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Large scale ocean models beyond the traditional approximation

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Abstract. — This work corresponds to classes given by A. Rousseau in February 2014 in Toulouse, in the framework of the CIMI labex. The objective is to describe and question the models that are traditionaly used for large scale oceanography, whether in 2D or 3D. Starting from fundamental equations (mass and momentum conservation), it is explained how (thanks to approximations for which we provide justifications) one can build simpler models that allow a realistic numerical implementation. We particularly focus on the so-called traditional approximation that neglects part of the Coriolis force.

Résumé. — Ce manuscrit retrace un cours donné par A. Rousseau en février 2014 à Toulouse dans le cadre du labex CIMI. Il s'agit de donner un aperçu, et de questionner, les modèles traditionnellement utilisés pour l'océanographie à grande échelle (qu'il s'agisse de modèles 2D ou 3D). En partant des équations complètes (conservation de la masse et de la quantité de mouvement), on explique comment (à partir d'approximations dont on donne les justifications physiques) on parvient à construire des modèles plus simples qui permettent une implémentation logicielle réaliste. Une focalisation particulière est effectuée sur l'approximation dite traditionnelle qui consiste à négliger une partie des termes de la force de Coriolis.

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1. Introduction

Oceans modeling is based on various equations which consist in different approximations of conservation laws. We start recalling these physical laws and explain which approximation leads to each model.

Let us start with the mass conservation, that expresses that the fluid mass can be neither created nor destroyed in terms of the mass density. It reads:

$$\frac{\partial \varrho}{\partial t} + (\underline{\mathbf{u}} \cdot \nabla)\, \varrho + \varrho \nabla \cdot \underline{\mathbf{u}} = 0\,. \tag{1.1}$$

In the following, we will denote

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + (\underline{\mathbf{u}} \cdot \nabla)\,,$$

such that (1.1) reads $\frac{D\varrho}{Dt} + \varrho \nabla \cdot \underline{\mathbf{u}} = 0$. The variable ϱ is the density of the fluid, and $\underline{\mathbf{u}} = (\underline{\mathbf{v}}, w) = (u, v, w)$ stands for its three dimensional velocity.

We also need the momentum balance, that relates the change rate of momentum of a particle to the sum of applied forces. In the case of the oceans, the reference frame is rotating with the angular velocity $\mathbf{\Omega}$, introducing the Coriolis force in the balance:

$$\varrho \frac{D\underline{\mathbf{u}}}{Dt} = -\nabla p - \varrho 2 \mathbf{\Omega} \times \underline{\mathbf{u}} - \varrho g e_z + F_{\underline{\mathbf{u}}}\,. \tag{1.2}$$

In this equation, $|\mathbf{\Omega}| = \Omega \approx 7.29 \times 10^{-5}\ \mathrm{s}^{-1}$ and $g \approx 9.81\ \mathrm{m.s}^{-2}$ is the gravitational acceleration. The vector e_z is defined by ${}^t(0, 0, 1)$ and the function $F_{\underline{\mathbf{u}}}$ stands for the external forces, including viscous terms.

These equations are complemented with a series of conservation laws for the temperature, the salinity, etc, depending on the quantities we consider. They satisfy the same type of equations, that can be written under the following unified formulation:

$$\frac{DT}{Dt} = Q_T\,, \tag{1.3}$$

where T is the variable (temperature, salinity, etc) and Q_T includes dissipative terms and external forces.

The last equation is an equation of state, relating the density ϱ to the pressure p, the temperature T (the salinity S, etc), that must be determined empirically:

$$\varrho = \mathcal{R}(p, T, S)\,. \tag{1.4}$$

In the oceans, the most popular state law reads $\varrho = \varrho_0\ (1 - \alpha\,(T - T_0))$, where T is the temperature, T_0 and ϱ_0 are reference values for the temperature and density respectively, and $\alpha > 0$ is a constant. A similar state law can be used to include both temperature and salinity, with a reference salinity S_0.

From equations (1.1)–(1.4), one can obtain the usual ocean models thanks to several approximations that will be recalled in the sequel. The only common approximation used for every model is the Boussinesq approximation.

Approximation (Boussinesq approximation, [2]). — *The Boussinesq approximation consists in neglecting the variations of the density when they are not multiplied by the gravity. In the mass equation, the density material derivative can then be removed whereas in the momentum balance, the mass density ϱ can be replaced by the reference density ϱ_0 in the inertial terms of the horizontal components.*

Consequently, applying Boussinesq approximation to equations (1.1)–(1.4) leads to the following Boussinesq equations:

$$\nabla \cdot \underline{\mathbf{u}} = 0\,, \tag{1.5a}$$

$$\varrho_0 \frac{D\underline{\mathbf{u}}}{Dt} = -\nabla p - \varrho_0 2\boldsymbol{\Omega} \times \underline{\mathbf{u}} - \varrho g e_z + F_{\underline{\mathbf{u}}}\,, \tag{1.5b}$$

$$\frac{DT}{Dt} = Q_T\,, \tag{1.5c}$$

$$\varrho = \varrho_0 \left(1 - \alpha \left(T - T_0\right)\right). \tag{1.5d}$$

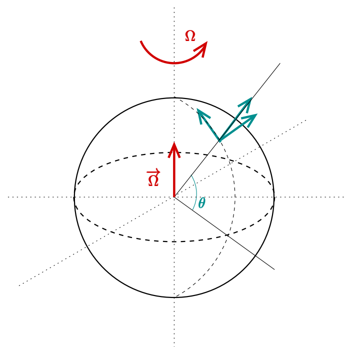
System (1.5) is the basis of the following work. The objective of this paper is to understand the various models that can be derived from this system, adding other assumptions. The outline is the following:

- First, one can assert the density to be constant (see details in Section 2): we obtain the Navier–Stokes equations from which we can also derive the Shallow-Water equations for a small aspect ratio.
- Another approach is to add the hydrostatic approximation (see Section 3) to lead to the so-called primitive equations.
- For the third model we assume the Rossby number to be small and perform an asymptotic expansion to get the three-dimensional quasi-geostrophic model, see Section 4.

