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Central limit theorem through expansion of the propagation of chaos for Bird and Nanbu systems

Sylvain Rubenthaler[(1)]

Résumé. — Les systèmes de Bird et Nanbu sont des systèmes de particules en interaction approchant la solution de l'équation de Boltzmann mollifiée. Ces systèmes vérifient la propagation du chaos. Dans l'esprit de [6, 7, 8], nous utilisons des techniques de couplage pour écrire un développement asymptotique dans la propagation du chaos, en terme du nombre de particules. Ce développement nous permet de démontrer la convergence p.s. de ces systèmes, ainsi qu'un théorème central-limite. Ce théorème central-limite s'applique à la mesure empirique du système. Comme dans [6, 7, 8], ces résultats s'appliquent aux trajectoires des particules sur un intervalle $[0; T]$.

Abstract. — The Bird and Nanbu systems are particle systems used to approximate the solution of the mollified Boltzmann equation. These systems have the propagation of chaos property. Following [6, 7, 8], we use coupling techniques to write a kind of expansion of the error in the propagation of chaos in terms of the number of particles. This expansion enables us to prove the a.s. convergence and the central-limit theorem for these systems. Notably, we obtain a central-limit theorem for the empirical measure of the system. As it is the case in [6, 7, 8], these results apply to the trajectories of particles on an interval $[0, T]$.

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1. Introduction

In [3], we obtained an expansion of the propagation of chaos for a Feynman-Kac particle system (which means, in the context [3], that the particles are interacting through a "selection of the fittest" process). This particle system approximates a particular Feynman-Kac measure, in the sense that the empirical measure associated to the system converges to the Feynman-Kac measure when the number of particles N goes to ∞. What is called propagation of chaos is the following double property of the particle system:

- q particles, amongst the total of N particles, looked upon at a fixed time, are asymptotically independent when $N \to +\infty$ (q is fixed)
- and their law is converging to the Feynman-Kac law.

In [3], we wrote an expansion, in powers of N, of the difference between the law of q independent particles, each of them of the Feynman-Kac law, and the law of q particles coming from the particle system. This expansion can be called a functional representation like in [3]; in the present paper, we call it an expansion of the error in the propagation of chaos. In the setting of [3], the time is discrete. In [3], we showed how to use this kind of expansion to derive a.s. convergence results (p. 824). In [4], we extend the result of [3] to the case where the time is continuous, still in the Feynman-Kac framework, and we establish central-limit theorems for U-statistics of these systems of particles. The proof of the central-limit theorems for U-statistics relies only on the exploitation of the expansion mentioned above.

In this paper, our aim is to establish a similar expansion for a family of particles systems including Bird and Nanbu systems. We do not go as far as obtaining an expansion in the terms of Theorem 1.6 and Corollary 1.8 of [4], but our expansion is sufficient to prove central-limit theorems (Theorem 2.8 and Corollary 2.9). Bird and Nanbu systems are used to approximate the solution of the mollified Boltzmann equation. We refer mainly to [7] and take into account models described in (2.5), (2.6) of [7] (a similar description can be found in [8], Section 3). Another reference paper on the subject is [6]. Our paper is mainly interesting in the following: it provides a sequel to the estimates on propagation of chaos of [7], [8] and it allows to apply the techniques of [3], [4] to Bird and Nanbu systems. In particular:

- In the present paper, we obtain a central-limit theorem for the empirical measure of the system (Th. 2.8) under less assumptions than in [11] Th. 4.2, 4.3. (we only make assumptions that are sufficient to ensure a solution to the problem 2.2 defined in Definition 2.2). Note that the results of [11] hold under the assumption that the operator L, describing the "free" trajectories of the particles (see below), has a certain form, and that its coefficients and their derivatives up to a certain order are bounded (see in particular (H_0'') p. 215 of [11]). These assumptions are stronger than our and are more than what is required to have existence of a solution to 2.2. Note also that the result in [11] is a functional CLT for the empirical process whereas our result is a Gaussian fluctuation field result for the empirical measure, it considers only a finite number of centered real test functions. A result similar to [11] can be found in [19] (with similar assumptions).

- Our convergence results (Theorem 2.7, Theorem 2.8, Corollary 2.9) hold for particles trajectories on any interval $[0, T]$.

Here, the proofs are radically different from those in [4] and this is why we decided to write them in a different paper. In [4], we deal with combinatorial problems related to the particle system studied there whereas in the present paper, we deal with coupling problems.

In Section 2, we will present Bird and Nanbu models, as they can be found in [7] and we will state our main results: Theorem 2.4 is a refinement of the propagation of chaos results for the above-cited models, Theorem 2.7 is an a.s. convergence result for these systems and Theorem 2.8 and Corollary 2.9 are central-limit theorems for these systems. In Section 3, we will introduce various particle systems which will be useful in the proofs and we will prove Th. 2.4. The proof of Th. 2.4 relies on estimates on population growth found in [1] and on coupling ideas. In Section 4, we will prove a convergence result for a particular kind of centered functions (Proposition 2.6), from which we will deduce Corollary 4.7. The kind of result found in Corollary 4.7 is called a Wick-type formula in [3] (see (3.6) p. 807 in [3] and [4], p.15 and Proposition 2.6). Corollary 4.7 and Proposition 2.6 are used in Section 5 to prove Th. 2.7 and Th. 2.8 and Cor. 2.9. Similar results can be found in [2, 15, 18, 20]. We will compare them to our result after the statement of Th. 2.8.

Note that CLT's of the same kind as our can be found in [16, 17]. The equation approximated by particles systems in these papers are quite different from our limit equation.

An important point is that here we want to discuss the mathematical properties of a certain class of particle systems. We will not discuss the physical models. Such a discussion can be found in [8].

2. Definition and main results

2.1. A first particle model

In the following, we deal with particles evolving in $\mathbb{R}^d$. We set the mappings $e_i : h \in \mathbb{R}^d \mapsto e_i(h) = (0,\ldots,0,h,0,\ldots,0) \in \mathbb{R}^{d\times N}$ (h at the i-th rank) ($1 \leqslant i \leqslant N$). We have a Markov generator L and a kernel $\widehat{\mu}(v,w,dh,dk)$ on $\mathbb{R}^{2d}$ which is symmetrical (that is $\widehat{\mu}(v,w,dh,dk) = \widehat{\mu}(w,v,dk,dh)$). We set $\mu(v,w,dh)$ to be the marginal $\widehat{\mu}(v,w,dh\times\mathbb{R}^d)$ up to mass at zero. Our assumptions are the same as in [7]:

Hypothesis 1. —

1. *We suppose that the generator L on $\mathbb{R}^d$ acts on a domain $\mathcal{D}(L)$ of $L^\infty(\mathbb{R}^d)$. (See [7] p. 119 for a discussion on $\mathcal{D}(L)$).*
2. *We suppose* $\sup_{x,a}\widehat{\mu}(x,a,\mathbb{R}^d\times\mathbb{R}^d) \leqslant \Lambda < \infty$.

In Nanbu and Bird systems, the kernel $\widehat{\mu}$ and the generator L have specific features coming from physical considerations. In these systems, the coordinates in $\mathbb{R}^d$ represent the position and speed of molecules. However, these considerations have no effect on our proof. That is why we claim to have a proof for systems more general than Bird and Nanbu systems.

The Nanbu and Bird systems are defined in (2.5) and (2.6) of [7], by the means of integrals over Poisson processes. Here, we give an equivalent definition.

Definition 2.1. — *The particle system described in [7] is denoted by*

$$(\overline{Z}_t)_{t\geqslant 0} = (\overline{Z}^i_t)_{t\geqslant 0, 1\leqslant i\leqslant N} \ .$$

It is a process of N particles in $\mathbb{R}^d$ and can be summarized by the following.

1. *Particles $(\overline{Z}^i_0)_{1\leqslant i\leqslant N}$ in $\mathbb{R}^d$ are drawn i.i.d. at time 0 according to a law $\widetilde{P}_0$.*
2. *Between jump times, the particles evolve independently of each other according to L.*
3. *We have a collection $(N_{i,j})_{1\leqslant i<j\leqslant N}$ of independent Poisson processes of parameter $\Lambda/(N-1)$ (the parameter Λ coming from Hypothesis 1.). For $i>j$, we set $N_{i,j} = N_{j,i}$. If $N_{i,j}$ has a jump at time t, we say that there is an <u>interaction</u> between particles i and j. If there*

is an interaction at time t, then the system undergoes a jump with probability $\frac{\widehat{\mu}(\overline{Z}^i_{t-},\overline{Z}^j_{t-},\mathbb{R}^{2d})}{\Lambda}$:

$$\overline{Z}_t = \overline{Z}_{t-} + e_i(H) + e_j(K)\,,\ \textit{with}\ (H,K) \sim \frac{\widehat{\mu}(\overline{Z}^i_{t-},\overline{Z}^j_{t-},.,.)}{\widehat{\mu}(\overline{Z}^{N,i}_{t-},\overline{Z}^{N,j}_{t-},\mathbb{R}^{2d})} \tag{2.1}$$

(independently of all the other variables).
And with probability $1 - \frac{\widehat{\mu}(\overline{Z}^i_{t-},\overline{Z}^j_{t-},\mathbb{R}^{2d})}{\Lambda}$, there is no jump at time t. We will use $(\overline{Z}^i_{0:t})_{1\leqslant i\leqslant N}$ to denote the system of the trajectories of particles on $[0,t]$ ($\forall t \geqslant 0$), that is for all i: $\overline{Z}^i_{0:t} = (\overline{Z}^i_s)_{0\leqslant s\leqslant t}$. We will use this notation "$0:t$" again in the following for the same purpose.

We denote the Skorohod space of processes in $\mathbb{R}^d$ by $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ (or $\mathbb{D}([0;t],\mathbb{R}^d)$, depending of the domain). As in [6], we define the total variation norm by the following: for all signed measures ν on a measurable space $(S,\mathcal{S})$,

$$\|\nu\|_{TV} = \sup\left\{\int_S f(x)\nu(dx), \|f\|_\infty \leqslant 1\right\}\,.$$

Definition 2.2. — *Let ξ be the canonical process on the Skorohod space $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)$. We say that $\widetilde{P} \in \mathcal{P}(\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d))$ is a solution to the martingale problem 2.2 with initial condition $\widetilde{P}_0$ if, for all $\phi \in \mathcal{D}(L)$ and for all $t \geqslant 0$,*

$$\phi(\xi_t) - \phi(\xi_0) - \int_0^t \int_{a,h\in\mathbb{R}^d} L\phi(\xi_s) + (\phi(\xi_s + h) - \phi(\xi_s))\mu(\xi_s,a,dh)\widetilde{P}_s(da)ds$$

is a $\widetilde{P}$-martingale and the marginal of $\widetilde{P}$ at time 0 is $\widetilde{P}_0$.

In view of the above equation, the reason why the mass of $\mu(v,w,.)$ in zero (for any v,w) is not important is clear. According to Theorem 3.1 of [7], there exists a solution $\widetilde{P}$ of the problem 2.2 defined above (under hypothesis 1). We denote the marginal of $\widetilde{P}$ on $\mathbb{D}([0,T],\mathbb{R}^d)$ by $\widetilde{P}_{0:T}$. We will work with this particular solution in the following. This theorem also proves that (for all q,t):

$$\|\mathcal{L}(\overline{Z}^1_{0:t},\ldots,\overline{Z}^q_{0:t}) - \mathcal{L}(\overline{Z}^1_{0:t})^{\otimes q}\|_{TV} \leqslant 2q(q-1)\frac{\Lambda t + \Lambda^2 t^2}{N-1}\,,$$

and

$$\|\mathcal{L}(\overline{Z}^1_{0:t}) - \widetilde{P}_{0:t}\|_{TV} \leqslant 6\frac{e^{\Lambda t} - 1}{N+1}\,. \tag{2.2}$$

Remark 2.3. — If μ is fixed, there exists different $\widehat{\mu}$'s having the proper marginal (that is, such that $\widehat{\mu}(.,.,.,\mathbb{R}^d) = \mu(.,.,.)$). In fact, it is the choice of $\widehat{\mu}$ that leads to having different systems such as the Bird and Nanbu systems. We refer the reader to [6, 8], [7] p. 119-120 for very good discussions on the difference between the Bird model and the Nanbu model. What matters here is that our result applies to any system satisfying Hypothesis 1. and having jumps of the form (2.1).

We can deduce propagation of chaos from the previous results, that is for all t, for all F bounded measurable,

$$\|\mathcal{L}(\overline{Z}^1_{0:t},\dots,\overline{Z}^q_{0:t})(F) - \widetilde{P}^{\otimes q}_{0:t}(F)\|_{TV} \leqslant \left(2q(q-1)\frac{\Lambda t + \Lambda^2 t^2}{N-1} + 6\frac{e^{\Lambda t}-1}{N+1}\right)\|F\|_\infty .$$

In Theorem 2.4, we will go further than the bound in the equation above by writing an expansion of the left-hand side term in powers of N (see the discussion below Theorem 2.4 concerning the nature of this expansion). We will use techniques introduced in [7]. The main point is that one should look at the processes backward in time.

2.2. Statement of main results

From now on, we will work with a fixed time horizon $T > 0$ and a fixed $q \in \mathbb{N}^*$.

2.2.1. Expansion of the propagation of chaos

We define for any $n, j \in \mathbb{N}^*$, $j \leqslant n$:

$$[n] = \{1,2,\dots,n\},\ \langle j,n\rangle = \{a : [j] \to [n], a \text{ injective }\},\ (n)_j = \#\langle j,n\rangle = \frac{n!}{(n-j)!}.$$

We take $q \in \mathbb{N}^*$ and $T > 0$. Let us set

$$\eta^N_{0:T} = \frac{1}{N}\sum_{1\leqslant i\leqslant N}\delta_{\overline{Z}^i_{0:t}}\ ,\ (\eta^N_{0:T})^{\odot q} = \frac{1}{(N)_q}\sum_{a\in\langle q,N\rangle}\delta_{(\overline{Z}^{a(1)}_{0:T},\dots,\overline{Z}^{a(q)}_{0:T})}\ .$$

For any function $F : \mathbb{D}([0,T],\mathbb{R}^d)^q \to \mathbb{R}$, we call $(\eta^N_{0:T})^{\odot q}(F)$ a U-statistic. Note that for all functions F,

$$\mathbb{E}(F(\overline{Z}^1_{0:T},\dots,\overline{Z}^q_{0:T})) = \mathbb{E}((\eta^N_{0:T})^{\odot q}(F)) \tag{2.3}$$

because $(\overline{Z}^1_{0:T},\dots,\overline{Z}^N_{0:T})$ is exchangeable. We define

$$F_{\text{sym}}(x^1,\dots,x^q) = \frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} F(x^{\sigma(1)},\dots,x^{\sigma(q)})\ ,$$

where the sum is taken over the set $\mathcal{S}_q$ of the permutations of $[q]$. We say that $F : \mathbb{D}([0,T],\mathbb{R}^d)^q \to \mathbb{R}$ is symmetric if for all σ in $\mathcal{S}_q$, $\forall x_1,\dots,x_q \in \mathbb{D}([0,T],\mathbb{R}^d)^q$, $F(x_{\sigma(1)},\dots,x_{\sigma(q)}) = F(x_1,\dots,x_q)$. If F is symmetric then $F_{sym} = F$. Note that for all F,

$$(\eta_{0:T}^N)^{\odot q}(F) = (\eta_{0:T}^N)^{\odot q}(F\mathrm{sym})\,.$$

Theorem 2.4. — *For all $q \geqslant 1$, for any bounded measurable symmetric F, for all $l_0 \geqslant 1$,*

$$\mathbb{E}((\eta_{0:T}^N)^{\odot q}(F)) = \sum_{0\leqslant l\leqslant l_0}\left[\frac{1}{(N-1)^l}\Delta_{q,T}^{N,l}(F)\right] + \frac{1}{(N-1)^{l_0+1}}\overline{\Delta}_{q,T}^{N,l_0+1}(F) \tag{2.4}$$

where the $\Delta_{q,T}^{N,l}$, $\overline{\Delta}_{q,T}^{N,l_0+1}$ are nonnegative measures uniformly bounded in N (defined in Equations (3.11), (3.12)).

We will give a bound on these measures Δ and $\overline{\Delta}$ in (3.14). Let us define $\mathbb{P}_{T,q}^N(F) = \mathbb{E}((\eta_T^N)^{\odot q}(F))$. Regarding the fact that the theorem above is or is not a proper expansion, what we can say is that according to the terminology of [3], p. 782, we cannot say that the sequence of measure $(\mathbb{P}_{T,q}^N)_{N\geqslant 1}$ is differentiable up to any order because the $\Delta_{q,T}^{N,l}$ appearing in the development depend on N.

2.2.2. Convergence results

The main interest of Th. 2.4 is that it gives us sufficient knowledge of the particle system to prove an almost sure convergence result and central-limit theorems. The key is to focus on functions centered in the right way.

Definition 2.5. — *We define a set of "centered" functions:*

$$\mathcal{B}_0^{sym}(q) = \Big\{F : \mathbb{D}([0,T],\mathbb{R}^{qd}) \to \mathbb{R}^+, F \textit{ measurable, symmetric, bounded,} \int_{x_1,\dots,x_q\in\mathbb{D}([0,T],E)} F(x_1,\dots,x_q)\widetilde{P}_{0:T}(dx_q) = 0\Big\}\,.$$

We set (for k even)

$$J_k = \frac{k!}{2^{k/2}(k/2)!}\,. \tag{2.5}$$

(this is the number of partitions of $[k]$ into $k/2$ pairs).

Proposition 2.6. *(Proof in Subsection 4.3). — For $q \geqslant 1$, $F \in \mathcal{B}_0^{sym}(q)$, we have:*

1. *for q odd, $N^{q/2}\mathbb{E}((\eta_{0:T}^N)^{\odot q}(F)) \underset{N\to+\infty}{\longrightarrow} 0$,*
2. *for q even,*

$$N^{q/2}\mathbb{E}((\eta_{0:T}^N)^{\odot q}(F)) \underset{N\to+\infty}{\longrightarrow} \sum_{1\leqslant k\leqslant q/2} J_q \begin{pmatrix} q/2 \\ k \end{pmatrix} (-1)^{\frac{q}{2}-k}$$

$$\times\mathbb{E}[\mathbb{E}_{\widetilde{\mathcal{K}}_T}(F(\widetilde{Z}_{0:T}^1,\dots,\widetilde{Z}_{0:T}^{2k},\widetilde{\widetilde{Z}}_{0:T}^{2k+1},\dots,\widetilde{\widetilde{Z}}_{0:T}^{q})-F(\widetilde{\widetilde{Z}}_{0:T}^{1},\dots,\widetilde{\widetilde{Z}}_{0:T}^{q})|\widetilde{L}_{1,q}) \times \prod_{1\leqslant i\leqslant q/2}\int_0^T \Lambda\widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds] \quad (2.6)$$

where the limit, indeed, does not depend on N (the notations $\widetilde{\widetilde{Z}}$, $\widetilde{\widetilde{Z}}$, $\widetilde{L}$, $\widetilde{K}$, $\mathbb{E}_{\widetilde{\mathcal{K}}_T}$ will be introduced in Subsection 3.2). This limit takes a particular form if $F = (f_1\otimes\cdots\otimes f_q)sym$ (with $f_i \in \mathcal{B}_0^{sym}(1)$, $\forall i$) (see Corollary 4.7).

