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On the first order asymptotics of partial Bergman kernels [(*)]

Dan Coman [(1)] and George Marinescu [(2)]

Abstract. — We show that under very general assumptions the partial Bergman kernel function of sections vanishing along an analytic hypersurface has exponential decay in a neighborhood of the vanishing locus. Considering an ample line bundle, we obtain a uniform estimate of the Bergman kernel function associated to a singular metric along the hypersurface. Finally, we study the asymptotics of the partial Bergman kernel function on a given compact set and near the vanishing locus.

Résumé. — Nous montrons, sous des hypothèses très générales, que le noyau de Bergman partiel des sections s'annulant sur une hypersurfaces analytique décroît exponentiellement dans un voisinage du lieu d'annulation. Pour un fibré ample, nous montrons une estimée uniforme du noyau de Bergman associé à une métrique singulière le long d'une hypersurface. Finalement nous étudions les asymptotiques du noyau de Bergman sur un compact près du lieu d'annulation.

1. Introduction

Partial Bergman kernels were recently studied in different contexts, especially Kähler geometry [11, 12, 13] or random polynomials [2, 15].

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Let us consider the following general setting.

(A) (X, ω) is a compact Hermitian manifold of dimension n, Σ is a smooth analytic hypersurface of X, and $t > 0$ is a fixed real number.
(B) (L, h) is a singular Hermitian holomorphic line bundle on X with singular metric h which has locally bounded weights.

We define the space

$$H^0_0(X, L^p) := H^0\big(X, L^p \otimes \mathcal{O}\big(- \lfloor tp \rfloor \Sigma\big)\big) \tag{1.1}$$

of holomorphic sections of the p-th tensor power L^p vanishing to order at least $\lfloor tp \rfloor$ along Σ, where $\lfloor x \rfloor$ denotes the integral part of $x \in \mathbb{R}$. Set $d_p = \dim H^0(X, L^p)$ and $d_{0,p} = \dim H^0_0(X, L^p)$. We introduce on $H^0(X, L^p)$ the L^2 inner product $(\,\cdot\,,\cdot\,)_p$ induced by the metric $h_p = h^{\otimes p}$ and the volume form $\omega^n/n!$, see (2.1). This inner product is inherited by $H^0_0(X, L^p)$. The (full) Bergman kernel function is defined by taking an orthonormal basis $\{S^p_j : 1 \leqslant j \leqslant d_p\}$ of $(H^0(X, L^p), (\,\cdot\,,\cdot\,)_p)$ and setting

$$P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2_{h_p}\,, \quad |S^p_j(x)|^2_{h_p} := \langle S^p_j(x), S^p_j(x)\rangle_{h_p}, \ \ x \in X. \tag{1.2}$$

By considering an orthonormal basis $\{S^p_j : 1 \leqslant j \leqslant d_{0,p}\}$ of $(H^0_0(X, L^p), (\cdot,\cdot)_p)$, we define the *partial Bergman kernel function* $P_{0,p}$ by

$$P_{0,p}(x) = \sum_{j=1}^{d_{0,p}} |S^p_j(x)|^2_{h_p}\,, \ x \in X. \tag{1.3}$$

Note that this definition is independent of the choice of basis, cf. (2.2).

The asymptotics of the Bergman kernel function for a positive line bundle (L, h) [4, 16], see also [10] for a comprehensive study, is very important in understanding the Yau–Tian–Donaldson conjecture. On the other hand, partial Bergman kernels are useful in connection to the slope semi-stability with respect to a submanifold [14]. On a toric variety X (and for a toric Σ) this study was carried out in [11]. In this context it is shown that the partial Bergman kernel has an asymptotic expansion, having rapid decay of order $p^{-\infty}$ (see §2.1 for this notation) in a neighborhood $U(\Sigma)$ of Σ, and giving the full Bergman kernel function to order $p^{-\infty}$ outside the closure of $U(\Sigma)$. Moreover [11] gives a complete distributional asymptotic expansion on X, whose leading term has an additional Dirac delta measure plus a dipole measure over $\partial U(\Sigma)$. These results were generalized in [13] and [17] to the case when the data in question are invariant under an S^1-action.

In general, if no symmetry is assumed, it was shown in [2, Theorem 4.3] that if the bundle $L \otimes \mathscr{O}(-\Sigma)$ is ample, there exists a neighborhood $U(\Sigma)$ of Σ, such that $P_{0,p}(x)$ has exponential decay on $U(\Sigma)$ and $p^{-n}P_{0,p}(x)$ converges to $c_1(L,h)^n/\omega^n$ in L^1 outside the closure of $U(\Sigma)$.

Our first result is that under the very general hypotheses (A) and (B) above (in particular, without any positivity condition), the partial Bergman kernel function decays exponentially in a neighborhood of the divisor Σ.

Theorem 1.1. — *Assume that conditions* (A)–(B) *are fulfilled. Then there exist a neighborhood U_t of Σ and a constant $a \in (0,1)$ such that $P_{0,p} \leqslant a^p$ on U_t for $p > 2t^{-1}$. In particular $P_{0,p} = O(p^{-\infty})$ as $p \to \infty$ on U_t.*

For more precise statements see Theorem 3.1 and Corollary 3.3. Theorem 1.1 can be formulated for non-compact manifolds X (see Theorem 3.4), in which case the exponential decay of the partial Bergman kernel holds in a neighborhood of the intersection of Σ with any given compact subset of X. This includes for instance the case of classical Bergman spaces of L^2-holomorphic functions on domains in $\mathbb{C}^n$.

An object which is closely related to the partial Bergman kernel is the Bergman kernel for a singular metric. The full asymptotic expansion on compact subsets of the regular part of the metric was established in [8, Theorem 1.8]. We are here concerned with asymptotics at arbitrary points with dependence on the distance to the singular set. More precisely, we will consider the following situation.