These derivations are quite classical (see [18] for example) under the traditional approximation for the Coriolis force that can be expressed as follows:

Approximation (Traditional approximation[1] for the Coriolis force). *In the rotating frame,* $\mathbf{\Omega} = \Omega\, {}^t(0, \cos\theta, \sin\theta)$ *where* θ *represents the latitude and will be first considered as a constant. The value* $2\mathbf{\Omega} \times \underline{\mathbf{u}}$ *reads*

$$2\mathbf{\Omega} \times \underline{\mathbf{u}} = 2\Omega \begin{pmatrix} -v\sin\theta + w\cos\theta \\ u\sin\theta \\ -u\cos\theta \end{pmatrix} = \begin{pmatrix} -fv + f^*w \\ fu \\ -f^*u \end{pmatrix}.$$



The traditional approximation consists in neglecting the cosine terms writing:

$$2\mathbf{\Omega} \times \underline{\mathbf{u}} \approx 2\Omega \begin{pmatrix} -v\sin\theta \\ u\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -fv \\ fu \\ 0 \end{pmatrix}.$$

The main point of this paper is to perform the derivation of the above-mentioned models without the traditional approximation, with careful analyses of each term, leading to modified models including cosine terms, see the discussion in [14, 15, 21, 22].

2. Constant density models

A first class of models is obtained assuming a constant density in System (1.5). We detail in this part how it leads to the Navier–Stokes equations and other approximations such as Shallow Water equations.

2.1. Euler, Navier–Stokes and tracers equations

We rewrite system (1.5) with the following approximation:

(1) The denomination of *traditional approximation* was introduced by Carl Eckart in his book [5].

Approximation (Constant density). — *We suppose the density to be constant, that is* $\varrho = \varrho_0$.

We get

$$\nabla \cdot \underline{\mathbf{u}} = 0\,, \tag{2.1a}$$

$$\frac{D\underline{\mathbf{u}}}{Dt} = -\frac{1}{\varrho_0}\nabla p - 2\mathbf{\Omega} \times \underline{\mathbf{u}} - g e_z + F_{\underline{\mathbf{u}}}\,, \tag{2.1b}$$

that are exactly the three dimensional Navier–Stokes equations, supplemented by tracers equations of the type

$$\frac{DT}{Dt} = Q_T,$$

for the temperature, salinity, etc, that can be solved independently.

In the following, we work on system (2.1) to get the Shallow Water equations.

Remark 2.1. — The kinematic viscosity is very small and it can be neglected for the water. In the following, for the sake of simplicity, we assume $F_{\underline{\mathbf{u}}} = 0$ and we consider the so-called incompressible Euler equations

$$\nabla \cdot \underline{\mathbf{u}} = 0\,, \tag{2.2a}$$

$$\frac{D\underline{\mathbf{u}}}{Dt} = -\frac{1}{\varrho_0}\nabla p - 2\mathbf{\Omega} \times \underline{\mathbf{u}} - g e_z\,. \tag{2.2b}$$

The drawback of this simplification is that we need to *assume* that the horizontal velocity $\underline{\mathbf{v}}$ does not depend on z. In the presence of viscosity at a certain scale, the same derivation could be performed with no additional assumption on the horizontal velocity, see for example [11].

2.2. Shallow Water equations

The two dimensional SW system is obtained from three dimensional NSE (2.2) in a shallow domain. We look for the equations satisfied by the horizontal mean velocity field and the free surface.

Figure 2.1 describes the computational domain, together with the bathymetry b, the water column height H and the free surface h. We supplement equations (2.2) with the following boundary conditions:

- At the free surface $z = h(t, x)$, we usually neglect atmospheric pressure and take $p = 0$.

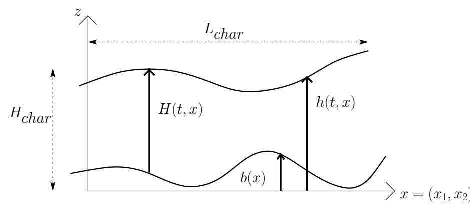


Figure 2.1. Notations used for the SW system

The normal velocity in the referential linked to a particle moving on the surface is zero:

$$\partial_t h + \underline{\mathbf{v}} \cdot \nabla_x h = w\,,$$

where ∇_x is the horizontal gradient.

- At the bottom $z = b(x)$, we have the nonpenetration condition

$$-\underline{\mathbf{v}} \cdot \nabla_x b + w = 0\,.$$

In what follows we write these equations under their nondimensional form, make an asymptotic development of $\underline{\mathbf{u}}$, and study its first orders. SWE are obtained after integrating the first momentum equation over the water height.

2.2.1. Dimensionless NSE

We write the NS system and the boundary conditions in a nondimensionalized form, using some characteristic scales specially chosen to get the SW model. We introduce the following dimensionless variables and numbers:

$$\begin{aligned}
x &= L_{char}\, x', & z &= H_{char}\, z', & &\text{with } \varepsilon = \frac{H_{char}}{L_{char}} \ll 1\,,\\
\underline{\mathbf{v}} &= u_{char}\, \underline{\mathbf{v}}', & w &= w_{char}\, w', & &\text{with } w_{char} = \varepsilon u_{char}\,,\\
t &= \frac{L_{char}}{u_{car}}\, t', & p &= p_{char}\, p', & &\text{with } p_{char} = \frac{u_{char}^2}{\varrho_0}\,,\\
\mathrm{Ro}^* &= \frac{u_{char}}{2L_{char}\Omega}\,, & \mathrm{Fr} &= \frac{u_{char}}{\sqrt{gH_{char}}}\,.
\end{aligned}$$

where ε is the aspect ratio, Ro^* the Rossby number[2] , Fr the Froude number. We drop the primes to rewrite the three dimensional NSE:

$$\partial_t \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \nabla_x \underline{\mathbf{v}} + w\partial_z \underline{\mathbf{v}} = -\nabla_x p - \frac{\sin\theta}{\mathrm{Ro}^*}\underline{\mathbf{v}}^\perp - \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*} w \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.3a}$$

$$\partial_t w + \underline{\mathbf{v}} \cdot \nabla_x w + w\partial_z w = -\frac{1}{\varepsilon^2}\partial_z p + \frac{1}{\varepsilon}\frac{\cos\theta}{\mathrm{Ro}^*} u - \frac{1}{\varepsilon^2 \mathrm{Fr}^2}, \tag{2.3b}$$

$$\mathrm{div}_x \underline{\mathbf{v}} + \partial_z w = 0. \tag{2.3c}$$

We also have to change the boundary conditions.

