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The local criteria for blowup of the Dullin-Gottwald-Holm equation and the two-component Dullin-Gottwald-Holm system

DUC-TRUNG HOANG⁽¹⁾

RÉSUMÉ. – Nous étudions les critères d’explosion pour l’équation de Dullin-Gottwald-Holm et pour le système de Dullin-Gottwald-Holm à deux composantes. Nous établissons un nouveau critère d’explosion pour le cas général $\gamma + c_0\alpha^2 \geq 0$, impliquant des conditions locales en espace sur les données initiales.

ABSTRACT. – We investigate wave breaking criteria for the Dullin-Gottwald-Holm equation and the two-component Dullin-Gottwald-Holm system. We establish a new blow-up criterion for the general case $\gamma + c_0\alpha^2 \geq 0$ involving local-in-space conditions on the initial data.

1. Introduction

The DGH equation is a nonlinear dispersive equation, modelling the propagation of unidirectional shallow waters over a flat bottom. It was proposed by Dullin, Gottwald and Holm [8] in 2001 and derived from the water wave theory by using the method of asymptotic analysis and a near identity normal transformation. The equation reads

$$\begin{cases} u_t - \alpha^2 u_{txx} + c_0 u_x + 3uu_x + \gamma u_{xxx} = \alpha^2 (2u_x u_{xx} + uu_{xxx}), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0. \end{cases} \quad (1.1)$$

Here, u stands for a fluid velocity in the x direction and c_0, α^2, γ are physical parameters: α^2 and γ/c are squares of length scales. The constant $c_0 = \sqrt{gh}$

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is the critical shallow water speed, while h is the mean fluid depth and g is the gravitational constant. In [8], the authors proved that the phase speed lies in the band $(-\frac{\gamma}{\alpha^2}, c_0)$ and longer linear wave are faster provided that $\gamma + c_0\alpha^2 \geq 0$.

Let $m = u - \alpha^2 u_{xx}$ be the momentum variable. Equation (1.1) can be rewritten in terms of the momentum as

$$\begin{cases} m_t + c_0 u_x + um_x + 2mu_x = -\gamma u_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ m(0, x) = m_0. \end{cases} \quad (1.2)$$

We denote by

$$p(x) = \frac{1}{2\alpha} e^{-|\frac{x}{\alpha}|}$$

the Green function for the operator $Q := (1 - \alpha^2 \partial_x^2)^{-1}$, in such a way that $Qf = (1 - \alpha^2 \partial_x^2)^{-1} f = p * f$ for $f \in L_2(\mathbb{R})$. The convolution relation $p * m = u$ allows to recover u from m .

The DGH equation can also be reformulated as a quasi-linear evolution equation of hyperbolic type :

$$\begin{cases} u_t + (u - \frac{\gamma}{\alpha^2})u_x = -\partial_x p * (\frac{\alpha^2}{2}u_x^2 + u^2 + (c_0 + \frac{\gamma}{\alpha^2})u), & t > 0, \quad x \in \mathbb{R} \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (1.3)$$

When $\gamma = 0$ and $\alpha = 1$ the system (1.1) boils down to the Camassa-Holm equation (the dispersionless Camassa-Holm equation if in addition $c_0 = 0$), which was derived by Camassa and Holm [7] by approximating directly the Hamiltonian for the Euler equations for an irrotational flow in the shallow water regime. In the past decades, a considerable number of papers investigated various properties of Camassa-Holm equation such as local well-posedness, blow-up phenomena, persistence properties of solutions, global existence of weak solutions.

Similarly to the Camassa-Holm equation, equation (1.2) preserves the bi-Hamiltonian structure and is completely integrable. It has solitary wave solutions [8], and the bi-Hamiltonian structure is described as follows:

$$m_t = -B_2 \frac{\delta E}{\delta m} = -B_1 \frac{\delta F}{\delta m},$$

where

$$\begin{aligned} B_1 &= \delta_x - \alpha^2 \delta_x^3, \\ B_2 &= \delta_x(m + \frac{c_0}{2}) + (m + \frac{c_0}{2})\delta_x + \gamma \delta_x^3. \end{aligned}$$

Important conserved quantities are:

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} (u^2 + \alpha^2 u_x^2),$$

$$F(u) = \frac{1}{2} \int_{\mathbb{R}} (u^3 + \alpha^3 u u_x^2 + c_0 u^2 - \gamma u_x^2).$$

In this paper we will also consider the two-component DGH equation that reads as follows:

$$\begin{cases} u_t - \alpha^2 u_{txx} + c_0 u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}) - \sigma \rho \rho_x, & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0 & t > 0, \quad x \in \mathbb{R}, \\ \rho(0, x) = \rho_0, \\ u(0, x) = u_0. \end{cases} \quad (1.4)$$

As shown by Constantin and Ivanov [6], system (1.4) can be derived from the shallow water theory. When $\alpha = 1$ and $\gamma = 0$, we get the two-component Camassa-Holm system.

From a geometrical meaning, the two-component DHG system corresponds to a geodesic flow on the semidirect product Lie group of diffeomorphisms acting on densities, respected to the H^1 norm of velocity and the L^2 norm of the density. In a hydrodynamical context, we consider $\sigma = 1$, and the natural boundary conditions are $u \rightarrow 0$ and $\rho \rightarrow 1$ as $x \rightarrow \infty$, for any t . Let $\tilde{\rho} = \rho - 1$, then $\tilde{\rho} \rightarrow 0$ as $x \rightarrow \infty$.