Let $S_\Sigma \in H^0(X, \mathcal{O}(\Sigma))$ be a canonical holomorphic section of the line bundle $\mathcal{O}(\Sigma)$, vanishing to first order on Σ. We fix a smooth Hermitian metric h_Σ on $\mathcal{O}(\Sigma)$ such that

$$\varrho := \log \left|S_\Sigma\right|_{h_\Sigma} < 0 \text{ on } X. \tag{1.4}$$

We consider a function $\xi : X \to \mathbb{R} \cup \{-\infty\}$, smooth on $X \setminus \Sigma$, such that $\xi = t\rho$ in a neighborhood U of Σ. Let $\operatorname{dist}(\,\cdot\,,\cdot\,)$ be the distance on X induced by ω. Our main result is the following:

Theorem 1.2. — *Let $(X,\omega), (L,h), \Sigma$ be as in* (A)–(B), *and assume ω is Kähler and h is smooth. Consider the singular Hermitian metric $\widetilde{h} = he^{-2\xi}$ on L and assume that $c_1(L,\widetilde{h}) \geqslant \varepsilon\omega$ for some constant $\varepsilon > 0$. Let $\widetilde{P}_p$ be the Bergman kernel function of $H^0_{(2)}(X, L^p, \widetilde{h}_p, \omega^n/n!)$, where $\widetilde{h}_p := \widetilde{h}^{\otimes p}$. Then there exists a constant $C > 1$ such that for every $x \in X \setminus \Sigma$ and every $p \in \mathbb{N}$ with $p \operatorname{dist}(x,\Sigma)^{8/3} > C$ we have*

$$\left| \frac{\widetilde{P}_p(x)}{p^n} \, \frac{\omega_x^n}{c_1(L,\widetilde{h})_x^n} - 1 \right| \leqslant C p^{-1/8}\,. \tag{1.5}$$

Theorem 1.2 can be interpreted in two ways. First, if x runs in a compact set $K \subset X \setminus \Sigma$, we have a concrete bound $p_0 = C \operatorname{dist}(K,\Sigma)^{-8/3}$ such that for $p > p_0$ the estimate (1.5) holds. By [8, Theorem 1.8] we have $\widetilde{P}_p(x) = \sum_{r=0}^{\infty} \mathbf{b}_r(x)p^{n-r} + O(p^{-\infty})$ as $p \to \infty$ locally uniformly on $X \setminus \Sigma$. Hence, there exists $p_0(K) \in \mathbb{N}$ and C_K such that for $p > p_0(K)$ we have

$$\left| \frac{\widetilde{P}_p(x)}{p^n} \, \frac{\omega_x^n}{c_1(L,\widetilde{h})_x^n} - 1 \right| \leqslant C_K p^{-1} \text{ on } K.$$

However, $p_0(K)$ is not easy to determine.

We can also recast Theorem 1.2 as a uniform estimate in p for the singular Bergman kernel on compact sets of $X \setminus \Sigma$ whose distance to Σ decreases as $p^{-3/8}$. Indeed, set $K_p = \{x \in X : \operatorname{dist}(x,\Sigma) \geqslant (C/p)^{3/8}\}$. Then (1.5) holds on K_p for every p.

We consider now the global behavior of the partial Bergman kernel. Given a compact set $K \subset X \setminus \Sigma$ we set

$$\begin{aligned} t_0(K) := \sup \Big\{ t > 0 : \exists\, \eta \in \mathscr{C}^\infty(X,[0,1]),\ \operatorname{supp}\eta \subset X \setminus K,\ \eta = 1 \text{ near } \Sigma, \\ \text{and } c_1(L,h) + t\, dd^c(\eta\varrho) \text{ is a Kähler current on } X \Big\}. \end{aligned} \tag{1.6}$$

A consequence of Theorems 1.1 and 1.2 is the following result about the asymptotics of the partial Bergman kernel:

Theorem 1.3. — *Let $(X,\omega),(L,h),\Sigma$ be as in* (A)–(B), *and assume ω is Kähler, h is smooth, and $c_1(L,h) \geqslant \varepsilon\omega$ for some constant $\varepsilon > 0$. Let $K \subset X \setminus \Sigma$ be a compact set and let $t \in (0, t_0(K))$. Then there exist constants $C > 1$, $M > 1$ and a neighborhood U_t of Σ, all depending on t, such that for $x \in U_t$ we have*

$$Me^{t\varrho(x)} < 1 \ \textit{ and } \ P_{0,p}(x) \leqslant (Me^{t\varrho(x)})^p \ \textit{ for } p > 2/t, \tag{1.7}$$

$$P_{0,p}(x) \geqslant \frac{p^n}{C} \exp(2tp\varrho(x)) \quad \textit{for } p \operatorname{dist}(x,\Sigma)^{8/3} > C, \tag{1.8}$$

where the function ϱ is defined in (1.4). *Moreover, we have uniformly on K,*

$$P_{0,p}(x) = P_p(x) + O(p^{-\infty}),\ \ p \to \infty, \tag{1.9}$$

and in particular,

$$P_{0,p}(x) = \mathbf{b}_0(x)p^n + \mathbf{b}_1(x)p^{n-1} + O(p^{n-2}),\ \ p \to \infty, \tag{1.10}$$

where

$$\mathbf{b}_0 = \frac{c_1(L,h)^n}{\omega^n},\ \ \mathbf{b}_1 = \frac{\mathbf{b}_0}{8\pi}\,(r^X - 2\Delta \log \mathbf{b}_0), \tag{1.11}$$

and r^X, Δ, are the scalar curvature, respectively the Laplacian, of the Riemannian metric associated to $c_1(L,h)$.

Hence, (1.7) and (1.8) show that on U_t the exponential decay estimate for the partial Bergman kernel function is sharp. Moreover, on K the partial Bergman kernel function has the same asymptotics as the full Bergman kernel function up to order $O(p^{-\infty})$. This was established in [13, Theorem 1.1] under the additional assumption that there is an S^1-action in a neighborhood of Σ. Our method is to estimate the partial Bergman kernel $P_{0,p}$ by above and below with the full Bergman kernel P_p and the singular Bergman kernel $\widetilde{P}_p$. On the set where the singular metric $\widetilde{h}$ equals h, the kernels $\widetilde{P}_p$ and P_p differ by $O(p^{-\infty})$. This is shown in Theorem 5.1, which gives a general localization result for singular Bergman kernels. Theorem 5.1 is a straightforward consequence of [8].

However, in Theorem 1.3 we do not necessarily obtain a *partition* of the manifold X in two sets, one with exponential decay (1.7) and one with "full asymptotics" (1.9), since in general $U_t \cup K \neq X$. In [2, 13, 11, 17] a partition with two different regimes was exhibited under further hypotheses. The approach introduced by Berman [2, Section 4.1] was to consider the equilibrium metric (or extremal envelope) h_t of h with poles along Σ (see also [13]). The metric h_t exists for t sufficiently small thanks to the positivity of (L, h). The local plurisubharmonic (psh) potentials of h_t have Lelong number t along Σ. It is shown in [13, Proposition 2.13] that the partial Bergman kernel function $P_{0,p}$ has exponential decay in the forbidden region $\{h_t > h\}$ which is a neighborhood of Σ. Moreover, under additional symmetry assumptions, it is shown in [13, Theorem 1.1] that $P_{0,p}$ is essentially equal to the full Bergman kernel outside the forbidden region.

In Theorem 1.1 we show that exponential decay holds near Σ with no assumption on the positivity of (L, h). Without positivity assumptions the equilibrium metric might not exist, but we give here a proof in the general case based only on the sub-average inequality for holomorphic functions.