- At the free surface $z = h(t, x)$, the horizontal variable h is rescaled as $h = H_{char} h'$ to get the dimensionless conditions:

$$p = 0, \tag{2.4a}$$

$$\partial_t h + \underline{\mathbf{v}} \cdot \nabla_x h = w. \tag{2.4b}$$

- At the bottom $z = b(x)$, if we write $b = H_{char} b'$, the nonpenetration condition reads

$$-\underline{\mathbf{v}} \cdot \nabla_x b + w = 0. \tag{2.5}$$

2.2.2. Average Shallow Water equations

We now perform the main approximation in (2.3) to get the Shallow Water equations:

Approximation (Shallow water). — *We suppose the aspect ratio ε to be small: the length of the considered domain is large compared to its height.*

We are led to study the system:

$$\partial_t \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \nabla_x \underline{\mathbf{v}} + w\partial_z \underline{\mathbf{v}} = -\nabla_x p - \frac{\sin\theta}{\mathrm{Ro}^*}\underline{\mathbf{v}}^\perp - \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*} w \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.6a}$$

$$\partial_z p = -\frac{1}{\mathrm{Fr}^2} + \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*} u + O(\varepsilon^2), \tag{2.6b}$$

$$\mathrm{div}_x \underline{\mathbf{v}} + \partial_z w = 0. \tag{2.6c}$$

We integrate (2.6b) from h to z, with z between b and h. The value of p at the free surface is given by (2.4a), and we find the pressure at order ε:

$$p(t, x, z) = \frac{1}{\mathrm{Fr}^2}\left(h(t, x) - z\right) + \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*}\int_h^z u + O(\varepsilon^2). \tag{2.7}$$

[2] In this section, we choose this 'unusual' definition of the Rossby number without $\sin\theta$ to keep the symmetry between the sine and cosine terms.

As we are looking for equations on the mean velocity and on the evolution of the free surface, we first integrate the momentum equation (2.6a) over the water height (between $z = b(x)$ and $z = h(t, x)$). We apply Leibniz formula and get

$$\partial_t \int_b^h \underline{\mathbf{v}} - \partial_t h \, \underline{\mathbf{v}}_{|z=h} + \mathrm{div}_x \int_b^h (\underline{\mathbf{v}} \otimes \underline{\mathbf{v}}) - ((\underline{\mathbf{v}} \cdot \nabla_x h) \, \underline{\mathbf{v}})_{|z=h} + ((\underline{\mathbf{v}} \cdot \nabla_x b) \, \underline{\mathbf{v}})_{|z=b}$$
$$+ (\underline{\mathbf{v}} \, w)_{|z=h} - (\underline{\mathbf{v}} \, w)_{|z=b} + \nabla_x \int_b^h p$$
$$= \nabla_x h \, p_{|z=h} - \nabla_x b \, p_{|z=b} - \frac{\sin\theta}{\mathrm{Ro}^*} \int_b^h \underline{\mathbf{v}}^\perp - \varepsilon \frac{\cos\theta}{\mathrm{Ro}^*} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_b^h w \, .$$

Then we use boundary conditions (2.4a), (2.4b), and (2.5) to simplify the expressions at the surface and at the bottom and obtain the integrated momentum equation

$$\partial_t \int_b^h \underline{\mathbf{v}} + \mathrm{div}_x \int_b^h (\underline{\mathbf{v}} \otimes \underline{\mathbf{v}}) + \nabla_x \int_b^h p$$
$$= -\nabla_x b \, p_{|z=b} - \frac{\sin\theta}{\mathrm{Ro}^*} \int_b^h \underline{\mathbf{v}}^\perp - \varepsilon \frac{\cos\theta}{\mathrm{Ro}^*} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_b^h w \, , \quad (2.8)$$

with a new Coriolis term (the last one).

We also want the evolution of the free surface: we integrate the divergence free equation (2.6c) from the bottom to the surface. Using Leibniz formula again, together with surface and bottom conditions (2.4b) and (2.5), we find

$$\partial_t h(t, x) + \mathrm{div}_x \int_b^h \underline{\mathbf{v}} = 0 \, . \quad (2.9)$$

In what follows we study the integrated momentum equation (2.8) and the free surface equation (2.9) when we approximate $\underline{\mathbf{v}}$ at the first order and at the second order. We recall that, as we neglected the viscous terms, we assume that $\underline{\mathbf{v}}$ does not depend on z.

2.2.3. Asymptotic expansion

We have already done the main assumption to get the SW system, that is, the depth is small compared to the length of the domain. Now we develop $\underline{\mathbf{v}}$, w, H, p, b in powers of ε, that is $\underline{\mathbf{v}} = \underline{\mathbf{v}}^0 + \varepsilon \underline{\mathbf{v}}^1 + \varepsilon^2 \underline{\mathbf{v}}^2 + \dots$ (and so on) with $H(t, x) = h(t, x) - b(x)$.

We look for the dynamics of $\underline{\mathbf{v}}^0$, the first order of the horizontal velocity, studying the previous equations at the first order in ε. Let us first rewrite the evolution equation (2.9):

$$\partial_t H^0 + \mathrm{div}_x(H^0 \underline{\mathbf{v}}^0) = 0\,. \tag{2.10}$$

Then we have p at the first order with (2.7):

$$p^0(t,x,z) = \frac{1}{\mathrm{Fr}^2}\left(h^0(t,x) - z\right);$$

we replace this value in the integrated momentum equation (2.8) and obtain

$$\partial_t(H^0\underline{\mathbf{v}}^0) + \mathrm{div}_x(H^0\underline{\mathbf{v}}^0 \otimes \underline{\mathbf{v}}^0) + \frac{1}{2\,\mathrm{Fr}^2}\nabla_x (H^0)^2 = -\frac{1}{\mathrm{Fr}^2} H^0\, \nabla_x b^0 - \frac{\sin\theta}{\mathrm{Ro}^*} H^0 \underline{\mathbf{v}}^{0\perp}. \tag{2.11}$$

