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Dimension free bounds for the Hardy–Littlewood maximal operator associated to convex sets [(*)]

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Abstract. — This survey is based on a series of lectures given by the authors at the working seminar "Convexité et Probabilités" at UPMC Jussieu, Paris, during the spring 2013. It is devoted to maximal functions associated to symmetric convex sets in high dimensional linear spaces, a topic mainly developed between 1982 and 1990 but recently renewed by further advances.

The series focused on proving these maximal function inequalities in $L^p(\mathbb{R}^n)$, with bounds independent of the dimension n and for all $p \in (1, +\infty]$ in the best cases. This program was initiated in 1982 by Elias Stein, who obtained the first theorem of this kind for the family of Euclidean balls in arbitrary dimension. We present several results along this line, proved by Bourgain, Carbery and Müller during the period 1986–1990, and a new one due to Bourgain (2014) for the family of cubes in arbitrary dimension. We complete the cube case with a negative answer to the possible dimensionless behavior of the weak type $(1, 1)$ constant, due to Aldaz, Aubrun and Iakovlev–Strömberg between 2009 and 2013.

Résumé. — Ces Notes reprennent et complètent une série d'exposés donnés par les auteurs au groupe de travail « Convexité et Probabilités » à l'UPMC Jussieu, Paris, au cours du printemps 2013. Elles sont consacrées à l'étude des fonctions maximales de type Hardy–Littlewood associées aux corps convexes symétriques dans $\mathbb{R}^n$. On s'intéresse tout particulièrement au comportement des constantes intervenant dans les estimations lorsque

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Article proposé par Xavier Tolsa.

la dimension n tend vers l'infini. Ce sujet a été développé principalement entre 1982 et 1990, mais a été relancé par des avancées récentes.

Le but de la série d'exposés était de prouver des inégalités maximales dans $L^p(\mathbb{R}^n)$ avec des bornes indépendantes de la dimension n, pour certaines familles de corps convexes. Dans les meilleurs cas, on a pu obtenir de tels résultats pour toutes les valeurs de p dans $(1,+\infty]$. Ce thème de recherche a été initié en 1982 par Elias Stein [75], qui a démontré le premier théorème de ce genre pour la famille des boules euclidiennes en dimension arbitraire, obtenant pour tout $p \in (1,+\infty]$ une borne dans $L^p(\mathbb{R}^n)$ indépendante de n. Nous présentons ce théorème de Stein ainsi que plusieurs autres résultats dans cette direction, démontrés par Bourgain, par Carbery et par Müller dans la période 1986–1990. En 1986, Bourgain [9] obtient une borne indépendante de n valable dans $L^2(\mathbb{R}^n)$ pour tout corps convexe symétrique dans $\mathbb{R}^n$, puis Bourgain [10] et Carbery [21] étendent le résultat $L^p(\mathbb{R}^n)$ de Stein aux corps convexes symétriques quelconques, mais sous la condition que $p > 3/2$. Müller [59] obtient un résultat valable pour tout $p > 1$ quand un certain paramètre géométrique, lié aux volumes des projections du corps convexe sur les hyperplans, reste borné. Ce paramètre ne reste pas borné pour tous les convexes, en particulier, il tend vers l'infini pour les cubes de grande dimension. Nous donnons un théorème récent (2014) dû à Bourgain [13] qui obtient pour tout $p > 1$ une borne dans $L^p(\mathbb{R}^n)$ indépendante de n pour la famille des fonctions maximales associées aux cubes en dimension arbitraire. Nous complétons l'étude du cas du cube par des résultats pour la constante de type faible $(1,1)$, dus à Aldaz [1], à Aubrun [3] et à Iakovlev–Strömberg [46] entre 2009 et 2013. À l'inverse du cas $L^p(\mathbb{R}^n)$, $1 < p \leqslant +\infty$, cette constante de type faible ne reste pas bornée quand la dimension tend vers l'infini.

Introduction

First defined by Hardy and Littlewood [44] in the one-dimensional setting, the Hardy–Littlewood maximal operator was generalized in arbitrary dimension by Wiener [83]. It turned out to be a powerful tool, for instance in harmonic or Fourier analysis, in differentiation theory or in singular integrals theory. It was extended to various situations, including not only homogeneous settings, as in the book of Coifman and Weiss [23], but also non-homogeneous, like noncompact symmetric spaces in works by Clerc and Stein [22] or Strömberg [78]. Also studied in vector-valued settings with the Fefferman–Stein type inequalities [33], it gave rise to several kinds of maximal operators which are now important in real analysis.

We shall denote by M the classical centered Hardy–Littlewood maximal operator, defined on the class of locally integrable functions f on $\mathbb{R}^n$ by

$$(\mathrm{M}f)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| \,\mathrm{d}y\,, \quad x \in \mathbb{R}^n, \tag{0.1}$$

where B_r is the Euclidean ball of radius r and center 0 in $\mathbb{R}^n$, and $|S|$ denotes here the n-dimensional Lebesgue volume of a Borel subset S of $\mathbb{R}^n$. It is well known that this nonlinear operator M is of *strong type* (p,p) when $1 < p \leqslant +\infty$ and of *weak type* $(1,1)$, as stated in the following famous theorem. We write $L^p(\mathbb{R}^n)$ for the L^p-space corresponding to the Lebesgue measure on $\mathbb{R}^n$.

Theorem 0.1 (Hardy–Littlewood maximal theorem). — *Let n be an integer $\geqslant 1$.*

(1) *For every function $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, the* weak type inequality
$$\big|\{x \in \mathbb{R}^n : (\mathrm{M}f)(x) > \lambda\}\big| \leqslant \frac{C(n)}{\lambda} \, \|f\|_{L^1(\mathbb{R}^n)} \tag{WT}$$
holds true, with a constant $C(n)$ depending only on the dimension n.

(2) *Let $1 < p \leqslant +\infty$. There exists a constant $C(n,p)$ such that for every function f in $L^p(\mathbb{R}^n)$, one has*
$$\|\mathrm{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C(n,p) \, \|f\|_{L^p(\mathbb{R}^n)} \, . \tag{ST}$$

The weak type inequality is optimal in the sense that $\mathrm{M}f$ is never in $L^1(\mathbb{R}^n)$, unless $f = 0$ almost everywhere. Zygmund introduced the so-called "$L \log L$ class" to give a sufficient condition for the local integrability of the Hardy–Littlewood maximal function, a condition that is actually necessary, as proved by Stein [72]. The proof of Theorem 0.1 by Hardy and Littlewood was combinatorial and used decreasing rearrangements. The authors said: "The problem is most easily grasped when stated in the language of cricket, or any other game in which a player compiles a series of scores of which an average is recorded". Passing through the Vitali covering lemma, which is recalled below, has become later a standard approach.

A natural question that can be raised is the following. Could we compute the best constant in both inequalities (WT) and (ST)? This question seems to be out of reach in full generality. There is a very remarkable exception to this statement, the one-dimensional case where Melas has shown in [57] by a mixture of combinatorial, geometric and analytic arguments, that the best constant in (WT) is $(11 + \sqrt{61})/12$. The case $p > 1$ is still open, even in the one-dimensional case, despite of substantial progress by Grafakos, Montgomery-Smith and Motrunich [41], who obtained by variational methods the best constant in (ST) for the class of positive functions on the line that are convex except at one point. The *uncentered* maximal operator $f \mapsto f^*$ is better understood [40], the uncentered maximal function f^* being defined for every $x \in \mathbb{R}^n$ by
$$f^*(x) = \sup_{B \in \mathcal{B}(x)} \frac{1}{|B|} \int_B |f(u)| \, \mathrm{d}u \, , \tag{0.2}$$

where $\mathcal{B}(x)$ denotes the family of Euclidean balls B containing x, with arbitrary center y and radius $> d(y,x)$. It is clear that $f^* \geqslant \mathrm{M}f$, and the maximal theorem also holds for f^* since any "uncentered" ball $B \in \mathcal{B}(x)$ of radius r is contained in $B(x, 2r)$, yielding the far from sharp pointwise inequality $f^* \leqslant 2^n \mathrm{M}f$.

Lacking for exact values, one may address the question of the asymptotic behavior of the constants when the dimension n tends to infinity. This program was initiated at the beginning of the 80s by Stein. In the usual proof of the Hardy–Littlewood maximal theorem based on the Vitali covering lemma, the dependence on the dimension n in the weak type result is exponential, of the form $C(n) = C^n$ for some $C > 1$. Then, by interpolation of Marcinkiewicz-type between the weak-L^1 case and the trivial L^∞ case, one can get for the strong type in $L^p(\mathbb{R}^n)$ a constant of the form $C(n,p) = pC^{n/p}/(p-1)$, when $1 < p \leqslant +\infty$ (see [39, Exercises, 1.3.3 (a)]). In [75], Stein has improved this asymptotic behavior in a spectacular fashion. Indeed, by using a spherical maximal operator together with a lifting method, he showed that for every $p > 1$, one can replace the bound $C(n,p)$ in (ST) by a bound $C(p)$ independent of n. The detailed proof appeared in the paper [77] by Stein and Strömberg.

The use of an appropriate spherical maximal operator is now a decisive approach for bounding the L^p norm of Hardy–Littlewood-type maximal operators independently of the dimension n, when $p > 1$. This is the case, for instance, for the Heisenberg group [84] or for hyperbolic spaces [54]. Moreover, Stein and Strömberg proved that the weak type $(1,1)$ constant grows at most like $\mathrm{O}(n)$, and it is still unknown whether or not this constant may be bounded independently of the dimension. The proof in [77] draws on the Hopf–Dunford–Schwartz ergodic theorem, about which Stein says in [73] that it is "one of the most powerful results in abstract analysis". The strategy, which exploits the relationship between averages on balls and either the heat semi-group or the Poisson semi-group, is well explained in [24], and has been applied in several different settings [27, 52, 53, 55].

In a large part of these Notes, we shall replace Euclidean balls in the definition (0.1) of the maximal operator by other centrally symmetric convex bodies in $\mathbb{R}^n$ (in what follows, we shall omit "centrally" and abbreviate it as *symmetric convex body*). For example, replacing averages over Euclidean balls B_r of radius r by averages over n-dimensional cubes Q_r with side $2r$ gives an operator M_Q which satisfies both the weak type and strong type maximal inequalities. Indeed, since $B_r \subset Q_r \subset \sqrt{n} B_r$, it is obvious that M_Q is bounded in $L^p(\mathbb{R}^n)$ with $C(n,p)$ replaced by $n^{n/2} C(n,p)$, but this painless route badly spoils the constants. Several results specific to the cube case have been obtained, as we shall indicate below.

More generally, as in Stein and Strömberg [77], one can give a symmetric convex body C in $\mathbb{R}^n$ and introduce the *maximal operator* M_C *associated to the convex set* C as follows: for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ one defines the function $\mathrm{M}_C f$ on $\mathbb{R}^n$ by

$$\begin{aligned}(\mathrm{M}_C f)(x) &= \sup_{t>0} \frac{1}{|tC|} \int_{x+tC} |f(y)| \,\mathrm{d}y \\ &= \sup_{t>0} \frac{1}{|C|} \int_C |f(x+tv)| \,\mathrm{d}v\,, \quad x \in \mathbb{R}^n\,,\end{aligned} \tag{0.3.\mathbf{M}}$$

where $x + tC := \{x + tc : c \in C\}$. One may also consider M_C when C is not symmetric but has its centroid at 0, see Fradelizi [34, Section 1.5]. The maximal operator M_C satisfies, again, a maximal theorem of Hardy–Littlewood type.

Let C be a symmetric convex body in $\mathbb{R}^n$. The weak type $(1,1)$ property for M_C can be deduced from the *Vitali covering lemma*: given a finite family of translated-dilated sets $x_i + r_i C$, $i \in I$, $x_i \in \mathbb{R}^n$, $r_i > 0$, one can extract a *disjoint* subfamily $(x_j + r_j C)_{j \in J}$, $J \subset I$, such that each set $x_i + r_i C$, $i \in I$, of the original family is contained in the *dilate* $x_j + 3r_j C$ of some member $x_j + r_j C$, $j \in J$, of the *extracted* disjoint family. One may explain the constant 3 by the use of the triangle inequality for the norm on $\mathbb{R}^n$ whose unit ball is C. Passing to the Lebesgue measure in $\mathbb{R}^n$, this statement naturally introduces a factor 3^n corresponding to the dilation factor 3. If f^*_C denotes the corresponding uncentered maximal function of f associated to C, then for every $\lambda > 0$, one has that

$$\big|\{x \in \mathbb{R}^n : f^*_C(x) > \lambda\}\big| \leqslant \frac{3^n}{\lambda} \int_{\{f^*_C > \lambda\}} |f(x)| \,\mathrm{d}x\,. \tag{0.4}$$

We briefly sketch a proof, very similar to that of Doob's maximal inequality presented in Section 1.1. It is convenient here to consider that C is an *open* subset of $\mathbb{R}^n$. Given an arbitrary compact subset K of the open set $U_\lambda = \{f^*_C > \lambda\}$, one applies the Vitali lemma to a finite covering of K by open sets $S_i = x_i + r_i C$ such that $\int_{S_i} |f| > \lambda |S_i|$. A simple feature of f^*_C is that each such S_i is actually *contained* in U_λ. If $J \subset I$ corresponds to the disjoint family given by Vitali, then

$$|K| \leqslant \sum_{j \in J} |x_j + 3r_j C| = 3^n \sum_{j \in J} |x_j + r_j C| \leqslant \frac{3^n}{\lambda} \int_{U_\lambda} |f(x)| \,\mathrm{d}x\,,$$

implying (0.4). Next, a direct argument involving only Fubini and Hölder can give an L^p bound, exactly as in the proof of Doob's Theorem 1.1 below, but giving a factor 3^n instead of $3^{n/p}$ obtained by interpolation. This Vitali method does not depend upon the symmetric body C, does not distinguish the centered and uncentered operators, and introduces a quite unsatisfactory exponential constant.

Stein and Strömberg have greatly improved this exponential dependence in [77]. By a clever covering argument with less overlap than in Vitali's lemma, they proved that the weak type constant admits a bound of the form $\mathrm{O}(n \log n)$, and by using the Calderón–Zygmund method of rotations, they obtained for the strong type property a constant which behaves as $np/(p-1)$. Concerning the weak type constant, Naor and Tao [60] have established the same $n \log n$ behavior for the large class of *n-strong micro-doubling* metric measure spaces (see also [25]). Several powerful results about the strong type constant for maximal functions associated to convex sets, beyond the one of Stein–Strömberg, have been established between 1986 and 1990. First of all, Bourgain proved a dimensionless theorem for general symmetric convex bodies in the L^2 case [9], applying geometrical arguments and methods from Fourier analysis. This result has been generalized to $L^p(\mathbb{R}^n)$, for all $p > 3/2$, by Bourgain [10] and Carbery [21] in two independent papers. They both bring into play an auxiliary dyadic maximal operator, but Bourgain uses it together with square function techniques while Carbery uses multipliers associated to fractional derivatives. Detlef Müller extended in [59] the L^p bound to every $p > 1$, but under an additional geometrical condition on the family of convex sets C under study. Müller also proved that for every fixed $q \in [1, +\infty)$, his condition is fulfilled by the family $\mathcal{F}_q$ of ℓ_n^q balls, $n \in \mathbb{N}^*$.

After Müller's article, activity in this area slowed down. Nevertheless, Bourgain recently proved in [13] that for all $p > 1$, the strong type constant can be bounded independently of the dimension when we average over cubes. In order to attack this problem, Bourgain applies an arsenal of techniques, including a holomorphic semi-group theorem due to Pisier [62] and ideas inspired by martingale theory. The cube case is rather well understood since Aldaz [1] has proved that the weak type $(1,1)$ constant $\kappa_{Q,n}$ for cubes must tend to infinity with the dimension n. The best lower bound known at the time of our writing is due to Iakovlev–Strömberg [46] who obtained $\kappa_{Q,n} \geqslant \kappa n^{1/4}$, improving a previous estimate $\kappa_{Q,n} \geqslant \kappa_\varepsilon (\log n)^{1-\varepsilon}$ for every $\varepsilon > 0$, which was obtained by Aubrun [3] following the Aldaz result.

In the present survey, except for Section 9 on the Aldaz "negative" result, we shall restrict ourselves to $p > 1$ and examine the strong type (p,p) behavior of maximal functions associated to symmetric convex bodies in $\mathbb{R}^n$. We shall present the dimensionless result of Stein for Euclidean balls, the works of Bourgain, Carbery and Müller during the 80s and the recent dimensionless theorem of Bourgain for cubes. As we shall see, the proofs require a lot of methods and tools, including multipliers, square functions, Littlewood–Paley theory, complex interpolation, holomorphic semi-groups and geometrical arguments involving convexity. The study of weak type inequalities for Hardy–Littlewood-type operators needs powerful methods as well: not only the aforementioned Hopf–Dunford–Schwartz ergodic theorem, but also sharp estimates for heat or Poisson semi-group, Iwasawa decomposition, K-bi-invariant convolution-type operators, expander-type estimates…

The first two sections contain general dimension free inequalities obtained respectively by probabilistic methods or by Fourier transform methods. The Poisson semi-group plays an important rôle in Stein's book [73], and appears also in Bourgain's articles [9, 10] and in Carbery [21]. We give a presentation of this semi-group, both on the probabilistic and Fourier analytic viewpoints. The third section is about some analytic tools that are employed later on, namely, estimates for the Gamma function in the complex plane, and the complex interpolation scheme for linear operators, as developed in Stein [70]. The Stein result for Euclidean balls in arbitrary dimension is our Theorem 4.1. Section 5 is about Bourgain's L^2-theorem in arbitrary dimension n, stating that there exists a constant κ_2 independent of n such that for any symmetric convex body C in $\mathbb{R}^n$, one has

$$\|\mathrm{M}_C f\|_{L^2(\mathbb{R}^n)} \leqslant \kappa_2 \, \|f\|_{L^2(\mathbb{R}^n)}$$

for every $f \in L^2(\mathbb{R}^n)$. The next section presents Carbery's proof of the generalization to L^p of the latter bound, obtained by Bourgain [10] and Carbery [21]. In both papers, the L^p result for general symmetric convex bodies is proved for $p > 3/2$ only. A theorem due to Detlef Müller [59] is given in Section 7; for families of symmetric convex sets C for which a certain parameter $q(C)$ remains bounded, it extends the dimensionless L^p bound to every $p > 1$. This parameter is related to the $(n-1)$-dimensional measure of hyperplane projections of a specific volume one linear image of C, the so-called *isotropic position*. Section 8 presents the result of Bourgain about cubes in arbitrary dimension. In this special case, an L^p bound independent of the dimension is valid for all $p > 1$, although the Müller condition is not satisfied. Bourgain's proof is highly dependent on the product structure of the cube. In Section 9, we prove the Aldaz result that the weak type $(1, 1)$ constant for cubes is not bounded when the dimension n tends to infinity. We mention the quantitative improvement by Aubrun [3], and give a proof for the lower bound $\kappa n^{1/4}$ due to Iakovlev–Strömberg [46].

We have put a notable emphasis on the notion of log-concavity. We shall see that with not much more effort, most maximal theorems for convex sets generalize to symmetric log-concave probability densities. This kind of extension from convex sets to log-concave functions has attracted a lot of attention in convex geometry in recent years, see [5, 42, 49, 50] among many others. In fact, Bourgain's estimate (5.17.**B**), which is crucial to all results in Section 5 and after, is only based on properties of log-concave distributions.

We have chosen a very elementary expository style. We shall give fully detailed proofs, except in the first two introductory sections. Most readers will know the contents of these sections and may start by reading Section 4. Some may be happy though to see a gentle introduction to a few points they are less familiar with. Our choice of topics in these two first sections owes a lot to Stein's monograph *Topics in harmonic analysis* [73]. In the next sections, we have chosen to recall and usually follow the methods from the original papers. This leads sometimes to unnecessary complications, but we shall try to give hints to other possibilities.

We believe that most of our notation is standard. We write $\lfloor x \rfloor$, $\lceil x \rceil$ for the *floor* and *ceiling* of a real number x, integers satisfying $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$. We pay a special attention to constants independent of the dimension, for instance those appearing in results about martingale inequalities, Riesz transforms, and try to keep specific letters for these constants throughout the paper, such as c_p, $\rho_p, \ldots$ We use the letter κ to denote a "universal" constant that does not deserve to be remembered. Most often in our Notes, "we" is a two-letter abbreviation for "the author", namely, Stein, Bourgain, Carbery, Müller and several others… We include an index and a notation index.

1. General dimension free inequalities, first part

This first section is devoted to general facts obtained by probabilistic methods, or merely employing the probabilistic language. We begin by reviewing the basic definitions. The functions here are real or complex valued, or they take values in a finite dimensional real or complex linear space F equipped with a norm denoted by $|x|$, for every vector $x \in F$. If Ω is a set, a *σ-field* $\mathcal{G}$ of subsets of Ω is a family of subsets that is closed under countable unions $\bigcup_{n \in \mathbb{N}} A_n$, closed under taking complement $A \mapsto A^c$, and such that $\emptyset \in \mathcal{G}$. If Ω is a set and $\mathcal{G}$ a σ-field of subsets of Ω, one says that a function g on Ω is *$\mathcal{G}$-measurable* when for every Borel subset B of the range space, the inverse image $g^{-1}(B)$, also denoted by

$$\{g \in B\} := \{\omega \in \Omega : g(\omega) \in B\},$$

belongs to the collection $\mathcal{G}$.

A *probability space* $(\Omega, \mathcal{F}, P)$ consists of a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω and a *probability measure* P on $(\Omega, \mathcal{F})$, i.e., a nonnegative σ-additive measure on $(\Omega, \mathcal{F})$ such that $P(\Omega) = 1$. If a function f is $\mathcal{F}$-measurable (we say then that f is a *random variable*) and if f is P-integrable, the *expectation of* f is the integral of f with respect to P, denoted by

$$\mathrm{E}\, f := \int_\Omega f(\omega)\, \mathrm{d}P(\omega)\,.$$

Random variables $(f_i)_{i \in I}$ on $(\Omega, \mathcal{F}, P)$ are *independent* if for any finite subset $J \subset I$, one has $\mathrm{E}\bigl(\prod_{j \in J} h_j \circ f_j\bigr) = \prod_{j \in J} \mathrm{E}(h_j \circ f_j)$ for all nonnegative Borel functions $(h_j)_{j \in J}$ on the range space. The *distribution* of the random variable f with values in $Y = \mathbb{R}$, $\mathbb{C}$ or F is the image probability measure $\mu = f_\# P$, defined on the Borel σ-field $\mathcal{B}_Y$ of Y by letting $\mu(B) = P\bigl(\{f \in B\}\bigr)$ for every $B \in \mathcal{B}_Y$. If μ is a distribution on the Euclidean space F, the *marginals* of μ on the linear subspaces F_0 of F are the distributions μ_{F_0} obtained from μ as images by orthogonal projection, i.e., one sets $\mu_{F_0} = (\pi_0)_\# \mu$ where π_0 is the orthogonal projection from F onto F_0. If f is F-valued and if μ is the distribution of f, then μ_{F_0} is that of $\pi_0 \circ f$.

If $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, the *conditional expectation* on $\mathcal{G}$ of an integrable function f is the unique element $\mathrm{E}(f|\mathcal{G})$ of $L^1(\Omega, \mathcal{F}, P)$ possessing a $\mathcal{G}$-measurable representative g such that

$$\mathrm{E}(\mathbf{1}_A f) = \mathrm{E}(\mathbf{1}_A g) = \mathrm{E}\bigl(\mathbf{1}_A\, \mathrm{E}(f|\mathcal{G})\bigr)$$

for every set $A \in \mathcal{G}$, where $\mathbf{1}_A$ denotes the *indicator function* of A, equal to 1 on A and 0 outside. It follows that

$$\mathrm{E}(hf) = \mathrm{E}\bigl(h\, \mathrm{E}(f|\mathcal{G})\bigr), \ \text{ and actually } \ \mathrm{E}(hf|\mathcal{G}) = h\, \mathrm{E}(f|\mathcal{G})$$

for every bounded $\mathcal{G}$-measurable scalar function h on Ω. When f is scalar and belongs to $L^2(\Omega, \mathcal{F}, P)$, the conditional expectation of f on $\mathcal{G}$ is the orthogonal projection of f onto the closed linear subspace $L^2(\Omega, \mathcal{G}, P)$ of $L^2(\Omega, \mathcal{F}, P)$ formed by $\mathcal{G}$-measurable and square integrable functions. When A is an *atom* of $\mathcal{G}$, i.e., a minimal non-empty element of $\mathcal{G}$, and if $P(A) > 0$, the value of $\mathrm{E}(f|\mathcal{G})$ on the atom A is the average of f on A, hence

$$\mathrm{E}(f|\mathcal{G})(\omega) = \frac{1}{P(A)} \int_A f(\omega')\, \mathrm{d}P(\omega')\,, \quad \omega \in A\,.$$

The conditional expectation operator $\mathrm{E}(\cdot|\mathcal{G})$ is linear and *positive*, i.e., it sends nonnegative functions to nonnegative functions. It follows that we have the inequality $\varphi(\mathrm{E}(f|\mathcal{G})) \leqslant \mathrm{E}(\varphi(f)|\mathcal{G})$ when the real-valued function φ is convex on the range space of f. In particular, one has that $\bigl|\mathrm{E}(f|\mathcal{G})\bigr| \leqslant \mathrm{E}\bigl(|f|\big|\mathcal{G}\bigr)$, and

$$\bigl\|\mathrm{E}(f|\mathcal{G})\bigr\|_{L^p(\Omega,\mathcal{F},P)} \leqslant \bigl\|f\bigr\|_{L^p(\Omega,\mathcal{F},P)}\,, \quad 1 \leqslant p < +\infty\,.$$

The inequality is true also when $p = +\infty$, it is easy and treated separately.

1.1. Doob's maximal inequality

A (discrete time) *martingale* on a probability space $(\Omega, \mathcal{F}, P)$ consists of a *filtration*, i.e., an increasing sequence $(\mathcal{F}_k)_{k\in I}$ of sub-σ-fields of $\mathcal{F}$ indexed by a subset I of $\mathbb{Z}$, and of a sequence $(M_k)_{k\in I}$ of integrable functions on Ω such that for all $k, \ell \in I$ with $k \leqslant \ell$, one has

$$M_k = \mathrm{E}\big(M_\ell \,\big|\, \mathcal{F}_k\big)\,.$$

Notice that each M_k, $k \in I$, is $\mathcal{F}_k$-measurable. If I has a maximal element N, the martingale is completely determined by its last element M_N, since we have then that $M_k = \mathrm{E}\big(M_N \,\big|\, \mathcal{F}_k\big)$ for every $k \in I$. In the case of a finite field $\mathcal{F}_k$, the martingale condition means that the value of M_k on each atom of $\mathcal{F}_k$ is the average of the values of M_ℓ on that atom, for every $\ell \in I$ with $\ell > k$. Clearly, any subsequence $(M_k)_{k\in J}$, $J \subset I$, is a martingale with respect to the filtration $(\mathcal{F}_k)_{k\in J}$.

Let us consider a finite martingale $(M_k)_{k=0}^N$ on $(\Omega, \mathcal{F}, P)$, with respect to a filtration $(\mathcal{F}_k)_{k=0}^N$. This martingale can be real or complex valued, or may take values in a finite dimensional normed space F. We introduce the *maximal process* $(M_k^*)_{k=0}^N$, which is defined by $M_k^* = \max_{0\leqslant j\leqslant k} |M_j|$ for $k = 0, \ldots, N$. In the vector-valued case, $|M_j|$ is the function assigning to each $\omega \in \Omega$ the norm of the vector $M_j(\omega) \in F$. We also employ the lighter notation $\|M\|_p$ for the norm $\|M\|_{L^p}$ of a function M in $L^p(\Omega, \mathcal{F}, P)$, when $1 \leqslant p \leqslant +\infty$.

Theorem 1.1 (Doob's inequality). — *Let $(M_k)_{k=0}^N$ be a martingale (real, complex or vector-valued). For every real number $c > 0$, one has that*

$$c\,P\big(\{M_N^* > c\}\big) \leqslant \int_{\{M_N^* > c\}} |M_N| \,\mathrm{d}P\,.$$

Furthermore, for every $p \in (1, +\infty]$, one has when $M_N \in L^p(\Omega, \mathcal{F}, P)$ that

$$\|M_N^*\|_p \leqslant \frac{p}{p-1}\, \|M_N\|_p\,. \tag{1.1}$$

Proof. — We cut the set $\{M_N^* > c\}$ into disjoint events $A_0, \ldots, A_N$, corresponding to the first time k when $|M_k| > c$. Let $A_0 = \{|M_0| > c\}$ and for each integer k between 1 and N, let A_k denote the set of $\omega \in \Omega$ such that $|M_k(\omega)| > c$ and $M_{k-1}^*(\omega) \leqslant c$. On the set A_k, we have $|M_k| > c$, and A_k belongs to the σ-field $\mathcal{F}_k$ since $|M_k|$ and M_{k-1}^* are $\mathcal{F}_k$-measurable, hence

$$\begin{aligned} c\,P(A_k) &\leqslant \int_{A_k} |M_k| \,\mathrm{d}P = \int_{A_k} \big|\mathrm{E}\big(M_N \,\big|\, \mathcal{F}_k\big)\big| \,\mathrm{d}P \\ &\leqslant \int_{A_k} \mathrm{E}\big(|M_N| \,\big|\, \mathcal{F}_k\big) \,\mathrm{d}P = \int_{A_k} |M_N| \,\mathrm{d}P\,. \end{aligned}$$

On the other hand, we see that $\{M_N^* > c\} = \bigcup_{k=0}^N A_k$, union of pairwise disjoint sets, therefore

$$
\begin{aligned}
c\,P\bigl(\{M_N^* > c\}\bigr) &= \sum_{k=0}^N c\,P(A_k) \\
&\leqslant \sum_{k=0}^N \int_{A_k} |M_N| \,\mathrm{d}P = \int_{\{M_N^* > c\}} |M_N| \,\mathrm{d}P\,. \qquad (1.2)
\end{aligned}
$$

The result for L^p when $1 < p < +\infty$ follows. For each value $t > 0$, we apply (1.2) with $c = t$, we use Fubini's theorem and Hölder's inequality, obtaining thus

$$
\begin{aligned}
\mathrm{E}\bigl((M_N^*)^p\bigr) &= \mathrm{E}\Bigl(\int_0^{M_N^*} pt^{p-1}\,\mathrm{d}t\Bigr) = \int_0^{+\infty} pt^{p-1} P\bigl(\{M_N^* > t\}\bigr)\,\mathrm{d}t \\
&\leqslant \int_0^{+\infty} pt^{p-2}\,\mathrm{E}\bigl(\mathbf{1}_{\{M_N^* > t\}}\,|M_N|\bigr)\,\mathrm{d}t = \mathrm{E}\Bigl(\frac{p}{p-1}(M_N^*)^{p-1}\,|M_N|\Bigr) \\
&\leqslant \frac{p}{p-1}\bigl(\mathrm{E}\bigl((M_N^*)^p\bigr)\bigr)^{1-1/p}\bigl(\mathrm{E}\bigl(|M_N|^p\bigr)\bigr)^{1/p},
\end{aligned}
$$

hence $\|M_N^*\|_p \leqslant p(p-1)^{-1}\|M_N\|_p$. The case $p = +\infty$ is straightforward. □

Remark 1.2. — In some contexts, it is useful to observe that the notion of conditional expectation on a sub-σ-field $\mathcal{F}_0$ of $\mathcal{F}$ remains well defined if we have a possibly infinite measure μ on $(\Omega, \mathcal{F})$, but which is *σ-finite* on $\mathcal{F}_0$, in other words, if Ω can be split in countably many sets A_i in $\mathcal{F}_0$ such that $\mu(A_i) < +\infty$ for each i. If this condition is fulfilled by μ and by the smallest sub-σ-field $\mathcal{F}_0$ of a filtration $(\mathcal{F}_k)_{k=0}^N$, we can also speak about martingales with respect to the infinite measure μ, and Theorem 1.1 remains true with the same proof, simply replacing the words “probability of an event” by “measure of a set”.

We can always consider the orthogonal projection π_0 from $L^2(\Omega, \mathcal{F}, \mu)$ onto $L^2(\Omega, \mathcal{F}_0, \mu)$, but $L^2(\Omega, \mathcal{F}_0, \mu) = \{0\}$ when $\mathcal{F}_0$ does not contain any set with finite positive measure. On the other hand, when $A \in \mathcal{F}_0$ has finite measure, the formula $\pi_0(\mathbf{1}_A f) = \mathbf{1}_A \pi_0(f)$ allows one to work on A as in the case of a probability measure.

1.2. The Hopf maximal inequality

We are given a measure space (X, Σ, μ) and a linear operator T from $L^1(X, \Sigma, \mu)$ to itself. We shall only consider σ-finite measures throughout these Notes, and we work in this section with the space $L^1(X, \Sigma, \mu)$ of real-valued functions. We assume that T is positive and nonexpansive, which means that for every nonnegative function $g \in L^1(X, \Sigma, \mu)$, Tg is nonnegative, and that the norm of T is $\leqslant 1$. We can sum up these two properties by saying that when $g \geqslant 0$, then $Tg \geqslant 0$ and $\int_X Tg \,\mathrm{d}\mu \leqslant \int_X g \,\mathrm{d}\mu$.

Let us consider a function f in $L^1(X, \Sigma, \mu)$, and for every integer $k \geqslant 0$ let

$$S_k(f) = f + Tf + T^2 f + \cdots + T^k f\,.$$

If N is a nonnegative integer, we set $S_N^*(f) = \max\{S_j(f) : 0 \leqslant j \leqslant N\}$.

Lemma 1.3 (Hopf). — *With the preceding notation, we have for every function $f \in L^1(X, \Sigma, \mu)$ and $N \geqslant 0$ that*

$$\int_{\{S_N^*(f)>0\}} f \,\mathrm{d}\mu \geqslant 0\,.$$

Proof, after Garsia [38]. — Let us simply write S_k for $S_k(f)$ and S^* for $S_N^*(f)$. By definition, we have $S_k \leqslant S^*$ for each integer $k \leqslant N$; since T is positive and linear, we see that

$$TS_k \leqslant TS^*, \quad \text{and} \quad S_{k+1} = f + TS_k \leqslant f + TS^*.$$

In order to get for $S_0 = f$ an inequality similar to $S_{k+1} \leqslant f + TS^*$, we replace S^* by its nonnegative part $S^{*+} = \max(S^*, 0) \geqslant S^*$. Using positivity, we can write

$$S_0 = f \leqslant f + T(S^{*+}), \quad S_{k+1} \leqslant f + TS^* \leqslant f + T(S^{*+})\,.$$

Taking the supremum of S_ks for $0 \leqslant k \leqslant N$, we obtain the crucial inequality

$$S^* \leqslant f + T(S^{*+}), \text{ or } f \geqslant S^* - T(S^{*+})\,. \tag{1.3}$$

Since T is positive and nonexpansive on $L^1(X, \Sigma, \mu)$, we have

$$\int_{\{S^*>0\}} S^* \,\mathrm{d}\mu = \int_X S^{*+} \,\mathrm{d}\mu \geqslant \int_X T(S^{*+}) \,\mathrm{d}\mu \geqslant \int_{\{S^*>0\}} T(S^{*+}) \,\mathrm{d}\mu\,,$$

and the result follows by (1.3), because

$$\int_{\{S^*>0\}} f \,\mathrm{d}\mu \geqslant \int_{\{S^*>0\}} \bigl(S^* - T(S^{*+})\bigr) \,\mathrm{d}\mu \geqslant 0\,. \qquad \square$$

We go on with the same linear operator T. For each integer $k \geqslant 0$, let us define the kth average operator $a_k = a_{k,T}$ associated to T by writing

$$a_k(f) = \frac{f + Tf + \cdots + T^k f}{k+1} = \frac{S_k(f)}{k+1}, \quad f \in L^1(X, \Sigma, \mu)\,.$$

For each integer $N \geqslant 0$, let $a_N^*(f) = \max\{a_j(f) : 0 \leqslant j \leqslant N\}$. It is clear that the set $\{a_N^*(f) > 0\}$ coincides with the set $\{S_N^*(f) > 0\}$ which appears in Lemma 1.3.

We continue in a simplified setting where we also assume that μ is finite and that $T\mathbf{1} = \mathbf{1}$. It follows that $a_k(\mathbf{1}) = \mathbf{1}$ for each $k \geqslant 0$ and $a_k(f-c) = a_k(f) - c$ for every $c \in \mathbb{R}$, thus $a_N^*(f-c) = a_N^*(f) - c$. Lemma 1.3 yields

$$\int_{\{a_N^*(f-c)>0\}} (f-c)\,\mathrm{d}\mu = \int_{\{S_N^*(f-c)>0\}} (f-c)\,\mathrm{d}\mu \geqslant 0\,.$$

Equivalently, for every $f \in L^1(X, \Sigma, \mu)$, we have

$$c\mu\big(\{a_N^*(f) > c\}\big) \leqslant \int_{\{a_N^*(f)>c\}} f\,\mathrm{d}\mu, \qquad N \geqslant 0,\ \ c \in \mathbb{R}\,. \tag{1.4}$$

This inequality makes sense also when μ is infinite. Note that if $c < 0$ and if μ is infinite, then $\mu\big(\{f \leqslant c\}\big) \leqslant \mu\big(\{|f| \geqslant |c|\}\big) < +\infty$, the measure of $\{a_N^*(f) > c\}$ is thus infinite and (1.4) is trivial. We can extend (1.4) to an infinite μ if there exists an increasing sequence $(C_\ell)_{\ell\geqslant 0}$ of subsets of X with finite measure such that

$$T^j\mathbf{1}_{C_\ell} \leqslant \mathbf{1} \text{ for all } j, \ell \geqslant 0\,,$$
$$T^j\mathbf{1}_{C_\ell} \underset{\ell\to+\infty}{\longrightarrow} \mathbf{1} \text{ pointwise for each } j \geqslant 0\,. \tag{1.5}$$

Let $c, \varepsilon > 0$ and abbreviate $\{a_N^*(f) > t\}$ as $D(t)$, for $t > 0$. Choose $c' > c$ such that $\int_{D(c)\setminus D(c')} \big(1 + |f|\big)\,\mathrm{d}\mu < \varepsilon$. Let $E(c', \ell) = \big\{\min_{0\leqslant j\leqslant N} T^j\mathbf{1}_{C_\ell} \leqslant c/c'\big\}$, choose a large ℓ such that $\mu(D(c') \setminus C_\ell) < \varepsilon$ and $\int_{E(c',\ell)} |f|\,\mathrm{d}\mu < \varepsilon$, then observe that

$$D(c') \subset \big\{a_N^*\big(f - c'\mathbf{1}_{C_\ell}\big) > 0\big\} \subset D(c) \cup E(c', \ell)$$

and apply Lemma 1.3 to $f - c'\mathbf{1}_{C_\ell}$. The assumption (1.5) is fulfilled when T is an operator of convolution with a probability measure on $\mathbb{R}^n$, acting on $L^1(\mathbb{R}^n)$.

For each function $f \in L^1(X, \Sigma, \mu)$, let us define

$$a^*(f) = \sup_{k\geqslant 0} a_k(f) = \sup_{k\geqslant 0} \frac{f + Tf + \cdots + T^k f}{k+1} = \lim_{N\to+\infty} a_N^*(f)\,.$$

The set $\{a^*(f) > c\}$ is the increasing union of the sets $\{a_N^*(f) > c\}$, $N \geqslant 0$, so, passing to the limit by dominated convergence, we deduce from (1.4) that

$$c\mu\big(\{a^*(f) > c\}\big) \leqslant \int_{\{a^*(f)>c\}} f\,\mathrm{d}\mu\,, \qquad c \in \mathbb{R}\,. \tag{1.6}$$

Following [29, Lemma VIII.6.7], we now get a variant of (1.6). Assume $c > 0$ in what follows. We define f_c by $f_c(x) = f(x)$ when $f(x) > c$ and $f_c(x) = 0$ otherwise, for $x \in X$. Note that $f \leqslant f_c + c$. If $a^*(f_c)(x) \leqslant c$, then $f_c(x) = a_0(f_c)(x) \leqslant c$ thus $f_c(x) = 0$ by construction. Hence f_c vanishes outside

$\{a^*(f_c) > c\}$ and

$$\int_{\{f>c\}} f \, \mathrm{d}\mu = \int_X f_c \, \mathrm{d}\mu = \int_{\{a^*(f_c)>c\}} f_c \, \mathrm{d}\mu \, .$$

Using the positivity of T and of a_k for each $k \geqslant 0$, we infer from $f_c \geqslant f - c$ that $a^*(f_c) \geqslant a^*(f-c) = a^*(f) - c$. Then, by (1.6) for f_c and since $c > 0$, we get

$$\int_{\{f>c\}} f \, \mathrm{d}\mu \geqslant c\mu\bigl(\{a^*(f_c) > c\}\bigr) \geqslant c\mu\bigl(\{a^*(f) - c > c\}\bigr) .$$

Finally, we have obtained

$$c\mu\bigl(\{a^*(f) > 2c\}\bigr) \leqslant \int_{\{f>c\}} f \, \mathrm{d}\mu , \quad c > 0 \, . \tag{1.7}$$

Let us define $A^*(f) = \sup_{k \geqslant 0} |a_k(f)| = \max(a^*(f), a^*(-f))$. Still assuming $c > 0$, we decompose the set $\{A^*(f) > c\} = \{a^*(f) > c\} \cup \{a^*(-f) > c\}$ into three disjoint pieces, $E_0 = \{a^*(f) > c \, , \, a^*(-f) \leqslant c\}$, $E_1 = \{a^*(f) > c \, , \, a^*(-f) > c\}$, and $E_2 = \{a^*(f) \leqslant c \, , \, a^*(-f) > c\}$. According to (1.6) we have

$$\begin{aligned} c\mu\bigl(\{A^*(f) > c\}\bigr) &\leqslant c\mu\bigl(\{a^*(f) > c\}\bigr) + c\mu\bigl(\{a^*(-f) > c\}\bigr) \\ &\leqslant \int_{\{a^*(f)>c\}} f \, \mathrm{d}\mu + \int_{\{a^*(-f)>c\}} (-f) \, \mathrm{d}\mu \\ &= \int_{E_0} f \, \mathrm{d}\mu + \int_{E_2} (-f) \, \mathrm{d}\mu \leqslant \int_{\{A^*(f)>c\}} |f| \, \mathrm{d}\mu \, , \end{aligned} \tag{1.8}$$

noting that the integrals of f and $-f$ on E_1 cancel each other. In the same way, we can get from (1.7) the variant form $c\mu\bigl(\{A^* f > 2c\}\bigr) \leqslant \int_{\{|f|>c\}} |f| \, \mathrm{d}\mu$. Notice that the latter "variant form" will be inherited by any linear operator S satisfying that $|S^k f| \leqslant T^k |f|$ for every $k \geqslant 0$, and see Remark 1.5.

When $1 < p < +\infty$, we deduce from (1.8) the L^p inequality

$$\Bigl\| \sup_{k \geqslant 0} \frac{|f + Tf + \cdots + T^k f|}{k+1} \Bigr\|_p \leqslant \frac{p}{p-1} \, \|f\|_p \tag{1.9}$$

as we have seen with Doob's inequality (1.1), while the variant form leads to a constant $2\bigl(p/(p-1)\bigr)^{1/p}$ which is larger than $p/(p-1)$ for every $p > 1$.

Let now $(T_t)_{t \geqslant 0}$ be a semi-group of linear operators on $L^1(X, \Sigma, \mu)$, i.e., operators satisfying $T_{s+t} = T_s \circ T_t$ for all $s, t \geqslant 0$. We assume in addition that each T_t is positive and nonexpansive on L^1. We also assume that T_t is actually defined on $L^1(X, \Sigma, \mu) + L^\infty(X, \Sigma, \mu)$ and that $T_t \mathbf{1} = \mathbf{1}$ for every

$t \geqslant 0$. This implies that T_t is continuous from L^∞ to L^∞, with norm 1. By interpolation, we get that the norm $\|T_t\|_{p\to p}$ on L^p, for $p \in [1, +\infty]$, is $\leqslant 1$. Suppose that the semi-group is *strongly continuous* on L^1, which means that $\|f - T_t f\|_1 \to 0$ as $t \to 0$, for each $f \in L^1$. Combined with our assumptions, this fact implies that $t \mapsto T_t f$ is continuous from $[0, +\infty)$ to L^p for every function $f \in L^p$ and $1 \leqslant p < +\infty$. For $f \in L^p(X, \Sigma, \mu)$ let

$$a^* f = \sup_{t>0} \frac{1}{t} \int_0^t T_s f \, \mathrm{d}s \,, \quad A^* f = \sup_{t>0} \Bigl| \frac{1}{t} \int_0^t T_s f \, \mathrm{d}s \Bigr| ,$$

where the supremum can be defined as an *essential supremum*, see the discussion in Section 3.3. Yet, for the main examples of semi-groups of interest in these Notes, namely, the Gaussian semi-group or the Poisson semi-group on $\mathbb{R}^n$, the function $t \mapsto (T_t f)(x)$ is continuous on $(0, +\infty)$ for each fixed $x \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$, so $a^* f$ and $A^* f$ have then a well defined pointwise value, possibly $+\infty$.

Suppose now that the measure μ is finite (or that a continuous analog of (1.5) is satisfied). When $1 < p < +\infty$, we obtain from (1.9) the L^p inequality

$$\bigl\| A^* f \bigr\|_p \leqslant \frac{p}{p-1} \, \|f\|_p \,. \tag{1.10}$$

If T_t is positive and $T_t \mathbf{1} = \mathbf{1}$, the case $p = +\infty$ in (1.10) is clear.

Since $t \mapsto a(t, f) := t^{-1} \int_0^t T_s f \, \mathrm{d}s$ is continuous from $(0, +\infty)$ to L^p, we can reach any $a(t, f)$, $t > 0$, as an almost everywhere limit of a sequence $(a(t_j, f))_{j \geqslant 0}$, where each t_j is rational and > 0. It follows that $A^* f$ can be defined as the supremum of $|a(t, f)|$ for $t > 0$ rational. For all integers $k \geqslant 0$ and $n \geqslant 1$, observe that

$$\begin{aligned} a\Bigl(\frac{k+1}{n}, f \Bigr) &= \frac{n}{k+1} \sum_{i=0}^{k} \int_{i/n}^{(i+1)/n} T_s f \, \mathrm{d}s \\ &= \Bigl(\frac{\sum_{i=0}^k T_{i/n}}{k+1} \Bigr) \Bigl(n \int_0^{1/n} T_s f \, \mathrm{d}s \Bigr) . \end{aligned}$$

Letting $f_n = n \int_0^{1/n} T_s f \, \mathrm{d}s = a(1/n, f)$ and $T = T_{1/n}$ we see that

$$a\Bigl(\frac{k+1}{n}, f \Bigr) = \frac{f_n + T f_n + \cdots + T^k f_n}{k+1} = a_{k,T}(f_n) \,.$$

Let Q_n be the set of positive multiples of $1/n$. By (1.9) applied to $T_{1/n}$ and f_n, and because $a(1/n, \cdot)$ is an average of operators with norm $\leqslant 1$ on L^p, we get

$$\Bigl\| \sup_{t \in Q_n} \bigl| a(t, f) \bigr| \Bigr\|_p = \Bigl\| \sup_{j \geqslant 1} \bigl| a(j/n, f) \bigr| \Bigr\|_p \leqslant \frac{p}{p-1} \bigl\| a(1/n, f) \bigr\|_p \leqslant \frac{p}{p-1} \, \|f\|_p \,.$$

We see that $Q_m \subset Q_{mn}$ for all $m, n \geqslant 1$. The sets Q_n corresponding to $n = \ell!$ for $\ell \geqslant 1$ are increasing with ℓ, and they cover the set of positive rationals. We can conclude by noticing that $A^* f$ is the increasing limit of $\sup_{t \in Q_{\ell!}} |a(t,f)|$.

Applying (1.6) we can obtain a version of Hopf's maximal inequality as

$$c\,\mu\big(\{a^* f > c\}\big) \leqslant \int_{\{a^* f > c\}} f \,\mathrm{d}\mu, \quad c \in \mathbb{R},\ f \in L^1(X, \Sigma, \mu),$$

and from (1.8), we have $c\,\mu\big(\{A^* f > c\}\big) \leqslant \int_{\{A^* f > c\}} |f| \,\mathrm{d}\mu$ when $c > 0$.

By the preceding remark about the sets $Q_{\ell!}$ it is enough to prove the inequality with $a_n^* := \sup_{t \in Q_n} a(t,f) = \sup_{k \geqslant 0} a_{k, T_{1/n}}(f_n)$ replacing $a^* f$, with $n \geqslant 1$ arbitrary and with a vanishing error term. By (1.6) we have $c\mu\big(\{a_n^* > c\}\big) \leqslant \int_{\{a_n^* > c\}} f_n$. Since the semi-group $(T_t)_{t \geqslant 0}$ is strongly continuous, we know that $\|f_n - f\|_1 \to 0$ and we can conclude because $\int_{\{a_n^* f > c\}} f_n \,\mathrm{d}\mu - \int_{\{a_n^* f > c\}} f \,\mathrm{d}\mu$ tends to zero.

We have made here assumptions more restrictive than those of the Hopf–Dunford–Schwartz statement [29, Chap. VIII] praised by Stein [73], which does not assume T_t positive, nor μ finite and $T_t \mathbf{1} = \mathbf{1}$. Theorem 1.4 below contains Lemma VIII.7.6 and Theorem VIII.7.7 from [29] in a slightly simplified form (the set U there has only one element here). The semi-group $(T_t)_{t \geqslant 0}$ on $L^1(X, \Sigma, \mu)$ is said to be *strongly measurable* if, for each f in $L^1(X, \Sigma, \mu)$, the mapping $t \mapsto T_t f \in L^1(X, \Sigma, \mu)$ is measurable with respect to the Lebesgue measure on $[0, +\infty)$.

Theorem 1.4 ([29]). — *Let $(T_t)_{t \geqslant 0}$ be a strongly measurable semi-group on the space $L^1(X, \Sigma, \mu)$, with $\|T_t\|_{1 \to 1} \leqslant 1$ and $\|T_t\|_{\infty \to \infty} \leqslant 1$ for all $t \geqslant 0$. For every function $f \in L^1(X, \Sigma, \mu)$ and every $c > 0$ one has*

$$c\mu\big(\{A^* f > 2c\}\big) \leqslant \int_{\{|f| > c\}} |f| \,\mathrm{d}\mu\,.$$

If $1 < p < +\infty$ and $f \in L^p(X, \Sigma, \mu)$, the function $A^* f$ is in $L^p(X, \Sigma, \mu)$ and

$$\|A^* f\|_p \leqslant 2 \Big(\frac{p}{p-1} \Big)^{1/p} \|f\|_p\,.$$

Remark 1.5. — In [29, Section VIII.6], the authors consider first a linear operator T acting from L^1 to L^1 with norm $\leqslant 1$ and also acting from L^∞ to L^∞ with norm $\leqslant 1$; in this discrete parameter case, they study

$$A_T^* f = \sup_{n \geqslant 1} \frac{1}{n} \Big| \sum_{k=0}^{n-1} T^k f \Big|,$$

before going to the continuous setting of a semi-group $(T_t)_{t \geqslant 0}$. One of the steps in their proof consists in introducing a *positive* operator P which acts from L^1 to L^1 and from L^∞ to L^∞, with norm $\leqslant 1$ in both cases, and such that

$$\forall\, n \geqslant 0\,, \quad |T^n f| \leqslant P^n(|f|)\,, \quad f \in L^\infty \cap L^1\,.$$

This step is easy when the measure is the uniform measure on a finite set. The assumptions imply that T is given by a matrix $(t_{i,j})$ such that the sum of absolute values in each row and in each column is $\leqslant 1$. It is then enough to take P to be the matrix with entries $p_{i,j}$ equal to the absolute values $|t_{i,j}|$ of the entries of T.

1.3. From martingales to semi-groups, via an argument of Rota

The arguments in this section, due to Rota [67], are presented in a more sophisticated manner in Stein's book [73, Chap. 4, §4]. We consider a Markov chain $X_0, \ldots, X_N$ with transition matrix P, assumed to be *symmetric*. We suppose for simplicity that the state space $\mathcal{E}$ is finite, with cardinality Z. For every $e_0 \in \mathcal{E}$, we have

$$\sum_{e \in \mathcal{E}} P(e_0, e) = 1\,.$$

For each integer k such that $0 \leqslant k < N$ and for all $e_0, e_1 \in \mathcal{E}$, the probability that $X_{k+1} = e_1$ knowing that $X_k = e_0$ is given by the entry $P(e_0, e_1)$ of the matrix P. This statement introduces implicitly the *Markov property*, which loosely speaking, prescribes that what happens after time k depends only on what we know at the instant k, regardless of the past positions at times $j < k$. For each integer $j \geqslant 2$, the power P^j of the matrix P controls the moves in j successive steps, the entry $P^j(e_0, e)$ giving the probability of moving from e_0 to e in exactly j steps. If Q is a transition matrix and f a scalar function on $\mathcal{E}$, we introduce the notation

$$(Qf)(x) = \sum_{y \in \mathcal{E}} Q(x,y) f(y)\,, \quad x \in \mathcal{E}\,.$$

When applied to a power P^j, the notation $P^j f$ corresponds to the semi-group notation $P_t f$, with $j \in \mathbb{N}$ replacing $t \geqslant 0$. If the transition matrix Q is symmetric, hence bistochastic, and if $1 \leqslant p \leqslant +\infty$, convexity implies that $\|Qf\|_p \leqslant \|f\|_p$ with respect to the uniform measure on $\mathcal{E}$. Let f be a function on $\mathcal{E}$ and let j, k be two nonnegative integers with $j+k \leqslant N$. If we fix $x_0 \in \mathcal{E}$, the mean of the values $f(y)$, when the chain makes j steps from the position x_0 at time k to the position y at time $k+j$, is equal to $(P^j f)(x_0)$.

A simple but important symmetric example is that of the *Bernoulli random walk* on $\mathbb{Z}$, where for all $x, y \in \mathbb{Z}$ we have $P(x,y) = 1/2$ when $|x-y| = 1$, and $P(x,y) = 0$ otherwise. This is not a finite example, but it can be "approximated" by considering on the finite set $\mathcal{E}_N = \{-N, \ldots, N\}$, for N large, the modified matrix P_N which still has $P_N(x,y) = 1/2$ when $|x-y| = 1$, for $x, y \in \mathcal{E}_N$, but where $P_N(N,N) = P_N(-N,-N) = 1/2$. One can also consider the Bernoulli random walk on $\mathbb{Z}^n$, for which $P(x,y) = 2^{-n}$ when $|x_i - y_i| = 1$ for all coordinates x_i, y_i, $i = 1, \ldots, n$, of the points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{Z}^n$.

Assume that the distribution of the initial position X_0 is uniform, that is to say, that $P(X_0 = e_0) = 1/Z$ for every $e_0 \in \mathcal{E}$. Then for each $e_1 \in \mathcal{E}$, we have

$$\begin{aligned} P(X_1 = e_1) &= \sum_{e \in \mathcal{E}} P(X_0 = e\,,\, X_1 = e_1) \\ &= \sum_{e \in \mathcal{E}} \frac{1}{Z} P(e, e_1) = \frac{1}{Z} \sum_{e \in \mathcal{E}} P(e_1, e) = \frac{1}{Z}\,, \end{aligned}$$

since the matrix P is symmetric. The distribution of the position X_1 of the chain at time $i = 1$ remains the uniform distribution, as well as that of $X_2, \ldots, X_N$. The uniform distribution is *invariant* under the action of P. Recalling the meaning of the transition matrix in terms of conditional probability, using Markov's property and letting $A_{N-1} = \{X_0 = e_0\,, X_1 = e_1\,, \ldots\,, X_{N-1} = e_{N-1}\}$, we have that

$$\begin{aligned} E &:= P(X_0 = e_0\,, X_1 = e_1\,, \ldots\,, X_N = e_N) \\ &= P(A_{N-1}\,, X_N = e_N) = P(A_{N-1})\, P(X_N = e_N \mid A_{N-1}) \\ &= P(A_{N-1})\, P(X_N = e_N \mid X_{N-1} = e_{N-1}) = P(A_{N-1})\, P(e_{N-1}, e_N)\,. \end{aligned}$$

We may go on, and by the symmetry property of the matrix we get

$$\begin{aligned} E = \cdots &= \frac{1}{Z} P(e_0, e_1) P(e_1, e_2) \ldots P(e_{N-2}, e_{N-1}) P(e_{N-1}, e_N) \\ &= \frac{1}{Z} P(e_N, e_{N-1}) P(e_{N-1}, e_{N-2}) \ldots P(e_2, e_1) P(e_1, e_0) \\ &= P(X_N = e_0\,, X_{N-1} = e_1\,, \ldots\,, X_1 = e_{N-1}\,, X_0 = e_N)\,. \end{aligned}$$

We see that the "reversed" chain has the same behavior as that of the original chain. Since the matrix is symmetric, we certainly have, whatever the distribution of X_0 can be, that the probability to arrive at a fixed y_0 at time N, starting from an arbitrary point x at time $k = N - j$, is given by $P^j(x, y_0) = P^j(y_0, x)$, the probability of moving from y_0 at time 0 to x at time j. But under the invariant distribution, we can say more: if g is a function on $\mathcal{E}$, the *mean* of the values $g(x)$ on all trajectories starting from x at time k and arriving at y_0 at time N is equal to $(P^j g)(y_0)$. Clearly, this statement is not true in general, since this mean value depends on the distribution of X_k, hence on that of X_0. Under the uniform distribution, we see by reversing the chain that the preceding mean is equal to the mean of

$g(x)$, when starting from y_0 at time 0 and arriving at x at time j, namely, this mean is equal to $(P^j g)(y_0)$.

Let us describe the situation more formally. Let $\Omega = \mathcal{E}^{N+1}$ denote the space of all possible trajectories $(e_0, e_1, \ldots, e_N) \in \mathcal{E}^{N+1}$ for the chain. On this model space Ω and for $k = 0, \ldots, N$, we set

$$X_k(\omega) = \omega_k \in \mathcal{E}\,, \qquad \omega = (\omega_0, \ldots, \omega_N) \in \mathcal{E}^{N+1}.$$

It is easy to determine the probability measure $\mathbf{P}$ on Ω that corresponds to the behavior of our Markov chain under the invariant distribution. For each singleton $\{\omega\} = \{(\omega_0, \ldots, \omega_N)\}$ in $\mathcal{P}(\Omega)$, we must have that

$$\mathbf{P}\big(\{(\omega_0, \ldots, \omega_N)\}\big) = \frac{1}{Z} P(\omega_0, \omega_1) P(\omega_1, \omega_2) \ldots P(\omega_{N-1}, \omega_N)\,.$$

For $k = 0, \ldots, N$, let $\mathcal{F}_k$ denote the finite field of subsets of Ω whose atoms A are of the following form: to any $e_0, \ldots, e_k$ fixed in $\mathcal{E}$ we associate $A_{\mathbf{e}} \in \mathcal{F}_k$ defined by

$$A = A_{\mathbf{e}} = \{\omega = (\omega_0, \ldots, \omega_N) : \omega_j = e_j,\ 0 \leqslant j \leqslant k\} \in \mathcal{F}_k, \quad \mathbf{e} = (e_0, \ldots, e_k)\,.$$

This $\mathcal{F}_k$ is the "field of past events" at time k, it increases with k. Let $\mathcal{G}_k$ denote the field of events occurring precisely at time k, whose atoms B are of the form

$$B = \{\omega = (\omega_0, \ldots, \omega_N) : \omega_k = e_k\} \in \mathcal{G}_k\,.$$

Clearly, we have $\mathcal{G}_k \subset \mathcal{F}_k$. A function on Ω which is $\mathcal{G}_k$-measurable depends only on the coordinate ω_k, and is thus of the form $g(X_k)$ with g a function on $\mathcal{E}$. If f is a function on $\mathcal{E}$, the Markov property yields

$$\mathrm{E}(f(X_N)|\mathcal{F}_k) = \mathrm{E}(f(X_N)|\mathcal{G}_k) = g(X_k)$$

where $g(x) = (P^{N-k} f)(x)$ for every $x \in \mathcal{E}$. The preliminary discussion shows that

$$(P^{N-k} f)(X_k) = \mathrm{E}(f(X_N)|\mathcal{F}_k)\,, \quad (P^{N-k} g)(X_N) = \mathrm{E}(g(X_k)|\mathcal{G}_N)\,. \tag{1.11}$$

We introduce the "canonical" martingale associated to a function f on $\mathcal{E}$, by letting

$$M_i = (P^{N-i} f)(X_i) = \mathrm{E}(f(X_N)|\mathcal{F}_i)\,, \quad 0 \leqslant i \leqslant N. \tag{1.12}$$

We see that in (1.11), one occurrence of P^{N-k} relates to the expectation at time $k < N$ of future positions $f(X_N)$, while the other is about expectation at time N of past positions $g(X_k)$. Combining the two equalities in (1.11) in a "back and forth" move, by taking $g = P^j f$ and $j = N - k$, we conclude that

$$(P^{2j} f)(X_N) = \mathrm{E}\big(M_{N-j}\,\big|\,\mathcal{G}_N\big)\,. \tag{1.13}$$

Since the conditional expectation operator on $\mathcal{G}_N$ is positive, we see that for every $j = N - k = 0, \ldots, N$, we have the inequality

$$\max_{0 \leqslant j \leqslant N} |(P^{2j} f)(X_N)| = \max_{0 \leqslant j \leqslant N} \bigl| \mathrm{E}\bigl(M_{N-j} \,\big|\, \mathcal{G}_N\bigr)\bigr| \leqslant \mathrm{E}\bigl(\max_{0 \leqslant i \leqslant N} |M_i| \,\big|\, \mathcal{G}_N\bigr).$$

It implies when $1 < p \leqslant +\infty$, according to Doob's inequality (1.1) and to the non-expansivity on L^p of conditional expectations, the chain of inequalities

$$\Bigl\| \max_{0 \leqslant j \leqslant N} |(P^{2j} f)(X_N)| \Bigr\|_p \leqslant \Bigl\| \mathrm{E}\bigl(\max_{0 \leqslant i \leqslant N} |M_i| \,\big|\, \mathcal{G}_N\bigr) \Bigr\|_p$$
$$\leqslant \Bigl\| \max_{0 \leqslant i \leqslant N} |M_i| \Bigr\|_p \leqslant \frac{p}{p-1} \|M_N\|_p = \frac{p}{p-1} \|f(X_N)\|_p . \quad (1.14)$$

We could recover the odd indices $2j+1$ by applying the latter inequality to Pf instead of f and using $\|Pf\|_p \leqslant \|f\|_p$, to the cost of an extra factor 2.

Estimating the maximal function of semi-groups is a central theme in [73]. The discrete case of (1.14) was obtained by Stein in the short article [71], independently of Rota [67], by methods preluding those of [73]. Theorem 1 in [71] applies to self-adjoint operators P on $L^2(X, \Sigma, \mu)$ satisfying also $\|P\|_{1 \to 1} \leqslant 1$ and $\|P\|_{\infty \to \infty} \leqslant 1$.

One can play the same game with convex functions other than the supremum function on $\mathbb{R}^{N+1}$. For example, let us begin with the convexity inequality

$$\Bigl(\sum_{0 \leqslant i \leqslant N} |\mathrm{E}(f_i | \mathcal{G})|^2 \Bigr)^{1/2} \leqslant \mathrm{E}\Bigl(\bigl(\sum_{0 \leqslant i \leqslant N} |f_i|^2 \bigr)^{1/2} \big| \mathcal{G} \Bigr),$$

and make use of the Burkholder–Gundy inequalities of Theorem 1.6, in order to obtain, when $0 \leqslant j_0 < j_1 < \ldots < j_r \leqslant N$, $1 < p < +\infty$, and with respect to the invariant measure μ, the inequality

$$\Bigl\| \Bigl(\sum_{k=1}^{r} |(P^{2j_k} f - P^{2j_{k-1}} f)|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mu)} \leqslant c_p \|f\|_{L^p(\mu)} . \quad (1.15)$$

Indeed, we have seen in (1.13) that $(P^{2j_k} f)(X_N)$ is the projection on $\mathcal{G}_N$ of the member $M_{N-j_k} = \mathrm{E}(f(X_N) | \mathcal{F}_{N-j_k})$ of the martingale $(M_j)_{j=0}^N$ in (1.12). Then $L_i = M_{N-j_{r-i}}$, $i = 0, \ldots, r$ is another martingale, and

$$(P^{2j_{k-1}} f)(X_N) - (P^{2j_k} f)(X_N) = \mathrm{E}(M_{N-j_{k-1}} - M_{N-j_k} | \mathcal{G}_N)$$

appears as projection on $\mathcal{G}_N$ of the *martingale difference* $d_{r-k+1} = L_{r-k+1} - L_{r-k}$ (see Section 1.4.2) when $1 \leqslant k \leqslant r$. This principle can be applied for bounding diverse convex functions of a semi-group, by considering them as projections of corresponding functions of a martingale, for which we may have an "L^p inequality".

Let us come back to (1.14). Since the distribution of X_N is uniform, we can restate (1.14) when $1 < p \leqslant +\infty$ as

$$\Big(\frac{1}{Z} \sum_{x \in \mathcal{E}} \max_{0 \leqslant j \leqslant N} |(P^{2j} f)(x)|^p \Big)^{1/p} \leqslant \frac{p}{p-1} \Big(\frac{1}{Z} \sum_{x \in \mathcal{E}} |f(x)|^p \Big)^{1/p},$$

or else, changing the normalization and letting N tend to infinity, we obtain

$$\Big(\sum_{x \in \mathcal{E}} \sup_{j \geqslant 0} |(P^{2j} f)(x)|^p \Big)^{1/p} \leqslant \frac{p}{p-1} \Big(\sum_{x \in \mathcal{E}} |f(x)|^p \Big)^{1/p}. \tag{1.16}$$

We can also write

$$\Big(\sum_{x \in \mathcal{E}} \sup_{j \geqslant 0} |(P^{j} f)(x)|^p \Big)^{1/p} \leqslant \frac{2p}{p-1} \Big(\sum_{x \in \mathcal{E}} |f(x)|^p \Big)^{1/p}.$$

If we want to deal with a countably infinite state space $\mathcal{E}$ such as $\mathcal{E} = \mathbb{Z}^n$, we may accept (as Stein [73] does) to work with an *infinite invariant measure*, uniform on $\mathcal{E}$, that gives measure 1 to each singleton $\{e\}$, $e \in \mathcal{E}$. We then obtain the same maximal inequality (1.16), applying Remark 1.2. If we do not accept an "infinite probability", we may, for example with the Bernoulli random walk, work with "boxes" finite but large enough: if f is finitely supported in $\mathbb{Z}^n$ and if N is fixed, we can find a finite box B in $\mathcal{E}$, so big that $P^j f$ vanishes outside B for every $j \leqslant 2N$. Changing the Bernoulli transition matrix $P(x, y)$ at the boundary of B, in order to force the Markov chain to remain inside, we are back to the finite case.