Our approach in Theorem 1.3 differs from the envelope approach above in that we first fix a compact K disjoint from Σ and then construct an interval of small $t > 0$ for which the corresponding partial Bergman kernel $P_{0,p}$ decays exponentially near Σ and is essentially equal to the full Bergman kernel on K.

2. Preliminaries

2.1. Bergman kernel function

Let (L, h) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (X, ω). We denote by $H^0(X, L^p)$ the space of holomorphic sections of $L^p := L^{\otimes p}$.

Let $H^0_{(2)}(X, L^p) = H^0_{(2)}(X, L^p, h_p, \omega^n/n!)$ be the Bergman space of L^2-holomorphic sections of L^p relative to the metric $h_p := h^{\otimes p}$ induced by h and the volume form $\omega^n/n!$ on X, endowed with the inner product

$$(S, S')_p := \int_X \langle S, S' \rangle_{h_p} \frac{\omega^n}{n!}\,, \quad S, S' \in H^0_{(2)}(X, L^p)\,. \tag{2.1}$$

Set $\|S\|_p^2 = (S, S)_p$, $d_p = \dim H^0_{(2)}(X, L^p)$. If h has locally bounded weights (e.g. h is smooth) we have of course $H^0_{(2)}(X, L^p) = H^0(X, L^p)$. We have the following variational characterization of the partial Bergman kernel

$$P_{0,p}(x) = \max\Big\{ |S(x)|^2_{h_p} : S \in H^0_0(X, L^p),\ \|S\|_p = 1 \Big\}, \tag{2.2}$$

and similar characterizations hold for the full and singular Bergman kernel functions P_p and $\widetilde{P}_p$.

Throughout the paper we also use the following terminology. For a sequence of continuous functions f_p on a manifold M we write $f_p = O(p^{-\infty})$ if for every compact subset $K \subset M$ and any $\ell \in \mathbb{N}$ there exists $C_{K,\ell} > 0$ such that for all $p \in \mathbb{N}$ we have $\|f_p\|_K \leqslant C_{K,\ell}\, p^{-\ell}$.

2.2. Geometric set-up

We prepare here the geometric set-up needed for the proofs of our results, by constructing a special neighborhood W of Σ.

Let (X, ω) be a compact Hermitian manifold of dimension n. Let (U, z), $z = (z_1, \ldots, z_n)$, be local coordinates centered at a point $x \in X$. For $r > 0$ and $y \in U$ we denote by

$$\Delta^n(y, r) = \{z \in U : |z_j - y_j| \leqslant r,\ j = 1, \ldots, n\}$$

the (closed) polydisk of polyradius $(r, \ldots, r)$ centered at y. If ω is a Kähler form, the coordinates (U, z) are called Kähler at $y \in U$ if

$$\omega_z = \frac{i}{2} \sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j + O(|z - y|^2) \text{ on } U.$$

Since Σ is compact, we can find an open cover $\mathcal{W} = \{W_j\}_{1\leqslant j\leqslant N}$ of Σ, where W_j are Stein contractible coordinate neighborhoods centered at points $y_j \in \Sigma$, such that

$$\begin{aligned} &\Delta^n(y_j,2) \subset W_j\,,\ \ \Sigma \subset W := \bigcup_{j=1}^{N} \Delta^n(y_j,1)\,, \\ &\Sigma \cap W_j = \big\{z \in W_j : z_1 = 0\big\}\,, \quad \text{for } j = 1\ldots,N\,, \end{aligned} \tag{2.3}$$

where $z = (z_1,\ldots,z_n)$ are the coordinates on W_j. Moreover, if ω is a Kähler form, we may also ensure that

$$\begin{aligned} &\forall\ x \in \Delta^n(y_j,1),\ \exists\ z = z(x) \text{ coordinates on } \Delta^n(y_j,2) \\ &\qquad\qquad\qquad\qquad \text{centered at } x \text{ and Kähler at } x\,. \end{aligned} \tag{2.4}$$

As in [5, §2.5, Lemma 2.7] one can easily prove the following:

Lemma 2.1. — *Let $(X,\omega),(L,h),\Sigma,\widetilde{h}$ be as in Theorem 1.2, and let $\mathcal{W} = \{W_j\}_{1\leqslant j\leqslant N}$ be an open cover of Σ verifying* (2.3) *and* (2.4). *There exist constants $C_1 > 1$, $C_2 > 0$ and $r_1 > 0$ with the following property: if $j \in \{1,\ldots,N\}$, $x \in \Delta^n(y_j,1)$ and $z = z(x)$ are the coordinates on $\Delta^n(y_j,2)$ given by* (2.4), *then:*

(1) $\Delta^n_z(x,r_1) \Subset \Delta^n(y_j,2)$ *and for $r \leqslant r_1$ we have*

$$n!\,\mathrm{d}m \leqslant (1+C_1r^2)\omega^n\,,\ \ \omega^n \leqslant (1+C_1r^2)n!\,\mathrm{d}m \ \ \text{on } \Delta^n_z(x,r)\,, \tag{2.5}$$

where $\mathrm{d}m = \mathrm{d}m(z)$ is the Euclidean volume and $\Delta^n_z(x,\cdot\,)$ is the open polydisk relative to the coordinates z.

(2) $(L,\widetilde{h})$ *has a weight φ_x on W_j with*

$$\begin{aligned} \varphi_x &= t\log|f| + \psi_x\,, \quad \psi_x \in \mathscr{C}^\infty(W_j)\,, \\ \psi_x(z) &= \operatorname{Re} F_x(z) + \psi'_x(z) + \widetilde{\psi}_x(z) \ \ \text{on } \Delta^n(y_j,2)\,, \end{aligned} \tag{2.6}$$

where f is a defining function for $\Sigma \cap W_j$, $F_x(z)$ is a holomorphic polynomial of degree $\leqslant 2$ in z, $\psi'_x(z) = \sum_{\ell=1}^n \lambda_\ell |z_\ell|^2$, $\lambda_\ell = \lambda_\ell(x)$, and

$$|\widetilde{\psi}_x(z)| \leqslant C_2|z|^3\,,\ \ z \in \Delta^n_z(x,r_1)\,. \tag{2.7}$$

3. Exponential decay

We prove here Theorem 1.1. Let $\mathcal{W} = \{W_j\}_{1\leqslant j\leqslant N}$ be the cover of Σ and $W \supset \Sigma$ be the neighborhood of Σ constructed in Section 2.2 (see (2.3)). For a function $\varphi \in L^\infty_{loc}(W_j)$ set

$$\|\varphi\|_{\infty,W_j} = \sup\big\{|\varphi(w)| :\ w \in \Delta^n(y_j,2)\big\}.$$

Let (L,h) be a singular Hermitian holomorphic line bundle on X, where the metric h has locally bounded weights. Since $L|_{W_j}$ is trivial, we fix a holomorphic frame e_j of $L|_{W_j}$, and denote by φ_j the corresponding weight of h on W_j, i.e. $|e_j|_h = e^{-\varphi_j}$. Set

$$\|h\|_\infty = \|h\|_{\infty,\mathcal{W}} := \max\big\{1, \|\varphi_j\|_{\infty,W_j} : 1 \leqslant j \leqslant N\big\}, \tag{3.1}$$

and let ϱ be the function defined in (1.4).