Equations (2.10)–(2.11) form the SW system at the first order in nondimensional variables. If we go back to dimensional variables we have the SW system at the first order:

$$\partial_t H + \mathrm{div}_x(H\underline{\mathbf{v}}) = 0\,, \tag{2.12a}$$

$$\partial_t(H\underline{\mathbf{v}}) + \mathrm{div}_x(H\underline{\mathbf{v}} \otimes \underline{\mathbf{v}}) + \frac{g}{2}\nabla_x H^2 = -gH\nabla_x b - 2\Omega\sin\theta H\underline{\mathbf{v}}^{\perp}. \tag{2.12b}$$

At this point, we get the usual SW system. The cosine part of the Coriolis force does not modify these equations at the first order. But if we want a better approximation (at the second order in ε), we will see new Coriolis terms in our SW system.

We denote by a bar the value of the variable at $O(\varepsilon^2)$:

$$\overline{u}^1 := u^0 + \varepsilon u^1, \text{ such that } u = \overline{u}^1 + O(\varepsilon^2)\,.$$

Let us rewrite the divergence condition at the second order:

$$\partial_t \overline{H}^1 + \mathrm{div}_x\left(\overline{H}^1 \overline{\underline{\mathbf{v}}}^1\right) = O(\varepsilon^2)\,. \tag{2.13}$$

We can get the value of p at the second order with (2.7):

$$p(t,x,z) = \frac{1}{\mathrm{Fr}^2}\left(h(t,x) - z\right) + \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*}\int_h^z u + O(\varepsilon^2)\,,$$

that is

$$\overline{p}^1(t,x,b) = \frac{1}{\mathrm{Fr}^2}\overline{H}^1(t,x) - \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*}u^0 H^0\,.$$

We replace the expression of the pression in the integrated momentum equation (2.8), and, using again the divergence free condition (2.6c) to express

w^0 as a function of h^0 and u^0, we get

$$\begin{aligned}
&\partial_t(\overline{H}^1\underline{\mathbf{v}}^1) + \mathrm{div}_x(\overline{H}^1\underline{\mathbf{v}}^1 \otimes \underline{\mathbf{v}}^1) + \frac{1}{2\mathrm{Fr}^2}\nabla_x\left(\overline{H}^1\right)^2 - \varepsilon\frac{\cos\theta}{2\mathrm{Ro}^*}\nabla_x\left(u^0\left(H^0\right)^2\right) \\
&= -\nabla_x b\left(\frac{1}{\mathrm{Fr}^2}\overline{H}^1 - \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*}u^0H^0\right) - \frac{\sin\theta}{\mathrm{Ro}^*}\overline{H}^1\underline{\mathbf{v}}^{1\perp} \\
&\quad + \varepsilon\frac{\cos\theta}{2\mathrm{Ro}^*}\left(H^0\right)^2\begin{pmatrix}1\\0\end{pmatrix}\mathrm{div}_x\underline{\mathbf{v}}^0 - \varepsilon\frac{\cos\theta}{\mathrm{Ro}^*}H^0\begin{pmatrix}1\\0\end{pmatrix}\nabla_x b^0\cdot\underline{\mathbf{v}}^0 + O(\varepsilon^2)\,.
\end{aligned} \tag{2.14}$$

Equations (2.13)–(2.14) form the SW system at the second order in nondimensional variables, with new cosine terms. Finally, let us go back to the dimensional form to get the SW system at the second order:

$$\partial_t H + \mathrm{div}_x(H\underline{\mathbf{v}}) = 0\,, \tag{2.15a}$$

$$\begin{aligned}
&\partial_t(Hu) + \mathrm{div}_x(H\underline{\mathbf{v}}\otimes\underline{\mathbf{v}}) + \frac{g}{2}\nabla_x H^2 \\
&\quad = -gH\,\nabla_x b - 2\Omega\,\sin\theta H\underline{\mathbf{v}}^\perp \\
&\qquad + \Omega\,\cos\theta\,\nabla_x\left(uH^2\right) + 2\Omega\,\cos\theta\,u\,H\,\nabla_x b \\
&\qquad + \Omega\,\cos\theta\,H^2\begin{pmatrix}1\\0\end{pmatrix}\mathrm{div}_x\underline{\mathbf{v}} - 2\Omega\,\cos\theta\,H\begin{pmatrix}1\\0\end{pmatrix}\nabla_x b\cdot\underline{\mathbf{v}}\,.
\end{aligned} \tag{2.15b}$$

System (2.15) is the second order Shallow Water system. Conversely to the first order approximation (2.12), the four cosine terms depending on the Coriolis force should be taken into account and cannot be neglected *a priori*. Note that a nontraditional multilayer Shallow Water model can also be derived, see [19] for example.

Remark 2.2. — If the latitude is not constant, the only difference with the previous development is that the term that reads $\Omega\,\cos\theta\,\nabla_x\left(uH^2\right)$ in the constant case must be replaced by $\Omega\,\nabla_x\left(\cos\theta uH^2\right)$ (with no additional difficulty), the other ones remaining unchanged.

Remark 2.3 (Quasi-Geostrophic model). — From the previous Shallow Water system (equations (2.12) at the first order or (2.15) at the second order), one can derive the Quasi-Geostrophic model, used for the abstract modeling of the ocean at midlatitudes (see [1]). It is obtained assuming that the Rossby and Froude numbers are very small (of the same order of magnitude), and performing an asymptotic development in powers of the Rossby number. It leads to a system that is usually written thanks to the stream function ψ, defined by $\underline{\mathbf{v}} = \nabla_x^\perp\psi$. From the SW equations at the second order (2.15), we obtain:

$$\frac{D}{Dt}\left(\left(\partial_{x_1}^2+\left(1+\delta^2\right)\partial_{x_2}^2\right)\psi-\frac{(2\Omega\sin\theta_0)^2}{gH_{char}}\psi\right.$$
$$\left.+\left(1-\frac{H_{char}}{2\tan\theta_0}\partial_{x_2}\right)\frac{2\Omega\sin\theta_0}{H_{char}}b+\beta x_2\right)=0, \quad (2.16)$$

where $\delta=\Omega\sqrt{H_{char}/g}\,\cos\theta_0$, and θ_0,β come from the β-plane approximation for the latitude: $2\Omega\sin\theta$ and $2\Omega\cos\theta$ are replaced by $2\Omega\sin\theta_0+\beta x_2$ and $2\Omega\cos\theta_0-\beta\tan\theta_0 x_2$ respectively, with θ_0 and β two constants.

