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On the definition of the Galois group of linear differential equations

Katsunori Saito[1]

Résumé. — Considérons une équation différentielle linéaire $Y' = AY$ sur un corps différentiel K, où $A \in \mathrm{M_n}(K)$. Soit $K(F)/K$ une extension différentielle de corps déterminée par un système fondamental F de solutions de l'équation. Ainsi donc, l'extension $K(F)/K$ dépend du choix de F. Nous montrons que le groupe de Galois selon la théorie de Galois générale d'Umemura est indépendant du choix de F et coïncide en particulier avec celui défini d'après la théorie de Galois de Picard-Vessiot. En appliquant ce résultat, nous pouvons démontrer les théorèmes de comparaison d'Umemura [5], [6], Malgrange [3] et Casale [1].

Abstract. — Let us consider a linear differential equation $Y' = AY$ over a differential field K, where $A \in \mathrm{M_n}(K)$. Let F be a fundamental system of solutions of the equation. So the differential field extension $K(F)/K$ depends on the choice of F. We show that Galois group according to the general Galois theory of Umemura is independent of the choice of F and, in particular, coincides with the Picard-Vessiot Galois group of the equation. Applying this result, we can prove comparison theorems of Umemura [5], [6], Malgrange [3] and Casale [1].

1. Introduction

There are two ways of understanding the Galois theory of algebraic equations.

In the first way, when we are given an algebraic equation, according to Galois's original paper, the Galois group is described as a permutation group of solutions of the equation.

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In the second way, Dedekind introduced a nice idea, an algebraic equation over a field k determines the normal extension F/k of fields. The Galois group is attached to the normal extension. Namely the Galois group is the automorphism group $\mathrm{Aut}(F/k)$ of the field extension. He replaced algebraic equations by field extension.

The similar situation arises in Picard-Vessiot theory. We consider a linear differential equation

$$Y' = AY \tag{1.1}$$

over a differential field K with the algebraically closed field of constants C.

In the first approach, linear differential equation (1.1) defines a $K[\partial]$-module M that generate a neutral tannakian category $\{\{M\}\}$. The Galois group of system (1.1) is the affine group scheme of the neutral tannakian category.

In the second approach, as Dedekind did, we define a normal extension L^{PV}/K of the system of linear differential equation (1.1) that is uniquely determined up to isomorphisms, called the Picard-Vessiot extension of (1.1). Then the field of constants of L^{PV} coincides with C and the Galois group $\mathrm{Gal}(L^{PV}/K)$ of system (1.1) is the differential automorphism group $\mathrm{Aut}^{\partial}(L^{PV}/K)$ that has an algebraic group structure over C.

In this note, we consider a differential field L over K that satisfies the following conditions.

1. There exists a matrix $F \in \mathrm{GL}_n(L)$ so that $F' = AF$.
2. The field L is generated over K by the entries f_{ij}'s of F.

We call the matrix F by fundamental system of solutions and the differential field L by generated by a fundamental system of solutions of linear differential equation (1.1).

We need a differential sub-ring $S := K[f_{ij},\ (\det F)^{-1}]_{1 \leqq i,j \leqq n}$ of L. So the differential field L is field of fractions $Q(S)$ of S.

For the Picard-Vessiot extension L^{PV}/K of linear differential equation (1.1), we similarly introduce a fundamental solution Z^{PV} and the Picard-Vessiot ring R so we have

$$(Z^{PV})' = AZ^{PV} \text{ and } R = K[Z^{PV},\ (\det Z^{PV})^{-1}]$$

(See section 2).

We consider Galois group Inf-gal (L/K) of the differential field extension L/K in general Galois theory of Umemura, and compare it to Galois group $\mathrm{Gal}(L^{PV}/K)$ of the Picard-Vessiot extension L^{PV}/K for equation (1.1). Then we get the following result (See Theorem 4.18).

Main Theorem. — *We assume that the field K is algebraically closed. Then we have an isomorphism*

$$\mathrm{Lie}\ (\mathrm{Inf\text{-}gal}\,(L/K)) \simeq \mathrm{Lie}\ (\mathrm{Gal}(L^{PV}/K)) \otimes_C L^\natural \tag{1.2}$$

where $L^\natural$ denotes the abstract field structure of the differential field L.

The main theorem says that the left hand side of (1.2) is independent of the fundamental system of solutions. So in this sense it directly dependent on linear equation (1.1) and hence close to the idea of Galois. The right hand side of (1.1) is the automorphism group of a kind of normalization, the Picard-Vessiot extension, therefor it belongs to Dedekind's idea.

The main theorem showing equivalence of the two approaches gives us comparison theorems (See section 5).

Theorem also holds for a G-primitive extension where G is a general algebraic group that might not be linear (Theorem 4.20).

In the particular case of the field L is the Picard-Vessiot field, that is $C_L = C_K$, Umemura [6] proved comparison theorem. We understand that Malgrange [3] and Casale [1] proved the theorem for tr.d.$L/K = n^2$, where A and F are $n \times n$ matrices and $K = \mathbb{C}(t)$.

The most important application of the main theorem are proofs of comparison theorems of Galois groups such as theorem of Umemura comparing his Galois group and Picard-Vessiot group and as well as the comparison theorem of Malgrange and Casale.

We are inspired of [6] and prove the main theorem in a similar way. But we simplified the argument by comparing the Galois group of Umemura and the Picard-Vessiot Galois group without using the infinitesimal deformation functor $\mathcal{F}_{L/K}$.

This paper is organized as follows. In section 2, we recall some definitions and results on Picard-Vessiot theory. In section 3, we give definitions of general Galois theory. In section 4, we prove the main results. In section 5, we have some examples.

The author thank H. Umemura who guided him to the subject, for valuable discussions.

2. Preliminaries

In this section, we recall the preceding results of differential Galois theory, Picard-Vessiot theory, strongly normal extension and G-primitive extension. All the ring that we consider, except for Lie algebras, are commutative and unitary $\mathbb{Q}$-algebras.

2.1. Picard-Vessiot theory

Picard-Vessiot theory is Galois theory of linear differential equations. For more details, we refer to [7] chapter 1.

Definition 2.1. — *A derivation on a ring R is a map $\partial : R \to R$ satisfying the following properties.*

(1) $\partial(a + b) = \partial(a) + \partial(b)$,

(2) $\partial(ab) = \partial(a)b + a\partial(b)$

for all a, $b \in R$.

We call a ring R equipped with a derivation ∂ on R a differential ring and similarly a field K equipped with a derivation ∂ a differential field.

We say a differential ring S is a differential extension of a differential ring R or a differential ring over R if the ring S is an over ring of R and the derivation ∂_S of S restricted on R coincides with the derivation ∂_R of R. We will often denote a differential ring equipped with derivation ∂ by (R, ∂) and $\partial(a)$ by a' for $a \in R$. A derivation ∂ will be sometimes called a differentiation.

Definition 2.2. — *Let (R, ∂) be a differential ring. An element $c \in R$ is called a constant if $c' = 0$ and a set C_R denotes the set of all constants of R.*

By definition, the set of constants C_R of (R, ∂) forms a sub-ring. Similarly C_K is a subfield for a differential field (K, ∂). We sometimes say the ring of constants C_R or the field of constants C_K.

Definition 2.3. — *A differential ideal I of a differential ring (R, ∂) is an ideal of R closed under the derivation. A simple differential ring is a differential ring whose differential ideals are only (0) and R.*

From now on we fix a differential field K, called the base field, of which *the field of constants $C_K = C$ is algebraically closed.* We consider a linear differential equation

$$Y' = AY, \quad A \in \mathrm{M}_n(K), \tag{2.1}$$

where $Y = (y_{ij})$ is an $n \times n$ matrix, their entries y_{ij}'s of which are differential variables and $Y' = (y'_{ij})$.