Theorem 3.1. — *In the setting of Theorem 1.1, there exists a constant $A \geqslant 1$ depending only on ρ and $\mathcal{W}$ such that for any $S \in H^0_0(X, L^p)$, $x \in W$, and $p \geqslant 1$, we have*

$$|S(x)|^2_{h_p} \leqslant (Ae^{\rho(x)})^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty} \|S\|^2_p \,.$$

Therefore, for every $x \in W$ and $p \geqslant 1$,

$$P_{0,p}(x) \leqslant (Ae^{\rho(x)})^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty} \,.$$

For the proof we need the following elementary lemma.

Lemma 3.2. — *If $k \geqslant 0$ and $f \in \mathcal{O}(\Delta(0,2))$, where $\Delta(0,2) \subset \mathbb{C}$ is the closed disk centered at 0 and of radius 2, then*

$$\int_{\Delta(0,2)} |f(\zeta)|^2 \,\mathrm{d}m(\zeta) \leqslant \frac{k+1}{2^{2k}} \int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \,\mathrm{d}m(\zeta)\,.$$

Proof. — Consider the power expansion $f(\zeta) = \sum_{j=0}^\infty a_j \zeta^j$ of f in $\Delta(0,2)$. Integrating in polar coordinates we obtain

$$\int_{\Delta(0,2)} |f(\zeta)|^2 \,\mathrm{d}m(\zeta) = 2\pi \sum_{j=0}^\infty |a_j|^2 \int_0^2 r^{2j+1} \,\mathrm{d}r = 2\pi \sum_{j=0}^\infty \frac{2^{2j+2}}{2j+2} |a_j|^2 \,.$$

On the other hand, $\zeta^k f(\zeta) = \sum_{j=k}^\infty a_{j-k} \zeta^j$, so

$$\begin{aligned}
&\int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \,\mathrm{d}m(\zeta) \\
&\qquad = 2\pi \sum_{j=k}^\infty \frac{2^{2j+2}}{2j+2} |a_{j-k}|^2 = 2\pi \sum_{j=0}^\infty \frac{2^{2j+2+2k}}{2j+2+2k} |a_j|^2 \\
&\qquad \geqslant \frac{2^{2k}}{k+1} 2\pi \sum_{j=0}^\infty \frac{2^{2j+2}}{2j+2} |a_j|^2 = \frac{2^{2k}}{k+1} \int_{\Delta(0,2)} |f(\zeta)|^2 \,\mathrm{d}m(\zeta)\,. \qquad \square
\end{aligned}$$

Proof of Theorem 3.1. — Let $x \in W$. Fix $j \in \{1,\dots,N\}$ such that $x \in \Delta^n(y_j, 1)$ and let e_j be the local frame of $L|_{W_j}$ and φ_j be the corresponding weight of h as considered in (3.1). Let $S \in H^0_0(X, L^p)$. On W_j we write

$S = se_j^{\otimes p}$, with $s \in \mathcal{O}(W_j)$. Then we have $s(z) = z_1^{\lfloor tp \rfloor}\widetilde{s}(z)$, with $\widetilde{s} \in \mathcal{O}(W_j)$. Using the sub-averaging inequality we get

$$\begin{aligned} |S(x)|^2_{h_p} &= |x_1|^{2\lfloor tp \rfloor}|\widetilde{s}(x)|^2 e^{-2p\varphi_j(x)} \\ &\leqslant |x_1|^{2\lfloor tp \rfloor} e^{-2p\varphi_j(x)} \frac{1}{\pi^n}\int_{\Delta^n(x,1)} |\widetilde{s}(z)|^2 \,\mathrm{d}m(z) \\ &\leqslant |x_1|^{2\lfloor tp \rfloor} e^{-2p\varphi_j(x)} \int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \,\mathrm{d}m(z)\,. \end{aligned} \tag{3.2}$$

Applying Fubini's theorem for the splitting $z = (z_1, z')$ and Lemma 3.2 for the variable z_1, we obtain

$$\begin{aligned} \int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \,\mathrm{d}m(z) &= \int_{\Delta^{n-1}(0,2)}\int_{\Delta(0,2)} |\widetilde{s}(z_1,z')|^2 \,\mathrm{d}m(z_1)\,\mathrm{d}m(z') \\ &\leqslant \frac{\lfloor tp \rfloor + 1}{2^{2\lfloor tp \rfloor}} \int_{\Delta^n(0,2)} |z_1|^{2\lfloor tp \rfloor}|\widetilde{s}(z)|^2 \,\mathrm{d}m(z) \\ &\leqslant C \exp\left(2p \sup_{\Delta^n(0,2)} \varphi_j\right) \int_{\Delta^n(0,2)} |s(z)|^2 e^{-2p\varphi_j(z)} \frac{\omega^n}{n!}\,, \end{aligned} \tag{3.3}$$

where $C = C(\mathcal{W}) \geqslant 1$ is chosen such that $\mathrm{d}m(z) \leqslant C\omega^n/n!$ on each $\Delta^n(y_j, 2)$ in the local coordinates of W_j, for $j = 1, \dots, N$. Combining (3.2) and (3.3) we get

$$|S(x)|^2_{h_p} \leqslant C\,|x_1|^{2\lfloor tp \rfloor} \exp\left(2p \sup_{\Delta^n(0,2)} \varphi_j - 2p\varphi_j(x)\right) \|S\|^2_p\,. \tag{3.4}$$

Note that there exists a constant $A' = A'(\rho, W) > 1$ such that

$$|x_1| \leqslant A' e^{\rho(x)}\,, \;\; x \in W\,. \tag{3.5}$$

Set $A = A'C$. The estimates (3.4) and (3.5) yield

$$|S(x)|^2_{h_p} \leqslant (C|x_1|)^{2\lfloor tp \rfloor} e^{4p\|h\|_\infty} \|S\|^2_p \leqslant (Ae^{\rho(x)})^{2\lfloor tp \rfloor} e^{4p\|h\|_\infty} \|S\|^2_p\,.$$

Taking into account (2.2) we immediately obtain the conclusion. □

Corollary 3.3. — *In the setting of Theorem 3.1 we let*

$$U_t := \left\{x \in W : \, (Ae^{\rho(x)})^t \, e^{4\|h\|_\infty} < 1\right\}. \tag{3.6}$$

Then for any $x \in U_t$ and $p > 2t^{-1}$ we have

$$P_{0,p}(x) \leqslant \left[(Ae^{\rho(x)})^t \, e^{4\|h\|_\infty}\right]^p. \tag{3.7}$$

In particular $P_{0,p} = O(p^{-\infty})$ as $p \to \infty$ on U_t.