The right hand side of (2.16) is usually composed of friction, viscosity and external forcing terms we did not take into account here.

We can notice that the cosine term of the Coriolis force has two different contributions. First, the Laplacian is modified in the second direction by the small coefficient δ. The other change is on the topography coefficient: we see the derivative of the topography in the second horizontal variable.

3. Primitive Equations

As for the Shallow Water Equations in Section 2.2, let us now consider the primitive equations with and without the full Coriolis terms. Let us go back to the original full system of equations (1.5) written as:

$$\frac{\partial u}{\partial t}+(\underline{\mathbf{u}}\cdot\nabla)u-fv+f^*w+\frac{\partial\phi}{\partial x}-\mu_{\underline{\mathbf{v}}}\Delta_h u-\nu_{\underline{\mathbf{v}}}\frac{\partial^2 u}{\partial z^2}=0, \quad (3.1a)$$

$$\frac{\partial v}{\partial t}+(\underline{\mathbf{u}}\cdot\nabla)v+fu+\frac{\partial\phi}{\partial y}-\mu_{\underline{\mathbf{v}}}\Delta_h v-\nu_{\underline{\mathbf{v}}}\frac{\partial^2 v}{\partial z^2}=0, \quad (3.1b)$$

$$\frac{\partial w}{\partial t}+(\underline{\mathbf{u}}\cdot\nabla)w-f^*u+\frac{\partial\phi}{\partial z}-\mu_{\underline{\mathbf{v}}}\Delta_h w-\nu_{\underline{\mathbf{v}}}\partial_z^2 w=-\frac{\rho}{\rho_0}\,g, \quad (3.1c)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad (3.1d)$$

$$\frac{\partial T}{\partial t}+(\underline{\mathbf{u}}\cdot\nabla)T-\mu_T\Delta_h T-\nu_T\frac{\partial^2 T}{\partial z^2}=F_T, \quad (3.1e)$$

supplemented with the state law for the density:

$$\rho=\rho_0\,\left(1-\alpha\left(T-T_0\right)\right). \quad (3.2)$$

3.1. Orders of magnitude

In order to perform a scale analysis and discriminate between the terms that can be neglected and those that should be retained in equations (3.1), we list in Table 3.1 the values that we consider for the physical parameters (length and height of the domain, Earth's rotation angular velocity, vertical and horizontal velocities, etc.). These orders of magnitude typically correspond to a planetary-scale motion ($WL/UH \approx 1$, see [3]), for example to the realistic configuration of the Northern Atlantic Ocean.

Height (H)	1000 m
Length (L)	1000 km
Horizontal Velocity (U)	1.0 m.s^{-1}
Vertical Velocity (W)	10^{-3} m.s^{-1}
Time ($T = L/U$)	10^6 s
Earth rot. velocity (Ω)	7.10^{-5} rad.s^{-1}

Table 3.1. Typical orders of magnitude for the Northern Atlantic Ocean.

Given those values, the aspect ratio $\varepsilon = H/L\ (= W/U) = 10^{-3}$ denotes the strong shallowness of the considered domain.

Let us compare the material derivative $Dw/Dt = \partial w/\partial t + (\underline{\mathbf{u}}.\nabla)w$ to the Coriolis term f^*u in the vertical momentum equation (3.1c). Thanks to Table 3.1, the ratio between these two terms ranges like

$$\frac{W/T}{2\Omega\cos\theta\, U} = \frac{\varepsilon}{T 2\Omega\cos\theta} < \varepsilon\,. \tag{3.3}$$

It is hence justified to neglect the vertical acceleration in (3.1c).

3.2. Traditional Hydrostatic Approximation

A simple comparison between $f^*\,w$ and fv in the zonal equation (3.1a) rapidly leads to the traditional approximation and to the hydrostatic primitive equations: indeed, since W scales as $\varepsilon\, U$, the $\cos\theta$ Coriolis term is neglected in the zonal equation, leading to the withdrawal of the term $f^*\,u$ in the vertical equation for conservation purposes. We finally come to the following traditional primitive equations, where the boxed equation (3.4c) is the so-called hydrostatic approximation:

$$\frac{\partial u}{\partial t} + (\underline{\mathbf{u}} \cdot \nabla) u - f v + \frac{\partial \phi}{\partial x} - \mu_{\underline{\mathbf{v}}} \Delta_h u - \nu_{\underline{\mathbf{v}}} \frac{\partial^2 u}{\partial z^2} = 0, \tag{3.4a}$$

$$\frac{\partial v}{\partial t} + (\underline{\mathbf{u}} \cdot \nabla) v + f u + \frac{\partial \phi}{\partial y} - \mu_{\underline{\mathbf{v}}} \Delta_h v - \nu_{\underline{\mathbf{v}}} \frac{\partial^2 v}{\partial z^2} = 0, \tag{3.4b}$$

$$\boxed{\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0} g,} \tag{3.4c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.4d}$$

$$\frac{\partial T}{\partial t} + (\underline{\mathbf{u}} \cdot \nabla) T - \mu_T \Delta_h T - \nu_T \frac{\partial^2 T}{\partial z^2} = 0. \tag{3.4e}$$

Over the last decades, this model has been widely used by oceanographers for operational computations and studied by applied mathematicians.