Using the above Proposition, some combinatorics and Borel-Cantelli Lemma, we prove the following theorem (see the proof in Subsection 5.1).

Theorem 2.7. — *For any measurable bounded f, $T \geqslant 0$,*

$$\eta_{0:T}^N(f) \underset{N\to+\infty}{\overset{a.s.}{\longrightarrow}} \widetilde{P}_{0:T}(f) \ .$$

Using the above results and a computation on characteristic functions, we then prove the following theorem (see the proof in Subsection 5.2).

Theorem 2.8. — *For all $f_1,\dots,f_q \in \mathcal{B}_0^{sym}(1)$, for all $T \geqslant 0$,*

$$N^{q/2}(\eta_{0:T}^N(f_1),\dots,\eta_{0:T}^N(f_q)) \underset{N\to+\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0,K) \ ,$$

(the matrix K is given in (5.27)).

A similar result can be found in [18], under the assumption that the initial law $\widetilde{P}_0$ falls in some particular set. This assumption makes it difficult to compare our covariance K to the ones found in [18] (the expressions of K varies according to the subset $\widetilde{P}_0$ is in). A similar result can also be found in [20], this time for particles moving in a set which can only be countable.

The result in [15] has common points with the theorem above, but the kernel $Q^{(n)}$ defined in [15] is asymmetric. The variance appearing in Th. 2.1 of [15] (Equation (2.7)) could be expressed as an expectation over random trees (if one uses Equation (2.22) of [15]) but the asymmetry of the kernel would make it different from our K anyway. The fact that we do not need an assumption of the kind of (2.4) p. 443 of [15] is another difference between our result and Th. 2.1 of [15]. [2] extends the result of [15] to a case where the jump rate is not bounded (without the second part of our Hypothesis 1.) but it is limited to processes in $\mathbb{Z}_+$.

Using classical techniques, we obtain the following Corollary.

Corollary 2.9. — *For any $q \in \mathbb{N}^*$, F bounded measurable and symmetric, we have*

$$\sqrt{N}\left((\eta_{0:T}^N)^{\odot q}(F) - \widetilde{P}_{0:T}(F)\right) \underset{N\to+\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0, q^2(\widetilde{P}_{0:T}((F^{(1)})^2) + V_{0:T}((F^{(1)})^2))\,,$$

where $F^{(1)}(x_1) = \int_{\mathbb{D}([0,T],\mathbb{R}^d)^{q-1}} F(x_1,\dots,x_q)\widetilde{P}_{0:T}(dx_2,\dots,dx_q)$ and $V_{0:T}$ is defined in (4.19).

3. Other systems of particles

In this section, we introduce the particle systems that we will need for the proofs of the main results.

3.1. Backward point of view

For $\lambda > 0$, we denote by $\mathcal{E}(\lambda)$ the exponential law of parameter λ. For any $x \in \mathbb{R}$, we define $\lfloor x \rfloor := \sup\{i \in \mathbb{Z}, i \leqslant x\}$, $\lceil x \rceil = \inf\{i \in \mathbb{Z}, i \geqslant x\}$.

We intend to construct a system of particles $(Z_{0:T}^i)_{1\leqslant i\leqslant N}$ such that the first q particles have the same law as $(\overline{Z}_{0:T}^1, \dots, \overline{Z}_{0:T}^q)$ (see Lemma 3.4). We use the fact that the processes $(N_{i,j}(T-t))_{0\leqslant t\leqslant T}$ are Poisson processes to construct the interaction graph for the first q particles moving backward in time. The system of particles $(Z_{0:T}^i)_{1\leqslant i\leqslant q}$ is indeed the central system in our paper, hence all other systems will be compared to it.

We start at $s = 0$ with $C_0^i = \{i\}$, for all $i \in [q]$. For $i \in [q]$, we want to define $(C_s^i)_{s\geqslant 0}$, $(K_s^i)_{s\geqslant 0}$ (respectively taking values in $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^*$). We take $(U_k)_{1\leqslant k}, (V_k)_{1\leqslant k}$ i.i.d. $\sim \mathcal{E}(1)$. In all the following, we will use the conventions: $\inf \emptyset = +\infty$ and $(\dots)_+$ is the nonnegative part. The processes (C^i), (K^i) are piecewise constant and make jumps. At any time t, we set $K_t^i = \# C_t^i$. For all $t \in [0,T]$, we set $K_t = \# \left(C_t^1 \cup \dots \cup C_t^q\right)$.

Before considering the technical details, let us explain our purpose. The population $C^1 \cup \dots \cup C^q$ is allowed to get a new particle from $[N] \backslash (C^1 \cup \dots \cup C^q)$ by growing a link to this particle (this particle is chosen uniformly). If such an event happens at time t_0, then the waiting time until the next such event is of law $\mathcal{E}\left(\frac{\Lambda K_{t_0}(N-K_{t_0})_+}{N-1}\right)$. To put it briefly, we say that this kind of event happens at a rate $\frac{\Lambda K_.(N-K_.)_+}{N-1}$. The population $C^1 \cup \dots \cup C^q$ is allowed to form links between particles of $C^1 \cup \dots \cup C^q$ (we will call "loops" these particular links), and this kind of event happens at a rate $\frac{\Lambda K_.(K_.-1)}{2(N-1)}$ (the newly linked particles are chosen uniformly). The processes C^1, ... , C^q are used in Definition 3.1 to define the process $(Z^i)_{1\leqslant i \leqslant N}$. As will be seen below, the link times correspond to the interaction times of some particles.

We define the jump times recursively by $T_0 = 0$ and:

$$\begin{aligned}
T_k' &= \inf\left\{T_{k-1} \leqslant s \leqslant T : (s - T_{k-1}) \times \frac{\Lambda K_{T_{k-1}}(N - K_{T_{k-1}})_+}{N-1} \geqslant U_k\right\} \\
T_k'' &= \inf\left\{T_{k-1} \leqslant s \leqslant T : (s - T_{k-1}) \times \frac{\Lambda K_{T_{k-1}}(K_{T_{k-1}} - 1)}{2(N-1)} \geqslant V_k\right\} \\
T_k &= \inf(T_k', T_k'') \ .
\end{aligned}$$

Here, we use a representation with inf's to emphasize the fact that these jump times are the jump times of Poisson processes with certain intensities. At T_k:

- If $T_k = T_k'$, we draw

$$r(k) \text{ uniformly in } C^1_{T_k-} \cup \dots \cup C^q_{T_k-}\,,\ j(k) \text{ uniformly in } [N]\backslash(C^1_{T_k-} \cup \dots \cup C^q_{T_k-})\,. \tag{3.1}$$

  For any i such that $r(k) \in C^i_{T_k-}$, we then perform the jump: $C^i_{T_k} = C^i_{T_k-} \cup \{j(k)\}$.

  Note that the $(\dots)_+$ in the definition of T_k' above prevents us from being in the situation where we would be looking for $j(k)$ in $\emptyset$.

- If $T_k = T''_k$, we draw

$$r(k) \text{ uniformly in } C^1_{T_k-} \cup \cdots \cup C^q_{T_k-}\,,\; j(k) \text{ uniformly in } C^1_{T_k-} \cup \cdots \cup C^q_{T_k-} \backslash \{r(k)\}\,. \tag{3.2}$$

For each k such that $T_k \leqslant T$, if l_1, l_2 are such that $r(k) \in C^{l_1}_{T_k-}$, $j(k) \in C^{l_2}_{T_k-}$, we say that there is a link between C^{l_1} and C^{l_2}. This whole construction is analogous to the construction of the interaction graph found in [7], p. 122. For all $t \leqslant T$, we set

$$\mathcal{K}_t = \left(K^i_s\right)_{1\leqslant i\leqslant q, 0\leqslant s\leqslant t}\,.$$

Let us now define an auxiliary process $(Z_s)_{0\leqslant s\leqslant T} = (Z^i_s)_{0\leqslant s\leqslant T, 1\leqslant i\leqslant N}$ of N particles in $\mathbb{R}^d$.

Definition 3.1. — *Let $k' = \sup\{k, T_k < \infty\}$. The interaction times of $(Z^i_s)_{1\leqslant s\leqslant T, 1\leqslant i\leqslant N}$ are $T - T_{k'} \leqslant T - T_{k'-1} \leqslant \cdots \leqslant T - T_1$. (We say that the interaction times are defined backward in time.)*

- *$Z^1_0, \dots, Z^N_0$ are i.i.d. $\sim \widetilde{P}_0$*
- *Between the times $(T - T_k)_{k\geqslant 1}$, the Z^i's evolve independently of each other according to the Markov generator L.*
- *At a time $T - T_k$, $(Z^i)_{1\leqslant i\leqslant N}$ undergoes an interaction that has the same law as in Definition 2.1, with (i, j) replaced by $(r(k), j(k))$.*

It is worth noting that for N large and $i \notin [q]$, it is very likely that the particle i has no interaction with the other particles.

For all $0 \leqslant t \leqslant T$, we set

$$L_t = \#\{k \in \mathbb{N} : T_k \leqslant t, T_k = T''_k\}\,.$$

Example 3.2. — Take $q = 2$. Suppose for example, that $T_0 = 0$, $T_1 = T/2$, $T_2 = 3T/4$, $T_3 = +\infty$, $r(1) = 1$, $j(1) = 2$, $r(2) = 2$, $j(2) = 3$.

Then

- for $s \in [0, T/2[$, $K_s = 2$, $L_s = 0$, $K^1_s = K^2_s = 1$,
- for $s \in [T/2, 3T/4[$, $K_s = 2$, $L_s = 1$, $K^1_s = K^2_s = 1$,
- for $s \in [3T/4, T]$, $K_s = 3$, $L_s = 1$, $K^1_s = 1$, $K^2_s = 2$.

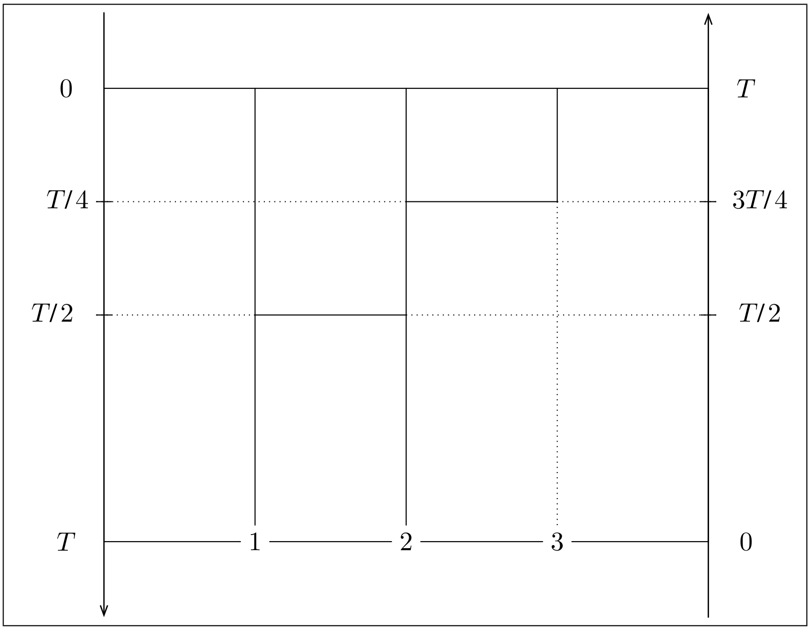

Figure 1. — Interaction graph for $(Z^1_{0:T}, Z^2_{0:T})$

Figure 1 is a pictorial representation of the example above. The time arrow for the particles is on the left. The time arrow for the processes (C^i), (K^i) is on the right. What we draw here is called the graph of interactions (for $Z^1_{0:T}$, $Z^2_{0:T}$) in [7, 8]. Suppose we want to simulate $Z^1_{0:T}$, $Z^2_{0:T}$. We first simulate the interaction times of the system. Suppose that these are exactly $T-T_1$, $T-T_2$ with T_1, T_2 coming from the example above. In Figure 1, solid vertical lines represent the trajectories we have to simulate to obtain $Z^1_{0:T}$, $Z^2_{0:T}$. The particle numbers are to be found at the bottom of the graph. The horizontal solid lines stand for the interaction we have to simulate in order to obtain $Z^1_{0:T}$, $Z^2_{0:T}$ (they may or may not induce jumps for the particles). For example, a horizontal solid line between the vertical solid lines representing the trajectories of $Z^1_{0:T}$, $Z^2_{0:T}$ stands for an interaction between particle 1 and particle 2. The interactions are simulated following Definition 3.1. The trajectory $Z^3_{0:T}$ is represented by a solid line between the times 0 and $T/4$ and by a dashed line between the times $T/4$ and T, with the number 3 at the bottom. As we want to simulate $Z^1_{0:T}$ and $Z^2_{0:T}$ and we have simulated the jumps as in the example above, then we are not interested in Z^3_t for $t > T/4$ and we are not interested in any $Z^i_{0:T}$ with $i \geqslant 4$. Again, the time for the particles should be read on the left.

The following lemma should be kept in mind throughout the whole paper.

Lemma 3.3. — *Let us denote an inhomogeneous Poisson process of rate* $(\lambda_t)_{t\geqslant 0}$ *by* $(N_t^\lambda)_{t\geqslant 0}$ *(λ is supposed to be piecewise constant).*

1. *Let us denote the jump times of* N^λ *by* $\tau_1 < \tau_2 < \dots$. *Then for all* $k \in \mathbb{N}^*$,
$$\mathcal{L}(\tau_1, \tau_2, \dots, \tau_k | \tau_k \leqslant T < \tau_{k+1})$$
is the law of the order statistics of k *independent variables of law of density* $t \mapsto \lambda_t / \int_0^T \lambda_s ds$ *on* $[0, T]$.

2. *For any* $j \in \mathbb{N}^*$, *take piecewise constant processes* $(\alpha_t^j)_{t\geqslant 0}$ *such that for all* t, $0 \leqslant \alpha_t^1 \leqslant \alpha_t^1 + \alpha_t^2 \leqslant \dots \leqslant \alpha_t^1 + \dots + \alpha_t^j \leqslant 1$. *Suppose we take* $(W_k)_{k\geqslant 0}$ *i.i.d. random variables of uniform law on* $[0,1]$ *independent of* N^λ. *Suppose we set* j *processes* $(N_t^i)_{t\geqslant 0, 1\leqslant i\leqslant j}$ *such that* $N_0^i = 0$ *for all* i, *the processes* N^i*'s are a.s. piecewise constant and may jump at the jump times of* N^λ *following this rule:* $\Delta N_t^i = 1$ *if and only if* $\Delta N_t^\lambda = 1$ *and* $\alpha_t^1 + \dots + \alpha_t^{i-1} \leqslant W_{N_{t-}^\lambda} < \alpha_t^1 + \dots + \alpha_t^i$. *Then the* N^i*'s are* j *independent inhomogeneous Poisson processes such that* N^i *has rate* $(\alpha_t^i \times \lambda_t)_{t\geqslant 0}$ *for all* i.

3. *Take* $j \in \mathbb{N}^*$. *Take* j *independent inhomogeneous Poisson processes* $(N^i)_{1\leqslant i\leqslant j}$ *respectively of rate* $(\lambda_t^i)_{t\geqslant 0}$ *(the* λ^i *are piecewise constant processes). Then* $N_t = N_t^1 + \dots + N_t^j$ *is an inhomogeneous Poisson process of rate* $(\lambda_t^1 + \dots + \lambda_t^j)_{t\geqslant 0}$ *and for all* i, s, $\mathbb{P}(\Delta N_t^i = \Delta N_t | \Delta N_t = 1) = \mathbb{P}(\Delta N_t^i = \Delta N_t | \Delta N_t = 1, (N_s^k)_{1\leqslant k\leqslant j, 0\leqslant s<t}) = \frac{\lambda_t^i}{\lambda_t^1 + \dots + \lambda_t^j}$.

The point 1. of the above Lemma is derived from the Mapping Theorem of [9] applied to Cox processes (see [9], p. 17 and 71). The point 2. of the above Lemma is derived from the Coloring Theorem of [9] (see p. 53). The point 3. of the above Lemma is derived from the Superposition Theorem of [9] (see p.16).

We then obtain the following Lemma.

Lemma 3.4. — *For all* $T \geqslant 0$, $(Z_{0:T}^1, \dots, Z_{0:T}^q) \overset{law}{=} (\overline{Z}_{0:T}^1, \dots, \overline{Z}_{0:T}^q)$.

The proof can be found in Section 6.1.

3.2. Auxiliary systems

We now define an auxiliary system $(\widetilde{Z}^i_{0:T})_{i\geqslant 1}$ with an infinite number of particles. We start at $s=0$ with $\widetilde{C}^i_0=\{i\}$, for all $i\in[q]$. For $1\leqslant i\leqslant N$, we define $(\widetilde{C}^i_s)_{s\geqslant 0,1\leqslant i\leqslant q}$, $(\widetilde{K}^i_s)_{s\geqslant 0,1\leqslant i\leqslant q}$ (respectively taking values in $\mathcal{P}(\mathbb{N}),\mathbb{N}$) by the following. The processes $(\widetilde{C}^i),(\widetilde{K}^i)$ are piecewise constant. At any time t, $\widetilde{K}^i_t=\#\widetilde{C}^i_t$.