Proof. — This follows immediately from Theorem 3.1, since $Ae^{\rho(x)} < 1$ for $x \in U_t$, and $2\lfloor tp\rfloor > 2tp - 2 > tp$ for $p > 2/t$. ☐

We conclude this section by giving a version of Theorem 3.1 in the case when X is not compact. Let (X, ω) be a Hermitian manifold of dimension n, Σ be a smooth analytic hypersurface of X, $t > 0$ a fixed real number, and (L, h) a singular Hermitian holomorphic line bundle on X with singular metric h which has locally bounded weights.

As in the case of a compact manifold X, we introduce the Bergman space $H^0_{(2)}(X, L^p) = H^0_{(2)}(X, L^p, h_p, \omega^n/n!)$ of L^2-holomorphic sections of L^p relative to the metric h_p induced by h and the volume form $\omega^n/n!$ on X, endowed with the inner product (2.1). Let $H^0_{(2),0}(X, L^p) \subset H^0_{(2)}(X, L^p)$ be the Bergman space of L^2-holomorphic sections of L^p vanishing to order at least $\lfloor tp\rfloor$ along Σ,

$$H^0_{(2),0}(X, L^p) := H^0_{(2)}\big(X, L^p \otimes \mathcal{O}\big(-\lfloor tp\rfloor\Sigma\big)\big).$$

The spaces $H^0_{(2)}(X, L^p)$ and $H^0_{(2),0}(X, L^p)$ are not necessarily finite dimensional but their Bergman kernel functions P_p and $P_{0,p}$ can be defined as in (1.2), (1.3), by means of at most countable orthonormal bases, see [6, Lemma 3.1].

We fix a compact set $E \subset X$ and consider an open cover $\mathcal{W} = \{W_j\}_{1\leqslant j\leqslant N}$ and a neighborhood W of the compact set $\Sigma \cap E$ constructed as in (2.3). Finally define $\|h\|_\infty = \|h\|_{\infty,\mathcal{W}}$ as in (3.1), and the function ϱ as in (1.4), such that $\varrho < 0$ in a neighborhood of E. The following theorem is proved exactly as Theorem 3.1:

Theorem 3.4. — *Let (X, ω) be a Hermitian manifold of dimension n, Σ be a smooth analytic hypersurface of X, $t > 0$ a fixed real number, and (L, h) a singular Hermitian holomorphic line bundle on X with singular metric h which has locally bounded weights. Then for any compact set $E \subset X$ there exists a neighborhood W of the compact set $\Sigma \cap E$ and a constant $A \geqslant 1$ such that for every $x \in W$ and $p \geqslant 1$,*

$$P_{0,p}(x) \leqslant (Ae^{\rho(x)})^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty}.$$

The constant A depends only on ρ and $\mathcal{W}$ above.

4. Singular Bergman kernel

In this section we prove Theorem 1.2 by using ideas of Berndtsson, who gave in [3, Section 2] a simple proof for the first order asymptotics of the

Bergman kernel function in the case of powers of an ample line bundle (see also [5, Theorem 1.3]).

We start by recalling the following version of Demailly's estimates for the $\overline{\partial}$ operator [7, Théorème 5.1] (see also [5, Theorem 2.5]) which will be needed in our proofs.

Theorem 4.1. — *Let (X,ω) be a compact Kähler manifold of dimension n, and let $B>0$ be a constant such that $\mathrm{Ric}_\omega \geqslant -2\pi B\omega$ on X. Let (L,h) be a singular Hermitian holomorphic line bundle on X such that $c_1(L,h)\geqslant \varepsilon\omega$, and fix p_0 such that $p_0\varepsilon \geqslant 2B$. Then for all $p>p_0$ and all $g\in L^2_{0,1}(X,L^p,loc)$ with $\overline{\partial}g=0$ and $\int_X |g|^2_{h_p}\,\omega^n<\infty$ there exists $u\in L^2_{0,0}(X,L^p,loc)$ such that $\overline{\partial}u=g$ and $\int_X |u|^2_{h_p}\,\omega^n \leqslant \frac{2}{p\varepsilon}\int_X |g|^2_{h_p}\,\omega^n$.*

Proof of Theorem 1.2. — Let $\mathcal{W}=\{W_j\}_{1\leqslant j\leqslant N}$ be an open cover of Σ verifying (2.3) and (2.4). If $j\in\{1,\dots,N\}$ and $x\in\Delta^n(y_j,1)$, let $z=z(x)$ be the coordinates on $\Delta^n(y_j,2)$ given by (2.4), and let $e_{j,x}$ be a holomorphic frame of L on W_j such that $|e_{j,x}|_{\widetilde{h}}=e^{-\varphi_x}$, where φ_x is given by (2.6).

Assume now that $x\in\Delta^n(y_j,1)\setminus\Sigma$ and define

$$r_x := \sup\Big\{r\in(0,r_1] : \Delta^n_z(x,r)\subset\Delta^n(y_j,2)\setminus\Sigma\Big\}.$$

We have

$$\begin{aligned} \omega_x &= \frac{i}{2}\sum_{\ell=1}^n dz_\ell\wedge d\bar z_\ell\,, \\ c_1(L,\widetilde{h})_x &= dd^c\varphi_x(0)=dd^c\psi_x(0)=dd^c\psi'_x(0)=\frac{i}{\pi}\sum_{\ell=1}^n \lambda_\ell\,dz_\ell\wedge d\bar z_\ell\,. \end{aligned} \tag{4.1}$$

Since $c_1(L,\widetilde{h})_x\geqslant\varepsilon\omega_x$ it follows that $\lambda_\ell\geqslant\varepsilon$, $\ell=1,\dots,n$. Moreover, there exists $H_x\in\mathcal{O}(\Delta^n_z(x,r_x))$ such that $\operatorname{Re}H_x=\operatorname{Re}F_x+t\log|f|$. We define a new frame for L over $\Delta^n_z(x,r_x)$ by $e_x=e^{H_x}e_{j,x}$. Hence

$$|e_x|_{\widetilde{h}}=\exp(\operatorname{Re}H_x)\exp(-\varphi_x)=\exp(-\psi'_x-\widetilde{\psi}_x)\,.$$