1.4. Brownian motion, and more on martingales

1.4.1. Gaussian distributions and Brownian motion

Let $|x|$ denote here the Euclidean norm of a vector x in $\mathbb{R}^n$. For every probability measure μ on $\mathbb{R}^n$ having a finite first order moment $\int_{\mathbb{R}^n} |x| \, \mathrm{d}\mu(x)$, one defines the *barycenter* of μ as

$$\operatorname{bar} \mu = \int_{\mathbb{R}^n} x \, \mathrm{d}\mu(x) \in \mathbb{R}^n.$$

To a probability measure μ on $\mathbb{R}^n$ with finite second order moment $\int_{\mathbb{R}^n} |x|^2 \, \mathrm{d}\mu(x)$, one associates the quadratic form

$$Q_\mu : \xi \mapsto \int_{\mathbb{R}^n} \big((x - \operatorname{bar} \mu) \cdot \xi \big)^2 \, \mathrm{d}\mu(x), \quad \xi \in \mathbb{R}^n.$$

The matrix Q of Q_μ with respect to the canonical basis of $\mathbb{R}^n$ is the *covariance matrix* of μ. The quadratic form Q_μ is positive definite when μ is not supported on any affine hyperplane, for example when μ is the uniform probability measure on a bounded convex set C with non empty interior, i.e., a *convex body* C. We say that μ is *centered* when $\operatorname{bar}\mu = 0$, and in this case the expression of Q_μ simplifies to $Q_\mu(\xi) = \int_{\mathbb{R}^n} (x \cdot \xi)^2 \,\mathrm{d}\mu(x)$ for every $\xi \in \mathbb{R}^n$.

When f is a probability density on $\mathbb{R}$ with finite second order moment, the *variance* σ^2 of $f(x)\,\mathrm{d}x$ is defined by

$$\sigma^2 = \int_{\mathbb{R}} \Bigl(x - \int_{\mathbb{R}} y f(y) \,\mathrm{d}y \Bigr)^2 f(x) \,\mathrm{d}x \,.$$

When f is centered, one has that $\sigma^2 = \int_{\mathbb{R}} x^2 f(x) \,\mathrm{d}x$.

A Gaussian random variable with distribution $N(0, \mathrm{I}_n)$ takes values in $\mathbb{R}^n$, its distribution γ_n is symmetric, thus centered, defined on $\mathbb{R}^n$ by

$$\mathrm{d}\gamma_n(x) = (2\pi)^{-n/2} \,\mathrm{e}^{-|x|^2/2} \,\mathrm{d}x \tag{1.17}$$

and γ_n admits the identity matrix I_n as covariance matrix. If F is an n-dimensional Euclidean space, we denote by γ_F the image of γ_n under an (any) isometry from $\mathbb{R}^n$ onto F. If X is a $N(0, \mathrm{I}_n)$ Gaussian random variable and $\sigma > 0$, then the multiple σX admits the distribution $\mathrm{d}\gamma_{n,\sigma}(x) = (2\pi)^{-n/2} \,\mathrm{e}^{-|x/\sigma|^2/2} \,\mathrm{d}(x/\sigma)$, called the $N(0, \sigma^2 \mathrm{I}_n)$ distribution, with $\sigma^2 \mathrm{I}_n$ as covariance matrix. One can consider that the *Dirac probability measure* δ_0 at the origin of $\mathbb{R}^n$ corresponds to $N(0, 0_n)$.

The (absolute) moments of the one-dimensional distribution γ_1 can be computed in terms of values of the Gamma function. For every $p > -1$, one has that

$$\int_{\mathbb{R}} |x|^p \,\mathrm{d}\gamma_1(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} |x|^p \,\mathrm{e}^{-x^2/2} \,\mathrm{d}x = 2^{p/2} \pi^{-1/2} \Gamma\bigl((p+1)/2\bigr) \,.$$

As p tends to $+\infty$, it follows from Stirling's formula that

$$g_p := \Bigl(\int_{\mathbb{R}} |x|^p \,\mathrm{d}\gamma_1(x) \Bigr)^{1/p} \simeq \sqrt{p/\mathrm{e}} \,. \tag{1.18}$$

An n-dimensional *Brownian motion* $(B_t)_{t \geqslant 0}$ starting at $x_0 \in \mathbb{R}^n$ is an $\mathbb{R}^n$-valued random process defined on some probability space $(\Omega, \mathcal{F}, P)$, such that $B_0 = x_0$, such that $B_t - B_s$ is a Gaussian random variable with distribution $N(0, (t-s)\mathrm{I}_n)$ whenever $0 \leqslant s \leqslant t$, and with *independent increments*: for every integer $k \geqslant 1$, when $0 \leqslant t_0 < \ldots < t_k$ are given, then

$$B_{t_0},\, B_{t_1} - B_{t_0},\, B_{t_2} - B_{t_1}, \ldots,\, B_{t_k} - B_{t_{k-1}}$$

are independent. The coordinates $(B_{t,i})_{t \geqslant 0}$, $i = 1, \ldots, n$, are independent one-dimensional Brownian motions. It is possible to choose *everywhere defined* measurable functions $(B_t)_{t \geqslant 0}$ satisfying the above properties in such a way that the trajectories $0 \leqslant t \mapsto B_t(\omega)$, or *random paths*, are continuous for (almost) every $\omega \in \Omega$. The Brownian motion is a martingale with continuous time parameter $t \geqslant 0$, with respect to a continuous time filtration $(\mathcal{F}_t)_{t \geqslant 0}$ where $\mathcal{F}_t$ is generated by the variables B_s, $0 \leqslant s \leqslant t$. See for example Durrett [31] for a detailed account.

It is well known that the Brownian motion on $\mathbb{R}^n$ is the limit of Markov chains with symmetric transition matrix, namely, a limit of suitably scaled Bernoulli random walks. Indeed, if $\delta > 0$ is given and if we consider a Bernoulli walk on the real line moving at each time $k\delta$, $k \in \mathbb{N}^*$, by a step $\pm\sqrt{\delta}$, so that

$$X_t^{(\delta)} = \sqrt{\delta} \sum_{k=1}^{\lfloor t/\delta \rfloor} \varepsilon_k \,, \quad t \geqslant 0 \,, \ \ \varepsilon_k = \pm 1 \,,$$

then the distribution of $(X_t^{(\delta)})_{t \geqslant 0}$ tends when $\delta \to 0$ to that of a one-dimensional Brownian motion. Here, $(\varepsilon_k)_{k=1}^{\infty}$ is a sequence of independent Bernoulli random variables, taking values ± 1 with probability $1/2$. If $(B_t)_{t \geqslant 0}$ is the Brownian motion in $\mathbb{R}^n$, starting at 0, and if we consider the associated Gaussian semi-group $(G_s)_{s \geqslant 0}$ defined for $f \in L^1(\mathbb{R}^n)$ and $s > 0$ by

$$\begin{aligned}(G_s f)(x) &= \mathrm{E}\, f(x + B_s) \\ &= (2\pi s)^{-n/2} \int_{\mathbb{R}^n} f(x+y)\, \mathrm{e}^{-|y|^2/(2s)} \,\mathrm{d}y \,, \quad x \in \mathbb{R}^n \,,\end{aligned} \tag{1.19}$$

we can show an inequality analogous to (1.16). For every p in $(1, +\infty]$ and for every function $f \in L^p(\mathbb{R}^n)$, we have a *maximal inequality for the Gaussian semi-group* with a bound independent of the dimension n, stating that

$$\Bigl(\int_{\mathbb{R}^n} \sup_{s>0} \bigl| (G_s f)(x) \bigr|^p \,\mathrm{d}x \Bigr)^{1/p} \leqslant \frac{p}{p-1} \Bigl(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Bigr)^{1/p} . \tag{1.20.G^*}$$

If we just need a maximal inequality possibly dimension dependent, there is an easy proof relating the Gaussian maximal function to the classical maximal function $\mathrm{M}f$, because the Gaussian kernel is radial and radially decreasing, see (4.6). Once Stein's Theorem 4.1 giving dimensionless estimates for $\mathrm{M}f$ is established, this easy bound of $G_s f$ by $\mathrm{M}f$ implies a dimensionless estimate for the Gaussian semi-group, or for the Poisson semi-group as well. With Bourgain, Carbery and Müller, we shall follow the opposite route, from the semi-group estimates to $\mathrm{M}f$ or $\mathrm{M}_C f$. We sketch an argument for obtaining (1.20.G^*) from the Bernoulli case.

Let us give some more details in dimension $n = 1$. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a sequence of independent Bernoulli random variables, taking values ± 1 with probability $1/2$. The associated semi-group (P_j), indexed by $j \in \mathbb{N}$, is defined by

$$(P_j g)(i) = \mathrm{E}\, g\biggl(i + \sum_{1 \leqslant k \leqslant j} \varepsilon_k \biggr), \quad j \geqslant 0,\ i \in \mathbb{Z}\,,$$

and it satisfies (1.16). As a consequence of the de Moivre–Laplace theorem and by classical tail estimates, we know that

$$(1+x^2)\, P\biggl(\Bigl\{ N^{-1/2} \sum_{1 \leqslant k \leqslant N} \varepsilon_k < x \Bigr\} \biggr)$$

tends to $(1+x^2)\,\gamma_1\bigl((-\infty, x)\bigr)$ when $N \to \infty$, uniformly in x real. It follows that $\mathrm{E}\, f\bigl(N^{-1/2} \sum_{1 \leqslant k \leqslant N} \varepsilon_k\bigr)$ tends to $\int_{\mathbb{R}} f(y)\, d\gamma_1(y)$, uniformly on Lipschitz functions having a Lipschitz constant bounded by some fixed C. If f is Lipschitz on $\mathbb{R}$, then

$$\mathrm{E}\, f\biggl(x + N^{-1/2} \sum_{1 \leqslant k < sN} \varepsilon_k \biggr) \underset{N}{\longrightarrow} (2\pi s)^{-1/2} \int_{\mathbb{R}} f(x+y)\, \mathrm{e}^{-y^2/(2s)}\, \mathrm{d}y\,,$$

uniformly in $x \in \mathbb{R}$ and $s \in [t_0, t_1]$, with $0 < t_0 \leqslant t_1$ fixed. This implies that for any given $\varepsilon > 0$ and N large enough, letting $g_N(i) = f(i/\sqrt{N})$ for $i \in \mathbb{Z}$ and assuming $sN - 1 \leqslant j_N < sN$, we have that

$$\Bigl| P_{j_N} g_N(i) - (G_s f)(i/\sqrt{N}) \Bigr| < \varepsilon\,, \quad i \in \mathbb{Z}\,,$$

for every $s \in [t_0, t_1]$. Applying (1.16) to g_N, we obtain when $s_0, s_1, \ldots, s_k$ and $a > 0$ are given that

$$\int_{-a}^{a} \max_{0 \leqslant j \leqslant k} \bigl| (G_{s_j} f)(x) \bigr|^p \,\mathrm{d}x \leqslant \eta^p(\varepsilon) + \Bigl(\frac{p}{p-1} \Bigr)^p \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{Z}} \biggl| f\Bigl(\frac{i}{\sqrt{N}} \Bigr) \biggr|^p,$$

where $\eta(\varepsilon)$ tends to 0 with ε, implying $(1.20.G^*)$ when $\varepsilon \to 0$, $N \to +\infty$, $a \to +\infty$ and if the sequence $\{s_j\}_{j \geqslant 0}$ is dense in $(0, +\infty)$. The same argument works in $\mathbb{R}^n$, thanks to the product structure of the Bernoulli and Gaussian measures and to the fact that the linear space generated by products $f(x_1, \ldots, x_n) = \prod_{j=1}^n f_j(x_j)$ is uniformly dense in the space of compactly supported Lipschitz functions on $\mathbb{R}^n$.

These considerations generalize to semi-groups of convolution with *symmetric* probability measures $(\mu_t)_{t \geqslant 0}$ on $\mathbb{R}^n$, that is to say, when $\mu_s * \mu_t = \mu_{s+t}$, $s, t \geqslant 0$, and $\mu_t(A) = \mu_t(-A)$ for every Borel subset $A \subset \mathbb{R}^n$. Given $k > 1$, one can find a finitely supported symmetric probability measure $\nu_{1/k}$ on $\mathbb{R}^n$ which is an approximation of $\mu_{1/k}$, in the sense that the integrals of a given finite family of functions f on $\mathbb{R}^n$ are nearly the same for $\mu_{j/k}$ and for $\nu_{1/k}^{*j}$ whenever $j \leqslant k^2$. We may assume that $\nu_{1/k}$ is supported in $\varepsilon\mathbb{Z}$, $\varepsilon > 0$. The symmetric Markov chain $(X_j)_{j \leqslant k^2}$ on $\mathcal{E} = \varepsilon\mathbb{Z}$ with transition governed by $\nu_{1/k}$ permits us to approximate the maximal function $\sup_t |\mu_t * f|$ of the semi-group, replacing it with $\max_{j \leqslant k^2} |\nu_{1/k}^{*j} * f|$.

It follows that some convex functions of the convolution semi-group can be estimated in L^p by projecting functions of a martingale. For example, the sum of squares of differences, already mentioned in the Gaussian case, can be studied also in the Poisson case by relating it to the square function of a martingale and applying the Burkholder–Gundy inequalities presented in the next section.

1.4.2. The Burkholder–Gundy inequalities

When $(M_k)_{k=0}^N$ is a martingale with respect to a filtration $(\mathcal{F}_k)_{k=0}^N$, one introduces the *difference sequence* $(d_k)_{k=0}^N$, which is defined by $d_0 = M_0$ and $d_k = M_k - M_{k-1}$ if $0 < k \leqslant N$. Observe that d_k is $\mathcal{F}_k$-measurable for $0 \leqslant k \leqslant N$ and that $\mathrm{E}(d_k | \mathcal{F}_{k-1}) = 0$ for $k > 0$. Conversely, given a sequence $(d_k)_{k=0}^N$ with these two properties, we obtain a martingale by setting $M_k = \sum_{j=0}^k d_j$, for $0 \leqslant k \leqslant N$. For a scalar martingale $(M_k)_{k=0}^N$, we define the *square function process* $(S_k)_{k=0}^N$ of the martingale by

$$S_k = \Bigl(\sum_{j=0}^{k} |d_j|^2 \Bigr)^{1/2}, \quad k = 0, \ldots, N\,.$$

For a real or complex martingale in L^2, the differences d_k and d_ℓ are orthogonal when $k \neq \ell$. If $k < \ell$ for example, then d_k and its complex conjugate $\overline{d_k}$ are $\mathcal{F}_{\ell-1}$-measurable, thus $\mathrm{E}(\overline{d_k}\, d_\ell) = \mathrm{E}\bigl(\overline{d_k}\, \mathrm{E}(d_\ell | \mathcal{F}_{\ell-1})\bigr) = 0$. It follows that

$$\mathrm{E}\,|M_N|^2 = \sum_{k=0}^{N} \mathrm{E}\,|d_k|^2 = \mathrm{E}\,|S_N|^2\,. \tag{1.21}$$

This equality $\|M_N\|_2 = \|S_N\|_2$ appears as an evident case of the following result.

Theorem 1.6 (Burkholder–Gundy [17]). — *For every p in $(1, +\infty)$, there exists a constant $c_p \geqslant 1$ such that for every integer $N \geqslant 1$, for every real or complex martingale $(M_k)_{k=0}^N$, one has*

$$c_p^{-1} \|M_N\|_p \leqslant \|S_N\|_p \leqslant c_p \|M_N\|_p\,.$$

The *Khinchin inequalities* (see for example Zygmund [85, vol. I, V.8, Th. 8.4]) are a very particular instance of the preceding theorem. Let $(\varepsilon_k)_{k=1}^N$ be a sequence of independent Bernoulli random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, taking the values ± 1 with probability $1/2$. For every p in $(0, +\infty)$, there exist constants $A_p, B_p > 0$ such that for every $N \geqslant 1$

and all scalars $(a_k)_{k=0}^N$, one has

$$A_p \Bigl(\sum_{k=1}^N |a_k|^2 \Bigr)^{1/2} \leqslant \Bigl(\mathrm{E} \Bigl| \sum_{k=1}^N a_k \varepsilon_k \Bigr|^p \Bigr)^{1/p} \leqslant \Bigl(\sum_{k=1}^N |a_k|^2 \Bigr)^{1/2}, \ \ 0 < p \leqslant 2 ,$$

$$\Bigl(\sum_{k=1}^N |a_k|^2 \Bigr)^{1/2} \leqslant \Bigl(\mathrm{E} \Bigl| \sum_{k=1}^N a_k \varepsilon_k \Bigr|^p \Bigr)^{1/p} \leqslant B_p \Bigl(\sum_{k=1}^N |a_k|^2 \Bigr)^{1/2}, \ \ 2 \leqslant p . \tag{1.22.K}$$

The exact values of the constants A_p, B_p are known ([79, 43]). In order to relate these inequalities to Theorem 1.6 when $1 < p < +\infty$, we consider a special filtration on $(\Omega, \mathcal{F}, P)$, generated by the sequence $(\varepsilon_k)_{k=1}^N$. Let $\mathcal{F}_0$ be the trivial field consisting of Ω and $\emptyset$, and for $k > 0$, let $\mathcal{F}_k$ be the finite field generated by $\varepsilon_1, \ldots, \varepsilon_k$. This field $\mathcal{F}_k$ has 2^k atoms of the form

$$A = A_{\mathbf{u}} = \{\omega \in \Omega : \varepsilon_j(\omega) = u_j, \, j = 1, \ldots, k\}, \quad \mathbf{u} = (u_1, \ldots, u_k), \tag{1.23}$$

where $u_j = \pm 1$. We shall call this particular sequence $(\mathcal{F}_k)_{k=0}^N$ of finite fields a *dyadic filtration.* In this framework, for $1 \leqslant k \leqslant N$, any scalar multiple $a_k \varepsilon_k$ of ε_k is a martingale difference d_k. For the associated martingale with $M_N = \sum_{k=1}^N a_k \varepsilon_k$, the square function S_N is the constant function equal to $(\sum_{k=1}^N |a_k|^2)^{1/2}$ and the Khinchin inequalities appear indeed as a simple example of application of Theorem 1.6. Of course, the latter sentence is historically totally inaccurate.

We shall prove only special cases of Theorem 1.6. We say that a sequence of random variables $(m_k)_{k=0}^N$ is *predictable* when

$$m_0 \text{ is } \mathcal{F}_0\text{-measurable, and } m_k \text{ is } \mathcal{F}_{k-1}\text{-measurable for } 0 < k \leqslant N . \tag{1.24}$$

If $(m_k)_{k=0}^N$ is scalar valued and predictable, and if $(d_k)_{k=0}^N$ is a martingale difference sequence, then $(m_k d_k)_{k=0}^N$ is again a martingale difference sequence since one has that $\mathrm{E}(m_k d_k \,|\, \mathcal{F}_{k-1}) = m_k \, \mathrm{E}(d_k \,|\, \mathcal{F}_{k-1}) = 0$. The new martingale $(L_k)_{k=0}^N$ defined by $L_k = \sum_{j=0}^k m_j d_j$ is said to be obtained as *martingale transform*, see [15, 16].

Consider a dyadic filtration $(\mathcal{F}_k)_{k=0}^N$ as defined above. Notice that each atom A of $\mathcal{F}_k$ as in (1.23) has probability 2^{-k}, and is split into two atoms $A_\pm$ of $\mathcal{F}_{k+1}$, $A_\pm := A \cap \{\varepsilon_{k+1} = \pm 1\}$, according to the value of ε_{k+1}. Let d_{k+1} be a martingale difference with respect to these dyadic fields. The function d_{k+1} should have mean 0 on the atom A of $\mathcal{F}_k$, and be constant on each of the two atoms $A_\pm$ of $\mathcal{F}_{k+1}$ contained in A, which have equal measure $P(A)/2$. It follows that d_{k+1} must take on A two opposite values $\pm v$. Consequently, the modulus (or the norm) of d_{k+1} is constant on A, thus $|d_{k+1}|$ is $\mathcal{F}_k$-measurable, so that $(|d_k|)_{k=0}^N$ is predictable, as defined in (1.24). We shall call *Bernoulli martingale* any martingale $(M_k)_{k=0}^N$ with respect to this dyadic filtration $(\mathcal{F}_k)_{k=0}^N$. A Bernoulli martingale with values in a vector space can be pictured as a *tree* $(v_{\varepsilon_1,\ldots,\varepsilon_k})$ of vectors, $0 \leqslant k \leqslant N$ and $\varepsilon_j = \pm 1$, such that each vector $v_{\varepsilon_1,\ldots,\varepsilon_k}$ in the tree is the midpoint of his two successors $v_{\varepsilon_1,\ldots,\varepsilon_k,1}$ and $v_{\varepsilon_1,\ldots,\varepsilon_k,-1}$. The vectors $v_{\varepsilon_1,\ldots,\varepsilon_k}$ are the values of the kth random variable M_k of the martingale, which can be defined by $M_k(\varepsilon_1, \ldots, \varepsilon_k) = v_{\varepsilon_1,\ldots,\varepsilon_k}$.

The next Lemma contains an easier case of a result due to Burgess Davis [26], namely, the left-hand inequality when $p = 1$. The rest of the statement presents a mixture of Doob's and Burkholder–Gundy's inequalities.

Lemma 1.7. — *For every p with $1 \leqslant p \leqslant 2$ and for every real or complex Bernoulli martingale $(M_k)_{k=0}^N$, one has that*

$$6^{-1}\|M_N^*\|_p \leqslant \|S_N\|_p \leqslant 6\|M_N^*\|_p \,.$$

Partial proof, after [56]. — We consider the case $p = 1$. The general strategy is to bring the problem to L^2, where $\|S_N\|_2 = \|M_N\|_2$ by (1.21), and this is essentially done by dividing $f = M_N \in L^1$ by a "parent" of $\sqrt{|f|}$, in order to get an element in L^2 "similar" to $\sqrt{|f|}$. One then applies known facts in L^2, and finally come back to L^1 by multiplication with a suitable L^2 function. We begin with the proof of the left-hand inequality in Lemma 1.7.

Let $(M_k)_{k=0}^N$ be a Bernoulli martingale. We know that $(|d_k|)_{k=0}^N$ is predictable, as well as $(S_k)_{k=0}^N$. Consider the martingale transform $L_k = \sum_{j=0}^k S_j^{-1/2} d_j$. In L^2 we know that $\mathrm{E}\,|L_N|^2 = \sum_{j=0}^N \mathrm{E}(S_j^{-1}|d_j|^2)$. We see that $S_0^{-1}|d_0|^2 = S_0$, and $S_j^{-1}|d_j|^2 \leqslant 2(S_j - S_{j-1})$ for $j \geqslant 1$ because, letting $t = S_{j-1}^2$ and $h = |d_j|^2$, we have

$$2(\sqrt{t+h} - \sqrt{t}) = \int_t^{t+h} u^{-1/2}\,\mathrm{d}u \geqslant h(t+h)^{-1/2} \,.$$

It follows that

$$\mathrm{E}\,|L_N|^2 \leqslant 2\,\mathrm{E}\,S_N \,. \tag{1.25}$$

Notice that $\left|\sum_{j=0}^s S_j^{-1/2} d_j\right| = |L_s| \leqslant L_N^*$ and $\left|\sum_{j=r+1}^s S_j^{-1/2} d_j\right| = |L_s - L_r| \leqslant 2L_N^*$ when $0 \leqslant r < s \leqslant N$. Multiplying termwise the sequence $(S_k^{-1/2} d_k)_{k=0}^N$ by the non-decreasing sequence $(S_k^{1/2})_{k=0}^N$, we obtain for every $s \leqslant N$ by Abel's summation method that

$$|M_s| = \left|\sum_{j=0}^s d_j\right| \leqslant S_s^{1/2} \sup_{0 \leqslant r \leqslant s} \left|\sum_{j=r}^s S_j^{-1/2} d_j\right| \leqslant 2 S_N^{1/2} L_N^* \,,$$

thus $M_N^* \leqslant 2S_N^{1/2} L_N^*$. By Cauchy–Schwarz, Doob's inequality (1.1) with $p = 2$, and by (1.25), we get the conclusion

$$\mathrm{E}\, M_N^* \leqslant 2(\mathrm{E}\, S_N)^{1/2} \|L_N^*\|_2 \leqslant 2^2 (\mathrm{E}\, S_N)^{1/2} \|L_N\|_2 \leqslant 2^{5/2}\, \mathrm{E}\, S_N \leqslant 6\, \mathrm{E}\, S_N\,.$$

We leave the rewriting of this proof when $1 < p < 2$ as an easy exercise for the reader, and we pass to the right-hand side inequality using the same method, with the help of the non-decreasing predictable sequence $(A_k)_{k=0}^N$ defined by

$$A_0 = |d_0| = |M_0|, \quad A_k = \max\bigl(A_{k-1}, M_{k-1}^* + |d_k|\bigr) \geqslant |M_k|, \ \ k = 1, \ldots, N\,,$$

and of the martingale transform $L_k = \sum_{j=0}^k A_j^{-1/2} d_j$, $k = 0, \ldots, N$. Observe that $|d_k| \leqslant |M_k| + |M_{k-1}| \leqslant 2M_N^*$, thus $A_N \leqslant 3M_N^*$. By Abel, writing $d_k = M_k - M_{k-1}$ for $k \geqslant 1$, we see that

$$\begin{aligned} |L_N| &= \Bigl| A_N^{-1/2} M_N + \sum_{k=0}^{N-1} M_k \Bigl(A_k^{-1/2} - A_{k+1}^{-1/2} \Bigr) \Bigr| \\ &\leqslant A_N^{1/2} + \sum_{k=0}^{N-1} A_k (A_k^{-1/2} - A_{k+1}^{-1/2}) \\ &\leqslant A_N^{1/2} + \sum_{k=0}^{N-1} \Bigl(\sqrt{A_{k+1}} - \sqrt{A_k} \Bigr) \leqslant 2A_N^{1/2}\,, \end{aligned}$$

where we make use of $u^2(u^{-1} - v^{-1}) \leqslant v - u$ when $0 < u \leqslant v$. In L^2 we know that $\mathrm{E}\bigl(\sum_{k=0}^N A_k^{-1} |d_k|^2\bigr) = \mathrm{E}\, |L_N|^2 \leqslant 4\, \mathrm{E}\, A_N$, and we go back to L^1 with Cauchy–Schwarz and the obvious inequality

$$\sum_{k=0}^N |d_k|^2 \leqslant A_N \sum_{k=0}^N A_k^{-1} |d_k|^2\,.$$

We obtain

$$\mathrm{E}\, S_N = \mathrm{E} \left(\sum_{k=0}^N |d_k|^2 \right)^{1/2} \leqslant (\mathrm{E}\, A_N)^{1/2}\, \|L_N\|_2 \leqslant 2\, \mathrm{E}\, A_N \leqslant 6\, \|M_N^*\|_1\,. \quad \square$$

Remark. — The Brownian martingales can be approximated by Bernoulli martingales, and we can obtain the analogous result for them. Actually, the preceding proof is even simpler to write in this case. Brownian martingales are defined by means of (Itō's) *stochastic integrals*

$$M_t(\omega) = \int_0^t m_s(\omega)\, \mathrm{d}B_s(\omega)\,, \quad t \geqslant 0\,,$$

where $(m_s)_{s \geqslant 0}$ is an *adapted process*, meaning essentially that each m_s, $s \geqslant 0$, is $\mathcal{F}_s$-measurable. The square function is then defined by $S_t^2(\omega) =$

$\int_0^t |m_s(\omega)|^2 \, \mathrm{d}s$ for every $t \geqslant 0$, and one can replace in the proof of Lemma 1.7 the Abel summation method by the more pleasant integration by parts.

Remark 1.8. — Together with Doob's inequality, Lemma 1.7 implies Theorem 1.6 for Bernoulli martingales when $1 < p \leqslant 2$. The Burkholder–Gundy inequalities are equivalent to saying that martingale difference sequences are *unconditional* in L^p when $1 < p < +\infty$, that is to say, that there exists a constant $\kappa_{u,p}$ such that for each integer $N \geqslant 0$, all scalars $(a_k)_{k=0}^N$ with $|a_k| \leqslant 1$ and all martingale differences $(d_k)_{k=0}^N$, we have

$$\Big\|\sum_{k=0}^N a_k d_k\Big\|_p \leqslant \kappa_{u,p} \Big\|\sum_{k=0}^N d_k\Big\|_p . \tag{1.26}$$

Going from Theorem 1.6 to unconditionality is simple, since the square function of the martingale at the left-hand side of (1.26) is less than that on the right-hand side, and we can take $\kappa_{u,p} = c_p^2$. The other direction follows from Khinchin, by averaging over signs $a_k = \pm 1$. Indeed, one obtains from (1.22.**K**) for $(f_k)_{k=1}^N$ in $L^p(X, \Sigma, \mu)$, $1 \leqslant p < +\infty$, that

$$\begin{aligned} A_p^p \Big\| \Big(\sum_{k=1}^N |f_k|^2 \Big)^{1/2} \Big\|_{L^p(\mu)}^p &\leqslant \mathrm{E} \int_X \Big|\sum_{k=1}^N \varepsilon_k f_k\Big|^p \, \mathrm{d}\mu \\ &\leqslant B_p^p \Big\| \Big(\sum_{k=1}^N |f_k|^2 \Big)^{1/2} \Big\|_{L^p(\mu)}^p . \end{aligned} \tag{1.27}$$

It is possible (see Pisier [64, Section 5.8]) to obtain the general case of unconditionality of martingale differences by approximating general martingale difference sequences by *blocks* of Bernoulli martingale differences. Also, one can see that (1.26) is self-dual and obtain by duality the Burkholder–Gundy inequalities for $2 \leqslant p < +\infty$.

The proof of Lemma 1.7 is valid with almost no change when the martingale takes values in a Hilbert space H, because $L^2(\Omega, \mathcal{F}, P, H)$ is a Hilbert space where the H-valued martingale differences are orthogonal. For values in a Banach space, two difficulties arise. First, the relevant "square function" has to be defined, and second, the Banach space-valued martingale differences are not unconditional in general. The Banach spaces where martingale differences are unconditional form a nice class of spaces, see Pisier [64, Chap. 5, The UMD property for Banach spaces].

Remark 1.9. — Let $f = \sum_{k=0}^N d_k$ be the sum of a Bernoulli martingale and let $g = \sum_{k=0}^N a_k d_k$ be obtained from f by a martingale transform operation, with $|a_k| \leqslant 1$ for $k = 0, \ldots, N$. By Lemma 1.7 and Doob's inequality (1.1), we have

$$\|g\|_p \leqslant \|g^*\|_p \leqslant 6\,\|S(g)\|_p \leqslant 6\,\|S(f)\|_p \leqslant 36\,\|f^*\|_p \leqslant \frac{36\,p}{p-1}\,\|f\|_p\,,\ \ 1<p\leqslant 2\,,$$

which shows that the constant $\kappa_{u,p}$ in (1.26) is of order $1/(p-1)$ in this case. Actually, Burkholder has found the exact value of the unconditional constant for general martingale transforms and for every $p \in (1, +\infty)$. It is given by

$$\kappa_{u,p} = p^* - 1\,, \ \text{ where } \ p^* := \max(p, p/(p-1))\,.$$

One can consult [16] and the references given there to several other articles by Burkholder. One can also find in [16, Section 5.4] a bound $c_p \leqslant p^* - 1$ for the constant c_p in Theorem 1.6.

1.4.3. A consequence of the "reflection principle"

Consider a Brownian motion $(B_s)_{s\geqslant 0}$ on $\mathbb{R}$, defined on a probability space $(\Omega, \mathcal{F}, P)$ and with respect to a filtration $(\mathcal{F}_s)_{s\geqslant 0}$. We assume that $B_0 = 0$, we fix a real number $v > 0$, and we let $S_v(\omega)$ denote the first time when the trajectory $s \mapsto B_s(\omega)$, $s \geqslant 0$, which is continuous for almost every $\omega \in \Omega$, reaches the point v. It is clear that if $s_0 > 0$ is given, one has $\{B_{s_0} \geqslant v\} \subset \{S_v \leqslant s_0\}$, thus

$$P\big(\{S_v \leqslant s_0\}\big) \geqslant P\big(\{B_{s_0} \geqslant v\}\big) = P\big(\{B_1 \geqslant v/\sqrt{s_0}\}\big) = \int_{v/\sqrt{s_0}}^{+\infty} \mathrm{e}^{-y^2/2}\,\frac{\mathrm{d}y}{\sqrt{2\pi}}\,.$$

From now on, we write $P(S_v \leqslant s_0)$ for $P\big(\{S_v \leqslant s_0\}\big)$. We will show that actually

$$P(S_v \leqslant s_0) = 2\,P(B_{s_0} \geqslant v) = 2\int_{v/\sqrt{s_0}}^{+\infty} \mathrm{e}^{-y^2/2}\,\frac{\mathrm{d}y}{\sqrt{2\pi}}\,,$$

which proves in passing that S_v is finite almost surely, since we have then

$$P(S_v < +\infty) = 2\int_0^{+\infty} \mathrm{e}^{-y^2/2}\,\frac{\mathrm{d}y}{\sqrt{2\pi}} = 1\,.$$

The reasoning makes use of the reflection of the Brownian motion after a stopping time τ. A *stopping time* is a random variable τ with values in $[0, +\infty]$, such that for every $t \geqslant 0$, the event $\{\tau \leqslant t\}$ belongs to the σ-field $\mathcal{F}_t$ of the past of time t. Intuitively, a stopping time corresponds to a decision to quit at time $\tau(\omega)$ that an observer, embarked on a path $t \mapsto X_t(\omega)$ of the random process $(X_t)_{t\geqslant 0}$ since the time $t = 0$, can take from his only knowledge of what happened on his way between 0 and the present time.

The random time S_v is an excellent example of stopping time, with a quite simple rule: I stop when I reach the point $v > 0$.

The Brownian reflected after the random time τ changes its direction, its trajectory becomes the symmetric of the original trajectory with respect to the point $(B_\tau)(\omega) := B_{\tau(\omega)}(\omega)$ that was reached at time $\tau(\omega)$. Let us denote by $(B^\tau_s)_{s\geqslant 0}$ the reflected Brownian, given by

$$\begin{aligned} B^\tau_s(\omega) &= B_s(\omega) && \text{if} \quad 0 \leqslant s \leqslant \tau(\omega)\,, \\ \frac{B^\tau_s(\omega) + B_s(\omega)}{2} &= B_{\tau(\omega)}(\omega) && \text{if} \quad s \geqslant \tau(\omega)\,. \end{aligned}$$

The reflected Brownian B^τ is still a Brownian motion. Consider first the simplest stopping time and reflection. Choosing a set A_1 in the σ-field $\mathcal{F}_{s_1}$ at time $s_1 > 0$, we define a stopping time τ_1 equal to s_1 on A_1 and to $+\infty$ outside. The corresponding reflection $(B^{\tau_1}_s)_{s\geqslant 0}$ is given by

$$\begin{aligned} B^{\tau_1}_s(\omega) &= B_s(\omega) && \text{if} \quad 0 \leqslant s \leqslant s_1 \quad \text{or} \quad \omega \notin A_1\,, \\ \frac{B^{\tau_1}_s(\omega) + B_s(\omega)}{2} &= B_{s_1}(\omega) && \text{if} \quad s \geqslant s_1 \quad \text{and} \quad \omega \in A_1\,. \end{aligned}$$

One shows easily that $(B^{\tau_1}_s)_{s\geqslant 0}$ is a Brownian motion. Iterating this operation, one can reach discrete stopping times, and pass to the limit for dealing with general stopping times. Indeed, a stopping time τ can be approximated by the first time $\tau_k > \tau$ such that $2^k\tau_k$ is an integer, i.e., $\tau_k = 2^{-k}(\lfloor 2^k\tau\rfloor + 1)$, for every $k \in \mathbb{N}$.

Another important property that can be checked following the same route is the following: if τ is an almost surely finite stopping time, the process "starting afresh at time τ", defined by $X_s = B_{\tau+s} - B_\tau$, i.e., $X_s(\omega) = B_{\tau(\omega)+s}(\omega) - B_{\tau(\omega)}(\omega)$, is also a Brownian motion.

Consider the Brownian reflected after the stopping time S_v, with $v > 0$. Since the Brownian paths are continuous and $B_0 = 0$, we have $B_{S_v(\omega)}(\omega) = v$ and for every $s_0 > 0$, the event $\{B_{s_0} > v\}$ is contained in $\{S_v < s_0\}$. Clearly, the event $\{B^{S_v}_{s_0} > v\}$ is also contained in $\{S_v < s_0\}$ and disjoint from $\{B_{s_0} > v\}$. Actually, since on the set $\{S_v < s_0\}$ one has $B^{S_v}_{s_0} + B_{s_0} = 2v$, one sees that

$$\{S_v < s_0\} \setminus \{B_{s_0} \geqslant v\} = \{B^{S_v}_{s_0} > v\}\,.$$

The event $\{B^{S_v}_{s_0} > v\}$ has the same probability as $\{B_{s_0} > v\}$, since $(B^{S_v}_s)_{s\geqslant 0}$ is another Brownian, and $P(S_v = s_0) \leqslant P(B_{s_0} = v) = 0$. We have therefore that

$$P(S_v \leqslant s_0) = P(S_v < s_0) = 2\,P(B_{s_0} > v) = 2\int_v^{+\infty} \mathrm{e}^{-u^2/(2s_0)}\,\frac{\mathrm{d}u}{\sqrt{2\pi s_0}}\,.$$

Consequently, for every $s > 0$, we obtain

$$P(S_v \leqslant s) = P\big(\sup_{0 \leqslant u \leqslant s} B_u \geqslant v\big) = 2 \int_{v/\sqrt{s}}^{+\infty} \mathrm{e}^{-y^2/2} \, \frac{\mathrm{d}y}{\sqrt{2\pi}} \,.$$

This allows us to find the density h_v of the distribution of S_v, which is given by

$$h_v(s) = \mathbf{1}_{s \geqslant 0} \, \frac{v s^{-3/2}}{\sqrt{2\pi}} \, \mathrm{e}^{-v^2/(2s)} \,, \quad s \in \mathbb{R} \,. \tag{1.28}$$

Remark. — A variant of the preceding reasoning applies to the exit time S from an open convex subset D of $\mathbb{R}^n$ containing the starting point x_0 of an n-dimensional Brownian motion. Suppose that this Brownian motion touches the boundary of D, for the first time, at the point $x = x(\omega)$ and at time $S(\omega)$. Let E_x be an affine half-space tangent to D at x, and exterior to D (this E_x is not unique in general). Starting again from x at time $S(\omega)$, there is a probability $1/2$ to end in E_x at time $s_0 > S(\omega)$, so there is at least one chance out of two to end up outside D at time s_0. The set $\{B_{s_0} \notin D\}$ is a subset of $\{S < s_0\}$ that occupies thus at least one half of it. We have therefore

$$P(S < s_0) \leqslant 2 P(B_{s_0} \notin D) \,.$$

This inequality says that the probability to be outside D at a time between 0 and s_0 is bounded by twice the probability to be outside D at time s_0. This can be readily interpreted in terms of maximal function. If $\| \cdot \|_C$ denotes the norm on $\mathbb{R}^n$ associated to a symmetric convex body C in $\mathbb{R}^n$, we deduce maximal inequalities in $L^p(\mathbb{R}^n)$ for the $\| \cdot \|_C$ norm of the martingale $(B_s)_{s \geqslant 0}$ that are better than Doob's inequality. Namely, for every $p > 0$ we have

$$\begin{aligned} \mathrm{E} \max_{0 \leqslant s \leqslant s_0} \|B_s\|_C^p &= p \int_0^{+\infty} t^{p-1} P\big(\max_{0 \leqslant s \leqslant s_0} \|B_s\|_C > t\big) \, \mathrm{d}t \\ &\leqslant 2p \int_0^{+\infty} t^{p-1} P\big(\|B_{s_0}\|_C > t\big) \, \mathrm{d}t = 2 \, \mathrm{E} \, \|B_{s_0}\|_C^p \,. \end{aligned}$$

For $p \leqslant 1$, there is no Doob's inequality in L^p, and when $p > 1$, one has always that $2^{1/p} < p/(p-1)$, because $(1-x)2^x < (1-x)\,\mathrm{e}^x \leqslant 1$ for $0 < x < 1$.

One could get a similar estimate when the set D is no longer convex, but has the property that for every boundary point x of D, there is a cone E_x based at x, disjoint from D and with a *solid angle* bounded below by $\delta > 0$ independent of x. If we measure the angle as the proportion of the unit sphere S^{n-1} of $\mathbb{R}^n$ intersected by the cone $E_x - x$ based at 0, then the constant 2 above has to be replaced by δ^{-1}.

1.5. The Poisson semi-group

Let us recall that the *Schwartz class* $\mathcal{S}(\mathbb{R}^n)$ consists of all C^∞ functions φ such that $(1+|x|^k)\varphi^{(\ell)}(x)$ is bounded on $\mathbb{R}^n$ for all integers $k, \ell \geqslant 0$. We shall denote by $(P_t)_{t\geqslant 0}$ the Poisson semi-group on $\mathbb{R}^n$, which can be defined, for f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, by

$$(P_t f)(x) = u(x,t), \quad x \in \mathbb{R}^n,\, t \geqslant 0\,, \tag{1.29}$$

where $u(x,t)$ is the (bounded) *harmonic extension* of f to the upper half-space H^+ of $\mathbb{R}^{n+1}$ formed by all (x,t) with $x \in \mathbb{R}^n$ and $t \geqslant 0$. For $x \in \mathbb{R}^n$ one has $u(x,0) = f(x)$, $\Delta u(x,t) = 0$ when $t > 0$, and u is continuous on H^+. The semi-group property $P_{t+s} = P_t P_s$ amounts to saying that the harmonic extension of the function f_s defined on $\mathbb{R}^n$ by $f_s(x) = u(x,s)$ is given by $v(x,t) = u(x,t+s)$.

The Poisson semi-group is intimately related to the Brownian motion $(B_s)_{s\geqslant 0}$ in $\mathbb{R}^{n+1}$. If the Brownian $(B_s)_{s\geqslant 0}$ starts at time $s = 0$ from the point (x_0, t_0), where $x_0 \in \mathbb{R}^n$ and $t_0 > 0$, we know that almost every path $s \mapsto B_s(\omega)$ will hit the hyperplane $H_0 = \{t = 0\}$ at some time $\tau_{t_0}(\omega) < +\infty$. If we decompose B_s into $(x_0 + X_s, t_0 + T_s)$, then T_s is a one-dimensional Brownian motion, starting from 0 at time 0, and X_s is a n-dimensional Brownian motion, starting from the point 0 in $\mathbb{R}^n$ and independent of T_s. The stopping time τ_{t_0} is the first time $s > 0$ when $T_s = -t_0$. If f is reasonable, for example continuous and bounded on $\mathbb{R}^n$, one sees that the (bounded) harmonic extension u of f to the upper half-space is given by

$$u(x_0,t_0) = \mathrm{E}\, F(B_{\tau_{t_0}}) = \mathrm{E}\, f(x_0 + X_{\tau_{t_0}}) = \int_\Omega f\bigl(x_0 + X_{\tau_{t_0}(\omega)}(\omega)\bigr)\, \mathrm{d}P(\omega)\,,$$

where F is defined on the hyperplane H_0 of $\mathbb{R}^{n+1}$ by $F(x,0) = f(x)$ for every $x \in \mathbb{R}^n$. The Poisson probability measure $P_{t_0}(x)\,\mathrm{d}x$ on $\mathbb{R}^n$ is the distribution of $X_{\tau_{t_0}}$, distribution of the Brownian motion (X_s) starting from $0 \in \mathbb{R}^n$ and stopped at time τ_{t_0}, when B_s reaches H_0. We shall employ the same notation P_t for the semi-group, for the Poisson distribution on $\mathbb{R}^n$, and for its density $P_t(x)$. The operator P_t is the convolution with the corresponding probability measure, it acts thus on $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant +\infty$. We shall say that t is the *parameter* of P_t.

The distribution of the stopping time τ_{t_0} is clearly the same as the distribution of the first time S_{t_0} when the one-dimensional Brownian motion starting from 0 reaches $t_0 > 0$, and we know by (1.28) the density h_t of the distribution of S_t. The Poisson distribution P_t on $\mathbb{R}^n$ is obtained by mixing Gaussian distributions on $\mathbb{R}^n$, distributions of X_s at various times s, the mixing being done according to the distribution of S_t. In the portion of the space Ω where $s_0 \leqslant \tau_t \leqslant s_0 + \delta s$, the coordinate x of the Brownian point

$B_s = (X_s, t + T_s)$ at time τ_t is approximately X_{s_0}, with probability of order $h_t(s_0)\,\delta s$, and $(X_s)_{s \geqslant 0}$ is independent of τ_t. The point $(x, 0) = (X_{s_0}, 0)$ is the point where the Brownian B_s touches the hyperplane H_0, knowing that $\tau_t = s_0$. This is the reason behind the *subordination principle* of the Poisson semi-group to the Gaussian semi-group, which implies in particular that the maximal function of the Poisson semi-group is bounded by that of the Gaussian semi-group $(G_s)_{s \geqslant 0}$ on $\mathbb{R}^n$. Indeed, we have by (1.28) that P_t is "in the (closed) convex hull" of the Gaussian semi-group, since

$$P_t = \int_0^{+\infty} G_s \frac{t s^{-3/2}}{\sqrt{2\pi}} \,\mathrm{e}^{-t^2/(2s)} \,\mathrm{d}s \,. \tag{1.30}$$

It follows that

$$|P_t * f| \leqslant \int_0^{+\infty} |G_s * f| \frac{t s^{-3/2}}{\sqrt{2\pi}} \,\mathrm{e}^{-t^2/(2s)} \,\mathrm{d}s \leqslant \sup_{u \geqslant 0} |G_u * f| \,.$$

We get a dimensionless estimate for the maximal function of the Poisson semi-group, consequence of the one in $(1.20.G^*)$ for the Gaussian case. We have

$$\Bigl\| \sup_{t > 0} |P_t f| \Bigr\|_{L^p(\mathbb{R}^n)} \leqslant \frac{p}{p-1} \Bigl(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Bigr)^{1/p} . \tag{1.31.P^*}$$

The remarks about comparing to $\mathrm{M}f$ are still in order here. Stein [73, Lemma 1, p. 48] proves $(1.31.P^*)$ with different constants and in a different way, capable of easier generalizations to non Euclidean settings. He does not deal with the Gaussian maximal function, but applies the Hopf maximal inequality (1.10) to the Gaussian semi-group together with the subordination principle. Using subordination, Stein shows that the Poisson maximal function $P^* f = \sup_{t > 0} |P_t f|$ is bounded by an average of expressions $t^{-1} \int_0^t (G_s f) \,\mathrm{d}s$ that are controlled by Hopf.

The formula (1.30) proves that the marginals of P_t are other Poisson distributions: indeed, the mixing distribution, which has density h_t, does not depend on the dimension n, and the projections on $\mathbb{R}^\ell$, $1 \leqslant \ell < n$, of Gaussian distributions $N(0, \sigma^2 \mathrm{I}_n)$ on $\mathbb{R}^n$ are $N(0, \sigma^2 \mathrm{I}_\ell)$ Gaussian distributions. We can also deduce the density of the distribution P_t for each $t > 0$, writing

$$\begin{aligned} P_t(x) &= \int_0^{+\infty} \mathrm{e}^{-|x|^2/(2s)} (2\pi s)^{-n/2} \frac{t s^{-3/2}}{\sqrt{2\pi}} \,\mathrm{e}^{-t^2/(2s)} \,\mathrm{d}s \\ &= t \int_0^{+\infty} (2\pi s)^{-n/2-1/2} \,\mathrm{e}^{-(t^2+|x|^2)/(2s)} \,\frac{\mathrm{d}s}{s} \,, \quad x \in \mathbb{R}^n \,. \end{aligned}$$

Setting $u = s/(t^2 + |x|^2)$, then $v = 1/(2u)$, we get

$$P_t(x) = t \bigl(\pi (t^2 + |x|^2) \bigr)^{-(n+1)/2} \int_0^{+\infty} \mathrm{e}^{-v} \, v^{(n+1)/2} \,\frac{\mathrm{d}v}{v} \,.$$

The *Poisson kernel* P_t on $\mathbb{R}^n$ is thus given by the formula

$$P_t(x) = P_t^{(n)}(x) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \frac{t}{(t^2+|x|^2)^{(n+1)/2}}\,, \quad x \in \mathbb{R}^n\,, t > 0\,. \tag{1.32}$$

In dimension $n = 1$, the Poisson kernel is the *Cauchy kernel*, equal to

$$P_t(x) = P_t^{(1)}(x) = \frac{t}{\pi(t^2+x^2)}\,, \quad x \in \mathbb{R}\,,\ t > 0\,. \tag{1.33.C}$$

The coefficient that comes into the n-dimensional formula (1.32) satisfies the asymptotic estimate

$$\frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \simeq \sqrt{\frac{2}{\pi n}}\,\frac{1}{\omega_n} = \sqrt{\frac{2n}{\pi}}\,\frac{1}{s_{n-1}}\,,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and s_{n-1} the $(n-1)$-dimensional measure of the unit sphere S^{n-1} in $\mathbb{R}^n$, given by

$$\omega_n = \frac{\pi^{n/2}}{(n/2)!} := \frac{\pi^{n/2}}{\Gamma\bigl((n/2)+1\bigr)}\,, \quad s_{n-1} = n\,\omega_n\,. \tag{1.34}$$

From this, we obtain estimates on the measure of Euclidean balls for the probability measure $P_1(x)\,\mathrm{d}x$ on $\mathbb{R}^n$. Writing $P_1(x) = F(|x|)$, we get an exact asymptotic estimate when the dimension n tends to infinity: for $\nu > 0$ fixed, we have

$$\begin{aligned}
&\int_{\{|x|>\sqrt{n}/\nu\}} P_1(x)\,\mathrm{d}x \\
&= s_{n-1}\int_{\sqrt{n}/\nu}^{+\infty} r^{n-1}F(r)\,\mathrm{d}r \simeq \sqrt{\frac{2}{\pi}}\int_{\sqrt{n}/\nu}^{+\infty} \frac{\sqrt{n}}{r}\Bigl(\frac{r^2}{1+r^2}\Bigr)^{(n+1)/2}\frac{\mathrm{d}r}{r} \\
&= \sqrt{\frac{2}{\pi}}\int_{1/\nu}^{+\infty}\Bigl(1+\frac{1}{nu^2}\Bigr)^{-(n+1)/2}\frac{\mathrm{d}u}{u^2} = \sqrt{\frac{2}{\pi}}\int_0^{\nu}\Bigl(1+\frac{y^2}{n}\Bigr)^{-(n+1)/2}\,\mathrm{d}y\,.
\end{aligned}$$

Therefore, when n tends to infinity, we see that

$$\int_{\{\nu|x|>\sqrt{n}\}} P_1(x)\ \mathrm{d}x \longrightarrow 2\int_0^{\nu} \mathrm{e}^{-y^2/2}\ \frac{\mathrm{d}y}{\sqrt{2\pi}}\,. \tag{1.35}$$

2. General dimension free inequalities, second part

In this section, we gather results that depend on the Fourier transform. In order that the Fourier transform be isometric on $L^2(\mathbb{R}^n)$, we set

$$\forall\, \xi \in \mathbb{R}^n, \quad \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\,\mathrm{e}^{-2\mathrm{i}\pi x\cdot\xi}\ \mathrm{d}x\,, \quad \widehat{\mu}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\mathrm{i}\pi x\cdot\xi}\ \mathrm{d}\mu(x)\,,$$

when f is in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ or when μ is a bounded measure on $\mathbb{R}^n$. By the Plancherel theorem (some say Parseval's theorem), we know that this defines a mapping from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ that extends to a unitary transformation $\mathcal{F}$ of $L^2(\mathbb{R}^n)$. The inverse mapping $\mathcal{F}^{-1}$ of $\mathcal{F}$ sends every square integrable function $\xi \mapsto g(\xi)$ to $\mathcal{F}\bigl(\xi \mapsto g(-\xi)\bigr)$, also expressible by $x \mapsto (\mathcal{F}g)(-x)$. We shall employ the notation $g^\vee = \mathcal{F}^{-1}g$ for the inverse Fourier transform.

The Plancherel–Parseval theorem extends to functions f with values in a Euclidean space F, giving then an isometry from $L^2(\mathbb{R}^n, F)$ to itself. This is clear for instance by looking at coordinates in an orthonormal basis of F.

With this normalization of the Fourier transform, we have that

$$\widehat{\gamma}_n(\xi) = \mathrm{e}^{-2\pi^2|\xi|^2}, \quad \xi \in \mathbb{R}^n,$$

and the Fourier transform of the Poisson kernel P_t on $\mathbb{R}^n$ is equal to $\mathrm{e}^{-2\pi t|\xi|}$, for every $\xi \in \mathbb{R}^n$. Indeed, as the marginals on $\mathbb{R}$ of P_t are Cauchy distributions with the same parameter t, we find by the residue theorem that

$$\widehat{P}_t(\xi) = \int_{\mathbb{R}} \frac{t\,\mathrm{e}^{-2\mathrm{i}\pi s|\xi|}}{\pi(t^2+s^2)}\,\mathrm{d}s = \mathrm{e}^{-2\pi t|\xi|}\,.$$

This information on the Fourier transform gives another way of checking the semi-group property $P_s * P_t = P_{s+t}$ of Poisson distributions. Using the Fourier inversion formula, we notice for future use that the harmonic extension $u(x,t) = (P_t f)(x)$ of $f \in \mathcal{S}(\mathbb{R}^n)$ considered in (1.29) can be written as

$$u(x,t) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi t|\xi|}\,\widehat{f}(\xi)\,\mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\,\mathrm{d}\xi\,, \quad x \in \mathbb{R}^n, t > 0\,. \tag{2.1}$$

2.1. Littlewood–Paley functions

The Littlewood–Paley function $g(f)$ associated to a function f on $\mathbb{R}^n$ is defined by

$$\forall\, x \in \mathbb{R}^n, \quad g(f)(x) = \Bigl(\int_0^{+\infty} \bigl|t\nabla u(x,t)\bigr|^2\,\frac{\mathrm{d}t}{t}\Bigr)^{1/2},$$

where u is the harmonic extension of f to the upper half-space in $\mathbb{R}^{n+1}$, and where ∇u is the gradient of u in $\mathbb{R}^{n+1}$. The classical theory, see for example Zygmund [85, vol. 2] for the circle case in Chap. 14, §3 and Chap. 15, §2, indicates that the norm of f in $L^p(\mathbb{R}^n)$, $1 < p < +\infty$, is equivalent to that of $g(f)$. One has that

$$\kappa_p^{-1}\|f\|_p \leqslant \|g(f)\|_p \leqslant \kappa_p\|f\|_p\,, \tag{2.2}$$

with a constant κ_p depending on p, but independent of the dimension n. A variant of this Littlewood–Paley function is defined by

$$g_1(f)^2 = \int_0^{+\infty} \Bigl| t \frac{\partial}{\partial t} P_t f \Bigr|^2 \frac{\mathrm{d}t}{t}\,. \tag{2.3}$$

It is clear that $g_1(f) \leqslant g(f)$, since $(\partial/\partial t)(P_t f)$ is a coordinate of the vector ∇u. The function g_1 is one of the variants studied by Stein [73]. More generally, for every integer $k \geqslant 1$, Stein sets

$$g_k(f)^2 = \int_0^{+\infty} \Bigl| t^k \frac{\partial^k}{\partial t^k} P_t f \Bigr|^2 \frac{\mathrm{d}t}{t}\,.$$

Let us define $Q_j = P_{2^j} - P_{2^{j+1}}$, for every $j \in \mathbb{Z}$. Since

$$\sum_{j\in\mathbb{Z}} |Q_j f|^2 = \sum_{j\in\mathbb{Z}} \Bigl| \int_{2^j}^{2^{j+1}} \Bigl(\frac{\partial}{\partial t} P_t f \Bigr) \mathrm{d}t \Bigr|^2,$$

we obtain by Cauchy–Schwarz that

$$\sum_{j\in\mathbb{Z}} |Q_j f|^2 \leqslant \sum_{j\in\mathbb{Z}} 2^j \int_{2^j}^{2^{j+1}} \Bigl| \frac{\partial}{\partial t} P_t f \Bigr|^2 \mathrm{d}t \leqslant \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \Bigl| \frac{\partial}{\partial t} P_t f \Bigr|^2 t\,\mathrm{d}t = g_1(f)^2\,.$$

The classical result (2.2) on $g(f)$ implies that for $1 < p < +\infty$, there exists a constant q_p independent of the dimension n such that

$$\Bigl\| \Bigl(\sum_{j\in\mathbb{Z}} |Q_j f|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mathbb{R}^n)} \leqslant \mathrm{q}_p \, \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n)\,. \tag{2.4}$$

Observe that the same proof implies that a similar inequality, with a different constant depending on $c > 1$, will hold for differences of the form $\widetilde{Q}_j = P_{t_j} - P_{t_{j-1}}$, where $(t_j)_{j\in\mathbb{Z}}$ is an increasing sequence of positive real numbers, provided that we have $t_{j+1} \leqslant c t_j$ for all js. On the other hand, by Rota's argument (1.15), one can obtain (2.4) from the Burkholder–Gundy inequalities of Theorem 1.6. Inequalities similar to (2.4) would hold for the Gaussian semi-group $(G_t)_{t\geqslant 0}$ defined in (1.19). Let us fix $T > 0$. We have seen that $G_{2t} f$, $0 \leqslant t \leqslant T$, is the projection on the σ-field $\mathcal{G}_T$ generated by B_T of the member M_{T-t} of the Brownian martingale $M_s = (P_{T-s} f)(B_s)$, $0 \leqslant s \leqslant T$, running under the infinite invariant measure given by the Lebesgue measure on $\mathbb{R}^n$. We then apply (1.15). Using Gaussian Q_js would allow us to avoid a few minor technical difficulties later, and this is essentially what Bourgain [13] does for the cube problem, see Section 8.

Relying on (1.15) and Remark 1.9 gives for the constant q_p in (2.4) an upper bound of order $p/(p-1)$ when $p \to 1$. This can also be obtained if one follows Stein [73, p. 48–51]. When $1 < p \leqslant 2$, the proof given there yields $\|g(f)\|_p \leqslant (p-1)^{-1/2} \mathrm{p}_p^{1-p/2} \, \|f\|_p$ for the right-hand side inequality in (2.2), where p_p is the constant in the maximal L^p-inequality for the Poisson semi-group. Since we have $\mathrm{p}_p \leqslant p/(p-1)$ by (1.31.P^*), we get that

$$\mathrm{q}_p \leqslant p/(p-1) \quad \text{when } \ 1 < p \leqslant 2\,. \tag{2.5}$$

Looking at the Fourier side, we see that $\sum_{j\in\mathbb{Z}} \widehat{Q}_j(\xi) = 1$ for every $\xi \neq 0$, since $\widehat{P_{2^j}}(\xi) = \mathrm{e}^{-2^{j+1}\pi|\xi|}$ tends to 1 when $j \to -\infty$ and to 0 when $j \to +\infty$. It implies for the convolution operators, still denoted by Q_j, that

$$\sum_{j\in\mathbb{Z}} Q_j = \mathrm{Id}\,. \tag{2.6}$$

2.1.1. Littlewood–Paley and maximal functions

Stein [73, Chap. III, §3, p. 75] explains how to get L^p estimates for several maximal functions related to semi-groups, by using the Littlewood–Paley functions. Consider a continuous function φ on the half-line $[0, +\infty)$, differentiable on $(0, +\infty)$, and denote by Φ its antiderivative vanishing at 0. For every $t > 0$, one has

$$t\varphi(t) = \int_0^t \bigl(s\varphi(s)\bigr)'\,\mathrm{d}s = \int_0^t \varphi(s)\,\mathrm{d}s + \int_0^t s\varphi'(s)\,\mathrm{d}s = \Phi(t) + \int_0^t s\varphi'(s)\,\mathrm{d}s\,.$$

Comparing L^1 and L^2 norms, one sees that

$$\int_0^t |s\varphi'(s)|\,\frac{\mathrm{d}s}{t} \leqslant \Bigl(\int_0^t |s\varphi'(s)|^2\,\frac{\mathrm{d}s}{t}\Bigr)^{1/2} \leqslant \Bigl(\int_0^t |s\varphi'(s)|^2\,\frac{\mathrm{d}s}{s}\Bigr)^{1/2}.$$

Therefore, one has

$$|\varphi(t)| \leqslant \frac{|\Phi(t)|}{t} + \Bigl(\int_0^{+\infty} |s\varphi'(s)|^2\,\frac{\mathrm{d}s}{s}\Bigr)^{1/2}, \quad t > 0\,.$$

One gets that

$$\sup_{t>0} |\varphi(t)| \leqslant \sup_{t>0} \frac{|\Phi(t)|}{t} + \Bigl(\int_0^{+\infty} |s\varphi'(s)|^2\,\frac{\mathrm{d}s}{s}\Bigr)^{1/2}.$$

If $\varphi(s) = (P_s f)(x)$ for a given $x \in \mathbb{R}^n$, the upper bound becomes

$$\sup_{t>0} |(P_t f)(x)| \leqslant \sup_{t>0} \frac{1}{t}\Bigl|\int_0^t (P_s f)(x)\,\mathrm{d}s\Bigr| + g_1(f)(x)\,.$$

One can (again) control the norm in L^p, $1 < p < +\infty$, of the maximal function of the Poisson semi-group, by the Hopf maximal inequality and the estimate for the Littlewood–Paley function. This control is easy in L^2, especially when L^2 admits an orthonormal basis (f_j) such that $P_t f_j = \mathrm{e}^{-t\lambda_j}\, f_j$ for every j, $\lambda_j \geqslant 0$, for example in the case of the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$. If $f = \sum_j a_j f_j$ in $L^2(\Omega)$, one has $P_t f = \sum_j a_j \,\mathrm{e}^{-\lambda_j t} f_j$, and

$$\begin{aligned}\int_\Omega g_1(f)(x)^2 \,\mathrm{d}x &= \int_0^{+\infty} \int_\Omega \Bigl| \sum_j a_j t\lambda_j \,\mathrm{e}^{-t\lambda_j} f_j(x) \Bigr|^2 \mathrm{d}x \,\frac{\mathrm{d}t}{t} \\ &= \int_0^{+\infty} \Bigl(\sum_j |a_j|^2 t^2 \lambda_j^2 \,\mathrm{e}^{-2t\lambda_j} \Bigr) \frac{\mathrm{d}t}{t} = \sum_j |a_j|^2 \int_0^{+\infty} t^2 \lambda_j^2 \,\mathrm{e}^{-2t\lambda_j} \,\frac{\mathrm{d}t}{t} \\ &= \Bigl(\int_0^{+\infty} u^2 \,\mathrm{e}^{-2u} \,\frac{\mathrm{d}u}{u} \Bigr) \sum_{\lambda_j > 0} |a_j|^2 \leqslant \frac{\Gamma(2)}{4} \|f\|_2^2 = \frac{1}{4} \|f\|_2^2 .\end{aligned}$$

For the other Littlewood–Paley functions $g_k(f)$, one has in the same way

$$\begin{aligned}\int_\Omega g_k(f)(x)^2 \,\mathrm{d}x &= \int_0^{+\infty} \int_\Omega \Bigl| \sum_j a_j t^k \lambda_j^k \,\mathrm{e}^{-t\lambda_j} f_j(x) \Bigr|^2 \mathrm{d}x \,\frac{\mathrm{d}t}{t} \\ &= \sum_j |a_j|^2 \int_0^{+\infty} t^{2k} \lambda_j^{2k} \,\mathrm{e}^{-2t\lambda_j} \,\frac{\mathrm{d}t}{t} \leqslant \frac{\Gamma(2k)}{4^k} \|f\|_2^2 .\end{aligned}$$

One can also work on $\mathbb{R}^n$ by Fourier transform with Parseval. One gets

$$\begin{aligned}\int_{\mathbb{R}^n} g_k(f)(x)^2 \,\mathrm{d}x &= (2\pi)^{2k} \int_0^{+\infty} \int_{\mathbb{R}^n} \bigl| \widehat{f}(\xi) t^k |\xi|^k \,\mathrm{e}^{-2\pi t|\xi|} \bigr|^2 \mathrm{d}\xi \,\frac{\mathrm{d}t}{t} \\ &= \frac{\Gamma(2k)}{4^k} \|f\|_2^2 .\end{aligned}$$

We have also other relations like

$$t^2 \varphi'(t) = \int_0^t \bigl(s^2 \varphi'(s) \bigr)' \,\mathrm{d}s = 2 \int_0^t s\varphi'(s) \,\mathrm{d}s + \int_0^t s^2 \varphi''(s) \,\mathrm{d}s$$

implying that

$$\sup_{t>0} |t\varphi'(t)| \leqslant 2 \int_0^{+\infty} |s\varphi'(s)|^2 \frac{\mathrm{d}s}{s} + \int_0^{+\infty} |s^2 \varphi''(s)|^2 \frac{\mathrm{d}s}{s} .$$

This brings back the successive maximal functions associated with each of the expressions $t^k \partial^k / \partial t^k (P_t f)$, $k \geqslant 1$, to quantities that can be estimated or are already estimated, as in

$$\sup_{t>0} \Bigl| t \frac{\partial}{\partial t} (P_t f)(x) \Bigr| \leqslant 2 g_1(f)(x) + g_2(f)(x) , \quad x \in \mathbb{R}^n .$$

2.2. Fourier multipliers

We introduce two dilation operators that appear in duality, for instance when dealing with the Fourier transform. Given a function g on $\mathbb{R}^n$ and

$\lambda > 0$, we use for these operations the notation

$$g_{(\lambda)}(x) = \lambda^{-n} g(\lambda^{-1}x)\,, \quad g_{[\lambda]}(x) = g(\lambda x)\,, \quad x \in \mathbb{R}^n. \tag{2.7}$$

If g already has a subscript, as in $g = g_1$, we shall use the heavier notation $(g_1)_{(\lambda)}$ or $(g_1)_{[\lambda]}$. One sees, for example when g is integrable and h bounded, that

$$\int_{\mathbb{R}^n} g_{(\lambda)}(x) h(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} g(y) h_{[\lambda]}(y)\,\mathrm{d}y\,, \text{ and } \widehat{(g_{(\lambda)})}(\xi) = \widehat{g}(\lambda\xi)\,, \ \ \xi \in \mathbb{R}^n,$$

that is to say, we have $\widehat{(g_{(\lambda)})} = (\widehat{g})_{[\lambda]}$. Clearly, $g_{(\lambda\mu)} = (g_{(\lambda)})_{(\mu)}$. The $g_{(\lambda)}$ dilation preserves the integral of g; it is extended to measures μ on $\mathbb{R}^n$ by setting $\mu_{(\lambda)}(f) = \mu(f_{[\lambda]})$, namely

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\mu_{(\lambda)}(x) = \int_{\mathbb{R}^n} f(\lambda x)\,\mathrm{d}\mu(x) \tag{2.8}$$

for every f in the space $\mathcal{K}(\mathbb{R}^n)$ of continuous and compactly supported functions. The measure $\mu_{(\lambda)}$ is the image of μ under the mapping $\mathbb{R}^n \ni x \mapsto \lambda x$. If $\mathrm{d}\mu(x) = g(x)\,\mathrm{d}x$, then $g_{(\lambda)}$ is the density of $\mu_{(\lambda)}$.