Before going into the technical details, let us explain the purpose of the construction. We intend to build a process $\widetilde{C}=\widetilde{C}^1\cup\cdots\cup\widetilde{C}^q$ that grows at a rate $\Lambda\widetilde{K}_.$ (by creating links to new particles) and that forms links between two particles of $\widetilde{C}^1\cup\cdots\cup\widetilde{C}^q$ at a rate $\frac{\Lambda\widetilde{K}_.(\widetilde{K}_.-1)}{2(N-1)}$ (in this case, the links will also be called loops). Note that for $k\in\mathbb{N}^*$, $\frac{\Lambda k(N-k)_+}{N-1}\underset{N\to+\infty}{\sim}\Lambda k$, and that, for all N, $\Lambda k\geqslant\frac{\Lambda k(N-k)_+}{N-1}$. Given the processes C^1, ..., C^q, we add to them jumps (and elements) so as to form populations $\widetilde{C}^1$, ..., $\widetilde{C}^q$ with the desired growth rate. It will be easier to write inequalities if the populations C^i and $\widetilde{C}^i$ are coupled (see for example the proof of Theorem 2.4). That is why we use the jump times T_k, T'_k, T''_k in the construction below.

se take $(\widetilde{U}_k)_{k\geqslant 1}$, $(\widetilde{U}'_k)_{k\geqslant 1}$ i.i.d. $\sim\mathcal{E}(1)$. We define the jump times recursively by $\widetilde{T}_0=0$ and

$$\widetilde{T}'_k=\inf\left\{\widetilde{T}_{k-1}\leqslant s\leqslant T,\,(s-\widetilde{T}_{k-1})\left(\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}-\frac{\Lambda K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}\right)\geqslant\widetilde{U}_k\right\}$$

$$\widetilde{T}''_k=\inf\left\{\widetilde{T}_{k-1}\leqslant s\leqslant T,(s-\widetilde{T}_{k-1})\times\frac{\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}(\widetilde{K}_{\widetilde{T}_{k-1}}-1)-\Lambda K_{\widetilde{T}_{k-1}}(K_{\widetilde{T}_{k-1}}-1)}{2(N-1)}\geqslant\widetilde{U}'_k\right\}$$

$$\widetilde{T}_k=\inf(\widetilde{T}'_k,\widetilde{T}''_k,\inf\{T_l:T_l>\widetilde{T}_{k-1}\})$$

(recall the definition of the process (K_t) and the T_k's from Subsection 3.1). Note that $\{T_k,k\geqslant 0\}\subset\{\widetilde{T}_k,k\geqslant 0\}$. At $\widetilde{T}_k$:

- If $\widetilde{T}_k=\widetilde{T}'_k$ (note that it implies that $\widetilde{K}_{\widetilde{T}_k-}-\frac{K_{\widetilde{T}_k-}(N-K_{\widetilde{T}_k-})_+}{N-1}>0$):
  - With probability

$$\frac{K_{\widetilde{T}_k-}-\frac{K_{\widetilde{T}_k-}(N-K_{\widetilde{T}_k-})_+}{N-1}}{\widetilde{K}_{\widetilde{T}_k-}-\frac{K_{\widetilde{T}_k-}(N-K_{\widetilde{T}_k-})_+}{N-1}}\,, \tag{3.3}$$

    we draw $\widetilde{r}(k)$ uniformly in $C^1_{\widetilde{T}_k-}\cup\cdots\cup C^q_{\widetilde{T}_k-}$ and

$$\widetilde{j}(k)=\min\left\{\mathbb{N}^*\backslash(\widetilde{C}^1_{\widetilde{T}_k-}\cup\cdots\cup\widetilde{C}^q_{\widetilde{T}_k-}\cup[N])\right\}.$$

The following jump is performed: for all l such that $\widetilde{r}(k) \in \widetilde{C}^l_{\widetilde{T}_k-}$, $\widetilde{C}^l_{\widetilde{T}_k} = \widetilde{C}^l_{\widetilde{T}_k-} \cup \left\{\widetilde{j}(k)\right\}$.

– With probability

$$\frac{\widetilde{K}_{\widetilde{T}_k} - K_{\widetilde{T}_k-}}{\widetilde{K}_{\widetilde{T}_k-} - \frac{K_{\widetilde{T}_k-}(N-K_{\widetilde{T}_k-})_+}{N-1}}\,, \tag{3.4}$$

we draw $\widetilde{r}(k)$ uniformly in $(\widetilde{C}^1_{\widetilde{T}_k-}\backslash C^1_{\widetilde{T}_k-})\cup\cdots\cup(\widetilde{C}^q_{\widetilde{T}_k-}\backslash C^q_{\widetilde{T}_k-})$ and $\widetilde{j}(k) = \min\left\{\mathbb{N}^*\backslash(\widetilde{C}^1_{\widetilde{T}_k-}\cup\cdots\cup\widetilde{C}^q_{\widetilde{T}_k-}\cup[N])\right\}$. The following jump is performed: for all l such that $\widetilde{r}(k) \in \widetilde{C}^l_{\widetilde{T}_k-}$, $\widetilde{C}^l_{\widetilde{T}_k} = \widetilde{C}^l_{\widetilde{T}_k-} \cup \left\{\widetilde{j}(k)\right\}$.

- If $\widetilde{T}_k = \widetilde{T}''_k$ (note that it implies $\widetilde{K}_{\widetilde{T}_k-} > K_{\widetilde{T}_k-}$):
  – With probability

$$\frac{(\widetilde{K}_{\widetilde{T}_k-} - K_{\widetilde{T}_k-})K_{\widetilde{T}_k-} + (\widetilde{K}_{\widetilde{T}_k-} - K_{\widetilde{T}_k-})(\widetilde{K}_{\widetilde{T}_k-} - K_{\widetilde{T}_k-} - 1)}{\widetilde{K}_{\widetilde{T}_k-}(\widetilde{K}_{\widetilde{T}_k-} - 1) - K_{\widetilde{T}_k-}(K_{\widetilde{T}_k-} - 1)}\,, \tag{3.5}$$

  we draw $\widetilde{r}(k)$ uniformly in $(\widetilde{C}^1_{\widetilde{T}_k-}\backslash C^1_{\widetilde{T}_k-})\cup\cdots\cup(\widetilde{C}^q_{\widetilde{T}_k-}\backslash C^q_{\widetilde{T}_k-})$ and $\widetilde{j}(k)$ uniformly in $(\widetilde{C}^1_{\widetilde{T}_k-}\cup\cdots\cup\widetilde{C}^q_{\widetilde{T}_k-})\backslash\{\widetilde{r}(k)\}$.

  – With probability

$$\frac{(\widetilde{K}_{\widetilde{T}_k-} - K_{\widetilde{T}_k-})K_{\widetilde{T}_k-}}{\widetilde{K}_{\widetilde{T}_k-}(\widetilde{K}_{\widetilde{T}_k-} - 1) - K_{\widetilde{T}_k-}(K_{\widetilde{T}_k-} - 1)}\,, \tag{3.6}$$

  we draw $\widetilde{r}(k)$ uniformly in $(\widetilde{C}^1_{\widetilde{T}_k-}\backslash C^1_{\widetilde{T}_k-})\cup\cdots\cup(\widetilde{C}^q_{\widetilde{T}_k-}\backslash C^q_{\widetilde{T}_k-})$ and $\widetilde{j}(k)$ uniformly in $C^1_{\widetilde{T}_k-}\cup\cdots\cup C^q_{\widetilde{T}_k-}$.

- If $\widetilde{T}_k = T'_l$ for some l, we take $\widetilde{r}(k) = r(l), \widetilde{j}(k) = j(l)$ as in (3.1). The following jump is performed: $\widetilde{C}^l_{\widetilde{T}_k} = \widetilde{C}^l_{\widetilde{T}_k-} \cup \{\widetilde{j}(k)\}$.

- If $\widetilde{T}_k = T''_l$ for some l, we take $\widetilde{r}(k) = r(l)$, $\widetilde{j}(k) = j(l)$ as in (3.2).

We define

$$\widetilde{L}_t = \#\left\{k : \widetilde{T}_k \in \left\{T''_l, \widetilde{T}''_l, l \geqslant 1\right\}, \widetilde{T}_k \leqslant T\right\}\,. \tag{3.7}$$

We set for all $s, t \leqslant T$, $i \in [q]$,

$$\widetilde{K}_s = \#(\widetilde{C}_s^1 + \cdots + \widetilde{C}_s^q) \ ,$$

$$\widetilde{\mathcal{K}}_t = (\widetilde{K}_s^j)_{1 \leqslant j \leqslant q, 0 \leqslant s \leqslant t} \, .$$

Notation 3.5. — *We set* $\mathbb{E}_{\mathcal{K}_t}(\dots) = \mathbb{E}(\dots|\mathcal{K}_t)$, $\mathbb{P}_{\mathcal{K}_t}(\dots) = \mathbb{P}(\dots|\mathcal{K}_t)$, $\mathbb{E}_{\widetilde{\mathcal{K}}_t}(\dots) = \mathbb{E}(\dots|\widetilde{\mathcal{K}}_t)$, $\mathbb{P}_{\widetilde{\mathcal{K}}_t}(\dots) = \mathbb{P}(\dots|\widetilde{\mathcal{K}}_t)$, $\mathbb{E}_{\mathcal{K}_t, \widetilde{\mathcal{K}}_t}(\dots) = \mathbb{E}(\dots|\mathcal{K}_t, \widetilde{\mathcal{K}}_t)$.

For all k, l_1, l_2 such that $\widetilde{r}(k) \in \widetilde{C}^{l_1}_{\widetilde{T}_k-}$, $\widetilde{j}(k) \in \widetilde{C}^{l_2}_{\widetilde{T}_k-}$, we say that there is a link between $\widetilde{C}^{l_1}$ and $\widetilde{C}^{l_2}$. We define for all I subset of $[q]$,

$$\begin{aligned}\mathcal{T}^I = &\left\{\widetilde{T}_k \leqslant T, k \geqslant 0, \, \widetilde{T}_k \in \left\{T'_l, \widetilde{T}'_l, l \geqslant 1\right\}, \widetilde{r}(k) \text{ or } \widetilde{j}(k) \in \cup_{i \in I} \widetilde{C}^i_{\widetilde{T}_k-}\right\} \\ &\cup \left\{\widetilde{T}_k \leqslant T, k \geqslant 0, \, \widetilde{T}_k \in \left\{T''_l, \widetilde{T}''_l, l \geqslant 1\right\}, \widetilde{r}(k) \text{ and } \widetilde{j}(k) \in \cup_{i \in I} \widetilde{C}^i_{\widetilde{T}_k-}\right\} . \end{aligned} \tag{3.8}$$

In other words, $\mathcal{T}^I$ is the set of jump times $\widetilde{T}_k$ such that

- if $\Delta \widetilde{L}_{\widetilde{T}_k} = 0$, $\exists l \in I$ such that $\widetilde{C}^l$ jumps in $\widetilde{T}_k$,
- if $\Delta \widetilde{L}_{\widetilde{T}_k} \neq 0$, $\exists l_1, l_2 \in I$ such that $\widetilde{r}(k) \in \widetilde{C}^{l_1}_{\widetilde{T}_k-}$, $\widetilde{j}(k) \in \widetilde{C}^{l_2}_{\widetilde{T}_k-}$.

We define for all t, j,

$$\widetilde{L}^I_t = \# \left\{ s \in \mathcal{T}^I \, , \, s \leqslant t \, , \, s \in \left\{T''_l, \widetilde{T}''_l, l \geqslant 1\right\}\right\} . \tag{3.9}$$

We have $\widetilde{L}_t = \widetilde{L}^{[q]}_t$. The following lemma is a consequence of Lemma 3.3. Its proof is elementary but quite long. It is written in Section 6.2.

Lemma 3.6. —

1. *The process* $(\widetilde{K}_s)_{0 \leqslant s \leqslant T}$ *is piecewise constant, has jumps of size* 1 *and satisfies, for all* s, t *such that* $0 \leqslant s \leqslant t$,

$$\mathbb{P}(\widetilde{K}_t = \widetilde{K}_s | \widetilde{K}_s) = \exp(-\Lambda(t-s)\widetilde{K}_s) \ .$$

The process $(\widetilde{L}_s)_{0 \leqslant s \leqslant T}$ *is piecewise constant, has jumps of size* 1 *and satisfies for all* s, t *such that* $0 \leqslant s \leqslant t$

$$\mathbb{P}(\widetilde{L}_t = \widetilde{L}_s | \widetilde{L}_s, (\widetilde{K}_u)_{0 \leqslant u \leqslant t}) = \exp\left(-\int_s^t \frac{\Lambda \widetilde{K}_u(\widetilde{K}_u - 1)}{N-1} du\right) .$$

Hence, knowing $(\widetilde{K}_u)_{0 \leqslant u \leqslant T}$, $(\widetilde{L}_t)_{t \geqslant 0}$ *is an inhomogeneous Poisson process of rate* $(\Lambda \widetilde{K}_u(\widetilde{K}_u - 1)/(N-1))_{0 \leqslant u \leqslant T}$.

2. *For all* t, $\widetilde{K}_t \geqslant K_t$ *and* $\widetilde{L}_t \geqslant L_t$ *a.s.* .

3. *If* $T_1 = \widetilde{T}_1, \dots, T_k = \widetilde{T}_k$ *then* $\widetilde{K}_{T_k} = K_{T_k}$, $\widetilde{L}_{T_k} = L_{T_k}$.

4. *The processes* $(\widetilde{K}^i_t)_{0\leqslant s\leqslant T}$ *are independent. They are piecewise constant, have jumps of size* 1 *and satisfy for all* s, t *such that* $0 \leqslant s \leqslant t$, *for all* $i \in [q]$,

$$\mathbb{P}(\widetilde{K}^i_t = \widetilde{K}^i_s | \widetilde{K}^i_s) = \exp(-\Lambda(t-s)\widetilde{K}^i_s)\ .$$

These processes are thus q *independent Yule processes (see [1], p. 102-109, p. 109 for the law of the Yule process). We have for all* $k \geqslant q$,

$$\mathbb{P}(\widetilde{K}_T = k) = \begin{pmatrix} k \\ q-1 \end{pmatrix} (e^{-\Lambda T})^q (1-e^{-\Lambda T})^{k-q}\,. \tag{3.10}$$

5. *Conditionally to* $\widetilde{\mathcal{K}}_T$, *the processes* $\left((\widetilde{L}^{\{2i-1,2i\}}_t)_{0\leqslant t\leqslant T}\right)_{i\in\{1,\dots,q/2\}}$ *are independent non homogeneous Poisson processes of rates, respectively,*

$$\left(\frac{\widetilde{K}^{2i-1}_t \widetilde{K}^{2i}_t}{N-1}\right)_{0\leqslant t\leqslant T}\,.$$

Let us carry on with the definition of $(\widetilde{Z})$.

Definition 3.7. — *Let* $\widetilde{k}' = \sup\left\{k, \widetilde{T}_k < \infty\right\}$. *The interaction times of the* $\widetilde{Z}^i$ *are* $T - \widetilde{T}_{\widetilde{k}'} \leqslant T - \widetilde{T}_{\widetilde{k}'-1} \leqslant \dots \leqslant T - \widetilde{T}_1$.

- *The* $(\widetilde{Z}^i_0)$ *are i.i.d.* $\sim \widetilde{P}_0$.
- *Between the jump times, the* $\widetilde{Z}^i$ *evolve independently of each other according to the Markov generator* L.
- *At a jump time* $T - \widetilde{T}_k$, $(\widetilde{Z})$ *undergoes a jump like in Definition 2.1, (3.), with* i, j *replaced by* $\widetilde{r}(k), \widetilde{j}(k)$.

In so doing, we have coupled the interaction times of the systems $(Z^i_{0:T})_{i\geqslant 0}$, $(\widetilde{Z}^i_{0:T})_{i\geqslant 0}$. *We can couple further and assume that for all* i, $\widetilde{Z}^i_{0:T}$ *and* $Z^i_{0:T}$ *coincide on the event* $\{\widetilde{T}_k, k\geqslant 1\} \cap \left\{\widetilde{T}'_k, k \geqslant 1\right\} = \emptyset$ *(in which case,* $\{T_k, k \geqslant 0\} = \{\widetilde{T}_k, k \geqslant 0\}$*).*

Definition 3.8. — *We define the auxiliary system* $(\widetilde{\widetilde{Z}}_{0:T})_{i\geqslant 0}$ *such that*

- *it has interactions at times* $\{T-\widetilde{T}_k, k\geqslant 1\}\setminus\{T-T''_k, T-\widetilde{T}''_k, k\geqslant 1\}$
- *the rest of the definition is the same as for* $(\widetilde{Z}^i_{0:T})_{i\geqslant 0}$.

In so doing, we have coupled the interaction times of the systems $(\widetilde{Z}^i_{0:T})_{i\geqslant 0}$, $(\widetilde{\widetilde{Z}}^i_{0:T})_{i\geqslant 0}$. *We can couple further and assume that for all* i, $\widetilde{Z}^i_{0:T}$ *and* $\widetilde{\widetilde{Z}}^i_{0:T}$ *coincide on the event* $\{\widetilde{T}_k, k\geqslant 1\}\cap\left\{T''_k, \widetilde{T}''_k, k\geqslant 1\right\}=\emptyset$.

It is worth noting that the laws of $(Z^1_{0:T},\dots,Z^q_{0:T})$ and $(\widetilde{Z}^1_{0:T},\dots,\widetilde{Z}^q_{0:T})$ and $(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})$ are exchangeable.

The q-uple $(\widetilde{Z}^i_{0:T})_{1\leqslant i\leqslant q}$ is obtained from $(Z^i_{0:T})_{1\leqslant i\leqslant q}$ by adding links. The q-uple $(\widetilde{\widetilde{Z}}^i_{0:T})_{1\leqslant i\leqslant q}$ is obtained from $(\widetilde{Z}^i_{0:T})_{1\leqslant i\leqslant q}$ by erasing the loops. When $N\to+\infty$, the probability that $(Z^1_{0:T},\dots,Z^q_{0:T})=(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})$ goes to 1. The law of $(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})$ does not depend on N (see its law in the theorem below). In fact, $(\widetilde{\widetilde{Z}}^i)_{1\leqslant i\leqslant q}$ is the asymptotic object appearing in the limit results (see Proposition 2.6, Corollary 4.7). The following result can be found in [7] (Section 3.4, p. 124) (or, equivalently [6], Section 5).

Theorem 3.9. — *The variable* $\widetilde{\widetilde{Z}}^1_{0:T}$ *has the law* $\widetilde{P}_{0:T}$ *(recall the definition* $\widetilde{P}$ *from below Definition 2.2)*

Suppose $q=2$. The figures 2, 3, 4 are realizations of the interaction graphs for $(Z^1_{0:T},Z^2_{0:T})$, $(\widetilde{Z}^1_{0:T},\widetilde{Z}^2_{0:T})$, $(\widetilde{\widetilde{Z}}^1_{0:T},\widetilde{\widetilde{Z}}^2_{0:T})$, $N=10$, for the same ω.