In this case, we have:

$$\begin{cases} u_t - \alpha^2 u_{txx} + c_0 u_x + 3uu_x + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + uu_{xxx}) - \tilde{\rho} \tilde{\rho}_x - \tilde{\rho}_x, & t > 0, \quad x \in \mathbb{R}, \\ \tilde{\rho}_t + (u\tilde{\rho})_x + u_x = 0 & t > 0, \quad x \in \mathbb{R}, \\ \tilde{\rho}(0, x) = \tilde{\rho}_0, \\ u(0, x) = u_0. \end{cases} \quad (1.5)$$

System (1.5) has two Hamiltonians:

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} (u^2 + \alpha^2 u_x^2 + \rho^2),$$

$$F(u) = \frac{1}{2} \int_{\mathbb{R}} (u^3 + \alpha^3 u u_x^2 + c_0 u^2 - \gamma u_x^2 + 2u\rho + u\rho^2).$$

As before, (1.5) is more conveniently reformulated as

$$\begin{cases} u_t + (u - \frac{\gamma}{\alpha^2})u_x = -\partial_x p * (\frac{\alpha^2}{2}u_x^2 + u^2 + (c_0 + \frac{\gamma}{\alpha^2})u + \frac{1}{2}\tilde{\rho}^2 + \tilde{\rho}), & t > 0, \quad x \in \mathbb{R} \\ \tilde{\rho}_t + u\tilde{\rho}_x = -u_x\tilde{\rho} - u_x, & t > 0, \quad x \in \mathbb{R} \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ \tilde{\rho}(0, x) = \tilde{\rho}_0(x) & x \in \mathbb{R} \end{cases} \quad (1.6)$$

The present paper addresses the problem of establishing wave breaking criteria for the DGH equation and the two-component DGH system. Previous results in this directions were obtained, *e.g.*, in [22] for the two-component DGH system and in [23] and [15] for DGH equation, where the authors dealt with the special case $\gamma + c_0\alpha^2 = 0$ and $\alpha > 0$ obtaining the finite time blowup of solution arising from certain initial profiles. But the blowup conditions in the above mentioned papers involve the computation of some global quantities associated with the initial datum (Sobolev norms, or integral conditions, or otherwise sign conditions, or antisymmetry relations, etc.). Motivated by the recent paper [1], we would like to establish a “local-in-space” blowup criterion, *i.e.* a criterion involving the properties of the initial datum only in a small neighborhood of a *single point*. Such criterion will be more general (and more natural), than earlier blowup results. In addition, our approach will also go through when $\gamma + c_0\alpha^2$ is not necessarily zero.

We will assume $\alpha > 0$. Notice that when $\alpha = 0$ the DGH equation reduces to the KdV equation, for which the solutions exist globally and no blowup result can be obtained. The following theorem represents our main result.

THEOREM 1.1. — *Let T^* be the maximal time of existence of a unique solution $u \in C([0, T^*]; H^s) \cap C^1([0, T^*]; H^{s-1})$ of the Cauchy problem for the DGH equation (1.3), arising from an initial datum $u_0 \in H^s(\mathbb{R})$, with $s > \frac{3}{2}$. If there exists $x_0 \in \mathbb{R}$ such that:*

$$u'_0(x_0) < -\frac{1}{\alpha}|u_0(x_0) + \frac{1}{2}(c_0 + \frac{\gamma}{\alpha^2})|,$$

then $T^ < \infty$.*

Our second result concerns the two-component DGH equation. It extends the recent blowup result [16] on the dispersionless two-component Camassa–Holm equations as well as several results quoted therein.

THEOREM 1.2. — *Suppose that $\gamma = 0$. Let T^* be the maximal time of the unique solution $(u, \tilde{\rho}) \in C([0, T^*]; H^s \times H^{s-1}) \cap C^1([0, T^*]; H^{s-1} \times H^{s-2})$*

of the integrable two-component DGH system (1.5), starting from $(u_0, \tilde{\rho}_0) \in H^s \times H^{s-1}$ with $s \geq \frac{5}{2}$. If there exists $x_0 \in \mathbb{R}$ such that:

- (i) $\tilde{\rho}_0(x_0) = -1$,
- (ii) $u'_0(x_0) < -\frac{1}{\alpha} |u_0(x_0) + \frac{1}{2}c_0|$,

then $T^* < \infty$.

2. Preliminaries

Using Kato's theory [13], one can establish the local well-posedness theorem for the DGH equation (1.1) and the two-component DGH system (1.4). For example, the equation (1.3) can be rewritten as

$$\begin{cases} \frac{du}{dt} + A(u) = H(u) \\ u(x, 0) = u_0(x), \end{cases} \quad (2.1)$$

with $A(u) = (u + \lambda)u_x$ and $H(u) = -\partial_x p^* (\frac{\alpha^2}{2}u_x^2 + u^2 + (c_0 + \frac{\gamma}{\alpha^2})u)$. Several proofs of local existence theory can be found in [4], [14], [18], [19] for the DGH equation, and in [11], [25] for the two-component DGH system. Here we recall the following result:

THEOREM 2.1 (See [20]). — *Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ and $\gamma + c_0\alpha^2 \geq 0$ of (1.1). Then there exists $T^* = T(\|u_0\|_{H^s}) > 0$ and a unique solution*

$$u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$$

of equation (1.3). Furthermore, the solution u depends continuously on the initial data u_0 .

THEOREM 2.2 (See [25]). — *Given $(u_0, \tilde{\rho}_0) \in H^s \times H^{s-1}$, $s \geq \frac{5}{2}$ and $\gamma + c_0\alpha^2 \geq 0$ of (1.5).*

Then there exists $T^ = T(\|u_0, \tilde{\rho}_0\|_{H^s \times H^{s-1}}) > 0$ and a unique solution*

$$(u, \tilde{\rho}) \in C([0, T^*]; H^s \times H^{s-1}) \cap C^1([0, T^*]; H^{s-1} \times H^{s-2})$$

of the system (3.4). Furthermore, the solution $(u, \tilde{\rho})$ depends continuously on the initial data $(u_0, \tilde{\rho}_0)$.

The maximal time of existence T^* is known to be independent of the parameter s . Moreover, if T^* is finite then $\lim_{t \rightarrow T^*} \|u(t)\|_{H^s} = \infty$. Next theorem tells something more about such blowup (or wave-breaking) scenario.