Let $\xi \mapsto m(\xi)$ belong to $L^\infty(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}^n)$, we have $\widehat{f} \in L^2(\mathbb{R}^n)$ by Plancherel, $\xi \mapsto m(\xi)\widehat{f}(\xi)$ is also in $L^2(\mathbb{R}^n)$ and is therefore the Fourier transform of some function $T_m f \in L^2(\mathbb{R}^n)$. We thus get a linear operator T_m on $L^2(\mathbb{R}^n)$ if we define $T_m f$, for every $f \in L^2(\mathbb{R}^n)$, by means of its Fourier transform, letting

$$(T_m f)\widehat{\ }(\xi) = m(\xi)\widehat{f}(\xi)\,, \quad \xi \in \mathbb{R}^n.$$

Let P_m be the operator of multiplication by m, defined by $P_m\varphi = m\varphi$. The operator $T_m = \mathcal{F}^{-1}P_m\mathcal{F}$ is bounded on $L^2(\mathbb{R}^n)$ since by Parseval, one has that

$$\int_{\mathbb{R}^n} |(T_m f)(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^n} |m(\xi)|^2 |\widehat{f}(\xi)|^2\,\mathrm{d}\xi \leqslant \|m\|_\infty^2\,\|f\|_2^2\,. \tag{2.9}$$

We shall say that T_m is the operator *associated to the multiplier* m.

One can ask whether T_m also operates as a bounded mapping on certain L^p spaces. In this survey, "bounded on L^p" will always mean *bounded from* L^p *to* L^p. Let q be the *conjugate exponent* of p, defined by $1/q + 1/p = 1$. Assuming that $1 < p < +\infty$, we see that T_m is bounded on $L^p(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} m(\xi)\widehat{\varphi}(\xi)\widehat{\psi}(\xi)\,\mathrm{d}\xi$ is uniformly bounded when $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ belong to the unit balls of $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ respectively, hence T_m is then also bounded on $L^q(\mathbb{R}^n)$ (and on $L^2(\mathbb{R}^n)$ by interpolation, so m has to be a bounded function, see the line after (2.12.**P**)).

We now observe that the multiplier m and its dilates $m_{[\lambda]} : \xi \mapsto m(\lambda\xi)$, $\lambda > 0$, define operators having equal norms on $L^p(\mathbb{R}^n)$. We see that

$$(T_{m_{[\lambda]}} f_{(\lambda)})\widehat{\ }(\xi) = m(\lambda\xi)\widehat{f}(\lambda\xi)$$

hence $T_{m_{[\lambda]}} f_{(\lambda)} = (T_m f)_{(\lambda)}$. Consider the operator $S_\lambda : f \mapsto f_{(\lambda)}$. For every $p \in [1, +\infty]$ and $1/q + 1/p = 1$, the multiple $S_{\lambda,p} := \lambda^{n/q} S_\lambda$ of S_λ is an isometric bijection of $L^p(\mathbb{R}^n)$ onto itself. The relation $S_\lambda \circ T_m = T_{m_{[\lambda]}} \circ S_\lambda$ becomes

$$T_{m_{[\lambda]}} = S_{\lambda,p} T_m S_{\lambda,p}^{-1} \tag{2.10}$$

and this implies that T_m and $T_{m_{[\lambda]}}$ have the same norm on $L^p(\mathbb{R}^n)$. More generally, let $\mathbf{m} = (m^{(j)})_{j \in J}$ be a family of multipliers and define $T_{\mathbf{m}} f = \sup_{j \in J} |T_{m^{(j)}} f|$. If we set $\mathbf{m}_{[\lambda]} = \bigl(m^{(j)}_{[\lambda]}\bigr)_{j \in J}$, then we have again that

$$T_{\mathbf{m}_{[\lambda]}} = S_{\lambda,p} T_{\mathbf{m}} S_{\lambda,p}^{-1} \tag{2.11}$$

because S_λ commutes with $f \mapsto |f|$ and $S_\lambda(\sup_{j \in J} f_j) = \sup_{j \in J} S_\lambda f_j$. Consequently, $T_{\mathbf{m}_{[\lambda]}}$ and $T_{\mathbf{m}}$ also have the same norm on $L^p(\mathbb{R}^n)$.

We shall speak of the *action on L^p of the multiplier m* and set

$$\|m\|_{p \to p} := \|T_m\|_{p \to p} \,.$$

If T_m is bounded on L^p, one says that m is a *multiplier on L^p*, or a *L^p-multiplier*. The next lemma will be useful, it is nothing but a direct consequence of the equality $\|m_{[\lambda]}\|_{p \to p} = \|m\|_{p \to p}$ for every $\lambda > 0$, and of the triangle inequality in L^p.

Lemma 2.1. — *Suppose that $1 \leqslant p \leqslant +\infty$ and that $m(\xi)$ is a $L^p(\mathbb{R}^n)$-multiplier. If the function ψ is integrable on $(0, +\infty)$, the multiplier N defined by*

$$N(\xi) = \int_0^{+\infty} \psi(\lambda) m(\lambda\xi) \, \mathrm{d}\lambda \,, \quad \xi \in \mathbb{R}^n,$$

is a $L^p(\mathbb{R}^n)$-multiplier and $\|N\|_{p \to p} \leqslant \|\psi\|_{L^1(0,+\infty)} \|m\|_{p \to p}$.

Note that clearly, multiplier operators commute to each other, and commute to translations and differentiations. We will apply many times the easy fact (2.9), which can be written as

$$\|m\|_{2 \to 2} = \|T_m\|_{2 \to 2} \leqslant \|m\|_{L^\infty(\mathbb{R}^n)} \,. \tag{2.12.$\mathbf{P}$}$$

The inequality is actually an equality, since by Parseval, the norm of T_m on $L^2(\mathbb{R}^n)$ is equal to that of P_m, the multiplication operator by m.

If K is a function integrable on $\mathbb{R}^n$, it acts by convolution on $L^p(\mathbb{R}^n)$ for all values $1 \leqslant p \leqslant +\infty$, and one gets easily by convexity of the L^p norm that

$$\|K * f\|_{L^p(\mathbb{R}^n)} \leqslant \|K\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \,. \tag{2.13}$$

This is an easy example of operator associated to a multiplier, since convolution of f with K corresponds to multiplication of $\widehat{f}$ by $\widehat{K}$. The Fourier transform $m = \widehat{K}$ of K is thus a multiplier on all spaces $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant +\infty$. Consider the Fourier transform m of the *convolution kernel* $K \in L^1(\mathbb{R}^n)$, equal to

$$m(\xi) = \int_{\mathbb{R}^n} K(x)\, \mathrm{e}^{-2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}x\,, \quad \xi \in \mathbb{R}^n.$$

For $\xi \neq 0$, let $\xi = |\xi|\theta$ and $x = y + s\theta$, where y is in the hyperplane $\theta^\perp$ orthogonal to $\theta \in S^{n-1}$, and $s \in \mathbb{R}$. By Fubini, we have for every real number u that

$$m(u\xi) = \int_{\mathbb{R}} \Bigl(\int_{\theta^\perp} K(y+s\theta)\, \mathrm{d}^{n-1}y \Bigr)\, \mathrm{e}^{-2\mathrm{i}\pi s u|\xi|}\, \mathrm{d}s\,,$$

where $\mathrm{d}^{n-1}y$ denotes the normalized Lebesgue measure on the Euclidean space $\theta^\perp \subset \mathbb{R}^n$. In what follows we associate to K and to θ in the unit sphere S^{n-1} the function $\varphi_{\theta,K}$ defined on $\mathbb{R}$ by

$$\forall\, s \in \mathbb{R}, \quad \varphi_{\theta,K}(s) := \int_{\theta^\perp} K(y+s\theta)\, \mathrm{d}^{n-1}y\,, \tag{2.14}$$

so that for $\xi \neq 0$ and $\theta = |\xi|^{-1}\xi$, letting $\varphi_\theta = \varphi_{\theta,K}$ we have

$$m(u\xi) = \int_{\mathbb{R}} \varphi_\theta(s)\, \mathrm{e}^{-2\mathrm{i}\pi s u|\xi|}\, \mathrm{d}s = \int_{\mathbb{R}} \frac{1}{|\xi|} \varphi_\theta\Bigl(\frac{v}{|\xi|}\Bigr)\, \mathrm{e}^{-2\mathrm{i}\pi v u}\, \mathrm{d}v\,. \tag{2.15}$$

The function $\mathbb{R} \ni u \mapsto m(u\theta)$ is the Fourier transform (in dimension 1) of φ_θ.

2.2.1. Multipliers "of Laplace type"

We consider a scalar function F on $(0,+\infty)$ that admits an expression of the form

$$\forall\, \lambda > 0\,, \quad F(\lambda) = \lambda \int_0^{+\infty} \mathrm{e}^{-\lambda t}\, a(t)\, \mathrm{d}t\,, \tag{2.16}$$

where a is a measurable function bounded on $(0,+\infty)$. The multiplier $m(\xi)$ "of Laplace type" associated to F is defined by $m(\xi) = F(|\xi|)$, for $\xi \in \mathbb{R}^n$. We note that $\|F\|_\infty \leqslant \|a\|_\infty$, thus by (2.12.**P**), this multiplier m is bounded on $L^2(\mathbb{R}^n)$ with operator norm $\leqslant \|a\|_\infty$. Stein proves the following result.

Proposition 2.2 ([73, Theorem 3′, p. 58]). — *Let F be defined on $(0,+\infty)$ by (2.16), for some function $a \in L^\infty(0,+\infty)$. The operator T_m associated to the multiplier $m(\xi) = F(|\xi|)$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < +\infty$ and*

$$\|T_m\|_{p\to p} \leqslant \lambda_p \|a\|_\infty\,,$$

where λ_p is a constant independent of the dimension n.

The identity operator belongs to this class (when $a(t) \equiv 1$), we thus see that $\lambda_p \geqslant 1$ for every p. It follows from the proposition that the imaginary powers of $(-\Delta)^{1/2}$ act on the spaces $L^p(\mathbb{R}^n)$ when $1 < p < +\infty$, with norms bounded independently of the dimension n. Indeed, we have the formula of Laplace type

$$\lambda^{\mathrm{i}b} = \frac{1}{\Gamma(1-\mathrm{i}b)}\,\lambda \int_0^{+\infty} \mathrm{e}^{-\lambda t}\, t^{-\mathrm{i}b}\,\mathrm{d}t\,, \quad \lambda > 0\,,\; a(t) = \frac{t^{-\mathrm{i}b}}{\Gamma(1-\mathrm{i}b)}\,, \tag{2.17}$$

hence $\|a\|_\infty = |\Gamma(1-\mathrm{i}b)|^{-1}$, for every $b \in \mathbb{R}$. According to the estimate (3.4) for the Gamma function, we get from Proposition 2.2 that

$$\forall\, b \in \mathbb{R},\quad \big\| |\xi|^{\mathrm{i}b} \big\|_{p \to p} \leqslant \lambda_p (1+b^2)^{-1/2}\, \mathrm{e}^{\pi|b|/2}\,, \quad 1 < p < +\infty\,. \tag{2.18}$$

Stein's proof of Proposition 2.2 draws on L^p inequalities for the Littlewood–Paley functions $g_1(f)$ and $g_2(f)$, and a comparison $g_1(T_m f) \leqslant \kappa\, g_2(f)$. We now sketch another possibility, which invokes martingale inequalities. If F is as in Proposition 2.2 and $m(\xi) = F(|\xi|)$, then $T_m f$, for $f \in \mathcal{S}(\mathbb{R}^n)$, can be expressed by

$$-\big(T_{m_{[2\pi]}} f\big) = \int_0^{+\infty} a(t) \Big(\frac{\partial}{\partial t} P_t f \Big)\,\mathrm{d}t\,. \tag{2.19}$$

Indeed, we know by (2.1) that

$$(P_t f)(x) = u(x,t) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi t|\xi|}\, \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\,\mathrm{d}\xi$$

and

$$\begin{aligned}
\Big(\int_0^{+\infty} a(t) \Big(\frac{\partial}{\partial t} P_t f \Big)\,\mathrm{d}t \Big)(x)
&= \int_0^{+\infty} a(t) \Big(\int_{\mathbb{R}^n} (-2\pi|\xi|)\, \mathrm{e}^{-2\pi t|\xi|}\, \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\,\mathrm{d}\xi \Big)\,\mathrm{d}t \\
&= -\int_{\mathbb{R}^n} F(2\pi|\xi|)\, \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\,\mathrm{d}\xi = -\big(T_{m_{[2\pi]}} f\big)(x)\,.
\end{aligned}$$

Suppose that a is a step function supported in $[t_0, t_N] \subset [0, +\infty)$. Then

$$a(t) = \sum_{j=1}^{N} a_j\, \mathbf{1}_{[t_{j-1}, t_j)}(t)\,,$$

with $0 = t_0 < t_1 < \ldots < t_N$. By (2.19), we obtain that

$$-T_{m_{[2\pi]}} f = \sum_{j=1}^{N} a_j (P_{t_j} - P_{t_{j-1}})(f)\,.$$

It follows that $T_m f$ can be considered as projection of a martingale transform by a conditional expectation $\mathrm{E}_{\mathcal{G}}$. Let $u_j = t_j/2$, $j = 0, \ldots, N$, and

$T := u_N$. We have seen in (1.13) that $P_{t_j} f = P_{2u_j} f$ is the image under the projection $\mathrm{E}_{\mathcal{G}}$ of the martingale member $M_{T-u_j} = (P_{u_j} f)(X_{T-u_j})$, so letting $L_i = M_{T-u_{N-i}}$, $i = 0, \ldots, N$, we see that $T_{m_{[2\pi]}} f$ is equal to

$$\mathrm{E}_{\mathcal{G}} \Bigl(\sum_{j=1}^{N} a_j (M_{T-u_{j-1}} - M_{T-u_j}) \Bigr) = \mathrm{E}_{\mathcal{G}} \Bigl(\sum_{i=1}^{N} a_{N-i+1} (L_i - L_{i-1}) \Bigr),$$

which is the transform of the martingale $(L_i)_{i=0}^N$ by the bounded non-random multipliers $(a_{N-i+1})_{i=1}^N$. Also, L_N is equal to $M_T = f(X_T)$ that has the distribution of f with respect to the (infinite) invariant measure, the Lebesgue measure on $\mathbb{R}^n$ (see Remark 1.2), hence $\|f\|_p = \|M_T\|_p$. In this simple case, one deduces Proposition 2.2 from Remark 1.8 about the Burkholder–Gundy inequalities, and it can be easily generalized, first to compactly supported continuous functions a. Using Remark 1.9, we find in this way that

$$\lambda_p \leqslant \kappa p^*, \qquad p^* := \max(p, p/(p-1)), \quad 1 < p < +\infty. \tag{2.20}$$

2.3. Riesz transforms

In dimension 1, there is only one Riesz transform R, which is called *the Hilbert transform H*. It is defined for $f \in L^2(\mathbb{R})$ by

$$\forall \xi \in \mathbb{R}, \quad (Rf)\widehat{\ }(\xi) = (Hf)\widehat{\ }(\xi) = -\frac{\mathrm{i}\xi}{|\xi|} \widehat{f}(\xi).$$

This is given by a multiplier of constant modulus 1 (almost everywhere), thus the transformation is isometric and invertible on $L^2(\mathbb{R})$ by Parseval, and H is a unitary operator on $L^2(\mathbb{R})$ with inverse $H^{-1} = -H$. If $\widetilde{u}(x,t)$ denotes the harmonic extension of Hf to the upper half-plane, then $u(x,t) + \mathrm{i}\widetilde{u}(x,t)$ is a holomorphic function of the complex variable $z = x + \mathrm{i}t$, because its Fourier transform vanishes for $\xi < 0$, implying by inverse Fourier transform that $u(x,t)$ is an integral in $\xi > 0$ of the holomorphic functions $\mathrm{e}^{-2\pi|\xi|t}\,\mathrm{e}^{2\mathrm{i}\pi\xi x} = \mathrm{e}^{2\mathrm{i}\pi\xi(x+\mathrm{i}t)}$. A classical theorem going back to Marcel Riesz [65] states that the Hilbert transform is bounded on $L^p(\mathbb{R})$ when $1 < p < +\infty$. This is also a consequence of the results on the Littlewood–Paley function $g(f)$, or of martingale inequalities as we shall see below. Some of the first deep connections between Brownian motion and classical Harmonic Analysis can be found in Burkholder–Gundy–Silverstein [18].

The Brownian argument is easier for the Hilbert transform $H_{\mathbb{T}}$ on the unit circle $\mathbb{T} \subset \mathbb{R}^2$. Let $(B_t)_{t \geqslant 0}$ be a plane Brownian motion defined on some $(\Omega, \mathcal{F}, P)$, starting from 0 in $\mathbb{R}^2$, and let τ be the first time t when B_t hits the circle $\mathbb{T}$. By rotational invariance, the distribution of B_τ is the uniform probability measure on the circle. Let f be a function in

$L^p(\mathbb{T})$ and let u be its harmonic extension to the unit disk. Assume that $2\pi u(0) = \int_0^{2\pi} f(\cos\theta, \sin\theta)\,\mathrm{d}\theta = 0$, and denote by $a \wedge b$ the minimum of a and b real. The random process $(M_t)_{t\geqslant 0} = (u(B_{t\wedge\tau}))_{t\geqslant 0}$ is a Brownian martingale, which can be expressed by the Itō integral

$$u(B_{t\wedge\tau}) = \int_0^{t\wedge\tau} \nabla u(B_s)\cdot \mathrm{d}B_s\,.$$

Suppose that $1 < p < +\infty$. By the continuous version of the Burkholder–Gundy inequalities, the norm $\|f\|_{L^p(\mathbb{T})} = \|u(B_\tau)\|_{L^p(\Omega,\mathcal{F},P)}$ is equivalent to the norm in $L^p(\Omega,\mathcal{F},P)$ of the square function of the martingale $(M_t)_{t\geqslant 0}$, given by

$$S(f) := \Bigl(\int_0^\tau |\nabla u(B_s)|^2\,\mathrm{d}s\Bigr)^{1/2}.$$

If $\widetilde{f} = H_{\mathbb{T}} f$ denotes the function on $\mathbb{T}$ conjugate to f and $\widetilde{u}$ its harmonic extension to the unit disk, then $|\nabla\widetilde{u}(x)| = |\nabla u(x)|$ for x in the unit disk, according to the Cauchy–Riemann equations for the function $u + \mathrm{i}\widetilde{u}$ holomorphic in the disk. It follows that $S(\widetilde{f}) = S(f)$ and the L^p-boundedness of the Hilbert transform for the circle is established via the Burkholder–Gundy inequalities of Theorem 1.6. The bound for the norm of $H_{\mathbb{T}}$ obtained in this manner is related to the constants in Burkholder–Gundy. The exact value of the L^p norm of H is known, this is due to Pichorides [61], see Remark 2.3 below.

In dimension n, there are n Riesz transforms R_j, defined on $L^2(\mathbb{R}^n)$ by

$$(R_j f)\widehat{\ }(\xi) = -\frac{\mathrm{i}\xi_j}{|\xi|}\,\widehat{f}(\xi), \quad j = 1,\dots,n.$$

Since $\bigl\|\bigl(\sum_{j=1}^n |R_j f|^2\bigr)^{1/2}\bigr\|_2^2 = \sum_{j=1}^n \|R_j f\|_2^2$, one has by Parseval that

$$\Bigl\|\Bigl(\sum_{j=1}^n |R_j f|^2\Bigr)^{1/2}\Bigr\|_2 = \|f\|_2\,. \tag{2.21}$$

The Riesz transforms are "collectively bounded" in $L^p(\mathbb{R}^n)$, by a constant ρ_p independent of the dimension n (Stein [76]), meaning that

$$\Bigl\|\Bigl(\sum_{j=1}^n |R_j f|^2\Bigr)^{1/2}\Bigr\|_p \leqslant \rho_p \|f\|_p\,, \quad 1 < p < +\infty\,. \tag{2.22}$$

Duoandikoetxea and Rubio de Francia [30] have connected in a few lines this inequality to the properties of the Hilbert transform (see also Pisier [63]).

Proof. — For each nonzero vector u in $\mathbb{R}^n$, let us introduce on $L^2(\mathbb{R}^n)$ the Hilbert transform H_u *in the direction* u by setting

$$\forall\, \xi \in \mathbb{R}^n, \quad (H_u f)\widehat{\ }(\xi) = -\frac{\mathrm{i}\, u\cdot\xi}{|u\cdot\xi|}\,\widehat{f}(\xi) = -\mathrm{i}\,\mathrm{sign}(u\cdot\xi)\,\widehat{f}(\xi)\,.$$

We deduce easily from the one-dimensional case that H_u acts on $L^p(\mathbb{R}^n)$, with the same norm as that of H on $L^p(\mathbb{R})$. It is enough to check the case when u is the first basis vector $\mathbf{e}_1$; if one writes the points x in $\mathbb{R}^n$ as $x = (t, y)$, $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, and if for f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ we set $f_y(t) = f(t, y)$, we can see that $(H_{\mathbf{e}_1} f)(t, y) = (H f_y)(t)$. Then, applying Fubini's theorem, we obtain

$$\begin{aligned}\iint |(H_{\mathbf{e}_1} f)(t,y)|^p \,\mathrm{d}t\mathrm{d}y &= \int_{\mathbb{R}^{n-1}} \Bigl(\int_{\mathbb{R}} |(H f_y)(t)|^p \,\mathrm{d}t \Bigr) \mathrm{d}y \\ &\leqslant \|H\|_{p\to p}^p \int_{\mathbb{R}^{n-1}} \Bigl(\int_{\mathbb{R}} |f_y(t)|^p \,\mathrm{d}t \Bigr) \mathrm{d}y = \|H\|_{p\to p}^p \|f\|_p^p \,.\end{aligned}$$

We can consider that $\mathcal{R}f = (R_1 f, \ldots, R_n f)$ is the operator associated to the vector-valued multiplier

$$\mathbf{m}(\xi) = -\mathrm{i}\,|\xi|^{-1}\xi \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n \,,$$

that is to say, the operator sending $f \in \mathcal{S}(\mathbb{R}^n)$ to the function $T_{\mathbf{m}} f$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose $\mathbb{R}^n$-valued Fourier transform is equal to $\widehat{f}(\xi)\mathbf{m}(\xi)$. For $f \in \mathcal{S}(\mathbb{R}^n)$, let us look at the vector-valued integral

$$(\mathcal{H}f)(x) = \int_{\mathbb{R}^n} (H_u f)(x) u \,\mathrm{d}\gamma_n(u) \in \mathbb{R}^n \,, \quad x \in \mathbb{R}^n \,,$$

where γ_n is the Gaussian probability measure from (1.17). The operator $\mathcal{H}$ corresponds to the vector-valued multiplier defined when $\xi \neq 0$ by

$$-\mathrm{i} \int_{\mathbb{R}^n} \operatorname{sign}(u \cdot \xi) u \,\mathrm{d}\gamma_n(u) = -\mathrm{i} \Bigl(\int_{\mathbb{R}} |v| \,\mathrm{d}\gamma_1(v) \Bigr) |\xi|^{-1}\xi = -\mathrm{i} \sqrt{\frac{2}{\pi}}\, |\xi|^{-1}\xi \,.$$

This can be seen by integrating on affine hyperplanes orthogonal to ξ. The "normalized" partial integral on the hyperplane $\xi^\perp + v|\xi|^{-1}\xi$, $v \in \mathbb{R}$, is equal to

$$\int_{\xi^\perp} \operatorname{sign}(v) (w + v|\xi|^{-1}\xi) \,\mathrm{d}\gamma_{\xi^\perp}(w) = |v|\, |\xi|^{-1}\xi \,.$$

It follows that $\mathcal{R}f = \sqrt{\pi/2}\,\mathcal{H}f$. For x fixed, the norm of $(\mathcal{H}f)(x)$ is the supremum of scalar products with vectors $\theta \in S^{n-1}$, and letting $1/q + 1/p = 1$, one has that

$$\begin{aligned}|(\mathcal{H}f)(x)| &\leqslant \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}^n} |\theta \cdot u| |(H_u f)(x)| \,\mathrm{d}\gamma_n(u) \\ &\leqslant \Bigl(\int_{\mathbb{R}} |v|^q \,\mathrm{d}\gamma_1(v) \Bigr)^{1/q} \Bigl(\int_{\mathbb{R}^n} |(H_u f)(x)|^p \,\mathrm{d}\gamma_n(u) \Bigr)^{1/p} .\end{aligned}$$

Using the notation g_q of (1.18) for the Gaussian moments, we get

$$\begin{aligned}\Bigl(\int_{\mathbb{R}^n} |(\mathcal{R}f)(x)|^p \,\mathrm{d}x\Bigr)^{1/p} &= \sqrt{\frac{\pi}{2}}\Bigl(\int_{\mathbb{R}^n} |(\mathcal{H}f)(x)|^p \,\mathrm{d}x\Bigr)^{1/p} \\ &\leqslant \sqrt{\frac{\pi}{2}}\, g_q \Bigl(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(H_u f)(x)|^p \,\mathrm{d}\gamma_n(u)\,\mathrm{d}x\Bigr)^{1/p} \\ &\leqslant \sqrt{\frac{\pi}{2}}\, g_q \,\|H\|_{p\to p}\,\|f\|_p\,. \end{aligned} \tag{2.23}$$

This argument yields $\rho_p \leqslant \sqrt{\pi/2}\, g_q \,\|H\|_{p\to p}$ for the constant ρ_p in (2.22). When $p = 2$, this gives $\rho_2 \leqslant \sqrt{\pi/2}$ instead of the correct value $\rho_2 = 1$ of (2.21). When p tends to 1, we obtain by (1.18) that

$$\Bigl(\int_{\mathbb{R}^n} |(\mathcal{R}f)(x)|^p \,\mathrm{d}x\Bigr)^{1/p} \leqslant \kappa \sqrt{q}\, \|H\|_{p\to p}\,\|f\|_p\,. \qquad \square$$

Remark 2.3. — The value $g_q = \bigl(\int_{\mathbb{R}} |v|^q \,\mathrm{d}\gamma_1(v)\bigr)^{1/q}$ tends to $\sqrt{2/\pi}$ when p tends to $+\infty$, and the asymptotic result $\rho_p \simeq \|H\|_{p\to p}$ obtained from (2.23) in this case is essentially best possible. Indeed, Iwaniec and Martin [47] have shown that the operator norm on $L^p(\mathbb{R}^n)$ of each individual Riesz transform R_j, $j = 1, \ldots, n$, is equal to the one of the Hilbert transform H on $L^p(\mathbb{R})$, hence $1 \leqslant \|H\|_{p\to p} \leqslant \rho_p$. According to Pichorides [61], the norm of the Hilbert transform is given by

$$\|H\|_{p\to p} = \cot\Bigl(\frac{\pi}{2p^*}\Bigr), \quad \text{with } \; p^* = \max\bigl(p, p/(p-1)\bigr)\,.$$

Iwaniec and Martin [47] also bound the "collective" norm in (2.22) by $\sqrt{2}\, H_p(1)$, where $H_p(1)$ is the norm on $L^p(\mathbb{C}) \simeq L^p(\mathbb{R}^2)$ of the "complex Hilbert transform", which corresponds to the multiplier $\mathbb{C} \ni \xi \mapsto \mathrm{i}\,|\xi|^{-1}\xi$, in other words, the operator $R_1 + \mathrm{i}\, R_2$ on $L^p(\mathbb{R}^2)$. Iwaniec and Sbordone [48, Appendix] add a few lines and give $H_p(1) \leqslant \frac{\pi}{2}\|H\|_{p\to p}$ so that finally

$$\rho_p \leqslant \sqrt{2}\, H_p(1) \leqslant \frac{\pi}{\sqrt{2}}\,\|H\|_{p\to p} \leqslant \kappa p^*\,. \tag{2.24}$$

Remark. — The proof from [30] is in the spirit of the *method of rotations*, which uses integration in polar coordinates to get directional operators in its radial part, see also Section 4.1. With this method, one can relate to the Hilbert transform not only the Riesz transforms, but also more general singular integrals with odd kernel, see [39, Section 5.2] for example.

3. Analytic tools

3.1. Some known facts about the Gamma function

From Euler's formula

$$\forall\, z \in \mathbb{C} \setminus (-\mathbb{N}), \quad \Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z\,(z+1) \ldots (z+n)}\,,$$

one passes to the Weierstrass infinite product for $1/\Gamma$, stating that

$$\frac{1}{\Gamma(z+1)} = \frac{1}{z\Gamma(z)} = \mathrm{e}^{\gamma z} \prod_{n=1}^{\infty} \Bigl((1 + z/n)\, \mathrm{e}^{-z/n} \Bigr),$$

where γ is the Euler–Mascheroni constant. It follows that $1/\Gamma$ is an entire function, with simple zeroes $z = 0, -1, -2, \ldots$. For the interpolation arguments to come, we need upper estimates on the modulus of $1/\Gamma(\sigma + \mathrm{i}\tau)$ for σ, τ real. From the preceding formula and from $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, we infer that

$$\Bigl| \frac{1}{\Gamma(1 + \mathrm{i}\tau)} \Bigr|^2 = \prod_{n=1}^{\infty} \Bigl(1 + \frac{\tau^2}{n^2} \Bigr) = \frac{\sinh(\pi\tau)}{\pi\tau}\,, \quad \tau \in \mathbb{R}\,, \tag{3.1}$$

according to another result due to Euler, the famous formula

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \Bigl(1 - \frac{z^2}{n^2} \Bigr). \tag{3.2.\textbf{E}}$$

The connoisseur has seen that we just came upon a special case of the "Euler reflection formula", stating that $\Gamma(z)^{-1}\Gamma(1-z)^{-1} = \sin(\pi z)/\pi$ for every $z \in \mathbb{C}$, or equivalently $\Gamma(1+z)^{-1}\Gamma(1-z)^{-1} = \sin(\pi z)/(\pi z)$. For every x real, one has

$$\frac{\sinh(\pi x)}{\pi x} \leqslant \frac{\mathrm{e}^{\pi|x|}}{1 + \pi|x|} \leqslant \frac{\mathrm{e}^{\pi|x|}}{(1+x^2)^{1/2}}\,. \tag{3.3}$$

The right-hand inequality is evident, the left-hand one is equivalent to saying that for every $y \geqslant 0$, we have $(1+y)\sinh(y) \leqslant y\,\mathrm{e}^y$ or $h(y) := (y-1)\,\mathrm{e}^{2y} + y + 1 \geqslant 0$, which is true because $h(0) = h'(0) = 0$ and $h''(y) = 4y\,\mathrm{e}^{2y} \geqslant 0$ when $y \geqslant 0$. Using (3.1) and (3.3), we get in particular that

$$\forall\, \tau \in \mathbb{R}\,, \quad \Bigl| \frac{1}{\Gamma(1 + \mathrm{i}\tau)} \Bigr| \leqslant \bigl(\sqrt{1+\tau^2} \bigr)^{-1/2} \mathrm{e}^{\pi|\tau|/2}\,. \tag{3.4}$$

More generally than in (3.1), for $\sigma \in [0,1]$, let us write

$$\begin{aligned}\left|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\right|^2 &= \mathrm{e}^{2\gamma\sigma} \prod_{n\geqslant 1} \Bigl(\bigl[(1+\sigma/n)^2 + (\tau/n)^2\bigr] \mathrm{e}^{-2\sigma/n} \Bigr) \\ &= \mathrm{e}^{2\gamma\sigma} \prod_{n\geqslant 1} \bigl((1+\sigma/n)^2 \mathrm{e}^{-2\sigma/n}\bigr) \prod_{n\geqslant 1} \left(1 + \left(\frac{\tau}{n+\sigma}\right)^2\right) \\ &= \Gamma(1+\sigma)^{-2} \prod_{n\geqslant 1} \left(1 + \left(\frac{\tau}{n+\sigma}\right)^2\right).\end{aligned}$$

We have by convexity of $\ln(1+x^{-2})$ for $x>0$ that

$$\begin{aligned}\prod_{n\geqslant 1} \left(1 + \left(\frac{\tau}{n+\sigma}\right)^2\right) &\leqslant \left(\prod_{n\geqslant 1} \left(1 + \left(\frac{\tau}{n}\right)^2\right)\right)^{1-\sigma} \left(\prod_{n\geqslant 1} \left(1 + \left(\frac{\tau}{n+1}\right)^2\right)\right)^{\sigma} \\ &= (1+\tau^2)^{-\sigma} \prod_{n\geqslant 1} \left(1 + \left(\frac{\tau}{n}\right)^2\right) \\ &= (1+\tau^2)^{-\sigma} \frac{\sinh(\pi\tau)}{\pi\tau}.\end{aligned}$$

It follows that

$$\left|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\right|^2 \leqslant \Gamma(1+\sigma)^{-2} (1+\tau^2)^{-\sigma} \frac{\sinh(\pi\tau)}{\pi\tau}$$

and applying (3.3) we obtain

$$\left|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\right| \leqslant \Gamma(1+\sigma)^{-1} \bigl(\sqrt{1+\tau^2}\bigr)^{1/2-1-\sigma} \mathrm{e}^{\pi|\tau|/2}. \tag{3.5}$$

We extend this bound by using the functional equation $z\Gamma(z) = \Gamma(z+1)$. When $z = k+1+\sigma+\mathrm{i}\tau$, with $\sigma \in (0,1)$ and $k \geqslant 1$ an integer, we have

$$\begin{aligned}\left|\frac{1}{\Gamma(k+1+\sigma+\mathrm{i}\tau)}\right| &= \left(\prod_{j=1}^{k} \bigl((j+\sigma)^2+\tau^2\bigr)^{-1/2}\right) \left|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\right| \\ &\leqslant \bigl(\sqrt{1+\tau^2}\bigr)^{-k} \left|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\right|. \end{aligned} \tag{3.6}$$

Letting $a \wedge b = \min(a,b)$ for $a, b \in \mathbb{R}$, we see that

$$\begin{aligned}\Gamma(1+\sigma) &= \int_0^{+\infty} u^{\sigma} \mathrm{e}^{-u} \,\mathrm{d}u \\ &\geqslant \int_0^{+\infty} (u \wedge 1) \mathrm{e}^{-u} \,\mathrm{d}u = \int_0^1 \mathrm{e}^{-u} \,\mathrm{d}u = 1 - \mathrm{e}^{-1} > \frac{1}{2}.\end{aligned}$$

Let us mention that the actual minimal value of Γ on $(0,+\infty)$ is reached at

$$x_\Gamma = 1.46163\dots \quad \text{and that} \quad \Gamma(x_\Gamma) > 0.88\,. \tag{3.7}$$

Note that on $(0,+\infty)$, the function $x \mapsto \ln\Gamma(x)$ is convex and $\ln\Gamma(1) = \ln\Gamma(2) = 0$, hence $\Gamma(x) \leqslant 1$ when $x \in [1,2]$ and $\Gamma(x) \geqslant 1$ on $(0,1]$ and $[2,+\infty)$.

We get consequently by (3.5) and (3.6) that

$$\Bigl|\frac{1}{\Gamma(k+1+\sigma+\mathrm{i}\tau)}\Bigr| \leqslant 2\,\bigl(\sqrt{1+\tau^2}\bigr)^{1/2-k-1-\sigma}\;\mathrm{e}^{\pi|\tau|/2}\,. \tag{3.8}$$

When $z = -k+\sigma+\mathrm{i}\tau$, with $k \geqslant 0$ an integer, we obtain by the functional equation

$$\Bigl|\frac{1}{\Gamma(z)}\Bigr| = \Bigl|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\Bigr| \prod_{j=-k}^{0} ((j+\sigma)^2+\tau^2)^{1/2}.$$

For $j = 0,-1$, the factors in the product are $\leqslant (1+\tau^2)^{1/2}$, thus

$$\Bigl|\frac{1}{\Gamma(-k+\sigma+\mathrm{i}\tau)}\Bigr| \leqslant \bigl(1+\tau^2\bigr)^{(k+1)/2}\Bigl|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\Bigr| \quad \text{when } k = 0,1\,, \tag{3.9}$$

and when $j \leqslant -2$, we have $((j+\sigma)^2+\tau^2)^{1/2} \leqslant (|j|-\sigma)\,(1+\tau^2)^{1/2}$. It follows for $z = -k+\sigma+\mathrm{i}\tau$, $k \geqslant 2$, that

$$\Bigl|\frac{1}{\Gamma(z)}\Bigr| \leqslant (k-\sigma)(k-1-\sigma)\dots(2-\sigma)\,\bigl(1+\tau^2\bigr)^{(k+1)/2}\Bigl|\frac{1}{\Gamma(1+\sigma+\mathrm{i}\tau)}\Bigr|\,. \tag{3.10}$$

By the functional equation and the convexity of $\ln\Gamma$ on $(0,+\infty)$, we have

$$\begin{aligned}\Gamma(1+\sigma)^{-1}(k-\sigma)(k-1-\sigma)\dots(2-\sigma) &= \frac{\Gamma(k+1-\sigma)}{\Gamma(2-\sigma)\Gamma(1+\sigma)}\\ &\leqslant \frac{\Gamma(k+1-\sigma)}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}\Gamma(k+1-\sigma) < 2\Gamma(k+1-\sigma)\,.\end{aligned}$$

Coming back to (3.10) and using (3.5), we conclude when $k \geqslant 2$ that

$$\Bigl|\frac{1}{\Gamma(-k+\sigma+\mathrm{i}\tau)}\Bigr| \leqslant 2\Gamma(k-\sigma+1)\,\bigl(\sqrt{1+\tau^2}\bigr)^{1/2+k-\sigma}\;\mathrm{e}^{\pi|\tau|/2}\,. \tag{3.11}$$

When $\operatorname{Re} z \geqslant -1$, it follows from (3.8) and (3.9) that

$$\Bigl|\frac{1}{\Gamma(z)}\Bigr| \leqslant 2\,\bigl(\sqrt{1+(\operatorname{Im} z)^2}\bigr)^{1/2-\operatorname{Re} z}\;\mathrm{e}^{\pi|\operatorname{Im} z|/2},$$

so, in every half-plane of the form $\operatorname{Re} z \geqslant a$, one has by (3.11) an upper bound

$$\Bigl|\frac{1}{\Gamma(z)}\Bigr| \leqslant \beta_a\bigl(\sqrt{1+|\operatorname{Im} z|^2}\bigr)^{1/2-a}\;\mathrm{e}^{\pi|\operatorname{Im} z|/2}, \tag{3.12.Γ}$$

with $\beta_a = 2\Gamma(|a|+1)$ when $a \leqslant -1$, and $\beta_a = 2$ otherwise.

Remark. — The rather crude estimate (3.12.Γ) is sufficient for our purposes. In [73], Stein refers to Titchmarsh [82, p. 259], for an exact asymptotic estimate. When σ is fixed and $|\tau| \to +\infty$, one has

$$|\Gamma(\sigma + \mathrm{i}\tau)| \simeq \sqrt{2\pi}\, \mathrm{e}^{-\pi|\tau|/2}\, |\tau|^{\sigma-1/2}.$$

When $\sigma \geqslant 1$, the preceding proof gives a *lower bound* $2^{-1}\sqrt{2\pi}\, \mathrm{e}^{-\pi|\tau|/2}\, |\tau|^{\sigma-1/2}$ for every τ. We can see it by replacing the inequality (3.3) with the evident inequality $\sinh(\pi x)/(\pi x) \leqslant (2\pi|x|)^{-1}\, \mathrm{e}^{\pi|x|}$. It is not possible to replace $\sqrt{1 + |\operatorname{Im} z|^2}$ by $|\operatorname{Im} z|$ in (3.12.Γ) when $\operatorname{Re} z \leqslant -1$, because the zeroes $-1, -2, \ldots$ of $1/\Gamma$ are simple. For more results on the Gamma function, we refer to Andrews–Askey–Roy [2].

3.2. The interpolation scheme

We begin with the classical *three lines lemma*, an easier version of which is the *Hadamard three-circle theorem.* After this, we shall turn to interpolation of holomorphic families of linear operators.

3.2.1. The three lines lemma

Lemma 3.1. — *Let S denote the open strip $\{z : 0 < \operatorname{Re} z < 1\}$ in the complex plane. Let f be a function holomorphic in S and continuous on the closure of S. Assume that f is bounded in S and that*

$$|f(0 + \mathrm{i}\tau)| \leqslant C_0, \quad |f(1 + \mathrm{i}\tau)| \leqslant C_1$$

for all $\tau \in \mathbb{R}$. Then, for every $\theta \in (0, 1)$, one has that $|f(\theta)| \leqslant C_0^{1-\theta} C_1^{\theta}$.

Remark 3.2. — Of course $f(\theta + \mathrm{i}\tau)$ admits the same bound for every $\tau \in \mathbb{R}$, by translating f vertically. The somewhat strange assumption that f must be *bounded* on the whole strip by a value which does not appear in the final result is not the finest assumption that makes the conclusion valid, see a better criterion below. However, when Lemma 3.1 applies, the function f *is bounded* at last. It is well known that some restriction on the size of f inside the strip is needed for the lemma to hold true. Indeed, in the strip $S_\pi = \{z : |\operatorname{Re} z| \leqslant \pi/2\}$, the function $f(z) = \mathrm{e}^{\cos z}$ has modulus one on the two lines $\operatorname{Re} z = \pm\pi/2$, but it is "very big" when $\operatorname{Re} z = 0$, since $|f(\mathrm{i}\tau)| = \mathrm{e}^{\cosh(\tau)}$. For a function f holomorphic in an open vertical strip S, continuous on the closure and bounded by 1 on the two boundary lines, either f is bounded by 1 on S, or else $\sup_{|\operatorname{Im} z| = |\tau|} |f(z)|$ must become extremely large when $|\tau|$ tends to infinity. This is the typical situation with the theorems of Phragmén–Lindelöf type, see [69, Chap 12, 12.7] for example.

Here is a sufficient criterion ensuring that $|f|$ is bounded by its supremum on the boundary ∂S_w of a vertical strip S_w of width w. If f is holomorphic on S_w, continuous on the closure with $|f| \leqslant 1$ on ∂S_w, and if for some $a < \pi/w$ one has

$$|f(z)| = O\big(\exp(\mathrm{e}^{a|\operatorname{Im} z|})\big)$$

when z tends to infinity in S_w, then $|f|$ is bounded by 1 on the strip. Let us prove it assuming $\ln|f(z)| \leqslant \kappa\,\mathrm{e}^{a|\operatorname{Im} z|}$ in $S_\pi = \{|\operatorname{Re} z| \leqslant \pi/2\}$, for an $a < 1 = \pi/w$. Set $g_\varepsilon(z) = \mathrm{e}^{-\varepsilon\cos(bz)}$, with $\varepsilon > 0$ and $a < b < 1$. If $z = \sigma + \mathrm{i}\tau$ and $|\sigma| \leqslant \pi/2$, we have

$$\begin{aligned}|g_\varepsilon(z)| &= \exp\big(-\varepsilon\operatorname{Re}\cos(bz)\big) = \exp\big(-\varepsilon\cos(b\sigma)\cosh(b\tau)\big)\\ &\leqslant \exp\big(-\varepsilon\cos(b\pi/2)\cosh(b\tau)\big) \leqslant \exp\big(-B_\varepsilon\,\mathrm{e}^{b|\tau|}\big) \leqslant 1\,, \quad B_\varepsilon > 0\,,\end{aligned}$$

hence $|f(z)g_\varepsilon(z)| \leqslant 1$ on ∂S_π, and if $|\tau| = |\operatorname{Im} z| \geqslant (b-a)^{-1}\ln(\kappa/B_\varepsilon)$ we get

$$\ln|f(z)g_\varepsilon(z)| \leqslant \kappa\,\mathrm{e}^{a|\tau|} - B_\varepsilon\,\mathrm{e}^{b|\tau|} \leqslant 0\,. \tag{3.13}$$

Given any $z_0 \in S_\pi$, we can find a rectangle $R_\varepsilon = \{|\operatorname{Re} z| \leqslant \pi/2,\ |\operatorname{Im} z| \leqslant \tau_0(\varepsilon)\}$ containing z_0 such that $|f(z)g_\varepsilon(z)| \leqslant 1$ on ∂R_ε. We then have $|f(z_0)g_\varepsilon(z_0)| \leqslant 1$ by the maximum principle, $|f(z_0)| \leqslant |\mathrm{e}^{\varepsilon\cos(bz_0)}|$ for every $\varepsilon > 0$, thus $|f(z_0)| \leqslant 1$.

Several times later on, we encounter situations where the function f is not bounded on the two lines limiting a vertical strip S, but has instead a growth exponential in $|\tau| = |\operatorname{Im} z|$. The next lemma generalizes the preceding. Our proof and estimate are not the "correct" ones, as we shall explain below after Corollary 3.4, but they give a reasonable explicit bound. In these Notes, we shall say that a function f defined on a vertical strip S has an *admissible growth* in the strip if for some $\kappa > 0$, the function f admits in S a bound of the form $|f(z)| \leqslant \kappa\,\mathrm{e}^{\kappa|\operatorname{Im} z|}$.

Lemma 3.3. — *Let f be a function holomorphic in the strip $S = \{z : 0 < \operatorname{Re} z < 1\}$, with admissible growth in S and continuous on the closure of S. Assume that there exist real numbers $a_0, a_1 \geqslant 0$ and b_0, b_1 such that for every $\tau \in \mathbb{R}$, one has*

$$|f(0 + \mathrm{i}\tau)| \leqslant \mathrm{e}^{a_0|\tau| + b_0}, \quad |f(1 + \mathrm{i}\tau)| \leqslant \mathrm{e}^{a_1|\tau| + b_1}\,.$$

For every $\theta \in (0, 1)$, it follows that

$$|f(\theta)| \leqslant \exp\left(\sqrt{\theta(1-\theta)}\,\sqrt{(1-\theta)a_0^2 + \theta a_1^2} + (1-\theta)b_0 + \theta b_1\right).$$

Proof. — We introduce the holomorphic function $g(z) := \mathrm{e}^{cz^2/2 + dz}$, with $c > 0$ and d real. If $z = \sigma + \mathrm{i}\tau$, we see that $|g(z)| = \mathrm{e}^{c(\sigma^2 - \tau^2)/2 + d\sigma}$. On the vertical side $\operatorname{Re} z = 0$ of S, we have that

$$|f(\mathrm{i}\tau)g(\mathrm{i}\tau)| \leqslant \mathrm{e}^{a_0|\tau| + b_0 - c\tau^2/2} \leqslant \mathrm{e}^{a_0^2/(2c) + b_0} =: E_0$$

and when $\operatorname{Re} z = 1$, we get the upper bound

$$|f(1+\mathrm{i}\tau)g(1+\mathrm{i}\tau)| \leqslant \mathrm{e}^{a_1|\tau|+b_1-c\tau^2/2+c/2+d} \leqslant \mathrm{e}^{a_1^2/(2c)+b_1+c/2+d} =: E_1 \,.$$

We choose d so that $E_0 = E_1$, and we need not mention the value of d.

It follows from the assumption $|f(z)| \leqslant \kappa\, \mathrm{e}^{\kappa|\operatorname{Im} z|} = \kappa\, \mathrm{e}^{\kappa|\tau|}$ that $f(z)g(z)$ tends to zero at infinity in S. Let us fix $\theta \in (0,1)$. If $f(\theta) \neq 0$, there exists $\tau_0 > 0$ such that $|f(z)g(z)| < |f(\theta)g(\theta)|$ when $|\operatorname{Im} z| \geqslant \tau_0$. By the maximum principle for the compact rectangle $R = \{0 \leqslant \operatorname{Re} z \leqslant 1,\, |\operatorname{Im} z| \leqslant \tau_0\}$, we know that the maximum of $|f(z)g(z)|$ is reached at the boundary of R, but it cannot be on the horizontal sides $|\operatorname{Im} z| = \tau_0$. Hence $|f(\theta)g(\theta)| \leqslant E_0 = E_1 = E_0^{1-\theta} E_1^{\theta}$, we get therefore

$$\begin{aligned}|f(\theta)| &\leqslant \mathrm{e}^{-c\theta^2/2-d\theta}\, E_0^{1-\theta} E_1^{\theta}\\ &= \exp\Bigl(\frac{(1-\theta)a_0^2+\theta a_1^2}{2c} + (1-\theta)b_0 + \theta b_1 + c\theta(1-\theta)/2\Bigr)\end{aligned}$$

and after optimizing in $c > 0$, we conclude that

$$|f(\theta)| \leqslant \exp\Bigl(\sqrt{(1-\theta)a_0^2+\theta a_1^2}\,\sqrt{\theta(1-\theta)} + (1-\theta)b_0 + \theta b_1\Bigr). \qquad \square$$

Corollary 3.4. — *Let f be a function holomorphic in $S = \{z : \alpha_0 < \operatorname{Re} z < \alpha_1\}$, with admissible growth in the strip S and continuous on the closure of S. Assume that there exist real numbers $u_0, u_1 \geqslant 0$ and v_0, v_1 such that*

$$|f(\alpha_0+\mathrm{i}\tau)| \leqslant \mathrm{e}^{u_0|\tau|+v_0}, \quad |f(\alpha_1+\mathrm{i}\tau)| \leqslant \mathrm{e}^{u_1|\tau|+v_1}$$

for every $\tau \in \mathbb{R}$. Let $\theta \in [0,1]$, set $\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1$, $u_\theta = (1-\theta)u_0 + \theta u_1$ and $v_\theta = (1-\theta)v_0 + \theta v_1$. For every $\tau \in \mathbb{R}$, one has

$$|f(\alpha_\theta+\mathrm{i}\tau)| \leqslant E_{w,\theta}(u_0,u_1)\, \mathrm{e}^{u_\theta|\tau|+v_\theta}\,,$$

where $w = \alpha_1 - \alpha_0$ denotes the width of the strip S and where

$$E_{w,\theta}(u_0,u_1) := \exp\Bigl(w\sqrt{\theta(1-\theta)}\,\sqrt{(1-\theta)u_0^2+\theta u_1^2}\Bigr).$$

Notice that $\sqrt{\theta(1-\theta)} \leqslant 1/2$ for every $\theta \in [0,1]$. When $0 \leqslant u_0, u_1 \leqslant u$, one can always employ the simpler bound $E_{w,\theta}(u,u) \leqslant \mathrm{e}^{wu/2}$.

Proof. — We begin with $S_1 := \{0 < \operatorname{Re} z < 1\}$. We bound the modulus of $f(\theta+\mathrm{i}\tau_0)$ for τ_0 in $\mathbb{R}$ by performing a vertical translation of f, then invoking Lemma 3.3. The function $F(z) = f(z+\mathrm{i}\tau_0)$ satisfies $|F(j+\mathrm{i}\tau)| \leqslant \mathrm{e}^{u_j|\tau|+(u_j|\tau_0|+v_j)}$, $j = 0,1$, and the bound for $|F(\theta)|$ given at Lemma 3.3 implies that

$$|f(\theta+\mathrm{i}\tau_0)| \leqslant E_{1,\theta}(u_0,u_1)\, \mathrm{e}^{u_\theta|\tau_0|+v_\theta}\,, \quad \tau_0 \in \mathbb{R}\,. \tag{3.14}$$

It is easy to pass to $S = \{\alpha_0 < \operatorname{Re} z < \alpha_1\}$ with the transform that replaces $f(z)$, defined for $z \in S$, by $F(Z) = f(\alpha_0 + Zw)$ for $Z \in S_1$. If $|f(\alpha_j + \mathrm{i}\tau)| \leqslant \mathrm{e}^{u_j|\tau|+v_j}$, $j = 0, 1$, then $|F(j + \mathrm{i}\tau)| \leqslant \mathrm{e}^{wu_j|\tau|+v_j}$ and by (3.14) we have that

$$|f(\alpha_\theta + \mathrm{i}\tau_0)| = |F(\theta + \mathrm{i}\tau_0/w)| \\ \leqslant E_{1,\theta}(wu_0, wu_1)\, \mathrm{e}^{(wu_\theta)|\tau_0/w|+v_\theta} = E_{w,\theta}(u_0, u_1)\, \mathrm{e}^{u_\theta|\tau_0|+v_\theta}. \qquad \square$$

Applying Corollary 3.4 in the case where $u_0 = u_1 = u \geqslant 0$ and $v_j = 0$, one sees that when f has an admissible growth in S, the hypothesis $|f(\alpha_j+\mathrm{i}\tau)| \leqslant \mathrm{e}^{u|\tau|}$ for all $\tau \in \mathbb{R}$ and $j = 0, 1$, implies $|f(\alpha + \mathrm{i}\tau)| \leqslant \mathrm{e}^{wu/2}\, \mathrm{e}^{u|\tau|}$ in the strip. It is not possible to replace the "bounding factor" $\mathrm{e}^{wu/2}$ by 1, as we shall understand below.

The "correct" proof of Lemma 3.3 uses a lemma given by Hirschman [45, Lemma 1], cited by Stein [70]. In our case, we consider the function U, harmonic in the open strip $S_1 = \{0 < \operatorname{Re} z < 1\}$ and continuous on the closed strip, equal to $a_j|\tau| + b_j$ at each boundary point $j + \mathrm{i}\tau$, with $a_j \geqslant 0$, $\tau \in \mathbb{R}$ and $j = 0, 1$. Let V be the *harmonic conjugate* of U in S_1, defined up to an additive constant by the fact that $\nabla V(z)$, for $z \in S_1$, is equal to $R\nabla U(z)$ where R is the rotation of angle $+\pi/2$ in $\mathbb{R}^2 \simeq \mathbb{C}$. Let us set $V(1/2) = 0$ in order to fix V entirely. Since U is harmonic, the 1-form $-U_y\, dx + U_x\, dy$ is closed and $V(z) = \int_0^1 R\nabla U(\gamma(s)) \cdot \gamma'(s)\, \mathrm{d}s$ for any C^1 path γ in S_1 such that $\gamma(0) = 1/2$ and $\gamma(1) = z$. Then $U + \mathrm{i}V$ is holomorphic, by the Cauchy–Riemann equations. Consider the holomorphic *outer function*

$$g(z) = \exp\bigl(-U(z) - \mathrm{i}V(z)\bigr), \quad z \in S_1,$$

for which $|g(z)| = \exp(-U(z))$ and $|g(z)| \leqslant \mathrm{e}^{-(b_0 \wedge b_1)}$ in S_1. If f is as in Lemma 3.3, then $|fg| \leqslant 1$ at the boundary of S_1 and fg has an admissible growth. It follows from an easy variation of Lemma 3.1 that $|(fg)(\theta)| \leqslant 1$ thus $|f(\theta)| \leqslant \mathrm{e}^{U(\theta)}$, and it remains to express $U(\theta)$, with the help of the harmonic measure at θ for S_1.

We shall obtain the harmonic measures for $S_\pi = \{z \in \mathbb{C} : |\operatorname{Re} z| < \pi/2\}$ from the case of the open unit disk D, by a conformal mapping (see also [39, proof of Lemma 1.3.8]). Let σ belong to $I_\pi = (-\pi/2, \pi/2) = S_\pi \cap \mathbb{R}$. The Poisson probability measure μ_σ at σ relative to S_π can be written as $\mu_\sigma = \mu_{\sigma,0} + \mu_{\sigma,1}$, where $\mu_{\sigma,0}$ is supported on $B_0 = -\pi/2 + \mathrm{i}\mathbb{R}$ and $\mu_{\sigma,1}$ on $B_1 = \pi/2 + \mathrm{i}\mathbb{R}$. If h is real, harmonic in S_π, bounded and continuous on the closure of S_π, the value of h at σ is equal to

$$h(\sigma) = \int_{\partial S_\pi} h\, \mathrm{d}\mu_\sigma = \int_{B_0} h\, \mathrm{d}\mu_{\sigma,0} + \int_{B_1} h\, \mathrm{d}\mu_{\sigma,1}. \tag{3.15}$$

The Poisson probability measure ν_r for D at $r \in (-1, 1)$ has density $g_r(\mathrm{e}^{\mathrm{i}\beta}) = (1 - r^2)/(1 - 2r\cos\beta + r^2)$ with respect to the invariant probability measure on the unit circle $\mathbb{T}$. Let Φ be the holomorphic bijection from S_π onto D given by $\Phi(z) = \tan(z/2)$ when $z \in S_\pi$, extended to $|\operatorname{Re} z| = \pi/2$ by the same formula. Then ∂S_π is sent to $\mathbb{T} \setminus \{\mathrm{i}, -\mathrm{i}\}$ and if $\Phi(\pi/2 + \mathrm{i}\tau) = \mathrm{e}^{\mathrm{i}\beta}$, we have $\beta \in (-\pi/2, \pi/2)$ and $\tanh(\tau/2) = \tan(\beta/2)$. For $r = \tan(\sigma/2)$ we see that $\nu_r = \Phi_\# \mu_\sigma$ and

$$\int_{B_1} h \,\mathrm{d}\mu_{\sigma,1} = \int_{\mathbb{R}} h(\pi/2 + \mathrm{i}\tau) f_\sigma(\tau)\,\mathrm{d}\tau \quad \text{with } f_\sigma(\tau) = \frac{\cos\sigma}{2\pi(\cosh\tau - \sin\sigma)}\,,$$

while $\int_{B_0} h\,\mathrm{d}\mu_{\sigma,0} = \int_{\mathbb{R}} h(-\pi/2 + \mathrm{i}\tau) f_{-\sigma}(\tau)\,\mathrm{d}\tau$. One finds $\|\mu_{\sigma,1}\|_1 = \theta := \sigma/\pi + 1/2$ and $\|\mu_{\sigma,0}\|_1 = 1 - \theta$ by harmonicity of $h(z) = \operatorname{Re} z$. When σ tends to $\pi/2$, the density f_σ resembles the Cauchy kernel $P^{(1)}_\varepsilon$ in (1.33.**C**) with $\varepsilon = \pi/2 - \sigma$, since

$$f_\sigma(\tau) = \frac{1}{2\pi} \frac{\sin\varepsilon}{\cosh\tau - \cos\varepsilon} \simeq \frac{\varepsilon}{\pi(\tau^2 + \varepsilon^2)}\,.$$

One can also comprehend f_σ as sum of the alternate series of Cauchy kernels

$$f_\sigma = P^{(1)}_{\pi/2-\sigma} - P^{(1)}_{\pi+\pi/2+\sigma} + P^{(1)}_{2\pi+\pi/2-\sigma} - P^{(1)}_{2\pi+\pi+\pi/2+\sigma} + P^{(1)}_{4\pi+\pi/2-\sigma} - P^{(1)}_{4\pi+\pi+\pi/2+\sigma} + \cdots\,,$$

indeed, if φ_σ denotes the sum of the series above and if g belongs to $\mathcal{K}(\mathbb{R})$, then $G(\sigma + \mathrm{i}\tau) = (\varphi_\sigma * g)(\tau)$ is harmonic in S_π, tends to $g(\tau)$ when $\sigma \to \pi/2$ and to 0 when $\sigma \to -\pi/2$, the same properties as for $(f_\sigma * g)(\tau)$.

Let h_* be a continuous function on ∂S_π, and suppose that the two functions $t \mapsto \mathrm{e}^{-|t|}\, h_*(\pm\pi/2 + \mathrm{i}t)$ are Lebesgue-integrable on the real line. Then, writing

$$\widetilde{h}_*(z) = \int_{\mathbb{R}} \bigl(h_*(\pi/2 + \mathrm{i}(\tau - t)) f_\sigma(t) + h_*(-\pi/2 + \mathrm{i}(\tau - t)) f_{-\sigma}(t)\bigr)\,\mathrm{d}t \qquad (3.16)$$

for every $z = \sigma + \mathrm{i}\tau \in S_\pi$, one defines a harmonic function $\widetilde{h}_*$ in S_π, continuous on the closure if one sets $\widetilde{h}_*(z_*) = h_*(z_*)$ for $z_* \in \partial S_\pi$. Let $\mathcal{H}_c(S_\pi)$ denote the class of functions harmonic in S_π and continuous on the closure. Not every $h \in \mathcal{H}_c(S_\pi)$ can be expressed by (3.16) from its restriction $h_* = h\big|_{\partial S_\pi}$. First, h_* must be μ_σ-integrable, but even then, $h(z) = \operatorname{Re}\cos(z) = \cos(\sigma)\cosh(\tau)$, for which $h_* = 0$, is a counterexample.

Let us say here that g defined on S_π, resp. ∂S_π, is *moderate* if there is $a < 1$ such that $g(z) = O(\mathrm{e}^{a|\operatorname{Im} z|})$ for $z \in S_\pi$, resp. ∂S_π. If h_* is moderate and continuous on ∂S_π, the extension $\widetilde{h}_*$ in (3.16) is in $\mathcal{H}_c(S_\pi)$, and it is moderate because

$$\begin{aligned}|\widetilde{h}_*(\sigma+\mathrm{i}\tau)| &\leqslant \kappa \int_{\mathbb{R}} \mathrm{e}^{a|\tau-t|}(f_\sigma+f_{-\sigma})(t)\,\mathrm{d}t \\ &\leqslant \kappa\Bigl(\mathrm{e}^a+\int_1^{+\infty}\frac{\mathrm{e}^{a|t|}}{\cosh t-1}\,\mathrm{d}t\Bigr)\,\mathrm{e}^{a|\tau|}\,.\end{aligned}$$

Lemma 3.5 (after [45]). — *If $h \in \mathcal{H}_c(S_\pi)$ is moderate and $h_* = h\big|_{\partial S_\pi}$, then $h = \widetilde{h}_*$.*

If one replaces S_π by a strip S_w of width w, then clearly the moderation condition in S_w must be formulated for $z \in S_w$ as $g(z) = O(\mathrm{e}^{a|\operatorname{Im} z|})$ with $a < \pi/w$.

Proof. — We have that h_* is moderate on ∂S_π, hence $U = h - \widetilde{h}_*$ is moderate on S_π and vanishes on ∂S_π. Given $z_0 \in S_\pi$, $a < 1$ such that $U = O(\mathrm{e}^{a|\tau|})$, $\varepsilon > 0$ and $b \in (a, 1)$, we see as in (3.13) that $U - \varepsilon \operatorname{Re}\cos(bz)$ is $\leqslant 0$ on the boundary of a rectangle containing z_0, hence $U(z_0) \leqslant \varepsilon \operatorname{Re}\cos(z_0)$ by the maximum principle. Doing it also with $-U$ and letting $\varepsilon \to 0$ we conclude that $h - \widetilde{h}_* = 0$. □

We now study the function h_1 defined by $h_1(\pi/2 + \mathrm{i}\tau) = |\tau|$, $h_1(-\pi/2 + \mathrm{i}\tau) = 0$ for every $\tau \in \mathbb{R}$ and its (moderate) harmonic extension given at $\sigma \in I_\pi$ by

$$h_1(\sigma) = \int_{\mathbb{R}} |\tau| f_\sigma(\tau)\,\mathrm{d}\tau = \frac{2}{\pi}\int_0^{+\infty} \arctan\Bigl(\frac{\cos\sigma}{\mathrm{e}^\tau - \sin\sigma}\Bigr)\,\mathrm{d}\tau\,.$$

Recall that $\|f_\sigma\|_{L^1(\mathbb{R})} = \theta = \sigma/\pi + 1/2$. When $\sigma = 0$, we have the easy bound

$$h_1(0) = \frac{2}{\pi}\int_0^{+\infty} \arctan(\mathrm{e}^{-\tau})\,\mathrm{d}\tau < \frac{2}{\pi}\int_0^{+\infty} \mathrm{e}^{-\tau}\,\mathrm{d}\tau = \frac{2}{\pi}\,.$$

One can find $h_1(0)$ by writing the power series expansion of $\arctan(x)$, letting then $x = \mathrm{e}^{-\tau}$ and integrating in $\tau \in (0, +\infty)$. One gets $h_1(0) = 2G/\pi < 0.584$, where $G = \sum_{k=0}^{+\infty} (-1)^k (2k+1)^{-2}$ is the *Catalan constant*, $0.915 < G < 0.916$. One has

$$h_1'(\sigma) = \frac{2}{\pi}\int_0^{+\infty} \frac{\mathrm{e}^{-2\tau} - \mathrm{e}^{-\tau}\sin\sigma}{\mathrm{e}^{-2\tau} - 2\,\mathrm{e}^{-\tau}\sin\sigma + 1}\,\mathrm{d}\tau = \frac{1}{\pi}\ln\bigl(2 - 2\sin\sigma\bigr)\,,$$

thus h_1 is concave on I_π and maximal when $\sigma = \pi/6$. One can find numerically that $0.646 < h_1(\pi/6) < 0.647$. By concavity, we obtain for each $\sigma \in I_\pi$ that

$$h_1(\sigma) = h_1(\sigma) - h_1(-\pi/2) \leqslant h_1'(-\pi/2)(\sigma + \pi/2) = \theta \ln 4\,. \tag{3.17}$$

One has $h_1(\pi/2) = 0$, the behavior of $h_1(\sigma)$ when $\varepsilon = \pi/2 - \sigma \to 0$ is given by

$$h_1(\sigma) = -\frac{1}{\pi} \int_0^{\varepsilon} \ln(2 - 2\cos s) \, \mathrm{d}s \simeq -\frac{1}{\pi} \int_0^{\varepsilon} \ln(s^2) \, \mathrm{d}s = \frac{2}{\pi} \bigl(\varepsilon \ln(1/\varepsilon) + \varepsilon \bigr) . \quad (3.18)$$

Since $h_1(\,\cdot + \mathrm{i}\tau) - h_1(\,\cdot\,)$ is bounded by $|\tau|$ on B_1 and vanishes on B_0, we have

$$0 < h_1(\sigma + \mathrm{i}\tau) \leqslant h_1(\sigma) + \theta|\tau| \leqslant h_1(\pi/6) + \theta|\tau| , \quad \sigma \in I_\pi , \quad \tau \in \mathbb{R} . \quad (3.19)$$

If $S_w = \{z \in \mathbb{C} : \alpha_0 < \operatorname{Re} z < \alpha_1\}$ has width $w = \alpha_1 - \alpha_0$ and if $\lambda = w/\pi$, we may associate to h, harmonic on S_w, the harmonic function $H(Z) = h(\alpha_{1/2} + \lambda Z)$ for $Z \in S_\pi$, where $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ when $t \in [0,1]$. If we set $h_{1,w}(\alpha_1 + \mathrm{i}\tau) = |\tau|$ and $h_{1,w} = 0$ on $\alpha_0 + \mathrm{i}\mathbb{R}$, then $H_{1,w} = \lambda h_1$, and we get from (3.19) that

$$h_{1,w}(\alpha_\theta + \mathrm{i}\tau) = h_{1,w}(\alpha_{1/2} + \lambda\sigma + \mathrm{i}\tau) = \lambda h_1(\sigma + \mathrm{i}\tau/\lambda) \leqslant w h_1(\pi/6)/\pi + \theta|\tau| .$$

We now comment on Corollary 3.4. If f is holomorphic in S_w with admissible growth, satisfies $|f(\alpha_j + \mathrm{i}\tau)| \leqslant \mathrm{e}^{u_j|\tau|}$ on ∂S_w, $u_j \geqslant 0$, $j = 0, 1$, the "correct" bound at $z \in S_w$ for f is $\mathrm{e}^{U_{\mathbf{u},w}(z)}$ where $U_{\mathbf{u},w} = u_0 h_{0,w} + u_1 h_{1,w}$, with $h_{0,w}(\alpha_{1/2} + \zeta) = h_{1,w}(\alpha_{1/2} - \zeta)$. One gets in particular $U_{\mathbf{u},w}(\alpha_{1/2}) = 2\lambda(u_0 + u_1)G/\pi$. When $u_0 = u_1 = 1$, this finer method gives at $\alpha_{1/2}$ a bounding factor $\mathrm{e}^{(4G/\pi)(w/\pi)}$ instead of $E_{w,1/2}(1,1) = \mathrm{e}^{w/2}$, and $4G/\pi^2 < 0.3713 < 1/2$.