DEFINITION 2.4. — *A Picard-Vessiot ring (R, ∂) over K for the equation* (2.1) *with $A \in \mathrm{M}_n(K)$ is a differential ring R over K satisfying:*

1. *R is a simple differential ring.*
2. *There exists a fundamental matrix $F = (f_{ij}) \in \mathrm{GL}_n(R)$ for the equation* (2.1) *so that $F' = (f'_{ij}) = AF$.*
3. *R is generated as a ring over K by entries f_{ij}'s of F and the inverse of the determinant of F, i.e., $R = K[f_{ij}, (\det F)^{-1}]$.*

LEMMA 2.5 (LEMMA 1.17 [7]). — *Let R be a simple differential ring over K. Then,*

1. *R has no zero divisors.*
2. *Suppose that R is finitely generated over K, then the field of fractions $Q(R)$ of R has C as a set of constants.*

By Lemma 2.5, Picard-Vessiot ring R is a domain and the field of fractions $Q(R)$ of R has C as the constant field. The following Proposition says existence and uniqueness of a Picard-Vessiot ring.

PROPOSITION 2.6 (PROPOSITION 1.20 [7]). — *For the linear differential equation* (2.1), *we have the following results.*

1. *There exists a Picard-Vessiot ring R for the equation.*
2. *Any two Picard-Vessiot rings for the equation are isomorphic.*
3. *The field of constants of the quotient field $Q(R)$ of a Picard-Vessiot ring is C.*

We also consider a Picard-Vessiot field.

DEFINITION 2.7. — *A Picard-Vessiot field for equation* (2.1) *is the field of fractions of Picard-Vessiot ring for the equation.*

The following Proposition characterizes a Picard-Vessiot field.

Proposition 2.8 (Proposition 1.22 [7]). — *A differential extension field L over K is the Picard-Vessiot field for equation* (2.1) *if and only if the following conditions are satisfied.*

(1) The field of constants of L is C.

(2) There exist a fundamental system of solutions $F \in \mathrm{GL}_n(L)$ for the equation, and

(3) the field L is generated over K by the entries of F.

We define the Galois group for Picard-Vessiot extension L/K.

Definition 2.9. — *The differential Galois group* $\mathrm{Gal}(L/K)$ *of equation* (2.1) *is defined as the group of differential K-automorphisms of the Picard-Vessiot field L for the equation.*

The following theorem says the Galois group has a linear algebraic group structure over the constants field C.

Theorem 2.10 (Theorem 1.27 [7]). — *Let L/K be the Picard-Vessiot extension for equation* (2.1)*, having differential Galois group* $G = \mathrm{Gal}(L/K)$*. Then*

(1) G, considered as a subgroup of $\mathrm{GL}_n(C)$*, is an algebraic group defined over C.*

(2) The Lie algebra of G coincides with the Lie algebra of the derivations of L/K that commute with the derivation ∂_L on L.

(3) The field L^G of G-invariant elements of L is equal to K.

2.2. Strongly normal extension

Strongly normal extension generalizes Picard-Vessiot extension. For more details, we refer to [6] or [2] Chapter VI section 1 to 4.

Definition 2.11. — *For a differential over field M/K, let L/K be a differential field extension and L is finitely generated over K as a field.*

1. Let $f, g\colon L \to M$ be two K-morphisms of differential field extensions of K. We say that the morphism f is strong over g if (i) $f(a) = g(a)$ for every constant a of L and (ii) the composite field $f(L).g(L)$ is generated over $g(L)$ by constants.

2. *We say that the extension L/K is strongly normal if for every differential field extension M of K and for arbitrary two K-morphisms $f, g\colon L \to M$ of differential fields, f is strong over g.*

LEMMA 2.12. — *If an differential field extension L/K is strongly normal, then the field of constants C_L of L coincides with the field of constants C_K of K.*

We can consider the Galois group of a strongly normal extension L/K.

DEFINITION 2.13. — *The Galois group $G(L/K)$ of a strongly normal extension L/K is defined as the group of K-automorphisms of L.*

THEOREM 2.14. — *The Galois group $G(L/K)$ has an algebraic group structure over C_K.*

2.3. G-primitive extension

For more details, we refer to [2] Chapter VI section 7.

Let $(L, \partial)/(K, \partial)$ be a differential field extension and $C_K = C$ be the field of constants of the base field K. Let G be an algebraic group defined over the field C and $p\colon \mathrm{Spec} L \to G$ be an L-valued point of G. $p^*\varphi = \varphi(p) \in L$ for $\varphi \in \mathcal{O}_p$. Let us put $\delta(p)(\varphi) = \partial(\varphi(p)) \in L$. The map $\delta(p)\colon \mathcal{O}_p \to L$ is a derivation so that

$$\delta(p)(\varphi\psi) = (\delta(p)(\varphi))\psi(p) + \varphi(p)\delta(p)(\psi)$$

for every $\varphi, \psi \in \mathcal{O}_p$. Then $\delta(p) \in T_pG$. The right translation $R_{p^*}^{-1}$ by the point $p \in G$ of the tangent vector $\delta(p)$ at $p \in G$, we get a right invariant derivation $l\delta(p) \in \mathrm{Lie}(G) \otimes_C L$. We call the derivation $l\delta(p)$ logarithmic derivation. The logarithmic derivation satisfies the following cocycle condition,

$$l\delta(pq) = l\delta(p) + \mathrm{Ad}(p)l\delta(q)$$

where $\mathrm{Ad}(p)\colon \mathrm{Lie}(G) \otimes_C L \to \mathrm{Lie}(G) \otimes_C L$ is adjoint representation of G and points p, q are L-valued point of G.

DEFINITION 2.15. — *Let L/K be a differential field extension, G be an algebraic group defined over C_K of constants of base field K and p be an L-valued point of G. We say the differential field extension L/K is G-primitive extension if (i) $L = K(p)$, (ii) $l\delta(p) \in \mathrm{Lie}(G) \otimes_C K$.*

If the field of constants C_L is equal to C_K, we can consider the Galois group of the G-primitive extension.

Theorem 2.16. — *Let L/K be G-primitive extension and C_L of L is equal to C_K. Then L is a strongly normal extension of K and the Galois group $G(L/K)$ is a closed subgroup of the algebraic group G.*

A Picard-Vessiot extension L/K is a special case of G-primitive extension for $G = \mathrm{GL}_n$ and $C_L = C_K$ so that the extension L/K is a strongly normal extension.

In general, G-primitive extension L/K has larger constant field C_L than the field C_K. Therefor we can not define the Galois group of L/K except for the Galois theory of Umemura.

3. General Galois theory

Let $(R, \{\partial_1, \partial_2, \cdots, \partial_d\})$ be a partial differential ring. So ∂_i are mutually commutative derivations of R such that we have

$$[\partial_i, \partial_j] = \partial_i\partial_j - \partial_j\partial_i = 0, \quad \text{for } 1 \leqslant i, j \leqslant d.$$

For example, the ring of power series

$$\left(S[[X_1, X_2, \cdots, X_d]], \left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\right\}\right)$$

is a partial differential ring for a $\mathbb{Q}$-algebra S.

We call a morphism

$$(R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \longrightarrow \left(S[[X_1, X_2, \cdots, X_d]], \left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\right\}\right) \tag{3.1}$$

of differential ring by a Taylor morphism. When we fix a partial differential ring $(R, \{\partial_1, \partial_2, \cdots, \partial_d\})$, there exists the universal one ι_R among Taylor morphisms (3.1).