THEOREM 2.3 (See [20]). — *Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Then the solution $u(t, x)$ of the DGH equation is uniformly bounded on $[0, T)$. Moreover, a blowup occurs at the time $T < \infty$ if and only if*

$$\lim_{t \rightarrow T} \inf_{x \in \mathbb{R}} (u_x(t, x)) = -\infty.$$

THEOREM 2.4 (See [25]). — *Given $(u_0, \tilde{\rho}_0) \in H^s \times H^{s-1}$, $s \geq \frac{5}{2}$. Then the solution $u, \tilde{\rho}$ of the two-component DGH equation is uniformly bounded on $[0, T)$. Moreover, a blowup occurs at the time $T < \infty$ if and only if*

$$\liminf_{t \rightarrow T} (\inf_{x \in \mathbb{R}} (u_x(t, x))) = -\infty.$$

Following McKean's approach for the Camassa-Holm equation [5], we introduce the particle trajectory $q(t, x) \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$, defined by

$$\begin{cases} q_t(t, x) = u(t, q(t, x)) - \frac{\gamma}{\alpha^2}, & t \in [0, T^*) \\ q(0, x) = x. \end{cases} \quad (2.2)$$

For every fixed $t \in [0, T)$, $q(t, \cdot)$ is an increasing diffeomorphism of the real line. In fact, taking the derivative with respect to x , yields

$$\frac{dq_t}{dx} = q_{xt} = u_x(t, q(t, x))q_x.$$

Then

$$q_x(x, t) = \exp \int_0^t u_x(q, s) ds.$$

The momentum m satisfies the fundamental identity,

$$m_0(x) + \frac{c_0}{2} + \frac{\gamma}{2\alpha^2} = \left(m(t, q(t, x)) + \frac{c_0}{2} + \frac{\gamma}{2\alpha^2} \right) q_x^2(t, x),$$

putting in evidence the specific properties of the momentum in the case $c_0 + \frac{\gamma}{\alpha^2} = 0$.

3. Convolution estimates

First of all, we state some useful results that will be used in the proof of Theorem 1.1. For convenience, let us set

$$\lambda = -\frac{\gamma}{\alpha^2} \quad \text{and} \quad k = \frac{1}{2} \left(c_0 + \frac{\gamma}{\alpha^2} \right).$$

The following lemma generalizes the estimates in [1]:

LEMMA 3.1. — *With the above notations, we have the following inequalities, for all $u \in H^s(\mathbb{R})$ where $s > \frac{3}{2}$,*

$$(p - \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \geq \frac{(u+k)^2}{2} - k^2 \quad (3.1)$$

and

$$(p + \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \geq \frac{(u+k)^2}{2} - k^2. \quad (3.2)$$

The above inequalities are sharp. The equality holds with the choice $u = ce^{-\frac{|x-y|}{\alpha}} - k$, with $c, y \in \mathbb{R}$.

Proof. — We denote by $1_{\mathbb{R}^+}$ and $1_{\mathbb{R}^-}$ the characteristic functions of \mathbb{R}^+ and \mathbb{R}^- respectively. Then:

$$\begin{aligned}
 (p - \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) &= (p + \text{sign}(x)p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \\
 &= 2p1_{\mathbb{R}^+} * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \\
 &= \frac{1}{\alpha} \int_{-\infty}^x e^{\frac{y-x}{\alpha}} \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right)(y) dy \\
 &= \frac{1}{\alpha} \int_{-\infty}^x e^{\frac{y-x}{\alpha}} \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right)(y) dy \quad (3.3) \\
 &\quad - \frac{k^2}{\alpha} \int_{-\infty}^x e^{\frac{y-x}{\alpha}} dy \\
 &= \frac{1}{\alpha} \int_{-\infty}^x e^{\frac{y-x}{\alpha}} \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right)(y) dy - k^2.
 \end{aligned}$$

Using Cauchy inequality, we have:

$$\begin{aligned}
 \int_{-\infty}^x e^{\frac{y}{\alpha}} (\alpha^2 u_x^2 + (u+k)^2)(y) dy &\geq 2\alpha \int_{-\infty}^x e^{\frac{y}{\alpha}} (u+k)u_x(y) dy \\
 &= \alpha e^{\frac{x}{\alpha}} (u+k)^2 - \int_{-\infty}^x e^{\frac{y}{\alpha}} (u+k)^2(y) dy.
 \end{aligned}$$

It follows that

$$\frac{1}{\alpha} \int_{-\infty}^x e^{\frac{y-x}{\alpha}} \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right)(y) dy \geq \frac{(u+k)^2}{2}.$$

So, (3.1) is obtained. Similarly, one proves (3.2). \square

LEMMA 3.2. — *With the above definitions, we have the following inequality:*

$$p * \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right) \geq \frac{(u+k)^2}{2}. \quad (3.4)$$

Proof. — Observe that

$$p * \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right) = p1_{\mathbb{R}^+} * \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right) + p1_{\mathbb{R}^-} * \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right).$$

Following Lemma 3.1, we have:

$$p1_{\mathbb{R}^+} * \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right) \geq \frac{(u+k)^2}{4}$$

and

$$p1_{\mathbb{R}^-} * \left(\frac{\alpha^2}{2} u_x^2 + (u+k)^2 \right) \geq \frac{(u+k)^2}{4}.$$

The result is obtained □

Now, for any $f \in H^1(\mathbb{R})$, we have the obvious inequality:

$$\min\{\alpha^2, 1\} \|f\|_{H^1}^2 \leq \int_{\mathbb{R}} (f^2 + \alpha^2 f_x^2) dx \leq \max\{\alpha^2, 1\} \|f\|_{H^1}^2.$$

If we define:

$$\|f\|_{H_\alpha^1}^2 = \int_{\mathbb{R}} (f^2 + \alpha^2 f_x^2) dx,$$

then $H_\alpha^1 \subset L^\infty$. The next lemma estimates the Sobolev constant related to the embedding $H_\alpha^1 \subset L^\infty$ in \mathbb{R}

LEMMA 3.3. — *We have the Sobolev embedding inequality:*

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2\alpha}} \|u\|_{H_\alpha^1}.$$

The equality can be attained. For example, we take $u = ce^{-\frac{|x-y|}{\alpha}}$ for some $c, y \in \mathbb{R}$.