3.3. Proof of Theorem 2.4 (asymptotic development in the propagation of chaos)

Proof. — By Lemma 3.4, we have for all $l_0\geqslant 0$:

$$\begin{aligned}\mathbb{E}(F(\overline{Z}^1_T,\dots,\overline{Z}^q_T)) &= \sum_{0\leqslant l\leqslant l_0}[\mathbb{E}(F(Z^1_T,\dots,Z^q_T)|L_T=l)\mathbb{P}(L_T=l)]\\ &\quad+\mathbb{E}(F(Z^1_T,\dots,Z^q_T)|L_T\geqslant l_0+1)\\ &\qquad\times\mathbb{P}(L_T\geqslant l_0+1).\end{aligned}$$

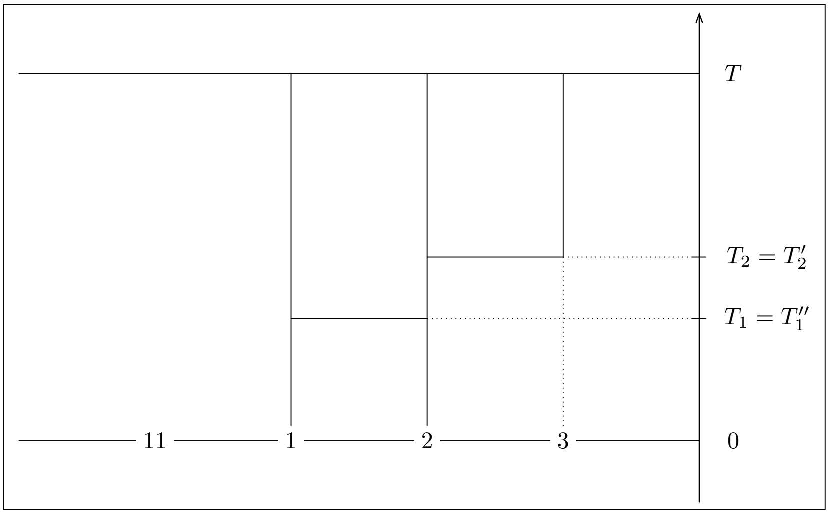

Figure 2. — Interaction graph for $(Z_{0:T}^{N,1}, Z_{0:T}^{N,2})$

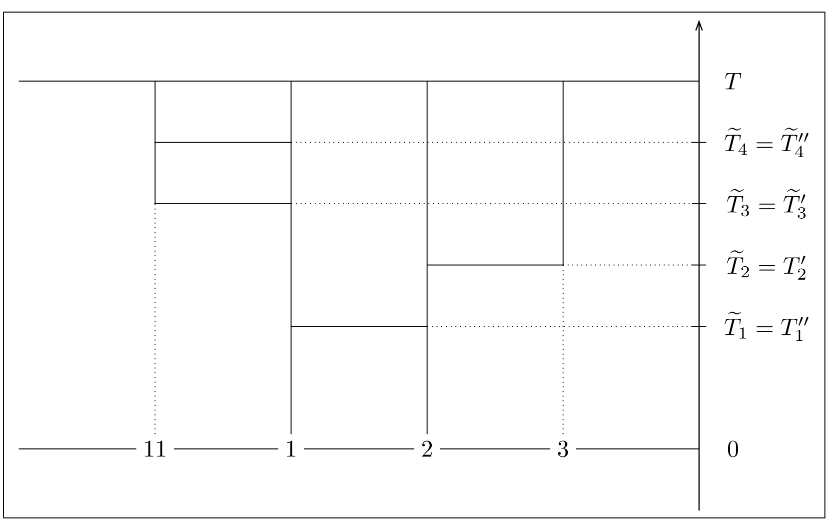

Figure 3. — Interaction graph for $(\widetilde{Z}_{0:T}^{N,1}, \widetilde{Z}_{0:T}^{N,2})$

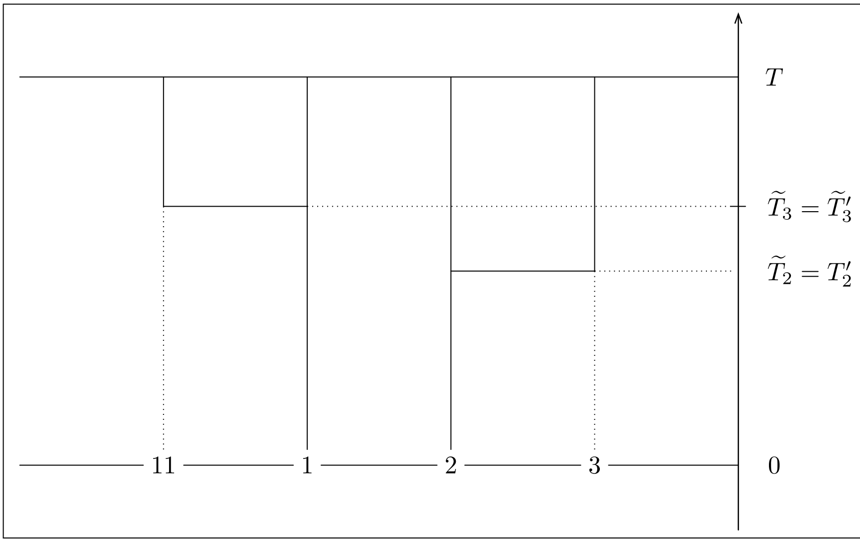

Figure 4. — Interaction graph for $(\widetilde{\widetilde{Z}}{}^{N,1}_{0:T}, \widetilde{\widetilde{Z}}{}^{N,2}_{0:T})$

Hence, we take, for any bounded measurable F,

$$\Delta^{N,l}_{q,T}(F) = \mathbb{E}(F(Z^1_{0:T}, \dots, Z^q_{0:T}) | L_T = l)\mathbb{P}(L_T = l)N^l \tag{3.11}$$

$$\overline{\Delta}^{N,l}_{q,T}(F) = \mathbb{E}(F(Z^1_{0:T}, \dots, Z^q_{0:T}) | L_T \geqslant l)\mathbb{P}(L_T \geqslant l)N^l \ . \tag{3.12}$$

It is sufficient for the proof to show that $\mathbb{P}(L_T \geqslant l)$ is of order $\leqslant 1/N^l$, for all $l \in \mathbb{N}^*$. We write N^λ for an inhomogeneous Poisson process of rate $(\lambda_s)_{s\geqslant 0}$. Using Lemma 3.6, we have

$$\begin{aligned}
\mathbb{P}(L_T \geqslant l) &\leqslant \mathbb{P}(\widetilde{L}_T \geqslant l) \\
&= \mathbb{E}(\mathbb{P}_{\widetilde{\mathcal{K}}_t}(N_T^{\Lambda \widetilde{K}_.(\widetilde{K}_.-1)/(N-1)} \geqslant l)) \\
&\leqslant \mathbb{E}(\mathbb{P}_{\widetilde{\mathcal{K}}_t}(N_T^{\Lambda \widetilde{K}_T(\widetilde{K}_T-1)/(N-1)} \geqslant l) \\
&\leqslant \mathbb{E}\left(\frac{1}{l!}\left(\frac{\Lambda T \widetilde{K}_T(\widetilde{K}_T - 1)}{N-1}\right)^l\right) \ .
\end{aligned} \tag{3.13}$$

And $\mathbb{E}((\widetilde{K}_T)^{2l}) < \infty$ by (3.10) of Lemma 3.6. □

Note that we have the following bounds (for all $F \in C_b^+(\mathbb{D}([0,T], \mathbb{R}^d)^q)$)

$$\sup(\Delta^{N,l}_{q,T}(F), \overline{\Delta}^{N,l}_{q,T}(F)) \leqslant \frac{(\Lambda T)^l}{l!}\mathbb{E}(\widetilde{K}_T^{2l})\|F\|_\infty e^{\frac{l}{N-1}} < \infty \,. \tag{3.14}$$

4. Rate of convergence for centered functions

4.1. Definitions and results

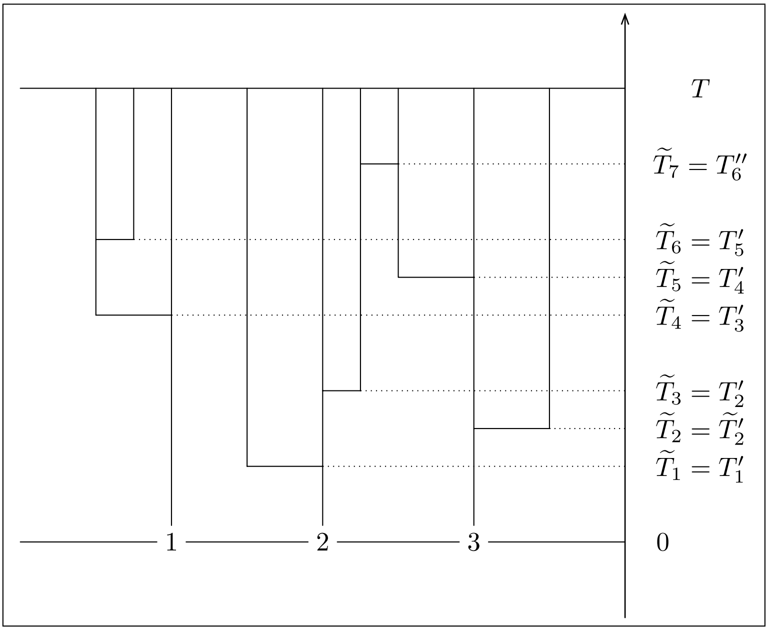

Figure 5. — $q = 3$, interaction graph for $(\widetilde{Z}^1_{0:T}, \widetilde{Z}^2_{0:T}, \widetilde{Z}^3_{0:T})$

For $i \in [q]$, $I \subset [q]$ such that $i \in I$, we define the event

$$A^I_i = \left\{ \left\{ \widetilde{T}_k : k \geqslant 0, \widetilde{T}_k \leqslant T, \widetilde{T}_k \in \mathcal{T}^I, \widetilde{r}(k) \text{ or } \widetilde{j}(k) \in \widetilde{C}^i_{\widetilde{T}_k -} \right\} \subset \{T'_k, k \geqslant 1\} \right\} ,$$

and we set

$$A_i = A^{[q]}_i .$$

Recall the definition of $\mathcal{T}^I$ from (3.8). For all i, l such that $1 \leqslant i < q$, $1 \leqslant l \leqslant q$, we define

$$\widetilde{L}_{i,i+i} = \left\{ \#\Big(\Big\{ \widetilde{T}_k : \widetilde{T}_k \leqslant T, \#\Big(\{\widetilde{r}(k), \widetilde{j}(k)\} \cap (\widetilde{C}^i_{\widetilde{T}_k -} \cup \widetilde{C}^{i+1}_{\widetilde{T}_k -}) \Big) = 2, k \geqslant 0 \Big\} \Big) = 1 \right\},$$

for q even, $\widetilde{L}_{1,q} = \widetilde{L}_{1,2} \cap \cdots \cap \widetilde{L}_{q-1,q} \cap \left\{ \#\Big(\Big\{ \widetilde{T}_k, k \geqslant 1 \Big\} \cap s \left\{ T''_l, \widetilde{T}''_l, l \geqslant 1 \right\} \Big) = q/2 \right\} s,$

$$E^l_T = \# \left(\{\widetilde{T}_k, k \geqslant 1, \widetilde{T}_k \leqslant T : \widetilde{r}(k) \in \widetilde{C}^l_{\widetilde{T}_k -}\} \cap \{\widetilde{T}'_i, i \geqslant 1\} \right) .$$

Note that we do not write anything about $\widetilde{j}$ in the last definition because $\widetilde{r}$ and $\widetilde{j}$ do not play symmetrical roles (see Section 3.2). For a fixed ω, we define an relation on $\mathbb{N}$ by

$$i \bowtie j \text{ if } \exists r \text{ such that } \widetilde{T}_r \in \{\widetilde{T}''_l, T''_l, l \geqslant 1\}$$
$$\text{and } \#\left(\{\widetilde{r}(r), \widetilde{j}(r)\} \cap \left(\widetilde{C}^i_{\widetilde{T}_r-} \cap \widetilde{C}^j_{\widetilde{T}_r-}\right)\right) = 2\,.$$

We extend it into an equivalence relation by imposing $i \bowtie i$, ($i \bowtie j$ and $j \bowtie k$) $\Longrightarrow i \bowtie k$. For all $i \in [k]$, $\mathcal{C}_i$ denote the class of i for the relation $\bowtie$. For $I \subset [q]$, $k \in [q]$, we define

$$L_I = \{\mathcal{C}_{\max(I)} \cap [q] = I\}\,, \tag{4.1}$$

$$L_I^k = L_I \cap (A^I_{\max(I)})^c \cap \left(\cap_{1 \leqslant i \leqslant k, i \in I} (A_i^{[k]})^c\right)$$
$$\cap \left(\cap_{k+1 \leqslant i \leqslant q, i \in I \setminus \max(I)} A_i^{([k] \cup I) \setminus \max(I)}\right)\,. \tag{4.2}$$

Let us have a look at Figure 5 to clarify the notions above. Suppose $\omega \in \Omega$ is such that the graph in Figure 5 occurs. Note that: $\omega \in A_1$, $\omega \in A_2^c$, $\omega \in A_3^c$, $\omega \in A_2^{\{1,2\}}$, $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2,3\}$, $E_T^3 = 1$.

Let us now write some sentences designed to illustrate the meaning of the definitions above. Let $I \subset [q]$. When a time of $\mathcal{T}^I$ is a jump time for $\widetilde{C}^{\widetilde{r}}, \widetilde{C}^{\widetilde{j}}$, we say that a link between $\widetilde{C}^{\widetilde{r}}$ and $\widetilde{C}^{\widetilde{j}}$ is formed at this time; and if $\widetilde{r}, \widetilde{j} \in I$, we say that this is an internal link (or a loop) of $(\widetilde{C}^i)_{i \in I}$. We can define the same notions for the processes $(C^k)_{1 \leqslant k \leqslant q}$. Using this terminology, the event A_l^I (with $l \in I$) is the event that the links happening at a time in $\mathcal{T}^I$ and between $\widetilde{C}^l$ and the rest are external links of $\widetilde{C}^l$ and do not belong to $\{\widetilde{T}'_k, k \geqslant 1\}$. The event $\widetilde{L}_{i,i+1}$ is the event that there is exactly one link between $\widetilde{C}^i$ and $\widetilde{C}^{i+1}$. The event $\widetilde{L}_{1,q}$ is the event that there is exactly one link between $\widetilde{C}^{2i-1}$ and $\widetilde{C}^{2i}$ for all $i \in [q/2]$, and that $\widetilde{L}_T$ is $q/2$. The relation $i \bowtie j$ expresses that $\widetilde{C}^i$ and $\widetilde{C}^j$ are linked by a string of links.

4.2. Technical lemmas

Before going into the proof of Proposition 2.6, we need some technical results.

Lemma 4.1. — *For all $l \in \mathbb{N}^*$,*

$$\mathbb{E}(\widetilde{K}_T^l) \leqslant \frac{(q+l-1)!}{(e^{-\Lambda T})^l (1 - e^{-\Lambda T})(q-1)!}\,.$$

Proof. — By (3.10), with $\alpha = e^{-\Lambda T}$:

$$\begin{aligned}
\mathbb{E}(\widetilde{K}_T^l) &= \sum_{k=q}^{+\infty} k^l \begin{pmatrix} k \\ q-1 \end{pmatrix} \alpha^q (1-\alpha)^{k-q} \\
&\leqslant \frac{\alpha^q}{(1-\alpha)(q-1)!} \sum_{k=q}^{+\infty} (k+l)(k+l-1)\dots(k-q+2)(1-\alpha)^{k-q+1} \\
&\leqslant \frac{\alpha^q}{(1-\alpha)(q-1)!} \frac{(q+l-1)!}{\alpha^{q+l}} .
\end{aligned}$$

□

Lemma 4.2. — *For all $r \in \{0,1,\dots,q\}$, $i_1,\dots,i_r \in [q]$, $l \geqslant 1$,*

$$\mathbb{P}(E_T^{i_1} \geqslant 1, \dots, E_T^{i_r} \geqslant 1, \widetilde{L}_T \geqslant l) \leqslant \frac{(\Lambda T)^{r+l}(q+2r+2l-1)!}{r^r (N-1)^{r+l} (e^{-\Lambda T})^{2r+2l} (1-e^{-\Lambda T})(q-1)! l!} .$$

Proof. — When $\mathcal{K}_T$ is fixed, for any i, the law of $\mathbb{1}_{E_T^i \geqslant 1}$ is Bernoulli of parameter

$$1 - \exp\left(- \int_0^T \Lambda K_t^i - \frac{\Lambda K_t^i (N-K_t)_+}{N-1} dt \right)$$

and $\mathbb{1}_{E_T^{i_1} \geqslant 1}, \dots, \mathbb{1}_{E_T^{i_r} \geqslant 1}$ are independent. So we have for all t, ω,

$$\begin{aligned}
\mathbb{E}_{\mathcal{K}_T}\left(\prod_{j=1}^r \mathbb{1}_{E_T^{i_j} \geqslant 1} \right) &\leqslant \prod_{j=1}^r \frac{\Lambda T K_T^{i_j} K_T}{N-1} \\
\text{(as } \sum_{j=1}^r K_T^{i_j} \leqslant \widetilde{K}_T) &\leqslant \frac{(\Lambda T)^r (\widetilde{K}_T)^{2r}}{r^r (N-1)^r} .
\end{aligned}$$

Basic estimates then leads to $\mathbb{E}_{\widetilde{\mathcal{K}}_T}\left(\prod_{j=1}^r \mathbb{1}_{E_T^{i_j} \geqslant 1} \right) \leqslant \frac{(\Lambda T)^r (\widetilde{K}_T)^{2r}}{r^r (N-1)^r}$. We have, as in (3.13),

$$\mathbb{P}(\widetilde{L}_T \geqslant l | \widetilde{\mathcal{K}}_T) \leqslant \frac{1}{l!} \left(\frac{\Lambda T \widetilde{K}_T (\widetilde{K}_T - 1)}{N-1} \right)^l .$$

Now, as $E_T^{i_1}, \dots, E_T^{i_r}, \widetilde{L}_T$ are independent conditionally to $\widetilde{\mathcal{K}}_T$, by Lemma 4.1, we obtain:

$$\mathbb{P}(E_T^{i_1} \geqslant 1, \dots, E_T^{i_r} \geqslant 1, \widetilde{L}_T \geqslant l) \leqslant \mathbb{E}\left(\frac{(\Lambda T)^{r+l}(\widetilde{K}_T)^{2r+2l}}{r^r(N-1)^{r+l}l!}\right)$$
$$\leqslant \frac{(\Lambda T)^{r+l}(q+2r+2l-1)!}{r^r(N-1)^{r+l}l!(e^{-\Lambda T})^{2r+2l}(1-e^{-\Lambda T})(q-1)!}.$$

□

We write $\mathcal{P}_m$ for the set of m-uples $(I_1, \dots, I_m)$ of subsets of $[q]$ partitioning $[q]$ and such that $\max(I_1) > \max(I_2) > \dots > \max(I_m)$.

Definition 4.3. — *We define the auxiliary particle systems* $(Z_{0:T}^{k,1}, Z_{0:T}^{k,2}, \dots, Z_{0:T}^{k,k})$ *(one system for each $k \in [q]$) by saying that*

- *it has interactions at times* $\{T - T_k, k \geqslant 1\} \cap \left\{T - t, t \in \mathcal{T}^{\{1,k\}}\right\}$,
- *the rest of the definition is the same as for* $(\widetilde{Z}_{0:T}^i)$.

