51. American Mathematical Society, Providence, RI, 2004. vi+375 pp. ISBN: 0-8218-3528-9

[6] Borel (A.) et al.. — Algebraic D-modules Academic Press, Inc. (1987).

[7] Borcherds (R.E.). — "Vertex algebras, Kac-Moody algebras, and the Monster." Proceedings of the National Academy of Sciences 83.10 p. 3068-3071 (1986).

[8] Frenkel (E.), Ben-Zvi (D.). — Vertex algebras and algebraic curves. Second edition. Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI (2004).

[9] Gorbounov (V.), Malikov (F.), Schechtman (V.). — Gerbes of chiral differential operators. II. Vertex algebroids, Invent. Math. 155., no. 3, p. 605-680 (2004).

[10] Heluani (R.). — Supersymmetry of the chiral de Rham complex. II. Commuting sectors, Int. Math. Res. Not. 2009, no. 6, p. 953-987.

[11] Hinich (V.). — unpublished manuscript.

[12] Huang (Yi.-Z.), Lepowski (J.). — On the D-module and formal variable approach to vertex operator algebras, in: Topics in geometry, p.175-202, Birkhäuser, Boston (1996).

[13] Kac (V.). — Vertex algebras for beginners. University Lecture Series, 10. American Mathematical Society, Providence, RI (1997). viii+141 pp. ISBN: 0-8218-0643-2

[14] Lada (T.), Markl (M.). — Strongly homotopy Lie algebras, Comm. in Algebra Vol. 23, Issue 6, p. 2147-2161 (1995).

[15] Linshaw (A.), Mathai (V.). — Twisted chiral de Rham complex, generalized geometry, and T-duality, posted on: arXiv:1412.0166

[16] Lada (T.) and Stasheff (J.). — Introduction to sh Lie algebras for physicists. International Journal of Theoretical Physics Vol. 32, No. 7, p. 1087-1103 (1993).

[17] Malikov (F.), Schechtman (V.), Vaintrob (A.). — Chiral de Rham complex, Comm. in Math. Phys. 204, p. 439-473 (1999).