Definition 3.1. — *The universal Taylor morphism ι_R is a differential morphism*

$$(R, \{\partial_1, \partial_2, \cdots, \partial_d\}) \longrightarrow \left(R^\natural[[X_1, X_2, \cdots, X_d]], \left\{\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \cdots, \frac{\partial}{\partial X_d}\right\}\right) \tag{3.2}$$

such that

$$\iota_R(a) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \partial^n(a) X^n$$

for an element $a \in R$, where we use the standard notation for multi-index. Namely, for $\alpha = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$,

$$n! = n_1! n_2! \cdots n_d!, \ \partial^n = \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_d^{n_d} \ \textit{and} \ X^n = X_1^{n_1} X_2^{n_2} \cdots X_d^{n_d}.$$

Then the universal Taylor morphism has following properties.

PROPOSITION 3.2 (UMEMURA [4] PROPOSITION (1.4)). — *(i) The universal Taylor morphism is an injection.*
(ii) The universal Taylor morphism is universal among the Taylor morphisms. Namely, for any Taylor morphism $\varphi : R \to S[[X]]$, there is one and only one Taylor morphism $\psi \colon R[[X]] \to S[[X]]$ such that the following diagram is commute.

$$\begin{tikzcd} R \arrow[r, "\iota_R"] \arrow[dr, "\varphi"'] & R[[X]] \arrow[d, "\psi"] \\ & S[[X]] \end{tikzcd}$$

Let $(L, \partial_L)/(K, \partial_K)$ be a differential field extension. We assume that the abstract field $L^\natural$ is finitely generated over the abstract field $K^\natural$. We have the universal Taylor morphism

$$\iota_L \colon L \to L^\natural[[X]]. \tag{3.3}$$

We choose a mutually commutative basis $\{D_1, D_2, \cdots, D_d\}$ of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$ of $K^\natural$-derivations of the abstract field $L^\natural$. We introduce partial differential field

$$L^\sharp := (L^\natural, \{D_1, D_2, \cdots, D_d\}).$$

Similarly the derivations $\{D_1, D_2, \cdots, D_d\}$ operate on coefficients of the ring $L^\natural[[X]]$. Then we introduce $\{D_1, D_2, \cdots, D_d\}$-differential structure on the ring $L^\natural[[X]]$. So the ring $L^\natural[[X]]$ has the differential structure defined by the differentiation d/dX and the set $\{D_1, D_2, \cdots, D_d\}$ of derivations. This differential ring is denoted by

$$L^\sharp[[X]] := \left(L^\natural[[X]], \left\{ \frac{d}{dX}, D_1, D_2, \cdots, D_d \right\} \right).$$

We replace the target space $L^\natural[[X]]$ of universal Taylor morphism (3.3) by $L^\sharp[[X]]$ so that we have

$$\iota_L\colon L \to L^\sharp[[X]].$$

In the definition below, we work in the differential ring $L^\sharp[[X]]$. We identify $L^\sharp$ with the partial differential field of constant power series

$$\left\{ \sum_{i=0}^{\infty} a_i X^i \in L^\sharp[[X]] \,\middle|\, a_i = 0 \text{ for } i \geqslant 1 \right\}$$

of the partial differential ring $L^\sharp[[X]]$. Therefore $L^\sharp$ is a partial differential sub-field of $L^\sharp[[X]]$ and the derivation d/dX acts trivially on $L^\sharp$.

We define the Galois hull $\mathcal{L}/\mathcal{K}$ as follows. The Galois hull $\mathcal{L}/\mathcal{K}$ might be considered as a normalization of the differential field extension L/K.

Definition 3.3. — *The Galois hull $\mathcal{L}/\mathcal{K}$ is a partial differential algebra extension in the partial differential ring $L^\sharp[[X]]$, where $\mathcal{L}$ is the partial differential sub-algebra generated by the image $\iota_L(L)$ and $L^\sharp$ in $L^\sharp[[X]]$ and $\mathcal{K}$ is the partial differential sub-algebra generated by the image $\iota_L(K)$ and $L^\sharp$ in $L^\sharp[[X]]$ so that*

$$\mathcal{K} = \iota_L(K).L^\sharp.$$

For the partial differential field $L^\sharp$, we have the universal Taylor morphism

$$\iota_{L^\sharp}\colon L^\sharp \to L^\natural[[W_1, W_2, \cdots, W_d]] = L^\natural[[W]], \tag{3.4}$$

where the variables $W_i's$ in (3.4) denote the variables X_i's. The morphism (3.4) gives a differential ring morphism

$$\left(L^\sharp[[X]], \left\{\frac{d}{dX}, D_1, D_2, \cdots, D_d\right\}\right) \longrightarrow \left(L^\natural[[W_1, W_2, \cdots, W_d]][[X]], \left\{\frac{d}{dX}, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \cdots, \frac{\partial}{\partial W_d}\right\}\right). \tag{3.5}$$

Restricting the differential morphism (3.5) to the differential sub-algebra $\mathcal{L}$, we get a differential morphism

$$\iota\colon \mathcal{L} \to L^\natural[[W_1, W_2, \ldots W_d]][[X]] = L^\natural[[W, X]]. \tag{3.6}$$

Similarly, for an $L^\natural$-algebra A, we have the partial differential morphism

$$L^\natural[[W, X]] \to A[[W, X]] \tag{3.7}$$

induced of the structural morphism $L^\natural \to A$ of $L^\natural$-algebra A. We get the differential morphism

$$\iota\colon \mathcal{L} \to A[[W, X]] \tag{3.8}$$

by composing (3.6) and (3.7). We define the infinitesimal deformation functor $\mathcal{F}_{L/K}$ and the Galois group Inf-gal (L/K)

Definition 3.4. — *We define the infinitesimal deformation functor*

$$\mathcal{F}_{L/K}\colon (Alg/L^\natural) \to (Sets)$$

from the category $(Alg/L^\natural)$ *of* $L^\natural$*-algebra to the category* $(Sets)$ *of sets as the set of infinitesimal deformations of the morphism* (3.6). *So*

$$\begin{aligned}\mathcal{F}_{L/K}(A) = \{f\colon \mathcal{L} \to A[[W, X]] \mid & f \textit{ is a partial differential morphism} \\ & \textit{congruent to the morphism } \iota \textit{ modulo nilpotent elements} \\ & \textit{such that } f|_{\mathcal{K}} = \iota|_{\mathcal{K}}\ \}.\end{aligned}$$

Definition 3.5. — *The Galois group in general Galois Theory is the group functor*

$$\text{Inf-gal}\,(L/K)\colon (Alg/L^\natural) \to (Grp)$$

from the category $(Alg/L^\natural)$ *of* $L^\natural$*-algebra to the category* (Grp) *of groups associating an* $L^\natural$*-algebra* A *with the automorphism group*

$$\begin{aligned}\text{Inf-gal}\,(L/K)(A) = \{\varphi\colon \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \to \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]] \mid & \varphi \textit{ is a differential} \\ \mathcal{K}\hat{\otimes}_{L^\sharp} A[[W]]\textit{-automorphism continuous with respect to the } & W\textit{-adic} \\ \textit{topology and congruent to the identity modulo nilpotent elements}\}. &\end{aligned}$$

Then the group functor Inf-gal (L/K) operates on the functor $\mathcal{F}_{L/K}$. The operation (Inf-gal (L/K), $\mathcal{F}_{L/K}$) is a principal homogeneous space. See Theorem (5.11) [5].