Let $V_{\mathbf{u},w}$ be the harmonic conjugate of $U_{\mathbf{u},w}$. Our first method in Corollary 3.4 applied to $f_0(z) = \mathrm{e}^{U_{\mathbf{u},w}(z) + \mathrm{i} V_{\mathbf{u},w}(z)}$ yields

$$U_{\mathbf{u},w}(\alpha_\theta + \mathrm{i}\tau) \leqslant \ln E_{w,\theta}(u_0, u_1) + u_\theta|\tau| . \quad (3.20)$$

If $u_0 = u_1 = u > 0$, we get $U_{\mathbf{u},w}(\alpha_\theta) = u(h_{0,w} + h_{1,w})(\alpha_\theta) \leqslant w\sqrt{\theta(1-\theta)}\,u$. This estimate (3.20) has the right order of magnitude in w and u, but not in θ when θ tends to 0 or 1. The correct order when $\theta \to 0$ is $\kappa\theta\log(1/\theta)$, according to (3.18).

Remark 3.6. — We shall have to deal with cases where the bounds on the lines limiting the strip $S_w = \{z \in \mathbb{C} : \alpha_0 < \operatorname{Re} z < \alpha_1\}$, $w = \alpha_1 - \alpha_0$, have the form

$$|f(\alpha_j + \mathrm{i}\tau)| \leqslant (1 + \tau^2)^{c_j} \, \mathrm{e}^{u_j|\tau| + v_j} , \quad c_j, u_j \geqslant 0 , \quad j = 0, 1 .$$

It is obviously possible to "absorb" the polynomial factor by replacing u_j in the exponential with $u_j + \varepsilon$, $\varepsilon > 0$ arbitrary, and modifying v_j accordingly, but one can work a little more carefully as follows.

Let $\ell_{1,w}$ be the moderate harmonic function on S_w such that $\ell_{1,w}(\alpha_1 + \mathrm{i}\tau) = \ln(1+\tau^2)$ for $\tau \in \mathbb{R}$ and $\ell_{1,w} = 0$ on $\alpha_0+\mathrm{i}\mathbb{R}$. Let $\alpha_\theta = (1-\theta)\alpha_0+\theta\alpha_1$, $\lambda = w/\pi$, $\sigma = \pi\theta - \pi/2$ and $L_{1,w}(Z) = \ell_{1,w}(\alpha_{1/2}+\lambda Z)$. By Lemma 3.5 and (3.16), we get

$$\begin{aligned}\ell_{1,w}(\alpha_\theta + \mathrm{i}\tau) &= \ell_{1,w}(\alpha_{1/2} + \lambda\sigma + \mathrm{i}\tau) = L_{1,w}(\sigma + \mathrm{i}\tau/\lambda)\\ &= \int_{\mathbb{R}} L_{1,w}(\pi/2+\mathrm{i}\tau/\lambda-\mathrm{i}t) f_\sigma(t)\,\mathrm{d}t \leqslant \int_{\mathbb{R}} \ln\bigl(1+(\lambda|t|+|\tau|)^2\bigr) f_\sigma(t)\,\mathrm{d}t\,.\end{aligned}$$

Applying Jensen's inequality to the probability density $\widetilde{f_\sigma} = \theta^{-1} f_\sigma$, one sees that

$$\begin{aligned}&\exp\Bigl(\int_{\mathbb{R}} \ln\bigl([1+(\lambda|t|+|\tau|)^2]^{1/2}\bigr)\widetilde{f_\sigma}(t)\,\mathrm{d}t\Bigr)\\ &\qquad\leqslant \int_{\mathbb{R}} [1+(\lambda|t|+|\tau|)^2]^{1/2}\widetilde{f_\sigma}(t)\,\mathrm{d}t \leqslant (1+\tau^2)^{1/2} + \lambda \int_{\mathbb{R}} |t|\widetilde{f_\sigma}(t)\,\mathrm{d}t\,,\end{aligned}$$

bounded by $(1+\tau^2)^{1/2}+\lambda\ln 4$ by (3.17). For every $\tau \in \mathbb{R}$, one has therefore

$$0 < \ell_{1,w}(\alpha_\theta + \mathrm{i}\tau) < 2\theta \ln\bigl((1+\tau^2)^{1/2} + \lambda\ln 4\bigr)\,. \tag{3.21}$$

Define a harmonic function U in S_w, continuous on the closure, by $U = c_0\ell_{0,w} + c_1\ell_{1,w}$, where $\ell_{0,w}(z) = \ell_{1,w}(2\alpha_{1/2} - z)$, so that $U(\alpha_j + \mathrm{i}\tau) = c_j \ln(1+\tau^2)$. Let V be conjugate to U in S_w. Then $g = \mathrm{e}^{-U-\mathrm{i}V}$ is holomorphic in S_w and $|(fg)(\alpha_j + \mathrm{i}\tau)| \leqslant \mathrm{e}^{u_j|\tau|+v_j}$ on ∂S_w. By (3.21), we can bound $|f(z)|$ at $z = \alpha_\theta + \mathrm{i}\tau$ by multiplying the inside bound of Corollary 3.4 for fg with the additional factor

$$\mathrm{e}^{U(\alpha_\theta+\mathrm{i}\tau)} \leqslant \bigl((1+\tau^2)^{1/2} + \ln(4)w/\pi\bigr)^{2c_\theta} \leqslant (1+\ln(4)w/\pi)^{2c_\theta}\,(1+\tau^2)^{c_\theta},$$

where $c_\theta = (1-\theta)c_0 + \theta c_1$. Since $\ln(4)/\pi < 1/2$, we may remember that

$$|f(\alpha_\theta + \mathrm{i}\tau)| \leqslant (1+w/2)^{2c_\theta}\, E_{w,\theta}(u_0,u_1)\,(1+\tau^2)^{c_\theta}\, \mathrm{e}^{u_\theta|\tau|+v_\theta}\,. \tag{3.22}$$

3.2.2. Interpolation of holomorphic families of linear operators

We now recall the classical complex interpolation method for bounding in the norm of $L^p(X,\Sigma,\mu)$, when $1 < p < +\infty$, a linear operator T_α that is a member of a holomorphic family of operators (T_z), for z in a vertical strip S containing α. We consider a linear space $\mathcal{E}$ which is a common subspace of all $L^r(X,\Sigma,\mu)$, $1 \leqslant r \leqslant +\infty$, and which is dense in $L^r(X,\Sigma,\mu)$ when $1 \leqslant r < +\infty$. This space $\mathcal{E}$ can be the space of simple Σ-measurable and μ-integrable functions, or for the specific spaces $L^r(\mathbb{R}^n)$, it can be $\mathcal{S}(\mathbb{R}^n)$ or the space $\mathcal{K}(\mathbb{R}^n)$. We consider a closed strip $\alpha_0 \leqslant \operatorname{Re} z \leqslant \alpha_1$ in $\mathbb{C}$, with $\alpha_0 < \alpha < \alpha_1$. We assume that each T_z, for z in this closed strip, is defined on $\mathcal{E}$ and linear with values in $L^1(X,\Sigma,\mu) + L^\infty(X,\Sigma,\mu)$. The holomorphy assumption means that for $f, g \in \mathcal{E}$, the function $z \mapsto \langle T_z f, g\rangle$ is holomorphic in the open strip $\alpha_0 < \operatorname{Re} z < \alpha_1$, but one also assumes that it extends as a continuous function on the closed strip. The above bracket is bilinear, given by $\int_X (T_z f) g \, \mathrm{d}\mu$. Later in these Notes, we shall abuse slightly and speak about holomorphic family of linear operators in the *closed* strip $\alpha_0 \leqslant \operatorname{Re} z \leqslant \alpha_1$.

We consider $1 \leqslant p_0, p_1 \leqslant +\infty$ and p between p_0 and p_1, so that $1 < p < +\infty$. We assume that when $\operatorname{Re} z = \alpha_j$, $j = 0, 1$, the T_zs are uniformly bounded from $\mathcal{E}$, equipped with the L^{p_j} norm, to $L^{p_j}(X, \Sigma, \mu)$, and we assume that for a certain $\theta \in (0, 1)$, we have both

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \alpha = (1-\theta)\,\alpha_0 + \theta\,\alpha_1 \,.$$

We want to show that T_α is bounded from $\mathcal{E}$, equipped with the L^p norm, to $L^p(X, \Sigma, \mu)$. Then, by the density of $\mathcal{E}$, we will be able to extend to $L^p(X, \Sigma, \mu)$ the bound obtained for the functions in $\mathcal{E}$.

We must of course bound $\langle T_\alpha f, g \rangle$, uniformly for f in the intersection of $\mathcal{E}$ with the unit ball of $L^p(X, \Sigma, \mu)$ and for g in the unit ball of the dual $L^q(X, \Sigma, \mu)$, $1/p + 1/q = 1$. Denote by q_0 the conjugate of p_0 and by q_1 that of p_1. Observe that we have also $1/q = (1-\theta)/q_0 + \theta/q_1$. We write $f(x) = u(x)|f(x)|$, $g(x) = v(x)|g(x)|$ for every $x \in X$, with $|u(x)| = |v(x)| = 1$. Next, for each $z \in \mathbb{C}$, we set

$$f_z(x) = u(x)|f(x)|^{p(sz+t)} \,, \quad g_z(x) = v(x)|g(x)|^{q(1-sz-t)} \,, \quad x \in X \,, \tag{3.23}$$

where s, t real are chosen such that $s\alpha_0 + t = 1/p_0$ and $s\alpha_1 + t = 1/p_1$. This yields $s\alpha + t = 1/p$. We see that $f_\alpha = f$, $g_\alpha = g$ and we also see that the exponents have been chosen so that the assumptions $\|f\|_p \leqslant 1$ and $\|g\|_q \leqslant 1$ imply

$$\forall\, \tau \in \mathbb{R} \,, \quad \|f_{\alpha_0 + \mathrm{i}\tau}\|_{p_0} \leqslant 1 \,, \quad \|f_{\alpha_1 + \mathrm{i}\tau}\|_{p_1} \leqslant 1 \,,$$
$$\|g_{\alpha_0 + \mathrm{i}\tau}\|_{q_0} \leqslant 1 \,, \quad \|g_{\alpha_1 + \mathrm{i}\tau}\|_{q_1} \leqslant 1 \,.$$

We notice for future reference that if f and g are bounded by M on X, then

$$|f_z| \leqslant \max(M^{p/p_0}, M^{p/p_1}), \quad |g_z| \leqslant \max(M^{q/q_0}, M^{q/q_1}) \tag{3.24}$$

when $\alpha_0 \leqslant \operatorname{Re} z \leqslant \alpha_1$, because $\operatorname{Re}(sz + t)$ stays between $1/p_0$ and $1/p_1$ and $\operatorname{Re}(1 - sz - t)$ between $1/q_0$ and $1/q_1$ when $z \in S$. We now apply the three lines Lemma 3.1 for bounding the value $H(\alpha) = \langle T_\alpha f, g \rangle$ of the holomorphic function

$$H : z \mapsto \langle T_z f_z, g_z \rangle \,, \quad z \in S \,, \tag{3.25}$$

from the bounds on the lines $\operatorname{Re} z = \alpha_0$ and $\operatorname{Re} z = \alpha_1$. When $\operatorname{Re} z = \alpha_j$, we get

$$|H(z)| = |\langle T_z f_z, g_z \rangle| \leqslant \|T_z\|_{p_j \to p_j} \, \|f_z\|_{p_j} \, \|g_z\|_{q_j} \leqslant \|T_z\|_{p_j \to p_j} \,,$$

for $j = 0, 1$. In addition, the holomorphic function H must be bounded on the strip, see Remark 3.2 above. If true, we know by Lemma 3.1 that

$$|H(\alpha)| = \bigl|\langle T_\alpha f, g\rangle\bigr| \leqslant \Bigl(\sup_{\tau\in\mathbb{R}} \|T_{\alpha_0+\mathrm{i}\tau}\|_{p_0\to p_0}\Bigr)^{1-\theta} \Bigl(\sup_{\tau\in\mathbb{R}} \|T_{\alpha_1+\mathrm{i}\tau}\|_{p_1\to p_1}\Bigr)^{\theta},$$

and by taking the supremum over f and g, we obtain

$$\|T_\alpha\|_{p\to p} \leqslant \Bigl(\sup_{\tau\in\mathbb{R}} \|T_{\alpha_0+\mathrm{i}\tau}\|_{p_0\to p_0}\Bigr)^{1-\theta} \Bigl(\sup_{\tau\in\mathbb{R}} \|T_{\alpha_1+\mathrm{i}\tau}\|_{p_1\to p_1}\Bigr)^{\theta}. \tag{3.26}$$

Finally, we can extend T_α from the dense subspace $\mathcal{E}$ to $L^p(X, \Sigma, \mu)$. Sometimes, rather than looking for extension, one obtains in this way a sharper estimate for the norm of an operator T_α already known to be bounded on $L^p(X, \Sigma, \mu)$.

This complex method, introduced for L^p spaces by Thorin [80, 81] for *one* linear operator, extended by Stein [70] to families, can also be extended (see [6]) to spaces of the form $L^p(L^r)$ and more generally, by the abstract complex interpolation method due to Calderón [19], to a pair of the form $(L^{p_0}(A_0), L^{p_1}(A_1))$. One then obtains estimates in $L^p(A_\theta)$, where A_θ is the space associated to the pair (A_0, A_1) by Calderón's method with parameter $\theta \in (0, 1)$.

In many cases later on, the norms of the operators $(T_z)_{z\in S}$ are not *uniformly* bounded on the boundary lines, but obey for some $\lambda > 0$ estimates of the form

$$\|T_{\alpha_0+\mathrm{i}\tau}\|_{p_0\to p_0} \leqslant C_0\, \mathrm{e}^{\lambda|\tau|}, \quad \|T_{\alpha_1+\mathrm{i}\tau}\|_{p_1\to p_1} \leqslant C_1\, \mathrm{e}^{\lambda|\tau|}, \quad \tau \in \mathbb{R}.$$

Using Corollary 3.4, we can handle this situation. We must simply check that the above function $H(z) = H_{f,g}(z)$ in (3.25) has an admissible growth in the strip. We have to find an *ad hoc* argument giving such a growth for each choice of f and g in suitable dense subsets, growth depending on f, g. Indeed, in general, we do not know yet bounds on the norm $\|T_z\|_{p_z\to p_z}$ for $z \in S$, where $w/p_z = (\alpha_1 - \operatorname{Re} z)/p_0 + (\operatorname{Re} z - \alpha_0)/p_1$ and where $w = \alpha_1 - \alpha_0$ is the width of S. If each function $H_{f,g}$ has an admissible growth in S, we obtain here at last that

$$\|T_\alpha\|_{p\to p} \leqslant C_0^{1-\theta} C_1^{\theta}\, \mathrm{e}^{\lambda w\sqrt{\theta(1-\theta)}}.$$

If an additional polynomial factor is present in the bound of $\|T_{\alpha_j+\mathrm{i}\tau}\|_{p_j\to p_j}$, $j = 0, 1$, then we make use of Remark 3.6 and of the estimate (3.22).

3.3. On the definition of maximal functions

Let us consider a family $(K_t)_{t>0}$ of integrable functions on $\mathbb{R}^n$ and define a related maximal function by the formula

$$\mu(f) = \sup_{t>0} |K_t * f| \tag{3.27}$$

for $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p < +\infty$. We are faced with a standard difficulty of processes with continuous time parameter. In this generality, the convolution $K_t * f$ is only defined almost everywhere, for each $t > 0$, and the preceding supremum is not a well defined equivalence class of measurable functions. However, if D is a countable subset of $(0, +\infty)$, there is no problem in considering

$$\mu_D(f) = \sup_{t \in D} |K_t * f| \,,$$

and a classical workaround for defining $\mu(f)$ consists in introducing the *essential supremum*: there is a countable subset $D_0 \subset (0, +\infty)$ such that $\mu_D(f) = \mu_{D_0}(f)$ almost everywhere, whenever $D \supset D_0$. In other words, for every $t > 0$, we then have $|K_t * f| \leqslant \mu_{D_0}(f)$ almost everywhere. The essential supremum is defined to be the equivalence class of $\mu_{D_0}(f)$. It is also the *least upper bound* of the family $(|K_t * f|)_{t>0}$ in the Banach lattice $L^p(\mathbb{R}^n)$.

Most often, we shall have the specific problem where one considers an integrable kernel K on $\mathbb{R}^n$ and defines a maximal function using the dilates of K, by

$$\mu(f) = \sup_{t>0} |K_{(t)} * f| \,.$$

If $f \in L^p(\mathbb{R}^n)$ and if K belongs to $L^q(\mathbb{R}^n)$, with $q < +\infty$ and $1/q + 1/p = 1$, then $K_{(t)} * f$ is defined pointwise and $t \mapsto K_{(t)}$ is continuous from $(0, +\infty)$ to L^q. It follows that $t \mapsto (K_{(t)} * f)(x)$ is continuous for every $f \in L^p(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and the aforementioned problem disappears. If $K \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ are nonnegative, then $(K_{(t)} * f)(x)$ is a definite value in $[0, +\infty]$ for every $x \in \mathbb{R}^n$, but it is not immediately clear that a direct application of (3.27) gives what we want. However, we can find an increasing sequence $(f_k)_{k \geqslant 0}$ of *bounded* nonnegative Borel functions tending almost everywhere to f. Then for every $x \in \mathbb{R}^n$ and $k \geqslant 0$, the map $t \mapsto (K_{(t)} * f_k)(x)$ is continuous from $(0, +\infty)$ to $[0, +\infty)$, because $t \mapsto K_{(t)}$ is continuous from $(0, +\infty)$ to $L^1(\mathbb{R}^n)$. It follows that $t \mapsto (K_{(t)} * f)(x)$ is lower semi-continuous, since it is an increasing limit of continuous functions. For every countable dense set D one has thus

$$\mu_D(f)(x) = \sup_{s \in D} (K_{(s)} * f)(x) = \sup_{t>0} (K_{(t)} * f)(x) \,.$$

This argument does not apply to kernels that can also assume negative values, and it is precisely the case that will appear later.

We will have to investigate maximal functions such as $\mu(f) = \sup_{t>0} |K_{(t)} * f|$, usually when $K \in L^1(\mathbb{R}^n)$, but also more generally when K is a bounded measure on $\mathbb{R}^n$. It will be often convenient to start the study with nice functions, for example functions φ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, for which $\mu(\varphi)$ is clearly defined. If a function $f \in L^p(\mathbb{R}^n)$ is given and since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we may find for every $\varepsilon > 0$ a sequence $(\varphi_k)_{k \geqslant 0}$ in $\mathcal{S}(\mathbb{R}^n)$ such that

$$f = \sum_{k=0}^{+\infty} \varphi_k \quad \text{in } L^p(\mathbb{R}^n) , \quad \text{and } \sum_{k=0}^{+\infty} \|\varphi_k\|_p < \|f\|_p + \varepsilon \, .$$

Since the convolution with $K_{(t)}$ is linear and continuous on $L^p(\mathbb{R}^n)$, we have

$$K_{(t)} * f = \sum_{k=0}^{+\infty} K_{(t)} * \varphi_k \quad \text{in } L^p(\mathbb{R}^n) , \quad \text{and } \sum_{k=0}^{+\infty} \|K_{(t)} * \varphi_k\|_p < +\infty \, ,$$

so the series $\sum_{k=0}^{+\infty} K_{(t)} * \varphi_k$ converges also almost everywhere to $K_{(t)} * f$, and we have almost everywhere

$$|K_{(t)} * f| \leqslant \sum_{k=0}^{+\infty} |K_{(t)} * \varphi_k| \leqslant \sum_{k=0}^{+\infty} \mu(\varphi_k) \, .$$

For any countable subset $D \subset (0, +\infty)$ we get $\mu_D(f) \leqslant \sum_{k=0}^{+\infty} \mu(\varphi_k)$, implying that $\mu(f)$, defined as essential supremum, is bounded by $\sum_{k=0}^{+\infty} \mu(\varphi_k)$. If we know that there exists κ such that $\|\mu(\varphi)\|_p \leqslant \kappa \|\varphi\|_p$ when $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it follows that

$$\|\mu(f)\|_p \leqslant \sum_{k=0}^{+\infty} \|\mu(\varphi_k)\|_p \leqslant \kappa \sum_{k=0}^{+\infty} \|\varphi_k\|_p \leqslant \kappa (\|f\|_p + \varepsilon) \, ,$$

for every $\varepsilon > 0$. In order to bound $\mu(f)$ in $L^p(\mathbb{R}^n)$, it is therefore enough to obtain a uniform bound for Schwartz functions. Clearly, any dense linear subspace of $L^p(\mathbb{R}^n)$ consisting of nice functions can be used instead of $\mathcal{S}(\mathbb{R}^n)$.

The classical maximal function $\mathrm{M} f$, as well as $\mathrm{M}_C f$ in (0.3.**M**), is actually defined by means of $\sup_{t>0} K_{(t)} * |f|$. This makes sense whenever the kernel K is nonnegative, but not for a general K. We shall distinguish

$$\mathrm{M}_K f := \sup_{t>0} K_{(t)} * |f| \quad \text{and} \quad M_K f := \sup_{t>0} \bigl| K_{(t)} * f \bigr|$$

by the tiny notational difference between the slanted or unslanted letter M. When the kernel K is nonnegative, we have obviously $M_K f \leqslant \mathrm{M}_K f = M_K(|f|)$.

4. The results of Stein for Euclidean balls

We prove here the remarkable fact due to Stein [75] that for $p > 1$, the maximal operator associated to Euclidean balls, i.e., the classical Hardy–Littlewood maximal operator M defined in (0.1), may be bounded in $L^p(\mathbb{R}^n)$ independently of the dimension n. Full details appeared in [77]. Other proofs have appeared since then, let us mention Auscher and Carro [4] who found the simple explicit bound $2+\sqrt{2}$ in $L^2(\mathbb{R}^n)$, extended by interpolation as $(2+\sqrt{2})^{2/p}$ for $p \geqslant 2$. It is not known whether or not the weak $(1,1)$ norm of the maximal operator M is also bounded independently of the dimension. Even if we shall not develop this weak type aspect mentioned in our introduction, let us recall that the best upper estimate that is known for the weak $(1,1)$ norm of M is the Stein–Strömberg $\mathrm{O}(n)$ bound [77].

Theorem 4.1 (Stein [75]). — *Let $1 < p \leqslant +\infty$. For every integer $n \geqslant 1$ and all functions $f \in L^p(\mathbb{R}^n)$, one has that*

$$\|\mathrm{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C(p)\|f\|_{L^p(\mathbb{R}^n)}\,,$$

where $C(p)$ is a constant independent of the dimension n.

4.1. Proof of Theorem 4.1

The main tool in the proof is the spherical maximal operator $\mathcal{M}$ defined by

$$(\mathcal{M}f)(x) = (M_\sigma f)(x) = \sup_{r>0} \left| \int_{S^{n-1}} f(x - r\theta)\,\mathrm{d}\sigma(\theta) \right|, \qquad x \in \mathbb{R}^n,$$

where σ is the normalized Haar measure on the unit sphere S^{n-1}. It is clear that $\mathcal{M}f$ is well defined when f is regular, but not when $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Theorem 4.2 below means in particular that for suitable p and n, $\mathcal{M}f$ can be defined when $f \in L^p(\mathbb{R}^n)$, for example by the method described at the end of Section 3.3. The maximal function $\mathcal{M}(|f|)$ controls $\mathrm{M}f$ pointwise, as one sees easily by using polar coordinates. The maximal operator $\mathcal{M}$ is bounded in $L^p(\mathbb{R}^N)$ for some p and N, with a bound depending on the dimension N, according to the following theorem also due to Stein. An extension by Bourgain of this result can be found in [8].

Theorem 4.2 (Stein [74]). — *Let $N \geqslant 3$ and assume that $N/(N-1) < p \leqslant +\infty$. There exists a constant $C(N,p)$ such that for every function $f \in L^p(\mathbb{R}^N)$, one has*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^N)} \leqslant C(N,p)\,\|f\|_{L^p(\mathbb{R}^N)}\,.$$

The condition $p > N/(N-1)$ can be easily seen necessary, and the case $p = +\infty$ is obvious, with $C(N, \infty) = 1$. We postpone the proof of this theorem to the next section. It requires a number of harmonic analysis methods, including square function, multipliers and Littlewood–Paley decomposition.

In order to prove Theorem 4.1, we first introduce the following weighted maximal operator, depending on a parameter $k \in \mathbb{N}$. For $f \in \mathcal{S}(\mathbb{R}^n)$, let

$$(\mathrm{M}_{n,k} f)(x) = \sup_{r>0} \frac{\int_{|y|\leqslant r} |f(x-y)|\, |y|^k \,\mathrm{d}y}{\int_{|y|\leqslant r} |y|^k \,\mathrm{d}y}\,, \quad x \in \mathbb{R}^n\,,$$

where $|y|$ denotes the Euclidean norm of $y \in \mathbb{R}^n$. Taking polar coordinates gives us the pointwise inequality

$$(\mathrm{M}_{n,k} f)(x) \leqslant (\mathcal{M}|f|)(x)\,, \quad x \in \mathbb{R}^n\,,$$

from which we can deduce by applying Theorem 4.2 that for every integer $N \geqslant 3$, for p such that $N/(N-1) < p \leqslant +\infty$ and for every f in $L^p(\mathbb{R}^N)$, we have

$$\|\mathrm{M}_{N,k} f\|_{L^p(\mathbb{R}^N)} \leqslant C(N,p)\, \|f\|_{L^p(\mathbb{R}^N)}\,, \tag{4.1}$$

where $C(N,p)$ is the constant in Theorem 4.2. We shall obtain Theorem 4.1 by lifting to $\mathbb{R}^n$ the inequality (4.1) obtained in a lower dimension $N = n-k$. This is done by integrating over the Grassmannian of $(n-k)$-planes in $\mathbb{R}^n$. This method of descent is in the spirit of the Calderón–Zygmund *method of rotations.*

We write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ and $x = (x_1, x_2)$ accordingly, for every $x \in \mathbb{R}^n$, with $x_1 \in \mathbb{R}^{n-k}$ and $x_2 \in \mathbb{R}^k$. For each U in the orthogonal group $\mathcal{O}(n)$, we introduce the auxiliary maximal operator

$$(\mathrm{M}_k^U f)(x) = \sup_{r>0} \frac{\int_{\mathbb{R}^{n-k}} \mathbf{1}_{\{|y_1|\leqslant r\}}\, |f(x - U(y_1,0))|\, |y_1|^k \,\mathrm{d}y_1}{\int_{\mathbb{R}^{n-k}} \mathbf{1}_{\{|y_1|\leqslant r\}}\, |y_1|^k \,\mathrm{d}y_1}\,, \quad x \in \mathbb{R}^n\,.$$

We need two lemmas.

Lemma 4.3. — *Let $n \geqslant k+3$ and $p > (n-k)/(n-k-1)$. Then for all $f \in L^p(\mathbb{R}^n)$ and $U \in \mathcal{O}(n)$, we have*

$$\|\mathrm{M}_k^U f\|_{L^p(\mathbb{R}^n)} \leqslant C(n-k,p)\, \|f\|_{L^p(\mathbb{R}^n)}\,,$$

where $C(n-k,p)$ is the constant appearing in Theorem 4.2.

Proof. — Let us set $f_{[U]}(x) = f(Ux)$, for every $x \in \mathbb{R}^n$. Since $U \in \mathcal{O}(n)$, the mapping $S_U : f \mapsto f_{[U]}$ is an isometry of $L^p(\mathbb{R}^n)$. Observe that

$$\int_{|y_1|\leqslant r} \bigl|f\bigl(Ux - U(y_1,0)\bigr)\bigr|\, |y_1|^k \,\mathrm{d}y_1 = \int_{|y_1|\leqslant r} \bigl|f_{[U]}(x - (y_1,0))\bigr|\, |y_1|^k \,\mathrm{d}y_1\,,$$

hence we have that $(\mathrm{M}_k^U f)(Ux) = (\mathrm{M}_k^{\mathrm{Id}} f_{[U]})(x)$, for every $x \in \mathbb{R}^n$. This means that $S_U \mathrm{M}_k^U = \mathrm{M}_k^{\mathrm{Id}} S_U$. It follows that we need only consider $\mathrm{M}_k^{\mathrm{Id}}$. Now, for every $x = (x_1, x_2) \in \mathbb{R}^n$, we have

$$\begin{aligned}(\mathrm{M}_k^{\mathrm{Id}} f)(x_1,x_2) &= \sup_{r>0} \frac{\int_{\mathbb{R}^{n-k}} \mathbf{1}_{\{|y_1|\leqslant r\}} \left|f(x_1-y_1,x_2)\right| |y_1|^k \,\mathrm{d}y_1}{\int_{\mathbb{R}^{n-k}} \mathbf{1}_{\{|y_1|\leqslant r\}} |y_1|^k \,\mathrm{d}y_1} \\ &= \big(\mathrm{M}_{n-k,k} f_{x_2}\big)(x_1)\end{aligned}$$

with $f_{x_2}(x_1) = f(x_1, x_2)$. Applying (4.1) to $\mathrm{M}_{n-k,k}$ for each $x_2 \in \mathbb{R}^k$ gives

$$\int_{\mathbb{R}^{n-k}} \left|(\mathrm{M}_k^{\mathrm{Id}} f)(x_1,x_2)\right|^p \mathrm{d}x_1 \leqslant C(n-k,p)^p \int_{\mathbb{R}^{n-k}} \left|f_{x_2}(x_1)\right|^p \mathrm{d}x_1 \,,$$

therefore

$$\begin{aligned}\|\mathrm{M}_k^{\mathrm{Id}} f\|_{L^p(\mathbb{R}^n)}^p &\leqslant C(n-k,p)^p \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^{n-k}} \left|f_{x_2}(x_1)\right|^p \mathrm{d}x_1\right) \mathrm{d}x_2 \\ &= C(n-k,p)^p \,\|f\|_{L^p(\mathbb{R}^n)}^p \,.\end{aligned}$$

□

Lemma 4.4. — *For every locally integrable function f on $\mathbb{R}^n$ and $1 \leqslant k \leqslant n$, one has the pointwise inequality*

$$(\mathrm{M}f)(x) \leqslant \int_{\mathcal{O}(n)} (\mathrm{M}_k^U f)(x)\,\mathrm{d}\mu_n(U)\,, \quad x \in \mathbb{R}^n\,,$$

where μ_n denotes the normalized Haar measure on $\mathcal{O}(n)$.

Proof. — The desired pointwise inequality follows from the next equality, true for every nonnegative Borel function g on $\mathbb{R}^n$, stating that

$$\frac{\int_{|y|\leqslant r} g(y)\,\mathrm{d}y}{\int_{|y|\leqslant r} \mathrm{d}y} = \frac{\int_{\mathcal{O}(n)} \int_{\mathbb{R}^{n-k}} \mathbf{1}_{\{|y_1|\leqslant r\}}\, g\big(U(y_1,0)\big)\, |y_1|^k \,\mathrm{d}y_1 \mathrm{d}\mu_n(U)}{\int_{\mathbb{R}^{n-k}} \mathbf{1}_{\{|y_1|\leqslant r\}} |y_1|^k\,\mathrm{d}y_1}\,. \tag{4.2}$$

Indeed, for each $r > 0$ and $x \in \mathbb{R}^n$, the previous equality allows us to write

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r} |f(x-y)|\,\mathrm{d}y &= \frac{\int_{\mathcal{O}(n)} \int_{|y_1|\leqslant r} \left|f\big((x-U(y_1,0)\big)\right| |y_1|^k \,\mathrm{d}y_1 \mathrm{d}\mu_n(U)}{\int_{|y_1|\leqslant r} |y_1|^k\,\mathrm{d}y_1} \\ &\leqslant \int_{\mathcal{O}(n)} (\mathrm{M}_k^U f)(x)\,\mathrm{d}\mu_n(U)\,,\end{aligned}$$

and we conclude by taking the supremum over all $r > 0$.

It remains to check (4.2). By standard measure-theoretic arguments about classes of functions generating the Borel σ-algebra of $\mathbb{R}^n$, we can suppose that g has the form $g(x) = g_0(|x|)g_1(x')$, with $x = |x|x'$ and $x' \in S^{n-1}$. By taking polar coordinates, we see that the left-hand side of (4.2) is equal to

$$\frac{n}{r^n}\left(\int_0^r g_0(t)t^{n-1}\,\mathrm{d}t\right)\left(\int_{S^{n-1}} g_1(y')\,\mathrm{d}\sigma_{n-1}(y')\right),$$

where σ_{n-1} is the invariant probability measure on S^{n-1}. The right-hand side is

$$\frac{n}{r^n}\Bigl(\int_0^r g_0(t)t^{n-1}\,\mathrm{d}t\Bigr)\Bigl(\int_{\mathcal{O}(n)}\int_{S^{n-k-1}} g_1\bigl(U(y_1',0)\bigr)\,\mathrm{d}\sigma_{n-k-1}(y_1')\,\mathrm{d}\mu_n(U)\Bigr).$$

Observe that for every $\theta_0 \in S^{n-1}$, we have

$$\int_{\mathcal{O}(n)} g_1(U\theta_0)\,\mathrm{d}\mu_n(U) = \int_{S^{n-1}} g_1(\theta)\,\mathrm{d}\sigma_{n-1}(\theta)\,,$$

since the left-hand side of this equality defines a probability measure on S^{n-1}, namely $\mathcal{B}_{S^{n-1}} \ni A \mapsto \int_{\mathcal{O}(n)} \mathbf{1}_A(U\theta_0)\,\mathrm{d}\mu_n(U)$, which is invariant under the left-action of $\mathcal{O}(n)$, hence equal to σ_{n-1}. We have therefore

$$\begin{aligned}\int_{\mathcal{O}(n)}\int_{S^{n-k-1}} &g_1\bigl(U(y_1',0)\bigr)\,\mathrm{d}\sigma_{n-k-1}(y_1')\mathrm{d}\mu_n(U)\\ &= \int_{S^{n-k-1}}\Bigl(\int_{\mathcal{O}(n)} g_1\bigl(U(y_1',0)\bigr)\,\mathrm{d}\mu_n(U)\Bigr)\,\mathrm{d}\sigma_{n-k-1}(y_1')\\ &= \int_{S^{n-k-1}}\Bigl(\int_{S^{n-1}} g_1(\theta)\,\mathrm{d}\sigma_{n-1}(\theta)\Bigr)\,\mathrm{d}\sigma_{n-k-1}(y_1')\\ &= \int_{S^{n-1}} g_1(y')\,\mathrm{d}\sigma_{n-1}(y')\,,\end{aligned}$$

completing the proof. □

Proof of Theorem 4.1. — Let $1 < p \leqslant +\infty$. There is obviously nothing to do if $n \leqslant 2$. When $n \leqslant p/(p-1)$, the "bad" Vitali-bound $C(n) = 3^n$ in the classical maximal inequality (ST) is less than a function of p alone, namely $3^{p/(p-1)}$. We can therefore assume that both inequalities $n > p/(p-1)$ and $n \geqslant 3$ hold. We then write $n = (n-k)+k$ with $n-k = \bigl\lfloor \max\bigl(p/(p-1), 2\bigr)\bigr\rfloor + 1$, and the result follows from Lemma 4.3 and Lemma 4.4 since with this choice, the bound $C(n-k, p)$ in Lemma 4.3 is now a function of p alone. □

4.2. Boundedness of the spherical maximal operator

In this section, we prove Theorem 4.2 following the approach of Rubio de Francia [68], see also Grafakos [39]. Let $n \geqslant 2$. The spherical maximal operator is expressed by

$$(\mathcal{M}f)(x) = \sup_{r>0}\bigl|\bigl[m_\sigma(r\,\cdot)\,\widehat{f}(\cdot)\bigr]^\vee(x)\bigr| = \sup_{r>0}\bigl|\bigl(\sigma_{(r)} * f\bigr)(x)\bigr|\,, \quad x \in \mathbb{R}^n,$$

where $h^\vee(x) = \widehat{h}(-x)$ denotes the inverse Fourier transform of a function h, m_σ is the Fourier transform of the uniform probability measure σ on the unit sphere S^{n-1}, and $\sigma_{(r)}$ is the dilated probability measure defined in (2.8). It is known that

$$m_\sigma(\xi) = \widehat{\sigma}(\xi) = (2\pi|\xi|)^{-(n-2)/2} \mathrm{J}_{(n-2)/2}(2\pi|\xi|)\,, \quad \xi \in \mathbb{R}^n, \tag{4.3}$$

with J_ν the Bessel function of order ν. This equality follows from the fact that the two functions $t \mapsto t^{-(n-2)/2} \mathrm{J}_{(n-2)/2}(t)$ and

$$t \mapsto F(t) = \int_{S^{n-1}} \mathrm{e}^{\mathrm{i}tx_1} \,\mathrm{d}\sigma(x) = \frac{2s_{n-2}}{s_{n-1}} \int_0^1 (1-s^2)^{(n-3)/2} \cos(st)\,\mathrm{d}s$$

are entire functions g satisfying $g(0) = 1$ and $t^2(g''(t)+g(t)) = -(n-1)tg'(t)$.

We shall rely on the Littlewood–Paley theory, decomposing multipliers into dyadic pieces with localized frequencies. More precisely, we shall dominate $\mathcal{M}$ by a series of maximal operators $\sum_{\ell=0}^{+\infty} M_{K_\ell}$, where each kernel K_ℓ is radial with a well localized Fourier transform m_ℓ. We establish that M_{K_ℓ} is of strong type when $p = 2$ and of weak type $(1,1)$. Then, we get an L^p bound for M_{K_ℓ} by interpolation, and the range of p in Theorem 4.2 is chosen for making the series of bounds convergent. For the case $p = 2$, we mainly use for m_ℓ both the decay at infinity and a support property, together with a precise upper bound for the $L^2(\mathbb{R}^n)$ norm of a related square function. When $p = 1$, we invoke the usual Hardy–Littlewood theorem. Before giving the proof of Theorem 4.2, we introduce the dyadic decomposition of $m_\sigma = \widehat{\sigma}$.

Let φ_0 be a smooth radial function on $\mathbb{R}^n$ satisfying for every $\xi \in \mathbb{R}^n$ that

$$\varphi_0(\xi) = \begin{cases} 1 & \text{if } |\xi| \leqslant 1 \\ 0 & \text{if } |\xi| \geqslant 2\,. \end{cases}$$

Let $\psi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ for ξ in $\mathbb{R}^n$. This function ψ is supported in the annulus $\{1/2 \leqslant |\xi| \leqslant 2\}$. For every integer $\ell \geqslant 1$ we define

$$\varphi_\ell(\xi) = \varphi_0(2^{-\ell}\xi) - \varphi_0(2^{1-\ell}\xi) = \psi_{[2^{-\iota}]}(\xi)\,, \quad \xi \in \mathbb{R}^n,$$

and for every $\ell \geqslant 0$, we consider the dyadic radial piece $m_\ell = \varphi_\ell m_\sigma$ associated to the multiplier m_σ. We can check that $\sum_{\ell=0}^{+\infty} \varphi_\ell = 1$, thus $m_\sigma = \sum_{\ell=0}^{+\infty} m_\ell$. For every $\ell \geqslant 0$, we introduce the integrable kernel $K_\ell = m_\ell^\vee = \varphi_\ell^\vee * \sigma$ and we set

$$(M_{K_\ell} f)(x) = \sup_{r>0} \bigl|\bigl[m_\ell(r\,\cdot)\widehat{f}(\cdot)\bigr]^\vee(x)\bigr| = \sup_{r>0} \bigl|\bigl[(\varphi_\ell^\vee)_{(r)} * \sigma_{(r)} * f\bigr](x)\bigr|, \ \ x \in \mathbb{R}^n,$$

when $f \in \mathcal{S}(\mathbb{R}^n)$. In particular, we have $M_{K_0} f = \sup_{r>0}\bigl|(\varphi_0^\vee)_{(r)} * \sigma_{(r)} * f\bigr|$ and

$$M_{K_\ell} f = \sup_{r>0} \bigl|(\psi^\vee)_{(2^{-\ell}r)} * \sigma_{(r)} * f\bigr|, \quad \ell \geqslant 1\,.$$

For every $x \in \mathbb{R}^n$ and $r > 0$, we see that $(\sigma_{(r)} * f)(x) = \sum_{\ell=0}^{+\infty} ((K_\ell)_{(r)} * f)(x)$ and we get the pointwise inequality

$$(\mathcal{M}f)(x) \leqslant \sum_{\ell=0}^{+\infty} (M_{K_\ell} f)(x)\,. \tag{4.4}$$

In a first subsection, we present some useful results on this type of maximal operators and associated square functions. Then, we shall prove that each M_{K_ℓ}, for $\ell \geqslant 0$, is of strong type when $p = 2$ and of weak type when $p = 1$, and we give the proof of Theorem 4.2 in a third subsection.

4.2.1. Maximal operator and square function

Let $m(\xi)$ be a multiplier that is a bounded continuous function on $\mathbb{R}^n$, vanishing at 0, with $|m(\xi)| = O(|\xi|)$ in a neighborhood of 0. For f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, set

$$\begin{aligned}(g_m f)(x) &= \Bigl(\int_0^{+\infty} |(T_{m_{[u]}} f)(x)|^2 \,\frac{\mathrm{d}u}{u} \Bigr)^{1/2} \\ &= \Bigl(\int_0^{+\infty} \Bigl| \int_{\mathbb{R}^n} m(u\xi) \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi} \,\mathrm{d}\xi \Bigr|^2 \,\frac{\mathrm{d}u}{u} \Bigr)^{1/2}.\end{aligned}$$

We obtain the Littlewood–Paley function $g_1(f)$ of (2.3) when $m(\xi) = 2\pi|\xi|\,\mathrm{e}^{-2\pi|\xi|}$.

Lemma 4.5. — *Assume that the multiplier $m(\xi)$ is a bounded function of $\xi \in \mathbb{R}^n$, supported in an annulus of the form $\{a \leqslant |\xi| \leqslant ra\}$, $a > 0$ and $r > 1$. For every function $f \in \mathcal{S}(\mathbb{R}^n)$, one has that*

$$\|g_m f\|_{L^2(\mathbb{R}^n)} \leqslant \sqrt{\ln r}\, \|m\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}\,.$$

Proof. — According to the Fubini theorem, followed by Parseval, Fubini again and setting finally $v = u|\xi|$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |(g_m f)(x)|^2 \,\mathrm{d}x \\ &\qquad = \int_0^{+\infty} \|T_{m_{[u]}} f\|_2^2 \,\frac{\mathrm{d}u}{u} = \int_0^{+\infty} \int_{\mathbb{R}^n} |m(u\xi)|^2 |\widehat{f}(\xi)|^2 \,\frac{\mathrm{d}u}{u}\,\mathrm{d}\xi \\ &\qquad \leqslant \|m\|_\infty^2 \int_{\mathbb{R}^n} \Bigl(\int_a^{ra} \frac{\mathrm{d}v}{v} \Bigr) |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi = \|m\|_\infty^2 \ln(r) \|f\|_2^2\,. \quad \square\end{aligned}$$

Lemma 4.6. — *Assume that $m(\xi)$ is of class C^1 on $\mathbb{R}^n$ and vanishes outside a compact subset of $\mathbb{R}^n \setminus \{0\}$. For every $t > 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\left| \int_{\mathbb{R}^n} m(t\xi)\widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi \right|^2 \leqslant 2\,(g_m f)(x)\,(g_{m^*} f)(x)\,, \quad x \in \mathbb{R}^n,$$

where we have set $m^*(\xi) = \xi \cdot \nabla m(\xi)$ for every $\xi \in \mathbb{R}^n$.

Proof. — For each $s \geqslant 0$ let us set

$$(g_{m,s} f)(x) = (T_{m_{[s]}} f)(x) = \int_{\mathbb{R}^n} m(s\xi)\widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi\,, \quad x \in \mathbb{R}^n.$$

We note that

$$s\frac{\mathrm{d}}{\mathrm{d}s}(g_{m,s} f)(x) = \int_{\mathbb{R}^n} s\xi\cdot\nabla m(s\xi)\widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi = \int_{\mathbb{R}^n} m^*(s\xi)\widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi\,,$$

which allows us to see this quantity as $(g_{m^*,s} f)(x)$. Since m vanishes in a neighborhood of 0, one has $(g_{m,0} f)(x) = 0$, thus

$$\begin{aligned} |(g_{m,t} f)(x)|^2 &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} |(g_{m,s} f)(x)|^2\, \mathrm{d}s \\ &= 2\,\mathrm{Re} \int_0^t \overline{(g_{m,s} f)(x)}\; s\frac{\mathrm{d}}{\mathrm{d}s}(g_{m,s} f)(x)\,\frac{\mathrm{d}s}{s} \\ &= 2\,\mathrm{Re} \int_0^t \overline{(g_{m,s} f)(x)}\,(g_{m^*,s} f)(x)\,\frac{\mathrm{d}s}{s}\,. \end{aligned}$$

By Cauchy–Schwarz, and bounding the integral on $[0, t]$ by the integral on $[0, +\infty)$, we obtain that

$$\begin{aligned} |(g_{m,t} f)(x)|^2 &\leqslant 2\left(\int_0^{+\infty} \left|(g_{m,s} f)(x)\right|^2 \frac{\mathrm{d}s}{s}\right)^{1/2} \left(\int_0^{+\infty} \left|(g_{m^*,s} f)(x)\right|^2 \frac{\mathrm{d}s}{s}\right)^{1/2} \\ &= 2\,(g_m f)(x)\,(g_{m^*} f)(x)\,. \end{aligned}$$

□

Lemma 4.7. — *Let K be an integrable kernel on $\mathbb{R}^n$. Suppose that m, the Fourier transform of K, is of class C^1 on $\mathbb{R}^n$ and supported in an annulus of the form $\{a \leqslant |\xi| \leqslant ra\}$, $a > 0$ and $r > 1$. For every function $f \in \mathcal{S}(\mathbb{R}^n)$, one has that*

$$\begin{aligned} \|M_K f\|^2_{L^2(\mathbb{R}^n)} &= \big\|\sup_{t>0} |K_{(t)} * f|\big\|^2_{L^2(\mathbb{R}^n)} \\ &\leqslant 2\ln(r)\,\|m\|_{L^\infty(\mathbb{R}^n)}\,\|m^*\|_{L^\infty(\mathbb{R}^n)}\,\|f\|^2_{L^2(\mathbb{R}^n)}\,, \end{aligned}$$

where $m^*(\xi) = \xi \cdot \nabla m(\xi)$ for $\xi \in \mathbb{R}^n$.

Proof. — By Lemma 4.6, we have for every $x \in \mathbb{R}^n$ and $t > 0$ that

$$\left|(K_{(t)} * f)(x)\right|^2 = \left| \int_{\mathbb{R}^n} m(t\xi)\widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi \right|^2 \leqslant 2\,(g_m f)(x)\,(g_{m^*} f)(x)\,.$$

This upper bound is independent of t, thus

$$\big((M_K f)(x)\big)^2 \leqslant 2\,(g_m f)(x)\,(g_{m^*} f)(x)\,,$$

and by Cauchy–Schwarz we get

$$\big\|M_K f\big\|^2_{L^2(\mathbb{R}^n)} \leqslant 2\,\|g_m f\|_{L^2(\mathbb{R}^n)}\,\|g_{m^*} f\|_{L^2(\mathbb{R}^n)}\,.$$

According to Lemma 4.5, we conclude that

$$\big\|M_K f\big\|^2_{L^2(\mathbb{R}^n)} \leqslant 2\,\ln(r)\,\|m\|_{L^\infty(\mathbb{R}^n)}\,\|m^*\|_{L^\infty(\mathbb{R}^n)}\,\|f\|^2_{L^2(\mathbb{R}^n)}\,. \qquad \square$$

The following proposition is nearly obvious.

Proposition 4.8. — *Let $K \in \mathcal{S}(\mathbb{R}^n)$ be a radial kernel. For every p in $(1, +\infty]$, the maximal operator M_K is bounded on $L^p(\mathbb{R}^n)$.*

One also gets the weak type $(1,1)$ for M_K, but we shall not use it.

Proof. — Since K is a Schwartz radial function, we can find an integrable function Ω, radial and radially decreasing, such that $|K| \leqslant \Omega$. It implies that

$$\sup_{r>0} \big|\big[K_{(r)} * f\big](x)\big| \leqslant \sup_{r>0} \big(\Omega_{(r)} * |f|\big)(x)\,, \quad x \in \mathbb{R}^n\,,$$

and Ω being radial and radially decreasing, we classically have

$$\sup_{r>0} \big(\Omega_{(r)} * |f|\big)(x) \leqslant \|\Omega\|_{L^1(\mathbb{R}^n)} (\mathrm{M} f)(x)\,, \quad x \in \mathbb{R}^n\,. \tag{4.5}$$

By Theorem 0.1, the usual maximal theorem for M, we get the conclusion.

For proving (4.5), it suffices to show that

$$\big|(\Omega * f)(x)\big| \leqslant \|\Omega\|_{L^1(\mathbb{R}^n)} (\mathrm{M} f)(x)\,, \quad x \in \mathbb{R}^n\,. \tag{4.6}$$

Suppose that $\Omega \leqslant 1$ for simplicity, and consider for each integer $k \geqslant 1$ the set

$$A_k = \{x \in \mathbb{R}^n : \Omega(x) > 2^{-k}\}\,.$$

This set A_k is a Euclidean ball, and if we define $g = \sum_{k \geqslant 1} 2^{-k} \mathbf{1}_{A_k}$, we can check that $g/2 \leqslant \Omega \leqslant g$. We rewrite g as

$$g = \sum_{k \geqslant 1} a_k \frac{\mathbf{1}_{A_k}}{|A_k|}\,,$$

with $a_k > 0$ for every $k \geqslant 1$. Since Ω is integrable, g is also integrable and

$$\sum_{k \geqslant 0} a_k = \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x \leqslant 2 \int_{\mathbb{R}^n} \Omega(x)\,\mathrm{d}x = 2\|\Omega\|_{L^1(\mathbb{R}^n)}\,.$$

We have for every $x \in \mathbb{R}^n$ that

$$\big|(\Omega * f)(x)\big| \leqslant (g * |f|)(x) = \sum_{k\geqslant 1} \frac{a_k}{|A_k|} \int_{x+A_k} |f(y)|\,\mathrm{d}y$$
$$\leqslant \Big(\sum_{k\geqslant 0} a_k\Big)(\mathrm{M}f)(x) \leqslant 2\|\Omega\|_{L^1(\mathbb{R}^n)}(\mathrm{M}f)(x)\,.$$

The inequality with constant 1 can be reached by refining the partition, replacing the values 2^{-k} by $(1+\varepsilon)^{-k}$, with $\varepsilon > 0$ tending to 0. One can also give a direct proof involving integration by parts, or the Fubini theorem and level sets of Ω. □

4.2.2. Strong and weak type results for M_{K_ℓ}, $\ell \geqslant 1$

We begin with the strong type result, when $p = 2$.

Proposition 4.9. — *For every integer $\ell \geqslant 1$ and every $f \in L^2(\mathbb{R}^n)$ one has that*

$$\big\|M_{K_\ell} f\big\|_{L^2(\mathbb{R}^n)} \leqslant C(n) 2^{-\ell(n-2)/2} \|f\|_{L^2(\mathbb{R}^n)}\,,$$

where $C(n)$ is a constant independent of ℓ.

Proof. — For each $\ell \geqslant 1$, the multiplier $m_\ell = \widehat{K_\ell}$ is C^1, supported in the annulus

$$I_\ell = \{\xi \in \mathbb{R}^n : 2^{\ell-1} \leqslant |\xi| \leqslant 2^{\ell+1}\}\,.$$

Applying Lemma 4.7 to K_ℓ, with $m_\ell^*(\xi) = \xi \cdot \nabla m_\ell(\xi)$ and $r = 4$, we obtain

$$\big\|M_{K_\ell} f\big\|^2_{L^2(\mathbb{R}^n)} \leqslant 2\ln(4)\, \|m_\ell\|_{L^\infty(\mathbb{R}^n)} \|m_\ell^*\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}\,.$$

The desired result will be consequence of the inequalities

$$\|m_\ell\|_{L^\infty(\mathbb{R}^n)} \leqslant C_1(n) 2^{-\ell(n-1)/2}\,, \quad \|m_\ell^*\|_{L^\infty(\mathbb{R}^n)} \leqslant C_2(n) 2^{-\ell(n-3)/2} \tag{4.7}$$

that we establish now, with $C_1(n)$ and $C_2(n)$ independent of ℓ. Thanks to well-known properties of Bessel functions (see for instance [2, p. 238]), we have

$$\sup_{t\geqslant 1} t^{1/2} |\mathrm{J}_\alpha(t)| < +\infty\,, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{J}_\alpha(t) = \frac{1}{2}\big(\mathrm{J}_{\alpha-1}(t) - \mathrm{J}_{\alpha+1}(t)\big)\,. \tag{4.8}$$

The first property follows from the fact that $u_\alpha(t) = \sqrt{t}\,\mathrm{J}_\alpha(t)$ satisfies a differential equation $u''_\alpha(t) + (1+\kappa_\alpha t^{-2}) u_\alpha(t) = 0$ for $t > 0$, hence $v_\alpha(t) := (u_\alpha(t)^2 + u'_\alpha(t)^2)/2$ satisfies $v'_\alpha(t) = -\kappa_\alpha t^{-2} u_\alpha(t) u'_\alpha(t) \leqslant |\kappa_\alpha| t^{-2} v_\alpha(t)$, yielding $v_\alpha(t) \leqslant \mathrm{e}^{|\kappa_\alpha|}\, v_\alpha(1)$ for every $t \geqslant 1$. The second property can be checked on the coefficients of the power series $\sum_{m\geqslant 0} (-1)^m \big(m!\Gamma(m+\alpha+1)\big)^{-1} (t/2)^{2m}$ of $t^{-\alpha}\mathrm{J}_\alpha(t)$, and when $\alpha = n \in \mathbb{N}$, it is even simpler to see it on the integral expression $2\pi \mathrm{J}_n(t) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(t \sin s - ns)}\,\mathrm{d}s$.

Since m_ℓ and m_ℓ^* are supported in the annulus I_ℓ, we need only bound $m_\ell(\xi)$ and $m_\ell^*(\xi)$ when $1 \leqslant 2^{\ell-1} \leqslant |\xi| \leqslant 2^{\ell+1}$ (we have $\ell \geqslant 1$). We then obtain (4.7) by recalling (4.3) and by applying (4.8) to $t = 2\pi|\xi| > 1$, which give that

$$|m_\ell(\xi)| \leqslant c_1(n)|\xi|^{-n/2+1/2} \quad \text{and } |m_\ell^*(\xi)| \leqslant c_2(n)|\xi|^{-n/2+3/2} . \qquad \square$$

We state in the next proposition a crucial weak type estimate for M_{K_ℓ}.

Proposition 4.10. — *Let $\ell \geqslant 1$. For all $f \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$, one has that*

$$\bigl|\bigl\{x \in \mathbb{R}^n : (M_{K_\ell} f)(x) > \lambda\bigr\}\bigr| \leqslant C(n)\, \frac{2^\ell}{\lambda} \|f\|_{L^1(\mathbb{R}^n)} ,$$

where $C(n)$ is a constant independent of ℓ and λ.

Proof. — We claim that it is enough to prove that for each $\ell \geqslant 1$, we have

$$\bigl|K_\ell(x)\bigr| \leqslant C(n)\, \frac{2^\ell}{\bigl(1+|x|\bigr)^{n+1}} , \quad x \in \mathbb{R}^n . \tag{4.9}$$

Indeed, since $(1+|x|)^{-n-1}$ is radial, radially decreasing and integrable, we will have for all $x \in \mathbb{R}^n$, as in (4.6), that

$$\sup_{r>0} \bigl|\bigl[(K_\ell)_{(r)} * f\bigr](x)\bigr| \leqslant \widetilde{C}(n)\, 2^\ell\, (\mathrm{M}f)(x) .$$

The result of Proposition 4.10 follows then from the weak estimate in Theorem 0.1, the standard maximal theorem. We now turn to the proof of (4.9). We want a bound for $K_\ell = \varphi_\ell^\vee * \sigma$, for $\ell \geqslant 1$, where σ is the uniform probability measure on S^{n-1} and $\varphi_\ell^\vee = (\psi^\vee)_{(2^{-\ell})}$. Since $\psi^\vee$ belongs to the Schwartz class, we can bound $|\psi^\vee|$ by a multiple $c_n g$ of the radial and radially decreasing integrable function $g(x) = (1+|x|)^{-n-1}$. In order to bound K_ℓ, we shall prove that

$$\begin{aligned} c_n^{-1} |K_\ell(x)| \leqslant (g_{(2^{-\ell})} * \sigma)(x) &= \int_{S^{n-1}} g_{(2^{-\ell})}(x-z)\, \mathrm{d}\sigma(z) \\ &\leqslant C(n) 2^\ell (1+|x|)^{-n-1} . \end{aligned}$$

This is easy when $|x| > 2$, because for each z in S^{n-1}, we have then $|x-z| \geqslant |x| - 1 \geqslant |x|/2$ and $1+|x| \leqslant 2|x|$. Recalling $g_{(2^{-\ell})}(y) = 2^{n\ell} g(2^\ell y)$, we get

$$\begin{aligned} G_\ell(x) := (g_{(2^{-\ell})} * \sigma)(x) &\leqslant \max_{z \in S^{n-1}} g_{(2^{-\ell})}(x-z) \leqslant 2^{n\ell}\, (1 + 2^\ell |x|/2)^{-n-1} \\ &\leqslant 2^{n\ell} 2^{-(\ell-1)(n+1)}\, |x|^{-n-1} = 2^{n+1-\ell}\, |x|^{-n-1} \\ &\leqslant 2^{2n+1}\, (1+|x|)^{-n-1} , \end{aligned}$$

even better than required. Suppose now that $|x| \leqslant 2$. It is enough to prove that $G_\ell(x) \leqslant C(n) 2^\ell$, since we have $1+|x| \leqslant 3$ in this second case, hence it will follow that $C(n)\,2^\ell \leqslant [C(n)3^{n+1}]2^\ell(1+|x|)^{-n-1}$. For $y \in \mathbb{R}^n$, we write $y=(v,t)$ with $v \in \mathbb{R}^{n-1}$ and t real. By the rotational invariance, we may restrict the study to $x=(0,s)$, $s \geqslant 0$. We write each $z \in S^{n-1}$ as $z=(v,t)$, and thus $x-z=(-v,s-t)$. Let π_0 be the orthogonal projection of $\mathbb{R}^n$ onto the hyperplane of vectors $(w,0)$, $w \in \mathbb{R}^{n-1}$. Since $g_{(2^{-\ell})}$ is radial and radially decreasing, we see that $g_{(2^{-\ell})}(x-z) \leqslant g_{(2^{-\ell})}(\pi_0(x-z)) = g_{(2^{-\ell})}(-v,0)$. This yields

$$\begin{aligned} G_\ell(x) = G_\ell(0,s) &= \int_{S^{n-1}} g_{(2^{-\ell})}(x-z)\,\mathrm{d}\sigma(z) \\ &\leqslant \int_{S^{n-1}} g_{(2^{-\ell})}(\pi_0(x-z))\,\mathrm{d}\sigma(z) \\ &= \int_{\mathbb{R}^{n-1}} g_{(2^{-\ell})}(-v,0)\,\mathrm{d}\nu(v)\,, \end{aligned}$$

where ν is the projection on $\mathbb{R}^{n-1}$ of the probability measure σ. We have that

$$\mathrm{d}\nu(v) = \frac{2}{s_{n-1}}\,\frac{\mathbf{1}_{\{|v|<1\}}}{\sqrt{1-|v|^2}}\,\mathrm{d}v = C(n)\frac{\mathbf{1}_{\{|v|<1\}}}{\sqrt{1-|v|^2}}\,\mathrm{d}v\,,$$

where s_{n-1} is the measure of S^{n-1} recalled in (1.34). We cut the integral with respect to ν into two parts, according to $|v|<1/2$ or not. In the part E_1 corresponding to $|v|<1/2$, we have $1-|v|^2 \geqslant 3/4$, hence

$$E_1 \leqslant \sqrt{\frac{4}{3}}\,C(n)\int_{|v|<1/2} g_{(2^{-\ell})}(v,0)\,\mathrm{d}v \leqslant 2C(n)\int_{\mathbb{R}^{n-1}} g_{(2^{-\ell})}(v,0)\,\mathrm{d}v\,.$$

We are integrating on $\mathbb{R}^{n-1}$ the function $g_{(2^{-\ell})}$ that is normalized for a change of variable in dimension n. This implies that

$$E_1 \leqslant 2C(n)2^\ell 2^{(n-1)\ell}\int_{\mathbb{R}^{n-1}} g(2^\ell v,0)\,\mathrm{d}v = 2C(n)2^\ell\int_{\mathbb{R}^{n-1}} g(u,0)\,\mathrm{d}u\,,$$

a bound of the expected form. In the second case, we have $|v|>1/2$ and

$$g_{(2^{-\ell})}(v,0) = 2^{n\ell}(1+2^\ell|v|)^{-n-1} \leqslant 2^{n\ell}2^{-(\ell-1)(n+1)} \leqslant 2^n.$$

It follows that the integral E_2 limited to $|v|>1/2$, with respect to the probability measure ν, is bounded by a function of n. □

4.2.3. Conclusion

Proof of Theorem 4.2. — Thanks to the results of the previous subsection, the proof is easy. Using the Marcinkiewicz theorem (see Zygmund [85, Chap. XII], or [64, Theorem 5.60]), we shall interpolate between the weak type $(1,1)$ and the strong type $(2,2)$. We apply Proposition 4.9, Proposition 4.10 in $\mathbb{R}^N$ and interpolation with parameter $\theta = 2-2/p$, where

$1 < p \leqslant 2$. For all $\ell \geqslant 1$ and all $f \in L^p(\mathbb{R}^N)$, since the chosen interpolation parameter θ satisfies $(1-\theta)/1 + \theta/2 = 1/p$, we have

$$\big\| M_{K_\ell} f \big\|_{L^p(\mathbb{R}^N)} \leqslant \kappa(1,2,p) C(N) \big(2^\ell\big)^{-1+2/p} \big(2^{-\ell(N-2)/2}\big)^{2-2/p} \|f\|_{L^p(\mathbb{R}^N)} \,,$$

where $\kappa(1,2,p)$ is independent of N and ℓ. We have thus obtained that

$$\big\| M_{K_\ell} f \big\|_{L^p(\mathbb{R}^N)} \leqslant C'(N,p)\, 2^{\ell\,[N/p-(N-1)]} \|f\|_{L^p(\mathbb{R}^N)} \,.$$

For $p > N/(N-1)$, the series $\sum_{\ell \geqslant 1} 2^{\ell\,[N/p-(N-1)]}$ converges. Moreover, we know by Proposition 4.8 that M_{K_0} maps $L^p(\mathbb{R}^N)$ to itself for all $1 < p < +\infty$. Therefore, in view of (4.4), we obtain that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^N)$ for every real number p such that $N/(N-1) < p \leqslant 2$. For $p > 2$, we proceed by interpolation between the $L^2(\mathbb{R}^N)$ case and the trivial $L^\infty(\mathbb{R}^N)$ case. □

5. The L^2 result of Bourgain

In an article published in 1986, Bourgain has generalized the L^2 case of the Stein result presented in Section 4. This L^2 case for Euclidean balls only required Proposition 4.9 and the “method of rotations”. The maximal operator M_C associated to a symmetric convex body C was defined in (0.3.**M**).

Theorem 5.1 (Bourgain [9]). — *There exists a universal constant κ_2 such that for every integer $n \geqslant 1$ and every symmetric convex body $C \subset \mathbb{R}^n$, one has*

$$\forall\, f \in L^2(\mathbb{R}^n)\,, \quad \|\mathrm{M}_C f\|_{L^2(\mathbb{R}^n)} \leqslant \kappa_2 \, \|f\|_{L^2(\mathbb{R}^n)} \,.$$

The rest of this section is devoted to the proof of this maximal theorem, together with the description of the general framework concerning maximal functions associated to convex sets. We shall in particular establish some geometric inequalities for log-concave distributions that will be applied in the subsequent sections.

5.1. The general setting

Let C be a symmetric convex body in $\mathbb{R}^n$. Throughout these Notes, we let K_C be the density of the uniform probability measure μ_C on C, and m_C denotes the Fourier transform of K_C or of μ_C. Hence, we have

$$K_C(x) = \frac{1}{|C|} \mathbf{1}_C(x)\,, \quad \mathrm{d}\mu_C(x) = K_C(x)\,\mathrm{d}x\,, \quad m_C(\xi) = \widehat{K_C}(\xi) = \widehat{\mu_C}(\xi)\,,$$

for all $x, \xi \in \mathbb{R}^n$. Notice that $K_{\lambda C} = (K_C)_{(\lambda)}$ and $m_{\lambda C}(\xi) = m_C(\lambda \xi)$ for each $\lambda > 0$ and $\xi \in \mathbb{R}^n$. We already know that the maximal operator M_C

acts boundedly on $L^p(\mathbb{R}^n)$, $1 < p \leqslant +\infty$, but the bounds we have so far depend on n.

This L^p result comes from the weak type estimate (0.4) given by the Vitali covering lemma. Except for the value of the constant, it is clear that this weak type $(1,1)$ result for M_C is optimal, as we can see by taking for f the indicator $\mathbf{1}_C$ of the symmetric convex body $C \subset \mathbb{R}^n$. Let C have volume 1, so that $\|f\|_1 = 1$. For any given $r > 0$ and $x \in rC$, we see that $x + (r+1)C$ contains C, therefore

$$(\mathrm{M}_C f)(x) \geqslant |(r+1)C|^{-1} \int_{x+(r+1)C} \mathbf{1}_C(y)\,\mathrm{d}y = |(r+1)C|^{-1} = (r+1)^{-n}$$

and $\{\mathrm{M}_C f \geqslant (r+1)^{-n}\} \supset rC$. Every value c in the interval $(0, 2^{-n}]$ can be written as $c = (r+1)^{-n}$ for some $r \geqslant 1$, hence

$$\forall\, c \in (0, 2^{-n}], \quad \bigl|\{\mathrm{M}_C f \geqslant c\}\bigr| \geqslant |rC| = \frac{(r+1)^{-n}}{c} r^n \geqslant \frac{2^{-n}}{c}.$$

The maximal function $\mathrm{M}_C \mathbf{1}_C$ is not integrable. It belongs to the space $L^{1,\infty}(\mathbb{R}^n)$, the so-called weak-$L^1$ space, and nothing better: any bounded radial and radially decreasing function belonging to $L^{1,\infty}(\mathbb{R}^n)$ is smaller than a multiple of $\mathrm{M}_C \mathbf{1}_C$.

The maximal function $\mathrm{M}_C f$ is given by $\mathrm{M}_C f = \sup_{t>0} \, (K_C)_{(t)} * |f|$, where $(K_C)_{(t)}$ is the dilate from (2.7). More generally, let K be a probability density on $\mathbb{R}^n$, resp. an integrable kernel K. We define the maximal function M_K or M_K by

$$\mathrm{M}_K f = \sup_{t>0} K_{(t)} * |f|\,, \quad \text{resp. } M_K f = \sup_{t>0} \bigl|K_{(t)} * f\bigr|\,.$$

If A is linear and bijective on $\mathbb{R}^n$, we can see that the maximal operators M_C and M_{AC} have the same norm on $L^p(\mathbb{R}^n)$. For a function f on $\mathbb{R}^n$ we define $f_{(A)}$ by

$$\forall\, x \in \mathbb{R}^n, \quad f_{(A)}(x) = |\det A|^{-1} f(A^{-1}x)\,.$$

We have $|f|_{(A)} = \bigl|f_{(A)}\bigr|$, $(\sup_i f_i)_{(A)} = \sup_i (f_i)_{(A)}$, and $(f * g)_{(A)} = f_{(A)} * g_{(A)}$ since

$$\int_{\mathbb{R}^n} |\det A|^{-2} f(A^{-1}(x-y)) g(A^{-1}y)\,\mathrm{d}y = |\det A|^{-1} \int_{\mathbb{R}^n} f(A^{-1}x - z) g(z)\,\mathrm{d}z\,.$$

It is clear that $\bigl(f_{(A)}\bigr)_{(t)} = f_{(tA)} = \bigl(f_{(t)}\bigr)_{(A)}$. If S_A is the mapping $f \mapsto f_{(A)}$, then $S_{A,p} := |\det A|^{1/q} S_A$, with q conjugate to p, is an onto isometry of $L^p(\mathbb{R}^n)$.

