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Diffusion-orthogonal polynomial systems of maximal weighted degree

Lev Soukhanov [(1)]

Abstract. — A diffusion-orthogonal polynomial system is a bounded domain Ω in $\mathbb{R}^d$ endowed with the measure μ and the second-order elliptic differential operator L, self adjoint w.r.t $L^2(\Omega, \mu)$, preserving the space of polynomials of degree $\leqslant n$ for any n. This notion was initially defined in [2], and 2-dimensional models were classified.

It turns out that the boundary of Ω is always an algebraic hypersurface of degree $\leqslant 2d$. It was pointed out in [2] that in dimension 2, when the degree is maximal (so, equals 4), the symbol of L (denoted by g^{ij}) is a cometric of constant curvature.

We present the self-contained classification-free proof of this property, and its multidimensional generalisation.

Résumé. — Un système de diffusion polynomial est la donnée d'un domaine borné $\Omega \subset \mathbb{R}^d$ équipé d'une mesure μ et d'un opérateur différentiel elliptique d'ordre deux L, autoadjoint sur $L^2(\Omega, \mu)$, tels que l'espace de polynômes de degré $\leqslant n$ est invariant par L pour tout n. Cette notion est introduite dans [2] et les modèles en dimension 2 y sont classifiés.

Une conséquence est que la frontière de Ω est toujours une hypersurface algébrique de degré $\leqslant 2d$. D'après [2], en dimension 2 et lorsque le degré est maximal (égal à 4) le symbole g^{ij} de L est une cométrique de courbure constante.

Nous présentons une démonstration indépendante de cette propriété valable en toute dimension.

1. Introduction

Definition 1.1. — *A* model *is a triple (Ω, μ, L), such that Ω is a bounded domain in $\mathbb{R}^d$, μ is a measure on Ω, L is a second-order differential elliptic operator, self-adjoint in $L^2(\Omega, \mu)$, preserving the space of polynomials of degree $\leqslant n$ for any n:*

$$L := g^{ij}\partial_i\partial_j + h^i\partial_i$$

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This notion naturally extends the notion of classical orthogonal polynomial systems (we restrict ourselves only to the bounded case). It was initially defined in [2], and full classification of these systems in dimension 2 was obtained.

Also, there is a weighted case: let us consider the natural numbers $w_1, \dots, w_d \geqslant 1$, called "weights" of the variables $x_1, \dots, x_d$.

The space W_k of polynomials of weighted degree $\leqslant k$ is the span of monomials of the form $x_1^{s_1} \dots x_d^{s_d}$ such that $\sum s_i w_i \leqslant k$. We will say that the polynomial has a weighted degree $\leqslant k$ if it is contained in the space W_k.

Then one can introduce the following definition:

Definition 1.2. — *A* weighted model *(for the given weights) is defined in a same way as a model, but L preserves the spaces W_k instead.*

At the moment, there is no known classification of weighted models even in degree 2.

The following important property holds: the boundary of Ω is a (part of) algebraic hypersurface and has a weighted degree at most $2(\sum w_i)$ (it is proved in [2] and will be explained throughout the text).

Definition 1.3. — *The weighted model is called* maximal *if the degree of boundary of Ω is exactly $2(\sum w_i)$.*

Remark 1.4. — In [2] it has been noted as a consequence of the classification that in the non-weighted case, the curvature of the metric $g_{ij} = (g^{ij})^{-1}$ is constant and non-negative for all maximal models.

We explain this fact and prove it in a much broader scope.

Theorem (Theorem 2.9). — *For a maximal weighted model of any dimension the riemannian metric $g_{ij} = (g^{ij})^{-1}$ is of constant scalar curvature.*

We are currently unable to prove (either in the weighted case, or even in non-weighted, but without using classification) that the curvature should be non-negative.

In dimension 2, the constant scalar curvature property implies that the metric is locally modelled on the round sphere, the euclidean plane or the hyperbolic plane. Then we make use of other invariants (covariant derivatives of the curvature tensor) and extend this property to higher dimensions:

Theorem (Corollary 3.10). — *The riemannian metric of the maximal model is locally homogeneous (in any dimension).*

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2. Scalar curvature of g

2.1. Necessary facts about diffusion-orthogonal models

Now we are going to present the list of some useful results of [1] and [2] about (weighted) models.

PROPERTY 2.1. — *Measure μ is always a smooth function multiple of the Lebesgue measure on Ω. We will denote this function as μ by abuse of notation.*

PROPERTY 2.2. — *Hence, by self-adjointness, L is of the form $\frac{1}{\mu}\partial_i \mu g^{ij}\partial_j$.*

PROPERTY 2.3. — *It is quite easy to see that g^{ij} is a symbol of L, and then g^{ij} must be a polynomial of weighted degree $\leqslant (w_i + w_j)$ (if not, L won't preserve W_k's).*

PROPERTY 2.4. — *The boundary of Ω is always contained in the hypersurface $\det(g^{ij}) = 0$, hence has a weighted degree at most $2(\sum w_i)$.*

Denote $\det(g^{ij})$ by D, $G = D^{-1}$.