4. Galois group of L/K generated by a fundamental system of solutions of linear differential equation over K in general Galois theory

In this section, we consider a differential field extension L/K generated by a fundamental system of a linear differential system

$$Y' = AY, \qquad A \in \mathrm{M}_n(K). \tag{4.1}$$

In general, since the field of constants C_L is larger than C_K, we can not treat the differential extension L/K in Picard-Vessiot theory. Therefore we will compare the Galois group of the extension L/K and the Galois group of Picard-Vessiot extension L^{PV}/K for equation (4.1) in general Galois theory of Umemura.

If L is a Picard-Vessiot field, we have an isomorphism

$$\mathrm{Lie}(\mathrm{Gal}(L/K)) \simeq \mathrm{Lie}(\mathrm{Gal}(L/\bar{K}))$$

where $\bar{K}$ is the algebraically closure of K in L. And also we have

$$\mathrm{Lie}(\text{Inf-gal}\,(L/K)) \simeq \mathrm{Lie}(\text{Inf-gal}\,(L/\bar{K})).$$

As we are interested in the Lie algebra of the Galois group of the differential field extension L/K, by replacing the base field K by its algebraic closure in L. Therefor we may assume that the base field K is algebraically closed in L.

We prove the main theorem (Theorem 4.18) in a similar way as Umemura's proof in [6].

Because L is a differential field generated by a system of solutions of linear differential equation (4.1), there exists a fundamental system of solutions $Z = (z_{ij}) \in \mathrm{GL}_n(L)$ such that

$$Z' = AZ \tag{4.2}$$

and $L = K(z_{ij})$. In the following, we work in the differential ring $L^\sharp[[X]]$. Since the universal Taylor morphism ι is a differential morphism, the image of the matrix Z by the ι satisfies

$$\frac{d}{dX}(\iota(Z)) = \iota(A)\iota(Z) \tag{4.3}$$

by (4.2).

Lemma 4.1. — *For an element $a \in L^\natural$, $a^\sharp$ denote the element a in $L^\sharp$. If we set $B := \iota(Z)(Z^\sharp)^{-1} \in \mathrm{GL}_n(L^\sharp[[X]])$, where $Z^\sharp = (z^\sharp_{ij}) \in \mathrm{GL}_n(L^\sharp)$, then the matrix B is in $\mathrm{GL}_n(K^\sharp[[X]])$.*

Proof. — We write

$$\iota(A) = \sum \frac{1}{k!} A_k X^k,\ A_k \in \mathrm{M}_n(K^\sharp)$$

and

$$B = \sum \frac{1}{k!} B_k X^k,\ B_k \in \mathrm{M}_n(L^\sharp).$$

It is sufficient to show that B_k is in $\mathrm{M}_n(K^\sharp)$ for $k \in \mathbb{N}$. We show this by induction on k. For $k = 0$, indeed $B_0 = I_n \in \mathrm{M_n}(K^\sharp)$ by definition of B. Assume $B_l \in \mathrm{M_n}(K^\sharp)$ for $l < k$. Since $Z^\sharp$ is constant matrix with respect to d/dX, it is follows from (4.3)

$$\frac{d}{dX} B = \iota(A)B. \tag{4.4}$$

We rewrite (4.4)

$$\frac{d}{dX}\left(\sum \frac{1}{k!} B_k X^k\right) = \left(\sum \frac{1}{k!} A_k X^k\right)\left(\sum \frac{1}{k!} B_k X^k\right). \tag{4.5}$$

Comparing coefficients of X^{k-1} of (4.5), we get

$$B_k = \sum_{l+m=k-1} \frac{(k-1)!}{l!m!} A_l B_m \in \mathrm{M_n}(K^\sharp).$$

□

In the construction of Galois hull $\mathcal{L}$ in general differential Galois theory, we consider a differential field extension L/K. However, we replace the differential field L by the differential sub-ring $S = K[Z, (\det Z)^{-1}]$. We consider the restriction of the universal Taylor morphism ι to the differential sub-ring S of L. And we replace the Galois hull $\mathcal{L}$ by the sub-algebra $\mathcal{S} := \iota(S).L^\sharp$ of $L^\sharp[[X]]$.

Lemma 4.2. — *In the differential ring $L^\sharp[[X]]$, the differential sub-ring*

$$\iota(K)[B, (\det B)^{-1}].L^\sharp$$

coincides with the differential sub-ring

$$\mathcal{S} = \iota(K[Z, (\det Z)^{-1}]).L^\sharp.$$

Proof. — From $B = \iota(Z)(Z^\sharp)^{-1}$,

$$\begin{aligned}\mathcal{S} = \iota(K[Z,\,(\det Z)^{-1}]).L^\sharp &= \iota(K)[\iota(Z),\,(\det \iota(Z))^{-1}].L^\sharp \\ &= \iota(K)[BZ^\sharp,\,(\det BZ^\sharp)^{-1}].L^\sharp = \iota(K)[B,\,(\det B)^{-1}].L^\sharp.\end{aligned}$$

□

Lemma 4.3. — *The sub-ring $\mathcal{S}$ of $L^\sharp[[X]]$ is a partial differential sub-ring.*

Proof. — We show that the ring $\mathcal{S}$ is closed under the derivations d/dX and D_i for $1 \leqslant i \leqslant d$. Since both $\iota(K[Z,\,(\det Z)^{-1}])$ and $L^\sharp$ are closed under the differentiation d/dX, the ring $\mathcal{S}$ is closed under the differentiation. To show that the ring $\mathcal{S}$ closed under the derivations D_i, by Lemma 4.2, we will show the ring $\iota(K)[B, (\det B)^{-1}].L^\sharp$ is closed under the derivations. The sub-ring $L^\sharp$ is closed obviously. By Lemma 4.1, the ring $\iota(K)[B, (\det B)^{-1}]$ is in the ring $K^\sharp[[X]]$ so that derivations D_i act trivially on $\iota(K)[B, (\det B)^{-1}]$. Then $\iota(K)[B, (\det B)^{-1}].L^\sharp$ closed under the derivations D_i. □

From Lemma 4.1 we get $\iota(K)[B,\,(\det B)^{-1}] \subset K^\sharp[[X]]$. So we have $C \subset C_{\iota(K)[B,\,(\det B)^{-1}]} \subset K^\sharp$.

Example 4.4. — We consider a differential ring $\mathbb{C}(x)[\exp x,\,(\exp x)^{-1}]$. The ring $\mathbb{C}(x)[\exp x,\,(\exp x)^{-1}]$ is a Picard-Vessiot ring over $\mathbb{C}(x)$ for a linear differential equation $Y' = Y$ with the fundamental system of solution $Z = \exp x$. The image of the fundamental system of solution $\iota(Z)$ is

$$\iota(Z) = \exp x + (\exp x)X + \frac{1}{2!}(\exp x)X^2 + \cdots = \exp(x + X).$$

Then the matrix B is

$$B = \iota(Z)(Z^\sharp)^{-1} = (\exp(x + X))(\exp x)^{-1} = \exp X.$$

So

$$\iota(K)[B,\,(\det B)^{-1}] = \iota(\mathbb{C}(x))[\exp X,\,(\exp X)^{-1}].$$

In this case $C_{\iota(\mathbb{C}(x))[\exp X,\,(\exp X)^{-1}]} = \mathbb{C}$.