The density K_{AC} is equal to $(K_C)_{(A)}$. For every integrable kernel K on $\mathbb{R}^n$, we see now that K and $K_{(A)}$ produce maximal functions that are conjugate by the isometry $S_{A,p}$ of $L^p(\mathbb{R}^n)$, and have therefore the same norm on $L^p(\mathbb{R}^n)$. We have

$$\begin{aligned} M_{K_{(A)}} f_{(A)} &= \sup_{t>0} \bigl| (K_{(A)})_{(t)} * f_{(A)} \bigr| = \sup_{t>0} \bigl| (K_{(t)})_{(A)} * f_{(A)} \bigr| \\ &= \sup_{t>0} \bigl| (K_{(t)} * f)_{(A)} \bigr| = (M_K f)_{(A)} \,. \end{aligned}$$

It follows that $M_{K_{(A)}} \circ S_{A,p} = S_{A,p} \circ M_K$. This remark allows us to assume that C is in *isotropic position*: one says that a symmetric convex body C is in isotropic position if the quadratic form

$$Q_C : \xi \mapsto Q_C(\xi) = \int_C (\xi \cdot x)^2 \, \mathrm{d}x \,, \quad \xi \in \mathbb{R}^n ,$$

is a multiple of the square $\xi \mapsto |\xi|^2$ of the Euclidean norm on $\mathbb{R}^n$. Since Q_C is positive definite for every symmetric convex body C, we can bring it to the form $\xi \mapsto \lambda |\xi|^2$, $\lambda > 0$, by a suitable linear change of coordinates. For an isotropic symmetric convex set C_0 of volume 1, one defines the *isotropy constant* $L(C_0)$ by

$$L(C_0)^2 = \int_{C_0} (\mathbf{e}_1 \cdot x)^2 \, \mathrm{d}x \,, \quad \text{and one has then} \quad \int_{C_0} (\xi \cdot x)^2 \, \mathrm{d}x = L(C_0)^2 |\xi|^2$$

for every $\xi \in \mathbb{R}^n$. For C_* isotropic of the form $C_* = r C_0$, $r > 0$, we get $|C_*| = r^n$ and for every $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} (\xi \cdot x)^2 K_{C_*}(x) \, \mathrm{d}x &= \frac{1}{|C_*|} \int_{C_*} (\xi \cdot x)^2 \, \mathrm{d}x = r^{-n} \int_{C_0} (\xi \cdot r u)^2 r^n \, \mathrm{d}u \\ &= r^2 L(C_0)^2 |\xi|^2 = |C_*|^{2/n} L(C_0)^2 |\xi|^2 \,. \end{aligned} \tag{5.1}$$

Let A linear and invertible put C_* in another isotropic position AC_*, so that $Q_{AC_*}(\xi) = \lambda |\xi|^2$ for some $\lambda > 0$ and all $\xi \in \mathbb{R}^n$. Letting $\nu = \lambda |AC_*|^{-1}$ we get

$$\begin{aligned} \nu |\xi|^2 = \int_{\mathbb{R}^n} (\xi \cdot y)^2 K_{AC_*}(y) \, \mathrm{d}y &= \int_{\mathbb{R}^n} (\xi \cdot Ax)^2 K_{C_*}(x) \, \mathrm{d}x \\ &= |C_*|^{2/n} L(C_0)^2 |A^T \xi|^2 \,, \end{aligned}$$

hence A is a multiple ρU of an isometry U, $|\det A| = \rho^n$ and $\nu = |C_*|^{2/n} L(C_0)^2 \rho^2 = |AC_*|^{2/n} L(C_0)^2$, thus $|AC_*|^{-2/n} \int_{\mathbb{R}^n} (\theta \cdot y)^2 K_{AC_*}(y) \, \mathrm{d}y = L(C_0)^2$ for every $\theta \in S^{n-1}$.

When C is isotropic, it follows that $L(C) := L(C_0)$ is well defined by

$$\begin{aligned} L(C)^2 &= |C|^{-2/n} \int_{\mathbb{R}^n} (\theta \cdot x)^2 K_C(x) \, \mathrm{d}x \\ &= |C|^{-1-2/n} \int_C (\theta \cdot x)^2 \, \mathrm{d}x \,, \quad \theta \in S^{n-1} \,. \end{aligned} \tag{5.2}$$

A well-known open question (see [58]) is to decide whether the isotropy constant is bounded above by a universal constant valid for all symmetric convex bodies and every n. The best upper bound that is known so far, due to Klartag [49] improving Bourgain [12], is $L(C) \leqslant \kappa n^{1/4}$ in dimension n. It is known that $L(C)$ is bounded below by a universal constant. However, neither this known fact nor the unsolved problem will interfere with the treatment of the maximal function problem.

Clearly, K_C and $(K_C)_{(\lambda)}$ have the same maximal function for every $\lambda > 0$, so we can choose any multiple among isotropic positions of C. Here, we do not follow Bourgain [9] who chooses the isotropic position of volume 1, we prefer the isotropic position such that μ_C has covariance matrix I_n. We thus assume that

$$\forall\, \theta \in S^{n-1}, \quad \int_{\mathbb{R}^n} (\theta \cdot x)^2 \,\mathrm{d}\mu_C(x) = \frac{1}{|C|} \int_C (\theta \cdot x)^2 \,\mathrm{d}x = 1\,. \tag{5.3}$$

This means that the one-dimensional marginals of μ_C, images of μ_C by $x \mapsto \theta \cdot x$ for $\theta \in S^{n-1}$, have all variance 1. We shall say in this case that C is isotropic and *normalized by variance*. We have then in addition that

$$\int_C |x|^2 \,\mathrm{d}x = n|C| \quad \text{and} \quad |C| = L(C)^{-n}\,.$$

If we look for a (centrally symmetric) Euclidean ball in $\mathbb{R}^n$ normalized by variance, its radius $r = r_{n,V}$ must therefore satisfy $\int_0^r t^{n+1} s_{n-1} \,\mathrm{d}t = n \int_0^r t^{n-1} s_{n-1} \,\mathrm{d}t$, giving

$$r_{n,V} = \sqrt{n+2}\,. \tag{5.4}$$

In the same way, we can bring to isotropy a symmetric probability density K on $\mathbb{R}^n$, i.e., such that $K(-x) = K(x)$ for $x \in \mathbb{R}^n$, by a linear change to $K_{(A)}$ for some A linear and invertible. When K is isotropic, there exists $\sigma > 0$ such that

$$\int_{\mathbb{R}^n} (\xi \cdot x)^2 K(x) \,\mathrm{d}x = \sigma^2 |\xi|^2, \quad \xi \in \mathbb{R}^n,$$

which means that all one-dimensional marginals of K have the same variance σ^2. We shall then say for brevity that K is *isotropic with variance* σ^2. The dilated density $K_{(1/\sigma)} : x \mapsto \sigma^n K(\sigma x)$ is normalized by variance. For example, the standard Gaussian γ_n in (1.17) is normalized by variance. For the study of maximal functions, we can always assume that K is normalized by variance.

5.2. On the volume of sections

We have seen in (2.14) that the Fourier transform m of a kernel $K \in L^1(\mathbb{R}^n)$ can be expressed as

$$m(u\xi) = \int_{\mathbb{R}} \varphi_{\theta,K}(s)\, \mathrm{e}^{-2\mathrm{i}\pi s u|\xi|} \,\mathrm{d}s\,, \quad u \in \mathbb{R},\ \xi \in \mathbb{R}^n \setminus \{0\}\,,$$

where one has set $\theta = |\xi|^{-1}\xi$ and $\varphi_\theta(s) = \varphi_{\theta,K}(s) = \int_{\theta^\perp} K(y+s\theta)\,\mathrm{d}^{n-1}y$ for every $s \in \mathbb{R}$. When K is the kernel K_C corresponding to a symmetric convex body C, the function φ_θ is the "normalized" function of $(n-1)$-dimensional volumes of hyperplane sections parallel to $\theta^\perp$, defined by

$$\varphi_{\theta,C}(s) = \int_{\theta^\perp} K_C(y+s\theta)\,\mathrm{d}^{n-1}y = \frac{\bigl|C \cap (\theta^\perp + s\theta)\bigr|_{n-1}}{|C|_n}\,.$$

We know by the Brunn–Minkowski inequality [37, Theorem 4.1] that $\varphi_{\theta,C}$ is log-concave on $\mathbb{R}$. Indeed, a form of this inequality states that

$$|(1-\lambda)A + \lambda B| \geqslant |A|^{1-\lambda}|B|^{\lambda}$$

whenever A, B are compact subsets of $\mathbb{R}^n$ and $\lambda \in [0,1]$. Recall that a function $K \geqslant 0$ on $\mathbb{R}^n$ is *log-concave* when

$$K\bigl((1-\alpha)x_0 + \alpha x_1\bigr) \geqslant K(x_0)^{1-\alpha} K(x_1)^{\alpha}\,, \quad x_0, x_1 \in \mathbb{R}^n,\ \alpha \in [0,1]\,,$$

in other words, when $\log K$ is concave on the convex set $\{K > 0\}$.

More generally than Brunn–Minkowski, the Prékopa–Leindler inequality [37, Theorem 7.1] implies that the function $\varphi_{\theta,K}$ defined in (2.14) is a log-concave probability density on the real line if K is a log-concave probability density on $\mathbb{R}^n$. The statement of Prékopa–Leindler is as follows: if α is in $(0,1)$, if f_0, f_1, f_α nonnegative and integrable Borel functions on $\mathbb{R}^n$ are such that

$$f_\alpha\bigl((1-\alpha)x_0 + \alpha x_1\bigr) \geqslant f_0(x_0)^{1-\alpha} f_1(x_1)^{\alpha}$$

for all $x_0, x_1 \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f_\alpha(x)\,\mathrm{d}x \geqslant \Bigl(\int_{\mathbb{R}^n} f_0(x)\,\mathrm{d}x\Bigr)^{1-\alpha} \Bigl(\int_{\mathbb{R}^n} f_1(x)\,\mathrm{d}x\Bigr)^{\alpha}.$$

Given $\theta \in S^{n-1}$, s_0, s_1 real and letting $f_j(y) = K(y + s_j\theta)$ for $y \in \theta^\perp$ and $j = 0, 1$, $s_\alpha = (1-\alpha)s_0 + \alpha s_1$ and $f_\alpha(y) = K(y + s_\alpha\theta)$, we obtain that $\varphi_{\theta,K}$ is log-concave by applying Prékopa–Leindler on $\theta^\perp \simeq \mathbb{R}^{n-1}$ to these functions f_0, f_1 and f_α. Similarly, one shows that convolutions of log-concave densities are log-concave. Without more effort, Bourgain's proof also gives the following theorem.

Theorem 5.2. — *There exists a constant $\kappa_2 < 140$ such that for every integer $n \geqslant 1$ and every symmetric log-concave probability density K on $\mathbb{R}^n$, one has*

$$\forall\, f \in L^2(\mathbb{R}^n)\,, \quad \|\mathrm{M}_K f\|_{L^2(\mathbb{R}^n)} \leqslant \kappa_2\, \|f\|_{L^2(\mathbb{R}^n)}\,.$$

We turn to the proof of the main inequalities about log-concave functions, which will be used throughout our Notes. We introduce the *right maximal function* f_r^* of a locally integrable function f on an interval $[\tau, +\infty)$ of the line by setting

$$f_r^*(x) = \sup_{t>0} \frac{1}{t} \int_x^{x+t} |f(s)|\,\mathrm{d}s\,, \quad x \geqslant \tau\,. \tag{5.5}$$

One sees that $f_r^* \leqslant f^* \leqslant 2\mathrm{M}f$, where f^* is the uncentered maximal function from (0.2), and $f_r^*(x) \geqslant |f(x)|$ at each *Lebesgue point* x of f, hence almost everywhere. When ψ is nonnegative, integrable and decreasing on $[x, +\infty)$, then

$$\int_x^{+\infty} |f(s)|\,\psi(s)\,\mathrm{d}s \leqslant \Bigl(\int_x^{+\infty} \psi(s)\,\mathrm{d}s \Bigr) f_r^*(x)\,. \tag{5.6}$$

One can get (5.6) as in (4.6), by approximating ψ by a combination of functions $t_k^{-1} \mathbf{1}_{[x, x+t_k]}$. We can also define in a similar way a *left* maximal function f_ℓ^*.

Lemma 5.3. — *Let φ be an integrable log-concave function on an interval $[\tau, +\infty)$, let p belong to $(0, +\infty)$ and let*

$$S_0(\tau) = \int_\tau^{+\infty} \varphi(s)\,\mathrm{d}s\,, \quad S_p(\tau) = \int_\tau^{+\infty} (s-\tau)^p \varphi(s)\,\mathrm{d}s\,.$$

Then $S_p(\tau)$ is finite. Furthermore, assuming $S_p(\tau) > 0$, we have

$$\varphi(\tau)^p \leqslant \frac{\Gamma(p+1)\, S_0(\tau)^{p+1}}{S_p(\tau)}\,, \quad \max_{s \geqslant \tau} \varphi(s)^p \geqslant \varphi_r^*(\tau)^p \geqslant \frac{S_0(\tau)^{p+1}}{(p+1)\, S_p(\tau)}\,. \tag{5.7}$$

Proof. — We have $\varphi \geqslant 0$ by definition of log-concavity. We assume $S_p(\tau) > 0$, hence $S_0(\tau) > 0$. We may suppose $\tau = 0$ by translating and $S_0 := S_0(0) = 1$ by homogeneity. We begin with the left-hand inequality in (5.7), assuming $a := \varphi(0) > 0$. Consider the log-affine probability density $\psi(s) = a\,\mathrm{e}^{-as}$ on $[0, +\infty)$, chosen so that $\psi(0) = \varphi(0)$. By log-concavity, the set $I = \{\varphi \geqslant \psi\}$ is an interval, such that $0 \in I \subset [0, +\infty)$. Since φ and ψ both have integral 1 on $[0, +\infty)$, the interval I is not reduced to $\{0\}$. If $I = [0, +\infty)$, the densities are equal and

$$S_p := \int_0^{+\infty} s^p \varphi(s)\,\mathrm{d}s = \int_0^{+\infty} s^p \psi(s)\,\mathrm{d}s = \frac{1}{a^p} \int_0^{+\infty} (as)^p\, \mathrm{e}^{-as}\, a\,\mathrm{d}s = \frac{\Gamma(p+1)}{a^p}\,.$$

Otherwise, the interval I is bounded, let $s_0 := \sup I > 0$. We have $\psi \leqslant \varphi$ on $[0, s_0)$ and $\varphi(s) < \psi(s)$ when $s > s_0$, implying that $S_p(0)$ is finite. The antiderivative F of $\varphi - \psi$ vanishing at 0 is first increasing, then decreasing on $[0, +\infty)$, and tends to 0 at infinity because φ and ψ have equal integrals. It follows that F is nonnegative on $[0, +\infty)$. Recalling that $0 \leqslant \varphi(s) < \psi(s)$ at infinity, we know that $|F(s)|$ is exponentially small at infinity, and integrating by parts we obtain

$$\int_0^{+\infty} s^p \bigl(\varphi(s) - \psi(s)\bigr) \, \mathrm{d}s = -p \int_0^{+\infty} s^{p-1} F(s) \, \mathrm{d}s \leqslant 0 \, .$$

One concludes the first part by writing

$$S_p = \int_0^{+\infty} s^p \varphi(s) \, \mathrm{d}s \leqslant \int_0^{+\infty} s^p \psi(s) \, \mathrm{d}s = \frac{\Gamma(p+1)}{a^p} \, .$$

For the right-hand inequality in (5.7), we let $b = \varphi_r^*(0) > 0$ and consider the probability density $\psi(s) = b \mathbf{1}_{[0,1/b]}(s)$ on $[0, +\infty)$. Let F be the antiderivative of $\varphi - \psi$ vanishing at 0. When $0 < x \leqslant 1/b$ we have by definition of $\varphi_r^*(0)$ that

$$\frac{F(x)}{x} = \frac{1}{x} \int_0^x (\varphi(s) - \psi(s)) \, \mathrm{d}s = \Bigl(\frac{1}{x} \int_0^x \varphi(s) \, \mathrm{d}s \Bigr) - b \leqslant 0 \, .$$

We see that $\psi(x) = 0 \leqslant \varphi(x)$ when $x \geqslant 1/b$. It follows that the function F is $\leqslant 0$ on $[0, 1/b]$, then increasing on $[1/b, +\infty)$, tends to 0 at infinity, thus F is $\leqslant 0$ on the half-line $[0, +\infty)$. Arguing as before, we have consequently

$$S_p = \int_0^{+\infty} s^p \varphi(s) \, \mathrm{d}s \geqslant b \int_0^{1/b} s^p \, \mathrm{d}s = \frac{1}{(p+1) \, b^p} \, . \qquad \square$$

For every $\theta \in S^{n-1}$, the function $\varphi_{\theta,C}$ associated to a symmetric convex set C is even, log-concave and has integral 1 by definition. We shall thus be in a position to apply to it the following Corollary 5.4.

Corollary 5.4. — *Suppose that φ is a symmetric log-concave probability density on $\mathbb{R}$ and let $\sigma^2 := \int_{\mathbb{R}} s^2 \varphi(s) \, \mathrm{d}s$. One has that*

$$\frac{1}{12 \sigma^2} \leqslant \varphi(0)^2 = \max_{s \in \mathbb{R}} \varphi(s)^2 \leqslant \frac{1}{2 \sigma^2} \, .$$

Proof. — Since φ is even and log-concave, we have $\varphi(0) = \max_{s \in \mathbb{R}} \varphi(s)$. We apply Lemma 5.3 with $p = 2$, $\tau = 0$, and observe that $S_0(0) = 1/2$, $S_2(0) = \sigma^2/2$. $\square$

The preceding result is sharp, as one sees with the two examples

$$\varphi_0(s) = \frac{1}{\sqrt{2}} \, \mathrm{e}^{-\sqrt{2}|s|} \, , \quad \varphi_1(s) = \frac{1}{2\sqrt{3}} \, \mathbf{1}_{[-\sqrt{3}, \sqrt{3}]}(s) \, , \quad s \in \mathbb{R} \, . \tag{5.8}$$

The next corollary is not very sharp, but easy to deduce from Lemma 5.3. When the function $\varphi \geqslant 0$ is defined on the line and $p \in [0, +\infty)$, we set

$$S_p^+(\tau) = \int_\tau^{+\infty} (s-\tau)^p \varphi(s)\,\mathrm{d}s\,, \quad S_p^-(\tau) = \int_{-\infty}^{\tau} |s-\tau|^p \varphi(s)\,\mathrm{d}s\,.$$

Corollary 5.5. — *Let φ be a* centered *log-concave probability density on $\mathbb{R}$ and let $\sigma^2 := \int_{\mathbb{R}} s^2 \varphi(s)\,\mathrm{d}s$. We have that*

$$\frac{1}{24\sigma^2} \leqslant \frac{\varphi_\ell^*(0)^2 + \varphi_r^*(0)^2}{2} \leqslant \max_{s\in\mathbb{R}} \varphi(s)^2 \leqslant \frac{4}{\sigma^2}\,.$$

Proof. — We begin with the rightmost inequality. Let us fix τ real. Since φ is a centered probability density, one has that

$$S_2^+(\tau) + S_2^-(\tau) = \int_{\mathbb{R}} (s-\tau)^2 \varphi(s)\,\mathrm{d}s = \sigma^2 + \tau^2 \geqslant \sigma^2\,, \quad \tau \in \mathbb{R}\,.$$

Up to a symmetry around τ, possibly replacing the function φ by $s \mapsto \varphi(2\tau - s)$, we may assume that $S_2^+(\tau) \geqslant \sigma^2/2$. We have $S_0^+(\tau) = \int_\tau^{+\infty} \varphi(s)\,\mathrm{d}s \leqslant 1$ since φ is a probability density on $\mathbb{R}$, thus by Lemma 5.3 with $p = 2$ we get

$$\varphi(\tau)^2 \leqslant \frac{2\,S_0^+(\tau)^3}{S_2^+(\tau)} \leqslant \frac{4}{\sigma^2}\,.$$

Since τ is arbitrary, we obtain the right-hand inequality. Let us pass to the other inequality. By Lemma 5.3 with $p = 2$ on the intervals $(0, +\infty)$ and $(-\infty, 0)$, we conclude using $S_2^\pm(0) \leqslant \sigma^2$ and $S_0^+(0) + S_0^-(0) = 1$ that

$$\varphi_r^*(0)^2 + \varphi_\ell^*(0)^2 \geqslant \frac{S_0^+(0)^3}{3S_2^+(0)} + \frac{S_0^-(0)^3}{3S_2^-(0)} \geqslant \frac{S_0^+(0)^3 + S_0^-(0)^3}{3\sigma^2} \geqslant \frac{1}{12\sigma^2}\,. \qquad \square$$

Lemma 5.6. — *Let φ be a symmetric log-concave probability density on $\mathbb{R}$, with variance σ^2. The function φ decays exponentially at infinity, with a rate depending on its variance and satisfying*

$$\forall\, s \in \mathbb{R}\,, \quad \sigma\varphi(\sigma s) \leqslant \min(2\,\mathrm{e}^{-|s|/2}, 11\,\mathrm{e}^{-|s|})\,.$$

Proof. — Without loss of generality, we may assume that $\sigma = 1$. It follows then from Corollary 5.4 that $1/(2\sqrt{3}) \leqslant a := \varphi(0) \leqslant 1/\sqrt{2}$. Consider the log-affine function $\psi_\beta(s) = a\,\mathrm{e}^{-\beta s}$ on $[0, +\infty)$, with $\beta > 0$, satisfying $\psi_\beta(0) = \varphi(0)$. If we have $\varphi(\tau_0) \leqslant \psi_\beta(\tau_0)$ for some $\tau_0 > 0$, it implies by log-concavity that $\varphi(s) \leqslant \psi_\beta(s) \leqslant \mathrm{e}^{-\beta s}/\sqrt{2}$ for $s \geqslant \tau_0$, and in order to obtain a bound for φ everywhere, we can apply for the values $0 \leqslant s \leqslant \tau_0$ the obvious inequalities

$$\varphi(s) \leqslant \varphi(0) = a \leqslant a\,\mathrm{e}^{\beta(\tau_0 - s)} \leqslant (\mathrm{e}^{\beta\tau_0}/\sqrt{2})\,\mathrm{e}^{-\beta s}\,.$$

For any $\tau_0 > 0$, we obtain since φ is even that

$$\varphi(\tau_0) \leqslant \psi_\beta(\tau_0) \;\Rightarrow\; \varphi(s) = \varphi(|s|) \leqslant \mathrm{e}^{\beta\tau_0 - \ln\sqrt{2}}\,\mathrm{e}^{-\beta|s|}\,, \quad s \in \mathbb{R}\,. \tag{5.9}$$

On the other hand, if $\varphi(s) > \psi_\beta(s)$ for every $s \in (0, \tau]$, then

$$1/2 = \int_0^{+\infty} s^2 \varphi(s)\,\mathrm{d}s > \int_0^\tau s^2 \psi_\beta(s)\,\mathrm{d}s = \frac{a}{\beta^3} \int_0^{\beta\tau} u^2\,\mathrm{e}^{-u}\,\mathrm{d}u$$
$$= \frac{a}{\beta^3} \Big[-\mathrm{e}^{-u}(u^2 + 2u + 2) \Big]_{u=0}^{\beta\tau} \geqslant \frac{1}{2\sqrt{3}\beta^3} \big(2 - \mathrm{e}^{-\beta\tau}(\beta^2\tau^2 + 2\beta\tau + 2) \big).$$

Equivalently, when $\varphi(s) > \psi_\beta(s)$ for every $s \in (0, \tau]$, we get that

$$\mathrm{e}^{-\beta\tau} \big(\beta^2\tau^2 + 2\beta\tau + 2 \big) > 2 - \sqrt{3}\,\beta^3\,. \tag{5.10}$$

Suppose that $\beta^3 < 2/\sqrt{3}$. Then (5.10) cannot be true if τ is large. For every such β, there exists $\tau_0 > 0$ such that $\varphi(\tau_0) \leqslant \psi_\beta(\tau_0)$ and by (5.9), there is a constant $c(\beta)$ such that $\varphi(s) \leqslant c(\beta)\,\mathrm{e}^{-\beta|s|}$ on the line. For numerical purposes, it is more convenient to express this as follows. If $0 < \sqrt{3}\,\beta^3 < 2$ and if

$$\mathrm{e}^{-x}(x^2 + 2x + 2) \leqslant 2 - \sqrt{3}\,\beta^3, \tag{5.11}$$

then $x > 0$, and letting $x_0(\beta) = x$, we know that $\varphi(s) \leqslant \psi_\beta(|s|) \leqslant \mathrm{e}^{-\beta|s|}/\sqrt{2}$ when $|s| \geqslant \tau_0(\beta) := x_0(\beta)/\beta$, and $\varphi(s) \leqslant c(\beta)\,\mathrm{e}^{-\beta|s|}$ for every $s \in \mathbb{R}$ by (5.9), with

$$c(\beta) = \mathrm{e}^{\beta\tau_0(\beta) - \ln\sqrt{2}} = \mathrm{e}^{x_0(\beta) - \ln\sqrt{2}}\,. \tag{5.12}$$

An almost optimal x satisfying (5.11) can be found numerically. We have for example that $\varphi(s) \leqslant 2.218\,\mathrm{e}^{-|s|/2}$ for all s when $\beta = 1/2$, with $x_0(0.5) = 1.143$. We also find $c(1) < 94.295$ with a choice $x_0(1) = 4.893$. We can then improve the first estimate given by (5.12) for $\beta = 1$. When $|s| \leqslant x_0(1) = \tau_0(1)$, we write

$$\varphi(s) \leqslant 2.218\,\mathrm{e}^{-|s|/2} = 2.218\,\mathrm{e}^{|s|/2}\,\mathrm{e}^{-|s|} \leqslant 2.218\,\mathrm{e}^{\tau_0(1)/2}\,\mathrm{e}^{-|s|} < 26\,\mathrm{e}^{-|s|}\,.$$

More generally, if we know a modified bound $c_m(\beta_1)$ such that $\varphi(s) \leqslant c_m(\beta_1)\,\mathrm{e}^{-\beta_1|s|}$ for every s and if $\varphi(s) \leqslant \mathrm{e}^{-\beta_2|s|}/\sqrt{2}$ when $|s| \geqslant \tau_0(\beta_2)$, with $\beta_1 < \beta_2$, then for $|s| \leqslant \tau_0(\beta_2)$ we can write

$$\varphi(s) \leqslant c_m(\beta_1)\,\mathrm{e}^{-\beta_1|s|} = c_m(\beta_1)\,\mathrm{e}^{(\beta_2-\beta_1)|s|}\,\mathrm{e}^{-\beta_2|s|}$$
$$\leqslant c_m(\beta_1)\,\mathrm{e}^{(\beta_2-\beta_1)\tau_0(\beta_2)}\,\mathrm{e}^{-\beta_2|s|},$$

so that

$$c_m(\beta_2) \leqslant \max\big(\mathrm{e}^{(\beta_2-\beta_1)\tau_0(\beta_2)}\,c_m(\beta_1), 1/\sqrt{2} \big)\,. \tag{5.13}$$

The following table displays admissible values for $x_0(\beta)$, $\tau_0(\beta)$, then the corresponding rough bound $c(\beta)$ from (5.12), and the modified bounds $c_m(\beta)$ obtained step by step applying (5.13), by dividing the interval $[0, 1]$ in ten equal segments, beginning with $c(0) = c_m(0) = \varphi(0) \leqslant 1/\sqrt{2} < 0.708$. We have replaced each higher precision value of x by the upper bound

$x_0(\beta) = \lceil 1000.x \rceil / 1000$, and used this replacement consistently in the further calculations of $\tau_0(\beta)$, $c(\beta)$ and $c_m(\beta)$.

β	$x_0(\beta)$	$\tau_0(\beta)$	$c(\beta)$	$c_m(\beta)$
0.0	0.000	0.000	0.708	0.708
0.1	0.182	1.820	0.849	0.850
0.2	0.381	1.906	1.036	1.029
0.3	0.603	2.010	1.293	1.259
0.4	0.854	2.135	1.662	1.559
0.5	1.143	2.287	2.218	1.960
0.6	1.484	2.474	3.119	2.511
0.7	1.903	2.719	4.742	3.296
0.8	2.451	3.064	8.203	4.478
0.9	3.255	3.617	18.328	6.430
1.0	4.893	4.893	94.295	10.489

We obtain the announced bounds when $\beta = 1/2$ and $\beta = 1$. One can obviously refine the previous argument and show that

$$\varphi(s) \leqslant c(0) \exp\Bigl(\int_0^1 \tau_0(\beta)\, \mathrm{d}\beta \Bigr) \mathrm{e}^{-|s|} \leqslant \frac{1}{\sqrt{2}} \exp\Bigl(\int_0^1 \tau_0(\beta)\, \mathrm{d}\beta \Bigr) \mathrm{e}^{-|s|} .$$

We may get in this way that $\varphi(s) < 9\,\mathrm{e}^{-|s|}$. An exact estimate could perhaps be obtained by an extreme point argument, as in [35]. Some numerical experiments suggest that for every $\beta \geqslant 0$, the maximum on $\mathbb{R}$ of $s \mapsto \mathrm{e}^{\beta|s|}\,\varphi(s)$, for φ symmetric log-concave probability density with variance 1, occurs for one of the two examples φ_0, φ_1 mentioned in (5.8). The example $\varphi_0(s)$ shows that $\mathrm{e}^{\beta|s|}\,\varphi(s)$ is unbounded when $\beta > \sqrt{2}$ and $\sigma = 1$. □

Our next estimate is so poor that it does not deserve to be given explicitly.

Corollary 5.7. — *There exists a numerical value $\kappa > 0$ such that for every centered log-concave probability density φ on $\mathbb{R}$ with variance $\sigma^2 = 1$, one has*

$$\forall\, s \in \mathbb{R}, \quad \varphi(s) \leqslant \kappa\, \mathrm{e}^{-|s|/\kappa} .$$

Proof. — Since φ is centered, we know that $\int_0^{+\infty} s\varphi(s)\,\mathrm{d}s = \int_{-\infty}^0 |s|\varphi(s)\,\mathrm{d}s$, and we can thus set $S_1 := S_1^+(0) = S_1^-(0)$. For $p \neq 1$, let us write $S_p^\pm$ instead of $S_p^\pm(0)$. We have that $S_2^+, S_2^- \leqslant \sigma^2 = 1$. By Corollary 5.5 and Lemma 5.3 with $p = 1$, applied on the intervals $[0, +\infty)$ and $(-\infty, 0]$, we get

$$2 \geqslant \max_{s \geqslant 0} \varphi(s) \geqslant \frac{(S_0^+)^2}{2 S_1} , \quad 2 \geqslant \max_{s \leqslant 0} \varphi(s) \geqslant \frac{(S_0^-)^2}{2 S_1} .$$

It follows that $8S_1 \geqslant (S_0^+)^2 + (S_0^-)^2 \geqslant 1/2$ so $S_1 \geqslant 1/16$. We also need a lower bound for $S_0^\pm$. Let $\kappa_1 = 16$. By Cauchy–Schwarz we have

$$\kappa_1^{-2} \leqslant S_1^2 \leqslant S_0^- S_2^- \leqslant S_0^- \,, \quad \kappa_1^{-2} \leqslant S_1^2 \leqslant S_0^+ S_2^+ \leqslant S_0^+ \,,$$

hence $S_0^-, S_0^+ \geqslant \kappa_1^{-2}$. Suppose that the maximum of φ is reached at $s_0 \geqslant 0$. Then φ is non-decreasing on $(-\infty, s_0]$ and by Lemma 5.3 with $p = 2$ we get

$$4 \geqslant \varphi(0)^2 = \max_{s \leqslant 0} \varphi(s)^2 \geqslant \frac{(S_0^-)^3}{3S_2^-} \geqslant \frac{\kappa_1^{-6}}{3} =: \kappa_2^{-2} \,. \tag{5.14}$$

The symmetric probability density $\varphi_1(s) = (2S_0^-)^{-1}\varphi(-|s|)$ on $\mathbb{R}$ is log-concave, has variance $\sigma_1^2 = S_2^-/S_0^- \leqslant \kappa_1^2$. By (5.14), we have $(S_0^-)^3/S_2^- \leqslant 12$. By Lemma 5.6, we know that

$$\varphi_1(s) \leqslant \frac{11}{\sigma_1}\,\mathrm{e}^{-|s|/\sigma_1} \,, \quad \text{and} \quad \varphi(s) \leqslant 22 \left(\frac{(S_0^-)^3}{S_2^-}\right)^{1/2} \mathrm{e}^{-|s|/\sigma_1} \leqslant 77\,\mathrm{e}^{-|s|/\kappa_1}$$

for $s \leqslant 0$. Let us pass to the positive side. We let $\widetilde{\varphi}$ be equal to $\varphi(s_0)$ on $[0, s_0]$ and to φ on $[s_0, +\infty)$. Then $\widetilde{S}_0^+ \geqslant S_0^+ \geqslant \kappa_1^{-2}$ and since $\kappa_2^{-1} \leqslant \varphi(0) \leqslant \varphi(x) \leqslant \varphi(s_0) \leqslant 2$ when $0 \leqslant x \leqslant s_0$, we have $\widetilde{\varphi} \leqslant 2\kappa_2\varphi$ on $[0, +\infty)$. The symmetrized function φ_1 corresponding to $\widetilde{\varphi}$ satisfies $\sigma_1^2 = \widetilde{S}_2^+/\widetilde{S}_0^+ \leqslant 2\kappa_1^2\kappa_2$. Also, we know that $(\widetilde{S}_0^+)^3/\widetilde{S}_2^+ \leqslant 3\max\widetilde{\varphi}(s)^2 \leqslant 12$. The rest is identical to the negative case. □

The next lemma is easy and classical. The *(total) mass* of a real valued (thus bounded) measure μ on $(\Omega, \mathcal{F})$ is defined by setting $\|\mu\|_1 = \mu^+(\Omega) + \mu^-(\Omega) = |\mu|(\Omega)$, where $\mu = \mu^+ - \mu^-$ is the Hahn decomposition of μ as difference of two nonnegative measures, and $|\mu| = \mu^+ + \mu^-$. On the line or on $\mathbb{R}^n$ we have

$$\|\mu\|_1 = \sup\left\{ \left| \int_{\mathbb{R}^n} \psi \,\mathrm{d}\mu \right| : \psi \in \mathcal{K}(\mathbb{R}^n),\ \|\psi\|_\infty \leqslant 1 \right\},$$

and when μ has a density f, one has that $\|f(x)\,\mathrm{d}x\|_1 = \|f\|_{L^1(\mathbb{R}^n)}$.

Lemma 5.8. — *Let μ be a real valued measure on $\mathbb{R}$ and let $m(t) = \widehat{\mu}(t)$ be its Fourier transform. For every $t \in \mathbb{R}$ we have*

$$|m(t)| \leqslant \|\mu\|_1 \,. \tag{5.15a}$$

If $\mathrm{d}\mu(s) = \psi(s)\,\mathrm{d}s$ with ψ integrable, then $m = \widehat{\mu} = \widehat{\psi}$ and $|m(t)| \leqslant \|\psi\|_{L^1(\mathbb{R}^n)}$.

Let us further assume that $\int_{\mathbb{R}} (1 + |s|)\,\mathrm{d}|\mu|(s) < +\infty$. Then m is C^1 on $\mathbb{R}$ and

$$\mathrm{i}\,m'(t) = 2\pi \int_{\mathbb{R}} s\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\mu(s) \,,$$

so $\mathrm{i}\,m'$ is the Fourier transform of the real valued measure $2\pi s\,\mathrm{d}\mu(s)$.

Let ν be a real valued measure on $\mathbb{R}$ and let $\psi(s) = \nu\big((-\infty, s]\big)$, for every $s \in \mathbb{R}$. The measure ν is the derivative of ψ in the sense of distributions and assuming ψ integrable, we have

$$2\mathrm{i}\pi t\widehat{\psi}(t) = 2\mathrm{i}\pi t \int_{\mathbb{R}} \psi(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s = \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\nu(s)\,, \tag{5.15b}$$

so $2\mathrm{i}\pi t\widehat{\psi}(t)$ is the Fourier transform of the derivative ν of ψ.

Let j, k be nonnegative integers. Suppose that ψ is of class C^{k-1} on the line, with a kth derivative $\psi^{(k)}$ in the sense of distributions that is a bounded measure ν_k on $\mathbb{R}$, and that $\lim_{|s|\to+\infty} \psi(s) = 0$, $\int_{\mathbb{R}} |s|^j\,\mathrm{d}|\nu_k|(s) < +\infty$. Then m is C^j and

$$(2\pi|t|)^k\,|m^{(j)}(t)| \leqslant (2\pi)^j\,\big\|\big(s^j\psi(s)\big)^{(k)}\big\|_1\,. \tag{5.15c}$$

Consequently, for $t \neq 0$, we have that

$$|m^{(j)}(t)| \leqslant \frac{(2\pi)^{j-k}}{|t|^k} \sum_{i=(k-j)^+}^{k-1} \binom{k}{i} \frac{j!}{(j+i-k)!} \int_{\mathbb{R}} |s|^{i+j-k}\,|\psi^{(i)}(s)|\,\mathrm{d}s + \frac{(2\pi)^{j-k}}{|t|^k} \int_{\mathbb{R}} |s|^j\,\mathrm{d}|\nu_k|(s)\,. \tag{5.15d}$$

In the line above, one can replace $\int_{\mathbb{R}} |s|^j\,\mathrm{d}|\nu_k|(s)$ with $\int_{\mathbb{R}} |s|^j\,|\psi^{(k)}(s)|\,\mathrm{d}s$, when ψ admits a derivative $\psi^{(k)}$ and $\mathrm{d}\nu_k(x) = \psi^{(k)}(x)\,\mathrm{d}x$.

Proof. — The first inequality (5.15a) is obvious. Assuming that $\int_{\mathbb{R}} |s|\,\mathrm{d}|\mu|(s)$ is finite, we write

$$m(t) = \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\mu(s) := \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\mu^+(s) - \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\mu^-(s)\,,$$

and we obtain by the dominated convergence theorem that

$$m'(t) = -2\mathrm{i}\pi \int_{\mathbb{R}} s\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\mu(s)\,.$$

If ν in (5.15b) has the form $\mathrm{d}\nu(x) = \psi'(x)\,\mathrm{d}x$ with ψ' a true derivative, we use integration by parts, otherwise we use Fubini's theorem for ν^+ and ν^-. We get

$$2\mathrm{i}\pi t \int_{\mathbb{R}} \psi(t)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s = \int_{\mathbb{R}} \mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}\nu(s)\,.$$

The verification of (5.15d) is left to the reader. Notice that by (5.16), the hypotheses imply that $\int_{\mathbb{R}} |s|^{i+j-k}\,|\psi^{(i)}(s)|\,\mathrm{d}s < +\infty$ when $(k-j)^+ \leqslant i < k$. Indeed, if $g^{(\ell+1)}$ is integrable on $[0, +\infty)$, then $g^{(\ell)}$ tends to a limit L at infinity and if g tends to 0 at infinity, it follows that $L = 0$, for example by the Taylor formula. □

The next lemma is straightforward.

Lemma 5.9. — *Let ν be a nonnegative measure on $(0,+\infty)$ and $\alpha > 0$. One has*

$$\alpha \int_0^{+\infty} s^{\alpha-1} \nu\bigl([s,+\infty)\bigr)\,\mathrm{d}s = \int_0^{+\infty} s^\alpha \,\mathrm{d}\nu(s)\,.$$

Let F be a function on $(0,+\infty)$ such that $|F(s)| \leqslant \int_s^{+\infty} \mathrm{d}\nu(s)$ for $s > 0$. One has

$$\alpha \int_0^{+\infty} s^{\alpha-1} |F(s)|\,\mathrm{d}s \leqslant \int_0^{+\infty} s^\alpha \,\mathrm{d}\nu(s)\,.$$

Suppose that the function g is differentiable on $\mathbb{R}$, with $\lim_{s\to\pm\infty} g(s) = 0$ and g' integrable on the line. It follows that

$$\alpha \int_{\mathbb{R}} |s|^{\alpha-1} |g(s)|\,\mathrm{d}s \leqslant \int_{\mathbb{R}} |s|^\alpha |g'(s)|\,\mathrm{d}s\,. \tag{5.16}$$

If in addition g is even and non-increasing on $[0,+\infty)$, one has

$$\int_{\mathbb{R}} |s|^\alpha |g'(s)|\,\mathrm{d}s = \alpha \int_{\mathbb{R}} |s|^{\alpha-1} g(s)\,\mathrm{d}s\,, \quad \textit{and} \quad \int_{\mathbb{R}} |g'(s)|\,\mathrm{d}s = 2g(0)\,.$$

Proof. — The first assertion is an immediate consequence of Fubini, because

$$\alpha \int_0^{+\infty} s^{\alpha-1} \nu\bigl([s,+\infty)\bigr)\,\mathrm{d}s = \alpha \iint \mathbf{1}_{\{0<s<t\}}\, s^{\alpha-1}\,\mathrm{d}\nu(t)\mathrm{d}s = \int_0^{+\infty} t^\alpha\,\mathrm{d}\nu(t)\,,$$

with integrals finite or not. The remaining facts are left to the reader. For (5.16), use $\mathrm{d}\nu(s) = |g'(s)|\,\mathrm{d}s$. □

We arrive to the main result of this section.

Proposition 5.10 ([9, §4]). — *Let K_{lc} be a symmetric log-concave probability density on $\mathbb{R}^n$, isotropic with variance σ^2. Let m_{lc} be the Fourier transform of K_{lc}. For every $\xi \in \mathbb{R}^n$ one has that*

$$\pi\sqrt{2}\sigma|\xi|\,|m_{lc}(\xi)| \leqslant 1\,, \quad |1-m_{lc}(\xi)| \leqslant 2\pi\sigma|\xi|\,, \quad |\xi\cdot\nabla m_{lc}(\xi)| \leqslant 2\,. \tag{5.17.$\mathbf{B}$}$$

The middle inequality follows from the fact that for every $\theta \in S^{n-1}$, one has

$$|\theta\cdot\nabla m_{lc}(t\theta)| \leqslant 2\pi\sigma\,, \quad t \in \mathbb{R}\,.$$

Remark. — These inequalities are valid for m_C, when C is a symmetric convex body, isotropic and normalized by variance. The case of convex bodies is the one given by Bourgain, but the proof is the same in the log-concave case.

Proof. — We have seen in (2.14) that for $\theta \in S^{n-1}$ and t real, one can write

$$m_{lc}(t\theta) = \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,,$$

where φ_θ is obtained by integrating K_{lc} on affine hyperplanes parallel to $\theta^\perp$. It is enough to prove the case $\sigma = 1$. We know that φ_θ is log-concave according to Prékopa–Leindler, it is even, has integral 1 and variance 1 by hypothesis. By Lemma 5.6, one has that $\varphi_\theta(s) \leqslant 2\,\mathrm{e}^{-|s|/2}$ for every $s \in \mathbb{R}$, but the desired estimates do not depend on this exponential decay, which ensures however absolute convergence for the integrals that follow. For every t, by (5.15d) with $j = 0$, $k = 1$ and since φ_θ is even and decreasing on $(0, +\infty)$, we have using Lemma 5.9 that

$$|m_{lc}(t\theta)| = \Bigl| \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s \Bigr| \leqslant \frac{1}{2\pi|t|} \int_{\mathbb{R}} |\varphi_\theta'(s)|\,\mathrm{d}s = \frac{\varphi_\theta(0)}{\pi|t|}\,.$$

The function φ_θ has variance 1 by our normalization assumption, and according to Corollary 5.4 we have the upper bound $\varphi_\theta(0) \leqslant 1/\sqrt{2}$. Writing $\xi = |\xi|\theta$, it follows that $\pi\sqrt{2}\,|\xi|\,|m_{lc}(\xi)| \leqslant 1$ for every ξ in $\mathbb{R}^n$.

Notice that our writing is not correct, because φ_θ might be discontinuous at the ends of its support, so that φ_θ' is a measure in that case, with two Dirac masses at the end points of the support. This happens for example with $\varphi_{\theta,C}$ when C is polyhedral and θ orthogonal to a facet. We leave the easy changes to the reader.

Given $\theta \in S^{n-1}$, the derivative of $t \mapsto m_{lc}(t\theta)$ is expressed by

$$\theta \cdot \nabla m_{lc}(t\theta) = \int_{\mathbb{R}} (-2\mathrm{i}\pi s)\varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,,$$

and

$$|\theta \cdot \nabla m_{lc}(t\theta)| \leqslant 2\pi \int_{\mathbb{R}} |s|\varphi_\theta(s)\,\mathrm{d}s \leqslant 2\pi \Bigl(\int_{\mathbb{R}} s^2 \varphi_\theta(s)\,\mathrm{d}s \Bigr)^{1/2} = 2\pi\,,$$

hence $|1 - m_{lc}(\xi)| = |m_{lc}(0) - m_{lc}(|\xi|\theta)| \leqslant 2\pi|\xi|$. We see also that

$$t\theta \cdot \nabla m_{lc}(t\theta) = \int_{\mathbb{R}} (-2\mathrm{i}\pi t)s\varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s = -\int_{\mathbb{R}} \bigl(s\varphi_\theta(s)\bigr)'\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,.$$

We estimate the two parts coming from $\bigl(s\varphi_\theta(s)\bigr)'$, first

$$\Bigl| \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s \Bigr| \leqslant \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{d}s = 1\,,$$

and as φ_θ is even and non-increasing on $[0, +\infty)$, we have by Lemma 5.9 that

$$\Bigl| \int_{\mathbb{R}} s\varphi_\theta'(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s \Bigr| \leqslant \int_{\mathbb{R}} |s\varphi_\theta'(s)|\,\mathrm{d}s = \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{d}s = 1\,.$$

We conclude that $|t\theta \cdot \nabla m_{lc}(t\theta)| \leqslant 2$ and get $|\xi \cdot \nabla m_{lc}(\xi)| \leqslant 2$ for every ξ. □

Lemma 5.11. — *Let K_{lc} be an even log-concave probability density on $\mathbb{R}^n$, normalized by variance, and m_{lc} its Fourier transform. For every $\theta \in S^{n-1}$ one has*

$$\Bigl|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\, m_{lc}(t\theta)\Bigr| \leqslant \delta_{j,c}\,\frac{1}{1+2\pi|t|}\,, \qquad j \geqslant 0\,,\ t \in \mathbb{R}\,,$$

where $\delta_{j,c}$ is a universal constant, estimated at (5.18).

Proof. — We know that $m_{lc}(t\theta) = \widehat{\varphi_\theta}(t)$. From Lemma 5.8, (5.15d) with $k = 0$, it follows that

$$\Bigl|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\, m_{lc}(t\theta)\Bigr| = \bigl|\widehat{\varphi_\theta}^{(j)}(t)\bigr| \leqslant (2\pi)^j \int_{\mathbb{R}} |s|^j \varphi_\theta(s)\,\mathrm{d}s\,,$$

and with $k = 1$,

$$\Bigl|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\, m_{lc}(t\theta)\Bigr| \leqslant \frac{(2\pi)^{j-1}}{|t|}\Bigl(j\int_{\mathbb{R}} |s|^{j-1}\varphi_\theta(s)\,\mathrm{d}s + \int_{\mathbb{R}} |s|^j\,|\varphi'_\theta(s)|\,\mathrm{d}s\Bigr).$$

The function φ_θ is a symmetric log-concave probability density on $\mathbb{R}$, with variance 1. By Corollary 5.4, we have for $j = 0$ that

$$\bigl(1+2\pi|t|\bigr)\,|m_{lc}(t\theta)| \leqslant \int_{\mathbb{R}} (\varphi_\theta(u) + |\varphi'_\theta(u)|)\,\mathrm{d}u \leqslant 1 + 2\,\varphi_\theta(0) \leqslant 1+\sqrt{2}\,.$$

For $j \geqslant 1$, we have $\int_{\mathbb{R}} |u|^j |\varphi_\theta{}'(u)|\,\mathrm{d}u = j\int_{\mathbb{R}} |u|^{j-1}\varphi_\theta(u)\,\mathrm{d}u$ by Lemma 5.9, and

$$\bigl(1+2\pi|t|\bigr)\Bigl|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\, m_{lc}(t\theta)\Bigr| \leqslant (2\pi)^j \int_{\mathbb{R}} \bigl(|u|^j + 2j|u|^{j-1}\bigr)\varphi_\theta(u)\,\mathrm{d}u\,.$$

The function φ_θ satisfies $\int_{\mathbb{R}} s^2 \varphi_\theta(s)\,\mathrm{d}s = 1$, implying that

$$\delta_{0,c} \leqslant 1+\sqrt{2} < 3\,;\quad \delta_{1,c} \leqslant 6\pi\,;\quad \delta_{2,c} \leqslant 20\pi^2\,. \tag{5.18a}$$

We know by Lemma 5.6 that $\varphi_\theta(s) \leqslant 11\,\mathrm{e}^{-|s|}$. This implies for $j > 2$ that

$$\delta_{j,c} \leqslant 22(2\pi)^j \int_0^{+\infty} (s^j + 2js^{j-1})\,\mathrm{e}^{-s}\ \mathrm{d}s = 66\,(2\pi)^j\,\Gamma(j+1)\,. \tag{5.18b}$$

□

Remarks 5.12. — One gets $\int_{\mathbb{R}} |s|^j \varphi_\theta(s)\,\mathrm{d}s \leqslant 3^{j/2}\Gamma(j+1)$ by applying Lemma 5.3 and Corollary 5.4; Lemma 5.6 yields the bound $22\Gamma(j+1)$, better when j is large.

If the log-concave probability density K on $\mathbb{R}^n$ is normalized by variance but is simply *centered*, then $\varphi_{\theta,K}$ is log-concave and centered for each θ, and satisfies the exponential decay of Corollary 5.7. If $\varphi_{\theta,K}$ reaches its maximum at s_0, then

$$\int_{\mathbb{R}} |s|^j\,|\varphi'_{\theta,K}(s)|\,\mathrm{d}s \leqslant 2|s_0|^j \varphi_{\theta,K}(s_0) + j\int_{\mathbb{R}} |s|^{j-1}\varphi_{\theta,K}(s)\,\mathrm{d}s$$

admits a universal bound κ_j. Lemma 5.11 remains valid in this extended case, with other constants $(\delta_j)_{j\geqslant 0}$ for which we do not have satisfactory explicit expressions. Fradelizi [34, Theorem 5] extended the $L^p(\mathbb{R}^n)$ result of Theorem 6.2 (Bourgain, Carbery) to *centered* bodies C in $\mathbb{R}^n$, not necessarily symmetric (unluckily, the word "centered" was forgotten in the statement given in [34]).

If C is an arbitrary convex body, then M_C is bounded on $L^p(\mathbb{R}^n)$, $p \in (1, +\infty]$, but for each fixed $n \geqslant 1$ and $p < +\infty$, there is no uniform bound for the family of arbitrary convex bodies in $\mathbb{R}^n$ (if $n = 1$, examine $\mathrm{M}_C f$ when $C = [1, 1+\varepsilon]$, $f = \mathbf{1}_C$ and $\varepsilon \to 0$). In a somewhat related direction, it is known that the $L^p(\mathbb{R}^n)$ norm of the uncentered operator in (0.2) is $\geqslant C_p^n$ for some $C_p > 1$, when $1 < p < +\infty$ [40].

Corollary 5.13. — *Let K_{lc} be a symmetric log-concave probability density on $\mathbb{R}^n$, isotropic with variance σ^2, and let m_{lc} be its Fourier transform. For every $\xi \in \mathbb{R}^n$ and $j \geqslant 0$ one has that*

$$\Bigl| \frac{\mathrm{d}^j}{\mathrm{d}t^j} m_{lc}(t\xi) \Bigr| \leqslant \delta_{j,c} \frac{|\sigma\xi|^j}{1 + 2\pi|t\sigma\xi|}\,, \quad t \in \mathbb{R}\,, \tag{5.19}$$

where $\delta_{j,c}$ is the universal constant of Lemma 5.11.

Proof. — The result is obvious when $\xi = 0$, otherwise we apply Lemma 5.11 with $\theta = |\xi|^{-1}\xi$ to the normalized Fourier transform $N(\xi) = m_{lc}(\xi/\sigma)$, obtaining thus

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} m_{lc}(t\xi) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} N(t|\sigma\xi|\theta) = |\sigma\xi|^j \frac{\mathrm{d}^j}{\mathrm{d}u^j} N(u\theta)\big|_{u=t|\sigma\xi|} \leqslant \delta_{j,c} \frac{|\sigma\xi|^j}{1 + 2\pi|t\sigma\xi|}\,.$$

□

5.3. Fourier analysis in $L^2(\mathbb{R}^n)$

Lemma 5.14 (Bourgain [9]). — *Let K be a kernel in $L^1(\mathbb{R}^n)$ and assume that its Fourier transform m is C^1 outside the origin. For every $j \in \mathbb{Z}$, define*

$$\alpha_j(m) = \sup_{2^{j-1}\leqslant|\xi|\leqslant 2^{j+1}} |m(\xi)| \quad \textit{and} \quad \beta_j(m) = \sup_{2^{j-1}\leqslant|\xi|\leqslant 2^{j+1}} |\xi \cdot \nabla m(\xi)|\,.$$

If $\Gamma_B(K) := \sum_{j\in\mathbb{Z}} \sqrt{\alpha_j(m)}\,\sqrt{\alpha_j(m) + \beta_j(m)} < +\infty$, then the maximal operator M_K associated to K is bounded on $L^2(\mathbb{R}^n)$. More precisely, one has that

$$\bigl\| M_K f \bigr\|_{L^2(\mathbb{R}^n)} = \bigl\| \sup_{t>0} |K_{(t)} * f| \bigr\|_{L^2(\mathbb{R}^n)} \leqslant 2\Gamma_B(K)\, \|f\|_{L^2(\mathbb{R}^n)}\,, \quad f \in L^2(\mathbb{R}^n)\,.$$

We shall simply write $\alpha_j = \alpha_j(m)$ and $\beta_j = \beta_j(m)$ in the rest of the section.

Remark. — Clearly, we have that

$$\sum_{j\in\mathbb{Z}} \sqrt{\alpha_j}\,\sqrt{\alpha_j+\beta_j} \leqslant \sum_{j\in\mathbb{Z}} \alpha_j + \sum_{j\in\mathbb{Z}} \sqrt{\alpha_j\,\beta_j}\,,$$

and each of the two terms in the right-hand side is less than the left-hand side. Bourgain employs both expressions as definitions of $\Gamma_B(K)$, one in [9] and the other in [10] or in [13]. The convergence of the series of α_js when j tends to $-\infty$ implies that $m(\xi)$ tends to 0 when ξ tends to 0, thus $m(0)=0$, which means that the integral of K on $\mathbb{R}^n$ is equal to 0. This lemma will not be applied to K_C or K_{lc}, but typically, to the difference of two kernels with equal integrals.

Proof. — We shall give a proof less rough than Bourgain's, relying on the tools introduced in Section 4. We consider a C^∞ function η on $\mathbb{R}$ such that

$$\eta(t)=1 \ \text{ if } \ t\leqslant 1\,, \quad \eta(t)=0 \ \text{ if } \ t\geqslant 2\,, \quad \text{and } \ 0\leqslant \eta\leqslant 1\,.$$

Next, we set $\rho(t)=\eta(t)-\eta(2t)$ for $t\in\mathbb{R}$. We see that ρ vanishes outside $[1/2,2]$. Also, $\rho(t)=1-\eta(2t)$ on $[1/2,1]$ and $\rho(t)=\eta(t)$ on $[1,2]$, so that $0\leqslant\rho(t)\leqslant 1$ and

$$d_0 := \sup_{t\in\mathbb{R}} |t\rho'(t)| = \sup_{t\in\mathbb{R}} |t\eta'(t)| = \sup_{t\in[1,2]} t|\eta'(t)|\,.$$

Let $\varepsilon>0$ be given. One can make sure that $d_0<(1+\varepsilon)/\ln 2$, choosing for η a C^∞ approximation of the function η_0 defined on $[0,2]$ by $\eta_0(t)=\min(1,1-\log_2 t)$, for which $t|\eta_0'(t)|=1/\ln(2)$ when $t\in[1,2]$.

For every $j\in\mathbb{Z}$ and $\xi\in\mathbb{R}^n$, let $\varphi_j(\xi)=\rho(2^{-j}|\xi|)$ and consider the annulus

$$C_j=\{\xi\in\mathbb{R}^n : 2^{j-1}\leqslant|\xi|\leqslant 2^{j+1}\}\subset\mathbb{R}^n.$$

From the properties of ρ, we have that $0\leqslant\varphi_j\leqslant 1$, φ_j vanishes outside C_j, and

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = \sum_{j\in\mathbb{Z}} \bigl(\eta(2^{-j}|\xi|)-\eta(2^{-j+1}|\xi|)\bigr) = 1$$

for every $\xi\neq 0$, because $\eta(2^{-j}|\xi|)=0$ when $j\leqslant\log_2(|\xi|)-1$ and $\eta(2^{-j}|\xi|)=1$ when $j\geqslant\log_2(|\xi|)$. We introduce for every $j\in\mathbb{Z}$ a multiplier m_j defined by

$$m_j(\xi)=\varphi_j(\xi)m(\xi), \quad \xi\in\mathbb{R}^n,$$

and we let $K_j=m_j^\vee=\varphi_j^\vee * K$. One has $\sum_{j\in\mathbb{Z}} K_j=K$, which allows us to write for $f\in\mathcal{S}(\mathbb{R}^n)$ and every $x\in\mathbb{R}^n$ the upper bound

$$(M_K f)(x)=\sup_{t>0}|(K_{(t)}*f)(x)| \leqslant \sup_{t>0}\sum_{j\in\mathbb{Z}}\bigl|[(K_j)_{(t)}*f](x)\bigr| \leqslant \sum_{j\in\mathbb{Z}}(M_{K_j}f)(x)\,.$$

By Lemma 4.7 with $r=4$, one has

$$\bigl\|M_{K_j}f\bigr\|^2_{L^2(\mathbb{R}^n)} \leqslant 2\ln 4\,\|m_j\|_{L^\infty(\mathbb{R}^n)}\,\|m_j^*\|_{L^\infty(\mathbb{R}^n)}\,\|f\|^2_{L^2(\mathbb{R}^n)}\,. \qquad (5.20)$$

We see that $\|m_j\|_\infty \leqslant \alpha_j$, since $|m_j| \leqslant |m|$ and since m_j is supported in the annulus C_j. On the other hand, $m_j^*(\xi) = \xi \cdot \nabla m_j(\xi)$ and we have

$$\nabla m_j(\xi) = \varphi_j(\xi) \nabla m(\xi) + m(\xi) \nabla \varphi_j(\xi) .$$

As φ_j is supported in C_j, we get $|\varphi_j(\xi) \xi \cdot \nabla m(\xi)| \leqslant \beta_j < (1+\varepsilon)\beta_j / \ln 2$, and

$$|m(\xi) \xi \cdot \nabla \varphi_j(\xi)| \leqslant \alpha_j \left| \xi \cdot 2^{-j} \rho'(2^{-j}|\xi|) \frac{\xi}{|\xi|} \right| \leqslant \alpha_j d_0 < (1+\varepsilon)\alpha_j / \ln 2 .$$

It follows that $\|m_j^*\|_\infty \leqslant (1+\varepsilon)(\alpha_j + \beta_j)/\ln 2$. By (5.20) we get

$$\left\| M_{K_j} f \right\|_{L^2(\mathbb{R}^n)} \leqslant 2 \sqrt{1+\varepsilon} \sqrt{\alpha_j} \sqrt{\alpha_j + \beta_j} \, \|f\|_{L^2(\mathbb{R}^n)} .$$

After summation in $j \in \mathbb{Z}$ and letting $\varepsilon \to 0$, we conclude that

$$\left\| M_K f \right\|_{L^2(\mathbb{R}^n)} \leqslant 2 \, \Gamma_B(K) \, \|f\|_{L^2(\mathbb{R}^n)} .$$

We pass from $f \in \mathcal{S}(\mathbb{R}^n)$ to $f \in L^2(\mathbb{R}^n)$ as explained in Section 3.3. □

5.3.1. Conclusion of Bourgain's argument

End of the proof of Theorem 5.1. — We begin with a version of the proof that illustrates well the fact that Lemma 5.14 is a comparison lemma: in vague terms, if we know that the conclusion of Theorem 5.1 is true for *one* family of convex sets, then it is true for all convex sets.

We rely here on Stein's Theorem 4.1 for the Euclidean ball B, asserting that the maximal operator M_B is bounded on $L^p(\mathbb{R}^n)$ for every p in $(1, +\infty]$, with a bound independent of the dimension n. In this paragraph, we only use the L^2 case of this result. Let us call $B = B_{n,V}$ the Euclidean ball in $\mathbb{R}^n$, centered at 0 and normalized by variance, which has radius $\sqrt{n+2}$ by (5.4). Let m_B denote the Fourier transform of K_B. Consider also a symmetric log-concave probability density K_{lc} on $\mathbb{R}^n$, isotropic and normalized by variance. The two functions m_{lc} and m_B satisfy the estimates (5.17.**B**) of Proposition 5.10. We apply Lemma 5.14 to the difference kernel $K = K_{lc} - K_B$. According to (5.17.**B**), for every $\xi \in \mathbb{R}^n$, the Fourier transform $m = m_{lc} - m_B$ satisfies

$$|\xi| \, |m(\xi)| \leqslant \sqrt{2}/\pi \, , \quad |m(\xi)| \leqslant |1 - m_{lc}(\xi)| + |1 - m_B(\xi)| \leqslant 4\pi |\xi| \, , \qquad |\xi \cdot \nabla m(\xi)| \leqslant 4 .$$

We deduce that $\beta_j = \sup_{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}} |\xi \cdot \nabla m(\xi)| \leqslant 4$ for $j \in \mathbb{Z}$. For $j < 0$ one has

$$\alpha_j = \sup_{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}} |m(\xi)| \leqslant 4\pi 2^{j+1} = 4\pi 2^{-|j|+1} \leqslant 32 \, . \, 2^{-|j|} ,$$

and for $j \geqslant 0$, we have $\alpha_j \leqslant \sqrt{2}\pi^{-1}2^{-j+1} \leqslant 2^{-j}$. It follows that the two series $\sum_{j\in\mathbb{Z}} \alpha_j$ and $\sum_{j\in\mathbb{Z}} \sqrt{\alpha_j\beta_j}$ converge, and

$$\sum_{j\in\mathbb{Z}} \alpha_j \leqslant 32 + 2\,, \quad \sum_{j\in\mathbb{Z}} \sqrt{\alpha_j\beta_j} \leqslant 20 + 10\sqrt{2}\,,$$

thus the maximal operator $f \mapsto \sup_{t>0} |K_{(t)} * f|$ is bounded on $L^2(\mathbb{R}^n)$ by a constant independent of the dimension, say, less than $2\Gamma_B(K) < 2(54 + 10\sqrt{2}) < 137$. Finally, for $f \geqslant 0$, we write

$$\begin{aligned} \mathrm{M}_{K_{lc}} f &= \sup_{t>0} |(K_{lc})_{(t)} * f| \\ &\leqslant \sup_{t>0} |(K_B)_{(t)} * f| + \sup_{t>0} |(K_{lc} - K_B)_{(t)} * f| = \mathrm{M}_B f + M_K f\,, \end{aligned}$$

and we can estimate $\mathrm{M}_{K_{lc}}$ by the sum of two operators that are bounded on $L^2(\mathbb{R}^n)$ by constants independent of the dimension n. □

The proof actually given by Bourgain [9] bypasses the L^2 result of Stein on Euclidean balls. The kernel K is now given as $K = K_{lc} - P$, where P is the Poisson kernel $P = P_1$ from (1.32) for the value $t = 1$ of the parameter. We know by (1.31.P^*) that the maximal operator $f \mapsto \sup_{t>0} |P_t f|$ associated to the Poisson kernel acts boundedly on $L^p(\mathbb{R}^n)$, $1 < p \leqslant +\infty$, with a bound $\leqslant 2$ when $p = 2$, thus independent of the dimension n. Now, everything is said: we replace the multiplier m_B by $\widehat{P}$ and it suffices to see that $\widehat{P}$ also satisfies good estimates similar to (5.17.**B**). But $\widehat{P}(\xi) = \mathrm{e}^{-2\pi|\xi|}$ clearly satisfies the even better estimates

$$|\xi|\,|\widehat{P}(\xi)| = |\xi|\,\mathrm{e}^{-2\pi|\xi|} \leqslant (2\pi\,\mathrm{e})^{-1}\,, \tag{5.21a}$$

$$|1 - \widehat{P}(\xi)| \leqslant 2\pi|\xi|\,, \quad |\xi\cdot\nabla\widehat{P}(\xi)| = 2\pi|\xi|\,\mathrm{e}^{-2\pi|\xi|} \leqslant \mathrm{e}^{-1}\,, \tag{5.21b}$$

where we made use of the inequality $x\,\mathrm{e}^{-x} \leqslant \mathrm{e}^{-1}$, true for every $x \geqslant 0$. This ends the second proof of Theorem 5.1, with different constants whose exact values are rather irrelevant. However, we found here an explicit bound $\kappa_2 < 2 + 137 < 140$, explicit but definitely not sharp.

6. The L^p results of Bourgain and Carbery

One gives again a symmetric convex body C in $\mathbb{R}^n$, and μ_C denotes the uniform probability measure on C. Beside the maximal function $\mathrm{M}_C f$ from (0.3.**M**), for every function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ we set

$$(\mathrm{M}_C^{(d)} f)(x) = \sup_{j\in\mathbb{Z}} \frac{1}{|2^j C|} \int_{x+2^j C} |f(y)|\,\mathrm{d}y = \sup_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |f(x + 2^j v)|\,\mathrm{d}\mu_C(v)\,.$$

One can call $\mathrm{M}_C^{(d)} f$ the *"dyadic" maximal function associated to the convex set* C. Obviously, $\mathrm{M}_C^{(d)} \leqslant \mathrm{M}_C$. More generally, we associate to every kernel K integrable on $\mathbb{R}^n$ the dyadic maximal function

$$(M_K^{(d)} f)(x) = \sup_{j \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} f(x + 2^j v) \, K(v) \, \mathrm{d}v \right| , \quad x \in \mathbb{R}^n \, .$$

In 1986, Bourgain and Carbery have obtained identical results for $L^p(\mathbb{R}^n)$. Somewhat surprisingly, the cases $\mathrm{M}_C^{(d)}$ and M_C are different, the boundedness of M_C on $L^p(\mathbb{R}^n)$ being obtained only when $p > 3/2$, as opposed to $p > 1$ for $\mathrm{M}_C^{(d)}$.