PROPERTY 2.5. — *For a maximal model the polynomial $grad_g(D) = g^{ij}\partial_i D$ is divisible by D.*

These properties are contained in Theorem 5.1 of [1].

2.2. Strategy of the proof

In the two following subsections we prove two facts about the scalar curvature $Sc(g)$. In the first one we obtain by inspection of the formula that $Sc(g)$ is a rational function of weighted degree $\leqslant 0$. In the second we prove that it is, in fact, polynomial, hence, constant.

In what follows the degree of a rational function P/Q is defined as $\deg(P) - \deg(Q)$.

2.3. Formula for $Sc(g)$

The metric is defined from the cometric as

$$g_{ij} = (g^{ij})^{-1}$$
$$g_{ij} = \hat{g}^{ij} D^{-1}$$

By the maximality of the model degree

$$\deg(D^{-1}) = -2 \sum w_i$$

and so

$$\deg(\hat{g}^{ij}) \leqslant (2 \sum w_i) - (w_i + w_j)$$
$$\deg(g_{ij}) \leqslant -w_i - w_j$$

The Christoffel symbols are defined as follows:

$$\Gamma_{ijk} = \partial_k g_{ij} - \partial_j g_{ki} - \partial_i g_{jk}$$
$$\Gamma_{ij}^k = g^{kl} \Gamma_{ij}^k$$

The degrees are as follows: (partial along i derivation lowers degree by w_i)

$$\deg(\Gamma_{ijk}) \leqslant -w_i - w_j - w_k$$
$$\deg(\Gamma_{ij}^k) \leqslant w_k - w_i - w_j$$

The formula of the Riemann curvature tensor is:

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ik}^s \Gamma_{js}^l - \Gamma_{is}^l \Gamma_{jk}^s$$

The degree is

$$\deg(R_{ijk}^l) \leqslant w_l - w_i - w_j - w_k$$

The scalar curvature is the following contraction:

$$Sc = R_{ijk}^j g^{ik}$$

So, its degree is

$$\deg(Sc) \leqslant 0$$

2.4. Scalar curvature is a polynomial

Claim 2.6. — *D vanishes on the boundary of Ω with multiplicity* 1.

Proof. — Otherwise, the degree of the boundary would be strictly less than the degree of D, which contradicts maximality of Ω. □

Recall also that, by the Property 2.5, $g^{ij}\partial_j D$ is divisible by D.

Now we are going to exploit this fact to substitute the coordinates in such a way that makes the metric g invertible. For the rest of this part we consider all equations to be defined over $\mathbb{C}$. We denote the complexification of the symmetric 2-form g by the same letter.

Consider a variety S ("double cover", defined by the equation $z^2 = D$ in the space $\mathbb{C}^{d+1}$ with coordinates $(x_1, \dots, x_d, z)$). It is obviously smooth at the points where $D \neq 0$, and it is also smooth at the smooth points of the set $\{z = 0, D = 0\}$ (because D vanishes with multiplicity 1). Let us denote by $S_{\mathbb{R}}$ the set $(\Omega \times \mathbb{R}) \cap S$ - set of real points of S projected into Ω. Also we denote the projection from S to $\mathbb{C}^d$ as π, the set of smooth points of S as S^{Sm}, the set $\{p \in \mathbb{C}^d | D(p) = 0\}$ as Δ.

Lemma 2.7. — *The pullback $\tilde{g} = \pi^\star g$ is an invertible (as a section of $S^2(TS^{Sm})$) regular symmetric 2-form on S^{Sm}. Its restriction on the smooth points of $S_{\mathbb{R}}$ is a smooth riemannian metric.*

Proof. — At first, notice that the pullback of g (either metric or cometric) is well defined on the preimage of $\mathbb{C}^d \backslash \{D = 0\}$ because π is a covering away from Δ. So, $\tilde{g}$ is continued from an open dense subset on S^{Sm} as a symmetric 2-form with rational coefficients on S^{Sm}, possibly having poles in the ramification divisor $\pi^{-1}\Delta$.

The rest is to check the absence of poles, which will be a local analysis in a neighborhood of the smooth point of $\pi^{-1}\Delta$.

Pick local coordinates $y_0, \dots, y_{d-1}$ in such a way that the local equation of the boundary is $y_0 = D = 0$ (this is possible by the implicit function theorem), and the point p is $(0, \dots, 0)$. Denote by J the Jacobian of the change of coordinates $x \to y$.

By the Property 2.5, $grad_g(y_0)/y_0$ is regular, which means that the coefficients $g^{y_0 y_*}$ are divisible by y_0. Also, $g^{y_0 y_0}$ is not divisible by y_0^2 (otherwise, $\det(g) = DJ^{-1}$, and, hence, D would vanish with multiplicity > 1).