Example 4.5. — We consider a differential ring $\mathbb{C}(x)[\log x]$. The ring $\mathbb{C}(x)[\log x]$ is a Picard-Vessiot ring over $\mathbb{C}(x)$ for a linear differential equation

$$Y' = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{x} \end{pmatrix} Y$$

with the fundamental system of solutions

$$Z = \begin{pmatrix} \log x & 1 \\ \frac{1}{x} & 0 \end{pmatrix}.$$

The image of the fundamental system of solutions $\iota(Z)$ is

$$\iota(Z) = \begin{pmatrix} \log x + \frac{1}{x}X - \frac{1}{2x^2}X^2 + \cdots & 1 \\ \frac{1}{x} - \frac{1}{x^2}X + \frac{1}{x^3}X^2 + \cdots & 0 \end{pmatrix} = \begin{pmatrix} \log(x+X) & 1 \\ \frac{1}{x+X} & 0 \end{pmatrix}.$$

So the matrix B is

$$B = \iota(Z)(Z^{\sharp})^{-1} = \begin{pmatrix} \log(x+X) & 1 \\ \frac{1}{x+X} & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 1 & -x\log x \end{pmatrix} = \begin{pmatrix} 1 & x(\log(1+\frac{X}{x})) \\ 0 & \frac{x}{x+X} \end{pmatrix}.$$

Then we get,

$$\frac{x}{x+X} \cdot \iota(x) = \left(\frac{x}{x+X}\right)(x+X) = x \in \iota(\mathbb{C}(x))[B,\, (\det B)^{-1}].$$

Since x is a constant with respect to derivations d/dX and D_i, the ring of constants $\iota(\mathbb{C}(x))[B, (\det B)^{-1}]$ is larger than the field $\mathbb{C}$.

Then we consider the sub-ring $\iota(K)[B,\, (\det B)^{-1}].K^{\sharp}$ of $\mathcal{S}$. The sub-ring is also a differential sub-ring. The following lemma is a famous result called linear disjointness theorem. For the definition of linear disjointness and basic properties, see Zariski-Samuel [8] Chapter II Section 15.

LEMMA 4.6 (KOLCHIN). — *Let (R, ∂) be a differential ring and let M be a differential sub-field of R. Then the field M and the ring of constants C_R of R are linearly disjoint over the field of constants C_M of M.*

Proof. — See Umemura [4] Lemma (1.1) or Kolchin [2] Chapter II, 1, Corollary 1 of Theorem 1. □

The lemma above as well as the lemma below are quite useful.

LEMMA 4.7. — *Let M be a field and let $(M[[X]], d/dX)$ be the differential ring of power series with coefficients in M. Let R be a differential sub-ring of $M[[X]]$ containing the field M. Then the ring R is a domain and the field of fractions $Q(R)$ has a differential field structure and we have*

$$C_{Q(R)} = M.$$

Proof. — Since $M[[X]]$ is a domain, it is clear that the sub-ring R is a domain. The field of fractions $Q(R)$ is a sub-field of $Q(M[[X]]) = M[[X]][X^{-1}]$ and contains M. So,

$$M \subset C_{Q(R)} \subset C_{M[[X]][X^{-1}]} = M.$$

□

Remark 4.8. — Lemma 4.6 and Lemma 4.7 are also true if rings are partial differential ring.

Applying Lemma 4.7 to $\iota(K)[B, (\det B)^{-1}].K^\sharp \subset K^\sharp[[X]]$ and $\mathcal{S} \subset L^\sharp[[X]]$, we have following corollaries.

Corollary 4.9. — *The field of constants $C_{Q(\iota(K)[B, (\det B)^{-1}].K^\sharp)}$ of the field of fractions $Q(\iota(K)[B, (\det B)^{-1}].K^\sharp)$ of $\iota(K)[B, (\det B)^{-1}].K^\sharp$ is $K^\sharp$.*

Corollary 4.10. — *The field of constants of the differential field $(Q(\mathcal{S}), d/dX)$ is $L^\sharp$.*

Lemma 4.11. — *The sub-ring $\iota(K)[B, (\det B)^{-1}].K^\sharp$ of $(\mathcal{S}, d/dX)$ and the sub-field $L^\sharp$ are linearly disjoint over $K^\sharp$. So we have a d/dX-differential isomorphism*

$$\iota(K)[B, (\det B)^{-1}].K^\sharp \otimes_{K^\sharp} L^\sharp \simeq \mathcal{S}.$$

Proof. — We work in the differential field $Q(\mathcal{S})$. To apply Lemma 4.6 to the differential sub-field

$$Q(\iota(K)[B, (\det B)^{-1}].K^\sharp)$$

of $Q(\mathcal{S})$, the field $Q(\iota(K)[B, (\det B)^{-1}].K^\sharp)$ and $C_{Q(\mathcal{S})}$ are linearly disjoint over $C_{Q(\iota(K)[B, (\det B)^{-1}].K^\sharp)}$. So the differential sub-ring $\iota(K)[B, (\det B)^{-1}].K^\sharp$ and $C_{Q(\mathcal{S})}$ also linearly disjoint over $C_{Q(\iota(K)[B, (\det B)^{-1}].K^\sharp)}$. Now Lemma follows from Corollary 4.9 and Corollary 4.10. □

From now on, we work in the partial differential ring $L^\natural[[W, X]]$ and identify a sub-ring R of $L^\sharp[[X]]$ with its image of the universal Taylor morphism $\iota_{L^\sharp}$.

Proposition 4.12. — *In the partial differential ring*

$$\left(L^\natural[[W_1, W_2, \cdots, W_d]][[X]], \left\{\frac{d}{dX}, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \cdots, \frac{\partial}{\partial W_d}\right\}\right),$$

the sub-ring $\mathcal{S}.L^\natural$ is a partial differential sub-ring. So we have a partial differential isomorphism

$$\mathcal{S}.L^\natural \simeq (\iota(K)[B, (\det B)^{-1}].K^\sharp \otimes_{K^\sharp} L^\sharp) \otimes_{K^\natural} L^\natural.$$

Proof. — Since the universal Taylor morphism $\iota_{L^\sharp}$ is differential morphism, the sub-ring $\mathcal{S}$ is closed under the differentiations d/dX and $\partial/\partial W_i$ by Lemma 4.3. Moreover the constants power series $L^\natural$ is clearly closed under the differentiations. So the sub-ring $\mathcal{S}.L^\natural$ is a partial differential sub-ring. In the same way as the proof of Lemma 4.11, we have $\{\partial/\partial W_i\}$-differential isomorphism

$$\mathcal{S}.L^\natural \simeq \mathcal{S} \otimes_{K^\natural} L^\natural. \tag{4.6}$$

Isomorphism (4.6) is also $\{d/dX,\, \partial/\partial W_i\}$-differential isomorphism because the sub-ring $\mathcal{S}$ and the sub-field $L^\natural$ are closed under the differentiations d/dX and $\partial/\partial W_i$. By Lemma 4.11, we have $\{d/dX\}$-differential isomorphism

$$\mathcal{S} \simeq \iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\sharp} L^\sharp. \tag{4.7}$$

Since the sub-ring $(\iota(K)[B,\, (\det B)^{-1}].K^\sharp)$ and $L^\sharp$ also are closed under the differentiations d/dX and $\partial/\partial W_i$, isomorphism (4.7) is $\{d/dX,\, \partial/\partial W_i\}$-differential isomorphism. So we get $\{d/dX,\, \partial/\partial W_i\}$-differential isomorphism

$$\mathcal{S} \otimes_{K^\natural} L^\natural \simeq (\iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\sharp} L^\sharp) \otimes_{K^\natural} L^\natural. \tag{4.8}$$

Then the Proposition follows from (4.6) and (4.8). □

Corollary 4.13. —

$$\mathcal{K}.L^\natural \simeq \iota(K).K^\sharp \otimes_{K^\sharp} L^\sharp \otimes_{K^\natural} L^\natural.$$

Corollary 4.14. — *For an $L^\natural$-algebra A,*

$$\mathcal{S}.A \simeq \iota(K).K^\sharp \otimes_{K^\sharp} L^\sharp \otimes_{K^\natural} A$$