In so doing, we have coupled the interaction times of $(Z_{0:T}^{k,i})_{1\leqslant i\leqslant k}$ *with the interaction times of the other systems. We can couple further and assume that* $(Z_{0:T}^1, \dots, Z_{0:T}^k)$ *and* $(Z_{0:T}^{k,1}, \dots, Z_{0:T}^{k,k})$ *coincide on the event* $\{T - T_k, k \geqslant 1\} \cap \left\{T - t, t \in \mathcal{T}^{\{1,k\}}\right\} = \{T - T_k, k \geqslant 1\}$. *Note that for all k, the law of* $(Z_{0:T}^{k,1}, \dots, Z_{0:T}^{k,k})$ *is exchangeable.*

Lemma 4.4. — *If q is odd then for all $k \in \{0, \dots, q\}$, for all $F \in \mathcal{B}_0^{sym}(q)$, $m \in [q]$, $(I_1, \dots, I_m) \in \mathcal{P}_m$,*

$$N^{q/2}\mathbb{E}(F(Z_{0:T}^{k,1}, \dots, Z_{0:T}^{k,k}, \widetilde{\widetilde{Z}}_{0:T}^{k+1}, \dots, \widetilde{\widetilde{Z}}_{0:T}^{q}) \prod_{i\in[m]} \mathbb{1}_{L_{I_i}^k}) \underset{N\to+\infty}{\longrightarrow} 0,$$

(we use the following convention: in the case $k = 0$,
$\mathbb{E}(F(Z_{0:T}^{k,1}, \dots, Z_{0:T}^{k,k}, \widetilde{\widetilde{Z}}_{0:T}^{k+1}, \dots, \widetilde{\widetilde{Z}}_{0:T}^{q})) = \mathbb{E}(\widetilde{\widetilde{Z}}_{0:T}^{1}, \dots, \widetilde{\widetilde{Z}}_{0:T}^{q})$*).*

If q is an even integer and $k \in [q]$,

$$N^{q/2}\mathbb{E}(F(Z_{0:T}^{k,1}, \dots, Z_{0:T}^{k,k}, \widetilde{\widetilde{Z}}_{0:T}^{k+1}, \dots, \widetilde{\widetilde{Z}}_{0:T}^{q}) \prod_{i\in[m]} \mathbb{1}_{L_{I_i}^k})$$
$$\underset{N\to+\infty}{\longrightarrow} \mathbb{E}\left(\mathbb{E}_{\widetilde{\mathcal{K}}_T}\left(F(\widetilde{Z}_{0:T}^1, \dots, \widetilde{Z}_{0:T}^k, \widetilde{\widetilde{Z}}_{0:T}^{k+1}, \dots, \widetilde{\widetilde{Z}}_{0:T}^{q})|\widetilde{L}_{1,q}\right.\right.$$
$$\left.\times \prod_{i=1}^{q/2}\int_0^T \Lambda \widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds\right) \qquad (4.3)$$

if

$$\forall j\,,\ \#I_j = 2\ and (I_j \subset [k]\ \ or\ I_j \subset \{k+1,\dots,q\})\,, \tag{4.4}$$

and

$$N^{q/2}\mathbb{E}(F(Z^{k,1}_{0:T},\dots,Z^{k,k}_{,0:T},\widetilde{\widetilde{Z}}{}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}{}^{q}_{0:T})\prod_{i\in[m]}\mathbb{1}_{L^k_{I_i}})\underset{N\to+\infty}{\longrightarrow}0$$

otherwise.

Moreover, these limits still hold if we replace $(Z^{k,1}_{0:T},\dots,Z^{k,k}_{0:T})$ *by* $(\widetilde{\widetilde{Z}}{}^{1}_{0:T},\dots,\widetilde{\widetilde{Z}}{}^{k}_{0:T})$ *and* $(\widetilde{Z}^{1}_{0:T},\dots,\widetilde{Z}^{k}_{0:T})$ *by* $(\widetilde{\widetilde{Z}}{}^{1}_{0:T},\dots,\widetilde{\widetilde{Z}}{}^{k}_{0:T})$ *in the formulas above.*

Note hat the expectation in (4.3) does not depend on N and that it cannot be simplified by the use of the tower formula because the conditional expectation is conditional with respect to $\widetilde{\mathcal{K}}_T, \widetilde{L}_{1,q}$.

Proof. — We define the event

$$B=\bigcup_{1\leqslant r\leqslant q}\ \bigcup_{1\leqslant i_1<\dots<i_r\leqslant q}\left(\left(\cap_{j=1}^r\{E^{i_j}_T\geqslant 1\}\right)\cap\{\widetilde{L}_T\geqslant\left\lceil\frac{q-r}{2}\right\rceil\}\right)\,. \tag{4.5}$$

Suppose that q is odd or that q is even and (4.4) does not hold. Then for all $m\in[q]$, $(I_1,\dots,I_m)\in\mathcal{P}_m$, $k\in[q]$,

$$\bigcap_{1\leqslant j\leqslant m}L^k_{I_j}\subset\left(B\cup\{\widetilde{L}_T\geqslant\frac{q}{2}+1\}\right)\,. \tag{4.6}$$

Let us write a short justification of the last formula. Suppose that $m\in[q]$, $(I_1,\dots,I_m)\in\mathcal{P}_m$, $\omega\in\bigcap_{1\leqslant j\leqslant m}L^k_{I_j}$. We note that, for all $I\subset[q]$, $\omega\in L_I\Rightarrow\widetilde{L}^I_T(\omega)\geqslant\#I-1$ (recall the definition of $\widetilde{L}^I_T$ from (3.9)). We note also that $\left(\omega\in L^k_I\text{ with }I=\{i\}\right)\Rightarrow E^i_T(\omega)\geqslant 1$. Indeed, for such an ω, $\omega\in\left(A^{\{i\}}_i\right)^c$ so there exists a $\widetilde{T}_r\in\mathcal{T}^{\{i\}}$ such that $\widetilde{T}_r\notin\{T'_l,l\geqslant 1\}$; as $\omega\in L_{\{i\}}$, i is not linked to any i' in $[q]$ by the relation $\bowtie$, so $E^i_T(\omega)\geqslant 1$. So, we have (4.6) for q odd. If q is even and we do not have (4.4), then (by the same reasoning as above):

- If there exists j such that $\#I_j\neq 2$, then $\omega\in B\cup\{\widetilde{L}_T\geqslant\frac{q}{2}+1\}$.

- If $\#I_j = 2$, for all j, and, say, for some l, $[k]\cap I_l \neq \emptyset$, $\{k+1,\dots,q\}\cap I_l \neq \emptyset$, then $\sum_{j=1}^m \widetilde{L}_T^{I_j} \geqslant q/2$. Let $i_0 = \min(I_l)$. As $\omega \in L_{I_l}^k$, we have $\omega \in (A_{i_0}^{[k]})^c$, so: either $E_T^{i_0} \geqslant 1$, either there exists $r \neq i_0$, $r\in[k]$ such that $\widetilde{L}_T^{\{r,i_0\}} \geqslant 1$. So, $\omega \in B \cup \{\widetilde{L}_T \geqslant \frac{q}{2}+1\}$.

Using Lemma 4.2 and Equation (4.6), we then have

$$N^{q/2}\mathbb{P}(\bigcap_{1\leqslant j\leqslant m} L_{I_j}^k) \underset{N\to+\infty}{\longrightarrow} 0\,.$$

Now, suppose that q is even and that (4.4) holds. Note that $(Z_{0:T}^{k,1},\dots,Z_{0:T}^{k,k})$ and $(\widetilde{Z}_{0:T}^1,\dots,\widetilde{Z}_{0:T}^k)$ coincide on the event $\widetilde{L}_{1,q}\cap\{E_T^1=\dots=E_T^k=0\}$. Then, using the symmetries of the problem and Lemma 4.2, we see that the limit we are looking for is the same as

$$\lim_{N\to+\infty} N^{q/2}\mathbb{E}(F(Z_{0:T}^{k,1},\dots,Z_{0:T}^{k,k},\widetilde{\widetilde{Z}}_{0:T}^{k+1},\dots,\widetilde{\widetilde{Z}}_{0:T}^{q})\mathbb{1}_{\widetilde{L}_{1,q}})$$
$$= \lim_{N\to+\infty} N^{q/2}\mathbb{E}(F(\widetilde{Z}_{0:T}^{1},\dots,\widetilde{Z}_{0:T}^{k},\widetilde{\widetilde{Z}}_{0:T}^{k+1},\dots,\widetilde{\widetilde{Z}}_{0:T}^{q})\mathbb{1}_{\widetilde{L}_{1,q}})\,.$$

We set for all j,t

$$\widetilde{L}'_{1,q} = \left\{\widetilde{L}_T^{\{1,2\}} \geqslant 1\right\}\cap\dots\cap\left\{\widetilde{L}_T^{\{q-1,q\}}\geqslant 1\right\}\,.$$

We set for all $j\in[q/2]$

$$\alpha(2j-1,2j) = \exp\left(-\int_0^T \frac{\Lambda \widetilde{K}_s^{2j-1}\widetilde{K}_s^{2j}}{N-1}ds\right)\,.$$

We have

$$\begin{aligned} N^{q/2}\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}'_{1,q}) &= \prod_{1\leqslant j\leqslant q/2}\left[N(1-\alpha(2j-1,2j))\right] \qquad (4.7)\\ &\underset{N\to+\infty}{\overset{\text{a.s.}}{\longrightarrow}} \prod_{1\leqslant j\leqslant q/2}\int_0^T \Lambda\widetilde{K}_s^{2j-1}\widetilde{K}_s^{2j}ds\,. \end{aligned}$$

We have

$$\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}'_{1,q}\backslash\widetilde{L}_{1,q}) \leqslant \mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}_T > q/2)\,.$$

So, by Lemma 4.2, $N^{q/2}\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}'_{1,q}\backslash\widetilde{L}_{1,q}) \underset{N\to+\infty}{\overset{\text{a.s.}}{\longrightarrow}} 0$. And so :

$$N^{q/2}\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}_{1,q}) \underset{N\to+\infty}{\overset{\text{a.s.}}{\longrightarrow}} \prod_{1\leqslant i\leqslant q/2}\int_0^T \Lambda\widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds\,.$$

Now,

$$\begin{aligned} N^{q/2}\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}_{1,q}) &\leqslant N^{q/2}\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}'_{1,q}) \\ \text{(by (4.7))} &\leqslant \left(\frac{N}{N-1}\right)^{q/2} \prod_{1\leqslant j\leqslant q/2} \int_0^T \Lambda \widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i} ds \\ &\leqslant 2^{q/2}T^{q/2}\Lambda^{q/2}(\widetilde{K}_T)^q , \end{aligned}$$

which is of finite expectation by (3.10). Thus, using the dominated convergence Theorem, we obtain

$$\lim_{N\to+\infty} \mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(F(\widetilde{Z}^1_{0:T},\dots,\widetilde{Z}^k_{0:T},\widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})|\widetilde{L}_{1,q})N^{q/2}\mathbb{P}_{\widetilde{\mathcal{K}}_T}(\widetilde{L}_{1,q}))$$
$$= \mathbb{E}\left(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(F(\widetilde{Z}^1_{0:T},\dots,\widetilde{Z}^k_{0:T},\widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})|\widetilde{L}_{1,q})\prod_{i=1}^{q/2}\int_0^T \Lambda\widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds\right) .$$

By Lemma 3.3, 1. and Lemma 3.6, 5., the law of

$$\mathbb{E}_{\widetilde{\mathcal{K}}_T}(F(\widetilde{Z}^1_{0:T},\dots,\widetilde{Z}^k_{0:T},\widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})|\widetilde{L}_{1,q})$$

does not depend on N.

We do not write the proof of the last point of the lemma because it is very similar to what is written above. □

4.3. Proof of Proposition 2.6

Proof. — Since $Z^i_{0:T} = \widetilde{\widetilde{Z}}^i_{0:T}$ on A_i (for all i), we have

$$\begin{aligned} &\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})) = \\ &\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) + \mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{A_1\cap\dots\cap A_q}) = \\ &\quad \mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) + \mathbb{E}(F(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})) \\ &\qquad - \mathbb{E}(F(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) = \\ &\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) - \mathbb{E}(F(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) , \end{aligned} \tag{4.8}$$

where, to obtain $\mathbb{E}(F(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})) = 0$, we have used the fact that $\widetilde{\widetilde{Z}}^q_{0:T}$ is independent of $\widetilde{\widetilde{Z}}^1_{0:T}\dots,\widetilde{\widetilde{Z}}^{q-1}_{0:T}$, and that $\int_{\mathbb{D}([0,T],\mathbb{R}^d)} F(z_1,\dots z_q)\widetilde{P}_{0:T}(dz_q)$

$= 0$, for all $z_1, \dots, z_{q-1}$. This kind of reasoning will be used again in the following. We have (using the symmetry of the problem)

$$\mathbb{E}(F(Z^1_{0:T}, \dots, Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) = \sum_{k=1}^{q}\binom{q}{k}\mathbb{E}(F(Z^1_{0:T}, \dots, Z^q_{0:T})\mathbb{1}_{A_1^c}\dots\mathbb{1}_{A_k^c}\mathbb{1}_{A_{k+1}}\dots\mathbb{1}_{A_q})$$

$$= \sum_{k=1}^{q}\binom{q}{k}\mathbb{E}(F(Z^{k,1}_{0:T},\dots, Z^{k,k}_{0:T}, \widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots, \widetilde{\widetilde{Z}}^{q}_{0:T})\mathbb{1}_{(A_1^{[k]})^c}\dots\mathbb{1}_{(A_k^{[k]})^c}\mathbb{1}_{A_{k+1}}\dots\mathbb{1}_{A_q}).$$

For all $H_1 \subset H_2 \subset H_3 \subset [q]$, such that $H_3\backslash H_2 \neq \emptyset$ (the equality being otherwise obvious), we have

$$\cap_{i\in H_3\backslash H_1} A_i^{H_3} = \left(\cap_{i\in H_2\backslash H_1} A_i^{H_2}\right)\cap\left(\cap_{i\in H_3\backslash H_2} A_i^{H_3}\right). \tag{4.9}$$

This will be used many times in this proof. We obtain already (with $H_1 = [k]$, $H_2 = [q-1]$, $H_3 = [q]$)

$$\mathbb{E}(F(Z^1_{0:T}, \dots, Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c}) = \mathbb{E}(F(Z^1_{0:T}, \dots, Z^q_{0:T})\mathbb{1}_{A_1^c\cap\dots\cap A_q^c} + \sum_{k=1}^{q-1}\binom{q}{k}\mathbb{E}(F(Z^{k,1}_{0:T},\dots, Z^{k,k}_{0:T}, \widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots, \widetilde{\widetilde{Z}}^{q}_{0:T})$$
$$\times\prod_{i=1}^{k}\mathbb{1}_{(A_i^{[k]})^c}\prod_{i=k+1}^{q-1}\mathbb{1}_{A_i^{[q-1]}}\times(1-\mathbb{1}_{A_q^c})). \tag{4.10}$$

We look at one term in the last sum above for a fixed $k \leqslant q-1$. As $\widetilde{\widetilde{Z}}^{q}_{0:T}$ is independent of $Z^{k,1}_{0:T},\dots, Z^{k,k}_{0:T}, \widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots, \widetilde{\widetilde{Z}}^{q-1}_{0:T}$ and $\mathbb{1}_{A_1^{[k]}}, \dots, \mathbb{1}_{A_k^{[k]}}, \mathbb{1}_{A_{k+1}^{[q-1]}}, \dots, \mathbb{1}_{A_{q-1}^{[q-1]}}$, we obtain that this quantity is equal to

$$\mathbb{E}(F(Z^{k,1}_{0:T},\dots, Z^{k,k}_{0:T}, \widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots, \widetilde{\widetilde{Z}}^{q}_{0:T})\mathbb{1}_{(A_1^{[k]})^c}\dots\mathbb{1}_{(A_k^{[k]})^c}\times\mathbb{1}_{A_{k+1}^{[q-1]}}\dots\mathbb{1}_{A_{q-1}^{[q-1]}}(-\mathbb{1}_{A_q^c})). \tag{4.11}$$

Let us set $i_1 = q$. Let $I_1 \subset [q]$. Recall the definition of $L^k_{I_1}$ from (4.2) and the definition of L_{I_1} from (4.1). For all $I_1 \subset [q]$ such that $q \in I_1$, we have

$$A_q^c \cap L_{I_1} = (A_q^{I_1})^c \cap L_{I_1}, \tag{4.12}$$

then (using (4.9) with $H_1 = [k]$, $H_2 = [k] \cup (I_1 \cap \{k+1, \dots, q-1\})$, $H_3 = [q-1]$)

$$\cap_{k+1\leqslant i\leqslant q-1}A_i^{[q-1]}=\left(\cap_{i\in I_1\cap\{k+1,\dots,q-1\}}A_i^{[k]\cup(I_1\cap\{k+1,\dots,q-1\})}\right)$$
$$\cap\left(\cap_{k+1\leqslant i\leqslant q-1,i\notin I_1}A_i^{[q-1]}\right)\,. \tag{4.13}$$

We have:

$$\left(\cap_{1\leqslant i\leqslant k}(A_i^{[k]})^c\right)\cap\left(\cap_{k+1\leqslant i\leqslant q-1}A_i^{[q-1]}\right)\cap A_q^c=$$
$$\bigsqcup_{I_1\subset[q]\,,\,q\in I_1}\left[\left(\cap_{1\leqslant i\leqslant k}(A_i^{[k]})^c\right)\cap\left(\cap_{k+1\leqslant i\leqslant q-1}A_i^{[q-1]}\right)\cap A_q^c\cap L_{I_1}\right]\,. \tag{4.14}$$

(the symbol $\sqcup$ means "disjoint union"). For each term in the union above, we have (using (4.12), (4.13))

$$\left(\cap_{1\leqslant i\leqslant k}(A_i^{[k]})^c\right)\cap\left(\cap_{k+1\leqslant i\leqslant q-1}A_i^{[q-1]}\right)\cap A_q^c\cap L_{I_1}$$
$$=\left(\cap_{i\in[k],i\notin I_1}(A_i^{[k]})^c\right)\cap\left(\cap_{k+1\leqslant i\leqslant q-1,i\notin I_1}A_i^{[q-1]}\right)\cap L_{I_1}^k\,. \tag{4.15}$$

For each $I_1\subset[q]$ such that $q\in I_1$, we set $i_2=\max([q]\backslash I_1)$, if it exists.