We fix now $j\in\{1,\dots,N\}$ and $x\in\Delta^n(y_j,1)\setminus\Sigma$ and we will estimate $\widetilde{P}_p(x)$. Let $r_p\in(0,r_x/2)$ be an arbitrary number which will be specified later. We start by estimating the norm of a section $S\in H^0_{(2)}(X,L^p,\widetilde{h}_p,\omega^n/n!)$ at x. Writing $S=se_x^{\otimes p}$, where $s\in\mathcal{O}(\Delta^n_z(x,r_x))$, we obtain by the sub-averaging inequality for psh functions on $\Delta^n_z(x,r_p)=\Delta^n(0,r_p)$,

$$|S(x)|^2_{\widetilde{h}_p}=|s(0)|^2\leqslant\frac{\int_{\Delta^n(0,r_p)}|s|^2e^{-2p\psi'_x}\,\mathrm{d}m}{\int_{\Delta^n(0,r_p)}e^{-2p\psi'_x}\,\mathrm{d}m}\,.$$

We have further by (2.5), (2.7),

$$\begin{aligned}\int_{\Delta^n(0,r_p)}&|s|^2e^{-2p\psi'_x}\,\mathrm{d}m\\&\leqslant(1+C_1r_p^2)\exp\big(2p\sup_{\Delta^n(0,r_p)}\widetilde{\psi}_x\big)\int_{\Delta^n(0,r_p)}|s|^2e^{-2p(\psi'_x+\widetilde{\psi}_x)}\frac{\omega^n}{n!}\\&\leqslant(1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)\|S\|_p^2\,.\end{aligned}$$

Set

$$E(r):=\int_{|\xi|\leqslant r}e^{-2|\xi|^2}\,\mathrm{d}m(\xi)=\frac{\pi}{2}\,\left(1-e^{-2r^2}\right).$$

Since $\lambda_\ell\geqslant\varepsilon$ we obtain

$$\frac{E(r_p\sqrt{p\varepsilon}\,)^n}{p^n\lambda_1\dots\lambda_n}\leqslant\int_{\Delta^n(0,r_p)}e^{-2p\psi'_x}\,\mathrm{d}m\leqslant\int_{\mathbb{C}^n}e^{-2p\psi'_x}\,\mathrm{d}m=\frac{(\pi/2)^n}{p^n\lambda_1\dots\lambda_n}\,.$$

Combining these estimates it follows that

$$|S(x)|^2_{\widetilde{h}_p}\leqslant\frac{(1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)}{E(r_p\sqrt{p\varepsilon})^n}\,p^n\lambda_1\dots\lambda_n\,\|S\|_p^2\,. \tag{4.2}$$

The singular Bergman kernel also satisfies a variational formula,

$$\widetilde{P}_p(x)=\max\Big\{|S(x)|^2_{\widetilde{h}_p}\,:\,S\in H^0_{(2)}(X,L^p,\widetilde{h}_p,\omega^n/n!),\ \|S\|_p=1\Big\}.$$

Hence (4.2) implies the following upper estimate for the singular Bergman kernel,

$$\frac{\widetilde{P}_p(x)}{p^n\lambda_1\dots\lambda_n}\leqslant\frac{(1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)}{E(r_p\sqrt{p\varepsilon})^n}\,,\quad\forall\ r_p\in(0,r_x/2). \tag{4.3}$$

For the lower estimate of $\widetilde{P}_p$, let $0\leqslant\chi\leqslant1$ be a smooth cut-off function on $\mathbb{C}^n$ with support in $\Delta^n(0,2)$ such that $\chi\equiv1$ on $\Delta^n(0,1)$, and set $\chi_p(z)=\chi(z/r_p)$. Then $F=\chi_pe_x^{\otimes p}$ is a section of L^p and $|F(x)|_{\widetilde{h}_p}=|e_x^{\otimes p}(x)|_{\widetilde{h}_p}=1$. We have

$$\begin{aligned}\|F\|_p^2&\leqslant\int_{\Delta^n(0,2r_p)}e^{-2p(\psi'_x+\widetilde{\psi}_x)}\,\frac{\omega^n}{n!}\\&\leqslant(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)\int_{\Delta^n(0,2r_p)}e^{-2p\psi'_x}\,\mathrm{d}m\\&\leqslant\left(\frac{\pi}{2}\right)^n\,\frac{(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)}{p^n\lambda_1\dots\lambda_n}\,.\end{aligned} \tag{4.4}$$

Set $\alpha = \overline{\partial} F$. Since $\|\overline{\partial}\chi_p\|^2 = \|\overline{\partial}\chi\|^2/r_p^2$, where $\|\overline{\partial}\chi\|$ denotes the maximum of $|\overline{\partial}\chi|$, we obtain as above

$$\begin{aligned}\|\alpha\|_p^2 &= \int_{\Delta^n(0,2r_p)} |\overline{\partial}\chi_p|^2 e^{-2p(\psi'_x+\widetilde{\psi}_x)}\,\frac{\omega^n}{n!}\\ &\leqslant \frac{\|\overline{\partial}\chi\|^2}{r_p^2}\left(\frac{\pi}{2}\right)^n \frac{(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)}{p^n\lambda_1\dots\lambda_n}\,.\end{aligned}$$

There exists $p_0\in\mathbb{N}$ such that for $p>p_0$ we can solve the $\overline{\partial}$–equation by Theorem 4.1. We get a smooth section G of L^p with $\overline{\partial}G = \alpha = \overline{\partial}F$ and

$$\|G\|_p^2 \leqslant \frac{2}{p\varepsilon}\|\alpha\|_p^2 \leqslant \frac{2\|\overline{\partial}\chi\|^2}{p\varepsilon r_p^2}\left(\frac{\pi}{2}\right)^n \frac{(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)}{p^n\lambda_1\dots\lambda_n}\,. \tag{4.5}$$

Note that G is holomorphic on $\Delta^n(0,r_p)$ since $\overline{\partial}G=\overline{\partial}F=0$ there. So the estimate (4.2) applies to G on $\Delta^n(0,r_p)$ and gives

$$\begin{aligned}|G(x)|^2_{\widetilde{h}_p} &\leqslant \frac{(1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)}{E(r_p\sqrt{p\varepsilon})^n}\,p^n\lambda_1\dots\lambda_n\|G\|_p^2\\ &\leqslant \frac{2\|\overline{\partial}\chi\|^2}{p\varepsilon r_p^2E(r_p\sqrt{p\varepsilon})^n}\left(\frac{\pi}{2}\right)^n(1+4C_1r_p^2)^2\exp\big(18C_2p\,r_p^3\big)\,.\end{aligned}$$