Proof. — The proof in Proposition 4.12 works also in these cases. □

We take a subset

$$(\iota(K).K^\sharp)^* = (\iota(K).K^\sharp)\backslash\{0\}$$

of the ring $\iota(K)[B,\, (\det B)^{-1}].K^\sharp$. The set $(\iota(K).K^\sharp)^*$ is a multiplicative set. The localization of $\iota(K)[B,\, (\det B)^{-1}].K^\sharp$ by $(\iota(K).K^\sharp)^*$ is equal to

$$Q(\iota(K).K^\sharp)[B,\, (\det B)^{-1}].$$

Lemma 4.15. — *The field of constants $C_{Q(\iota(K).K^\sharp)}$ of $Q(\iota(K).K^\sharp)$ and the field of constants $C_{Q(\iota(K).K^\sharp)[B,\,(\det B)^{-1}]}$ of $Q(\iota(K).K^\sharp)[B,\, (\det B)^{-1}]$ are equal to $K^\natural$.*

Proof. — We can apply Lemma 4.7 to $\iota(K).K^\sharp \subset K^\sharp[[X]]$ then $C_{Q(\iota(K).K^\sharp)} = K^\sharp$.

$$K^\sharp \subset Q(\iota(K).K^\sharp)[B,\, (\det B)^{-1}] \subset K^\sharp[[X]][X^{-1}].$$

We have $C_{Q(\iota(K).K^\sharp)[B,\,(\det B)^{-1}]} = K^\sharp$. The derivations D_i's act trivially on $K^\sharp$, then $K^\sharp = K^\natural$. □

Lemma 4.16. — *$(Q(\iota(K).K^\sharp)[B,\, (\det B)^{-1}], d/dX)$ is a Picard-Vessiot ring over the field $(Q(\iota(K).K^\sharp), d/dX)$ for the equation $Y' = \iota(A)Y$ if the field $K^\natural$ is algebraically closed.*

Proof. — T denotes $Q(\iota(K).K^\sharp)[B,\, (\det B)^{-1}]$. By the proof of Lemma 4.15, T is a sub-ring of $K^\sharp[[X]][X^{-1}]$. Then the field of fractions $Q(T)$ of the ring T has also the same constants field $K^\natural$. $Q(T)$ has the fundamental system of solutions B of the equation $Y' = \iota(A)Y$ and generated over $Q(\iota(K).K^\sharp)$ by the entries of B. Since by Proposition 2.8 of section 2.1, the field $Q(T)$ is a Picard-Vessiot field. Therefore T is a Picard-Vessiot ring. □

Lemma 4.17. — *We assume that the field $K^\natural$ is algebraically closed. Let*

$$(R = K[Z^{PV}, (\det Z^{PV})^{-1}],\, \partial)$$

be a Picard-Vessiot ring for the equation $Y' = AY$ over the field (K, ∂). Then $Q(\iota(K).K^\sharp)[\iota(Z^{PV}), (\det \iota(Z^{PV}))^{-1}]$ and $Q(\iota(K).K^\sharp)[B,\, (\det B)^{-1}]$ are differentially isomorphic.

Proof. — Since ι is a differential isomorphism, the image $\iota(R)$ is a Picard-Vessiot ring over the field $\iota(K)$ for the equation $Y' = \iota(A)Y$. We consider the ring $\iota(R) \otimes_C K^\sharp$. The quotient field

$$Q(\iota(R) \otimes_C K^\sharp) \simeq Q(\iota(K).K^\sharp)(\iota(Z^{PV})) \subset K^\sharp[[X]][X^{-1}]$$

is a Picard-Vessiot field over the field $Q(\iota(K).K^\sharp)$ for the equation $Y' = \iota(A)Y$ because the field of constants $C_{Q(\iota(R)\otimes_C K^\sharp)}$ coincides with $K^\sharp$ by the same argument of Lemma 4.15. Since

$$Q(\iota(K).K^\sharp)(\iota(Z^{PV})) = Q(Q(\iota(K).K^\sharp)[\iota(Z^{PV}), (\det \iota(Z^{PV}))^{-1}]),$$

the ring

$$Q(\iota(K).K^\sharp)[\iota(Z^{PV}), (\det \iota(Z^{PV}))^{-1}]$$

that is the localization of $\iota(R) \otimes_C K^\sharp$ by $(\iota(K).K^\sharp)^*$ is also a Picard-Vessiot ring over the field $Q(\iota(K).K^\sharp)$ for the equation $Y' = \iota(A)Y$. So this lemma follows from Proposition 2.6 (2) of section 2.1. □

The following theorem is the main result.

Theorem 4.18. — *We assume that the field K is algebraically closed. Let L/K be a differential field extension generated by a fundamental system of solutions of a linear differential equation* (4.1) *and let $L^{PV} = K(Z^{PV})$ be a Picard-Vessiot field for the equation* (4.1) *so that $(Z^{PV})' = AZ^{PV}$. Then we have an isomorphism*

$$\mathrm{Lie}\,(\mathrm{Inf\text{-}gal}\,(L/K)) \simeq \mathrm{Lie}\,(\mathrm{Gal}(L^{PV}/K)) \otimes_C L^\natural.$$

Proof. — In the proof, we directly compare $\mathrm{Inf\text{-}gal}\,(L/K)$ and $\mathrm{Gal}(L^{PV}/K)$ by Proposition 4.12 without using the functor $\mathcal{F}_{L/K}$ which simplify the proof.

We have to show

$$\begin{aligned}\mathrm{Inf\text{-}aut}(\mathcal{S}\hat{\otimes}_{L^\sharp}A[[W]]/\mathcal{K}\hat{\otimes}_{L^\sharp}A[[W]]) \\ \simeq \mathrm{Inf\text{-}aut}(K[Z^{PV}, (\det Z^{PV})^{-1}] \otimes_C A/K \otimes_C A)\end{aligned}$$

for $A = L^\natural[\varepsilon]$ with $\varepsilon^2 = 0$, where for a ring $\mathfrak{B}$ and $\mathfrak{B}$-algebra $\mathfrak{A}$ we denote by $\mathrm{Inf\text{-}aut}(\mathfrak{A}/\mathfrak{B})$ the set of $\mathfrak{B}$-automorphisms of $\mathfrak{A}$ congruent to the identity map of $\mathfrak{A}$ modulo nilpotent element. The following argument works for every $L^\natural$-algebra A. We have the following isomorphisms

$$\begin{aligned}K[Z^{PV}, (\det Z^{PV})^{-1}] \otimes_C A &\simeq (K[Z^{PV}, (\det Z^{PV})^{-1}] \otimes_C K^\sharp) \otimes_{K^\natural} A \\ &\simeq (\iota(K[Z^{PV}, (\det Z^{PV})^{-1}]) \otimes_C K^\sharp) \otimes_{K^\natural} A \\ &\simeq (\iota(K)[\iota(Z^{PV}), (\det \iota(Z^{PV}))^{-1}].K^\sharp) \otimes_{K^\natural} A,\end{aligned} \tag{4.9}$$

over $K \otimes_C A$. Given an infinitesimal automorphism f of $L^{PV} \otimes_C A$ over $K \otimes_C A$, by isomorphisms (4.9) defines an infinitesimal automorphism $\tilde{f}$ of

$$(\iota(K)[\iota(Z^{PV}), (\det \iota(Z^{PV}))^{-1}].K^\sharp) \otimes_{K^\natural} A$$