In the inviscid case, it is known (see [12]) that the primitive equations cannot be well-posed with any set of local boundary conditions. Temam, Tribbia and Rousseau have proposed alternative models and/or boundary conditions (of nonlocal type) to overcome this difficulty: see [20]; [16] and the review paper [17]. But thanks to the dissipative terms in equations (3.4), it is possible to state a nice well-posedness theorem with local (Dirichlet) boundary conditions on the cylindrical domain $\mathcal{M}$ (see Figure 3.1), even for the nonlinear equations, as it was done in [4] (see also [13]):

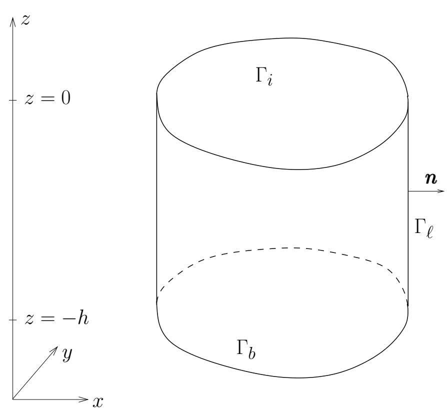


Figure 3.1. Cylindrical domain $\mathcal{M}$.

THEOREM 3.1 (Existence of strong solutions, see [4]). — *With appropriate initial conditions for* $\underline{\mathbf{u}}$ *and* T*, with Dirichlet boundary conditions on* *the side of the cylindrical domain* $\mathcal{M}$ *(see Figure 3.1) and a given* $t_{max} > 0$*, there exists a unique strong solution* $(\underline{\mathbf{u}}\,, T)$ *of the system* (3.4) *on the interval* $[0, t_{max}]$ *which depends continuously on the initial data.*

3.3. Beyond the traditional approximation

We want to show in the sequel that the $\cos\theta$ Coriolis terms are by far the largest of the omitted terms. The following alternate scale analysis, together with the theoretical clues for the well-posedness of the corresponding mathematical problem, makes us believe that the quasi-hydrostatic primitive equations realize the best compromize between physical representativeness and mathematical solvability.

Let us proceed to an alternate scale analysis of the zonal equation, and consider the term $f^* w$ in relation to the material derivative Du/Dt in the zonal equation (3.1a). The ratio scales like

$$\frac{2\Omega\cos\theta\, W}{U/T} = \frac{2\varepsilon\Omega}{T}\cos\theta = 14\%\cos\theta\,, \tag{3.5}$$

and thus the retention of the term $f^* w$ seems desirable. For conservation purposes[(3)], it is also desirable to retain the $2\Omega\cos\theta\, u$ term in the vertical momentum equation (3.1c), so that the Coriolis force remains orthogonal to the fluid velocity. We finally end up with the following quasi-hydrostatic primitive equations

$$\frac{\partial u}{\partial t} + (\underline{\mathbf{u}}\cdot\nabla)u - fv + f^*w + \frac{\partial\phi}{\partial x} - \mu_{\underline{\mathbf{v}}}\Delta_h u - \nu_{\underline{\mathbf{v}}}\frac{\partial^2 u}{\partial z^2} = 0\,, \tag{3.6a}$$

$$\frac{\partial v}{\partial t} + (\underline{\mathbf{u}}\cdot\nabla)v + fu + \frac{\partial\phi}{\partial y} - \mu_{\underline{\mathbf{v}}}\Delta_h v - \nu_{\underline{\mathbf{v}}}\frac{\partial^2 v}{\partial z^2} = 0\,, \tag{3.6b}$$

$$-f^*u + \frac{\partial\phi}{\partial z} = -\frac{\rho}{\rho_0}\,g\,, \tag{3.6c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \tag{3.6d}$$

$$\frac{\partial T}{\partial t} + (\underline{\mathbf{u}}\cdot\nabla)T - \mu_T\Delta_h T - \nu_T\frac{\partial^2 T}{\partial z^2} = 0\,. \tag{3.6e}$$

The work that has been done in [10] generalizes the result established by Cao and Titi [4] for the traditional PEs. Thanks to some additional *a priori* estimates on the velocity, we obtain the following theorem for the nontraditional quasi-hydrostatic primitive equations:

(3) It can also be shown (see [23]) that the $2\Omega\cos\theta\, u$ term may be retained in Equation (3.1c), regardless of conservation arguments.

Theorem 3.2 (See [10]). — *With appropriate initial conditions for* $\underline{\mathbf{u}}$ *and* T*, with Dirichlet boundary conditions on the side of the cylindrical domain* $\mathcal{M}$ *(see Figure 3.1) and a given* $t_{max} > 0$*, there exists a unique strong solution* $(\underline{\mathbf{u}}, T)$ *of the system* (3.6) *on the interval* $[0, t_{max}]$ *which depends continuously on the initial data.*

4. Quasi-geostrophic models

In this section we present the derivation of the quasi-hydrostatic quasi-geostrophic (QHQG) equations. The derivation follows classical principles (as in [1]): scaling, asymptotic expansion with respect to a small parameter, equations satisfied at order zero and one. Here, the small parameter (denoted Ro in the sequel) is the Rossby number, so that we underline the effect of rotating terms (see also [8]). In order to account for the complete Coriolis force (see e.g., [10] and references therein), we retain all the rotating terms in the original equations, including the terms that are usually neglected in the traditional approximation.

4.1. Scaling Parameters and Scaled Equations

We consider a three-dimensional domain with periodic boundary conditions in the horizontal directions, rigid lid and flat bottom in the vertical. We rewrite the incompressible Boussinesq equations including the complete Coriolis force (1.5), as:

$$\frac{Du}{Dt} - fv + f^* w = -\frac{\partial \varphi}{\partial x}, \tag{4.1a}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \varphi}{\partial y}, \tag{4.1b}$$

$$\frac{Dw}{Dt} - f^* u + \frac{g\,\varrho}{\varrho_0} = -\frac{\partial \varphi}{\partial z}, \tag{4.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.1d}$$

$$\frac{D\varrho}{Dt} = 0. \tag{4.1e}$$

Here φ is the renormalized pressure, $\varphi = p/\varrho_0$. The density and the pressure may be classically decomposed as

$$\varrho(x,y,z,t) = \overline{\rho}(z) + \rho(x,y,z,t) \text{ and } \varphi(x,y,z,t) = \overline{\varphi}(z) + \phi(x,y,z,t), \tag{4.2}$$

where $\overline{\rho}$ and $\overline{\varphi}$ are the (known) background density and potential, depending only on the vertical variable. We also denote by $N^2(z) = -\overline{\rho}'(z)$ the buoyancy frequency, assuming that $\overline{\rho}'(z)$ is bounded away from zero.