Let $S = F-G\in H^0_{(2)}(X,L^p,\widetilde{h}_p,\omega^n/n!)$. Then

$$\begin{aligned}|S(x)|^2_{\widetilde{h}_p} &\geqslant \Big(|F(x)|_{\widetilde{h}_p} - |G(x)|_{\widetilde{h}_p}\Big)^2 = \Big(1-|G(x)|_{\widetilde{h}_p}\Big)^2\\ &\geqslant \left[1-\left(\frac{\pi}{2}\right)^{n/2}\frac{\sqrt{2}\,\|\overline{\partial}\chi\|(1+4C_1r_p^2)}{r_p\sqrt{p\varepsilon}\,E(r_p\sqrt{p\varepsilon})^{n/2}}\,\exp\big(9C_2p\,r_p^3\big)\right]^2 =: K_1(r_p)\,.\end{aligned}$$

Moreover, by (4.4) and (4.5)

$$\|S\|_p^2\leqslant(\|F\|_p+\|G\|_p)^2\leqslant\left(\frac{\pi}{2}\right)^n\frac{K_2(r_p)}{p^n\lambda_1\dots\lambda_n}\,,$$

where

$$K_2(r_p) = (1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)\left(1+\frac{\sqrt{2}\,\|\overline{\partial}\chi\|}{r_p\sqrt{p\varepsilon}}\right)^2.$$

Therefore

$$\widetilde{P}_p(x)\geqslant\frac{|S(x)|^2_{\widetilde{h}_p}}{\|S\|_p^2}\geqslant\left(\frac{2}{\pi}\right)^n p^n\lambda_1\dots\lambda_n\frac{K_1(r_p)}{K_2(r_p)}\,. \tag{4.6}$$

Using now (4.1), (4.3) and (4.6) we deduce that for every $x \in \bigcup_{j=1}^{N} \Delta^n(y_j,1)\backslash\Sigma$, $r_p < r_x/2$ and $p > p_0$,

$$\frac{K_1(r_p)}{K_2(r_p)} \leqslant \widetilde{P}_p(x)\,\frac{\omega_x^n}{p^n c_1(L,\widetilde{h})_x^n} \leqslant K_3(r_p)\,, \tag{4.7}$$

where

$$K_3(r_p) = \left(\frac{\pi/2}{E(r_p\sqrt{p\varepsilon})}\right)^n (1 + C_1 r_p^2)\exp\bigl(2C_2 p\, r_p^3\bigr)\,.$$

We take now $r_p = p^{-3/8}$, so $p\, r_p^3 = p^{-1/8} \to 0$ and $p\, r_p^2 = p^{1/4} \to \infty$ as $p \to \infty$. Note that there exists a constant $C_3 > 0$ such that

$$K_1(p^{-3/8}) \geqslant 1 - C_3 p^{-1/8}\,,$$
$$K_2(p^{-3/8}) \leqslant 1 + C_3 p^{-1/8}\,,\;\; K_3(p^{-3/8}) \leqslant 1 + C_3 p^{-1/8}\,.$$

It follows by (4.7) that there exists a constant $C_4 > 0$ such that

$$1 - C_4\, p^{-1/8} \leqslant \widetilde{P}_p(x)\,\frac{\omega_x^n}{p^n c_1(L,\widetilde{h})_x^n} \leqslant 1 + C_4\, p^{-1/8} \tag{4.8}$$

holds for every $x \in \bigcup_{j=1}^N \Delta^n(y_j,1) \setminus \Sigma$, $p^{-3/8} < r_x/2$ and $p > p_0$. Now $r_x > c\,\mathrm{dist}(x,\Sigma)$, for some constant $c > 0$, so there exists a constant $C_5 > 0$ such that (4.8) holds for $p > C_5\,\mathrm{dist}(x,\Sigma)^{-8/3}$. This concludes the proof of (1.5) for $x \in \bigcup_{j=1}^N \Delta^n(y_j,1) \setminus \Sigma$.

By [8, Theorem 1.8] there exist $C_6 > 0$ and $p_0' \in \mathbb{N}$ such that

$$\left|\widetilde{P}_p(x)\,\frac{\omega_x^n}{p^n c_1(L,\widetilde{h})_x^n} - 1\right| \leqslant \frac{C_6}{p}\,,$$

for $x \in X \setminus \bigcup_{j=1}^N \Delta^n(y_j,1)$ and $p > p_0'$. The proof of Theorem 1.2 is complete. □

5. Estimates for the partial Bergman kernel

In this section we prove Theorem 1.3. Let $t < t_0(K)$. By the definition (1.6) of $t_0(K)$, there exist $\eta \in \mathscr{C}^\infty(X,[0,1])$ and $\delta > 0$ such that $\operatorname{supp}\eta \subset X \setminus K$, $\eta = 1$ near Σ, and $c_1(L,h) + t dd^c(\eta\varrho) \geqslant \delta\omega$ in the sense of currents on X. Define

$$\widetilde{h}_t = h\exp(-2t\eta\varrho)\,,\;\; \widetilde{h}_{t,p} = \widetilde{h}_t^{\otimes p}\,.$$

Note that $\widetilde{h}_t = h$ in a neighborhood of K and $\widetilde{h}_t \geqslant h$ on X. Since Σ is smooth, it follows by (1.4) that $H^0_0(X, L^p) = H^0_{(2)}(X, L^p, \widetilde{h}_{t,p}, \omega^n/n!)$. We denote the norm on $H^0_{(2)}(X, L^p, \widetilde{h}_{t,p}, \omega^n/n!)$ by

$$\|S\|^2_{t,p} = \int_X |S|^2_{\widetilde{h}_{t,p}} \frac{\omega^n}{n!} = \int_X |S|^2_{h_p} \exp(-2tp\eta\varrho) \frac{\omega^n}{n!} \, .$$

Let $\widetilde{P}_{t,p}$ be the Bergman kernel function of $H^0_{(2)}(X, L^p, \widetilde{h}_{t,p}, \omega^n/n!)$. Recall that $\|S\|_p$ is the norm given by the scalar product (2.1) on $H^0_0(X, L^p)$. Since $\varrho < 0$ we have $\|S\|^2_{t,p} \geqslant \|S\|^2_p$ for any $S \in H^0_0(X, L^p)$. Let $S \in H^0_0(X, L^p)$ with $\|S\|^2_{t,p} \leqslant 1$. Then $\|S\|^2_p \leqslant 1$, too, hence

$$|S|^2_{\widetilde{h}_{t,p}} = |S|^2_{h_p} \exp(-2tp\eta\varrho) \leqslant P_{0,p} \exp(-2tp\eta\varrho) \, ,$$

and thus

$$\widetilde{P}_{t,p} \leqslant P_{0,p} \exp(-2tp\eta\varrho) \, .$$

Denote now by P_p the Bergman kernel function of $H^0(X, L^p)$ endowed with the scalar product (2.1). Since $H^0_0(X, L^p)$ is isometrically embedded in $H^0(X, L^p)$ we have $P_{0,p} \leqslant P_p$. Consequently we have shown:

$$\begin{aligned} \widetilde{P}_{t,p} \exp(2tp\eta\varrho) \leqslant P_{0,p} \leqslant P_p \ \text{ on } X, \\ \widetilde{P}_{t,p} \leqslant P_{0,p} \leqslant P_p \ \text{ near } K. \end{aligned} \tag{5.1}$$