If $i_2\geqslant k+1$, we can then write (using (4.9) with $H_3=[q-1]$, $H_2=[q-1]\backslash\{i_2\}$, $H_1=[k]\cup(I_1\cap[q-1])$)

$$\left(\cap_{i\in[k],i\notin I_1}(A_i^{[k]})^c\right)\cap\left(\cap_{k+1\leqslant i\leqslant q-1,i\notin I_1}A_i^{[q-1]}\right)\cap L_{I_1}^k=$$
$$\left(\cap_{i\in[k],i\notin I_1}(A_i^{[k]})^c\right)\cap\left(\cap_{k+1\leqslant i\leqslant q-1,i\notin(I_1\cup\{i_2\})}A_i^{[q-1]\backslash\{i_2\}}\right)\cap A_{i_2}^{[q-1]}\cap L_{I_1}^k\,. \tag{4.16}$$

$\widetilde{\widetilde{Z}}{}^{i_2}_{0:T}$ is independent of $Z_{0:T}^{k,1}$, … , $Z_{0:T}^{k,k}$, $(\widetilde{\widetilde{Z}}{}^{j}_{0:T})_{j\in\{k+1,\dots,q\}\backslash\{i_2\}}$, $\mathbb{1}_{L_{I_1}^k}$, $(\mathbb{1}_{(A_j^{[k]})^c})_{1\leqslant j\leqslant k,j\notin I_1}$, $(\mathbb{1}_{A_j^{[q-1]\backslash\{i_2\}}})_{k+1\leqslant j\leqslant q-1,j\notin\{I_1\cup\{i_2\}\}}$. So, for I_1 such that $i_2\geqslant k+1$, we obtain (using that $F\in\mathcal{B}_0^{sym}(q)$) and (4.15), (4.16))

$$\mathbb{E}(F(Z_{0:T}^{k,1},\dots,Z_{0:T}^{k,k},\widetilde{\widetilde{Z}}{}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}{}^{q}_{0:T})\prod_{i\in[k]}\mathbb{1}_{(A_i^{[k]})^c}\prod_{k+1\leqslant i\leqslant q-1}\mathbb{1}_{A_i^{[q-1]}}\times\mathbb{1}_{A_q^c}\mathbb{1}_{L_{I_1}})=$$
$$\mathbb{E}(F(\dots)\prod_{i\in[k],i\notin I_1}\mathbb{1}_{(A_i^{[k]})^c}\prod_{k+1\leqslant i\leqslant q-1\,,\,i\notin I_1\cup\{i_2\}}\mathbb{1}_{A_i^{[q-1]\backslash\{i_2\}}}\times(1-\mathbb{1}_{(A_{i_2}^{[q-1]})^c})\mathbb{1}_{L_{I_1}^k})=$$
$$-\mathbb{E}(F(\dots)\prod_{i\in[k],i\notin I_1}\mathbb{1}_{(A_i^{[k]})^c}\prod_{k+1\leqslant i\leqslant q-1\,,\,i\notin I_1\cup\{i_2\}}\mathbb{1}_{A_i^{[q-1]\backslash\{i_2\}}}\times\mathbb{1}_{(A_{i_2}^{[q-1]})^c}\mathbb{1}_{L_{I_1}^k})\,.$$

For I_1 such that $i_2 \in [k]$, we have (using (4.15))

$$\mathbb{E}(F(Z^{k,1}_{0:T},\dots,Z^{k,k}_{0:T},\widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})\prod_{i\in[k]}\mathbb{1}_{(A_i^{[k]})^c}\prod_{k+1\leqslant i\leqslant q-1}\mathbb{1}_{A_i^{[q-1]}}\times\mathbb{1}_{A_q^c}\mathbb{1}_{L_{I_1}})$$
$$=\mathbb{E}(F(\dots)\prod_{i\in[k],i\notin I_1}\mathbb{1}_{(A_i^{[k]})^c}\times\mathbb{1}_{L^k_{I_1}})\,.$$

Recall $\mathcal{P}_m$ defined in Section 4.2. Starting from (4.10), (4.11), (4.14), and proceeding recursively, we obtain that

$$\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c})=\sum_{k=1}^{q}\sum_{m=1}^{q}\binom{q}{k}\sum_{(I_1,\dots,I_m)\in\mathcal{P}_m}(-1)^{s(I_1,\dots,I_m)}$$
$$\times\,\mathbb{E}(F(Z^{k,1}_{0:T},\dots,Z^{k,k}_{0:T},\widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})\prod_{j\in[m]}\mathbb{1}_{L^k_{I_j}})\,,\quad(4.17)$$

where $s(I_1,\dots,I_m)=\#\{j\in[m],\max(I_j)\geqslant k+1\}$. And in the same way

$$\mathbb{E}(F(\widetilde{\widetilde{Z}}^{1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c})=$$
$$\sum_{k=1}^{q}\sum_{m=1}^{q}\binom{q}{k}\sum_{(I_1,\dots,I_m)\in\mathcal{P}_m}(-1)^{s(I_1,\dots,I_m)}\times\mathbb{E}(F(\widetilde{\widetilde{Z}}^{1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})\prod_{j\in[m]}\mathbb{1}_{L^k_{I_j}})\,,\quad(4.18)$$

By Lemma 4.4, we then obtain that, for q even, using the symmetry of the problem (recall the definition of J_k from (2.5)),

$$N^{q/2}\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c})\underset{N\to+\infty}{\longrightarrow}$$
$$\sum_{1\leqslant k\leqslant q\,,\,k\text{ even}}(-1)^{\frac{q-k}{2}}\begin{pmatrix}q\\k\end{pmatrix}J_kJ_{q-k}$$
$$\times\mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(F(\widetilde{Z}^1_{0:T},\dots,\widetilde{Z}^k_{0:T},\widetilde{\widetilde{Z}}^{k+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})|\widetilde{L}_{1,q})\prod_{1\leqslant i\leqslant q/2}\int_0^T\Lambda\widetilde{K}^{2i-1}_s\widetilde{K}^{2i}_s ds)\,,$$

$$N^{q/2}\mathbb{E}(F(\widetilde{\widetilde{Z}}^{1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{q}_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c})\underset{N\to+\infty}{\longrightarrow}$$
$$\sum_{1\leqslant k\leqslant q,\,k\text{ even}}(-1)^{\frac{q-k}{2}}\begin{pmatrix}q\\k\end{pmatrix}J_kJ_{q-k}$$

$$\times\, \mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(F(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})|\widetilde{L}_{1,q})\prod_{1\leqslant i\leqslant q/2}\int_0^T \Lambda \widetilde{K}^{2i-1}_s\widetilde{K}^{2i}_s ds)\,,$$

and for q odd, these limits are 0. We then use the equality $\begin{pmatrix} q \\ k \end{pmatrix} J_k J_{q-k} = \begin{pmatrix} q/2 \\ k/2 \end{pmatrix} J_q$ to finish the proof. □

Corollary 4.5. — *For $F\in\mathcal{B}^{sym}_0(q)$, we have*

$$\left|\mathbb{E}((\eta^N_{0:T})^{\odot q}(F))\right|\leqslant \frac{2^{2q+1}(\Lambda T\vee 1)^{q+1}(3q)!}{(e^{-\Lambda T})^{2q+1}(1-e^{-\Lambda T})q!(q-1)!}\frac{\|F\|_\infty}{(N-1)^{q/2}}\left(\frac{1}{(\lceil\frac{q}{4}\rceil)!}+\frac{1}{(N-1)^{\frac{q}{4}}}\right).$$

Proof. — For $m,m'\in[q]$ and $(I_1\dots,I_m)\in\mathcal{P}_m$, $(I'_1\dots,I'_{m'})\in\mathcal{P}_{m'}$, we have that $(\cap_{j\in[m]}L_{I_j})\cap(\cap_{j\in[m']}L_{I'_j})=\emptyset$ if $(I_1\dots,I_m)\neq(I'_1\dots,I'_{m'})$. Note, also, that for all $m,k\in[q]$, $(I_1\dots,I_m)\in\mathcal{P}_m$, we have that $\cap_{j\in[m]}L^k_{I_j}\subset\left(B\cup\{\widetilde{L}_T\geqslant\lceil q/2\rceil\}\right)$ (by (4.6) and the inequality $\frac{q}{2}+1\geqslant\lceil\frac{q}{2}\rceil$). Hence, by (4.17) and Lemma 4.2,

$$\begin{aligned}
&\left|\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T})\mathbb{1}_{(A_1\cap\dots\cap A_q)^c})\right|\\
&\qquad\leqslant\sum_{k=1}^q\begin{pmatrix} q \\ k \end{pmatrix}\mathbb{P}(B\cup\{\widetilde{L}_T\geqslant\lceil q/2\rceil\})\|F\|_\infty\\
&\qquad\leqslant 2^q\mathbb{P}(B\cup\{\widetilde{L}_T\geqslant\lceil q/2\rceil\})\|F\|_\infty\\
&\leqslant 2^q\sum_{r=0}^q\begin{pmatrix} q \\ r \end{pmatrix}\mathbb{P}(E^1_T\geqslant 1,\dots,E^r_T\geqslant 1,\widetilde{L}_T\geqslant\left\lceil\frac{q-r}{2}\right\rceil)\|F\|_\infty
\end{aligned}$$

(note that $\forall r\in\{0,\dots,q\}$, using the notation $l=\left\lceil\frac{q-r}{2}\right\rceil$:

$$2r+2l\leqslant 2q+1\,,\ r+l\leqslant q+1,$$
$$\frac{(q+2r+2l-1)!}{(q-1)!r^r l!}\leqslant\frac{(2q+r)!}{(q-1)!r!l!}\leqslant\frac{(3q)!}{(q-1)!q!}\ )$$

$$\begin{aligned}
&\leqslant 2^q\sum_{r=0}^q\begin{pmatrix} q \\ r \end{pmatrix}\frac{(\Lambda T\vee 1)^{q+1}(3q)!\|F\|_\infty}{(N-1)^{r+\lceil\frac{q-r}{2}\rceil}(e^{-\Lambda T})^{2q+1}(1-e^{-\Lambda T})(q-1)!q!\left(\lceil\frac{q-r}{2}\rceil\right)!}\\
&\leqslant 2^q\frac{(\Lambda T\vee 1)^{q+1}(3q)!\|F\|_\infty}{(e^{-\Lambda T})^{2q+1}(1-e^{-\Lambda T})(q-1)!q!}\left(\sum_{r=0}^{\lfloor q/2\rfloor}\begin{pmatrix} q \\ r \end{pmatrix}\frac{1}{(N-1)^{q/2}\left(\lceil\frac{q}{4}\rceil\right)!}\right.\\
&\qquad\left.+\sum_{r=\lfloor q/2\rfloor+1}^q\begin{pmatrix} q \\ r \end{pmatrix}\frac{1}{(N-1)^{q/2}(N-1)^{\frac{q}{4}}}\right)
\end{aligned}$$

$$\leqslant \frac{2^{2q}(\Lambda T\vee 1)^{q+1}(3q)!}{(e^{-\Lambda T})^{2q+1}(1-e^{-\Lambda T})(q-1)!q!}\frac{\|F\|_\infty}{(N-1)^{q/2}}\left(\frac{1}{(\lceil\frac{q}{4}\rceil)!}+\frac{1}{(N-1)^{\frac{q}{4}}}\right).$$

The same is true if we replace the Z^i's by $\widetilde{\widetilde{Z}}^i$'s. Thus (4.8) gives us the desired bound. □

4.4. Wick formula

We suppose here that q is even.

Definition 4.6. — *We introduce an auxiliary infinite system of particles* $(\check{Z}^1_{0:T},\check{Z}^2_{0:T},\dots)$ *such that*

- *it has interaction times*
$$\left\{T-\widetilde{T}_k, k\geqslant 1\right\}\cap\left\{T-t, t\in\mathcal{T}^{\{1,2\}}\cup\dots\cup\mathcal{T}^{\{q-1,q\}}\right\}$$
- *the rest of the definition is the same as for* $(\widetilde{Z}^i_{0:T})_{i\geqslant 1}$.

By doing this, we have coupled the interaction times of the system $(\check{Z}^i_{0:T})_{i\geqslant 1}$ *and of the system* $(\widetilde{Z}^i_{0:T})_{i\geqslant 1}$. *We can couple further and assume that* $(\check{Z}^1_{0:T},\dots,\check{Z}^q_{0:T})$ *and* $(\widetilde{\widetilde{Z}}^1_{0:T},\dots,\widetilde{\widetilde{Z}}^q_{0:T})$ *coincide on the event* $\left\{\widetilde{T}_k, k\geqslant 1\right\}\cap\left(\mathcal{T}^{\{1,2\}}\cup\dots\cup\mathcal{T}^{\{q-1,q\}}\right)=\left\{\widetilde{T}_k,k\geqslant 1\right\}$.

The system $(\check{Z}^i_{0:T})_{1\leqslant i}$ is obtained from $(\widetilde{Z}^i_{0:T})_{1\leqslant i\leqslant k}$ by stripping off the links which are not internal to $\widetilde{C}^1,\widetilde{C}^2$ or $\widetilde{C}^3,\widetilde{C}^4,\dots$.

We set for all f,g bounded $\mathbb{D}(\mathbb{R}_+,\mathbb{R}^d)\to\mathbb{R}$

$$V_{0:T}(f,g)=\mathbb{E}\left(\mathbb{E}(f(\check{Z}^1_{0:T})g(\check{Z}^2_{0:T})-f(\widetilde{\widetilde{Z}}^1_{0:T})g(\widetilde{\widetilde{Z}}^2_{0:T})|\widetilde{L}_{1,2},\widetilde{K}^1_{0:T},\widetilde{K}^2_{0:T})\int_0^T\Lambda\widetilde{K}^1_s\widetilde{K}^2_s ds\right). \tag{4.19}$$

Note that for all f,g, $V_{0:T}(f,g)=V_{0:T}(g,f)$. Note that the formula above cannot be simplified by the use of the tower formula. For all $k\in\mathbb{N}^*$, we set $\mathcal{I}_k$ to be the set of partitions of $[k]$ into subsets of cardinality 2.

Corollary 4.7. *[Wick formula].* — *For* $F\in\mathcal{B}_0^{sym}(q)$ *of the form* $F=(f_1\otimes\dots\otimes f_q)_{sym}$ *and q even,*

$$N^{q/2}\mathbb{E}(F(Z^1_{0:T},\dots,Z^q_{0:T}))\underset{N\to+\infty}{\longrightarrow}\sum_{J\in\mathcal{I}_q}\prod_{\{a,b\}\in J}V_{0:T}(f_a,f_b).$$

The name "Wick formula" comes from the Wick formula on the expectation of a product of Gaussians. In this formula, there is a sum over pairings, just as in the above Corollary. See Theorem 22.3, p. 360 in [13] for the Wick formula.

Proof. — To shorten the notations, we will write:

$$\prod_{i=1}^{q/2}\int_0^T \Lambda\widetilde{K}_s^{2i-1}\widetilde{K}_s^{2i}ds = p\,.$$

With this particular form for F, the limit in (2.6) of Proposition 2.6 becomes

$$\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}\sum_{k=1}^{q/2} J_q\begin{pmatrix} q/2 \\ k\end{pmatrix}(-1)^{\frac{q}{2}-k}$$

$$\times\,\mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(f_{\sigma(1)}(\widetilde{Z}^1_{0:T})\dots f_{\sigma(2k)}(\widetilde{Z}^{2k}_{0:T})f_{\sigma(2k+1)}(\widetilde{\widetilde{Z}}{}^{2k+1}_{0:T})\dots f_{\sigma(q)}(\widetilde{\widetilde{Z}}{}^{q}_{0:T})$$

$$-\,f_{\sigma(1)}(\widetilde{\widetilde{Z}}{}^{1}_{0:T})\dots f_{\sigma(q)}(\widetilde{\widetilde{Z}}{}^{q}_{0:T})|\widetilde{L}_{1,q})p)$$

$$=\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}\sum_{k=0}^{q/2} J_q\begin{pmatrix} q/2 \\ k\end{pmatrix}(-1)^{\frac{q}{2}-k}$$

$$\times\mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(f_{\sigma(1)}(\widetilde{Z}^1_{0:T})\dots f_{\sigma(2k)}(\widetilde{Z}^{2k}_{0:T})f_{\sigma(2k+1)}(\widetilde{\widetilde{Z}}{}^{2k+1}_{0:T})\dots f_{\sigma(q)}(\widetilde{\widetilde{Z}}{}^{q}_{0:T})|\widetilde{L}_{1,q})p)\,,$$

because $\sum_{k=1}^{q/2}\begin{pmatrix} q/2 \\ k\end{pmatrix}(-1)^{\frac{q}{2}-k} = (-1)^{\frac{q}{2}+1}$. Using the exchangeability property of the particle systems, we can transform the last expression into

$$\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}\sum_{k=0}^{q/2} J_q(-1)^{\frac{q}{2}-k}\sum_{I\subset[q/2],\#I=k}\mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(\prod_{i\in I} f_{\sigma(2i-1)}(\widetilde{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{Z}^{2i}_{0:T})$$

$$\prod_{i\notin I} f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}{}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{\widetilde{Z}}{}^{2i}_{0:T})|\widetilde{L}_{1,q})p) =$$

$$\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} J_q\mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(\prod_{i=1}^{q/2}(f_{\sigma(2i-1)}(\widetilde{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{Z}^{2i}_{0:T}) - f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}{}^{2i-1}_{0:T})$$

$$f_{\sigma(2i)}(\widetilde{\widetilde{Z}}{}^{2i}_{0:T}))|\widetilde{L}_{1,q})p) =$$

$$\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q} J_q\mathbb{E}(\mathbb{E}_{\widetilde{\mathcal{K}}_T}(\prod_{i=1}^{q/2}(f_{\sigma(2i-1)}(\check{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\check{Z}^{2i}_{0:T}) - f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}{}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{\widetilde{Z}}{}^{2i}_{0:T}))$$

$$|\widetilde{L}_{1,2},\dots,\widetilde{L}_{q-1,q})p)\,, \quad (4.20)$$

the last equality being true because

$$\mathcal{L}((\widetilde{Z}^i_{0:T})_{i\geqslant 1},(\widetilde{\widetilde{Z}}^i_{0:T})_{i\geqslant 1}|\widetilde{\mathcal{K}}_T,\widetilde{L}_{1,q})=\mathcal{L}((\check{Z}^i_{0:T})_{i\geqslant 1},(\widetilde{\widetilde{Z}}^i_{0:T})_{i\geqslant 1}|\widetilde{\mathcal{K}}_T,\widetilde{L}_{1,2},\dots,\widetilde{L}_{q-1,q}).$$

By Lemma 3.6, 5., the processes $\widetilde{L}^{\{1,2\}},\dots,\widetilde{L}^{\{q,q-1\}}$, defined in (3.9), are independent, conditionally to $\widetilde{\mathcal{K}}_T$. And for all $i\in[q/2]$,

$$\begin{aligned}&\mathbb{E}_{\widetilde{\mathcal{K}}_T}(f(\check{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\check{Z}^{2i}_{0:T})-f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{\widetilde{Z}}^{2i}_{0:T})|\widetilde{L}_{1,2},\dots,\widetilde{L}_{q-1,q})\\&=\mathbb{E}(f(\check{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\check{Z}^{2i}_{0:T})-f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{\widetilde{Z}}^{2i}_{0:T})|\widetilde{L}_{2i-1,2i},\widetilde{K}^{2i-1}_{0:T},\widetilde{K}^{2i}_{0:T}).\end{aligned}$$

So, the quantity in (4.20) is equal to

$$\begin{aligned}&\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}J_q\mathbb{E}(\prod_{i=1}^{q/2}[\mathbb{E}((f_{\sigma(2i-1)}(\check{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\check{Z}^{2i}_{0:T})\\&-f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{\widetilde{Z}}^{2i}_{0:T}))|\widetilde{L}_{2i-1,2i},\widetilde{K}^{2i-1}_{0:T},\widetilde{K}^{2i}_{0:T})\times\int_0^T\Lambda\widetilde{K}^{2i-1}_s\widetilde{K}^{2i-1}_s ds)])\\&=\text{(by Lemma 3.6, 4)}\\&\frac{1}{q!}\sum_{\sigma\in\mathcal{S}_q}J_q\prod_{i=1}^{q/2}\mathbb{E}(\mathbb{E}((f_{\sigma(2i-1)}(\check{Z}^{2i-1}_{0:T})f_{\sigma(2i)}(\check{Z}^{2i}_{0:T})\\&-f_{\sigma(2i-1)}(\widetilde{\widetilde{Z}}^{2i-1}_{0:T})f_{\sigma(2i)}(\widetilde{\widetilde{Z}}^{2i}_{0:T}))|\widetilde{L}_{2i-1,2i},\widetilde{K}^{2i-1}_{0:T},\widetilde{K}^{2i}_{0:T})\int_0^T\Lambda\widetilde{K}^{2i-1}_s\widetilde{K}^{2i-1}_s ds)\\&=\sum_{J\in\mathcal{I}_q}\prod_{\{a,b\}\in J}V_{0:T}(f_a,f_b).\end{aligned}$$

□

5. Proof of convergence theorems

5.1. Proof of Theorem 2.7 (almost sure convergence)

Proof. — We recall the notations of [3]. For any empirical measure $m(x)=\frac{1}{N}\sum_{i=1}^N\delta_{x^i}$ (based on N points $x^1,x^2,\dots,x^N$),

$$m(x)^{\otimes q}:=\frac{1}{N^q}\sum_{a\in[N]^{[q]}}\delta_{(x^{a(1)},\cdots,x^{a(q)})}\ ,$$

where $[N]^{[q]}=\{a:[q]\to[N]\}$. Note that for any F,

$$m(x)^{\otimes q}(F)=m(x)^{\otimes q}(F_{\text{sym}})\ .$$

We define, for all $p \in [q]$, $[q]_p^{[q]} := \{a \in [q]^{[q]}, \#\mathrm{Im}(a) = p\}$, and $(\forall k \leqslant q)$,

$$\partial^k L_q = \sum_{q-k \leqslant p \leqslant q} s(p, q-k) \frac{1}{(q)_p} \sum_{a \in [q]_p^{[q]}} a \,,$$

(the $s(.,.)$ are the Stirling numbers of the first kind), and for all F (of q variables), for all $b \in [q]^{[q]}$,

$$D_b(F)(x^1, \dots, x^q) = F(x^{b(1)}, \dots, x^{b(q)}) \;,$$

$$D_{\partial^k L_q}(F) = \sum_{q-k \leqslant p \leqslant q} s(p, q-k) \frac{1}{(q)_p} \sum_{a \in [q]_p^{[q]}} D_a(F) \;.$$

The derivative-like notation $\partial^k L$ comes from [3], where it makes sense to think of a derivative at this point. We keep the same notation in order to be consistent, but it has no particular meaning in our setting. We then have, by Corollary 2.3 p. 789 of [3], for any empirical measure $m(x)$ (based on N points) and for any F of q variables,

$$m(x)^{\otimes q}(F) = m(x)^{\odot q} \left(\sum_{0 \leqslant k < q} \frac{1}{N^k} D_{\partial^k L_q}(F) \right) \;.$$

Suppose $F \in \mathcal{B}_0^{sym}(q)$, we then obtain

$$\mathbb{E}\left((\eta_{0:T}^N)^{\otimes q}(F)\right) = \sum_{0 \leqslant k < q} \frac{1}{N^k} \sum_{q-k \leqslant p \leqslant q} s(p, q-k) \sum_{a \in [q]_p^{[q]}} \mathbb{E}\left((\eta_{0:T}^N)^{\odot q}(D_a(F))\right) \;. \tag{5.21}$$

We take $a \in [q]^{[q]}$ with $p = \#\mathrm{Im}(a) \geqslant q-k$ and $k < q/2$. Note that $\#\{i \in [q], \#a^{-1}(\{i\}) = 1\} \geqslant q - 2k > 0$.