For the neighborhood of S around p pick the local coordinates

$$\tilde{y}_0^2 = z^2 = y_0$$
$$\tilde{y}_i = y_i.$$

By the chain rule:

$$\partial_{\tilde{y}_i} = \partial_{y_i}$$
$$\partial_{\tilde{y}_0} = 2\tilde{y}_0 \partial_{y_0}$$

Hence, in the $\tilde{y}$-coordinates,

$$g^{\tilde{y}_*\tilde{y}_*} = g^{y_*y_*},\ \ * \neq 0$$
$$g^{\tilde{y}_0\tilde{y}_*} = g^{y_0y_*}/(2\tilde{y}_0),\ \ * \neq 0$$
$$g^{\tilde{y}_0\tilde{y}_0} = g^{y_0y_0}/(4\tilde{y}_0^2)$$

Notice that, as $g^{y_0y_*}$ and $g^{y_*y_*}$ are divisible by y_0, these coefficients do not blow up. Also notice that the determinant of the matrix g is divided by $\tilde{y}_0^2 = y_0$ totally (first row and first column are divided by $\tilde{y}_0$ each), so it becomes a non-vanishing function (in the y-coordinates it was yJ). Hence, the form g is regular and invertible in the $\tilde{y}$-coordinates.

Moreover, the form g is positive-definite at smooth points of $S_{\mathbb{R}}$ (as it is invertible and positive definite in an open dense subset), so it is a smooth riemannian metric on the smooth part of $S_{\mathbb{R}}$. □

Theorem 2.8. — *The scalar curvature is a polynomial.*

Proof. — Let us denote the irreducible components of the polynomial D as $D = D_1 \dots D_k$. There are no multiple components, because the degree of the boundary is maximal. Also, for each D_i there is smooth point on the boundary such that D_i vanishes in it (otherwise we could exclude this D_i from the equation of the boundary).

By the formula of the scalar curvature, it is a rational function of the irreducible form $P/D_1^{a_1} \dots D_n^{a_n}$ (because all the divisions performed are divisions on the determinant of g^{ij} in the inverse matrix formula).

For each D_i choose the smooth point p_i on the boundary at which $D_i(p_i) = 0$ but $P(p_i) \neq 0$ (it can be done by a general position argument). It suffices to check that the scalar curvature is finite at p_i's.

Scalar curvature is an invariant of the metric, so it can be calculated on the preimages of the points p_i's on S instead (in the $\tilde{y}$-coordinates). There, by the previous lemma, we know that the metric is invertible smooth symmetric 2-form, hence the scalar curvature is finite. □

Theorem 2.9. — *For a maximal degree model (g_{ij}) has constant scalar curvature.*

Proof. — Scalar curvature is a polynomial of degree 0 or less, so it is constant. □

For the dimension 2, this theorem guarantees that the metric is locally modelled on one of the three classical spaces: sphere, euclidean plane or hyperbolic plane. In the higher dimensions the various possible refinements of this theorem are possible.

3. Local homogeneity of g

The scalar curvature is important, but it is only the first in the big list of *local metric invariants.*

Definition 3.1. — *A* function of g *is a rational function, depending on* $\partial_{s_1} \dots \partial_{s_k} g^{ij}$ *with only (possibly)* $\det(g)^n$ *in a denominator.*

Remark 3.2. — For example, the coefficients of (g_{ij}) and all of their derivatives are functions of g (as the inverse matrix is $\hat{g}/\det(g)$, and $\hat{g}$ is polynomial in g).

Definition 3.3. — *A* local metric invariant $I(g)$ *is a function of* g *invariant under change of coordinates.*

Remark 3.4. — For example, the scalar curvature, or, generally, any contraction of the Riemann curvature tensor and the metric tensor is a local metric invariant.

Definition 3.5. — *The metric is called* locally homogeneous *if any two points have isometric neighborhoods.*

Definition 3.6. — *An invariant is called a* scalar Weyl invariant *if it is a full tensor contraction of* g^{ij}*'s,* g_{ij}*'s and covariant derivatives of Riemann curvature tensor* $\nabla^{s_1} \dots \nabla^{s_r} R^l_{ijk}$*'s.*

Theorem 3.7 (Prufer, Tricerri, Vanhecke). — *A metric is locally homogeneous if and only if all its scalar Weyl invariants are constant.*

This is a version of a famous result of Singer. It is proved in [3].

Lemma 3.8. — *Any scalar Weyl invariant on a maximal model is a rational function of degree* 0*.*

Proof. — It is by the same inspection of the formula as for scalar curvature:

$$\deg(R^l_{ijk}) = w_l - w_i - w_j - w_k$$
$$\deg(g^{ij}) \leqslant w_i + w_j$$
$$\deg(g_{ij}) \leqslant -w_i - w_j$$
$$\deg(\Gamma^k_{ij}) \leqslant w_k - w_i - w_j$$
$$\deg(\partial_i) = -w_i$$

So, the degree of any formula with these symbols $\leqslant$ sum of weights of upper indices minus sum of weights of lower indices. But, as the scalar invariant is a full contraction, each lower index is paired with an upper index, so the degree is $\leqslant 0$ □

Lemma 3.9. — *Any local invariant on a maximal model is polynomial.*

Proof. — The same proof as for scalar curvature, literally: it is a rational function with only possibly powers of D in the denominator, and it does not tend to infinity at smooth points of boundary. ☐

Corollary 3.10 (of the theorems above). — *For a maximal model the metric is locally homogeneous.*

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