Theorem 6.1 (Bourgain [10], Carbery [21]). — *For every* p *in* $(1, +\infty]$, *there exists a constant* $\kappa^{(d)}(p)$ *such that for every integer* $n \geqslant 1$ *and every symmetric convex body* $C \subset \mathbb{R}^n$, *one has*

$$\forall \, f \in L^p(\mathbb{R}^n) \, , \ \ \|\mathrm{M}_C^{(d)} f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa^{(d)}(p) \, \|f\|_{L^p(\mathbb{R}^n)} \, .$$

Theorem 6.2 (Bourgain [10], Carbery [21]). — *For every* p *in* $(3/2, +\infty]$, *there exists a constant* $\kappa(p)$ *such that for every integer* $n \geqslant 1$ *and for every symmetric convex set* $C \subset \mathbb{R}^n$, *one has that*

$$\forall \, f \in L^p(\mathbb{R}^n), \ \ \|\mathrm{M}_C f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa(p) \, \|f\|_{L^p(\mathbb{R}^n)} \, .$$

We recalled in the Introduction that the maximal theorem of strong type is not true for $p = 1$, even with a constant depending on n, and even for the smaller function $\mathrm{M}_C^{(d)} f$, since $\mathrm{M}_C f \leqslant 2^n \mathrm{M}_C^{(d)} f$. Note that Theorems 6.1 and 6.2 are obvious for $L^\infty(\mathbb{R}^n)$, with $\kappa^{(d)}(\infty) = \kappa(\infty) = 1$. By Bourgain [9], we have the result in $L^2(\mathbb{R}^n)$, so we obtain it for $p \in [2, +\infty]$ by interpolation. Consequently, our work will be limited to values of p in the interval $(1, 2]$. We shall follow Carbery's approach to both theorems. This approach has been applied later in the Detlef Müller article [59] (see Section 7), on which relies Bourgain's recent article [13] devoted to the maximal function associated to high dimensional cubes (see Section 8).

The proof will use the inequalities (5.17.**B**) and (5.19), which are also true for log-concave densities, and by simply following the proofs of Bourgain or Carbery, we can extend the results to the log-concave setting. As suggested in [10], one can actually take one more step, forget convexity and exploit only the inequalities on the Fourier transform given by Lemma 5.11. In this more general framework, we consider a probability density K_g on $\mathbb{R}^n$, or merely a kernel K_g integrable on $\mathbb{R}^n$ and having a Fourier transform m_g which satisfies the following: there exist $\delta_{0,g}, \delta_{1,g} > 0$ such that for every

$\theta \in S^{n-1}$, we have

$$\Bigl| m_g(t\theta) \Bigr| \leqslant \frac{\delta_{0,g}}{1+|t|} \,, \quad \Bigl| \frac{\mathrm{d}}{\mathrm{d}t} m_g(t\theta) \Bigr| = \bigl| \theta \cdot \nabla m_g(t\theta) \bigr| \leqslant \frac{\delta_{1,g}}{1+|t|} \,, \quad t \in \mathbb{R} \,. \tag{6.1.$\mathbf{H}$}$$

The form of the $\delta_{0,g}$-bound of m_g has been chosen for the sake of uniformity, but when K_g is a probability density, we know of course that $\|m_g\|_{L^\infty(\mathbb{R}^n)} = 1$ and in particular we have $\delta_{0,g} \geqslant 1$ in that case.

Proposition 6.3. — *Theorems 6.1 and 6.2 are also valid for any symmetric log-concave probability density K_{lc} on $\mathbb{R}^n$, namely*

$$\begin{aligned} &\|\mathrm{M}^{(d)}_{K_{lc}} f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa^{(d)}(p) \, \|f\|_{L^p(\mathbb{R}^n)} \,, && 1 < p \leqslant +\infty \,, \\ &\|\mathrm{M}_{K_{lc}} f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa(p) \, \|f\|_{L^p(\mathbb{R}^n)} \,, && 3/2 < p \leqslant +\infty \,. \end{aligned}$$

If a probability density K_g satisfies (6.1.**H**), *then for $3/2 < p \leqslant 2$ we have*

$$\|\mathrm{M}_{K_g}\|_{p \to p} \leqslant \kappa_p \, (\delta_{0,g} + \delta_{1,g})^{2-2/p} \,,$$

and this result extends to every $p \in (1, 2]$ in the case of the dyadic operator $\mathrm{M}^{(d)}_{K_g}$.

All these results are obvious when $p = +\infty$, and easy when $p > 2$ by interpolation (L^2, L^∞) after the case $p = 2$ is obtained. When $p \leqslant 2$, the log-concave statements follow from the "general" one. Indeed, for the study of maximal functions, we may assume that the convex set C or the symmetric log-concave probability density K_{lc} is isotropic and normalized by variance. Then, by (5.17.**B**) or by Lemma 5.11, m_C or m_{lc} satisfy (6.1.**H**) with universal constants $\delta_{0,c}$ and $\delta_{1,c}$.

6.1. *A priori* estimate and interpolation

Suppose that a family $(T_j)_{j \in \mathbb{Z}}$ of operators on $L^p(X, \Sigma, \mu)$ is given, for a set of values of p and on a certain measure space (X, Σ, μ) (further down, it will be $\mathbb{R}^n$, equipped with the Lebesgue measure). These operators can be linear operators, or nonlinear operators of the form

$$T_j f = \sup_{v \in V} |T_{j,v} f| \,,$$

where each $T_{j,v}$ is linear and where v runs over a certain set V of indices. We want to study the maximal function

$$T^* f = \sup_{j \in \mathbb{Z}} |T_j f| = \sup_{j \in \mathbb{Z}, v \in V} |T_{j,v} f| \,.$$

We also consider later a kernel K integrable on $\mathbb{R}^n$. In the application to the geometrical problem, this kernel will be (as in Section 5.3.1) the difference $K = K_1 - K_2$ of two kernels, where K_1 is the uniform probability density on an isotropic convex set C or a probability density K_g satisfying (6.1.**H**), and K_2 is a kernel for which the dimensionless maximal inequality is already known. We have to deal with two cases. In the first one, T_j will be the convolution with the dilate $K_{(2^j)}$ from (2.7) of K, and the maximal function $T^* f = M_K^{(d)} f$ will then permit us to relate the dyadic maximal function $\mathrm{M}_C^{(d)} f$ to a maximal function whose bounded character on $L^p(\mathbb{R}^n)$ is already known. In the second one, the operator $T_{j,v}$ will be the convolution with $K_{(v2^j)}$ with $v \in [1, 2] = V$, in which case

$$T_j f = \sup_{2^j \leqslant t \leqslant 2^{j+1}} |K_{(t)} * f| \,, \tag{6.2}$$

and $T^* f = M_K f$ allows us to study the "global" maximal function $\mathrm{M}_C f$ or $\mathrm{M}_{K_g} f$.

We assume that linear operators $(Q_j)_{j\in\mathbb{Z}}$ such that $\sum_{j\in\mathbb{Z}} Q_j = \mathrm{Id}$ are given. In the applications to come, these operators will be those of Equation (2.6), in the Section 2.1 on Littlewood–Paley functions.

Definition 6.4 (Carbery [21]). — *Given families $(T_j)_{j\in\mathbb{Z}}$ and $(Q_j)_{j\in\mathbb{Z}}$ as above, we say that T^* is* weakly bounded *on $L^p(X, \Sigma, \mu)$ if there exists a constant A such that*

$$\forall\, f \in L^p(X, \Sigma, \mu)\,, \quad \forall\, k \in \mathbb{Z}\,, \quad \Big\| \sup_{j\in\mathbb{Z}} |T_j Q_{j+k} f| \Big\|_{L^p(\mu)} \leqslant A \|f\|_{L^p(\mu)} \,. \quad (\mathbf{W}_p)$$

We say that T^* is strongly bounded *on $L^p(X, \Sigma, \mu)$ if there exists a real nonnegative sequence $(a_k)_{k\in\mathbb{Z}}$, satisfying $\sum_{k\in\mathbb{Z}} a_k^r < +\infty$ for every $r > 0$, and such that*

$$\forall\, f \in L^p(X, \Sigma, \mu)\,, \quad \forall\, k \in \mathbb{Z}\,, \quad \Big\| \sup_{j\in\mathbb{Z}} |T_j Q_{j+k} f| \Big\|_{L^p(\mu)} \leqslant a_k \|f\|_{L^p(\mu)} \,. \quad (\mathbf{S}_p)$$

By $T_j Q_{j+k} f$, we mean of course $T_j(Q_{j+k} f)$.

Remarks 6.5. — In this generality, the supremum for $v \in V$ in $T_j f = \sup_{v\in V} |T_{j,v} f|$ must be understood as essential supremum, as explained in Section 3.3. In our cases of application, the function $v \mapsto T_{j,v}(x)$, $x \in X$, will be a continuous function on an interval V of the line, in which case the pointwise supremum coincides with the supremum on any countable dense subset of V.

It is evident that $(\mathbf{S}_p)$ implies $(\mathbf{W}_p)$, and $(\mathbf{S}_p)$ implies that T^* is bounded, because

$$|T_{j,v} f| = \Bigl| \sum_{k \in \mathbb{Z}} T_{j,v} Q_{j+k} f \Bigr| \leqslant \sum_{k \in \mathbb{Z}} |T_{j,v} Q_{j+k} f| \leqslant \sum_{k \in \mathbb{Z}} |T_j Q_{j+k} f| \, ,$$

thus

$$|T_j f| = \sup_{v \in V} |T_{j,v} f| \leqslant \sum_{k \in \mathbb{Z}} |T_j Q_{j+k} f|, \quad \text{then} \quad T^* f \leqslant \sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f|$$

and

$$\|T^* f\|_{L^p(\mu)} \leqslant \sum_{k \in \mathbb{Z}} \Bigl\| \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f| \Bigr\|_{L^p(\mu)} \leqslant \Bigl(\sum_{k \in \mathbb{Z}} a_k \Bigr) \|f\|_{L^p(\mu)} \, . \tag{6.3}$$

If one has $(\mathbf{W}_{p_0})$ and $(\mathbf{S}_{p_1})$ and if $1/p = (1-\theta)/p_0 + \theta/p_1$, with $0 < \theta \leqslant 1$, then as in (3.26) we obtain by interpolation

$$\forall\, f \in L^p(\mu) \, , \; \forall\, k \in \mathbb{Z} \, , \quad \Bigl\| \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f| \Bigr\|_{L^p(\mu)} \leqslant A^{1-\theta} a_k^\theta \, \|f\|_{L^p(\mu)} \, ,$$

and $\sum_{k \in \mathbb{Z}} A^{(1-\theta) r} a_k^{\theta r} < +\infty$ for every $r > 0$, so $(\mathbf{S}_p)$ is satisfied. In order to obtain this, we apply the complex interpolation of linear operators between spaces $L^p(\ell^q)$ [7, Chap. 5, Th. 5.1.2]. Here, the range space is of the form $L^p(\mu, \ell^\infty(\mathbb{Z}))$, a case covered by complex interpolation. Indeed, in the simpler case where the T_js are linear, we obtain the result by considering for each $k \in \mathbb{Z}$ the linear operator

$$f \mapsto (T_j Q_{j+k} f)_{j \in \mathbb{Z}} \in L^p(X, \Sigma, \mu, \ell^\infty(\mathbb{Z})) \, , \quad f \in L^p(\mu) \, .$$

If V has more than one element, the range space will be $L^p(\mu, \ell^\infty(\mathbb{Z} \times V))$. The nonlinear operator $f \mapsto \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f|$ belongs to the class of *linearizable operators* considered in [36].

We now describe the assumptions that will be made in the main result of this section. First of all, we assume that there exist constants C_p, $1 < p \leqslant 2$, such that

$$\forall\, p \in (1, 2], \; \forall\, f \in L^p(\mu), \quad \Bigl\| \Bigl(\sum_{j \in \mathbb{Z}} |Q_j f|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\mu)} \leqslant C_p \|f\|_{L^p(\mu)} \, . \tag{$A_\mathbf{0}$}$$

If the $(Q_j)_{j \in \mathbb{Z}}$ are those of (2.4), then we can take $C_p = \mathrm{q}_p$ which behaves as $1/(p-1)$ when $p \to 1$, according to (2.5).

We assume that $T_{j,v} = U_{j,v} - S_{j,v}$, where $U_{j,v}$ and $S_{j,v}$ are positive linear operators, and we assume for S^*, defined by $S^* f = \sup_{j \in \mathbb{Z},\, v \in V} |S_{j,v} f|$, that there exist $p_{\min}$ in the open interval $(1, 2)$ and constants C'_p, $p_{\min} < p \leqslant 2$, such that

$$\forall\, p \in (p_{\min}, 2] \, , \quad \|S^*\|_p \leqslant C'_p \, , \tag{$A_\mathbf{1}$}$$

where $\|R\|_p$ is a shorter notation for the norm $\|R\|_{p\to p}$ of an operator R. The condition "$U_{j,v}$ *positive*" will be the only reason for requiring that the kernel K_g in Proposition 6.3 be a probability density rather than an arbitrary integrable kernel. The $U_{j,v}$s will correspond to the kernel K_g under study, while the $S_{j,v}$s will often refer to Poisson kernels for which the maximal function estimates in $L^p(\mathbb{R}^n)$ are already known by (1.31.P^*).

We assume that for every $p \in (p_{\min}, 2]$, there exists a constant C''_p such that

$$\forall\, j \in \mathbb{Z}\,, \quad \|T_j\|_p \leqslant C''_p\,. \tag{$A_{\mathbf{2}}$}$$

We shall assume that T^* satisfies $(\mathbf{S}_2)$, hence we have that

$$\forall\, f \in L^2(\mu)\,,\ \forall\, k \in \mathbb{Z}\,,\ \Bigl\| \sup_{j\in\mathbb{Z}} |T_j Q_{j+k} f| \Bigr\|_{L^2(\mu)} \leqslant a_k \|f\|_{L^2(\mu)}\,, \tag{$A_{\mathbf{3}}$}$$

where $\sum_{k\in\mathbb{Z}} a_k^r < +\infty$ for every $r > 0$.

Proposition 6.6 (Carbery [21]). — *Under the assumptions* $(A_{\mathbf{0}})$, $(A_{\mathbf{1}})$, $(A_{\mathbf{2}})$ *and* $(A_{\mathbf{3}})$, *the maximal operator* T^* *is bounded on* $L^p(X, \Sigma, \mu)$ *for every real number* p *such that* $p_{\min} < p \leqslant 2$. *For every* p_0 *such that* $p_{\min} < p_0 < p \leqslant 2$, *we have*

$$\|T^*\|_p \leqslant (C_{r_0})^{2\gamma/p_0} (C''_{p_0})^{\gamma} \Bigl(\sum_{k\in\mathbb{Z}} a_k^{(1-\gamma)p/2} \Bigr)^{2/p} + 2\,C'_p\,, \tag{6.4}$$

with $r_0 = 2p/(p+2-p_0) \in (p_0, p)$ *and* $\gamma = [1/p - 1/2]/[1/p_0 - 1/2]$.

Our main interest in applications will be the maximal operator U^*, which is also bounded on $L^p(X, \Sigma, \mu)$ since S^* is bounded on $L^p(X, \Sigma, \mu)$ according to $(A_{\mathbf{1}})$.

Proof. — Under the assumption $(A_{\mathbf{3}})$, one already knows by (6.3) that T^* is bounded on $L^2(X, \Sigma, \mu)$. We fix $p_1 = p$ such that $p_{\min} < p_1 < 2$ and we try to prove that T^* is bounded on $L^{p_1}(X, \Sigma, \mu)$. For doing this, it is enough to show that for every *finite* subfamily $(T_j)_{j\in J}$ of $(T_j)_{j\in\mathbb{Z}}$, the corresponding maximal operator

$$f \mapsto \max_{j\in J} |T_j f|$$

is L^{p_1}-bounded by a constant independent of the chosen finite subset $J \subset \mathbb{Z}$.

We thus consider a family (T_j) that has only a finite number of nonzero terms, implying that $\|T^*\|_{p_1} < +\infty$ by Property $(A_{\mathbf{2}})$. We choose p_0 arbitrary such that $p_{\min} < p_0 < p_1$, and we introduce r_0 such that $p_{\min} < p_0 < r_0 < p_1 < r_1 := 2$, defined in this way: if $\theta \in (0,1)$ is such that

$$\frac{1}{2} = \frac{1-\theta}{p_0} + \frac{\theta}{\infty}\,, \tag{6.5a}$$

that is to say, if $\theta = 1 - p_0/2$, then we set

$$\frac{1}{r_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \Bigl(= \frac{1}{2} + \frac{1}{p_1} - \frac{p_0}{2p_1}\,, \quad r_0 = \frac{2p_1}{p_1 + 2 - p_0} \Bigr). \tag{6.5b}$$

Here is the plan: by a first interpolation between p_0 and p_1, we will show that T^* satisfies $(\mathbf{W}_{r_0})$ with a constant bounded by a function of $\|T^*\|_{p_1}$. Next, we will interpolate between $(\mathbf{W}_{r_0})$ and $(\mathbf{S}_{r_1}) = (\mathbf{S}_2)$ and obtain $(\mathbf{S}_{p_1})$, giving a new bound for the norm $\|T^*\|_{p_1}$, whose particular form

$$\|T^*\|_{p_1} \leqslant A(\|T^*\|_{p_1} + B)^{\beta}\,, \quad \text{for some} \;\; \beta \in (0,1)\,,$$

implies that $\|T^*\|_{p_1}$ is bounded by a constant independent of the chosen finite subfamily. This will complete the proof.

For $1 \leqslant r, s \leqslant +\infty$, let $\kappa(r,s)$ be the smallest constant such that

$$\Bigl\| \Bigl(\sum_{j \in \mathbb{Z}} |T_j g_j|^s \Bigr)^{1/s} \Bigr\|_{L^r} \leqslant \kappa(r,s) \Bigl\| \Bigl(\sum_{j \in \mathbb{Z}} |g_j|^s \Bigr)^{1/s} \Bigr\|_{L^r}$$

for every sequence $(g_j)_{j \in \mathbb{Z}}$ in $L^r(X, \Sigma, \mu)$.

— One sees that $\kappa(p_0, p_0) \leqslant C''_{p_0}$, by $(A_{\mathbf{2}})$ and the simple sum-integral inversion

$$\begin{aligned} \Bigl\| \Bigl(\sum_{j \in \mathbb{Z}} |T_j g_j|^{p_0} \Bigr)^{1/p_0} \Bigr\|_{L^{p_0}}^{p_0} &= \sum_{j \in \mathbb{Z}} \|T_j g_j\|_{L^{p_0}}^{p_0} \\ &\leqslant (C''_{p_0})^{p_0} \Bigl\| \Bigl(\sum_{j \in \mathbb{Z}} |g_j|^{p_0} \Bigr)^{1/p_0} \Bigr\|_{L^{p_0}}^{p_0}. \end{aligned}$$

— One has also $\kappa(p_1, +\infty) \leqslant \|T^*\|_{p_1} + 2C'_{p_1}$. Indeed, when $(W_j)_{j \in \mathbb{Z}}$ is a family of positive operators and $g = \sup_{j \in \mathbb{Z}} |g_j|$, one has

$$|W_j g_j| \leqslant W_j |g_j| \leqslant W_j g\,, \quad \sup_{j \in \mathbb{Z}} |W_j g_j| \leqslant \sup_{j \in \mathbb{Z}} W_j g\,.$$

Because $S_{j,v}$ is positive, we have $\sup_{j \in \mathbb{Z}} |S_{j,v} g_j| \leqslant \sup_{j \in \mathbb{Z}} S_{j,v} g$ for every $v \in V$, and letting $S_j g_j = \sup_{v \in V} |S_{j,v} g_j|$ we see according to $(A_{\mathbf{1}})$ that

$$\sup_{j \in \mathbb{Z}} S_j g_j \leqslant S^* g\,, \quad \bigl\| \sup_{j \in \mathbb{Z}} S_j g_j \bigr\|_{L^{p_1}} \leqslant \|S^* g\|_{L^{p_1}} \leqslant C'_{p_1} \bigl\| \sup_{j \in \mathbb{Z}} |g_j| \bigr\|_{L^{p_1}}\,.$$

Since $U_{j,v} = T_{j,v} + S_{j,v}$ is positive, we obtain also for $U_j f = \sup_{v \in V} |U_{j,v} f|$ that

$$\begin{aligned} \bigl\| \sup_{j \in \mathbb{Z}} |U_j g_j| \bigr\|_{L^{p_1}} \leqslant \|U^* g\|_{L^{p_1}} &\leqslant \|T^* g\|_{L^{p_1}} + \|S^* g\|_{L^{p_1}} \\ &\leqslant (\|T^*\|_{p_1} + C'_{p_1})\, \|g\|_{L^{p_1}}\,, \end{aligned}$$

and finally $\bigl\| \sup_{j \in \mathbb{Z}} |T_j g_j| \bigr\|_{L^{p_1}} \leqslant (\|T^*\|_{p_1} + 2C'_{p_1}) \bigl\| \sup_{j \in \mathbb{Z}} |g_j| \bigr\|_{L^{p_1}}$, which proves the inequality $\kappa(p_1, +\infty) \leqslant \|T^*\|_{p_1} + 2C'_{p_1}$.

We apply complex interpolation between spaces $L^p(\ell^q)$ [7, Chap. 5, Th. 5.1.2], namely between the spaces $L^{p_0}(\ell^{p_0})$ and $L^{p_1}(\ell^\infty)$, which gives the space $L^{r_0}(\ell^2)$ for the chosen value θ of the interpolation parameter, by (6.5a) and (6.5b). We already explained that the case where T_j is not linear can also be covered by complex interpolation. It follows from (3.26) that

$$\kappa(r_0, 2) \leqslant \kappa(p_0, p_0)^{1-\theta} \kappa(p_1, +\infty)^\theta \leqslant (C''_{p_0})^{1-\theta} (\|T^*\|_{p_1} + 2C'_{p_1})^\theta .$$

Remark (in passing). — It is exactly in this manner that Stein [73, Chap. VI, Th. 8, p. 103] shows the inequality (6.6) on the square function $(\sum_n |E_n f_n|^2)^{1/2}$ of a sequence (E_n) of conditional expectations with respect to an increasing sequence of σ-fields, stating that

$$\Big\| \Big(\sum_n |E_n f_n|^2 \Big)^{1/2} \Big\|_q \leqslant \kappa_q \Big\| \Big(\sum_n |f_n|^2 \Big)^{1/2} \Big\|_q , \quad 1 < q < +\infty . \tag{6.6}$$

When $1 < q < 2$, the proof applies inversion for a pair (q_0, q_0), and Doob's maximal theorem for a pair $(q_1, +\infty)$ with $q_0 < q < q_1$ and $q(q_1 - q_0) = 2(q_1 - q)$.

Thus, with $g_j = Q_{j+k} f$ for a fixed $k \in \mathbb{Z}$, one has

$$\begin{aligned} \Big\| \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f| \Big\|_{L^{r_0}} &\leqslant \Big\| \Big(\sum_{j \in \mathbb{Z}} |T_j Q_{j+k} f|^2 \Big)^{1/2} \Big\|_{L^{r_0}} \\ &\leqslant \kappa(r_0, 2) \Big\| \Big(\sum_{j \in \mathbb{Z}} |Q_{j+k} f|^2 \Big)^{1/2} \Big\|_{L^{r_0}} \\ &= \kappa(r_0, 2) \Big\| \Big(\sum_{j \in \mathbb{Z}} |Q_j f|^2 \Big)^{1/2} \Big\|_{L^{r_0}} \leqslant C_{r_0} \, \kappa(r_0, 2) \, \|f\|_{L^{r_0}} . \end{aligned}$$

We have proved the property $(\mathbf{W}_{r_0})$, since we got that

$$\forall\, f \in L^{r_0}, \ \forall\, k \in \mathbb{Z}, \quad \Big\| \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f| \Big\|_{L^{r_0}} \leqslant C_{r_0} \, \kappa(r_0, 2) \, \|f\|_{L^{r_0}} .$$

If for a certain $\rho \in (0, 1)$, we write

$$\frac{1}{p_1} = \frac{1-\rho}{r_0} + \frac{\rho}{r_1} = \frac{1-\rho}{r_0} + \frac{\rho}{2} \qquad \Big(\rho = \frac{p_1 - p_0}{2 - p_0} \Big) ,$$

we get $(\mathbf{S}_{p_1})$ by interpolating between $(\mathbf{W}_{r_0})$ and $(\mathbf{S}_2) = (\mathbf{S}_{r_1})$, obtaining thus

$$\forall\, k \in \mathbb{Z} , \quad \Big\| \sup_{j \in \mathbb{Z}} |T_j Q_{j+k} f| \Big\|_{L^{p_1}} \leqslant (C_{r_0} \, \kappa(r_0, 2))^{1-\rho} a_k^\rho \, \|f\|_{L^{p_1}} .$$

By (6.3), it follows that

$$\Big\| \sup_{j \in \mathbb{Z}} |T_j f| \Big\|_{L^{p_1}} \leqslant (C_{r_0} \, \kappa(r_0, 2))^{1-\rho} \Big(\sum_{k \in \mathbb{Z}} a_k^\rho \Big) \|f\|_{L^{p_1}} .$$

One has finally an implicit inequality about $\|T^*\|_{p_1}$, namely

$$\begin{aligned}\|T^*\|_{p_1} &\leqslant \bigl[C_{r_0}\,\kappa(r_0,2)\bigr]^{1-\rho}\Bigl(\sum_{k\in\mathbb{Z}} a_k^\rho\Bigr)\\ &\leqslant \bigl[C_{r_0}\,(C''_{p_0})^{1-\theta}(\|T^*\|_{p_1}+2C'_{p_1})^\theta\bigr]^{1-\rho}\Bigl(\sum_{k\in\mathbb{Z}} a_k^\rho\Bigr)\\ &= \bigl(C_{r_0}\,(C''_{p_0})^{1-\theta}\bigr)^{1-\rho}\Bigl(\sum_{k\in\mathbb{Z}} a_k^\rho\Bigr)(\|T^*\|_{p_1}+2C'_{p_1})^{\theta(1-\rho)}\,,\end{aligned}$$

implying that $\|T^*\|_{p_1}$ is bounded by a constant depending only upon C_{r_0}, C'_{p_1}, C''_{p_0} and the a_ks. Indeed, suppose that $C \geqslant 0$ satisfies $C \leqslant A(C+B)^\beta$, where $A, B > 0$ and $0 < \beta < 1$. We write

$$C \leqslant \bigl(A^{1/(1-\beta)}\bigr)^{1-\beta}(C+B)^\beta \leqslant (1-\beta)A^{1/(1-\beta)} + \beta(C+B)\,,$$

yielding

$$C \leqslant A^{1/(1-\beta)} + \frac{\beta}{1-\beta}\,B\,.$$

This bound is essentially correct when B is small, and we shall use it below with $A = \bigl(C_{r_0}\,(C''_{p_0})^{1-\theta}\bigr)^{1-\rho}\Bigl(\sum_{k\in\mathbb{Z}} a_k^\rho\Bigr)$, $B = 2C'_{p_1}$ and $\beta = \theta(1-\rho)$.

However, when $B \geqslant A^{1/(1-\beta)}$, a better bound $(1-\beta)^{-1}AB^\beta$ is available. In this case, $A \leqslant B^{1-\beta}$, thus $C \leqslant B^{1-\beta}(C+B)^\beta \leqslant B + \beta C$, hence

$$C \leqslant A\Bigl(\frac{B}{1-\beta}+B\Bigr)^\beta = \Bigl(\frac{2-\beta}{1-\beta}\Bigr)^\beta AB^\beta \leqslant \frac{1}{1-\beta}\,AB^\beta\,,$$

because $(2-\beta)^\beta(1-\beta)^{1-\beta} \leqslant \beta(2-\beta)+(1-\beta)^2 = 1$.

Recall that $\rho = (p_1-p_0)/(2-p_0)$, so $\beta = \theta(1-\rho) = 1-p_1/2 < 1$. We find an explicit bound for $\|T^*\|_{p_1}$, independent of the finite subfamily $(T_j)_{j\in J}$ of (T_j)s that was chosen at the beginning, of the form

$$\begin{aligned}\|T^*\|_{p_1} &\leqslant \bigl(C_{r_0}\,(C''_{p_0})^{1-\theta}\bigr)^{2(1-\rho)/p_1}\Bigl(\sum_{k\in\mathbb{Z}} a_k^\rho\Bigr)^{2/p_1} + \frac{2-p_1}{p_1}\,2\,C'_{p_1}\\ &\leqslant (C_{r_0})^{2\gamma/p_0}\,(C''_{p_0})^\gamma\Bigl(\sum_{k\in\mathbb{Z}} a_k^\rho\Bigr)^{2/p_1} + 2\,C'_{p_1}\,,\end{aligned}$$

with $\gamma = [1/p_1-1/2]/[1/p_0-1/2]$. Observe that $\rho = [p_1/(2p_0)-1/2]/[1/p_0-1/2] = (1-\gamma)p_1/2$. We get in particular a bound of C''_{p_1} by a power $\gamma < 1$ of C''_{p_0}. There is no miracle: this power γ is the one corresponding to interpolation between C''_{p_0} and the value C''_2 hidden in the assumption $(A_\mathbf{3})$. □

6.2. Fractional derivatives

If a function h is given in the Schwartz space $\mathcal{S}(\mathbb{R})$, one can express it as Fourier transform of another function $k \in \mathcal{S}(\mathbb{R})$ and write

$$\forall\, t \in \mathbb{R}\,, \quad h(t) = \int_{\mathbb{R}} k(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s\,.$$

One has then an expression for the derivatives of h by means of (unbounded) multipliers. For every integer $j \geqslant 1$ and every $t \in \mathbb{R}$, one sees that

$$(-1)^j h^{(j)}(t) = \int_{\mathbb{R}} (2\mathrm{i}\pi s)^j\, k(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s\,.$$

It is tempting to extend the notion of derivative, from the integer case $j \in \mathbb{N}$ to every complex value z such that $\operatorname{Re} z > -1$, by setting

$$\forall\, t \in \mathbb{R}\,, \quad (D^z h)(t) = \int_{\mathbb{R}} (2\mathrm{i}\pi s)^z k(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s\,. \tag{6.7}$$

Note that $D^1 h = -h'$ with this definition. We define complex powers by

$$(2\mathrm{i}\pi s)^z = \mathrm{e}^{z \ln(2\mathrm{i}\pi s)} = \mathrm{e}^{z(\ln(2\pi|s|) + \mathrm{i}\operatorname{Arg}(2\mathrm{i}\pi s))} = |2\pi s|^z\, \mathrm{e}^{\mathrm{i}\pi z \operatorname{sign}(s)/2}\,,$$

and we have that $(\lambda \mathrm{i} s)^z = \lambda^z (\mathrm{i}s)^z$ when $\lambda > 0$. If we dilate the function h to $h_{[\lambda]}$, with $\lambda > 0$ as in (2.7), we know that $h_{[\lambda]} = \mathcal{F}(k_{(\lambda)})$, therefore

$$\begin{aligned}(D^z h_{[\lambda]})(t) &= \int_{\mathbb{R}} (2\mathrm{i}\pi s)^z \lambda^{-1} k(\lambda^{-1} s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s \\ &= \lambda^z \int_{\mathbb{R}} (2\mathrm{i}\pi u)^z k(u)\, \mathrm{e}^{-2\mathrm{i}\pi u\lambda t}\, \mathrm{d}u\,.\end{aligned}$$

This means that

$$D^z(h_{[\lambda]}) = \lambda^z \big(D^z h\big)_{[\lambda]}\,, \quad \text{or} \quad D_t^z h(\lambda t) = \lambda^z (D^z h)(\lambda t)\,, \tag{6.8}$$

where we use the notation $D_t^z h(\lambda t)$ when the function of t does not have an explicit name, as in $t \mapsto h(\lambda t)$. For a specific value, we shall write for example $D_t^z h(\lambda t)\big|_{t=1}$.

If we would like to extend D^z to $h = \mathbf{1}$, we might consider the function $\mathbf{1}$ as the limit of $h_{[\lambda]}$ when $h(0) = 1$ and $\lambda \searrow 0$. Then (6.8) suggests that $D^z \mathbf{1} = 0$ when $\operatorname{Re} z > 0$, and that $D^z \mathbf{1}$ is undefined if $\operatorname{Re} z < 0$.

When z is not a nonnegative integer, the operator D^z is not local. We will see later however that $(D^z h)(t_0)$ depends only on the values of h on $[t_0, +\infty)$. This could be checked right now by arguments involving holomorphic functions.

When $-1 < \operatorname{Re} z < 0$, the differentiation D^z is in fact a *fractional integration.* We shall see below that $(D^z h)(t) = (I^{-z} h)(t)$, where I^w is given for $\operatorname{Re} w > 0$ by

$$(I^w h)(t) = \frac{1}{\Gamma(w)} \int_t^{+\infty} (u-t)^{w-1} h(u)\,\mathrm{d}u\,. \tag{6.9}$$

The next lemma provides the tool that relates the definitions (6.7) and (6.9).

Lemma 6.7. — *Let ζ be a complex number such that $\operatorname{Re}\zeta < 0$ and let $\varepsilon > 0$. The inverse Fourier transform of the function $t \mapsto \Gamma(-\zeta)^{-1} \mathbf{1}_{(-\infty,0)}(t)\,(-t)^{-\zeta-1}\,\mathrm{e}^{\varepsilon t}$ is equal to $s \mapsto (\varepsilon + 2\mathrm{i}\pi s)^\zeta$, namely*

$$\frac{1}{\Gamma(-\zeta)} \int_{\mathbb{R}} \mathbf{1}_{(-\infty,0)}(t)\,(-t)^{-\zeta-1}\,\mathrm{e}^{\varepsilon t}\,\mathrm{e}^{2\mathrm{i}\pi s t}\,\mathrm{d}t = (\varepsilon + 2\mathrm{i}\pi s)^\zeta\,, \quad s \in \mathbb{R}\,.$$

Proof. — By a contour integral of $(-z)^{-\zeta-1}\,\mathrm{e}^z$, running along the negative real half-line and along the half-line $H_s = \{(\varepsilon + 2\mathrm{i}\pi s)t \in \mathbb{C} : t < 0\}$, we obtain

$$\Gamma(-\zeta) = \int_{-\infty}^0 (-t)^{-\zeta-1}\,\mathrm{e}^t\,\mathrm{d}t = (\varepsilon + 2\mathrm{i}\pi s)^{-\zeta} \int_{-\infty}^0 (-t)^{-\zeta-1}\,\mathrm{e}^{(\varepsilon+2\mathrm{i}\pi s)t}\,\mathrm{d}t\,,$$

giving the announced result. ☐

Integrating (6.9) by parts, we see that

$$(I^w h)(t) = -\frac{1}{\Gamma(w+1)} \int_t^{+\infty} (u-t)^w h'(u)\,\mathrm{d}u\,.$$

This new formula makes sense for $\operatorname{Re} w > -1$ and could be used for defining the fractional derivative D^z if $z = -w$ and $\operatorname{Re} w \in (-1, 0)$, by setting for t real

$$(D^z h)(t) = -\frac{1}{\Gamma(1-z)} \int_t^{+\infty} (u-t)^{-z} h'(u)\,\mathrm{d}u\,. \tag{6.10}$$

This is proved in Lemma 6.8. It is coherent with the fact that D^α, for $0 < \alpha < 1$, can be considered as the antiderivative of order $1-\alpha$ of the derivative $D^1 h = -h'$,

$$D^\alpha h = D^{\alpha-1} D^1 h = -D^{\alpha-1} h' = -I^{1-\alpha} h'\,.$$

The operation D^z is not symmetric on $\mathbb{R}$; this is obvious from the formulas for I^w. The choice that was done of $(2\mathrm{i}\pi s)^z$ instead of $(-2\mathrm{i}\pi s)^z$ in (6.7) induces the direction in which the fractional antiderivative is computed. This direction, to $+\infty$, is well adapted to the "radial" Carbery's method introduced in [20].

Lemma 6.8. — *Let $\alpha \in (0,1)$, $t_0 \in \mathbb{R}$ be given and let k be a function on $\mathbb{R}$ such that $(1+|s|^\alpha)k(s)$ is integrable on the real line. Assume that $h = \widehat{k}$*

is Lipschitz with $|h'(t)| \leqslant \kappa_1(1+|t|)^{-1}$ *for almost every* $t \geqslant t_0$*. Then, for every* $t > t_0$ *and* z *such that* $\operatorname{Re} z = \alpha$*, we have*

$$-\frac{1}{\Gamma(1-z)} \int_t^{+\infty} (u-t)^{-z} h'(u)\,\mathrm{d}u = \int_{\mathbb{R}} (2\mathrm{i}\pi s)^z k(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,.$$

Proof. — Let η be a nonnegative C^∞ function on $\mathbb{R}$, with integral 1 and with compact support in $[-1,1]$. Consider $\varepsilon \in (0,1)$ and

$$k_\varepsilon(s) = k(s)(\eta^\vee)_{[\varepsilon]}(s) = k(s)\eta^\vee(\varepsilon s)\,, \quad s \in \mathbb{R}\,.$$

Then $\eta^\vee \in \mathcal{S}(\mathbb{R})$, $sk_\varepsilon(s)$ is integrable and $h_\varepsilon := \widehat{k_\varepsilon} = h * \eta_{(\varepsilon)}$ is C^1. We can write

$$-\,h'_\varepsilon(t) = \int_{\mathbb{R}} 2\mathrm{i}\pi s k_\varepsilon(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,, \quad t \in \mathbb{R}\,. \tag{6.11}$$

Since h is Lipschitz, we also know that $h'_\varepsilon = h' * \eta_{(\varepsilon)}$. Fix $t \geqslant t_0 + \varepsilon$. When $|\tau| \leqslant 1$ and $u \geqslant t$, we have $u - \varepsilon\tau \geqslant t_0$, $1+|u| \leqslant 1+\varepsilon|\tau|+|u-\varepsilon\tau| \leqslant 2+2|u-\varepsilon\tau|$, so

$$|h'_\varepsilon(u)| = \left| \int_{-1}^1 h'(u-\varepsilon\tau)\eta(\tau)\,\mathrm{d}\tau \right| \leqslant \kappa_1 \int_{-1}^1 \frac{\eta(\tau)}{1+|u-\varepsilon\tau|}\,\mathrm{d}\tau \leqslant \frac{2\kappa_1}{1+|u|}\,. \tag{6.12}$$

Applying (6.11) and $|(u-t)^{-z}| = (u-t)^{-\alpha}$, Fubini's theorem and the inverse Fourier transform of $v \mapsto [(-v)_+]^{-z}\,\mathrm{e}^{\varepsilon v}$ given by Lemma 6.7 with $\zeta = z-1$, we get

$$\begin{aligned}
&-\frac{1}{\Gamma(1-z)} \int_t^{+\infty} (u-t)^{-z}\,\mathrm{e}^{\varepsilon(t-u)}\,h'_\varepsilon(u)\,\mathrm{d}u \\
&= \frac{1}{\Gamma(1-z)} \int_t^{+\infty} (u-t)^{-z}\,\mathrm{e}^{\varepsilon(t-u)} \left(\int_{\mathbb{R}} 2\mathrm{i}\pi s k_\varepsilon(s)\,\mathrm{e}^{-2\mathrm{i}\pi su}\,\mathrm{d}s \right) \mathrm{d}u \\
&= \frac{1}{\Gamma(1-z)} \iint \mathbf{1}_{\{t-u<0\}}(u-t)^{-z}\,\mathrm{e}^{\varepsilon(t-u)}\,2\mathrm{i}\,\pi s k_\varepsilon(s)\,\mathrm{e}^{2\mathrm{i}\pi s(t-u)}\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,\mathrm{d}u \\
&= \int_{\mathbb{R}} (\varepsilon + 2\mathrm{i}\pi s)^{z-1}(2\mathrm{i}\pi s)k_\varepsilon(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,.
\end{aligned}$$

Letting ε tend to 0, by a double application of Lebesgue's dominated convergence, using (6.12) and since $h'_\varepsilon(u) \to h'(u)$ at every Lebesgue point u of h', we obtain

$$-\frac{1}{\Gamma(1-z)} \int_t^{+\infty} (u-t)^{-z} h'(u)\,\mathrm{d}u = \int_{\mathbb{R}} (2\mathrm{i}\pi s)^z k(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,. \qquad \square$$

It is quite comforting to have two possible ways of defining $D^z h$. However, we will have to handle cases where the Fourier transform $h(t)$ is well controlled, but where the estimates on $k(s)$ are not so good. We shall therefore concentrate on the integral definition (6.10) of $D^z h$. We have to check that the properties obtained with the first definition remain true when only the second applies.

When $\alpha \in (0,1)$ tends to 1, one has $\Gamma(1-\alpha) \simeq (1-\alpha)^{-1}$ and for $\varepsilon > 0$ we get

$$- \frac{1}{\Gamma(1-\alpha)} \int_{t+\varepsilon}^{+\infty} (u-t)^{-\alpha} h'(u) \, \mathrm{d}u \to 0 \, ,$$

$$(1-\alpha) \int_t^{t+\varepsilon} (u-t)^{-\alpha} \, \mathrm{d}u = \varepsilon^{1-\alpha} \to 1 \, .$$

We recover the fact that $(D^1 h)(t) = -h'(t)$, already known by Fourier.

Let us mention the case of $h(t) = \mathrm{e}^{-\lambda|t|}$, the Fourier transform of a Cauchy kernel. When $t \geqslant 0$ and $0 < \operatorname{Re} z < 1$, we have

$$\begin{aligned} D_t^z \, \mathrm{e}^{-\lambda|t|} &= \frac{1}{\Gamma(1-z)} \, \mathrm{e}^{-\lambda t} \int_t^{+\infty} (u-t)^{-z} \lambda \, \mathrm{e}^{-\lambda(u-t)} \, \mathrm{d}u \\ &= \lambda^z \, \mathrm{e}^{-\lambda|t|} \, . \end{aligned} \tag{6.13}$$

The dilation relation (6.8) follows from a simple change of variable similar to the one in the line above, and is left to the reader.

We have introduced in (5.5) the right maximal function h_r^* of h. Notice that for h Lipschitz on $(t_0, +\infty)$ and for every $t \geqslant t_0$, $\delta > 0$, we have

$$|h(t+\delta)| \leqslant |h(t)| + \int_t^{t+\delta} |h'(u)| \, \mathrm{d}u \leqslant h_r^*(t) + \delta (h')_r^*(t) \, . \tag{6.14}$$

Lemma 6.9. — *Let h be Lipschitz on $(t_0, +\infty)$, $\alpha \in (0,1)$ and $h(t) = o(t^\alpha)$ at $+\infty$. Let $h_0 = h_r^*$ be the right maximal function of h and $h_1 = (h')_r^*$ that of h'. Then*

$$|(D^\alpha h)(t)| \leqslant 6 \, h_0(t)^{1-\alpha} h_1(t)^\alpha \, , \quad t \geqslant t_0 \, .$$

If w is complex and $\operatorname{Re} w = \alpha$, then for every $t \geqslant t_0$ we have

$$|(D^w h)(t)| \leqslant \frac{2}{\alpha(1-\alpha)} \frac{(1+|w|)^{1-\alpha}}{|\Gamma(1-w)|} h_0(t)^{1-\alpha} h_1(t)^\alpha \, .$$

Proof. — For $t \geqslant t_0$ and $\delta > 0$, we express $E_\alpha := -\Gamma(1-\alpha) \, (D^\alpha h)(t)$ as

$$\int_t^{t+\delta} (u-t)^{-\alpha} h'(u) \, \mathrm{d}u + \int_{t+\delta}^{+\infty} (u-t)^{-\alpha} h'(u) \, \mathrm{d}u \, .$$

Applying (5.6) and integration by parts, we bound each of the two pieces

$$\begin{aligned}|E_\alpha| &\leqslant \frac{\delta^{1-\alpha}}{1-\alpha}\,h_1(t) + \left|\Big[(u-t)^{-\alpha}h(u)\Big]_{t+\delta}^{+\infty}\right| + \alpha\left|\int_{t+\delta}^{+\infty}(u-t)^{-\alpha-1}h(u)\,\mathrm{d}u\right| \\ &\leqslant \frac{\delta^{1-\alpha}}{1-\alpha}\,h_1(t) + \delta^{-\alpha}|h(t+\delta)| \\ &\qquad + \alpha\Big(\int_t^{t+\delta}\delta^{-\alpha-1}|h(u)|\,\mathrm{d}u + \int_{t+\delta}^{+\infty}(u-t)^{-\alpha-1}|h(u)|\,\mathrm{d}u\Big).\end{aligned}$$

By (6.14), by (5.6) for the non-decreasing function ψ defined by $\psi(u) = \delta^{-\alpha-1}$ when $u \in [t, t+\delta]$ and $\psi(u) = (u-t)^{-\alpha-1}$ for $u \geqslant t+\delta$, we obtain

$$\begin{aligned}|E_\alpha| &\leqslant \frac{\delta^{1-\alpha}}{1-\alpha}\,h_1(t) + \delta^{-\alpha}\big(h_0(t) + \delta h_1(t)\big) + (1+\alpha)\delta^{-\alpha}h_0(t) \\ &= \frac{2-\alpha}{1-\alpha}\,\delta^{1-\alpha}\,h_1(t) + (2+\alpha)\delta^{-\alpha}h_0(t) \\ &\leqslant \frac{2}{1-\alpha}\,\delta^{1-\alpha}\,h_1(t) + 3\delta^{-\alpha}h_0(t)\,.\end{aligned}$$

We choose $\delta = \delta_0 = h_0(t)/h_1(t)$ and get that

$$|E_\alpha| \leqslant \Big(\frac{2}{1-\alpha} + 3\Big)h_0(t)^{1-\alpha}h_1(t)^\alpha\,.$$

Recalling $\Gamma(1-\alpha) \geqslant 1$ and the minimal value $\Gamma(x_\Gamma) > 0.88$ in (3.7) we have

$$\begin{aligned}|(D^\alpha h)(t)| &\leqslant \Big(\frac{2}{\Gamma(2-\alpha)} + \frac{3}{\Gamma(1-\alpha)}\Big)h_0(t)^{1-\alpha}h_1(t)^\alpha \\ &\leqslant \Big(\frac{2}{\Gamma(x_\Gamma)} + 3\Big)h_0(t)^{1-\alpha}h_1(t)^\alpha \leqslant 6\,h_0(t)^{1-\alpha}h_1(t)^\alpha\,.\end{aligned}$$

When w is complex and $\operatorname{Re} w = \alpha$, we use $|(u-t)^{-w}| = (u-t)^{-\alpha}$, the same integration by parts, $|(u-t)^{-w-1}| = (u-t)^{-\alpha-1}$ and we get

$$\begin{aligned}|E_w| &\leqslant \frac{\delta^{1-\alpha}}{1-\alpha}\,h_1(t) + \delta^{-\alpha}\big(h_0(t) + \delta h_1(t)\big) + |w|\Big(1 + \frac{1}{\alpha}\Big)\delta^{-\alpha}h_0(t) \\ &\leqslant \frac{2\delta^{1-\alpha}}{1-\alpha}\,h_1(t) + \frac{2}{\alpha}\,(1+|w|)\delta^{-\alpha}h_0(t)\,.\end{aligned}$$

Choosing $\delta = (1+|w|)h_0(t)/h_1(t)$ we obtain the announced result. ☐

In what follows, we shall consider the following assumptions on a function h:

$$\begin{cases} h \text{ is Lipschitz on } [t_0, +\infty)\,, \\ |h(t)| \leqslant \kappa_0(1+|t|)^{-1} \text{ for } t \geqslant t_0\,, \\ |h'(t)| \leqslant \kappa_1(1+|t|)^{-1} \text{ for almost every } t \geqslant t_0\,. \end{cases} \tag{6.15}$$

Corollary 6.10. — *Suppose that the function h defined on $(t_0, +\infty)$, $t_0 \geqslant 0$, satisfies* (6.15)*. Then for every $\alpha \in (0,1)$, we have*

$$|(D^\alpha h)(t)| \leqslant 6 \, \frac{\kappa_0^{1-\alpha} \kappa_1^\alpha}{1 + |t|} \, , \quad t \geqslant t_0 \, .$$

Proof. — The two upper bounds in (6.15) are decreasing functions of $t \in [t_0, +\infty)$, hence they also bound h_r^* or $(h')_r^*$. We conclude by applying Lemma 6.9. □

Assuming that h has enough derivatives and continuing integrations by parts, starting from (6.10), we get successive formulas for $D^z h$ for each integer $j > 0$, which make sense when $\operatorname{Re} z < j$. Let $z = j - 1 + w$, with $j \geqslant 1$ and $\operatorname{Re} w \in (0,1)$. We obtain that

$$(D^z h)(t) = (-1)^{j-1} (D^w h^{(j-1)})(t) = \frac{(-1)^j}{\Gamma(1-w)} \int_t^{+\infty} (u-t)^{-w} h^{(j)}(u) \, \mathrm{d}u \, ,$$

and for every $z \in \mathbb{C}$ such that $\operatorname{Re} z < j$, we have

$$(D^z h)(t) = \frac{(-1)^j}{\Gamma(j-z)} \int_t^{+\infty} (u-t)^{-z+j-1} h^{(j)}(u) \, \mathrm{d}u \, . \tag{6.16}$$

By gluing the successive definitions, we define entire functions of z for every fixed t and $h \in \mathcal{S}(\mathbb{R})$. By the principle of analytic continuation, we conclude that the integral formula for $D^z h$ coincides when $\operatorname{Re} z > -1$ with the one obtained by Fourier transform (a fact that we have checked in Lemma 6.8 when $0 < \operatorname{Re} z < 1$).

Lemma 6.11. — *Let α be in $(0,1)$. Suppose that the function h satisfies the assumptions* (6.15) *on $[t_0, +\infty)$, $t_0 \geqslant 0$, and define $D^\alpha h$ by* (6.10)*. We have that*

$$(I^\alpha D^\alpha h)(t) = t \, , \quad t \geqslant t_0 \, .$$

Proof. — We first assume in addition that

$$\int_{t_0}^{+\infty} |h'(u)| \, \mathrm{d}u < +\infty \, , \quad \text{thus} \quad h(t) = - \int_t^{+\infty} h'(u) \, \mathrm{d}u$$

for every $t \geqslant t_0$ since h is Lipschitz. For $u \geqslant t_0$, accepting possibly infinite integrals of nonnegative measurable functions, set

$$G(u) = \frac{1}{\Gamma(1-\alpha)} \int_u^{+\infty} (v-u)^{-\alpha} |h'(v)| \, \mathrm{d}v \, .$$

When h is decreasing on $(t_0, +\infty)$, the function G is equal to $D^\alpha h$, and $|D^\alpha h| \leqslant G$ in general. Then, consider F, equal to $I^\alpha G$ in good cases, defined for $t \geqslant t_0$ by

$$\begin{aligned}F(t) &:= \frac{1}{\Gamma(\alpha)}\int_t^{+\infty}(u-t)^{\alpha-1}G(u)\,\mathrm{d}u\\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_t^{+\infty}(u-t)^{\alpha-1}\int_u^{+\infty}(v-u)^{-\alpha}|h'(v)|\,\mathrm{d}v\,\mathrm{d}u\\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_t^{+\infty}\Big(\int \mathbf{1}_{t\leqslant u\leqslant v}\,(u-t)^{\alpha-1}(v-u)^{-\alpha}\,\mathrm{d}u\Big)|h'(v)|\,\mathrm{d}v\,.\end{aligned}$$

Setting $u = t + y(v-t)$, one gets with $\gamma_\alpha = \Gamma(\alpha)\Gamma(1-\alpha)$ that

$$\begin{aligned}F(t) &= \gamma_\alpha^{-1}\Big(\int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\,\mathrm{d}y\Big)\int_t^{+\infty}|h'(v)|\,\mathrm{d}v\\ &= \int_t^{+\infty}|h'(v)|\,\mathrm{d}v < +\infty\,.\end{aligned}$$

The last equality can be deduced from (6.13) by applying the preceding computation to $h(v) = \mathrm{e}^{-|v-t_0|}$, or one can check directly that $\gamma_\alpha = \int_0^1 y^{\alpha-1}(1-y)^{-\alpha}\,\mathrm{d}y$. From the Fubini theorem and the same calculation without absolute values, it follows that if $\int_{t_0}^{+\infty}|h'(u)|\,\mathrm{d}u < +\infty$, then for every $t \geqslant t_0$ we have

$$(I^\alpha D^\alpha h)(t) = -\int_t^{+\infty} h'(u)\,\mathrm{d}u = h(t)\,.$$

Under (6.15), we introduce $h_\varepsilon(t) = \mathrm{e}^{-\varepsilon|t-t_0|}\,h(t)$ with $\varepsilon > 0$, for which we use the preceding case and convergence when $\varepsilon \to 0$. When $\varepsilon \in (0,1)$ and $t > t_0$ we have

$$|h_\varepsilon(t)| \leqslant |h(t)| \leqslant \frac{\kappa_0}{1+|t|}\,,\qquad |h_\varepsilon'(t)| \leqslant (\varepsilon|h(t)| + |h'(t)|) \leqslant \frac{\kappa_0+\kappa_1}{1+|t|}\,.$$

By Corollary 6.10, we have $|D^\alpha h_\varepsilon| \leqslant \kappa(1+|t|)^{-1}$, and we can apply twice dominated convergence when $\varepsilon \to 0$ in

$$\int_t^{+\infty}(u-t)^{\alpha-1}\Big(\int_u^{+\infty}(v-u)^{-\alpha}h_\varepsilon'(v)\,\mathrm{d}v\Big)\,\mathrm{d}u = h_\varepsilon(t)\,. \qquad \square$$

Assuming (6.15) and $\operatorname{Re} z > 0$, we have

$$D_t^z(th(t)) = t(D^z h)(t) - z(D^{z-1}h)(t)\,. \tag{6.17}$$

This is obtained when $0 < \operatorname{Re} z < 1$ with an integration by parts, writing

$$\begin{aligned}\Gamma(1-z)\big(-D_t^z(th(t)) + t(D^z h)(t)\big) &= \int_t^{+\infty} (u-t)^{-z}\big((u-t)h'(u) + h(u)\big)\,\mathrm{d}u \\ &= \int_t^{+\infty} (u-t)^{-z+1} h'(u)\,\mathrm{d}u + \int_t^{+\infty} (u-t)^{-z} h(u)\,\mathrm{d}u \\ &= z\int_t^{+\infty} (u-t)^{-z} h(u)\,\mathrm{d}u = z\Gamma(1-z)(D^{z-1}h)(t)\,.\end{aligned}$$

6.2.1. Multipliers associated to fractional derivatives

If K is a kernel integrable on $\mathbb{R}^n$, we know by (2.15) that its Fourier transform m is expressed for $\xi \neq 0$ as

$$m(u\xi) = \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi s u|\xi|}\,\mathrm{d}s = \int_{\mathbb{R}} \frac{1}{|\xi|}\varphi_\theta\Big(\frac{v}{|\xi|}\Big)\,\mathrm{e}^{-2\mathrm{i}\pi v u}\,\mathrm{d}v\,, \quad u \in \mathbb{R}\,,$$

where $\theta = |\xi|^{-1}\xi$ and where the function φ_θ is defined on $\mathbb{R}$ by (2.14). Letting $\alpha > 0$ and assuming that $x \mapsto |x|^\alpha K(x)$ is integrable on $\mathbb{R}^n$, this yields

$$\begin{aligned}D_u^\alpha m(u\xi) &= \int_{\mathbb{R}} (2\mathrm{i}\pi v)^\alpha \frac{1}{|\xi|}\varphi_\theta\Big(\frac{v}{|\xi|}\Big)\,\mathrm{e}^{-2\mathrm{i}\pi v u}\,\mathrm{d}v \\ &= \int_{\mathbb{R}} (2\mathrm{i}\pi s|\xi|)^\alpha \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi s|\xi|u}\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n} (2\mathrm{i}\pi x\cdot\xi)^\alpha K(x)\,\mathrm{e}^{-2\mathrm{i}\pi u x\cdot\xi}\,\mathrm{d}x\,,\end{aligned}$$

which is naturally extended by 0 when $\xi = 0$. We set in what follows

$$\begin{aligned}(\xi\cdot\nabla)^\alpha m(\xi) := D_u^\alpha m(u\xi)\big|_{u=1} &= \int_{\mathbb{R}^n} (2\mathrm{i}\pi x\cdot\xi)^\alpha\, K(x)\,\mathrm{e}^{-2\mathrm{i}\pi x\cdot\xi}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (2\mathrm{i}\pi s|\xi|)^\alpha \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi s|\xi|}\,\mathrm{d}s\,.\end{aligned} \tag{6.18.∇^α}$$

When $\alpha = 1$ and $\xi \neq 0$, the quantity $(\xi\cdot\nabla)^1 m(\xi)$ is equal to $-\xi\cdot\nabla m(\xi)$, which is the product by $-|\xi|$ of the usual directional derivative of the function m in the direction of the norm-one vector $\theta = |\xi|^{-1}\xi$. When $0 < \alpha < 1$, under the assumptions (6.15), we can give according to Lemma 6.8 the integral formula

$$(\xi\cdot\nabla)^\alpha m(\xi) = -\frac{1}{\Gamma(1-\alpha)}\int_1^{+\infty} (u-1)^{-\alpha}\,\frac{\mathrm{d}}{\mathrm{d}u} m(u\xi)\,\mathrm{d}u\,. \tag{6.19}$$

We shall use the integral formula (6.19) when $m(\xi)$ is Lipschitz outside the origin and when for every $u_0 > 0$ and $u \geqslant u_0$, we have for every $\theta \in S^{n-1}$ that

$$|m(u\theta)| + \Bigl|\frac{\mathrm{d}}{\mathrm{d}u} m(u\theta)\Bigr| \leqslant \frac{\kappa(\theta, u_0)}{1 + |u|} \,.$$

If K is an isotropic log-concave probability density with variance σ^2, we know by Corollary 5.13 that $|(\mathrm{d}/\mathrm{d}u)\, m(u\xi)| \leqslant \delta_{1,c}|\sigma\xi|/(1 + 2\pi|u\sigma\xi|) \leqslant \delta_{1,c}/(2\pi|u|)$, thus

$$|(\xi \cdot \nabla)^\alpha m(\xi)| \leqslant \frac{\delta_{1,c}}{2\pi|\Gamma(1-\alpha)|} \int_1^{+\infty} (u-1)^{-\alpha} u^{-1} \,\mathrm{d}u = \kappa_\alpha \delta_{1,c} \,, \tag{6.20}$$

and the bounded function $\xi \mapsto (\xi \cdot \nabla)^\alpha m(\xi)$ defines an L^2 multiplier. We reach of course the same conclusion under (6.1.**H**) for a "general" kernel K_g.

We have seen in (2.10) that the multiplier norm of $m(\xi)$ on $L^p(\mathbb{R}^n)$ is the same as that of the dilate $m(\lambda\xi)$, for every $\lambda > 0$. It is thus natural to look for a norm invariant by dilation, if we want a norm capable to control the action on L^p of a multiplier. Since we shall work radially with Carbery's approach, we begin with a smooth function h compactly supported in $(0, +\infty)$, and when $\alpha \in (0,1)$ we set with Carbery [21]

$$\|h\|_{L^2_\alpha} := \biggl(\int_0^{+\infty} \Bigl| t^{\alpha+1} D_t^\alpha \Bigl(\frac{h(t)}{t}\Bigr) \Bigr|^2 \frac{\mathrm{d}t}{t} \biggr)^{1/2} . \tag{6.21}$$

One verifies that this norm is invariant by dilation. By (6.8), we have

$$t^{\alpha+1} D_t^\alpha \Bigl(\frac{h_{[\lambda]}(t)}{t} \Bigr) = t^{\alpha+1} \lambda D_t^\alpha \Bigl(\frac{h(\lambda t)}{\lambda t} \Bigr) = (\lambda t)^{\alpha+1} D_v^\alpha \Bigl(\frac{h(v)}{v} \Bigr)\Big|_{v=\lambda t} , \tag{6.22}$$

and the change of variable $u = \lambda t$ in (6.21) completes the proof. Let h be Lipschitz on $(t_0, +\infty)$ for all $t_0 > 0$. Applying (6.17) to $\widetilde{h}(t) = h(t)/t$, we get for all $t > 0$

$$\begin{aligned} D_t^\alpha \Bigl(\frac{h(t)}{t} \Bigr) &= \frac{\alpha}{t} D_t^{\alpha-1} \Bigl(\frac{h(t)}{t} \Bigr) + \frac{1}{t} (D^\alpha h)(t) \\ &= \frac{1}{t} D_t^{\alpha-1} \Bigl(\frac{\alpha h(t)}{t} - h'(t) \Bigr). \end{aligned} \tag{6.23}$$

Remark 6.12. — When $1/2 < \alpha < 1$, the L^2_α norm dominates the $L^\infty(0, +\infty)$ norm of the function h. For a justification, let us assume in addition that h is bounded and Lipschitz on each interval $(t, +\infty)$ with $t > 0$. Then $H : u \mapsto h(u)/u$ satisfies (6.15) on $(t, +\infty)$ and we can apply

Lemma 6.11, giving $I^\alpha D^\alpha H = H$, thus

$$\begin{aligned} \frac{h(t)}{t} &= \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (u-t)^{\alpha-1} D_u^\alpha \Bigl(\frac{h(u)}{u} \Bigr) \, \mathrm{d}u \\ &= \frac{1}{t\Gamma(\alpha)} \int_t^{+\infty} (t/u)(1-t/u)^{\alpha-1} \Bigl[u^{\alpha+1} D_u^\alpha \Bigl(\frac{h(u)}{u} \Bigr) \Bigr] \, \frac{\mathrm{d}u}{u} \, . \end{aligned}$$

Applying Cauchy–Schwarz, $\Gamma(\alpha) > 1$ for $\alpha \in (0,1)$ and letting $y = t/u$, we get

$$\begin{aligned} h(t)^2 &\leqslant \Bigl(\int_t^{+\infty} (t/u)^2 (1-t/u)^{2\alpha-2} \frac{\mathrm{d}u}{u} \Bigr) \Bigl(\int_t^{+\infty} \Bigl[u^{\alpha+1} D_u^\alpha \Bigl(\frac{h(u)}{u} \Bigr) \Bigr]^2 \frac{\mathrm{d}u}{u} \Bigr) \\ &\leqslant \Bigl(\int_0^1 y(1-y)^{2\alpha-2} \, \mathrm{d}y \Bigr) \|h\|^2_{L^2_\alpha} \leqslant \frac{1}{2\alpha-1} \|h\|^2_{L^2_\alpha} \, . \end{aligned}$$

The latter calculation is the basis for the L^2 part of Carbery's Proposition 6.14.

Remark 6.13. — Using the second expression in (6.23), we see that $\|h\|^2_{L^2_\alpha}$ is the integral on $(0, +\infty)$, and with respect to $(\mathrm{d}t)/t$, of the square of the modulus of

$$\begin{aligned} t^\alpha D_t^{\alpha-1} \Bigl(\frac{\alpha h(t)}{t} - h'(t) \Bigr) &= \frac{1}{\Gamma(1-\alpha)} \int_t^{+\infty} (u/t-1)^{-\alpha} \bigl(\alpha h(u) - u h'(u) \bigr) \, \frac{\mathrm{d}u}{u} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_1^{+\infty} (v-1)^{-\alpha} \bigl(\alpha h(tv) - tv h'(tv) \bigr) \, \frac{\mathrm{d}v}{v} \, . \end{aligned}$$

In most cases, this expression tends to $\kappa h(0)$ when $t \to 0$, with $\kappa > 0$, and then we have that $\|h\|_{L^2_\alpha}$ is finite only if $h(0) = 0$, as for Bourgain's criterion $\Gamma_B(K)$.

We do not see an easy way to compare the L^2_α norm and the quantity appearing in the Γ_B criterion. However, in the very special case where $H(t) = h(t)/t$ is $\geqslant 0$, convex and decreasing on $(0, +\infty)$, the function $|H'| = -H'$ is decreasing and it follows from Lemma 6.9 that $(D^{1/2} H)(t)$ is bounded by $\kappa \sqrt{|H(t) H'(t)|}$, hence

$$\begin{aligned} \|h\|^2_{L^2_{1/2}} &\leqslant \kappa \int_0^{+\infty} t^3 \, \frac{|h(t)|}{t} \Bigl(\frac{|h'(t)|}{t} + \frac{|h(t)|}{t^2} \Bigr) \, \frac{\mathrm{d}t}{t} \\ &\leqslant \kappa \int_0^{+\infty} \Bigl(|h(t)| \, |t h'(t)| + |h(t)|^2 \Bigr) \, \frac{\mathrm{d}t}{t} \\ &\leqslant \kappa' \sum_{j \in \mathbb{Z}} \Bigl(\alpha_j(h) \beta_j(h) + \alpha_j(h)^2 \Bigr). \end{aligned}$$

We obtain then (in this very special situation) that $\|h\|_{L^2_{1/2}} \leqslant \kappa \Gamma_B(h^\vee)$.

6.3. Fourier criteria for bounding the maximal function

In the next proposition due to Carbery, we impose conditions that fit into our presentation but are certainly too restrictive.

Proposition 6.14 (Carbery [21]). — *Let K be a kernel integrable on $\mathbb{R}^n$ with integral equal to 0, let m be the Fourier transform of K. Assume that $m_\theta := u \mapsto m(u\theta)$ is differentiable on $(0, +\infty)$ for every $\theta \in S^{n-1}$, and that $m'_\theta(u)$, $u > 0$, is bounded by a constant independent of θ.*

(1) *If there exists $\alpha \in (1/2, 1)$ such that*

$$C_\alpha(m) := \sup_{\theta \in S^{n-1}} \bigl\| t \mapsto m(t\theta) \bigr\|_{L^2_\alpha} < +\infty\,, \tag{6.24}$$

then for every function $f \in L^2(\mathbb{R}^n)$ one has

$$\|M_K f\|_{L^2(\mathbb{R}^n)} = \bigl\| \sup_{t>0} |K_{(t)} * f| \bigr\|_{L^2(\mathbb{R}^n)} \leqslant \frac{1}{\sqrt{2\alpha - 1}}\, C_\alpha(m) \|f\|_{L^2(\mathbb{R}^n)}\,.$$

(2) *Suppose that $p < +\infty$ and $1/p < \alpha < 1$. If the multiplier $(\xi \cdot \nabla)^\alpha m(\xi)$ from (6.18.∇^α) is bounded on $L^p(\mathbb{R}^n)$, then for every f in $L^p(\mathbb{R}^n)$ one has that*

$$\bigl\| \sup_{1 \leqslant t \leqslant 2} |K_{(t)} * f| \bigr\|_{L^p(\mathbb{R}^n)} \leqslant \kappa_{\alpha,p} \bigl(2\|m\|_{p\to p} + \|(\xi \cdot \nabla)^\alpha m(\xi)\|_{p\to p} \bigr) \|f\|_{L^p(\mathbb{R}^n)}\,, \tag{6.25}$$

with $\kappa_{\alpha,p} \leqslant (2\alpha)^{1-1/p}(p-1)^{1-2/p}(\alpha - 1/p)^{1/p-1}$.

When $1 < p \leqslant 2$, one has the simpler larger bound $\kappa_{\alpha,p} \leqslant \sqrt{2}\bigl(\alpha - 1/p\bigr)^{-1/p}$. Indeed, for $0 < \alpha < 1$, we have that $2^{1/2-1/p}\alpha^{1-1/p}(p-1)^{1-2/p}(\alpha - 1/p)^{2/p-1}$ is less than $\bigl([\alpha - 1/p] / [\sqrt{2\alpha}\,(p-1)]\bigr)^{2/p-1}$. When $1 < p \leqslant 2$, this expression increases with $\alpha \in (1/p, 1]$, and for $\alpha = 1$, one has $(1 - 1/p)/(\sqrt{2}\,(p-1)) = 1/(\sqrt{2}p) \leqslant 1$.