Let now W be the neighborhood of Σ defined in (2.3) and let U_t be defined as in (3.6), so that the exponential estimate (3.7) holds on U_t for $p > 2t^{-1}$. By shrinking U_t we can assume that $\eta = 1$ on U_t. Setting $M := e^{4\|h\|_\infty} A^t$ we obtain (1.7). By Theorem 1.2 we have

$$\widetilde{P}_{t,p}(x) \geqslant (1 - Cp^{-1/8}) p^n \, \frac{c_1(L, \widetilde{h}_t)^n_x}{\omega^n_x}$$

for every $p \in \mathbb{N}$ with $p \, \mathrm{dist}(x, \Sigma)^{8/3} > C$. Note that $c_1(L, \widetilde{h}_t) \geqslant \delta\omega$ in the sense of currents on X. Since $c_1(L, \widetilde{h}_t)$ is smooth on $X \setminus \Sigma$ we have $\dfrac{c_1(L, \widetilde{h}_t)^n}{\omega^n} \geqslant \delta^n$ on $X \setminus \Sigma$. By increasing C if necessary, it follows that

$$\widetilde{P}_{t,p}(x) \geqslant \frac{p^n}{C} \quad \text{for } p > C \, \mathrm{dist}(x, \Sigma)^{-8/3}.$$

Hence

$$P_{0,p}(x) \geqslant \frac{p^n}{C} \exp(2tp\varrho(x)) \quad \text{for } x \in U_t \text{ and } p > C \, \mathrm{dist}(x, \Sigma)^{-8/3}.$$

This proves (1.8). In order to prove (1.9) we need the following localization theorem for the Bergman kernel for singular Hermitian metrics. We refer to [1, Theorem 2.2] for a localization principle in the case of smooth metrics.

Theorem 5.1. — *Let (X,ω) be a compact Hermitian manifold and $L \to X$ be a holomorphic line bundle. Consider two singular Hermitian metrics h_1 and h_2 on L, which are smooth outside a proper analytic set $\Sigma \subset X$ and such that $c_1(L,h_1)$, $c_1(L,h_2)$ are Kähler currents. Let $P_p^{(j)}$ be the Bergman projection on $H^0(X, L^p, h_j^p, \omega^n/n!)$, $j=1,2$. We assume that there exists an open set $U \Subset X \setminus \Sigma$ such that $h_1 = h_2$ on U. Then the Bergman kernels satisfy $P_p^{(1)}(z,w) - P_p^{(2)}(z,w) = O(p^{-\infty})$ on U in any $\mathscr{C}^\ell$-topology, $\ell \in \mathbb{N}$, as $p \to \infty$.*

Proof. — The proof follows essentially from the analysis in [8] (see also [9]). Let h_0 be any singular Hermitian metric on L, smooth on $X \setminus \Sigma$ and satisfying $c_1(L,h_0) \geqslant \varepsilon\omega$ in the sense of currents on X, for some $\varepsilon > 0$. Let $P_p^{(0)}$ be the Bergman projection on $H^0(X, L^p, h_0^p, \omega^n/n!)$.

Consider an open set $D \subset U$ such that $L|_D$ is trivial. Let $s : D \to L$ be a holomorphic frame and let $\varphi \in \mathscr{C}^\infty(D)$ be the weight of h_0 corresponding to s, that is, $|s|_{h_0} = e^{-\varphi}$. Let us denote by $\mathscr{E}'(D)$ the space of distributions with compact support in D and by $L^2(D)$ the space of square-integrable functions with respect to the volume form $\omega^n/n!$. The localized Bergman projection with respect to s is the operator $P_{p,s}^{(0)} : L^2(D) \cap \mathscr{E}'(D) \to L^2(D)$, defined by $P_p^{(0)}(ue^{p\varphi}s^{\otimes p}) = P_{p,s}^{(0)}(u)e^{p\varphi}s^{\otimes p}$. It is easy to see that

$$P_p^{(0)}(z,w) = P_{p,s}^{(0)}(z,w)e^{p(\varphi(z)-\varphi(w))}s^{\otimes p}(z) \otimes (s^{\otimes p})^*(w) \in L_z^p \otimes (L_w^p)^*, \quad z,w \in D. \tag{5.2}$$

By [8, Theorem 9.2] the kernel of $P_{p,s}^{(0)}$ satisfies

$$P_{p,s}^{(0)}(z,w) = \mathcal{S}_p(z,w) + O(p^{-\infty}) \text{ on } D, \tag{5.3}$$

where $\mathcal{S}_p$ is the localized approximate Szegő kernel defined in [8, (3.43)]. Note that by [8, Theorem 3.12] we have

$$\mathcal{S}_p(z,w) = e^{ip\Psi(z,w)}b(z,w,p) + O(p^{-\infty}) \text{ on } D, \tag{5.4}$$

where $\Psi : D \times D \to \mathbb{C}$ is a phase function depending on the eigenvalues of $c_1(L,h_0)$ with respect to ω and described precisely in [8, Theorem 3.8]. Moreover, $b(\cdot,\cdot,p) : D \times D \to \mathbb{C}$ is a semi-classical symbol of order $n = \dim X$, depending only on the restriction of h and ω to D.

We apply now these results for $h_0 = h_1$ and $h_0 = h_2$. Since $h_1|_D = h_2|_D$ we deduce that the weight φ, the phase Ψ and the symbol $b(\cdot,\cdot,p)$ above are the same for h_1 and h_2. We infer from (5.3) and (5.4) that $P_{p,s}^{(1)}(z,w) - P_{p,s}^{(2)}(z,w) = O(p^{-\infty})$ on D. Finally, (5.2) yields $P_p^{(1)}(z,w) - P_p^{(2)}(z,w) = O(p^{-\infty})$ on D. The proof of Theorem 5.1 is complete. $\square$

We apply now Theorem 5.1 to the metrics $\widetilde{h}_t$ and h, which are equal on a neighborhood V of K and infer that

$$\widetilde{P}_{t,p} - P_p = O(p^{-\infty}) \text{ locally uniformly on } V. \tag{5.5}$$

This fact combined with (5.1), (5.5) yields (1.9). Finally, (1.10) and (1.11) follow from the expansion of the Bergman kernel P_p (see [10, Theorems 4.1.1–3]) or of the singular Bergman kernel (see [8, Theorem 1.8]).

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