Before going further in the derivation of the corresponding QG model, let us insist on the fact that we keep in equations (4.1a) and (4.1c) the Coriolis terms f^*w and f^*u. With all these terms (on which we want to focus, as in [6]), we will finally lead to a slightly modified QG model (see (4.19)). We think that it is a relevant modification, since the QG approximation aims at underlying the Earth's rotation effects: one should thus include every rotation term in the primitive equations prior to an asymptotic expansion with respect to the Rossby number.

In the context of the β-plane approximation, we have[(4)], with θ_0 the average latitude:

$$f = f_0(1 + \mathrm{Ro}\,\beta_0 y) = 2\Omega \sin\theta_0(1 + \mathrm{Ro}\,\beta_0 y)\,, \qquad f^* = f_0^* = 2\Omega\cos\theta_0\,.$$

We now introduce the following dimensionless variables, as it is classically done in QG modeling (see [1]):

$$\begin{array}{lll} (x,y) = L\,(x',y')\,, & z = H\,z', & t = \dfrac{L}{U}t', \\ u = U\,u', & v = U\,v', & w = \dfrac{UH}{L}w', \\ \overline{\rho} = P\overline{\rho}', & \rho = \dfrac{\varrho_0 f_0 UL}{gH}\rho', & \phi = f_0 UL\phi'. \end{array}$$

The Rossby number $\mathrm{Ro} = U/f_0L$ is the fundamental ordering parameter in the following asymptotic expansion. A secondary ordering parameter is the scale ratio of ρ to ϱ: this ratio is assumed to be Ro, that is, we assume:

$$\frac{\varrho_0 f_0 UL}{gH} = P\mathrm{Ro}\,.$$

Finally, the density may be expressed in terms of nondimensionnal quantities:

$$\varrho = P\Big(\overline{\rho}'(z) + \mathrm{Ro}\,\rho'\Big).$$

Another usual nondimentional number is the aspect ratio $\varepsilon = H/L$; ε also appears in the ratio between the two Coriolis terms in the zonal momentum equation (4.1a), which scales as

$$\lambda = \varepsilon\cot(\theta_0)\,. \tag{4.3}$$

(4) For simplicity we do not consider any Taylor expansion of f^* with respect to θ. In fact we only keep the 0^{th} order term, as suggested in [7].

When we considered the scaling numbers introduced above, we have implicitly assumed that the leading term at the left-hand-side of Equation (4.1a) was fv, which means that $\varepsilon\cot(\theta_0)$ should not be too large:

$$\lambda \lesssim 1. \tag{4.4}$$

Fortunately, because the aspect ratio ε is rather small in large ocean models, the condition (4.4) is easily satisfied. However, the objective of the present work is to draw the reader's attention on the fact that λ is not necessarily small, and that it may have some physical repercussions.

We end this section with the non-hydrostatic scaled equations (we naturally drop the primes):

$$\mathrm{Ro}\,\frac{Du}{Dt} - (1+\mathrm{Ro}\,\beta_0 y)v + \lambda w = -\frac{\partial\phi}{\partial x}, \tag{4.5a}$$

$$\mathrm{Ro}\,\frac{Dv}{Dt} + (1+\mathrm{Ro}\,\beta_0 y)u = -\frac{\partial\phi}{\partial y}, \tag{4.5b}$$

$$\mathrm{Ro}\,\varepsilon^2\frac{Dw}{Dt} - \lambda u + \rho = -\frac{\partial\phi}{\partial z}, \tag{4.5c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.5d}$$

$$\mathrm{Ro}\,\frac{D\varrho}{Dt} + w\overline{\rho}_z = 0, \tag{4.5e}$$

where we recall that Ro is the Rossby number (meant to go to zero), $\varepsilon = H/L$ is the domain aspect ratio, and $\lambda = \varepsilon\cot(\theta_0)$.

Remark 4.1. — Actually, the term $\mathrm{Ro}\,\varepsilon^2 Dw/Dt$ could be set to zero in (4.5c) above with no modification in the sequel: indeed the reader will see below that $w^{(0)} = 0$, hence $\mathrm{Ro}\,\varepsilon^2 Dw/Dt = O(\mathrm{Ro}^2)$ can be neglected prior to the QG approximation.The new model is thus called QHQG since the differences between the new QG model and the traditional one rely only on the terms related to the Coriolis force.

4.2. Geostrophic Balance

We now consider an asymptotic expansion of all variables with respect to the Rossby number: for every unknown function ξ, we write the formal asymptotic expansion

$$\xi = \xi^{(0)} + \mathrm{Ro}\,\xi^{(1)} + \mathrm{Ro}^2\xi^{(2)} + \dots$$

where $(\xi^{(j)})_{j\geqslant 0}$ behave as $O(1)$ as Ro goes to zero. Equations (4.5a)-(4.5c) give, keeping only the order zero terms in Ro:

$$-v^{(0)} + \lambda w^{(0)} = -\phi_x^{(0)}, \tag{4.6}$$

$$u^{(0)} = -\phi_y^{(0)}, \tag{4.7}$$

$$-\lambda u^{(0)} + \rho^{(0)} = -\phi_z^{(0)}. \tag{4.8}$$

The incompressibility condition reads $w_z^{(0)} = -u_x^{(0)} - v_y^{(0)}$ and this *traditionally* leads to $w^{(0)} = 0$, thanks to equations (4.6), (4.7) and boundary conditions on w (see [1]). Here, the incompressibility condition does not provide $w_z^{(0)} = 0$, but we have, denoting $\partial_Z = \partial_z + \lambda\partial_y$:

$$\begin{aligned} w_Z^{(0)} &= w_z^{(0)} + \lambda w_y^{(0)} \\ &= -u_x^{(0)} - v_y^{(0)} + \lambda w_y^{(0)} \\ &= \mathrm{curl}(\phi_y, \phi_x) \\ w_Z^{(0)} &= 0\,. \end{aligned} \tag{4.9}$$

Thanks to (4.9) and to homogeneous boundary conditions on $w^{(0)}$, we finally obtain[5] that $w^{(0)} = 0$.