Proof. — For any $y \in \mathbb{R}$, we have:

$$\begin{aligned} (u)^2(y) &= \int_{-\infty}^y uu_x dx - \int_y^{+\infty} uu_x dx \\ &\leq \frac{1}{2\alpha} \left(\int_{-\infty}^y (u^2 + \alpha^2 u_x^2) dx + \int_y^{+\infty} (u^2 + \alpha^2 u_x^2) dx \right) \\ &= \frac{1}{2\alpha} \|u\|_{H_\alpha^1}^2. \end{aligned} \tag{3.5}$$

So, we obtain:

$$\|u\|_{L^\infty} \leq \frac{1}{\sqrt{2\alpha}} \|u\|_{H_\alpha^1}.$$

□

4. The local-in-space criterion for blow-up of the DGH equation

Recall that, by definition, $(1 - \alpha^2 \partial_x^2)Qf = f$ for any $f \in L^2(\mathbb{R})$ so that $p * f - f = \alpha^2 \partial_x^2(p * f)$. Taking the derivative with respect to x in (1.3) yields:

$$\begin{cases} u_{tx} + (u + \lambda)u_{xx} = \frac{-u_x^2}{2} + \frac{u^2 + 2ku}{\alpha^2} - \frac{1}{\alpha^2}p * \left(\frac{\alpha^2}{2}u_x^2 + u^2 + 2ku\right) \\ u(x, 0) = u_0(x). \end{cases}$$

For any $0 < T < T^*$, we see that u and u_x are continuous on $[0, T) \times \mathbb{R}$, and $u(t, x)$ is Lipschitz, uniformly respected to t . So, the flow map $q(t, x)$ introduced in (2.2):

$$\begin{cases} q_t(t, x) = u(t, q(t, x)) + \lambda & t \in [0, T^*) \\ q(0, x) = x, \end{cases} \quad (4.1)$$

is indeed well defined in the interval $[0, T)$ with $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$.

We have

$$\begin{aligned} \frac{d}{dt}[u_x(t, q(t, x))] &= [u_{tx} + u_{xx}(u + \lambda)](t, q(t, x)) \\ &= \frac{-u_x^2}{2} + \frac{u^2 + 2ku}{\alpha^2} - \frac{1}{\alpha^2}p * \left(\frac{\alpha^2}{2}u_x^2 + u^2 + 2ku\right) \\ &= \frac{-u_x^2}{2} + \frac{(u + k)^2}{\alpha^2} - \frac{1}{\alpha^2}p * \left(\frac{\alpha^2}{2}u_x^2 + (u + k)^2\right), \end{aligned}$$

where we used the fact that $\int_{\mathbb{R}} p(t, x) = 1$. By the inequality (3.2), $\frac{1}{\alpha^2}p * \left(\frac{\alpha^2}{2}u_x^2 + (u + k)^2\right) \geq \frac{(u+k)^2}{2\alpha^2}$. Hence,

$$\begin{aligned} \frac{d}{dt}[u_x(t, q(t, x))] &\leq \left(-\frac{1}{2}u_x^2 + \frac{(u + k)^2}{\alpha^2} - \frac{(u + k)^2}{2\alpha^2}\right)(t, q(t, x)) \\ &= \left(-\frac{1}{2}u_x^2 + \frac{1}{2\alpha^2}(u + k)^2\right)(t, q(t, x)). \end{aligned}$$

Inspired by [1], we now introduce

$$A(t, x) = e^{\frac{q(t, x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{1}{\alpha}(u + k) - u_x\right)(t, q(t, x))$$

and

$$B(t, x) = e^{-\frac{q(t, x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(\frac{1}{\alpha}(u + k) + u_x\right)(t, q(t, x)).$$

Then,

$$\frac{d}{dt}[u_x(t, q(t, x))] \leq \frac{1}{2}(AB)(t, q(t, x)).$$

The following result plays an important role:

LEMMA 4.1. — For all $x \in \mathbb{R}$, the map $A(t, x)$ is monotonically increasing and $B(t, x)$ is monotonically decreasing with respect to $t \in [0, T]$.

Proof. — Let us calculate $\frac{d}{dt}A(t, x)$:

$$\begin{aligned}
 \frac{d}{dt}A(t, x) &= e^{\frac{q(t,x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{1}{\alpha}(u + \lambda) \left(\frac{1}{\alpha}(u + k) - u_x \right) - \frac{k - \lambda}{\alpha} \left(\frac{1}{\alpha}(u + k) - u_x \right) \right. \\
 &\quad \left. + \left(\frac{1}{\alpha}(u + k) - u_x \right)_t + (u + \lambda) \left(\frac{1}{\alpha}(u + k) - u_x \right)_x \right) \\
 &= e^{\frac{q(t,x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{u + k}{\alpha} \left(\frac{1}{\alpha}(u + k) - u_x \right) + \frac{1}{\alpha}(u_t + (u + \lambda)u_x) \right. \\
 &\quad \left. - (u_{xt} + (u + \lambda)u_{xx}) \right) \\
 &= e^{\frac{q(t,x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{u + k}{\alpha} \left(\frac{1}{\alpha}(u + k) - u_x \right) - \frac{1}{\alpha} \partial_x p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \right. \\
 &\quad \left. + \frac{u_x^2}{2} - \frac{u^2 + 2ku}{\alpha^2} + \frac{1}{\alpha^2} p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \right) \\
 &= e^{\frac{q(t,x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{u_x^2}{2} - \frac{(u + k)u_x}{\alpha} + \frac{k^2}{\alpha^2} \right. \\
 &\quad \left. + \frac{1}{\alpha^2} (p - \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \right).
 \end{aligned}$$