over $\iota(K) \otimes_C A$. As every element of

$$\iota(K) \otimes_C K^\sharp \simeq \iota(K).K^\sharp$$

is invariant the infinitesimal automorphism $\tilde{f}$, the morphism $\tilde{f}$ extends to the localization of $\iota(K)[\iota(Z^{PV}), (\det \iota(Z^{PV}))^{-1}].K^\natural$ by the multiplicative set $(\iota(K) \otimes_C K^\sharp)^*$. By Lemma 4.17, we get an infinitesimal automorphism $\bar{f}$ of

$$Q(\iota(K).K^\sharp)[B, (\det B)^{-1}] \otimes_{K^\natural} A$$

over $Q(\iota(K).K^\sharp) \otimes_C A$. Since B is a fundamental system of solutions of linear differential equation, the infinitesimal automorphism $\bar{f}$ induces an infinitesimal automorphism $\bar{f}'$ of

$$\iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\natural} A$$

over $\iota(K).K^\sharp \otimes_{K^\natural} A$. Since the W_i's are variable, $\bar{f}'$ defines an infinitesimal automorphism of

$$\iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\natural} A[[W]]$$

over $\iota(K).K^\sharp \otimes_{K^\natural} A[[W]]$. Therefore an infinitesimal automorphism of $\mathcal{S} \otimes_{L^\sharp} A[[W]]$ over $\mathcal{K} \otimes_{L^\sharp} A[[W]]$ by Lemma 4.11. So consequently an infinitesimal automorphism of $\mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]]$ over $\mathcal{K}\hat{\otimes}_{L^\sharp} A[[W]]$.

To prove the converse, we notice that we have

$$\begin{aligned}&\text{Inf-aut}(\mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]]/\mathcal{K}\hat{\otimes}_{L^\sharp} A[[W]])\\ &\qquad\qquad \simeq \text{Inf-aut}(\iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\sharp} A/\iota(K) \otimes_{K^\sharp} A)\end{aligned}$$

In fact, given an $g \in \text{Inf-aut}(\mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]]/\mathcal{K}\hat{\otimes}_{L^\sharp} A[[W]])$ so that

$$g\colon \mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]] \to \mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]].$$

By Corollary 4.14 the restriction $\tilde{g} := g|_{\iota(K)[B,\,(\det B)^{-1}].K^\sharp.A}$ to the sub-ring

$$\iota(K)[B,\, (\det B)^{-1}].K^\sharp.A \simeq \iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\sharp} A$$

of $\mathcal{S} \otimes_{L^\sharp} A[[W]]$ maps $\iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\natural} A$ to

$$\iota(K)[B,\, (\det B)^{-1}].K^\sharp.A \simeq \iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\natural} A.$$

Therefore we have a commutative diagram

$$\begin{array}{ccc}\iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\natural} A & \xrightarrow{\tilde{g}} & \iota(K)[B,\, (\det B)^{-1}].K^\sharp \otimes_{K^\natural} A\\ \downarrow & & \downarrow\\ \mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]] & \xrightarrow[g]{} & \mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]].\end{array}$$

Now by isomorphisms (4.9),

$$\begin{aligned}&\text{Inf-aut}(\mathcal{S}\hat{\otimes}_{L^\sharp} A[[W]]/\mathcal{K}\hat{\otimes}_{L^\sharp} A[[W]])\\ &\qquad\qquad \simeq \text{Inf-aut}(K[Z^{PV},\, (\det Z^{PV})^{-1}] \otimes_C A/K \otimes_C A).\end{aligned}$$

□

The proof above simplified and clarified the previous proofs for the special cases.

When the base field K is not algebraically closed, the above argument allows us to prove the following result.

Theorem 4.19. — *Let L/K be a differential field extension generated by a fundamental system of solutions of a linear differential equation* (4.1) *and let L^{PV} be a Picard-Vessiot ring for the equation* (4.1). *There exists a finite field extension $\tilde{L}$ of $L^\natural$, we have an isomorphism*

$$\mathrm{Lie}\,(\mathrm{Inf\text{-}gal}\,(L/K)) \otimes_{L^\natural} \tilde{L} \simeq \mathrm{Lie}\,(\mathrm{Gal}(L^{PV}/K)) \otimes_C \tilde{L}.$$

Proof. — We replace the base field K by its algebraic closure $\bar{K}$ and L by $L \otimes_K \bar{K}$. Then we can apply the argument of proof of Theorem 4.18. □

We can get similar result for a G-primitive extension L/K.

Theorem 4.20. — *We assume that the base differential field K is algebraically closed. Let L/K be a G-primitive extension. There exists a G-primitive extension L' over K with the field of constants $C_{L'}$ is equal to C_K, so that the extension L'/K is strongly normal, such that*

$$\mathrm{Lie}\,(\mathrm{Inf\text{-}gal}\,(L/K)) \simeq \mathrm{Lie}\,(\mathrm{Gal}(L'/K)) \otimes_C L^\natural$$

where $Gal(L'/K)$ is the Galois group of the strongly normal extension L'/K.

Proof. — Since L is G-primitive extension over K, there exists an L-valued point p of G such that $K = L(p)$. Through the universal Taylor morphism $\iota\colon L \to L^\sharp[[X]]$ we have

$$L^\natural.\iota(L) = K(p^\natural).\iota(K)(\iota(p)) = K.\iota(K)(p^\natural, \iota(p)(p^\natural)^{-1})$$

and $\iota(p)(p^\natural)^{-1} \in G(K^\natural[[X]])$. So the same argument also works in the proof of Theorem 4.18. □

5. Comparison theorems

The main theorem allows us more comparison theorems.

Example 5.1. — We consider a linear differential equation $y'' = xy$ over $K := \mathbb{C}(t)$ called Airy equation. We consider a differential domain

$$S := \mathbb{C}(x)[y_{11},\ y_{12},\ y_{21},\ y_{22}, \det{}^{-1}]$$

where y_{ij} are variables over K and det denotes the determinant $y_{11}y_{22} - y_{12}y_{21}$ of the matrix

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

We extend the differentiation of the base field K to K-algebra S according to the linear differential equation

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

In the differential domain S, we can show easily that the element det is a transcendental constant over K. So the domain S is not a Picard-Vessiot ring but a generated by a fundamental system of solutions of linear differential equations over K.

We also consider another differential domain

$$R := \mathbb{C}(x)[\mathrm{Ai}(x),\ \mathrm{Bi}(x),\ \mathrm{Ai}'(x),\ \mathrm{Bi}'(x)].$$

The functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are two solutions holomorphic over $\mathbb{C}$ of Airy equation which form a basis of the $\mathbb{C}$-vector space of the solutions. The initial value of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ and their derivatives $\mathrm{Ai}'(x)$ and $\mathrm{Bi}'(x)$ are given by

$$\mathrm{Ai}(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})},\ \mathrm{Ai}'(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})},\ \mathrm{Bi}(0) = \frac{1}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})},\ \mathrm{Bi}'(0) = \frac{3^{\frac{1}{6}}}{\Gamma(\frac{1}{3})}$$

where $\Gamma(x)$ is the gamma function. We have

$$\det \begin{pmatrix} \mathrm{Ai}(x) & \mathrm{Bi}(x) \\ \mathrm{Ai}'(x) & \mathrm{Bi}'(x) \end{pmatrix} = \mathrm{Ai}(x)\mathrm{Bi}'(x) - \mathrm{Ai}'(x)\mathrm{Bi}(x) = \frac{1}{\pi}.$$

The differential domain R is simple differential domain, so that R is a Picard-Vessiot ring for the Airy differential equation. Then the differential field extension $L := Q(R)/K$ is a Picard-Vessiot extension with the Galois group $\mathrm{SL}_2(\mathbb{C})$, which is the special linear group.