Observe that if we set $\xi = |\xi|\theta$ for some nonzero vector $\xi \in \mathbb{R}^n$, we have

$$\bigl\| t \mapsto m(t\xi) \bigr\|_{L^2_\alpha} = \bigl\| t \mapsto m(t\theta) \bigr\|_{L^2_\alpha}$$

according to the invariance by dilation (6.22) of the norm L^2_α. So the supremum in (1) is also the supremum on $\xi \in \mathbb{R}^n$. We shall need the following Lemma, slightly more general than the conclusion (1) in Proposition 6.14.

Lemma 6.15. — *Let $(K_t)_{t>0}$ be a family of integrable kernels on $\mathbb{R}^n$, and denote by $\xi \mapsto m(\xi, t)$ the Fourier transform of K_t. Assume that for* *every $u_0 > 0$, there exist N and $\kappa(u_0)$ satisfying the following: for every ξ in $\mathbb{R}^n$, the function $g_\xi : u \mapsto m(\xi, u)/u$, for $u \in [u_0, +\infty)$, is Lipschitz and*

$$|g_\xi(u)| + |g'_\xi(u)| \leqslant \kappa(u_0) \frac{(1+|\xi|)^N}{1+|u|}\,, \quad \xi \in \mathbb{R}^n,\ u \geqslant u_0\,. \tag{6.26}$$

If there is $\alpha \in (1/2, 1)$ such that $c_\alpha := \sup_{\xi \in \mathbb{R}^n} \big\| t \mapsto m(\xi, t) \big\|_{L^2_\alpha} < +\infty$, then

$$\forall\, f \in \mathcal{S}(\mathbb{R}^n)\,, \quad \big\| \sup_{t>0} |K_t * f| \big\|_{L^2(\mathbb{R}^n)} \leqslant \frac{1}{\sqrt{2\alpha - 1}}\, c_\alpha \|f\|_{L^2(\mathbb{R}^n)}\,.$$

Proof. — By the assumptions, the function g_ξ satisfies (6.15). As in Remark 6.12, we obtain by Lemma 6.11 for all $\xi \in \mathbb{R}^n$ and $t > 0$ that

$$\frac{m(\xi, t)}{t} = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (u-t)^{\alpha-1} D_u^\alpha \Big(\frac{m(\xi, u)}{u} \Big)\, \mathrm{d}u\,.$$

For $f \in \mathcal{S}(\mathbb{R}^n)$, according to (6.26) and Corollary 6.10, we can use Fubini and get

$$\begin{aligned}
(K_t * f)(x) &= \int_{\mathbb{R}^n} m(\xi, t) \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi \\
&= \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} t(u-t)^{\alpha-1} \int_{\mathbb{R}^n} D_u^\alpha \Big(\frac{m(\xi, u)}{u} \Big) \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi\, \mathrm{d}u \\
&= \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (t/u)(1-t/u)^{\alpha-1} \Big(\int_{\mathbb{R}^n} u^{\alpha+1} D_u^\alpha \Big(\frac{m(\xi, u)}{u} \Big) \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi \Big) \frac{\mathrm{d}u}{u}\,.
\end{aligned}$$

For $u > 0$ and $x \in \mathbb{R}^n$, let us set

$$(P_u^\alpha f)(x) = \int_{\mathbb{R}^n} u^{\alpha+1} D_u^\alpha \Big(\frac{m(\xi, u)}{u} \Big) \widehat{f}(\xi)\, \mathrm{e}^{2\mathrm{i}\pi x\cdot\xi}\, \mathrm{d}\xi\,.$$

This operator P_u^α is associated to the multiplier

$$p_u^\alpha(\xi) = u^{\alpha+1} D_v^\alpha \Big(\frac{m(\xi, v)}{v} \Big)\Big|_{v=u}\,, \quad \xi \in \mathbb{R}^n.$$

One can rewrite

$$(K_t * f)(x) = \frac{1}{\Gamma(\alpha)} \int_t^{+\infty} (t/u)(1-t/u)^{\alpha-1} (P_u^\alpha f)(x)\, \frac{\mathrm{d}u}{u}\,. \tag{6.27}$$

By Cauchy–Schwarz and since $\Gamma(\alpha) > 1$ when $\alpha \in (0, 1)$, we get

$$|(K_t * f)(x)|^2 \leqslant \Big(\int_t^{+\infty} (t/u)^2 (1-t/u)^{2(\alpha-1)}\, \frac{\mathrm{d}u}{u} \Big) \Big(\int_0^{+\infty} \big|(P_u^\alpha f)(x)\big|^2\, \frac{\mathrm{d}u}{u} \Big).$$

For $\alpha > 1/2$, one has $2(\alpha-1) > -1$ and letting $y = t/u$, one sees that

$$\int_t^{+\infty} (t/u)^2 (1-t/u)^{2(\alpha-1)}\, \frac{\mathrm{d}u}{u} = \int_0^1 y(1-y)^{2(\alpha-1)}\, \mathrm{d}y < \frac{1}{2\alpha - 1}\,.$$

We have obtained for $|(K_t * f)(x)|^2$ a bound independent of t, hence

$$\sup_{t>0} |(K_t * f)(x)|^2 \leqslant \kappa_\alpha^2 \Bigl(\int_0^{+\infty} \bigl|(P_u^\alpha f)(x)\bigr|^2 \frac{\mathrm{d}u}{u} \Bigr),$$

with $\kappa_\alpha^{-2} = 2\alpha - 1$. By Fubini and Parseval, we have

$$\begin{aligned}
\Bigl\| \sup_{t>0} |K_t * f| \Bigr\|_{L^2(\mathbb{R}^n)}^2 &\leqslant \kappa_\alpha^2 \int_{\mathbb{R}^n} \Bigl(\int_0^{+\infty} \bigl|(P_u^\alpha f)(x)\bigr|^2 \frac{\mathrm{d}u}{u} \Bigr) \mathrm{d}x \\
&= \kappa_\alpha^2 \int_0^{+\infty} \|P_u^\alpha f\|_{L^2(\mathbb{R}^n)}^2 \frac{\mathrm{d}u}{u} \\
&= \kappa_\alpha^2 \int_{\mathbb{R}^n} \int_0^{+\infty} \Bigl| u^{\alpha+1} D_u^\alpha \Bigl(\frac{m(\xi, u)}{u} \Bigr) \widehat{f}(\xi) \Bigr|^2 \frac{\mathrm{d}u}{u} \, \mathrm{d}\xi \\
&\leqslant \kappa_\alpha^2 \int_{\mathbb{R}^n} c_\alpha^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi = \kappa_\alpha^2 c_\alpha^2 \|f\|_{L^2(\mathbb{R}^n)}^2 . \qquad \square
\end{aligned}$$

Remark 6.16. — If $|a(t)| \leqslant c(t_0)$ when $t \geqslant t_0 > 0$ and if $b(t) = a(t)/t$, then we have $(1+t)|b(t)| = (t^{-1}+1)|a(t)| \leqslant c(t_0)(1+t_0^{-1})$ when $t \geqslant t_0$. If we add that $|a'(t)| \leqslant c(t_0)$ for $t \geqslant t_0$, we have also $(1+t)|a'(t)/t| \leqslant c(t_0)(1+t_0^{-1})$, $t \geqslant t_0$, and

$$|b'(t)| \leqslant \Bigl(\frac{|a'(t)|}{t} + \frac{|b(t)|}{t} \Bigr) \leqslant \frac{c(t_0)(1+t_0^{-1})^2}{1+t}, \quad t \geqslant t_0 > 0 .$$

If we know that for every $u_0 > 0$, there is $c(u_0)$ such that

$$|m(\xi, u)| + \Bigl| \frac{\mathrm{d}}{\mathrm{d}u} m(\xi, u) \Bigr| \leqslant c(u_0)\,(1+|\xi|)^N, \quad \xi \in \mathbb{R}^n, \ u \geqslant u_0 ,$$

it follows that (6.26) is true, with $\kappa(u_0) \leqslant 2c(u_0)(1+t_0^{-1})^2$.

Proof of Proposition 6.14. — We apply Lemma 6.15 to the family $K_t = K_{(t)}$ of dilates of K, $t > 0$. Under the assumptions of Proposition 6.14, we first have that $|m(t\xi)| + |(\mathrm{d}/\mathrm{d}t) m(t\xi)| \leqslant \kappa(1+|\xi|)$. Remark 6.16 implies then that the family of functions $g_\xi : t \mapsto m(t\xi)/t$ satisfies (6.26). We thus obtain by Lemma 6.15 the L^2-maximal inequality when $f \in \mathcal{S}(\mathbb{R}^n)$, and we may extend it to all functions in $L^2(\mathbb{R}^n)$ by the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, as explained in Section 3.3.

Let us pass to the proof of (2), the L^p case. We use the notation of the proof of Lemma 6.15, adapted to $m(\xi, t) = m(t\xi)$. Denote by q the conjugate exponent of p, and observe that $q-2 > -1$ because $p < +\infty$. When $\alpha \in (1/p, 1)$ and $t \geqslant 1$, by applying Hölder to (6.27) and since $\alpha - 1 > -1/q$,

$\Gamma(\alpha) > 1$, we obtain

$$\begin{aligned}
&|(K_{(t)} * f)(x)| \\
&\leqslant \Gamma(\alpha)^{-1} \Bigl(\int_t^{+\infty} (t/u)^q (1-t/u)^{q(\alpha-1)} \,\mathrm{d}u \Bigr)^{1/q} \Bigl(\int_t^{+\infty} |(P_u^\alpha f)(x)|^p \,\frac{\mathrm{d}u}{u^p} \Bigr)^{1/p} \\
&\leqslant t^{1/q} \Bigl(\int_t^{+\infty} (t/u)^q (1-t/u)^{q(\alpha-1)} \,\frac{\mathrm{d}u}{t} \Bigr)^{1/q} \Bigl(\int_t^{+\infty} |(P_u^\alpha f)(x)|^p \,\frac{\mathrm{d}u}{u^p} \Bigr)^{1/p} \\
&\leqslant t^{1/q} \Bigl(\int_1^{+\infty} v^{-q\alpha} (v-1)^{q(\alpha-1)} \,\mathrm{d}v \Bigr)^{1/q} \Bigl(\int_1^{+\infty} |(P_u^\alpha f)(x)|^p \,\frac{\mathrm{d}u}{u^p} \Bigr)^{1/p} \\
&\leqslant \Bigl(\int_1^2 (v-1)^{q\alpha-q} \,\mathrm{d}v + \int_2^{+\infty} (v-1)^{-q} \,\mathrm{d}v \Bigr)^{1/q} \\
&\qquad\qquad\qquad \times t^{1/q} \Bigl(\int_1^{+\infty} |(P_u^\alpha f)(x)|^p \,\frac{\mathrm{d}u}{u^p} \Bigr)^{1/p} .
\end{aligned}$$

With $c_{\alpha,p}^q = 1/(q\alpha - q + 1) + 1/(q-1) = \alpha(p-1)/(\alpha - 1/p)$, it follows that

$$\begin{aligned}
\Bigl\| \sup_{1 \leqslant t \leqslant 2} |K_{(t)} * f| \Bigr\|_{L^p(\mathbb{R}^n)}^p &\leqslant c_{\alpha,p}^p 2^{p/q} \int_1^{+\infty} \Bigl(\int_{\mathbb{R}^n} |(P_u^\alpha f)(x)|^p \,\mathrm{d}x \Bigr) \frac{\mathrm{d}u}{u^p} \\
&= c_{\alpha,p}^p 2^{p/q} \int_1^{+\infty} \|P_u^\alpha f\|_p^p \,\frac{\mathrm{d}u}{u^p} \\
&\leqslant \frac{c_{\alpha,p}^p 2^{p/q}}{p-1} \sup_{u \geqslant 1} \|P_u^\alpha f\|_p^p ,
\end{aligned}$$

and we shall see that $\|P_u^\alpha\|_{p\to p} \leqslant 2\,\|m\|_{p\to p} + \|(\xi\cdot\nabla)^\alpha m(\xi)\|_{p\to p}$. The multipliers p_u^α are dilates of one another, indeed, for every $\lambda > 0$, we have by (6.22) that

$$\begin{aligned}
p_u^\alpha(\lambda\xi) &= u^{\alpha+1} D_v^\alpha \Bigl(\frac{m(v\lambda\xi)}{v} \Bigr) \Bigr|_{v=u} \\
&= u^{\alpha+1} \lambda^{\alpha+1} D_v^\alpha \Bigl(\frac{m(v\xi)}{v} \Bigr) \Bigr|_{v=\lambda u} = p_{\lambda u}^\alpha(\xi) .
\end{aligned}$$

It suffices therefore to consider p_1^α. According to (6.23), one has

$$p_1^\alpha(\xi) = D_t^\alpha \Bigl(\frac{m(t\xi)}{t} \Bigr) \Bigr|_{t=1} = \alpha\, D_t^{\alpha-1} \Bigl(\frac{m(t\xi)}{t} \Bigr) \Bigr|_{t=1} + D_t^\alpha m(t\xi) \bigr|_{t=1} .$$

The multiplier $D_t^\alpha m(t\xi)\bigr|_{t=1}$ is precisely equal to $(\xi\cdot\nabla)^\alpha m(\xi)$. The other term, since $\alpha - 1 < 0$, can be written by (6.9) as

$$U(\xi) = \alpha\, D_t^{\alpha-1} \Bigl(\frac{m(t\xi)}{t} \Bigr) \Bigr|_{t=1} = \frac{\alpha}{\Gamma(1-\alpha)} \int_1^{+\infty} (v-1)^{-\alpha} \Bigl(\frac{m(v\xi)}{v} \Bigr) \,\mathrm{d}v .$$

By Lemma 2.1, we have $\|U\|_{p\to p} \leqslant 2\,\|m\|_{p\to p}$ because

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_1^{+\infty} (v-1)^{-\alpha} \,\frac{\mathrm{d}v}{v} \leqslant \frac{\alpha}{\Gamma(1-\alpha)} \Bigl(\frac{1}{1-\alpha} + \frac{1}{\alpha} \Bigr) = \frac{1}{\Gamma(2-\alpha)} \leqslant 2\,,$$

cutting $\int_1^{+\infty}$ at $v = 2$, and using (3.7). □

6.4. Proofs of Theorems 6.1 and 6.2, and Proposition 6.3

We need only show Proposition 6.3, and we can limit ourselves to $1 < p \leqslant 2$. As in Bourgain's proof of the L^2 theorem for M_C at the end of Section 5.3.1, the kernel K to which we shall apply Proposition 6.6 is given by $K = K_g - P$, where P is the Poisson kernel P_1 from (1.32), and K_g is a probability density on $\mathbb{R}^n$ satisfying (6.1.$\mathbf{H}$) with two constants $\delta_{0,g} \geqslant 1$ and $\delta_{1,g}$ controlling the Fourier transform m_g and its gradient. We know by (1.31.P^*) that the maximal operator associated to the Poisson kernel acts on $L^r(\mathbb{R}^n)$, $1 < r \leqslant +\infty$, with constants independent of the dimension n. Letting B denote the Euclidean ball normalized by variance in $\mathbb{R}^n$, we could replace P by K_B and invoke Stein's Theorem 4.1 instead.

We shall apply Proposition 6.6 in the two cases corresponding to Theorems 6.1 and 6.2, in order to show that the maximal function (or the dyadic maximal function) associated to the kernel K is bounded on L^p for $p > 3/2$ (or for $p > 1$). We shall get by difference that the maximal function for K_g (or K_{lc}, K_C) is bounded as well. In the "dyadic" case of Theorem 6.1, the operator T_j, for $j \in \mathbb{Z}$, is the convolution with the dilate $K_{(2^j)}$ of K. For Theorem 6.2, $T_{j,v}$ is the convolution with $K_{(v2^j)}$, $1 \leqslant v \leqslant 2$, and T_j is given by

$$T_j f = \sup_{1 \leqslant v \leqslant 2} |T_{j,v} f| = \sup_{2^j \leqslant t \leqslant 2^{j+1}} |K_{(t)} * f|\,.$$

One has to check that the assumptions of Proposition 6.6, namely, $(A_{\mathbf{0}})$, $(A_{\mathbf{1}})$, $(A_{\mathbf{2}})$ and $(A_{\mathbf{3}})$, are satisfied in these two cases. If the (Q_j) are those of Littlewood–Paley, defined by

$$\widehat{Q}_j(\xi) = \mathrm{e}^{-2\pi 2^j |\xi|} - \mathrm{e}^{-2\pi 2^{j+1} |\xi|}\,, \quad \xi \in \mathbb{R}^n\,,$$

then the assumption $(A_{\mathbf{0}})$ is satisfied according to (2.4), with $C_p = \mathrm{q}_p$.

For $(A_{\mathbf{1}})$, we write $T_{j,v} = U_{j,v} - S_{j,v}$, where the $U_{j,v} = (K_g)_{(v2^j)}$ are related to K_g and the $S_{j,v} = P_{(v2^j)}$ to the Poisson kernel. The operators $U_{j,v}$ and $S_{j,v}$ are positive, as convolutions with probability densities. As mentioned before, this is the only place where we need K_g to be a probability density rather than a general integrable kernel. We know by (1.31.P^*) that the maximal operator S^* associated to the Poisson kernel is bounded on

$L^p(\mathbb{R}^n)$, $1 < p < +\infty$, by a constant C'_p independent of the dimension n. Consequently, the property $(A_{\mathbf{1}})$ is satisfied.

Let us consider $(A_{\mathbf{2}})$. The first case is when $T_j = K_{(2^j)}$ and in this case, according to (2.13), the operator T_j is bounded on all the spaces $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant +\infty$, by the L^1 norm of K and we get that

$$\|T_j\|_{p\to p} \leqslant \|K\|_{L^1(\mathbb{R}^n)} \leqslant 2\,. \tag{6.28}$$

In the second case, we have to use the part (2) of Proposition 6.14. This will be discussed below.

Finally, we must show $(A_{\mathbf{3}})$, i.e., prove that T^* satisfies the property $(\mathbf{S}_2)$. For k fixed in $\mathbb{Z}$, we shall bound the maximal operator of the kernel $N_k := K * Q_k$ using the conclusion (1) of Proposition 6.14. We show in Section 6.5 that for every value $\alpha \in (1/2, 1)$, the norm $b_k := C_\alpha(\widehat{N_k})$ decays exponentially with $|k|$, with constants depending on α and (linearly) on $\delta_{0,g} + \delta_{1,g}$. In the "dyadic case", the bound obtained in this way by (1) for the maximal function of N_k implies that

$$\begin{aligned}\big\|\sup_{j\in\mathbb{Z}} |T_j Q_{j+k} f|\big\|_2 &= \big\|\sup_{j\in\mathbb{Z}} |K_{(2^j)} * (P_{(2^{j+k})} - P_{(2^{j+1+k})}) * f|\big\|_2 \\ &= \big\|\sup_{j\in\mathbb{Z}} |(N_k)_{(2^j)} * f|\big\|_2 \leqslant \big\|\sup_{t>0} |(N_k)_{(t)} * f|\big\|_2 \leqslant \kappa_\alpha b_k\,,\end{aligned}$$

which proves $(\mathbf{S}_2)$ in this case. The case of the global maximal function requires a small adaptation, Carbery says: "This is not exactly what being strongly bounded on L^2 means, but a slight modification of this argument will give precisely what we require". Indeed, there is now a gap between what we get from Proposition 6.14 and the assumption we need for applying Proposition 6.6. We shall discuss it in the subsection 6.4.1 and resolve this "gap question" in the subsection 6.5.1. We obtain at last by Lemma 6.19 and by Lemma 6.15 that there exist universal coefficients $(a_k)_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} a_k^s < +\infty$ for every $s > 0$, and such that

$$\big\|\sup_{j\in\mathbb{Z}} |T_j Q_{j+k} f|\big\|_2 \leqslant (\delta_{0,g} + \delta_{1,g}) a_k\,, \quad k \in \mathbb{Z}\,. \tag{6.29}$$

For $(A_{\mathbf{2}})$ in the "global" case, we study the operators $(W_t)_{t>0}$ defined by

$$W_t f = \sup_{t\leqslant u\leqslant 2t} |K_{(u)} * f|\,, \quad t > 0\,,$$

and we want to prove $(A_{\mathbf{2}})$ for the family of $T_j = W_{2^j}$ from (6.2), with $j \in \mathbb{Z}$. Using the invariance by dilation (2.11) of multiplier norms, we see that the operators W_t have the same norm when t varies, hence we need to find a bound for $T_0 = W_1$ only. For this, we want to apply the conclusion (2) of Proposition 6.14, so we must show that the multipliers m and $(\xi\cdot\nabla)^\alpha m(\xi)$

are bounded on $L^p(\mathbb{R}^n)$ for some $\alpha \in (1/p, 1)$. For m it is clear by the elementary fact (2.13).

For $(\xi \cdot \nabla)^\alpha m(\xi)$ we shall use complex interpolation between $(\xi \cdot \nabla)^0 m(\xi) = m(\xi)$ that acts on $L^1(\mathbb{R}^n)$, and $(\xi \cdot \nabla) m(\xi)$ that acts on $L^2(\mathbb{R}^n)$ since it is a bounded function on $\mathbb{R}^n$ by (6.20) and (6.1.**H**). We get by interpolation that the multiplier $(\xi \cdot \nabla)^\alpha m(\xi)$ is bounded on $L^p(\mathbb{R}^n)$, with p given by

$$\frac{1}{p} = \frac{1-\alpha}{1} + \frac{\alpha}{2} = 1 - \frac{\alpha}{2}\,,$$

and we need $1 - \alpha/2 = 1/p < \alpha$ for applying (2), thus $1 < 3\alpha/2 = 3 - 3/p$. We must therefore have $p > 3/2$ in order to conclude. We see that the reason for the restriction on the values of p in Theorem 6.2 is to be found precisely here.

This sketch is not fully accurate. For being able to interpolate, one must control in L^2 the values $\alpha = 1 + \mathrm{i}\tau$, for every τ real, which causes no difficulty, but also the values $\alpha = 0 + \mathrm{i}\tau$ in L^1, and this is more technical. The precise work, involving a slight modification of the strategy described here, is done in Section 7.3 when we are well embedded by Müller [59] in the mood for interpolation. For every $p \in (3/2, 2]$, we shall then obtain for some $\alpha > 1/p$, function of p, a bound of the form $\|(\xi \cdot \nabla)^\alpha m(\xi)\|_{p \to p} \leqslant \kappa_p (\delta_{0,g} + \delta_{1,g})^{2-2/p}$. By Proposition 6.14, we deduce

$$\big\|T_0 f\big\|_{L^p(\mathbb{R}^n)} = \big\| \sup_{1 \leqslant t \leqslant 2} |K_{(t)} * f| \big\|_{L^p(\mathbb{R}^n)} \leqslant \kappa'_p (\delta_{0,g} + \delta_{1,g})^{2-2/p} \|f\|_{L^p(\mathbb{R}^n)}$$

for every function $f \in L^p(\mathbb{R}^n)$. We get $(A_{\mathbf{2}})$ with $p_{\min} = 3/2$, since

$$\|T_j\|_{p \to p} = \|T_0\|_{p \to p} \leqslant \kappa'_p (\delta_{0,g} + \delta_{1,g})^{2-2/p}, \quad j \in \mathbb{Z},\ 3/2 < p \leqslant 2\,. \tag{6.30}$$

Applying Proposition 6.6, we finish the proof of Proposition 6.3. For $p \in (3/2, 2]$, we shall bound $T^* = M_K$ in $L^p(\mathbb{R}^n)$, thus also M_{K_g}. We choose a value p_0, function of p, such that $3/2 < p_0 < p$, and we let $\delta = \delta_{0,g} + \delta_{1,g}$. We have by (6.30) that $C''_{p_0} \leqslant \kappa''_p \delta^{2-2/p_0}$. Then, applying (6.4), (6.29), (6.30) and $\delta_{0,g} \geqslant 1$, we obtain

$$\begin{aligned} \|\mathrm{M}_{K_g}\|_{p \to p} \leqslant \|T^*\|_{p \to p} + \kappa_{p,0} &\leqslant \kappa_{p,1} (C''_{p_0})^\gamma \Big(\sum_{k \in \mathbb{Z}} (\delta a_k)^{(1-\gamma)p/2} \Big)^{2/p} + \kappa_{p,2} \\ &\leqslant \kappa_p \delta^{2-2/p} \end{aligned}$$

as announced, observing that $1 - \gamma = [1/p_0 - 1/p]/[1/p_0 - 1/2]$ is the interpolation parameter for L^p and the pair (L^{p_0}, L^2), and that the powers of δ under the exponents γ and $1 - \gamma$ are of the form $2 - 2/r$, $r = p_0$ or 2. In the dyadic case, we may replace (6.30) by (6.28) and obtain the result for $\mathrm{M}^{(d)}_{K_g}$ when $p \in (1, 2]$.

Remark 6.17. — Bringing back the question to the Poisson kernel leads to some complications, because the function $\varphi_\theta(s)$ associated to the Poisson kernel, i.e., the Cauchy kernel (1.33.**C**), does not have decay properties as good as that of the function $\varphi_{\theta,C}$ of a convex set. This approach however does not depend on the L^p result of Stein for the Euclidean ball.

Why not employ the Gaussian semi-group instead? In some non Euclidean situations, like Heisenberg groups or Grushin operators for instance, and especially for the weak type $(1,1)$ property of associated maximal functions, the Poisson kernel is preferable. Indeed, some asymptotic estimates, uniform in the dimension, are required on the kernel and are easier to obtain for the Poisson kernel. But in the Euclidean case, we cannot see a compelling obstacle to the use of the Gaussian kernel. We would get an excellent decay, both in the space variable and in the Fourier variable. We have chosen to stick to the original proofs, but we urge the reader to rewrite them with Gaussian kernels instead. We shall see in Section 8 that Bourgain uses Gaussian kernels.

6.4.1. Where is the gap?

As was said above, we will arrive for $N_k = K * Q_k$ at

$$C_\alpha(N_k) := \sup_{\theta \in S^{n-1}} \bigl\| t \mapsto N_k(t\theta) \bigr\|_{L^2_\alpha} \leqslant \kappa_\alpha 2^{-\gamma|k|}, \quad k \in \mathbb{Z},$$

for some $\gamma > 0$. This implies by Proposition 6.14 (1), that

$$\bigl\| \sup_{t>0} |(N_k)_{(t)} * f| \bigr\|_2 \leqslant \kappa_\alpha 2^{-\gamma|k|}.$$

Translating the definition of N_k gives

$$\bigl\| \sup_{t>0} |(K * Q_k)_{(t)} * f| \bigr\|_2 \leqslant \kappa_\alpha 2^{-\gamma|k|}$$

where $K = K_g - P$, or

$$\bigl\| \sup_{v \in [1,2]} \sup_{j \in \mathbb{Z}} |(K_{(v2^j)} * (P_{(v2^{j+k})} - P_{(v2^{j+k+1})}) * f| \bigr\|_2 \leqslant \kappa_\alpha 2^{-\gamma|k|}.$$

This must be compared to bounding the expression

$$\bigl\| \sup_{v \in [1,2]} \sup_{j \in \mathbb{Z}} |(K_{(v2^j)} * (P_{(2^{j+k})} - P_{(2^{j+k+1})}) * f| \bigr\|_2,$$

which is what we are waiting for, in the definition of Property $(\mathbf{S}_2)$ for the family of operators $(T_{j,v})$, $j \in \mathbb{Z}$, $v \in [1,2]$.

6.5. A proof for the property $(\mathbf{S}_2)$

In what follows, $m = m_g - \widehat{P}$ is the Fourier transform of the kernel $K = K_g - P$ that appears in the proof of Proposition 6.3, where K_g is a probability density on $\mathbb{R}^n$ satisfying (6.1.**H**). We have

$$\widehat{P}(\xi) = \mathrm{e}^{-2\pi|\xi|} \ \text{ and we let } \ \rho(\xi) = \widehat{P}(\xi) - \widehat{P}(2\xi)\,, \quad \xi \in \mathbb{R}^n .$$

For every $k \in \mathbb{Z}$, every $\xi \in \mathbb{R}^n$ and $u > 0$, we set

$$m_k(\xi) = \widehat{N_k}(\xi) = m(\xi)\big(\mathrm{e}^{-2^{k+1}\pi|\xi|} - \mathrm{e}^{-2^{k+2}\pi|\xi|}\big) = m(\xi)\rho(2^k\xi)\,,$$
$$h_k^\xi(u) = \frac{m_k(u\xi)}{u} .$$

One must show that for any given $\alpha \in (1/2, 1)$, the quantity

$$C_\alpha(m_k)^2 = \sup_{\theta \in S^{n-1}} \big\|u \mapsto m_k(u\theta)\big\|^2_{L^2_\alpha} = \sup_{\theta \in S^{n-1}} \int_0^{+\infty} \big(u^{\alpha+1}(D^\alpha h_k^\theta)(u)\big)^2 \frac{\mathrm{d}u}{u}$$

introduced in (6.24) decays exponentially to 0 when $|k|$ tends to infinity. We fix therefore $\theta \in S^{n-1}$ and for $u \in \mathbb{R}$, we set

$$\phi(u) = m(u\theta)\,, \quad \chi(u) = \mathrm{e}^{-2\pi|u|} - \mathrm{e}^{-4\pi|u|} = \widehat{P}(u\theta) - \widehat{P}(2u\theta) = \rho(u\theta) .$$

Let $\delta = \delta_{0,g} + \delta_{1,g} \geqslant 1$, where $\delta_{0,g}, \delta_{1,g}$ are the constants in (6.1.**H**). We know that

$$|u|\,|m_g(u\theta)| \leqslant \delta_{0,g} \leqslant \delta\,, \ \ |\theta \cdot \nabla m_g(u\theta)| \leqslant \delta_{1,g} \leqslant \delta\,,$$
$$|u\theta \cdot \nabla m_g(u\theta)| \leqslant \delta\,, \ \ u \in \mathbb{R} .$$

On the other hand, the derivative with respect to $u > 0$ of $\widehat{P}(u\theta) = \mathrm{e}^{-2\pi|u|}$ is bounded by 2π, and according to (5.21a), (5.21b), we have

$$|u\widehat{P}(u\theta)| \leqslant (2\pi\,\mathrm{e})^{-1} < 1 \leqslant \delta\,, \qquad \Big|u\frac{\mathrm{d}}{\mathrm{d}u}\widehat{P}(u\theta)\Big| \leqslant \mathrm{e}^{-1} < \delta .$$

For $\phi(u) = m(u\theta) = m_g(u\theta) - \widehat{P}(u\theta)$ we get $|\phi'(u)| \leqslant \delta + 2\pi$. Using again $\delta > 1$, we simplify this bound as $|\phi'(u)| < 8\delta$. It follows first that $|\phi(u)| \leqslant 8\delta|u|$, and

$$|\phi(u)| \leqslant 8\delta\big(|u| \wedge |u|^{-1}\big)\,, \quad |\phi'(u)| \leqslant 8\delta\big(1 \wedge |u|^{-1}\big) . \tag{6.31a}$$

For $\chi(u)$, we see when $u > 0$ that $0 \leqslant \chi(u) \leqslant \mathrm{e}^{-2\pi u}$ and

$$-2\pi\,\mathrm{e}^{-2\pi u} \leqslant \chi'(u) = -2\pi\,\mathrm{e}^{-2\pi u} + 4\pi\,\mathrm{e}^{-4\pi u} \leqslant 2\pi\,\mathrm{e}^{-2\pi u}\,,$$

implying that $|\chi'(u)| \leqslant 2\pi$ for $u \neq 0$ and

$$|\chi(u)| \leqslant (2\pi|u|) \wedge |2\pi\,\mathrm{e}\,u|^{-1}\,, \quad |\chi'(u)| \leqslant (2\pi) \wedge |\,\mathrm{e}\,u|^{-1} . \tag{6.31b}$$

We obtain a symmetric treatment of the two functions χ and $\phi_\delta := \delta^{-1}\phi$ since, up to some *universal* multiple κ (we express this by the sign $\lesssim$), we have

$$|\phi_\delta(u)|,\, |\chi(u)| \lesssim |u| \wedge |u|^{-1}\,, \quad |\phi'_\delta(u)|,\, |\chi'(u)| \lesssim 1 \wedge |u|^{-1}\,. \tag{6.32}$$

We set $p_k(u) = m_k(u\theta) = \phi(u)\chi(2^k u)$, $h_k(u) = p_k(u)/u$ and we want to estimate $\|p_k\|_{L^2_\alpha}$ for every $k \in \mathbb{Z}$. Notice that

$$p_{-k}(2^k v) = \chi(v)\phi(2^k v)\,.$$

The L^2_α norm is invariant by dilation and the assumptions on ϕ_δ and χ are identical, we may therefore restrict the verification to the case $k \geqslant 0$. Let us fix an integer $k \geqslant 0$. We have the following table, divided into the three regions where the chosen bounds (6.32) for the functions h_k and h'_k keep the same analytical expression, namely, the intervals $(0, 2^{-k})$, $(2^{-k}, 1)$ and $(1, +\infty)$. We consider that h'_k is the derivative of the product of $u^{-1}\phi(u)$ and $\chi(2^k u)$, we bound therefore $|h'_k|$ by the sum of $\big|(u^{-1}\phi(u))'\big|\,|\chi(2^k u)|$ and $|u^{-1}\phi(u)|\,2^k|\chi'(2^k u)\big|$.

u :		0	2^{-k}	1
$u^{-1}\|\phi_\delta(u)\|$	$\lesssim$	1	1	u^{-2}
$\|\chi(2^k u)\|$	$\lesssim$	$2^k u$	$2^{-k}u^{-1}$	$2^{-k}u^{-1}$
$u^{-1}\|\phi'_\delta(u)\| + u^{-2}\|\phi_\delta(u)\|$	$\lesssim$	$u^{-1} + u^{-1}$	$u^{-1} + u^{-1}$	$u^{-2} + u^{-3} \lesssim u^{-2}$
$2^k\|\chi'(2^k u)\|$	$\lesssim$	2^k	u^{-1}	u^{-1}
$\delta^{-1}\|h_k(u)\|$	$\lesssim$	$2^k u \leqslant 2^{-k}u^{-1}$	$2^{-k}u^{-1}$	$2^{-k}u^{-3}$
$\delta^{-1}\|h'_k(u)\|$	$\lesssim$	$2^k + 2^k \lesssim u^{-1}$	$2^{-k}u^{-2} + u^{-1} \lesssim u^{-1}$	$2^{-k}u^{-3} + u^{-3} \lesssim u^{-3}$

We see that $\delta^{-1}|h'_k(u)| \lesssim H_1(u) := u^{-1} \wedge u^{-3}$. This function H_1 is non-increasing on $(0, +\infty)$ and independent of k, and $\delta^{-1}|h_k(u)| \lesssim H_{0,k}(u) = 2^{-k}H_1(u)$. It follows from Lemma 6.9 that for $t > 0$, we have

$$\delta^{-1}|(D^\alpha h_k)(t)| \lesssim H_{0,k}(t)^{1-\alpha}\, H_1(t)^\alpha \lesssim 2^{-(1-\alpha)k} H_1(t)\,,$$

and the conclusion is reached since we obtain then

$$\begin{aligned}\|\phi\,\chi_{[2^k]}\|^2_{L^2_\alpha} = \|p_k\|^2_{L^2_\alpha} &= \int_0^{+\infty} \big|t^{\alpha+1}(D^\alpha h_k)(t)\big|^2\, \frac{\mathrm{d}t}{t} \\ &\lesssim \delta^2 2^{-2(1-\alpha)k}\Big(\int_0^1 (t^{\alpha+1}t^{-1})^2\,\frac{\mathrm{d}t}{t} + \int_1^\infty (t^{\alpha+1}t^{-3})^2\,\frac{\mathrm{d}t}{t}\Big)\end{aligned}$$

and

$$\int_0^1 t^{2\alpha-1}\,\mathrm{d}t + \int_1^\infty t^{2\alpha-5}\,\mathrm{d}t = \frac{1}{2\alpha} + \frac{1}{4-2\alpha} = \frac{1}{\alpha(2-\alpha)} < \frac{1}{\alpha} < +\infty\,,$$

thus $\|p_k\|_{L^2_\alpha} \lesssim \delta\alpha^{-1/2} 2^{-(1-\alpha)k}$ when $k \geqslant 0$, and $\|p_k\|_{L^2_\alpha} \leqslant \kappa\alpha^{-1/2}\delta\, 2^{-(1-\alpha)|k|}$ when $k \in \mathbb{Z}$. This implies by Proposition 6.14 (1) that

$$\big\|\sup_{r>0}\big([(m_k)_{[r]}\widehat{f}\,]^\vee\big)\big\|_{L^2(\mathbb{R}^n)} \leqslant \kappa\delta\, 2^{-(1-\alpha)|k|}\|f\|_{L^2(\mathbb{R}^n)} \tag{6.33}$$

for every $\alpha \in (1/2, 1)$, giving the property ($\mathbf{S}_2$) (see Definition 6.4) in the dyadic case.

It would be just as simple to work with the $\Gamma_B(K)$ criterion of Bourgain given in Section 5.3. We prove a general Lemma that will be invoked again in Section 8 for the cube problem.

Lemma 6.18. — *Suppose that two integrable kernels K_1 and K_2 on $\mathbb{R}^n$ satisfy, for a certain κ and every $\theta \in S^{n-1}$, that*

$$|\widehat{K}_j(u\theta)| \leqslant \kappa(|u| \wedge |u|^{-1})\,, \quad |\theta\cdot\nabla\widehat{K}_j(u\theta)| \leqslant \kappa(1 \wedge |u|^{-1})\,, \quad j = 1, 2\,, \quad u \in \mathbb{R}\,.$$

It follows that $\Gamma_B\big(K_1 * (K_2)_{(2^k)}\big) \leqslant C(\kappa) 2^{-|k|/2}$ for $k \in \mathbb{Z}$.

Proof. — We fix $\theta \in S^{n-1}$, and in order to remind us about the preceding case, we let m be the Fourier transform of K_1 and ρ that of K_2. We will modify the table above, in order to emphasize now $\phi(u) := m(u\theta)$ and $u\theta\cdot\nabla m(u\theta) = u\phi'(u)$ that appear in the components $\alpha_j(m)$ and $\beta_j(m)$ of $\Gamma_B(K)$, and we proceed similarly for $\chi(u) := \rho(u\theta)$.

Let m_k be the Fourier transform of the kernel $K_1 * (K_2)_{(2^k)}$. We have that $m_k(u\theta) = m(u\theta)\rho(2^k u\theta)$ and we may again restrict ourselves to $k \geqslant 0$, since a dilation by 2^i on a multiplier $g(\xi)$ produces a shift of i places on the indices j of the sequences $(\alpha_j(g))_{j\in\mathbb{Z}}$, $(\beta_j(g))_{j\in\mathbb{Z}}$, leaving $\sum_{j\in\mathbb{Z}}$ unchanged. The bounds below do not depend on $\theta \in S^{n-1}$, so we will be able to estimate

$$A_k(u) := \sup_{\theta\in S^{n-1}} |m_k(u\theta)| \quad \text{and} \quad B_k(u) := \sup_{\theta\in S^{n-1}} |u\theta\cdot\nabla m_k(u\theta)|\,.$$

Note that $B_k(u)$ is controlled by $\phi(u)2^k u\chi'(2^k u)$ and $u\phi'(u)\chi(2^k u)$. We have $\alpha_j(m_k) \sim A_k(2^j)$, $\beta_j(m_k) \sim B_k(2^j)$, for every $j \in \mathbb{Z}$. The new table is divided into the same three regions as before.

$u:$		0	2^{-k}	1
$\lvert\phi(u)\rvert$	$\lesssim$	u	u	u^{-1}
$\lvert\chi(2^k u)\rvert$	$\lesssim$	$2^k u$	$2^{-k}u^{-1}$	$2^{-k}u^{-1}$
$u\lvert\phi'(u)\rvert$	$\lesssim$	u	u	1
$2^k u\lvert\chi'(2^k u)\rvert$	$\lesssim$	$2^k u$	1	1
$A_k(u)$	$\lesssim$	$2^k u^2$	2^{-k}	$2^{-k}u^{-2}$
$B_k(u)$	$\lesssim$	$2^k u^2 + 2^k u^2$	$u + 2^{-k} \lesssim u$	$u^{-1} + 2^{-k}u^{-1} \lesssim u^{-1}$
$\sqrt{A_k(u)B_k(u)}$	$\lesssim$	$2^k u^2$	$2^{-k/2}u^{1/2}$	$2^{-k/2}u^{-3/2}$

It follows that for every $j \in \mathbb{Z}$, we have

$$\alpha_j(m_k) \lesssim \begin{cases} 2^{k+2j} & \text{if } j \leqslant -k\,, \\ 2^{-k} & \text{if } -k \leqslant j \leqslant 0\,, \\ 2^{-k-2j} & \text{if } 0 \leqslant j\,, \end{cases} \quad \text{so} \quad \sum_{j \in \mathbb{Z}} \alpha_j(m_k) \lesssim (k+1) 2^{-k}\,,$$

and

$$\sqrt{\alpha_j(m_k)\beta_j(m_k)} \lesssim \begin{cases} 2^{k+2j} & \text{if } j \leqslant -k\,, \\ 2^{-k/2+j/2} & \text{if } -k \leqslant j \leqslant 0\,, \\ 2^{-k/2-3j/2} & \text{if } 0 \leqslant j\,, \end{cases}$$

$$\text{so} \quad \sum_{j \in \mathbb{Z}} \sqrt{\alpha_j(m_k)\beta_j(m_k)} \lesssim 2^{-k/2}\,.$$

Taking the supremum, we obtain $\Gamma_B\bigl(K_1 * (K_2)_{(2^k)}\bigr) \leqslant C(\kappa) 2^{-|k|/2}$, for $k \in \mathbb{Z}$. □

Coming back to Carbery's situation, we obtain in this way by Lemma 5.14 that

$$\|m_k\|_{2 \to 2} \leqslant \kappa \delta\, 2^{-|k|/2}, \quad k \in \mathbb{Z},$$

slightly better than what we got with $C_\alpha(m_k)$. Indeed, we must choose $\alpha > 1/2$ with Carbery, and we have obtained for $C_\alpha(m_k)$ a bound of order $\delta 2^{-(1-\alpha)|k|}$.

6.5.1. A solution to the gap question

The gap question has been exposed in Section 6.4.1. Instead of the function studied precedently, equal to

$$\widehat{N_k}(\xi) : t \mapsto m_{[t]}(\xi)(\widehat{P}_{[t 2^k]} - \widehat{P}_{[t 2^{k+1}]})(\xi)\,, \quad t > 0,\ \xi \in \mathbb{R}^n\,,$$

we need to study the family of multipliers defined by

$$\widehat{n}_k(\xi, t) = m_{[t]}(\xi)(\widehat{P}_{[2^{j+k}]} - \widehat{P}_{[2^{j+k+1}]})(\xi)\,, \quad j \in \mathbb{Z} \text{ and } 2^j \leqslant t \leqslant 2^{j+1}\,,$$

which are the Fourier transforms of the kernels $K_{(t)} * (P_{(2^{j(t)+k})} - P_{(2^{j(t)+k+1})})$ with $j(t) = \lfloor \log_2 t \rfloor$. They do not fit into the setting of Proposition 6.14, but can be treated using Lemma 6.15. We do the following: for every $j \in \mathbb{Z}$, let $x_j = 2^j + 2^{j-1}$ be the midpoint of the interval $I_j = [2^j, 2^{j+1}]$. Let the "new" function be

$$t \mapsto m_{[2^j + 2(t - 2^j)]}(\xi)(\widehat{P}_{[2^{j+k}]} - \widehat{P}_{[2^{j+k+1}]})(\xi)$$

for t in the first half $[2^j, x_j]$ of the interval I_j, and

$$t \mapsto m_{[2^{j+1}]}(\xi)(\widehat{P}_{[2^k(2^j + 2(t - x_j))]} - \widehat{P}_{[2^{k+1}(2^j + 2(t - x_j))]})(\xi)$$

in the second half. The first half "contains" the family $\widehat{n}_k(\xi,t)$ that we have to study, and adjoining the second half will allow us to exploit easily what has been done in Section 6.5 for the regular setting. We can describe more compactly the new setting if we define two motions going along $(0,+\infty)$ according to

$$X(t) = \begin{cases} 2^j + 2(t-2^j)\,, & 2^j \leqslant t \leqslant x_j\,, \\ 2^{j+1}\,, & x_j \leqslant t \leqslant 2^{j+1}\,, \end{cases}$$

and

$$Y(t) = \begin{cases} 2^j\,, & 2^j \leqslant t \leqslant x_j\,, \\ 2^j + 2(t-x_j)\,, & x_j \leqslant t \leqslant 2^{j+1}\,. \end{cases}$$

Then, the new function can be written as

$$\widetilde{m}_k(\xi,t) := m_{[X(t)]}(\xi)\big(\widehat{P}_{[2^kY(t)]} - \widehat{P}_{[2^{k+1}Y(t)]}\big)(\xi)\,, \tag{6.34}$$

corresponding to the family of kernels $K_t = K_{(X(t))} * (P_{(2^kY(t))} - P_{(2^{k+1}Y(t))})$. The two functions X, Y are non-decreasing, continuous, piecewise linear, and we have $X(2^j) = Y(2^j) = 2^j$ for j in $\mathbb{Z}$. Notice that $X(2t) = 2X(t)$ and $Y(2t) = 2Y(t)$ (make use of $2x_j = x_{j+1}$). Also, $0 \leqslant X'(t), Y'(t) \leqslant 2$. Applying Remark 6.16, one sees easily that the functions $g_\xi(t) = \widetilde{m}_k(\xi,t)/t$ satisfy (6.26).

In the "dilation case" where $m_0(\xi,t) = m(t\xi)$, we have that $m_0(s\xi,t) = m_0(\xi,st)$ for every $s > 0$, and it allowed us to restrict the study of the functions $t \mapsto m_0(\xi,t)$, $\xi \in \mathbb{R}^n$, to the case $|\xi| = 1$. This is not true anymore, but we still have that $m(2\xi,t) = m(\xi,2t)$ for the two components Φ and Ψ of $\widetilde{m}_k(\xi,t)$, defined by

$$\Phi(\xi,t) = m_{[X(t)]}(\xi), \quad \Psi(\xi,t) = \big(\widehat{P}_{[Y(t)]} - \widehat{P}_{[Y(2t)]}\big)(\xi)\,,$$

and this permits us to restrict to the case $1 \leqslant |\xi| < 2$. Indeed,

$$\Phi(2\xi,t) = m_{[X(t)]}(2\xi) = m\big(2X(t)\xi\big) = m\big(X(2t)\xi\big) = \Phi(\xi,2t)\,.$$

The same property holds true for $\Psi(\xi,t)$, with Y replacing X.

Let us fix ξ such that $1 \leqslant |\xi| < 2$, and consider now

$$\phi_1(u) = \Phi(\xi,u) = m(X(u)\xi)\,, \;\; \chi_1(u) = \Psi(\xi,u) = \mathrm{e}^{-2\pi Y(u)|\xi|} - \mathrm{e}^{-4\pi Y(u)|\xi|}\,.$$

Letting $\xi = |\xi|\theta$, we compare $\phi(u) = m(u\theta)$ with $\phi_1(u) = \phi(X(u)|\xi|)$. For every $u > 0$, we have $u \leqslant X(u) \leqslant 2u$ and $u/2 \leqslant Y(u) \leqslant u$. We have therefore that $u \leqslant X(u)|\xi| \leqslant 4u$ and $u/2 \leqslant Y(u)|\xi| \leqslant 2u$. Recall that m, difference of m_g and $\widehat{P}$, satisfies (6.31a). It follows that

$$\begin{aligned}\delta^{-1}|\phi_1(u)| = |\phi_\delta(X(u)|\xi|)| &\leqslant 8\big[(X(u)|\xi|) \wedge (X(u)^{-1}|\xi|^{-1})\big] \\ &\leqslant 32\big(|u| \wedge |u|^{-1}\big) \lesssim |u| \wedge |u|^{-1}\,.\end{aligned}$$

We also have $\phi_1'(u) = X'(u)\,\phi'(X(u)|\xi|)$, and since $X'(u) \leqslant 2$,

$$\delta^{-1}|\phi_1'(u)| \leqslant 2|\phi_\delta'(X(u)|\xi|)| \leqslant 16\big[1 \wedge (X(u)^{-1}|\xi|^{-1})\big] \leqslant 16\big(1 \wedge |u|^{-1}\big)\,,$$

which can be written as $\delta^{-1}|\phi_1'(u)| \lesssim 1 \wedge |u|^{-1}$. Using (6.31b), we have the same kind of inequalities for χ_1. The proof in Section 6.5 depended only on these two bounds, so the result in (6.33) is also valid in the modified setting and gives the following lemma.

Lemma 6.19. — *Suppose that K_g is a probability density on $\mathbb{R}^n$ satisfying* (6.1.**H**)*, that $m = m_g - \widehat{P}$ and that $\widetilde{m}_k$ is defined by* (6.34)*. For $\alpha \in (1/2, 1)$, one has*

$$\sup_{\xi \in \mathbb{R}^n} \big\| t \mapsto \widetilde{m}_k(\xi, t) \big\|_{L^2_\alpha} \leqslant \kappa(\delta_{0,g} + \delta_{1,g}) 2^{-(1-\alpha)|k|}\,, \quad k \in \mathbb{Z}\,.$$

6.6. Appendix: proof of Bourgain's L^2 theorem by Carbery's criterion

Proof. — This section is intended to illustrate the Fourier definition (6.7) of D^α, and we shall have to perform some contortions in order to enter into the suitable setting. The kernel K on $\mathbb{R}^n$ to which we want to apply the conclusion (1) of Carbery's Proposition 6.14 is again $K = K_{lc} - P$, as in Section 6.4, where K_{lc} is a symmetric log-concave probability density on $\mathbb{R}^n$ normalized by variance. Let us fix a norm one vector $\theta \in \mathbb{R}^n$; here, the function $\varphi_\theta(s) = \int_{\theta^\perp} K(y + s\theta)\,\mathrm{d}^{n-1}y$, for $s \in \mathbb{R}$, is the difference of two symmetric probability densities ϕ_j, associated respectively to K_{lc} and to the Poisson kernel P. The function ϕ_1 of integrals of K_{lc} on affine hyperplanes parallel to $\theta^\perp$ satisfies, according to Lemma 5.6, an estimate of exponential decay $\phi_1(s) \leqslant \kappa\,\mathrm{e}^{-|s|/\kappa}$, for $s \in \mathbb{R}$ and for a certain $\kappa > 0$ universal. On the other hand, $\phi_2(s)$ is the Cauchy kernel (1.33.**C**) equal to $\pi^{-1}(1 + s^2)^{-1}$, for which one has only $\phi_2(s) \leqslant 1 \wedge s^{-2}$, where $a \wedge b$ denotes the minimum of two real numbers a and b. This estimate is valid also for ϕ_1, up to some universal factor κ, and we shall remember for the absolute value of φ_θ that

$$\forall\, s \in \mathbb{R}\,, \quad |\varphi_\theta(s)| \leqslant \kappa\Big(1 \wedge \frac{1}{s^2}\Big)\,. \tag{6.35}$$

The Fourier transform m of K is given by

$$m(t\theta) = \int_{\mathbb{R}} \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,.$$

Denote by Φ the antiderivative of φ_θ vanishing at 0. The function Φ is odd, it vanishes also at infinity because φ_θ is even with integral zero. We deduce from (6.35), for some $\kappa > 0$ and every $s \in \mathbb{R}$, that

$$|\Phi(s)| \leqslant \kappa(|s| \wedge |s|^{-1})\,. \tag{6.36}$$

For $t \neq 0$, we could, performing an integration by parts, express $m(t\theta)$ by a simply converging integral

$$m(t\theta) = 2\mathrm{i}\pi t \int_{-\infty}^{+\infty} \Phi(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,,$$

but we prefer to work with absolutely converging integrals, for example in this way: let us denote by $\widetilde{P}_0$ the L^1-normalized truncation $\widetilde{P}_0 = \|\mathbf{1}_B P\|^{-1}_{L^1(\mathbb{R}^n)}\,\mathbf{1}_B P$ of the Poisson kernel P at a sufficiently large Euclidean ball B in $\mathbb{R}^n$, so that $\|\mathbf{1}_B P\|_1 > 1/2$. We can see according to (1.35) that the radius of B must be at least of order $\kappa\sqrt{n}$. Another possibility is to introduce a modified Poisson kernel

$$\widetilde{P}(x) = 2\,P(x)\,\mathrm{e}^{-\varepsilon_0|x|^2/2}\,,$$

where $\varepsilon_0 > 0$ is chosen so that the integral of $\widetilde{P}$ is equal to 1. With both choices, one has $\widetilde{P}_0, \widetilde{P} \leqslant 2P$, and the estimates of the maximal function for the kernel P are thus clearly true for $\widetilde{P}$, with a bound simply doubled. For the same fixed θ of norm one, the modified function ϕ_2 defined by

$$\phi_2(s) = 2\int_{\theta^\perp} P(y+s\theta)\,\mathrm{e}^{-\varepsilon_0(|y|^2+s^2)/2}\,\mathrm{d}^{n-1}y \leqslant C(n)\,\mathrm{e}^{-\varepsilon_0 s^2/2}$$

decays exponentially at infinity, and since $\phi_2(s) \leqslant 2\pi^{-1}(1+s^2)^{-1}$, the modified function ϕ_2 satisfies (6.35) and (6.36). The modified antiderivative Φ inherits now at infinity of the exponential decay of ϕ_1 and of ϕ_2, and this makes the integrals that follow absolutely convergent. However, the “universal” estimates remain given by (6.35) and (6.36).

The situation would be simpler using a Gaussian kernel, letting

$$K(x) = K_C(x) - G(x)\,, \quad x \in \mathbb{R}^n,$$

with G being the $N(0, \mathrm{I}_n)$ density (1.17) on $\mathbb{R}^n$.

We apply here the Fourier definition (6.7) for D^α. For every $t > 0$ we write

$$\frac{m(t\theta)}{t} = 2\mathrm{i}\pi \int_{\mathbb{R}} \Phi(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,,$$

where $|\Phi|$ decays exponentially at infinity. This ensures that $t \mapsto m(t\theta)/t$ is C^∞ on the line, with bounded derivatives. By (6.7), we can express the fractional derivative appearing in Carbery's criterion as

$$D_t^\alpha\Bigl(\frac{m(t\theta)}{t}\Bigr) = 2\mathrm{i}\pi(2\pi)^\alpha \int_{\mathbb{R}} (\mathrm{i}s)^\alpha \Phi(s)\,\mathrm{e}^{-2\mathrm{i}\pi st}\,\mathrm{d}s\,.$$

For $0 < \alpha < 1$, we write

$$\int_0^\infty s^\alpha \Phi(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s = \frac{1}{2\mathrm{i}\pi t} \int_0^\infty \bigl(s^\alpha \Phi(s)\bigr)' \, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s \,,$$

and because $\bigl(s^\alpha \Phi(s)\bigr)'$ vanishes at 0, we see that

$$\int_0^\infty s^\alpha \Phi(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s = -\frac{1}{4\pi t^2} \int_0^\infty \bigl(s^\alpha \Phi(s)\bigr)'' \, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s \,.$$

The integrals on the side of negative s ask for an analogous treatment, essentially already seen in Section 5.2, Lemma 5.8. We estimate the various parts (five parts) issued from the differentiations of $s^\alpha \Phi(s)$ to the first and second order, by applying the upper bounds (6.35) and (6.36) and the fact that $0 < \alpha < 1$. Notice that

$$\int_0^\infty (s^{\alpha-1} + s^{\alpha-2})(s \wedge s^{-1})\, \mathrm{d}s = \frac{1}{1+\alpha} + \frac{1}{\alpha} + \frac{1}{1-\alpha} + \frac{1}{2-\alpha} =: \kappa_\alpha \,.$$

Grouping two of the terms issued from $(s^\alpha \Phi)'$, $(s^\alpha \Phi)''$ and using (6.36), we have

$$\Bigl| \int_0^\infty s^{\alpha-1} \Phi(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s \Bigr| + \Bigl| \int_0^\infty s^{\alpha-2} \Phi(s)\, \mathrm{e}^{-2\mathrm{i}\pi st}\, \mathrm{d}s \Bigr| \leqslant \kappa \kappa_\alpha \,,$$

we also have $\int_0^\infty (s^\alpha + s^{\alpha-1}) |\varphi_\theta(s)|\, \mathrm{d}s \leqslant \kappa \kappa_\alpha$ for two other terms by (6.35), and finally for each $\phi = \phi_j$, $j = 1, 2$, decreasing on the positive side of the real line, we know by Lemma 5.9 that

$$\int_0^\infty s^\alpha |\phi'(s)|\, \mathrm{d}s = \alpha \int_0^{+\infty} s^{\alpha-1} \phi(s)\, \mathrm{d}s < +\infty \,,$$

which permits us to close this list of estimates for $\varphi_\theta = \phi_1 - \phi_2$. It follows that for every $t > 0$, we have

$$\Bigl| D_t^\alpha \Bigl(\frac{m(t\theta)}{t} \Bigr) \Bigr| \leqslant \kappa'_\alpha (t^{-1} \wedge t^{-2}) \,,$$

with $\kappa'_\alpha \leqslant \kappa' (2\pi)^\alpha \kappa_\alpha$ independent of the direction θ. Recalling the definition (6.24) and since $0 < \alpha < 1$, we get

$$\begin{aligned} C_\alpha(m)^2 &= \sup_{\theta \in S^{n-1}} \bigl\| t \mapsto m(t\theta) \bigr\|_{L^2_\alpha}^2 = \sup_{\theta \in S^{n-1}} \int_0^{+\infty} \Bigl| t^{\alpha+1} D_t^\alpha \Bigl(\frac{m(t\theta)}{t} \Bigr) \Bigr|^2 \frac{\mathrm{d}t}{t} \\ &\leqslant (\kappa'_\alpha)^2 \Bigl(\int_0^1 (t^{\alpha+1} t^{-1})^2 \frac{\mathrm{d}t}{t} + \int_1^{+\infty} (t^{\alpha+1} t^{-2})^2 \frac{\mathrm{d}t}{t} \Bigr) \\ &= (\kappa'_\alpha)^2 \Bigl(\int_0^1 t^{2\alpha-1}\, \mathrm{d}t + \int_1^{+\infty} t^{2\alpha-3}\, \mathrm{d}t \Bigr) \\ &= (\kappa'_\alpha)^2 \Bigl(\frac{1}{2\alpha} + \frac{1}{2-2\alpha} \Bigr) < +\infty \,. \end{aligned}$$

One thus chooses $\alpha \in (1/2, 1)$ arbitrary and applies Carbery's Proposition 6.14 (1), which gives the boundedness on $L^2(\mathbb{R}^n)$ of the maximal operator associated to the difference kernel $K = K_{lc} - \widetilde{P}$. We get in this way that the maximal operator $\mathrm{M}_{K_{lc}}$ is bounded on $L^2(\mathbb{R}^n)$ by a constant independent of the dimension n. $\square$

7. The Detlef Müller article

Müller [59] introduces a geometrical parameter $Q(C)$ associated to every symmetric convex body C in $\mathbb{R}^n$. When C is isotropic of volume 1, this parameter $Q(C)$ is equal to the maximum of $(n-1)$-dimensional volumes of hyperplane projections of C. Müller shows that in the class $\mathcal{C}(\lambda)$ consisting of Cs for which $Q(C)$ and the isotropy constant $L(C)$ are bounded by a given λ, the existence for the maximal operator M_C associated to C of an $L^p(\mathbb{R}^n)$ bound, uniform in n, can be pushed to every value $p > 1$ with a constant $\kappa(p, \lambda)$ depending on p and λ only. This removes — in a way — the restriction $p > 3/2$ imposed by Bourgain and Carbery.

We have seen in (5.1) and (5.3) that when C_0 is isotropic of volume 1 in $\mathbb{R}^n$, then the dilate $C_1 = r_0 C_0$ with $r_0 = L(C_0)^{-1}$ is isotropic and normalized by variance. The proof of Müller will actually make use of a parameter $q(C_1)$ equal to the supremum in $\theta \in S^{n-1}$ of the masses of the signed measures $\theta \cdot \nabla K_{C_1}$. We shall see that for θ of norm one, the mass of the measure $\theta \cdot \nabla K_{C_1}$, the directional derivative in the sense of distributions of the probability measure μ_{C_1}, is given by

$$\frac{2|P_\theta C_1|_{n-1}}{|C_1|_n} = 2 r_0^{-n} r_0^{n-1} |P_\theta C_0|_{n-1} \leqslant \frac{2}{r_0} Q(C_0) = 2L(C_0)Q(C_0)\,,$$

where P_θ is the orthogonal projection onto the hyperplane $\theta^\perp$. For every symmetric convex set C, we let C_0 be an isotropic position of volume 1 for C and we set

$$q(C) = 2L(C_0)Q(C_0)\,. \tag{7.1}$$

Müller [59, Section 3] proves that $q(C)$ is uniformly bounded for the family of unit balls B_n^q of ℓ_n^q, $1 \leqslant q < +\infty$ fixed and $n \in \mathbb{N}^*$. This is easy when $q = 2$. By (5.4), we know that the Euclidean ball $B_{n,V}$ in $\mathbb{R}^n$ normalized by variance has a radius $r_{n,V}$ equal to $\sqrt{n+2}$, hence by the log-convexity of the Gamma function we get

$$\begin{aligned} q(B_n^2) = \sup_{\theta \in S^{n-1}} \frac{2|P_\theta B_{n,V}|_{n-1}}{|B_{n,V}|_n} &= \frac{2\omega_{n-1}}{r_{n,V}\,\omega_n} = \frac{2\Gamma(n/2+1)}{\sqrt{\pi(n+2)}\,\Gamma(n/2+1/2)} \\ &\leqslant \frac{2\Gamma(n/2+1/2)^{1/2}\,\Gamma(n/2+3/2)^{1/2}}{\sqrt{\pi(n+2)}\,\Gamma(n/2+1/2)} = 2\sqrt{\frac{n+1}{2\pi(n+2)}} < \sqrt{\frac{2}{\pi}}\,. \end{aligned}$$

Given a kernel K integrable on $\mathbb{R}^n$ and having partial derivatives $\partial_j K$ in the sense of distributions that are (signed) measures μ_j, for $j = 1, \ldots, n$, we define the *directional variation* $V(K)$ of K by

$$V(K) := \sup_{\theta \in S^{n-1}} \bigl\| \theta \cdot \nabla K \bigr\|_1 = \sup_{\theta \in S^{n-1}} \Bigl\| \sum_{j=1}^n \theta_j \mu_j \Bigr\|_1 . \tag{7.2}$$

We will show at Lemma 7.10 that $V(K_C) = q(C)$ when C is an isotropic symmetric convex body normalized by variance. For the $N(0, \mathrm{I}_n)$ Gaussian density γ_n, we see that $V(\gamma_n) = \int_{\mathbb{R}^n} |x \cdot \mathbf{e}_1| \, \mathrm{d}\gamma_n(x) = \int_{\mathbb{R}} |u| \, \mathrm{d}\gamma_1(u) = \sqrt{2/\pi}$. Notice that

$$V(K_{(t)}) = t^{-1} V(K), \quad t > 0, \quad \text{ and } \ V(K * \mu) \leqslant V(K) \tag{7.3}$$

for any probability measure μ on $\mathbb{R}^n$. Since $V(\gamma_n)$ is independent of n, it follows from the subordination formula (1.30) that the same is true for the Poisson kernel $P_1^{(n)}$ on $\mathbb{R}^n$ expressed in (1.32). Precisely, because G_s in (1.30) is a $N(0, s\mathrm{I}_n)$ Gaussian measure, we have $V(G_s) = s^{-1/2} V(\gamma_n)$ by (7.3) and we first get

$$V(P_1^{(n)}) \leqslant \int_0^{+\infty} V(G_s) \frac{s^{-3/2}}{\sqrt{2\pi}} \, \mathrm{e}^{-1/(2s)} \, \mathrm{d}s = \int_0^{+\infty} \frac{\mathrm{e}^{-1/(2s)}}{\pi} \frac{\mathrm{d}s}{s^2} = \frac{2}{\pi} , \tag{7.4}$$

but actually $V(P_1^{(n)}) = 2/\pi$ since for each $x \in \mathbb{R}^n$, all gradients $\nabla G_s(x)$, $s > 0$, are nonnegative multiples of the same vector $-x$. This equality is of course also easy to derive by a direct calculation on the Poisson density.

Besides the appearance of the parameter $q(C)$, Müller's proof draws on estimates such as (6.1.**H**), but extended to more derivatives of the Fourier transform m_C of K_C. That bounding more derivatives leads to improved results was already seen in Bourgain [11], who obtained a dimension free bound in $L^p(\mathbb{R}^n)$ for all $p > 1$ in the case of the maximal operator M_C of ℓ_n^q balls when q is an even integer. We shall consider a probability density K_g on $\mathbb{R}^n$ or more generally an integrable kernel K_g, with a Fourier transform m_g satisfying that for every integer $j \geqslant 0$, there exists a constant $\delta_{j,g}$ such that

$$\Bigl| \frac{\mathrm{d}^j}{\mathrm{d}t^j} m_g(t\theta) \Bigr| \leqslant \frac{\delta_{j,g}}{1 + t} , \quad \theta \in S^{n-1}, \ t > 0 . \tag{7.5.$\mathbf{H}_\infty$}$$

Actually, for each specific value $p \in (1, 3/2]$, bounding M_C in $L^p(\mathbb{R}^n)$, knowing that $q(C) \leqslant \lambda$, requires a certain finite number of estimates from the infinite list (7.5.$\mathbf{H}_\infty$), and this number increases to infinity when p tends to 1. We let

$$\Delta_k = \sum_{j=0}^k \delta_{j,g} . \tag{7.6}$$

The "radial" estimate (7.5.$\mathbf{H}_\infty$) implies $|\mathrm{d}^j/(\mathrm{d}t^j) m_g(t\xi)| \leqslant \delta_{j,g} |\xi|^j/(1+|t\xi|)$ for $\xi \neq 0$. It is natural to disregard $\xi = 0$ in a radial method, but when $j > 0$, we can extend continuously $\xi \mapsto \mathrm{d}^j/(\mathrm{d}t^j) m_g(t\xi)$ by giving the value 0 at $\xi = 0$.