We have now to use the Hoeffding's decomposition (see [14, 10], or [4], Section 4, for the details). For any symmetrical $G : \mathbb{D}([0,T], (\mathbb{R}^d)^q) \to \mathbb{R}$, we define

$$\theta = \int G(x_1, \dots x_q) \widetilde{P}_{0:T}(dx_1, \dots, dx_q) \,,$$

$$G^{(j)}(x_1, \dots, x_j) = \int G(x_1, \dots, x_q) \widetilde{P}_{0:T}^{\otimes (q-j)}(dx_{j+1}, \dots, dx_q) \,,$$

and recursively

$$h^{(1)}(x_1) = G^{(1)}(x_1) - \theta \,,$$

$$h^{(k)}(x_1,\dots,x_k)=G^{(j)}(x_1,\dots,x_k)-\sum_{i=1}^{j-1}\sum_{(j,i)}h^{(i)}-\theta\,,$$

where $\sum_{(j,i)}h^{(i)}$ is an abbreviation for the function

$$(x_1,\dots,x_j)\mapsto\sum_{1\leqslant r_1<\dots<r_i\leqslant j}h^{(i)}(x_{r_1},\dots,x_{r_i})\,.$$

For all j, $h^{(j)}$ is in $\mathcal{B}_0^{sym}(j)$. We have the formula

$$G(x_1,\dots,x_q)=h^{(q)}(x_1,\dots,x_q)+\sum_{j=1}^{q-1}\sum_{(q,j)}h^{(j)}\,.$$

We take now $G(x_1,\dots,x_q)=D_a(F)$ (still with $F\in\mathcal{B}_0^{sym}(q)$). For $j<q-2k$, $G^{(j)}=0$. So we can show by recurrence that $h^{(j)}=0$ for $j<q-2k$. So

$$G(x_1,\dots,x_q)=h^{(q)}(x_1,\dots,x_q)+\sum_{j=q-2k}^{q-1}\sum_{(q,j)}h^{(j)}\,.$$

So, by Corollary 4.5, we have for some constant C

$$\mathbb{E}(D_aF(Z^1_{0:T},\dots,Z^q_{0:T}))\leqslant\frac{C}{N^{(q-2k)/2}}\,.$$

And so, by (5.21),

$$\mathbb{E}\left((\eta^N_{0:T})^{\otimes q}(F)\right)\leqslant\frac{C}{N^{\frac{q}{2}}}\,.$$

Suppose that we take a bounded function $f:\mathbb{D}([0,T],\mathbb{R}^d)\to\mathbb{R}$. We set $\bar f=f-\widetilde{P}_{0:T}(f)$. We then have (with the notation $\bar f^{\otimes q}(x^1,\dots,x^q):=\bar f(x^1)\times\dots\times\bar f(x^q)$)

$$\begin{aligned}\mathbb{E}(((\eta^N_{0:T}(f)-\widetilde{P}_{0:T}(f))^q)&=\mathbb{E}((\eta^N_{0:T}(\bar f))^q)\\&=\mathbb{E}((\eta^N_{0:T})^{\otimes q}(\bar f^{\otimes q}))\\&=\mathbb{E}((\eta^N_{0:T})^{\otimes q}(\overline{f}^{\otimes q})\mathrm{sym}\ )\\&\leqslant\frac{C}{N^{\frac{q}{2}}}\,.\end{aligned}\tag{5.22}$$

Provided we take $q=4$, we can apply Borel-Cantelli Lemma to finish the proof. □

5.2. Proof of Theorem 2.8 (central-limit theorem)

Proof. — To simplify, we suppose here that $\|f_1\|_\infty \leqslant 1, \dots, \|f_q\|_\infty \leqslant 1$. For any $u_1, \dots, u_q \in \mathbb{R}$, we have:

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(N\eta^N_{0:T}\left(\log\left(1+\frac{iu_1f_1+\dots+iu_qf_q}{\sqrt{N}}\right)\right)\right)\right) \qquad (5.23)\\
&= \mathbb{E}\left(\prod_{j=1}^N\left(1+\frac{iu_1f_1(Z^j_{0:T})+\dots+iu_qf_q(Z^j_{0:T})}{\sqrt{N}}\right)\right)\\
&= \mathbb{E}(\sum_{0\leqslant k\leqslant N}\frac{1}{N^{k/2}}\sum_{1\leqslant j_1,\dots,j_k\leqslant q} i^k u_{j_1}\dots u_{j_k}\times\sum_{1\leqslant i_1<\dots<i_k\leqslant N} f_{j_1}(Z^{i_1}_{0:T})\dots f_{j_k}(Z^{i_k}_{0:T}))\\
&= \sum_{0\leqslant k\leqslant N}\frac{(N)_k}{N^{k/2}}\sum_{1\leqslant j_1,\dots,j_k\leqslant q} i^k u_{j_1}\dots u_{j_k}\frac{1}{k!}\mathbb{E}\left((\eta^N_{0:T})^{\odot k}(f_{j_1}\otimes\dots\otimes f_{j_k})\right)\\
&= \sum_{0\leqslant k\leqslant N}\frac{(N)_k}{N^{k/2}}\sum_{1\leqslant j_1,\dots,j_k\leqslant q} i^k u_{j_1}\dots u_{j_k}\frac{1}{k!}\mathbb{E}\left((\eta^N_{0:T})^{\odot k}(f_{j_1}\otimes\dots\otimes f_{j_k})_{sym}\right).
\end{aligned}$$

By Corollary 4.5, for all $k \in [N]$, we have (computing very roughly)

$$\begin{aligned}
&\left|\frac{(N)_k}{N^{k/2}}\sum_{1\leqslant j_1,\dots,j_k\leqslant q} i^k u_{j_1}\dots u_{j_k}\frac{1}{k!}\mathbb{E}\left((\eta^N_{0:T})^{\odot k}(f_{j_1}\otimes\dots\otimes f_{j_k})_{sym}\right)\right|\\
&\quad\leqslant \frac{q^k(\max(|u_1|,\dots,|u_q|))^k}{k!}\times\frac{2^{2k+1}(3k)!(\Lambda T\vee 1)^{k+1}}{(k-1)!k!(e^{-\Lambda T})^{2k+1}(1-e^{-\Lambda T})}\\
&\qquad\times\left(\frac{1}{(\lceil\frac{k}{4}\rceil)!}+\frac{1}{(N-1)^{\frac{k}{4}}}\right)\frac{(N)_k}{(N-1)^{k/2}N^{k/2}}\\
&\leqslant \frac{q^k2^{2k+1}3^{3k}(3k)(\max(|u_1|,\dots,|u_q|))^k(\Lambda T\vee 1)^{k+1}}{(e^{-\Lambda T})^{2k+1}(1-e^{-\Lambda T})}\left(\frac{1}{(\lceil\frac{k}{4}\rceil)!}+\frac{1}{(N-1)^{\frac{k}{4}}}\right),
\end{aligned}$$

and this last term is summable in k if N is big enough. Using Corollary 4.7 and Proposition 2.6, we then obtain:

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(N\eta^N_{0:T}\left(\log\left(1+\frac{iu_1f_1+\dots iu_qf_q}{\sqrt{N}}\right)\right)\right)\right)\\
&\underset{N\to+\infty}{\longrightarrow}\sum_{k\geqslant 0,k\text{ even}}(-1)^{k/2}\sum_{1\leqslant j_1,\dots,j_k\leqslant q}\frac{u_{j_1}\dots u_{j_k}}{k!}\sum_{I_k\in\mathcal{I}_k}\prod_{\{a,b\}\in I_k}V_{0:T}(f_{j_a},f_{j_b})\\
&= \sum_{k\geqslant 0,k\text{ even}}\frac{(-1)^{k/2}}{2^{k/2}(k/2)!}\times\sum_{1\leqslant j_1,\dots,j_k\leqslant q}u_{j_1}\dots u_{j_k}V_{0:T}(f_{j_1},f_{j_2})\dots V_{0:T}(f_{j_{k-1}},f_{j_k})
\end{aligned}$$

$$= \sum_{k\geqslant 0, k \text{ even}} \frac{(-1)^{k/2}}{2^{k/2}(k/2)!} \left(\sum_{1\leqslant j_1,j_2\leqslant q} u_{j_1} u_{j_2} V_{0:T}(f_{j_1}, f_{j_2}) \right)^{k/2}$$

$$= \exp\left(-\frac{1}{2} \sum_{1\leqslant j_1,j_2\leqslant q} u_{j_1} u_{j_2} V_{0:T}(f_{j_1}, f_{j_2}) \right) \quad (5.24)$$

We can also write a series development of the log in (5.23) and obtain:

$$\mathbb{E}\left(\exp\left(N\eta^N_{0:T}(\log(1+\frac{iu_1f_1+\cdots+iu_qf_q}{\sqrt{N}}))\right)\right)$$
$$= \mathbb{E}\left(\exp\left(\sum_{k\geqslant 1}\frac{(-1)^{k+1}}{k}N^{1-k/2}\eta^N_{0:T}((iu_1f_1+\cdots+iu_qf_q)^k)\right)\right) . \quad (5.25)$$

We have, for all k, $\|\frac{(iu_1f_1+\cdots+iu_qf_q)^k}{N^{k/2}}\|_\infty \leqslant \frac{(|u_1|+\cdots+|u_q|)^k}{N^{k/2}}$ for some constant C (independent of k). So, the remaining term in the series development of the log can be bounded by

$$\|\sum_{k\geqslant 3}\frac{(-1)^{k+1}}{k}N^{1-k/2}(iu_1f_1+\cdots+iu_qf_q)^k\|_\infty \leqslant \frac{C(|u_1|+\cdots+|u_q|)^4}{N} ,$$

for some constant C, if $N \geqslant 2(|u_1|+\cdots+|u_q|)^2$. So, the limit when $N\to+\infty$ of (5.25) is the same as the limit of

$$\mathbb{E}\left(\exp\left(\sqrt{N}(iu_1\eta^N_{0:T}(f_1)+\ldots+iu_q\eta^N_{0:T}(f_q)\right)\exp\left(\frac{1}{2}\eta^N_{0:T}((u_1f_1+\ldots+u_qf_q)^2)\right)\right) .$$

We have, for some constant C and $f := iu_1f_1+\cdots+iu_qf_q$ (recall $x\in\mathbb{R}\Rightarrow |e^{ix}|=1$),

$$|\mathbb{E}(e^{\sqrt{N}\eta^N_{0:T}(f)}e^{-\frac{1}{2}\eta^N_{0:T}(f^2)})-\mathbb{E}(e^{\sqrt{N}\eta^N_{0:T}(f)}e^{-\frac{1}{2}\widetilde{P}_{0:T}(f^2)})|$$
$$\leqslant C\mathbb{E}(\left|\widetilde{P}_{0:T}(f^2)-\eta^N_{0:T}(f^2)\right|) \quad (5.26)$$

So, by Theorem 2.7, the left-hand side of (5.26) goes to 0 as $N\to+\infty$. So

$$\lim_{N\to 0}\exp\left(\sqrt{N}\eta^N_{0:T}\left(\log(1+\frac{iu_1f_1+\cdots+iu_qf_q}{\sqrt{N}})\right)\right)$$
$$= \lim_{N\to 0}\mathbb{E}\left(e^{\sqrt{N}\eta^N_{0:T}(iu_1f_1+\cdots+iu_qf_q)}e^{\frac{1}{2}\widetilde{P}_{0:T}((u_1f_1+\cdots+iu_qf_1)^2)}\right) ,$$

(meaning that if these limits exist, they are equal), which concludes the proof with, $\forall i,j$,

$$K(i,j) = \widetilde{P}_{0:T}(f_if_j) + V_{0:T}(f_i,f_j) . \quad (5.27)$$

□

Note that we can bound the two terms of rhs above. Take $f_1,\dots,f_q$ as above and such that $\|f_1\|_\infty \leqslant 1, \dots, \|f_q\|_\infty \leqslant 1$. For all i,j:

$$|\widetilde{P}_{0:T}(f_i f_j)| \leqslant 1\,,$$

$$\begin{aligned} |V_{0:T}(f_i,f_j)| &\leqslant 2\mathbb{E}(T\Lambda \widetilde{K}^1_T \widetilde{K}^2_T) \\ \text{(by Lemma 4.1)} &\leqslant \frac{2T\Lambda e^{2T\Lambda}(q+1)q}{(1-e^{\Lambda T})^2}\,. \end{aligned}$$

5.3. Proof of Corollary 2.9

Proof. — The result is a consequence of Theorem 2.8 from this paper and of Theorem 4.1 from [4]. We only have to prove that for all $j \geqslant 2$, $f \in \mathcal{B}_0^{sym}(j)$,

$$\mathbb{E}\left(\left((\eta^N_{0:T})^{\odot j}(f)\right)^2\right) \leqslant \frac{C}{N^j}\,,$$

for some constant C which may depend on j, f, T. Looking at the proof of Lemma 4.3 of [4], we see that we need only to prove that for all $k \in \{j+1,\dots,2j\}$, $r \in [k]$, for all $F : \mathbb{D}([0,T],\mathbb{R}^d)^k \to \mathbb{R}$ bounded measurable, symmetric in the $k-r$ last variables and such that $\int_{\mathbb{D}([0,T],E)} F(z_1,\dots,z_k)\widetilde{P}_{0:T}(dx_i) = 0$, for all $i \in \{r+1,\dots,k\}$, we have

$$\left|\mathbb{E}((\eta^N_{0:T})^{\odot k}(F))\right| \leqslant \frac{C}{N^{\frac{k-r}{2}}}\,,$$

for some constant C depending on F, k, r. The proof of this inequality follows the outline of the proof of Proposition 2.6. Here, we write only the beginning of the decomposition. We have (for F, k, r as above)

$$\begin{aligned} &\mathbb{E}((\eta^N_{0:T})^{\odot k}(F)) = \mathbb{E}(F(Z^1_{0:T},\dots,Z^k_{0:T})) \\ &= \mathbb{E}(F(Z^1_{0:T},\dots,Z^k_{0:T})\mathbb{1}_{(A_{r+1}\cap\dots\cap A_k)^c}) + \mathbb{E}(F(Z^1_{0:T},\dots,Z^k_{0:T})\mathbb{1}_{A_{r+1}\cap\dots\cap A_k}) \\ &= \mathbb{E}(F(Z^1_{0:T},\dots,Z^k_{0:T})\mathbb{1}_{(A_{r+1}\cap\dots\cap A_k)^c}) \\ &\qquad - \mathbb{E}(F(Z^1_{0:T},\dots,Z^r_{0:T},\widetilde{\widetilde{Z}}^{r+1}_{0:T},\dots,\widetilde{\widetilde{Z}}^{k}_{0:T})\mathbb{1}_{(A_{r+1}\cap\dots\cap A_k)^c})\,. \end{aligned}$$

□

6. Appendix

6.1. Proof of Lemma 3.4

Proof. — Let us here give a brief explanation of why this equality is true. We start the construction of the link times at time 0. We first look at the times $\{T_k, k \geqslant 1\} \cap \{T'_l, l \geqslant 1\}$. The law of $\tau = \inf\{T_k, k \geqslant 1\} \cap \{T'_l, l \geqslant 1\}$ is $\mathcal{E}(\frac{\Lambda q(N-q)_+}{N-1})$. At time τ, we choose $r(k)$ in $C^1_{\tau-} \cup \dots \cup C^q_{\tau-}$ and $j(k) \in [N] \backslash C^1_{\tau-} \cup \dots \cup C^q_{\tau-}$ and the jump $C^{r(k)}_{\tau} = C^{r(k)}_{\tau-} \cup \{j(k)\}$is performed. For example, in Figure 1, we wait $3T/4$ and then we add 3 to the set $C^2_{(3T/4)-} = \{2\}$.

The situation in Definition 2.1 is the following. We have Poisson processes $N_{i,j}$ like in Subsection 2. Let us start at the bottom of the interaction graph and then move upward. As the processes $(N_{i,j}(T-t))_{0 \leqslant t \leqslant T}$ are Poisson processes, we wait for $\tau' = \inf\{t : \text{ jump time of } N_{i,j}(T-.), i \in [q], j \notin [q]\}$. And then, if τ' is a jump time for $N_{r',j'}$ with $r' \in [q]$, we add a branch corresponding to j' to the branch corresponding to r' (in the same way as in Figure 1, where we added the branch with the label 3 to the branch with the label 2). The random times τ and τ' have the same law due to Lemma 3.3, 3.. The random couple of indexes $(r(k), j(k))$, (r', j') have the same law due to Lemma 3.3, 2..