By (3.1), $(p - \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \geq \frac{(u+k)^2}{2} - k^2$. Then we deduce that

$$\begin{aligned}
 \frac{d}{dt}A(t, x) &\geq e^{\frac{q(t,x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{u_x^2}{2} - \frac{(u + k)u_x}{\alpha} + \frac{k^2}{\alpha^2} + \frac{1}{\alpha^2} \left(\frac{(u + k)^2}{2} - k^2 \right) \right) \\
 &= e^{\frac{q(t,x)}{\alpha} + \frac{(k-\lambda)t}{\alpha}} \left(\frac{u_x^2}{2} - \frac{(u + k)u_x}{\alpha} + \frac{(u + k)^2}{2\alpha^2} \right).
 \end{aligned} \tag{4.2}$$

So, for all x , $t \mapsto A(t, x)$ is monotonically increasing. Similarly

$$\begin{aligned}
 \frac{d}{dt}B(t, x) &= e^{-\frac{q(t,x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(-\frac{1}{\alpha}(u + \lambda) \left(\frac{1}{\alpha}(u + k) + u_x \right) \right. \\
 &\quad \left. + \frac{-k + \lambda}{\alpha} \left(\frac{1}{\alpha}(u + k) + u_x \right) \right. \\
 &\quad \left. + \left(\frac{1}{\alpha}(u + k) + u_x \right)_t + (u + \lambda) \left(\frac{1}{\alpha}(u + k) + u_x \right)_x \right) \\
 &= e^{-\frac{q(t,x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(-\frac{u + k}{\alpha} \left(\frac{1}{\alpha}(u + k) + u_x \right) \right. \\
 &\quad \left. + \frac{1}{\alpha} (u_t + (u + \lambda)u_x) + (u_{xt} + (u + \lambda)u_{xx}) \right)
 \end{aligned}$$

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$$\begin{aligned}
&= e^{-\frac{q(t,x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(-\frac{u+k}{\alpha} \left(\frac{1}{\alpha} (u+k) + u_x \right) \right. \\
&\quad \left. - \frac{1}{\alpha} \partial_x p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) - \frac{u_x^2}{2} + \frac{u^2 + 2ku}{\alpha^2} \right. \\
&\quad \left. - \frac{1}{\alpha^2} p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \right) \\
&= -e^{-\frac{q(t,x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(\frac{u_x^2}{2} + \frac{(u+k)u_x}{\alpha} + \frac{k^2}{\alpha^2} \right. \\
&\quad \left. + \frac{1}{\alpha^2} (p + \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right) \right).
\end{aligned}$$

Applying now the second estimate of Lemma 3.1 we obtain:

$$\begin{aligned}
\frac{d}{dt} B(t, x) &\leq -e^{-\frac{q(t,x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(\frac{u_x^2}{2} + \frac{(u+k)u_x}{\alpha} + \frac{k^2}{\alpha^2} + \frac{(u+k)^2}{2} - \frac{k^2}{\alpha^2} \right) \\
&= -e^{-\frac{q(t,x)}{\alpha} + \frac{(-k+\lambda)t}{\alpha}} \left(\frac{u_x^2}{2} + \frac{(u+k)u_x}{\alpha} + \frac{(u+k)^2}{2\alpha^2} \right) \\
&\leq 0.
\end{aligned}$$

So, for all $x, t \mapsto B(t, x)$ is monotonically decreasing. \square

Now, we are ready to prove the main theorem.

Proof of Theorem 1.1. — Let x_0 be such that $u'_0(x_0) < -\frac{1}{\alpha}|u_0(x_0) + \frac{1}{2}(c_0 + \frac{\gamma}{\alpha^2})| = -\frac{1}{\alpha}|u_0(x_0) + k|$. We denote

$$g(t) = u_x(t, q(t, x_0)),$$

$A(t) = A(t, x_0)$ and $B(t) = B(t, x_0)$. For all $t \in [0, T^*)$, we have: $\frac{d}{dt} A(t) \geq 0$ and $\frac{d}{dt} B(t) \leq 0$.

So, $A(t) \geq A(0) = e^{\frac{x_0}{\alpha}} \left(\frac{1}{\alpha} (u_0(x_0) + k) - u'_0(x_0) \right) > 0$, and $B(t) \leq B(0) = e^{-\frac{x_0}{\alpha}} \left(\frac{1}{\alpha} (u_0(x_0) + k) + u'_0(x_0) \right) < 0$. Thus $AB(t) \leq AB(0) < 0$. Then for all $t \in [0, T)$: $g'(t) < 0$.

Assume by contradiction $T^* = \infty$, then $g(t) \leq g(0) - \alpha_0 t$ where $\alpha_0 = \frac{1}{2}(u'_0(x_0)^2 - \frac{1}{\alpha^2}(u_0 + k)^2)(x_0)$. We choose t_0 such that $g(0) - \alpha_0 t_0 \leq 0$ and $(g(0) - \alpha_0 t_0)^2 \geq \frac{1}{\alpha^3} (\|u_0\|_{H^1_\alpha} + k\sqrt{2\alpha})^2$. For $t \geq t_0$, we have:

$$\begin{aligned}
g'(t) &\leq \frac{1}{2} \left(\frac{1}{\alpha^2} (u+k)^2 - u_x^2 \right) (t, q(t, x_0)) \\
&\leq \frac{1}{2} \left(\frac{1}{\alpha^2} (\|u(t, \cdot)\|_{L^\infty} + k)^2 - g(t)^2 \right).
\end{aligned}$$

Using Sobolev embedding inequality in lemma (3.3) and the energy conservation identity, one has:

$$\begin{aligned} g'(t) &\leq \frac{1}{2} \left(\frac{1}{\alpha^2} \left(\frac{\|u_0\|_{H^1_\alpha}}{\sqrt{2\alpha}} + k \right)^2 - g(t)^2 \right) \\ &\leq -\frac{1}{4} g(t)^2 \end{aligned}$$

for all $t \in (t_0, \infty)$. Dividing both sides by $g(t)^2$ and integrating, we get

$$\frac{1}{g(t_0)} - \frac{1}{g(t)} + \frac{1}{4}(t - t_0) \leq 0 \quad t \geq t_0.$$

This is a contradiction since $-\frac{1}{g(t)} > 0$ and $\frac{1}{4}(t - t_0) \rightarrow \infty$ as $t \rightarrow \infty$. Thus $u_x(t, g(t, x_0))$ blows up in finite time and $T^* \leq t_0 + \frac{4}{|g(t_0)|} < \infty$. \square

Remark. — Local-in-space blowup criterion in the particular case $\gamma = c_0 = 0$ and $\alpha = 1$ (corresponding to the Camassa-Holm equation) has been first built in [1] and later extended in [2] to a class of possibly non-quadratic nonlinearities. See also [3] for improvements specific to the periodic case. Our Theorem 1.1 improves the result of [1] in a different direction, by extending the blowup result to arbitrary values of γ , c_0 and $\alpha > 0$.