The geostrophic equations read:

$$-v^{(0)} = -\phi_x^{(0)}, \tag{4.10a}$$

$$u^{(0)} = -\phi_y^{(0)}, \tag{4.10b}$$

$$\rho^{(0)} = -\phi_z^{(0)} - \lambda\phi_y^{(0)} = -\phi_Z^0\,, \tag{4.10c}$$

$$w^{(0)} = 0\,. \tag{4.10d}$$

4.3. Quasi-Geostrophic Equations

Now we need the first order equations in order to determine the evolution of $\phi^{(0)}$. We denote by d_g the zero-order material derivative:

$$d_g = \partial_t + u^{(0)}\partial_x + v^{(0)}\partial_y\,.$$

[5] Alternatively, we have equation (4.5e) which (written to the order zero and since $\overline{\rho}(z)$ never vanishes) leads to $w^{(0)} = 0$.

The first order equations are:

$$d_g u^{(0)} - \beta_0 y v^{(0)} - v^{(1)} + \lambda w^{(1)} = -\phi_x^{(1)}, \tag{4.11a}$$
$$d_g v^{(0)} + \beta_0 y u^{(0)} + u^{(1)} = -\phi_y^{(1)}, \tag{4.11b}$$
$$-\lambda u^{(1)} + \rho^{(1)} = -\phi_z^{(1)}, \tag{4.11c}$$
$$u_x^{(1)} + v_y^{(1)} + w_z^{(1)} = 0, \tag{4.11d}$$
$$d_g \rho^{(0)} + w^{(1)} \overline{\rho}_z = 0. \tag{4.11e}$$

We now take the curl of equations (4.11a)–(4.11b) to obtain, thanks to Equation (4.11d)

$$d_g(v_x^{(0)} - u_y^{(0)}) - w_z^{(1)} - \lambda w_y^{(1)} + \beta_0 v^{(0)} = 0. \tag{4.12}$$

We notice, as for the traditional QG equations, that $\beta_0 v^{(0)} = d_g(\beta_0 y)$. We thus try to express $-w_z^{(1)} - \lambda w_y^{(1)} = w_Z^1$ as $d_g(\Gamma)$ where Γ is a function to be defined. To this aim, we will extensively make use of Equation (4.11e) that we reformulate:

$$w^{(1)} = N^{-2} d_g \rho^{(0)} = d_g(N^{-2}\rho^{(0)}). \tag{4.13}$$

Given (4.13), we may compute the required quantity

$$w_z^{(1)} + \lambda w_y^{(1)} = \Big(d_g(N^{-2}\rho^{(0)}\Big)_z + \lambda\Big(d_g(N^{-2}2\rho^{(0)})\Big)_y. \tag{4.14}$$

We remark that for any function ξ and any variable $*$ we have the identity

$$\Big(d_g(\xi)\Big)_* = d_g(\xi_*) + u_*^{(0)}\partial_x \xi + v_*^{(0)}\partial_y \xi\,,$$

so that we can write

$$w_z^{(1)} + \lambda w_y^{(1)} = d_g\Big((N^{-2}\rho^{(0)})_z\Big) + \lambda d_g\Big((N^{-2}\rho^{(0)})_y\Big) + R, \tag{4.15}$$

where the remainder R, according to the remark above, is

$$\begin{aligned} R &= u_z^{(0)}(N^{-2}\rho^{(0)})_x + v_z^{(0)}(N^{-2}\rho^{(0)})_y + \lambda u_y^{(0)}(N^{-2}\rho^{(0)})_x + \lambda v_y^{(0)}(N^{-2}\rho^{(0)})_y \\ &= (N^{-2}\rho^{(0)})_x\,(u_z^{(0)} + \lambda u_y^{(0)}) + (N^{-2}\rho^{(0)})_y\,(v_z^{(0)} + \lambda v_y^{(0)}). \end{aligned} \tag{4.16}$$

Using (4.10a)-(4.10c) again, we have

$$R = N^{-2}\rho_x^{(0)}\rho_y^{(0)} - N^{-2}\rho_y^{(0)}\rho_x^{(0)} = 0,$$

which simplifies Equation (4.15) as follows:

$$w_Z^{(1)} = w_z^{(1)} + \lambda w_y^{(1)} = d_g\Big((N^{-2}\rho^{(0)})_z + \lambda(N^{-2}\rho^{(0)})_y\Big). \tag{4.17}$$

Back to Equation (4.12), we obtain the quasi-hydrostatic quasi-geostrophic equation:

$$d_g\Big(v_x^{(0)} - u_y^{(0)} - (N^{-2}\rho^{(0)})_z - \lambda(N^{-2}\rho^{(0)})_y + \beta_0 y\Big) = 0\,. \tag{4.18}$$

The potential vorticity

$$v_x^{(0)} - u_y^{(0)} - (N^{-2}\rho^{(0)})_z - \lambda(N^{-2}\rho^{(0)})_y + \beta_0 y$$

is thus conserved along material paths.

Let us now rewrite Equation (4.18), expressing everything in terms of $\phi^{(0)}$. We have

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big) \times \Big(\Delta\phi^{(0)} + N^{-2}(\partial_z + \lambda\partial_y)^2\phi^{(0)} + N_z^{-2}(\partial_z + \lambda\partial_y)\phi^{(0)} + \beta_0 y\Big) = 0\,, \tag{4.19}$$

where Δ is horizontal Laplacian operator. We could write this more compactly with $\partial_Z = \partial_z + \lambda\partial_y$,

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\Big(\Delta\phi^{(0)} + (N^{-2}\phi_Z^{(0)})_Z + \beta_0 y\Big) = 0\,. \tag{4.20}$$

One can thus easily recognize the traditional QG equation (see [1, (2.23)]), except that the differential operator ∂_z is replaced by $\partial_Z = \partial_z + \lambda\partial_y$. We recall here that $\lambda = \varepsilon\cot\theta_0$ is proportional to the domain aspect ratio. In particular, we recover the traditional QG equation when setting $\varepsilon = 0$ in equations (4.18), (4.19) or (4.20). A detailed proof of the convergence as well as a physical analysis of the QHQG equations can be found in [9].

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