We now look at the horizontal lines between existing branches (such as the line between 1 and 2 in Figure 1). Let $0 \leqslant t \leqslant T$. We set $j = \#\{k, T_k \leqslant t\}$. We compute

$$\begin{aligned}
&\mathbb{P}(\forall k \leqslant j, T_k \neq T''_k | j, T'_1, T'_2, \dots, T'_j, (K_u)_{0 \leqslant u \leqslant t}) \\
&= \mathbb{P}(T'_1 \frac{\Lambda q(q-1)}{2(N-1)} < V_1, \dots, (t - T'_k) \frac{\Lambda(q+j)(q+j-1)}{2(N-1)} \\
&\qquad\qquad < V_{j+1} | j, T'_1, T'_2, \dots, T'_j, (K_u)_{0 \leqslant u \leqslant t}) \\
&= \exp\left(- \int_0^t \frac{\Lambda K_u(K_u - 1)}{2(N-1)} du \right) .
\end{aligned}$$

So, conditionally to $(K_u)_{0 \leqslant u \leqslant T}$, $(\#\{T_k = T''_k, T_k \leqslant t\})_{t \geqslant 0}$ is an inhomogeneous Poisson process of rate $\left(\frac{\Lambda K_u(K_u-1)}{2(N-1)}\right)_{0 \leqslant u \leqslant t}$. When a jump time of the form $T_k = T''_k$ occurs, we choose $r(k)$ uniformly in $C^1_{T_k-} \cup \dots \cup C^q_{T_k-}$ and $j(k)$ uniformly in $C^1_{T_k-} \cup \dots \cup C^q_{T_k-} \backslash \{r(k)\}$. We then add a horizontal line between branches $r(k)$ and $j(k)$. Due to Lemma 3.3, 2., this is the way horizontal branches are added to existing vertical branches in Definition 2.1. □

6.2. Proof of Lemma 3.6

Proof. — The process $(\widetilde{K}_t)_{0\leqslant t\leqslant T}$ is piecewise constant and has jumps of size 1. The jump times of $\widetilde{K}_t$ belong to $\{T'_k, k\geqslant 1\}$ or to $\left\{\widetilde{T}'_k, k\geqslant 1\right\}$. The jump times of K_t belong to $\{T'_k, k\geqslant 1\}$. Suppose we are at time s, and we know $\widetilde{K}_s$. We set $\widetilde{j}=\#\{\widetilde{T}_k, s<\widetilde{T}_k\leqslant t\}$, $\widetilde{k}_0=\sup\{k,\widetilde{T}_k\leqslant s\}$, $j=\#\{T_k, s<T_k\leqslant t\}$, $k_0=\sup\{k,T_k\leqslant s\}$. We compute

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{\widetilde{K}_t=\widetilde{K}_s}|\widetilde{K}_s) &= \mathbb{E}(\mathbb{E}(\mathbb{1}_{\widetilde{K}_t=\widetilde{K}_s}|\widetilde{K}_s,K_s,T_{k_0},\dots,T_{k_0+j},\widetilde{T}_{\widetilde{k}_0},\dots,\widetilde{T}_{\widetilde{k}_0+\widetilde{j}})|\widetilde{K}_s)\\ &= \mathbb{E}\Bigg[\mathbb{E}\Bigg[\prod_{i=1}^{j}\mathbb{1}_{U_{k_0+i}>(T_{k_0+i}-T_{k_0+i-1})\frac{\Lambda K_s(N-K_s)_+}{N-1}}\\ &\qquad\times\mathbb{1}_{U_{k_0+j}>(t-T_{k_0+j})\frac{\Lambda K_s(N-K_s)_+}{N-1}}\\ &\quad\times\prod_{i=1}^{\widetilde{j}}\mathbb{1}_{\widetilde{U}_{\widetilde{k}_0+i}>(\widetilde{T}_{\widetilde{k}_0+i}-\widetilde{T}_{\widetilde{k}_0+i-1})\left(\Lambda\widetilde{K}_s-\frac{\Lambda K_s(N-K_s)_+}{N-1}\right)}\\ &\qquad\times\mathbb{1}_{\widetilde{U}_{\widetilde{k}_0+\widetilde{j}}>(t-\widetilde{T}_{\widetilde{k}_0+\widetilde{j}})\left(\Lambda\widetilde{K}_s-\frac{\Lambda K_s(N-K_s)_+}{N-1}\right)}\\ &\quad|\widetilde{K}_s,K_s,T_{k_0},\dots,T_{k_0+j},\widetilde{T}_{\widetilde{k}_0},\dots,\widetilde{T}_{\widetilde{k}_0+\widetilde{j}}\Bigg]|\widetilde{K}_s\Bigg]=\\ &\qquad\qquad\qquad\exp(-(t-s)\Lambda\widetilde{K}_s)\,.\end{aligned}$$

The process $(\widetilde{L}_t)_{0\leqslant t\leqslant T}$ is piecewise constant and has jumps of size 1. The jump times of this process belong to $\{T''_k, k\geqslant 1\}$, or to $\{\widetilde{T}''_k, k\geqslant 1\}$. Suppose we are at time s and we know $\widetilde{L}_s$, $(\widetilde{K}_u)_{0\leqslant u\leqslant T}$. Let $t\geqslant s$. Due to the properties of the exponential law, the probability $\mathbb{P}(\widetilde{L}_t=\widetilde{L}_s|\widetilde{L}_s,(\widetilde{K}_u)_{0\leqslant u\leqslant T})$ is equal to This probability is equal to

$$\begin{aligned}&\mathbb{E}(\mathbb{1}_{\widetilde{L}_t=\widetilde{L}_s}|\widetilde{L}_s,(\widetilde{K}_u)_{0\leqslant u\leqslant T})=\\ &\mathbb{E}(\mathbb{E}(\mathbb{1}_{\widetilde{L}_t=\widetilde{L}_s}|\widetilde{L}_s,(\widetilde{K}_u)_{0\leqslant u\leqslant T},T_{k_0},\dots,T_{k_0+j},\widetilde{T}_{\widetilde{k}_0},\dots,\widetilde{T}_{\widetilde{k}_0+\widetilde{j}})|\widetilde{L}_s,(\widetilde{K}_u)_{0\leqslant u\leqslant T})\\ &\quad=\mathbb{E}\Bigg[\mathbb{E}\Bigg[\prod_{i=1}^{j}\mathbb{1}_{V_{k_0+i}>(T_{k_0+i}-T_{k_0+i-1})\frac{\Lambda K_{T_{k_0+i}}(K_{T_{k_0+i}}-1)}{N-1}}\\ &\qquad\times\mathbb{1}_{V_{k_0+j}>(t-T_{k_0+j})\frac{\Lambda K_{T_{k_0+j}}(K_{T_{k_0+j}}-1)}{N-1}}\\ &\times\prod_{i=1}^{\widetilde{j}}\mathbb{1}_{\widetilde{U}'_{\widetilde{k}_0+i}>(\widetilde{T}_{\widetilde{k}_0+i}-\widetilde{T}_{\widetilde{k}_0+i-1})\left(\frac{\Lambda\widetilde{K}_{\widetilde{T}_{k_0+i}}(\widetilde{K}_{\widetilde{T}_{k_0+i}}-1)}{N-1}-\frac{\Lambda K_{\widetilde{T}_{k_0+i}}(K_{\widetilde{T}_{k_0+i}}-1)_+}{N-1}\right)}\end{aligned}$$

$$\times \mathbb{1}_{\widetilde{U}'_{\widetilde{k}_0+\widetilde{j}}>(t-\widetilde{T}_{\widetilde{k}_0+\widetilde{j}})}\left(\frac{\Lambda\widetilde{K}_{\widetilde{T}_{k_0+i}}(\widetilde{K}_{\widetilde{T}_{k_0+i}}-1)}{N-1}-\frac{\Lambda K_{\widetilde{T}_{k_0+i}}(K_{\widetilde{T}_{k_0+i}}-1)_+}{N-1}\right)$$
$$|\widetilde{L}_s,(\widetilde{K}_u)_{0\leqslant u\leqslant T},T_{k_0},\dots,T_{k_0+j},\widetilde{T}_{\widetilde{k}_0},\dots,\widetilde{T}_{\widetilde{k}_0+\widetilde{j}}\Big]\,|\widetilde{L}_s,(\widetilde{K}_u)_{0\leqslant u\leqslant T}\Big]=$$
$$\exp(-(t-s)\Lambda\widetilde{K}_s)\,.$$

This probability is equal to $\mathbb{P}\left(\int_s^t \frac{\Lambda\widetilde{K}_u(\widetilde{K}_u-1)}{N-1}du \leqslant V_1'|(\widetilde{K}_u)_{0\leqslant u\leqslant T}\right)$ (for some V_1' of law $\mathcal{E}(1)$). This proves the point 1. of the lemma.

We have for all ω, t, $\Delta K_t(\omega)=1 \Rightarrow \Delta\widetilde{K}_t(\omega)=1$ and $\Delta L_t(\omega)=1 \Rightarrow \Delta\widetilde{L}_t(\omega)=1$, so we have the point 2. of the Lemma. The point 3. of the Lemma is immediate.

Let $k\geqslant 1$. Suppose we are at time $\widetilde{T}_{k-1}$, with $\widetilde{T}_{k-1}<T$. The variables $\inf\{T_l',T_l'\geqslant\widetilde{T}_{k-1}\}-\widetilde{T}_{k-1}$, $\inf\{T_l'',T_l''\geqslant\widetilde{T}_{k-1}\}-\widetilde{T}_{k-1}$, $\inf\{\widetilde{T}_l',\widetilde{T}_l'\geqslant\widetilde{T}_{k-1}\}-\widetilde{T}_{k-1}$, $\inf\{\widetilde{T}_l'',\widetilde{T}_l''\geqslant\widetilde{T}_{k-1}\}-\widetilde{T}_{k-1}$ are of exponential law (recall the definition from sections 3.1, 3.2). The infimum of four independent exponential variables $E_1,\dots,E_4$ of parameters, respectively, $\lambda_1,\dots,\lambda_4$ satisfies $\mathbb{P}(E_1=\inf(E_1,\dots,E_4)|\inf(E_1,\dots,E_4)<t)=\frac{\lambda_1}{\lambda_1+\dots+\lambda_4}$ ($\forall t>0$) (see Th. 2.3.3. of [12]). So,

$$\mathbb{P}(\widetilde{T}_k\in\{\widetilde{T}_l',l\geqslant 1\}|\mathcal{K}_{\widetilde{T}_{k-1}},\widetilde{\mathcal{K}}_{\widetilde{T}_{k-1}},\widetilde{T}_k<T) = \frac{\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}-\frac{\Lambda K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}}{\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}+\frac{\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}(\widetilde{K}_{\widetilde{T}_{k-1}}-1)}{(N-1)}},$$

$$\mathbb{P}(\widetilde{T}_k\in\{T_l',l\geqslant 1\}|\mathcal{K}_{\widetilde{T}_{k-1}},\widetilde{\mathcal{K}}_{\widetilde{T}_{k-1}},\widetilde{T}_k<T) = \frac{\frac{\Lambda K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}}{\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}+\frac{\Lambda\widetilde{K}_{\widetilde{T}_{k-1}}(\widetilde{K}_{\widetilde{T}_{k-1}}-1)}{(N-1)}},$$

so, recalling (3.1), (3.3), (3.4),

$$\mathbb{P}(\Delta\widetilde{K}^i_{\widetilde{T}_k}=1|\widetilde{T}_k\in\{T_l',\widetilde{T}_l',l\geqslant 1\},\mathcal{K}_{\widetilde{T}_{k-1}},\widetilde{\mathcal{K}}_{\widetilde{T}_{k-1}},\widetilde{T}_k<T)=$$
$$\left(\frac{\widetilde{K}_{\widetilde{T}_{k-1}}-\frac{K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}}{\widetilde{K}_{\widetilde{T}_{k-1}}}\right)\times\left[\frac{K_{\widetilde{T}_{k-1}}-\frac{K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}}{\widetilde{K}_{\widetilde{T}_{k-1}}-\frac{K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}}\times\frac{K^i_{\widetilde{T}_{k-1}}}{K_{\widetilde{T}_{k-1}}}\right.$$
$$\left.+\frac{\widetilde{K}_{\widetilde{T}_{k-1}}-K_{\widetilde{T}_{k-1}}}{\widetilde{K}_{\widetilde{T}_{k-1}}-\frac{K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}}\times\frac{\widetilde{K}^i_{\widetilde{T}_{k-1}}-K^i_{\widetilde{T}_{k-1}}}{\widetilde{K}_{\widetilde{T}_{k-1}}-K_{\widetilde{T}_{k-1}}}\right]$$

$$+\frac{\left(\frac{K_{\widetilde{T}_{k-1}}(N-K_{\widetilde{T}_{k-1}})_+}{N-1}\right)}{\widetilde{K}_{\widetilde{T}_{k-1}}}\times\frac{K^i_{\widetilde{T}_{k-1}}}{K_{\widetilde{T}_{k-1}}}=\frac{\widetilde{K}^i_{\widetilde{T}_{k-1}}}{\widetilde{K}_{\widetilde{T}_{k-1}}}\,,\quad(6.28)$$

and this last expression depends only on $\widetilde{\mathcal{K}}_{\widetilde{T}_k-}$. By point 1 of the lemma, the process $(\widetilde{K}_s)_{s\geqslant 0}$ is equal in law to the sum of q independent Yule processes $Y^{(1)}_s, \dots, Y^{(q)}_s$, and its law is thus independent of N (see [1], p. 102-109, p. 109 for the law of the Yule process). We have, for all s,

$$\mathbb{P}(Y^{(1)}_s=k)=e^{-s\Lambda}(1-e^{-s\Lambda})^{k-1}\quad(6.29)$$

and so (see for example [5], p. 288),

$$\mathbb{P}(\widetilde{K}_t=k)=\mathbb{P}(Y^{(1)}_t+\dots+Y^{(q)}_t=k)=\begin{pmatrix}k\\q-1\end{pmatrix}(e^{-\Lambda t})^q(1-e^{-\Lambda t})^{k-q}\,.\quad(6.30)$$

Using the point 1. of the lemma, Equation (6.28) and the point 2. of Lemma 3.3, we obtain the point 4. of the Lemma.

Reasoning as above, we can show that, conditionally to $\mathcal{K}_T$, $\widetilde{\mathcal{K}}_T$, the process $(\#\{T_k, T_k=T''_k, T_k\leqslant t\})_{0\leqslant t\leqslant T}$ is a homogeneous Poisson process of rate $\left(\frac{\Lambda K_t(K_t-1)}{2(N-1)}\right)_{0\leqslant t\leqslant T}$ and the process $(\#\{\widetilde{T}_k, \widetilde{T}_k=\widetilde{T}''_k, \widetilde{T}_k\leqslant t\})_{0\leqslant t\leqslant T}$ is a homogeneous Poisson process of rate

$$\left(\frac{\Lambda\widetilde{K}_t(\widetilde{K}_t-1)-\Lambda K_t(K_t-1)}{2(N-1)}\right)_{0\leqslant t\leqslant T}\,.$$

So, by Lemma 3.3, 3.,

$$\mathbb{P}(t\in\{T''_l, l\geqslant 1\}|\mathcal{K}_T,\widetilde{\mathcal{K}}_T,\Delta\widetilde{L}_t=1)=\frac{\Lambda K_t(K_t-1)}{\Lambda\widetilde{K}_t(\widetilde{K}_t-1)}\,.$$

We have, for all i (recalling (3.2)),

$$\mathbb{P}(\Delta\widetilde{L}^{\{i,i+1\}}_t=1|\mathcal{K}_T,\widetilde{\mathcal{K}}_T,t\in\{T''_l,l\geqslant 1\})=\frac{2K^i_tK^{i+1}_t}{K_t(K_t-1)}\,,$$

and (recalling (3.5), (3.6))

$$\mathbb{P}(\Delta\widetilde{L}^{\{i,i+1\}}_t=1|\mathcal{K}_T,\widetilde{\mathcal{K}}_T,t\in\{\widetilde{T}''_l,l\geqslant 1\})$$
$$=\frac{(\widetilde{K}_t-K_t)K_t+(\widetilde{K}_t-K_t)(\widetilde{K}_t-K_t-1)}{\widetilde{K}_t(\widetilde{K}_t-1)-K_t(K_t-1)}$$

$$\times\left(\frac{(\widetilde{K}_t^i-K_t^i)\widetilde{K}_t^{i+1}}{(\widetilde{K}_t-K_t)(\widetilde{K}_t-1)}+\frac{(\widetilde{K}_t^{i+1}-K_t^{i+1})\widetilde{K}_t^i}{(\widetilde{K}_t-K_t)(\widetilde{K}_t-1)}\right)$$

$$+\frac{(\widetilde{K}_t-K_t)K_t}{\widetilde{K}_t(\widetilde{K}_t-1)-K_t(K_t-1)}\times\left(\frac{(\widetilde{K}_t^i-K_t^i)K_t^{i+1}}{(\widetilde{K}_t-K_t)K_t}+\frac{(\widetilde{K}_t^{i+1}-K_t^{i+1})K_t^i}{(\widetilde{K}_t-K_t)K_t}\right)=$$

$$\frac{(\widetilde{K}_t^i-K_t^i)(\widetilde{K}_t^{i+1}+K_t^{i+1})+(\widetilde{K}_t^{i+1}-K_t^{i+1})(\widetilde{K}_t^i+K_t^i)}{\widetilde{K}_t(\widetilde{K}_t-1)-K_t(K_t-1)}\,.$$

So,

$$\mathbb{P}(\Delta\widetilde{L}_t^{\{i,i+1\}}=1|\mathcal{K}_T,\widetilde{\mathcal{K}}_T,\Delta\widetilde{L}_t=1)=\left(\frac{\widetilde{K}_t(\widetilde{K}_t-1)-K_t(K_t-1)}{\widetilde{K}_t(\widetilde{K}_t-1)}\right)$$

$$\times\left(\frac{(\widetilde{K}_t^i-K_t^i)(\widetilde{K}_t^{i+1}+K_t^{i+1})+(\widetilde{K}_t^{i+1}-K_t^{i+1})(\widetilde{K}_t^i+K_t^i)}{\widetilde{K}_t(\widetilde{K}_t-1)-K_t(K_t-1)}\right)$$

$$+\left(\frac{K_t(K_t-1)}{\widetilde{K}_t(\widetilde{K}_t-1)}\right)\left(\frac{2K_t^iK_t^{i+1}}{K_t(K_t-1)}\right)=\frac{2\widetilde{K}_t^i\widetilde{K}_t^{i+1}}{\widetilde{K}_t(\widetilde{K}_t-1)}\,,$$

so,

$$\mathbb{P}(\Delta\widetilde{L}_t^{\{i,i+1\}}=1|\widetilde{\mathcal{K}}_T,\Delta\widetilde{L}_t=1)=\frac{2\widetilde{K}_t^i\widetilde{K}_t^{i+1}}{\widetilde{K}_t(\widetilde{K}_t-1)}\,.$$

So, using Lemma 3.3, 2. we have the point 5. of the lemma. □

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