5. Blow-up for two-component DGH system

When $\gamma = 0$, the equation (1.5) becomes

$$\begin{cases} u_t - \alpha^2 u_{txx} + c_0 u_x + 3uu_x = \alpha^2(2u_x u_{xx} + uu_{xxx}) - \tilde{\rho} \tilde{\rho}_x - \tilde{\rho}_x, & t > 0, \quad x \in \mathbb{R}, \\ \tilde{\rho}_t + (u \tilde{\rho})_x + u_x = 0 & t > 0, \quad x \in \mathbb{R}, \\ \tilde{\rho}(0, x) = \tilde{\rho}_0, \\ u(0, x) = u_0. \end{cases} \quad (5.1)$$

This can be rewritten as

$$\begin{cases} u_t + uu_x = -\partial_x p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + c_0 u + \frac{1}{2} \tilde{\rho}^2 + \tilde{\rho} \right), & t > 0, \quad x \in \mathbb{R} \\ \tilde{\rho}_t + u \tilde{\rho}_x = -u_x \tilde{\rho} - u_x, & t > 0, \quad x \in \mathbb{R} \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ \tilde{\rho}(0, x) = \tilde{\rho}_0(x) & x \in \mathbb{R} \end{cases} \quad (5.2)$$

Here we give the proof for Theorem 1.2.

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Proof. Again, using the identity $p*f - f = \alpha^2 \partial_x^2(p*f)$, we take the derivative with respect to x in (1.6) which yields:

$$u_{tx} + uu_{xx} = \frac{-u_x^2}{2} + \frac{u^2 + c_0 u}{\alpha^2} + \frac{1}{\alpha^2} \left(\frac{\tilde{\rho}^2}{2} + \tilde{\rho} \right) - \frac{1}{\alpha^2} p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + c_0 u + \frac{\tilde{\rho}^2}{2} + \tilde{\rho} \right).$$

As before, we make use of the flow map, defined as in (4.1). When $\gamma = 0$, the map becomes

$$\begin{cases} q_t(t, x) = u(t, q(t, x)) & t \in [0, T^*) \\ q(0, x) = x. \end{cases} \quad (5.3)$$

Notice that $q \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$. We have

$$\begin{aligned} \frac{d}{dt} [u_x(t, q(t, x))] &= [u_{tx} + uu_{xx}](t, q(t, x)) \\ &= \frac{-u_x^2}{2} + \frac{u^2 + c_0 u}{\alpha^2} + \frac{1}{\alpha^2} \left(\frac{\tilde{\rho}^2}{2} + \tilde{\rho} \right) \\ &\quad - \frac{1}{\alpha^2} p * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + c_0 u + \frac{\tilde{\rho}^2}{2} + \tilde{\rho} \right) \\ &= \frac{-u_x^2}{2} + \frac{(u + \frac{c_0}{2})^2}{\alpha^2} + \frac{1}{2\alpha^2} (\tilde{\rho} + 1)^2 \\ &\quad - \frac{1}{\alpha^2} p * \left(\frac{\alpha^2}{2} u_x^2 + (u + \frac{c_0}{2})^2 + (\tilde{\rho} + 1)^2 \right). \end{aligned}$$

Applying Lemma 3.2: $\frac{1}{\alpha^2} p * \left(\frac{\alpha^2}{2} u_x^2 + (u + \frac{c_0}{2})^2 \right) \geq \frac{(u + \frac{c_0}{2})^2}{2\alpha^2}$, and the obvious estimate $p * (\tilde{\rho} + 1)^2 \geq 0$, we get

$$\begin{aligned} \frac{d}{dt} [u_x(t, q(t, x))] &\leq \left(-\frac{1}{2} u_x^2 + \frac{(u + \frac{c_0}{2})^2}{\alpha^2} + \frac{1}{2\alpha^2} (\tilde{\rho} + 1)^2 - \frac{(u + \frac{c_0}{2})^2}{2\alpha^2} \right) (t, q(t, x)) \\ &= \left(-\frac{1}{2} u_x^2 + \frac{1}{2\alpha^2} (u + \frac{c_0}{2})^2 + \frac{1}{2\alpha^2} (\tilde{\rho} + 1)^2 \right) (t, q(t, x)). \end{aligned}$$

We also have the following identity:

$$\begin{aligned} \frac{d}{dt} [(\tilde{\rho}(t, q(t, x)) + 1)q_x(t, x)] &= (\tilde{\rho}_t(t, q(t, x)) + \tilde{\rho}_x(t, q(t, x))q_t(t, x))q_x(t, x) + (\tilde{\rho}(t, q(t, x)) + 1)q_{xt}(t, x) \\ &= \left(\tilde{\rho}_t(t, q(t, x)) + \tilde{\rho}_x(t, q(t, x))u(t, q(t, x)) + \tilde{\rho}(t, q(t, x))u_x(t, q(t, x)) \right. \\ &\quad \left. + u_x(t, q(t, x)) \right) q_x(t, x) \\ &= 0. \end{aligned}$$