We put

$$V'_{\mathbb{C}[x]} := \mathrm{Spec}\ \mathbb{C}[x,\ \mathrm{Ai}(x),\ \mathrm{Bi}(x),\ \mathrm{Ai}'(x),\ \mathrm{Bi}'(x)]$$

and

$$V_{\mathbb{C}[x]} := \mathrm{Spec}\ \mathbb{C}[x,\ y_{11},\ y_{12},\ y_{21},\ y_{22}, \det{}^{-1}]$$

The variety $V'_{\mathbb{C}[x]}$ is a closed subvariety of $V_{\mathbb{C}[x]}$.

$$\begin{tikzcd} V'_{\mathbb{C}[x]} \arrow[r, "f"] \arrow[d, "\varphi"'] & V_{\mathbb{C}[x]} \arrow[d, "\varphi"] \\ \operatorname{Spec}\mathbb{C}[x] \arrow[r, equal] & \operatorname{Spec}\mathbb{C}[x] \end{tikzcd}$$

Airy equation $y'' - xy = 0$ defines a foliation on $V'_{\mathbb{C}[x]}$ and $V_{\mathbb{C}[x]}$ so that they generate Galois groupoid $\mathcal{G}'$ and $\mathcal{G}$ over $V'_{\mathbb{C}[x]}$ and $V_{\mathbb{C}[x]}$. According to Malgrange [3] page 222, section 3 Example (1) or Casale [1] page 75, Theorem 5.3.1, the Galois groupoid $\mathcal{G}'|_{\varphi^{-1}(0)}$, $\mathcal{G}|_{\varphi^{-1}(0)}$ transverses over the point $0 \in \mathbb{C} = \operatorname{Spec}\,\mathbb{C}[x]$ are isomorphic.
If we translate this comparison theorem into the language of Umemura [5], noticing that we work at the generic point of $V := \operatorname{Spec} S$ and $V' := \operatorname{Spec} R$. We get following proposition by our theorem.

Proposition 5.2. —

$$\operatorname{Lie}(\text{Inf-gal}\,(L/K)) \otimes_{L^\natural} \tilde{L} \simeq \operatorname{Lie}(\operatorname{Gal}(L^{PV}/K)) \otimes_C \tilde{L},$$

where $L = Q(S)$, $L^{PV} = Q(R)$ and $\tilde{L}$ is a finite extension of $L^\natural$.

Example 5.3. — Let K be a differential field of which the field of constants $C_K = C$ that we assume to be algebraically closed. We consider a differential extension field $L = K(z_1,\ z_2,\ \cdots,\ z_n)$ over K such that $\boldsymbol{z} = {}^t(z_1,\ z_2,\ \cdots,\ z_n)$ is a solution of a linear differential equation

$$\boldsymbol{y}' = A\boldsymbol{y},\ \ A \in \mathrm{M}_n(K). \tag{5.1}$$

We assume that the z_i's are transcendental over K so that $tr.d\ L/K = n$. The extension $K[z_1,\ z_2,\ \cdots,\ z_n]/K$ defines the fibration

$$\operatorname{Spec}\ K[z_1,\ z_2,\ \cdots,\ z_n] \to \operatorname{Spec}\ K.$$

If the field K is finitely generated over C, equation (5.1) defines a foliation over an appropriate geometrical model of K. In particular if $K = \mathbb{C}$ or $K = \mathbb{C}(t)$, a foliation defined by equation (5.1) on Spec $K[z_1,\ z_2,\ \cdots,\ z_n]$ induces the Galois groupoid in the sense of Malgrange on the fibre that is a transverse. The Galois groupoid expected to be isomorphic to the Galois groupoid determined by the linear differential equation (5.1).

In our terminology,

Theorem 5.4. — *We have*

$$\operatorname{Lie}(\text{Inf-gal}\,(L/K)) \otimes_{L^\natural} \tilde{L} \simeq \operatorname{Lie}(\operatorname{Gal}(L^{PV}/K)) \otimes_C \tilde{L},$$

where $\tilde{L}$ is a finite extension of $L^\natural$.

We choose a basis $\{\partial_{z_i}\}$ of the $L^\natural$-vector space $\mathrm{Der}(L/K)$ of the derivations of L over K and construct the Galois hull $\mathcal{L}/\mathcal{K}$ of the extension L/K.

It follows from the definition of the universal Taylor morphism $\iota : L \to L^\sharp[[X]]$, the image

$$\iota(\boldsymbol{z}) = \boldsymbol{z} + \iota(A)\boldsymbol{z}X + \frac{1}{2!}\iota(A' + A^2)\boldsymbol{z}X^2 + \cdots . \tag{5.2}$$

As $\iota : \mathcal{L} \to L^\sharp[[X]]$ is a differential homomorphism, $\mathcal{Z} = \iota(\boldsymbol{z})$ satisfies

$$\frac{d}{dX}\mathcal{Z} = \iota(A)\mathcal{Z}.$$

or $\mathcal{Z}$ is a solution to linear differential equation

$$\frac{d}{dX}\boldsymbol{y} = \iota(A)\boldsymbol{y}. \tag{5.3}$$

Since in $L^\sharp[[X]]$ the derivations ∂_{z_i} and d/dX are mutually commutative and since $\partial_{z_i}\iota(A) = O$, applying the derivations ∂_{z_i}'s to (5.2), we get

$$\frac{d}{dX}\mathcal{Z}_i = \iota(A)\mathcal{Z}_i$$

where $\mathcal{Z}_i = \partial_{z_i}\mathcal{Z}$ for $i = 1,\ 2,\ \cdots,\ n$. Namely, the vectors $\mathcal{Z}_i$'s are n-solutions to (5.4). We are going to see they are linearly independent over $\mathcal{K}$. In fact, if we set $F(X) := (\mathcal{Z}_1,\ \cdots \mathcal{Z}_n) \in \mathrm{M}_n(L^\sharp[[X]])$, since the constant term $F(0)$ of $F(X)$ is the identity matrix I_n by (5.2), $\det F(X)$ is invertible in $L^\sharp[[X]]$ and we can follow the argument in the part of main theorem replacing B by $F(X)$.

Example 5.5. — Let $(K, d/dt)$ be a differential field of meromorphic functions over a complex domain U of $\mathbb{C}$ where t is a coordinate on $\mathbb{C}$. We assume $C_K = \mathbb{C}$. We consider a linear differential equation

$$\boldsymbol{y}' = A\boldsymbol{y},\ A \in \mathrm{M}_n(K), \tag{5.4}$$

where the entries of A are regular at a point $p \in U \subset \mathbb{C}$. For $\boldsymbol{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{C}^n$, let ${}^t(y_1(\boldsymbol{c};t), y_2(\boldsymbol{c};t) \cdots, y_n(\boldsymbol{c};t))$ be a solution of (5.4) such that

$$(y_1(\boldsymbol{c};p), y_2(\boldsymbol{c};p) \cdots, y_n(\boldsymbol{c};p)) = \boldsymbol{c}.$$

Hence there exists a neighborhood V_p of $p \in U$ that the $y_i(\boldsymbol{c};t)$'s are regular functions on $\mathbb{C}^n \times V_p$ for $1 \leqq i \leqq n$. This satisfies assumption of Example 5.3. Then we have the following corollary of Theorem 5.4.

Corollary 5.6. —

$$\mathrm{Lie}(\mathrm{Inf}\text{-}\mathrm{gal}\,(L/K)) \otimes_{L^\natural} \tilde{L} \simeq \mathrm{Lie}(\mathrm{Gal}(L^{PV}/K)) \otimes_C \tilde{L},$$

where $L := K(y_1, y_2, \cdots, y_n)$ *and* $\tilde{L}$ *is a finite extension of* $L^\natural$.

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