This implies that $(\tilde{\rho}(t, q(t, x)) + 1)q_x(t, x) = (\tilde{\rho}_0(x) + 1)$. The initial condition implies $\tilde{\rho}_0(x_0) + 1 = 0$, then $\tilde{\rho}(t, q(t, x_0)) + 1 = 0$ for all t . Therefore,

$$\frac{d}{dt}[u_x(t, q(t, x_0))] \leq \left(-\frac{1}{2}u_x^2 + \frac{1}{2\alpha^2}\left(u + \frac{c_0}{2}\right)^2\right)(t, q(t, x_0)).$$

Using now similar calculations as for the DHG equation, we factorize $\left(-\frac{1}{2}u_x^2 + \frac{1}{2\alpha^2}\left(u + \frac{c_0}{2}\right)^2\right)(t, q(t, x_0)) = \frac{1}{2}(AB)(t, q(t, x_0))$ where $A(t, x_0) = e^{\frac{q(t, x_0)}{\alpha} + \frac{c_0 t}{2\alpha}}\left(\frac{1}{\alpha}\left(u + \frac{c_0}{2}\right) - u_x\right)(t, q(t, x_0))$ and $B(t, x_0) = e^{-\frac{q(t, x_0)}{\alpha} - \frac{c_0 t}{2\alpha}}\left(\frac{1}{\alpha}\left(u + \frac{c_0}{2}\right) + u_x\right)(t, q(t, x_0))$. Using Lemma 4.2, we see that $A(t, x_0)$ is monotonically increasing and $B(t, x_0)$ monotonically decreasing with respect to t .

But at x_0 we have, by our assumption, $u'_0(x_0) < -\frac{1}{\alpha}|u_0(x_0) + \frac{c_0}{2}|$. We denote $g(t) = u_x(t, q(t, x_0))$, $A(t) = A(t, x_0)$ and $B(t) = B(t, x_0)$. For all $t \in [0, T^*)$, we have:

$$\frac{d}{dt}A(t) \geq 0$$

and

$$\frac{d}{dt}B(t) \leq 0.$$

So, $A(t) \geq A(0) = e^{\frac{x_0}{\alpha}}\left(\frac{1}{\alpha}(u_0(x_0) + \frac{c_0}{2}) - u'_0(x_0)\right) > 0$, and $B(t) \leq B(0) = e^{-\frac{x_0}{\alpha}}\left(\frac{1}{\alpha}(u_0(x_0) + \frac{c_0}{2}) + u'_0(x_0)\right) < 0$. Thus $AB(t) \leq AB(0) < 0$. Then for all $t \in [0, T)$: $g'(t) < 0$.

Assume by contradiction $T^* = \infty$, then $g(t) \leq g(0) - \alpha_0 t$ where $\alpha_0 = \frac{1}{2}(u'(0)^2 - \frac{1}{\alpha}(u_0 + \frac{c_0}{2})^2)(x_0)$. We choose t_0 such that $g(0) - \alpha_0 t_0 \leq 0$ and $(g(0) - \alpha_0 t_0)^2 \geq \frac{1}{\alpha^3}(\|u_0\|_{H^1_\alpha} + \|\tilde{\rho}_0\|_{L^2} + \frac{c_0}{2}\sqrt{2\alpha})^2$. For $t \geq t_0$, we have:

$$\begin{aligned} g'(t) &\leq \frac{1}{2}\left(\frac{1}{\alpha^2}\left(u + \frac{c_0}{2}\right)^2 - u_x^2\right)(t, q(t, x_0)) \\ &\leq \frac{1}{2}\left(\frac{1}{\alpha^2}(\|u(t, \cdot)\|_{L^\infty} + \frac{c_0}{2})^2 - g(t)^2\right) \end{aligned}$$

Using Sobolev embedding inequality in Lemma (3.3) and the energy conservation identity, one has:

$$\begin{aligned} g'(t) &\leq \frac{1}{2}\left(\frac{1}{\alpha^2}\left(\frac{\|u\|_{H^1_\alpha}}{\sqrt{2\alpha}} + \frac{c_0}{2}\right)^2 - g(t)^2\right) \\ &\leq \frac{1}{2}\left(\frac{1}{\alpha^2}\left(\frac{\|u\|_{H^1_\alpha}}{\sqrt{2\alpha}} + \frac{\|\tilde{\rho}\|_{L^2}}{\sqrt{2\alpha}} + \frac{c_0}{2}\right)^2 - g(t)^2\right) \\ &= \frac{1}{2}\left(\frac{1}{\alpha^2}\left(\frac{\|u_0\|_{H^1_\alpha}}{\sqrt{2\alpha}} + \frac{\|\tilde{\rho}_0\|_{L^2}}{\sqrt{2\alpha}} + \frac{c_0}{2}\right)^2 - g(t)^2\right) \\ &\leq -\frac{1}{4}g(t)^2 \end{aligned}$$

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for all $t \in (t_0, \infty)$. Dividing both sides by $g^2(t)$ and integrating, we get

$$\frac{1}{g(t_0)} - \frac{1}{g(t)} + \frac{1}{4}(t - t_0) \leq 0, \quad t \geq t_0.$$

This is a contradiction since $-\frac{1}{g(t)} > 0$ and $\frac{1}{4}(t - t_0) \rightarrow \infty$ as $t \rightarrow \infty$. Thus $u_x(t, q(t, x_0))$ blows up in finite time and $T^* \leq t_0 + \frac{4}{|g(t_0)|} < \infty$. \square

APPENDIX

We have already indicated that the upper estimate on the time blow up depends on a nonlocal in space quantity related to u_0 . In this part, we give another proof for Theorem 1.1 that the estimate on the time blow up depends on a local space related to u_0 . A similar technique can be applied to Theorem 1.2.

Proof. — Using the argument:

$$\frac{d}{dt}[u_x(t, q(t, x))] \leq \left(-\frac{1}{2}u_x^2 + \frac{1}{2\alpha^2}(u+k)^2\right)(t, q(t, x)).$$

We factorize $\left(-\frac{1}{2}u_x^2 + \frac{1}{2\alpha^2}(u+k)^2\right)(t, q(t, x)) = \frac{1}{2}(AB)(t, q(t, x))$ where $A(t, x) = \frac{1}{\alpha}(u+k) - u_x$ and $B(t, x) = \frac{1}{\alpha}(u+k) + u_x$. So, it follows that:

$$\frac{d}{dt}[u_x(t, q(t, x))] \leq \frac{1}{2}(AB)(t, x).$$

We have:

$$\begin{aligned} \frac{d}{dt}A(t, x) &= \frac{d}{dt}\left(\frac{u+k}{\alpha} - u_x\right)(t, q(t, x)) \\ &= \left(\frac{u_t}{\alpha} - u_{xt}\right) + \left(\frac{u_x}{\alpha} - u_{xx}\right)q_t(t, x) \\ &= \left(\frac{u_t}{\alpha} - u_{xt}\right) + \left(\frac{u_x}{\alpha} - u_{xx}\right)(u + \lambda) \\ &= \frac{1}{\alpha}(u_t + (u + \lambda)u_x) - (u_{tx} + (u + \lambda)u_{xx}) \\ &= \frac{u_x^2}{2} - \frac{u^2 + 2ku}{\alpha^2} + \frac{1}{\alpha^2}(p - \alpha\partial_x p) * \left(\frac{\alpha^2}{2}u_x^2 + u^2 + 2ku\right). \end{aligned}$$

By lemma (3.1): $(p - \alpha \partial_x p) * (\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku) \geq \frac{(u+k)^2}{2} - k^2$, then we have:

$$\begin{aligned} \frac{d}{dt} A(t, x) &\geq \frac{u_x^2}{2} - \frac{u^2 + 2ku}{\alpha^2} + \frac{1}{\alpha^2} \left(\frac{(u+k)^2}{2} - k^2 \right) \\ &= \frac{u_x^2}{2} - \frac{u^2 + 2ku}{\alpha^2} \\ &= -\frac{1}{2} (AB)(t, x). \end{aligned}$$

Similarly, computing for $B(t, x)$ yields

$$\begin{aligned} \frac{d}{dt} B(t, x) &= \frac{d}{dt} \left(\frac{u+k}{\alpha} + u_x \right) (t, q(t, x)) \\ &= \left(\frac{u_t}{\alpha} + u_{xt} \right) + \left(\frac{u_x}{\alpha} + u_{xx} \right) q_t(t, x) \\ &= \left(\frac{u_t}{\alpha} + u_{xt} \right) + \left(\frac{u_x}{\alpha} + u_{xx} \right) (u + \lambda) \\ &= \frac{1}{\alpha} (u_t + (u + \lambda) u_x) + (u_{tx} + (u + \lambda) u_{xx}) \\ &= -\frac{u_x^2}{2} + \frac{u^2 + 2ku}{\alpha^2} - \frac{1}{\alpha^2} (p + \alpha \partial_x p) * \left(\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku \right). \end{aligned}$$

By lemma (3.1): $(p + \alpha \partial_x p) * (\frac{\alpha^2}{2} u_x^2 + u^2 + 2ku) \geq \frac{(u+k)^2}{2} - k^2$, then we have:

$$\begin{aligned} \frac{d}{dt} B(t, x) &\leq -\frac{u_x^2}{2} + \frac{u^2 + 2ku}{\alpha^2} - \frac{1}{\alpha^2} \left(\frac{(u+k)^2}{2} - k^2 \right) \\ &= -\frac{u_x^2}{2} + \frac{u^2 + 2ku}{\alpha^2} \\ &= \frac{1}{2} (AB)(t, x). \end{aligned}$$

The initial condition $u'_0(x_0) < -\frac{1}{\alpha} |u_0(x_0) + \frac{1}{2}(c_0 + \frac{\gamma}{\alpha^2})|$ is equivalent to $A(0, x_0) > 0$ and $B(0, x_0) < 0$. Let:

$$\omega = \sup\{t \in [0, T^*) : A(\cdot, x_0) > 0 \text{ and } B(\cdot, x_0) < 0 \text{ on } [0, t]\}.$$

Then $\omega > 0$. If $\omega < T^*$ then at least one of the inequalities $A(\omega, x_0) \leq 0$ and $B(\omega, x_0) \geq 0$ must be true. This is a contradiction with the fact that $AB(\cdot, x_0) < 0$ on the interval $[0, \omega]$ i.e: $AB(\omega, x_0) \leq AB(0, x_0) < 0$, then $A(\omega, x_0) \geq A(0, x_0) > 0$ and $B(\omega, x_0) \leq B(0, x_0) < 0$. Hence, $\omega = T^*$.

To conclude the proof, we argue as in [3], considering

$$h(t) = \sqrt{-(AB)(t, x_0)}.$$

Then the time derivative of h

$$\begin{aligned} \frac{d}{dt}h(t) &= -\frac{A_t B + AB_t}{2\sqrt{-AB}}(t, x_0) \\ &\geq \frac{(-AB)(A - B)}{4\sqrt{-AB}}(t, x_0). \end{aligned}$$

By the geometric-arithmetic mean inequality $(A - B)(t, x_0) \geq 2\sqrt{-(AB)(t, x_0)} = 2h(t)$, it follows

$$\frac{d}{dt}h(t) \geq \frac{1}{2}h^2(t).$$

But $h(0) = \sqrt{-(AB)(0, x_0)} > 0$. Hence the solution blows up in finite time and $T^* < \frac{2}{h(0)}$. Or it can be rewritten as:

$$T^* < \frac{2}{\sqrt{u'_0(x_0)^2 - \frac{1}{\alpha^2}(u_0(x_0) + k)^2}}.$$

□

We conclude by observing that we do not know if Theorem 1.2 remains valid when $\gamma \neq 0$. The main difficulty arises from the fact that when $\gamma \neq 0$, the underlying nonlinear transport equations associated with u and ρ travel with different speeds (the two speeds are $u - \gamma/\alpha^2$ and u respectively). This makes difficult to use the characteristics method to derive the ordinary differential system leading to the blowup.

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