Theorem 7.1 (Müller [59]). — *For every $p \in (1, +\infty]$ and $\lambda > 0$, there exists a constant $\kappa(p, \lambda)$ independent of n such that*

$$\|\mathrm{M}_{K_{lc}} f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa(p, \lambda) \, \|f\|_{L^p(\mathbb{R}^n)}$$

if K_{lc} is an isotropic symmetric log-concave probability density on $\mathbb{R}^n$, normalized by variance and with $V(K_{lc}) \leqslant \lambda$. In particular, for every symmetric convex body C in $\mathbb{R}^n$ such that $q(C) \leqslant \lambda$, one has $\|\mathrm{M}_C f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa(p, \lambda) \, \|f\|_{L^p(\mathbb{R}^n)}$. When $p \in (1, 2]$, we can write more precisely

$$\|\mathrm{M}_{K_{lc}} f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa(p)(1 + \lambda^{2/p-1}) \, .$$

If a probability density K_g satisfies (7.5.$\mathbf{H}_\infty$) *and if $p \in (1, 2]$, then we have*

$$\|\mathrm{M}_{K_g} f\|_{L^p(\mathbb{R}^n)} \leqslant \kappa_p \, \Delta_{k_0(p)}^{1-1/p} \Delta_1^{1-1/p} (1 + V(K_g)^{2/p-1}), \ \ \textit{with} \ \ k_0(p) < p/(p-1) \, .$$

The subsequent proof furnishes for the constant κ_p in the line above an order exponential in $q = p/(p-1)$ that is certainly not right, see Remarks 7.13 and 7.14. The case $p > 3/2$ is already known, with $\kappa(p, \lambda)$ independent of λ, see Theorem 6.2 and Proposition 6.3. We know by Lemma 5.11 that isotropic symmetric log-concave probability densities satisfy (7.5.$\mathbf{H}_\infty$) with absolute constants $(\delta_{j,c})_{j=0}^\infty$. We shall thus concentrate on the K_g case and on values $p \in (1, 3/2]$. Taking Carbery's results into account, the following proposition will be (essentially) enough for proving Müller's theorem.

Proposition 7.2 ([59, Proposition 1]). — *Let K_g be an integrable kernel on $\mathbb{R}^n$ satisfying* (7.5.$\mathbf{H}_\infty$) *and let m_g be its Fourier transform. For every $\alpha \in (0, 1)$ and every $p \in (1, +\infty)$, the multiplier $(\xi \cdot \nabla)^\alpha m_g(\xi)$ in* (6.18.∇^α) *admits on $L^p(\mathbb{R}^n)$ a bound that depends upon p, α, $\mathbf{d} = (\delta_{j,g})_{j=0}^\infty$ and $V(K_g)$, but not on the dimension n. When $p \in (1, 2]$ and if $\|K_g\|_{L^1(\mathbb{R}^n)} \leqslant 1$, we can write*

$$\|(\xi \cdot \nabla)^\alpha m_g(\xi)\|_{p \to p} \leqslant 1 + \kappa(\alpha, p) \, \Delta_{k(p)}^{(4/3)(1-1/p)} \big(1 + \delta_{0,g}^{(2/3)(1-1/p)} V(K_g)^{2/p-1}\big) \, ,$$

with $k(p) = \lceil 3p/(4p-4) \rceil$.

The case $p = 2$ follows easily from Parseval (2.12.$\mathbf{P}$) by (6.1.$\mathbf{H}$) and (6.20). The result for $p \geqslant 2$ can be obtained by duality from the case $1 < p \leqslant 2$.

Proof of Theorem 7.1. — Let $p \in (1, 2)$ be given. We then choose $p_0 \in (1, p)$ and $\alpha \in (1/p_0, 1)$ as being functions of p, for example $p_0 = (2p + 2)/(5 - p)$ and $\alpha = (p + 7)/(4p + 4)$. We apply in $L^{p_0}(\mathbb{R}^n)$ the part (2) of Proposition 6.14 to the kernel $K = K_g - P$. We know by Proposition 7.2

that $(\xi \cdot \nabla)^\alpha m_g(\xi)$ is bounded on $L^{p_0}(\mathbb{R}^n)$ by a function of $V(K_g)$ and we will check in Section 7.2 that $(\xi \cdot \nabla)^\alpha \widehat{P}(\xi)$ is also bounded on $L^{p_0}(\mathbb{R}^n)$ by some $\pi_{\alpha,p_0} = \pi(p)$. It follows for $m = m_g - \widehat{P}$ that

$$\|(\xi \cdot \nabla)^\alpha m(\xi)\|_{p_0 \to p_0} \leqslant \kappa_0(p, \mathbf{d})(1 + V(K_g)^{2/p_0 - 1}) \leqslant \kappa_0(p, \mathbf{d})(1 + \lambda^{2/p_0 - 1}),$$

with $\kappa_0(p, \mathbf{d}) \leqslant \kappa(p) \Delta_{k(p_0)}^{(4/3)(1-1/p_0)} \delta_{0,g}^{(2/3)(1-1/p_0)}$, where $\Delta_j \geqslant \delta_{0,g} \geqslant 1$ because K_g here is a probability density. We obtain in this way that

$$f \mapsto W_1 f := \sup_{1 \leqslant u \leqslant 2} |K_{(u)} * f|$$

is bounded on $L^{p_0}(\mathbb{R}^n)$. This was the only missing information for deducing from Proposition 6.6 that M_K is bounded on $L^p(\mathbb{R}^n)$ when $1 < p \leqslant 3/2$. Indeed, with the notation of Section 6.1, let $T_{j,v}$ be the convolution with $K_{(2^j v)}$, $v \in [1, 2]$ and let T_j be as in (6.2). By Proposition 6.14(2), we have for every $j \in \mathbb{Z}$ that

$$\begin{aligned} \|T_j\|_{p_0 \to p_0} = \|T_0\|_{p_0 \to p_0} &= \|W_1\|_{p_0 \to p_0} \\ &\leqslant \kappa_{\alpha,p_0} \big(2 + \|(\xi \cdot \nabla)^\alpha m(\xi)\|_{p_0 \to p_0}\big), \end{aligned} \tag{7.7}$$

with κ_{α,p_0} from (6.25). We bound it by $C''_{p_0}(\lambda) := \kappa_{\alpha,p_0}\big(2 + \kappa_0(p, \mathbf{d})(1 + \lambda^{2/p_0 - 1})\big)$. By (6.4), with p_0 already set and $r_0 = 2p/(p + 2 - p_0)$ function of p and p_0, we get

$$\|M_K\|_{p \to p} \leqslant (C_{r_0})^{2\gamma/p_0} C''_{p_0}(\lambda)^\gamma \Big(\sum_{k \in \mathbb{Z}} a_k^{(1-\gamma)p/2}\Big)^{2/p} + 2C'_p, \tag{7.8}$$

where $\gamma = [1/p - 1/2]/[1/p_0 - 1/2] = (p+1)/(2p)$. The constants C_{r_0} in $(A_\mathbf{0})$, C'_p in $(A_\mathbf{1})$ depend only on p, p_0 and r_0, hence on p alone, and they exist regardless of $p > 3/2$ or not. By Section 6.5, we know that under (6.1.**H**), the $(a_k)_{k \in \mathbb{Z}}$ in $(A_\mathbf{3})$ satisfy $a_k \leqslant (\delta_{0,g} + \delta_{1,g}) a_{\alpha,k}$ with $(a_{\alpha,k})_{k \in \mathbb{Z}}$ universal. We obtain

$$\begin{aligned} \|\mathrm{M}_{K_g}\|_{p \to p} &\leqslant \|M_K\|_{p \to p} + \kappa_p \\ &\leqslant \kappa(p, \mathbf{d})(1 + \lambda^{2(1/p_0 - 1/2)\gamma}) = \kappa(p, \mathbf{d})(1 + \lambda^{2/p - 1}), \end{aligned}$$

with $1 - 1/p_0 = (3p - 3)/(2p + 2)$, $k(p_0) = \lceil (p + 1)/(2p - 2) \rceil < p/(p - 1)$, and

$$\begin{aligned} \kappa(p, \mathbf{d}) &\leqslant \kappa(p) \big(\Delta_{k(p_0)}^{(4/3)(1-1/p_0)} \delta_{0,g}^{(2/3)(1-1/p_0)}\big)^\gamma \Delta_1^{1-\gamma} \\ &\leqslant \kappa(p) \Delta_{k_0(p)}^{1-1/p} \Delta_1^{1-1/p}. \end{aligned}$$

□

7.1. The Müller strategy

Müller prefers to work with another version i^w of the fractional integral I^w from (6.9). This version is defined when $\operatorname{Re} w > 0$, beginning this time with $f \in C^\infty(\mathbb{R})$, by the formula

$$(i^w f)(t) = \frac{1}{\Gamma(w)} \int_t^2 (u-t)^{w-1} f(u) \, \mathrm{d}u \,, \quad t \leqslant 2 \,.$$

The chosen limit 2 is rather arbitrary, but will be quite convenient for the computations that follow, in particular because $(2-1)^w = 1$ for every w. Integrating by parts as we did for I^w in Section 6.2, we get

$$(i^w f)(t) = \frac{(2-t)^w f(2)}{\Gamma(w+1)} - \frac{1}{\Gamma(w+1)} \int_t^2 (u-t)^w f'(u) \, \mathrm{d}u \,.$$

This new formula makes sense for $\operatorname{Re} w > -1$ and defines a fractional derivative d^z if $z = -w$ and $\operatorname{Re} z < 1$, by setting

$$(d^z f)(t) = \frac{(2-t)^{-z} f(2)}{\Gamma(1-z)} - \frac{1}{\Gamma(1-z)} \int_t^2 (u-t)^{-z} f'(u) \, \mathrm{d}u \,, \quad t \leqslant 2 \,. \tag{7.9}$$

Notice that $(d^0 f)(t) = f(2) - \int_t^2 f'(u) \, \mathrm{d}u = f(t)$. Continuing integration by parts as in Section 6.2, we get successive formulas defining $d^z f$, for each integer k, which make sense for $\operatorname{Re} z < k$ and extend each other. Gluing them together, we can define entire functions of z for every t fixed and every given function $f \in C^\infty(\mathbb{R})$, for example $(d^z \mathbf{1})(1) = 1/\Gamma(1-z)$ if $f = \mathbf{1}$. Suppose that $\operatorname{Re} z < 0$. From

$$(d^z f)(t) = \frac{1}{\Gamma(-z)} \int_t^2 (u-t)^{-z-1} f(u) \, \mathrm{d}u \,,$$

we get for every integer $k \geqslant 1$ that

$$(d^z f)(t) = E_k(z,t) + (-1)^k \frac{1}{\Gamma(k-z)} \int_t^2 (u-t)^{-z+k-1} f^{(k)}(u) \, \mathrm{d}u \,, \tag{7.10}$$

a formula to be compared with (6.16), and where $E_k(z,t)$ is equal to

$$E_k(z,t) = \sum_{j=0}^{k-1} (-1)^j \, \frac{(2-t)^{-z+j} f^{(j)}(2)}{\Gamma(j+1-z)} \,.$$

If z is in $\mathbb{C}$, $t \leqslant 2$ and $\operatorname{Re} z < k$, we can take (7.10) as definition for $(d^z f)(t)$.

When $-1 < \operatorname{Re} z < 0$, $f \in \mathcal{S}(\mathbb{R}^n)$ and $t < 2$, we see that

$$\begin{aligned} (D^z f)(t) - (d^z f)(t) &= ([I^{-z} - i^{-z}] f)(t) \\ &= \frac{1}{\Gamma(-z)} \int_2^{+\infty} (u-t)^{-z-1} f(u) \, \mathrm{d}u \,. \end{aligned} \tag{7.11}$$

This equality can be extended by analytic continuation to every $z \in \mathbb{C}$ with $\operatorname{Re} z > -1$, or it can be proved by successive integrations by parts. In particular, one has $(d^N f)(t) = (D^N f)(t) = (-1)^N f^{(N)}(t)$ for every integer $N \geqslant 0$ because $\Gamma(-N)^{-1} = 0$. As we did for D^α, when the function of t does not have an explicit name, we use the notation $d_t^\alpha f(2t)$, and $d_t^\alpha f(2t)\big|_{t=1}$ for the value at $t = 1$.

Lemma 7.3 (Müller [59]). — *Let m denote the Fourier transform of a kernel K integrable on $\mathbb{R}^n$. For every $\alpha \in (0, 1)$, the difference*

$$(\xi \cdot \nabla)^\alpha m(\xi) - d_t^\alpha m(t\xi)\big|_{t=1}, \quad \xi \in \mathbb{R}^n,$$

is a multiplier on $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant +\infty$, with a norm bounded by $\|K\|_{L^1(\mathbb{R}^n)}$.

Proof. — By (6.18.∇^α) we have $(\xi \cdot \nabla)^\alpha m(\xi) = D_t^\alpha m(t\xi)\big|_{t=1}$. From (7.11), we get

$$(\xi \cdot \nabla)^\alpha m(\xi) - d_t^\alpha m(t\xi)\big|_{t=1} = \frac{1}{\Gamma(-\alpha)} \int_2^{+\infty} (u-1)^{-\alpha-1} m(u\xi)\, \mathrm{d}u\,.$$

The result follows by Lemma 2.1, since

$$\frac{1}{|\Gamma(-\alpha)|} \int_2^{+\infty} (u-1)^{-\alpha-1}\, \mathrm{d}u = \frac{1}{|-\alpha\Gamma(-\alpha)|} = \frac{1}{\Gamma(1-\alpha)} < 1\,. \qquad \square$$

Thanks to the reduction from $(\xi \cdot \nabla)^\alpha m(\xi)$ to $d_t^\alpha m(t\xi)\big|_{t=1}$ given by Lemma 7.3, one can transform the condition (2) of Proposition 6.14. The objective now is to control the action on $L^p(\mathbb{R}^n)$ of the multiplier $d_t^\alpha m_g(t\xi)\big|_{t=1}$, for some fixed $\alpha \in (1/p, 1)$ denoted by $\alpha = 1 - \varepsilon$, where $\varepsilon > 0$ gets arbitrarily small when p tends to 1. Müller embeds the "objective" into the holomorphic family of multipliers

$$m_z^\varepsilon(\xi) = (1 + |\xi|)^{1-\varepsilon-z}\, d_t^z m_g(t\xi)\big|_{t=1}, \quad \operatorname{Re} z > -1, \tag{7.12}$$

and applies the complex interpolation scheme described in Section 3.2. For the value $z = \alpha = 1 - \varepsilon$, one has

$$m_\alpha^\varepsilon(\xi) = m_{1-\varepsilon}^\varepsilon(\xi) = d_t^{1-\varepsilon} m_g(t\xi)\big|_{t=1} = d_t^\alpha m_g(t\xi)\big|_{t=1},$$

which is the objective to be controlled. Müller studies this holomorphic family for $z \in \mathbb{C}$ varying in a strip of the form $-\varepsilon \leqslant \operatorname{Re} z \leqslant \nu$, with $\nu > 0$ real. He shows by rather long and delicate calculations that the multipliers $m_z^\varepsilon(\xi)$ are bounded functions of $\xi \in \mathbb{R}^n$, for all z in this strip, not uniformly in z, but with a $L^\infty(\mathbb{R}^n)$ norm of order $\Gamma(z)^{-1}$. This allows him to control the action on $L^2(\mathbb{R}^n)$, which is used for one end of the interpolation scale, the one corresponding to $\operatorname{Re} z = \nu$.

The other end of the scale is $\operatorname{Re} z = -\varepsilon$, where the operator associated to

$$\begin{aligned} m^{\varepsilon}_{-\varepsilon+\mathrm{i}\tau} &= (1+|\xi|)^{1-\mathrm{i}\tau}\, d_t^{-\varepsilon+\mathrm{i}\tau} m_g(t\xi)\big|_{t=1} \\ &= (1+|\xi|)^{-\mathrm{i}\tau}(1+|\xi|)\, d_t^{-\varepsilon+\mathrm{i}\tau} m_g(t\xi)\big|_{t=1} \end{aligned}$$

involves a "small" fractional integration $d^{-\varepsilon+\mathrm{i}\tau}$ of order ε, and a multiplication on the Fourier side by $1+|\xi|$. We will show that these multipliers $m^{\varepsilon}_{-\varepsilon+\mathrm{i}\tau}$ are bounded on all the spaces $L^r(\mathbb{R}^n)$, $1 < r < +\infty$. In order to do it, we shall have to work mainly on the multiplier $|\xi| m_g(\xi)$. The parameter $V(K_g)$ appears when bounding the action of this multiplier on $L^r(\mathbb{R}^n)$, and the proof will use the dimensionless estimates for the Riesz transforms given in (2.22). Next, given p in $(1,2]$, we choose $p_0 \in (1,p)$, $\alpha \in (1/p_0, 1)$, and $\nu > \alpha$ which is a function of p, p_0, α. By interpolation between $L^2(\mathbb{R}^n)$ (when $\operatorname{Re} z = \nu$) and $L^{p_0}(\mathbb{R}^n)$ (when $\operatorname{Re} z = -\varepsilon$), we shall obtain for the value $\alpha = 1-\varepsilon$ the boundedness on $L^p(\mathbb{R}^n)$ of the multiplier $m^{\varepsilon}_{\alpha}(\xi)$ that is our "objective", thus proving Proposition 7.2.

Let us comment on the formulas for the Müller multipliers. We know by (5.19) in Corollary 5.13 that differentiating N times the function $t \mapsto m_g(t\xi)$ introduces a factor of order $(1+|\xi|)^{N-1}$, which must be compensated for being in a position to apply Parseval for the L^2 bound, using (2.12.**P**) as usual. This is done by multiplying by $(1+|\xi|)^{1-\varepsilon-\nu}$ when $z = \nu$. On the other hand, we do not want a compensating factor when $z = \alpha$, where we want to precisely recover our objective. The compensation will thus be of the form $(1+|\xi|)^{az+b}$, with $a\nu + b = 1-\varepsilon-\nu$ and $a\alpha + b = 0$. We then get a "compensating factor" with a positive power of $|\xi|$ for $\operatorname{Re} z < \alpha$, which becomes an additional problem and requires more work.

The interpolation strip technique has been often employed by Stein. For example, in [73, Chap. III, §3], for studying the maximal function $\sup_{t>0} |P_t f|$ of general semi-groups, Stein works on a strip S of the form $-1 \leqslant \operatorname{Re} z \leqslant N$. If $z = -1$, he considers that the maximal inequality of Hopf concerns the derivative of order -1 of the semi-group, that is to say, its antiderivative (multiplied by $t^z = t^{-1}$)

$$t^{-1} D_t^{-1}(P_t f) = \frac{1}{t}\int_0^t (P_s f)\,\mathrm{d}s\,.$$

By Hopf, this operator is known to be L^p bounded, $1 < p < +\infty$. Stein must check in addition that the extension to complex values in the vertical line $z = -1 + \mathrm{i}\tau$ also gives bounded operators on $L^p(\mathbb{R}^n)$.

Stein's objective is to study the maximal function of the semi-group itself, which corresponds to the derivative of order $z = 0$. In order to do this, he interpolates between Hopf in L^{p_0}, $p_0 < p < 2$, for $\operatorname{Re} z = -1$, and an L^2 estimate of derivatives of the semi-group, for $\operatorname{Re} z = N$.

For each integer k, the quantity $t^k D_t^k(P_t f)$ appears in the Littlewood–Paley function $g_k(f)$, so one can control in L^2 its maximal function, see Section 2.1.1. The holomorphic family is then defined by $z \mapsto t^z D_t^z(P_t f)$, $z \in S$, for a suitable version D^z of fractional differentiation.

The general strategy above was already applied in [71] to the discrete case.

7.2. Model of proof: the Poisson case

For proving Theorem 7.1, we have to apply Carbery's Proposition 6.14 (2) to the difference $K = K_g - P$. Müller shows that $(\xi \cdot \nabla)^\alpha m_g(\xi)$ acts on $L^p(\mathbb{R}^n)$ when $0 < \alpha < 1$ and $1 < p < +\infty$, and we need to verify that the corresponding multiplier $(\xi \cdot \nabla)^\alpha \widehat{P}(\xi)$ for the Poisson kernel P also acts on $L^p(\mathbb{R}^n)$, $1 < p < +\infty$, with bounds independent of the dimension n. This could be covered by Proposition 7.2, by observing that the Poisson kernel $P_1^{(n)}$ in (1.32), with Fourier transform $\mathrm{e}^{-2\pi|\xi|}$, clearly satisfies (7.5.$\mathbf{H}_\infty$) and has $V(P_1^{(n)})$ bounded independently of n according to (7.4). We actually prefer to take an opportunity to examine the structure of Müller's proof in a simple case. When $\alpha \in (0, 1)$, we could find a shorter specific proof, but the longer one that is given below provides a better introduction to what follows in this Section 7.

One sees that $(\xi \cdot \nabla)^\alpha \widehat{P}(\xi) = (2\pi|\xi|)^\alpha \,\mathrm{e}^{-2\pi|\xi|}$, either by applying (6.13) that gives $D_t^\alpha \,\mathrm{e}^{-\lambda|t|} = \lambda^\alpha \,\mathrm{e}^{-\lambda|t|}$ for $\lambda, t > 0$, or by making use of the residue theorem.

Indeed, according to (6.18.∇^α) with $\xi = |\xi|\theta$, one has

$$(\xi \cdot \nabla)^\alpha \widehat{P}(\xi) = \int_{\mathbb{R}} (2\mathrm{i}\pi s|\xi|)^\alpha \varphi_\theta(s)\,\mathrm{e}^{-2\mathrm{i}\pi s|\xi|}\,\mathrm{d}s = \int_{\mathbb{R}} (2\mathrm{i}\pi s|\xi|)^\alpha \frac{\mathrm{e}^{-2\mathrm{i}\pi s|\xi|}}{\pi(1+s^2)}\,\mathrm{d}s\,,$$

that can be computed using a contour formed of $[-R, R]$ with $R > 1$, and of a half-circle of radius R centered at 0, located in the lower complex half-plane.

We are going to bound the action on $L^p(\mathbb{R}^n)$ of the multiplier $|\xi|^\alpha \,\mathrm{e}^{-|\xi|}$ by the interpolation scheme of Section 3.2. Consider the holomorphic family of multipliers

$$\mathcal{P}_z(\xi) = |\xi|^z \,\mathrm{e}^{-|\xi|}, \quad \operatorname{Re} z \geqslant 0\,,\ \xi \in \mathbb{R}^n.$$

We will interpolate between $L^2(\mathbb{R}^n)$ and $L^{p_0}(\mathbb{R}^n)$, $p_0 > 1$ close to 1. For proving the boundedness on $L^2(\mathbb{R}^n)$, it is enough by (2.12.$\mathbf{P}$) to see that the function $\xi \mapsto |\xi|^z \,\mathrm{e}^{-|\xi|}$ is bounded when ξ varies in $\mathbb{R}^n$, and since this function is radial, its supremum is independent of n. If we write $z = a + \mathrm{i}b$, $a \geqslant 0$, we have

$$\begin{aligned}\sup_{\xi \in \mathbb{R}^n} \sup_{\operatorname{Re} z = a} |\mathcal{P}_z(\xi)| &= \sup_{\xi \in \mathbb{R}^n, b \in \mathbb{R}} \bigl\{ \bigl| |\xi|^{a+\mathrm{i}b} \bigr| \,\mathrm{e}^{-|\xi|} \bigr\} \\ &= \sup_{r \geqslant 0} \{ r^a \,\mathrm{e}^{-r} \} = a^a \,\mathrm{e}^{-a} \,. \end{aligned} \tag{7.13}$$

We work on a line $\operatorname{Re} z = \nu$, with ν "large", for dealing with the L^2 boundedness, and the other line is $\operatorname{Re} z = 0$. For the values $z = 0 + \mathrm{i}b$, b real, we know by (2.18) when $1 < r < +\infty$ that the norm on $L^r(\mathbb{R}^n)$ of the multiplier $|\xi|^{\mathrm{i}b}$ is bounded by $\lambda_r \,\mathrm{e}^{\pi |b|/2}$, with λ_r independent of the dimension n. The multiplier $\mathrm{e}^{-|\xi|}$ corresponds to the convolution with a Poisson probability measure, so it is bounded by 1 on $L^r(\mathbb{R}^n)$ when $1 \leqslant r \leqslant +\infty$ by (2.13).

Let $\alpha \in (0,1)$ be given. Consider $p \in (1,2)$, introduce $p_0 = 2p/(p+1) \in (1,p)$, making $1/p_0$ the midpoint between 1 and $1/p$. Then with $\theta = p - 1 \in (0,1)$ we can check that $1/p = (1-\theta)/p_0 + \theta/2$, and we define ν by the condition $\alpha = (1-\theta).0 + \theta\nu$, namely, we set $\nu = \alpha/(p-1)$. Let T_z be the operator associated to the multiplier $\mathcal{P}_z$. We have to estimate the norm of T_α on L^p by bounding $\langle T_\alpha f, g \rangle$ uniformly for f in the unit ball of $L^p(\mathbb{R}^n)$ and g in the unit ball of the dual $L^q(\mathbb{R}^n)$, where $1/q + 1/p = 1$. Consider the holomorphic function

$$H : z \mapsto \langle T_z f_z, g_z \rangle$$

where f_z, g_z are as in (3.23). The bounds obtained for the family T_z do not allow us to apply directly the three lines Lemma 3.1, but Corollary 3.4 will do the job. We got at the boundary of the strip, for the norms $\|T_z\|_{p_0 \to p_0}$ when $\operatorname{Re} z = 0$, a bound of the form $O(\mathrm{e}^{\kappa |\operatorname{Im} z|})$. For every real number τ, the function H satisfies

$$|H(0 + \mathrm{i}\tau)| \leqslant \lambda_{p_0} \,\mathrm{e}^{\pi|\tau|/2} \quad \text{and also} \quad |H(\nu + \mathrm{i}\tau)| \leqslant \nu^\nu \,\mathrm{e}^{-\nu} \,.$$

By Corollary 3.4, the value $H(\alpha)$ is bounded uniformly by a quantity η depending on p_0, θ and on the width $w = \nu$ of the strip, hence on α, p only. As explained in (3.26), this gives then for the action of T_α on $L^p(\mathbb{R}^n)$ a bound $\|T_\alpha\|_{p \to p} \leqslant \eta$.

For applying Corollary 3.4, it remains to check that H has an admissible growth in $S = \{z : 0 < \operatorname{Re} z < \nu\}$. We may actually reduce the discussion to a function H bounded in the strip (but without universal estimate). Indeed, one can observe that all operators T_z, $z \in S$, are uniformly bounded on $L^2(\mathbb{R}^n)$, since $|\mathcal{P}_z(\xi)|$ is bounded by ν^ν for all $\xi \in \mathbb{R}^n$ and z in S by (7.13). We may limit ourselves to f, g continuous with compact support, so that f_z, g_z, $z \in S$, stay in a bounded subset of L^2, according to (3.24), implying that $H = H_{f,g}$ is bounded in the strip.

7.3. The interpolation part of Carbery's proof for Theorem 6.2

Proof. — In order to complete the proof of Theorem 6.2 and Proposition 6.3, it remains to show that the multiplier $(\xi \cdot \nabla)^\alpha m(\xi)$, where m is the Fourier transform of $K = K_g - P$, is bounded on $L^p(\mathbb{R}^n)$ for at least one value $\alpha > 1/p$ when $p > 3/2$. We have seen in the preceding Section 7.2 that $(\xi \cdot \nabla)^\alpha \widehat{P}(\xi)$ is bounded on $L^p(\mathbb{R}^n)$, we need only consider now $(\xi \cdot \nabla)^\alpha m_g(\xi)$. We will obtain the result by interpolating between the boundedness on $L^1(\mathbb{R}^n)$, for $\alpha_0 = -\varepsilon$, and the boundedness on $L^2(\mathbb{R}^n)$, for $\alpha_1 = 1 - \varepsilon$, of a certain holomorphic family $N_z(\xi)$ such that $N_\alpha(\xi)$ controls $(\xi \cdot \nabla)^\alpha m_g(\xi)$. If $p > 3/2$ is fixed, its conjugate q is < 3. We write

$$\frac{2}{3} > \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2}, \quad \text{thus} \quad \frac{\theta}{2} = 1 - \frac{1}{p} = \frac{1}{q}$$

and $\theta = 2/q > 2/3 > 1 - \theta/2$. One can then find $\varepsilon \in (0,1)$ small enough, and independent of the dimension n, so that

$$\alpha := (1-\theta)(-\varepsilon) + \theta(1-\varepsilon) = \theta - \varepsilon > 1 - \frac{\theta}{2} = \frac{1}{p}.$$

We need $0 < \varepsilon < 3\theta/2 - 1$, we can set for example $\varepsilon = 3\theta/4 - 1/2 = (p-3/2)/p$. By Lemma 7.3, it is enough to show that $d_t^\alpha m_g(t\xi)\big|_{t=1}$ is bounded on L^p. Consider the holomorphic family of multipliers (N_z), simpler than that of Müller, namely, $N_z(\xi) := d_t^z m_g(t\xi)\big|_{t=1}$ in the strip $-\varepsilon \leqslant \operatorname{Re} z \leqslant 1-\varepsilon$. When $\operatorname{Re} z < 0$, we have

$$N_z(\xi) = \frac{1}{\Gamma(-z)} \int_1^2 (u-1)^{-z-1} m_g(u\xi) \, \mathrm{d}u, \tag{7.14}$$

and in particular

$$N_{-\varepsilon+\mathrm{i}\tau}(\xi) = \frac{1}{\Gamma(\varepsilon - \mathrm{i}\tau)} \int_1^2 (u-1)^{\varepsilon-\mathrm{i}\tau-1} m_g(u\xi) \, \mathrm{d}u.$$

We see that

$$\int_1^2 \big|(u-1)^{\varepsilon-\mathrm{i}\tau-1}\big| \, \mathrm{d}u = \int_1^2 (u-1)^{\varepsilon-1} \, \mathrm{d}u = \varepsilon^{-1} < +\infty,$$

thus $N_{-\varepsilon+\mathrm{i}\tau}$ acts on L^1, with norm $\leqslant 2\varepsilon^{-1}(1+\tau^2)^{1/4-\varepsilon/2} \, \mathrm{e}^{\pi|\tau|/2}$, according to Lemma 2.1, to the inequality (3.12.Γ) for the Gamma function and since the L^1 norm of the kernel K_g is equal to 1. When $\operatorname{Re} z = 1 - \varepsilon$, we have by (7.9) that

$$N_{1-\varepsilon+\mathrm{i}\tau}(\xi) = \frac{m_g(2\xi)}{\Gamma(\varepsilon - \mathrm{i}\tau)} - \frac{1}{\Gamma(\varepsilon - \mathrm{i}\tau)} \int_1^2 (u-1)^{\varepsilon-\mathrm{i}\tau-1} \xi \cdot \nabla m_g(u\xi) \, \mathrm{d}u.$$

The kernel $N_{1-\varepsilon+\mathrm{i}\tau}$ is a bounded function of ξ, because we have $|m_g(2\xi)| \leqslant \delta_{0,g}$ and $\big|u\xi \cdot \nabla m_g(u\xi)\big| \leqslant \delta_{1,g}$ by (6.1.**H**). Using (3.12.Γ) we obtain

$$\begin{aligned}\big|N_{1-\varepsilon+\mathrm{i}\tau}(\xi)\big| &\leqslant \frac{1}{|\Gamma(\varepsilon-\mathrm{i}\tau)|}\Big(\delta_{0,g} + \int_1^2 \big|(u-1)^{\varepsilon-\mathrm{i}\tau-1}\big| \,\frac{\delta_{1,g}}{u}\,\mathrm{d}u\Big)\\ &\leqslant 2(\delta_{0,g}+\delta_{1,g})\,\varepsilon^{-1}(1+\tau^2)^{1/4-\varepsilon/2}\,\mathrm{e}^{\pi|\tau|/2}\,.\end{aligned}$$

This shows that the operator associated to $N_{1-\varepsilon+\mathrm{i}\tau}(\xi)$ is bounded on $L^2(\mathbb{R}^n)$ with a bound $O(\mathrm{e}^{\kappa|\tau|})$. We deal with this estimate as in the preceding Section 7.2, and we obtain by interpolation that $N_\alpha(\xi)$ is a $L^p(\mathbb{R}^n)$-multiplier. Remark 3.6 takes care of the polynomial factor $(1+\tau^2)^{1/4-\varepsilon/2} \leqslant (1+\tau^2)^{1/4}$. By Lemma 7.3 and (3.22) with $w=1$, $c_j = 1/4$, $u_j = \pi/2$, and since $\delta_{0,g} \geqslant 1$ we get

$$\begin{aligned}\|(\xi\cdot\nabla)^\alpha m_g(\xi)\|_{p\to p} &\leqslant 1 + \frac{2p}{p-3/2}\Big(\frac{3}{2}\Big)^{1/2}\,\mathrm{e}^{\pi/4}(\delta_{0,g}+\delta_{1,g})^{2-2/p}\\ &\leqslant \kappa_p \Delta_1^{2-2/p}\,.\end{aligned}$$

We now check that the function $H(z) = \langle N_z f_z, g_z\rangle$ of (3.25) has an admissible growth in $S = \{-\varepsilon \leqslant \operatorname{Re} z \leqslant 1-\varepsilon\}$. We may again observe that all kernels $N_z(\xi)$ are bounded functions of ξ. Indeed, $N_z(\xi)$ can be expressed in the whole strip by

$$N_z(\xi) = \frac{m_g(2\xi)}{\Gamma(-z+1)} - \frac{1}{\Gamma(-z+1)}\int_1^2 (u-1)^{-z}\xi\cdot\nabla m_g(u\xi)\,\mathrm{d}u\,, \quad \xi\in\mathbb{R}^n,$$

so that $|N_z(\xi)| \leqslant \kappa_{\varepsilon,\delta}(1+\tau^2)^{1/4}\,\mathrm{e}^{\pi|\tau|/2}$. Next, we can assume that the two functions f, g appearing in the definition of H are bounded with bounded support, and argue with (3.24) as at the end of Section 7.2, obtaining that $|H(z)| \leqslant \kappa\|N_z\|_{2\to 2} \leqslant \kappa'_\varepsilon(1+\tau^2)^{1/4}\,\mathrm{e}^{\pi|\tau|/2}$, a growth admissible for applying Corollary 3.4. □

We see pretty well why Müller finds a better result than the one given by the preceding argument, which suffices for Carbery's theorem. It is because Müller is able to make use of multipliers more difficult to handle, which contain an extra factor $|\xi|$ on the line $\operatorname{Re} z = -\varepsilon$, for example $m^\varepsilon_{-\varepsilon}(\xi) = (1+|\xi|)N_{-\varepsilon}(\xi)$ when $z=-\varepsilon$. This factor $|\xi|$ is precisely the one that will be treated by the geometrical parameter $q(C)$. On the other hand, Müller's approach is not better when $p > 3/2$, since the result is known in this case without assumption on $q(C)$.

Remark. — The factor $1/\Gamma(-z)$ in (7.14) is not purely decorative. Without it, $N_z(\xi)$ would have a "pole" at $z=0$, which is compensated by the zero of $1/\Gamma(z)$ at 0. One could perhaps get away here with a less sophisticated factor such as $z/(a-z)$, with a real and > 1. See also Remark 7.13.

7.4. Upper bounds for the functions $\xi \mapsto m_z^\varepsilon(\xi)$

We present a version of Müller's upper bounds for the functions m_z^ε defined in (7.12). Müller's bounds in [59] are not fully explicit since they use asymptotic estimates, but they do not contain the annoying factor ε^{-1} that our somewhat shorter proof introduces below.

Lemma 7.4 ([59, Lemma 2]). — *Assume that the kernel K_g integrable on $\mathbb{R}^n$ satisfies (7.5.$\mathbf{H}_\infty$), let $\varepsilon \in (0,1)$, let $\nu \geqslant 1-\varepsilon$ and set $\ell = \lceil \nu + \varepsilon \rceil$. For every $z \in \mathbb{C}$ such that $-\varepsilon \leqslant \operatorname{Re} z \leqslant \nu$, one has that*

$$\forall\, \xi \in \mathbb{R}^n, \quad |m_z^\varepsilon(\xi)| \leqslant \kappa_\nu \varepsilon^{-1} \Delta_\ell \big(1 + (\operatorname{Im} z)^2\big)^{\nu/2-1/4}\, \mathrm{e}^{\pi |\operatorname{Im} z|/2},$$

where $\kappa_\nu = 4\Gamma\big(\max(\nu,2)\big)\,(3/2)^{\nu-1/2}\,\mathrm{e}^{\pi/4}$ and where Δ_ℓ is defined at (7.6).

One of the difficulties in Müller's article is the following: with the operator D^α, we have been able to compute certain integrals by the residue theorem, on entire half-lines. The corresponding values for d^α are less pleasant, because they involve bounded segments, and quarters of circle at finite distance whose contribution is not zero. Let us mention another difficulty, somewhat related to the latter. If we know that $|D_t^z m(t\theta)| \leqslant \kappa(1+|t|)^{-1}$ for every t real and $\theta \in S^{n-1}$, then by the homogeneity relation (6.8) we get $|D_t^z m(t\xi)| \leqslant \kappa|\xi|^{\operatorname{Re} z}(1+|t\xi|)^{-1}$ for $\xi \in \mathbb{R}^n$, but this kind of behavior is not clear for d^z. The more delicate analysis of [59] will not be given here, but some special cases are rather easy. Indeed, the computation is not difficult when $\operatorname{Re} z = k - \varepsilon$, for every integer $k \geqslant 0$. We will however be able to deduce Lemma 7.4 from the easy cases that are treated in the next lemma.

Lemma 7.5. — *Assume that K_g is integrable on $\mathbb{R}^n$ and satisfies (7.5.$\mathbf{H}_\infty$). For every $\varepsilon \in (0,1)$, every integer $k \geqslant 0$ and $z \in \mathbb{C}$ such that $\operatorname{Re} z = k - \varepsilon$, one has*

$$\forall\, \xi \in \mathbb{R}^n, \quad |m_z^\varepsilon(\xi)| \leqslant 2\kappa'_k \varepsilon^{-1} \Delta_k \big(1 + (\operatorname{Im} z)^2\big)^{k^*/2-\varepsilon/2-1/4}\, \mathrm{e}^{\pi |\operatorname{Im} z|/2},$$

where $k^* = \max(k,1)$, $\kappa'_k \leqslant \Gamma(k-\varepsilon)$ for $k \geqslant 3$ and $\kappa'_0, \kappa'_1, \kappa'_2 \leqslant 1$.

Proof. — We first give the proof for $k = 0$, when $z = -\varepsilon + \mathrm{i}\tau$. We have

$$m_{-\varepsilon+\mathrm{i}\tau}^\varepsilon(\xi) = (1+|\xi|)^{1-\varepsilon-(-\varepsilon+\mathrm{i}\tau)} \frac{1}{\Gamma(\varepsilon-\mathrm{i}\tau)} \int_1^2 (u-1)^{\varepsilon-\mathrm{i}\tau-1} m_g(u\xi)\,\mathrm{d}u$$

and it follows that

$$|m_{-\varepsilon+\mathrm{i}\tau}^\varepsilon(\xi)| \leqslant \left|\frac{1}{\Gamma(\varepsilon-\mathrm{i}\tau)}\right| \int_1^2 (u-1)^{\varepsilon-1}(1+|\xi|)\,|m_g(u\xi)|\,\mathrm{d}u\,.$$

By (7.5.$\mathbf{H}_\infty$), we know that $|m_g(u\xi)| \leqslant \delta_{0,g}(1+|\xi|)^{-1}$ when $u \geqslant 1$, thus

$$|\Gamma(\varepsilon-\mathrm{i}\tau)\, m_{-\varepsilon+\mathrm{i}\tau}^\varepsilon(\xi)| \leqslant \delta_{0,g} \int_1^2 (u-1)^{\varepsilon-1}\,\mathrm{d}u = \frac{\delta_{0,g}}{\varepsilon}\,.$$

Using also (3.12.Γ), this simplest case reads as

$$\|m^{\varepsilon}_{-\varepsilon+\mathrm{i}\tau}\|_\infty \leqslant 2\delta_{0,g}\varepsilon^{-1}(1+\tau^2)^{1/4-\varepsilon/2}\,\mathrm{e}^{\pi|\tau|/2}, \quad \tau\in\mathbb{R},$$

and $1/4 = k^*/2 - 1/4$ here. For $k > 0$, we have by (7.10) with $z = k-\varepsilon+\mathrm{i}\tau$ that

$$d_t^{k-\varepsilon+\mathrm{i}\tau} m_g(t\xi)\big|_{t=1} = E_k + (-1)^k \frac{1}{\Gamma(\varepsilon-\mathrm{i}\tau)} \int_1^2 (u-1)^{\varepsilon-\mathrm{i}\tau-1} \frac{\mathrm{d}^k}{\mathrm{d}u^k} m_g(u\xi)\,\mathrm{d}u,$$

where

$$E_k = \sum_{j=0}^{k-1} (-1)^j \frac{\frac{\mathrm{d}^j}{\mathrm{d}u^j} m_g(u\xi)\big|_{u=2}}{\Gamma(j+1-k+\varepsilon-\mathrm{i}\tau)}.$$

By our assumption (7.5.$\mathbf{H}_\infty$), the function $u\mapsto m_g(u\xi)$ satisfies

$$\forall\, u\geqslant 1, \quad \Big|\frac{\mathrm{d}^j}{\mathrm{d}u^j} m_g(u\xi)\Big| = \Big|\frac{\mathrm{d}^j}{\mathrm{d}u^j} m_g\big(u|\xi|\theta\big)\Big| \leqslant \delta_{j,g}\frac{|\xi|^j}{1+|\xi|} \tag{7.15}$$

for each integer $j\geqslant 0$, if $\xi\neq 0$ and $\theta = |\xi|^{-1}\xi$. This yields

$$\begin{aligned}\Big|\int_1^2 (u-1)^{\varepsilon-\mathrm{i}\tau-1}\frac{\mathrm{d}^k}{\mathrm{d}u^k} m_g(u\xi)\,\mathrm{d}u\Big| &\leqslant \delta_{k,g}\int_1^2 (u-1)^{\varepsilon-1}\frac{|\xi|^k}{1+|\xi|}\,\mathrm{d}u \\ &= \frac{\delta_{k,g}}{\varepsilon}\frac{|\xi|^k}{1+|\xi|}.\end{aligned}$$

For the terms in the expression E_k, we have by (7.15) that

$$\Big|\frac{\mathrm{d}^j}{\mathrm{d}u^j} m_g(u\xi)\big|_{u=2}\Big| \leqslant \delta_{j,g}\frac{|\xi|^j}{1+|\xi|} \leqslant \delta_{j,g}\frac{(1+|\xi|)^j}{1+|\xi|} \leqslant \delta_{j,g}(1+|\xi|)^{k-1}, \quad j=0,\ldots,k-1.$$

Recalling $\Delta_k = \sum_{j=0}^k \delta_{j,g}$ and (3.12.Γ) with $a = -k+1+\varepsilon$, we get

$$\begin{aligned}|m^{\varepsilon}_{k-\varepsilon+\mathrm{i}\tau}(\xi)| &= (1+|\xi|)^{1-\varepsilon-(k-\varepsilon)}\,\big|d_t^{k-\varepsilon+\mathrm{i}\tau} m_g(t\xi)\big|_{t=1}\big| \\ &\leqslant \Delta_k\,\varepsilon^{-1}(1+|\xi|)^{1-k}\,(1+|\xi|)^{k-1} \\ &\qquad \times \max\{|\Gamma(\varepsilon-\mathrm{i}\tau-j_1)|^{-1} : 0\leqslant j_1\leqslant k-1\} \\ &\leqslant \beta_a\,\Delta_k\,\varepsilon^{-1}(1+\tau^2)^{1/4+(k-1-\varepsilon)/2}\,\mathrm{e}^{\pi|\tau|/2}.\end{aligned} \tag{7.16}$$

We may take $\beta_a = 2$ when $k\leqslant 2$ and $\beta_a = 2\Gamma(k-\varepsilon)$ otherwise. □

Remark. — One could not make the same simple computation for $k-\varepsilon'$ when $\varepsilon' > \varepsilon$. Indeed, we have then

$$m^{\varepsilon}_{k-\varepsilon'}(\xi) = (1+|\xi|)^{1-(k-\varepsilon')-\varepsilon} d_t^{k-\varepsilon'+\mathrm{i}\tau} m_g(t\xi)\big|_{t=1},$$

so $|m^\varepsilon_{k-\varepsilon'}(\xi)|$ contains the factor $(1+|\xi|)^{1-k+\varepsilon'-\varepsilon}$, that is not controllable by the preceding proof when $\varepsilon' > \varepsilon$. With one more integration by parts in the log-concave case we obtain

$$\frac{\mathrm{d}^j}{\mathrm{d}u^j} m_{lc}(u\xi) = \frac{(-2\mathrm{i}\pi|\xi|)^{j-2}}{u^2} \int_{\mathbb{R}} \bigl(s^j \varphi_\theta(s)\bigr)'' \,\mathrm{e}^{-2\mathrm{i}\pi s u|\xi|}\,\mathrm{d}s$$

that seems to give an additional improvement, able to swallow the bad factor $|\xi|^{\varepsilon'-\varepsilon}$ above. However, we would need now for $\int_{\mathbb{R}} |s|^j |\varphi''_\theta(s)|\,\mathrm{d}s$ a universal bound that does not exist, and actually, this integral does not make sense in general.

When $k \geqslant 1$, the kernel $\widetilde{m}^\varepsilon_{k-\varepsilon+\mathrm{i}\tau} := (1+|\xi|)^{1-k-\mathrm{i}\tau} D_u^{k-\varepsilon+\mathrm{i}\tau} m_g(u\xi)\big|_{u=1}$ is even easier to bound since we can write directly

$$\Bigl| \int_1^{+\infty} (u-1)^{\varepsilon-\mathrm{i}\tau-1} \frac{\mathrm{d}^k}{\mathrm{d}u^k} m_g(u\xi)\,\mathrm{d}u \Bigr| \leqslant \delta_{k,g} \Bigl(\int_1^{+\infty} (u-1)^{\varepsilon-1} \frac{\mathrm{d}u}{u} \Bigr) |\xi|^{k-1},$$

but $D_u^{-\varepsilon+\mathrm{i}\tau} m_g(u\xi)\big|_{u=1}$ is not a bounded function of ξ in the neighborhood of $\xi = 0$. For example, we have $D_u^{-\varepsilon}\,\mathrm{e}^{-u|\xi|}\,\big|_{u=1} = |\xi|^{-\varepsilon}\,\mathrm{e}^{-|\xi|}$. Thus $\widetilde{m}^\varepsilon_{-\varepsilon}$ is not an L^2 multiplier, nor an L^p multiplier for any $p \neq 2$, and this justifies working with d^z_t instead.

Proof of Lemma 7.4. — Let $\nu > 1-\varepsilon$ and $\ell = \lceil \nu + \varepsilon \rceil \geqslant 2$, so that $\nu \leqslant \ell - \varepsilon$. If $\operatorname{Re} z < \ell$, we have by (7.10) that

$$d^z_t m_g(t\xi)\big|_{t=1} = E_\ell(z) + (-1)^\ell \frac{1}{\Gamma(\ell - z)} \int_1^2 (u-1)^{-z+\ell-1} \frac{\mathrm{d}^\ell}{\mathrm{d}u^\ell} m_g(u\xi)\,\mathrm{d}u\,,$$

with $E_\ell(z) = \sum_{i=0}^{\ell-1} (-1)^i\,\Gamma(i+1-z)^{-1} \frac{\mathrm{d}^i}{\mathrm{d}u^i} m_g(u\xi)\big|_{u=2}$. We fix $\xi \in \mathbb{R}^n$ and consider the holomorphic function

$$H_\xi : z \mapsto m^\varepsilon_z(\xi) = (1+|\xi|)^{1-\varepsilon-z} d^z_t m_g(t\xi)\big|_{t=1}$$

in the strip $-\varepsilon < \operatorname{Re} z < \ell - \varepsilon$. We have $\bigl|(\mathrm{d}^i/\mathrm{d}u^i) m_g(u\xi)\bigr| \leqslant \delta_{i,g}|\xi|^i$ by (7.5.$\mathbf{H}_\infty$), and it follows from (3.12.Γ) that $|H_\xi(z)| \leqslant \kappa\,\mathrm{e}^{\kappa|\tau|}$ in the strip, with κ depending on $|\xi|$. Consider an arbitrary z_0 such that $1-\varepsilon < \nu_0 := \operatorname{Re} z_0 \leqslant \nu$. Let k integer be such that $k - \varepsilon < \nu_0 \leqslant k+1-\varepsilon$, thus $1 \leqslant k < \ell$. By Lemma 7.5, when $\operatorname{Re} z = k - \varepsilon$ or $\operatorname{Re} z = k+1-\varepsilon$, we have for $H_\xi(z)$ the good bound

$$|H_\xi(z)| \leqslant 2\kappa'_{\operatorname{Re} z+\varepsilon} \Delta_{\operatorname{Re} z+\varepsilon}\,\varepsilon^{-1} (1+(\operatorname{Im} z)^2)^{\operatorname{Re} z/2-1/4}\,\mathrm{e}^{\pi|\operatorname{Im} z|/2}. \qquad (7.17)$$

We write $\Delta_k < \Delta_{k+1} \leqslant \Delta_\ell$ and $\nu_0 + \varepsilon = (1-\theta)k + \theta(k+1)$. When $k \geqslant 3$, we have

$$\kappa'^{1-\theta}_k \kappa'^{\theta}_{k+1} \leqslant \Gamma(k{-}\varepsilon)^{1-\theta}\Gamma(k{+}1{-}\varepsilon)^\theta = \frac{\Gamma(k{+}1{-}\varepsilon)}{(k-\varepsilon)^{1-\theta}}$$
$$\leqslant \frac{\Gamma(\nu_0)^\theta\Gamma(\nu_0+1)^{1-\theta}}{(k-\varepsilon)^{1-\theta}} = \frac{\nu_0^{1-\theta}}{(k-\varepsilon)^{1-\theta}}\,\Gamma(\nu_0) < 2\Gamma(\nu)\,,$$

and $\kappa'^{1-\theta}_1\kappa'^{\theta}_2 \leqslant 1$, $\kappa'^{1-\theta}_2\kappa'^{\theta}_3 \leqslant 2$. By Corollary 3.4 and Remark 3.6, (3.22) with $w = 1$ and $c_j = (k+j-\varepsilon)/2 - 1/4$, $j = 0, 1$, we get for $|H_\xi(z_0)|$ a bound

$$4\Gamma(\max(\nu,2))(3/2)^{\operatorname{Re} z_0 - 1/2}\,\mathrm{e}^{\pi/4}\,\varepsilon^{-1}\Delta_\ell(1+(\operatorname{Im} z_0)^2)^{\operatorname{Re} z_0/2-1/4}\,\mathrm{e}^{\pi|\operatorname{Im} z_0|/2}\,.$$

This proves Lemma 7.4 when $1-\varepsilon < \operatorname{Re} z_0 \leqslant \nu$. The case $-\varepsilon \leqslant \operatorname{Re} z_0 \leqslant \nu = 1-\varepsilon$ is left to the reader, one has $k = 0$ and the polynomial component of the bound is then $(1+\tau^2)^{\nu/2-1/4}$ on both sides of the strip $-\varepsilon < \operatorname{Re} z < 1-\varepsilon$. □

An alternative proof could go like this: divide the integral $\int_1^2$ in the definition of $d_t^z m_g(t\xi)\big|_{t=1}$ into $\int_1^{1+a}$ and $\int_{1+a}^2$, for some suitable $a \in [0,1]$. For the first integral $\int_1^{1+a}$, we modify (7.10) and get when $-1 < \operatorname{Re} z < 0$ that

$$d_{z,1}(a) := \frac{1}{\Gamma(-z)}\int_1^{1+a}(u-1)^{-z-1}m_g(u\xi)\,\mathrm{d}u$$
$$= E_{k+2,a}(z) + (-1)^{k+2}\frac{1}{\Gamma(k+2-z)} \times \int_1^{1+a}(u-1)^{-z+k+1}\frac{\mathrm{d}^{k+2}}{\mathrm{d}u^{k+2}}m_g(u\xi)\,\mathrm{d}u$$

for every integer $k \geqslant -1$, where $E_{k+2,a}(z)$ is equal to

$$E_{k+2,a}(z) = \sum_{j=0}^{k+1}(-1)^j\,\frac{a^{-z+j}\,\frac{\mathrm{d}^j}{\mathrm{d}u^j}\,m_g\big(u\xi\big)\big|_{u=1+a}}{\Gamma(j+1-z)}\,.$$

Let now $-1 < \operatorname{Re} z \leqslant \nu$ and write $z = k + \sigma + \mathrm{i}\tau$ with k integer and $0 < \sigma \leqslant 1$. Applying the preceding formulas it follows by (7.5.$\mathbf{H}_\infty$) that

$$|d_{z,1}(a)| \leqslant \sum_{j=0}^{k+1}\frac{a^{-k-\sigma+j}\,\delta_{j,g}\,|\xi|^j}{|\Gamma(j+1-z)|\,(1+|\xi|)} + \frac{(2-\sigma)^{-1}a^{2-\sigma}\,\delta_{k+2,g}\,|\xi|^{k+2}}{|\Gamma(k+2-z)|\,(1+|\xi|)}\,.$$

When $|\xi| \leqslant 1$, we choose $a = 1$ and obtain $|d_{z,1}(a)| \leqslant C_k(z)(1+|\xi|)^{-1}$ where

$$C_k(z) = \Delta_{k+2}\,\max\big\{|\Gamma(i-z)|^{-1} : 0 \leqslant i \leqslant k+2\big\}.$$

When $|\xi| > 1$, we let $a = |\xi|^{-1}$ and get $|d_{z,1}(a)| \leqslant C_k(z)|\xi|^{k+\sigma}(1+|\xi|)^{-1}$. The other term $d_{z,2}(a)$, corresponding to $\int_{1+a}^2$, is zero when $|\xi| \leqslant 1$ since

$a = 1$ in this case. Otherwise, we have $a = |\xi|^{-1}$ and assuming $k + \sigma \neq 0$, we get

$$|d_{z,2}(a)| = \left| \frac{1}{\Gamma(-z)} \int_{1+|\xi|^{-1}}^{2} (u-1)^{-z-1} m_g(u\xi)\,\mathrm{d}u \right| \leqslant \frac{1}{|\Gamma(-z)|} \frac{|\xi|^{k+\sigma} - 1}{k+\sigma} \frac{\delta_{0,g}}{1+|\xi|}\,.$$

There is no problem as long as $\operatorname{Re} z = k + \sigma$ is not close to 0. Otherwise, we can apply $\big| |\xi|^{k+\sigma} - 1 \big| \leqslant |k+\sigma|\, |\xi|^{(k+\sigma)^+} \ln |\xi|$, where $t^+ = \max(t, 0)$. Summing up and letting $L(\xi) = 1 + (\ln |\xi|)^+$, we have when $-1 < \operatorname{Re} z =: s < \nu$ that

$$\left| d_t^z m_g(t\xi)\big|_{t=1} \right| \leqslant C_\nu(z) \big([1 + |s|^{-1}] \wedge L(\xi) \big) (1 + |\xi|)^{s^+ - 1},$$

giving bounds multiple of $(1+|\xi|)^{-1}$, $(1+|\xi|)^{-1} L(\xi)$ for s in $[-1, -\varepsilon/2]$ and $[-\varepsilon/2, 0]$ respectively, $(1+|\xi|)^{s-1} L(\xi)$ and $(1+|\xi|)^{s-1}$ in $[0, \varepsilon/2]$ and $[\varepsilon/2, \nu]$ respectively. For $m_z^\varepsilon(\xi)$, we get bounds multiple of 1, $(1+|\xi|)^{-\varepsilon/2} L(\xi)$ and $(1+|\xi|)^{-\varepsilon}$ for s in $[-\varepsilon, -\varepsilon/2]$, $[-\varepsilon/2, \varepsilon/2]$ and $[\varepsilon/2, \nu]$ respectively. This shows that $m_z^\varepsilon(\xi)$ is a bounded function of $\xi \in \mathbb{R}^n$.

7.5. Lemma 4 of Müller's article

We must control the action on $L^p(\mathbb{R}^n)$, $p > 1$ close to 1, of multipliers m_z^ε when $\operatorname{Re} z = -\varepsilon$. If $z = -\varepsilon + \mathrm{i}\tau$, we have

$$\Gamma(\varepsilon - \mathrm{i}\tau)\, m_{-\varepsilon+\mathrm{i}\tau}^\varepsilon(\xi) = (1 + |\xi|)^{1-\mathrm{i}\tau} \int_1^2 (s-1)^{\varepsilon - \mathrm{i}\tau - 1} m_g(s\xi)\,\mathrm{d}s\,.$$

Since $\int_1^2 \big|(s-1)^{\varepsilon-\mathrm{i}\tau-1}\big|\,\mathrm{d}s = \varepsilon^{-1}$, it is enough to bound uniformly in $s \in [1, 2]$ the norm of $n_s(\xi) := (1+|\xi|)^{1-\mathrm{i}\tau} m_g(s\xi)$. This multiplier can be decomposed into several parts: first $(1+|\xi|)^{-\mathrm{i}\tau}$, which is taken care of by Proposition 2.2 on multipliers of Laplace type. Indeed, replacing λ by $1 + \lambda$ in (2.17) and integrating by parts, one finds that $(1+\lambda)^{-\mathrm{i}\tau} = \lambda \int_0^{+\infty} \mathrm{e}^{-\lambda t}\, a_\tau(t)\,\mathrm{d}t$, with

$$a_\tau(t) = \frac{1}{\Gamma(1+\mathrm{i}\tau)} \left(t^{\mathrm{i}\tau}\, \mathrm{e}^{-t} + \int_0^t s^{\mathrm{i}\tau}\, \mathrm{e}^{-s}\,\mathrm{d}s \right) \tag{7.18}$$

and $|a_\tau(t)| \leqslant \big|\Gamma(1+\mathrm{i}\tau)\big|^{-1} \leqslant (1+\tau^2)^{-1/4}\, \mathrm{e}^{\pi|\tau|/2}$ according to (3.4). Next, in $n_s(\xi)$, we have $(1+|\xi|) m_g(s\xi)$, which is formed of $m_g(s\xi)$, multiplier bounded by $\|K_g\|_{L^1(\mathbb{R}^n)}$ on all $L^p(\mathbb{R}^n)$ spaces, and of $s^{-1}|s\xi| m_g(s\xi)$, $s > 1$, with a multiplier norm less than that of $|\xi| m_g(\xi)$, according to (2.10).

Given an integrable kernel K on $\mathbb{R}^n$ and its Fourier transform m, the question boils down to handling the crucial multiplier

$$m^{\#}(\xi) := |\xi| m(\xi)\,. \tag{7.19}$$

We summarize the latter discussion in the lemma that follows, where we include the bound $2(1+\tau^2)^{1/4}\,\mathrm{e}^{\pi|\tau|/2}$ from (3.12.Γ) for the factor $|\Gamma(\varepsilon-\mathrm{i}\tau)|^{-1}$ that was left apart above. So far, the kernel K can be arbitrary in $L^1(\mathbb{R}^n)$.

LEMMA 7.6. — *Let p belong to $(1,2]$. One has that*

$$\|m^{\varepsilon}_{-\varepsilon+\mathrm{i}\tau}\|_{p\to p} \leqslant 2\varepsilon^{-1}\lambda_p\,\mathrm{e}^{\pi|\tau|}\,(\|K_g\|_{L^1(\mathbb{R}^n)} + \|m^{\#}\|_{p\to p})\,, \quad \tau\in\mathbb{R}\,,$$

where λ_p is the constant appearing in Proposition 2.2.

The serious work will be done in the proof of the following essential lemma.

LEMMA 7.7. — *Let K_g be a kernel integrable on $\mathbb{R}^n$ satisfying* (6.1.**H**), *and m_g its Fourier transform. Let $m_g^{\#}$ be defined by* (7.19) *and $p\in(1,2]$. One has that*

$$\|m_g^{\#}\|_{p\to p} \leqslant (2\pi)^{1-2/p}\,\rho_p\,\delta_{0,g}^{2-2/p}\,V(K_g)^{-1+2/p}\,,$$

where ρ_p is the constant from (2.22) *and where $V(K_g)$ is defined at* (7.2).

The proof of Lemma 7.7 will be broken into several easy statements. Some of them are used again in Section 8. To begin with, we merely assume that K is an integrable kernel on $\mathbb{R}^n$ having partial derivatives $\partial_j K$ in the sense of distributions that are (signed) measures μ_j, and we let $m=\widehat{K}$. We can express $m^{\#}(\xi)$ with the help of the Riesz transforms $(R_j)_{j=1}^n$ introduced in Section 2.3, writing

$$2\pi\,m^{\#}(\xi) = \sum_{j=1}^n \frac{-\mathrm{i}\xi_j}{|\xi|}\,(2\mathrm{i}\pi\xi_j)m(\xi)\,.$$

The functions $(2\mathrm{i}\pi\xi_j)m(\xi)$, $j=1,\ldots,n$, are the Fourier transforms of the measures $\mu_j=\partial_j K$. When K is the uniform probability density K_C on a symmetric convex set C, the μ_js are supported on the boundary of C, and we shall see below that $V(K_C)=q(C)$ if C is isotropic and normalized by variance.

The convolution operator $T_{m^{\#}}$ can thus be written under the form

$$T_{m^{\#}} : f\mapsto T_{m^{\#}}f = (2\pi)^{-1}\sum_{j=1}^n R_j\mu_j * f\,.$$

Riesz transforms commute with convolutions. If g is in the dual L^q of L^p, we have

$$2\pi|\langle T_{m^\#}f,g\rangle| = \Bigl|\sum_{j=1}^n \langle R_j\mu_j * f, g\rangle\Bigr| = \Bigl|\sum_{j=1}^n \langle (R_j f) * \mu_j, g\rangle\Bigr| = \Bigl|\sum_{j=1}^n \langle R_j f, \widetilde{\mu}_j * g\rangle\Bigr|$$
$$\leqslant \int_{\mathbb{R}^n} \Bigl(\sum_{j=1}^n |R_j f|^2\Bigr)^{1/2} \Bigl(\sum_{j=1}^n |\widetilde{\mu}_j * g|^2\Bigr)^{1/2},$$

where $\widetilde{\mu}_j$ denotes the image of μ_j under the symmetry $x \mapsto -x$ of $\mathbb{R}^n$. By (2.22), the Riesz transforms are "collectively bounded" in $L^p(\mathbb{R}^n)$ by a constant ρ_p independent of the dimension n, and we obtain therefore that

$$2\pi|\langle T_{m^\#}f,g\rangle| \leqslant \rho_p \|f\|_p \Bigl\|\Bigl(\sum_{j=1}^n |\widetilde{\mu}_j * g|^2\Bigr)^{1/2}\Bigr\|_q .$$

Noticing that $\widetilde{\mu}_j * g = (\mu_j * \widetilde{g})^{\sim}$ and $\bigl(\sum_{j=1}^n |\mu_j * g|^2\bigr)^{1/2} = |\nabla K * g|$, we are led to study the operator

$$U_K : g \in L^q(\mathbb{R}^n) \mapsto \nabla K * g \in L^q(\mathbb{R}^n, \mathbb{R}^n) \tag{7.20}$$

given by the vector-valued convolution with ∇K. Let us state what we have got.

Lemma 7.8. — *Let K be an integrable kernel on $\mathbb{R}^n$, m its Fourier transform and let $m^\#$ be defined by* (7.19)*. For every $p \in (1,2]$ and $q = p/(p-1)$, one has*

$$\|T_{m^\#}\|_{p\to p} \leqslant (2\pi)^{-1}\rho_p \sup_{\|g\|_q \leqslant 1} \|\nabla K * g\|_{L^q(\mathbb{R}^n)} = (2\pi)^{-1}\rho_p \|U_K\|_{q\to q} .$$

When $K = K_g$ satisfies (6.1.**H**), we shall estimate $\|U_{K_g}\|_{q\to q}$ by interpolation between L^2 and L^∞. Contrary to the L^2 estimate which will make use of (6.1.**H**), the L^∞ estimate is a straightforward observation following from the definition of $V(K)$. In the special case $K_g = K_C$ of a convex body C, this L^∞ case will bring in the geometrical parameter $q(C) = 2Q(C_0)L(C_0)$, equal to $V(K_C)$.

Lemma 7.9. — *Let K be an integrable kernel on $\mathbb{R}^n$ having a finite directional variation $V(K)$, and let U_K be defined by* (7.20)*. One has that*

$$\|U_K\|_{\infty\to\infty} \leqslant V(K) . \tag{7.21}$$

Proof. — For each $x \in \mathbb{R}^n$, the Euclidean norm of the vector $(\nabla K * g)(x) \in \mathbb{R}^n$ is given by the supremum over $\theta \in S^{n-1}$ of

$$\Bigl|\theta \cdot \Bigl(\int_{\mathbb{R}^n} g(x-y)\,\mathrm{d}(\nabla K)(y)\Bigr)\Bigr| = \Bigl|\int_{\mathbb{R}^n} g(x-y)\,\mathrm{d}(\theta\cdot\nabla K)(y)\Bigr|$$
$$\leqslant \|g\|_\infty \,\|\theta \cdot \nabla K\|_1 \leqslant V(K)\,\|g\|_\infty .$$

□

Lemma 7.10. — *For every symmetric convex body C, isotropic and normalized by variance, one has that $V(K_C) = q(C)$.*

Proof. — Let θ belong to S^{n-1} and let y be in $\theta^\perp$. For each line $y + \mathbb{R}\theta$ that meets the set C, the jumps of the density $K_C = |C|^{-1}\mathbf{1}_C$ of μ_C, when traveling on the line in the direction of increasing real numbers, are equal to $|C|^{-1}$ when we enter C, and to $-|C|^{-1}$ when leaving C, implying that the mass of the directional derivative is equal to $2/|C|$ times the measure of the projection of C onto $\theta^\perp$. More precisely, suppose without loss of generality that θ is the first basis vector $\mathbf{e}_1$ of $\mathbb{R}^n$ and let π_1 be the orthogonal projection onto $\mathbf{e}_1^\perp$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be given, and write each $x \in \mathbb{R}^n$ as $x = (s, y)$ with $s \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Using Fubini, we get

$$\begin{aligned} -\langle \mathbf{e}_1 \cdot \nabla \mu_C, \psi \rangle &= -\langle \frac{\partial \mu_C}{\partial x_1}, \psi \rangle = \langle \mu_C, \frac{\partial \psi}{\partial x_1} \rangle \\ &= \int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} |C|^{-1} \mathbf{1}_C(s, y) \frac{\partial \psi}{\partial x_1}(s, y) \, \mathrm{d}s \Big) \, \mathrm{d}y \,. \end{aligned}$$

The inside integral is 0 if $L_y = y + \mathbb{R}\mathbf{e}_1$ does not meet the convex set C. Otherwise, the line L_y cuts C along a segment $[y + s_1(y)\mathbf{e}_1, y + s_2(y)\mathbf{e}_1]$, $s_1(y) \leqslant s_2(y)$, and

$$-\langle \mathbf{e}_1 \cdot \nabla \mu_C, \psi \rangle = \frac{1}{|C|_n} \int_{\pi_1 C} \big(\psi(s_2(y), y) - \psi(s_1(y), y) \big) \, \mathrm{d}y \leqslant \frac{2|\pi_1 C|_{n-1}}{|C|_n} \, \|\psi\|_\infty \,.$$

Going back to a general $\theta \in S^{n-1}$ and according to (7.1), we conclude that

$$\|\theta \cdot \nabla \mu_C\|_1 \leqslant \frac{2}{|C|_n} \, |P_\theta C|_{n-1} \leqslant 2Q(C_0)L(C_0) = q(C) \,.$$

We get $V(K_C) \leqslant q(C)$, which suffices for our purpose. Müller [59, Lemma 3] shows that this inequality is actually an equality. □

When $K_g = K_C$, we have $\|U_{K_C}\|_{\infty \to \infty} \leqslant q(C)$, specifying the estimate (7.21) obtained in the general case. We complete now the interpolation for U_{K_g}. We formulate the next Lemma so that we can apply it again in Section 8.

Lemma 7.11. — *Let K be an isotropic log-concave probability density on $\mathbb{R}^n$ with variance σ^2. For $2 \leqslant q \leqslant +\infty$, one has that*

$$\|U_K f\|_q = \|\nabla K * f\|_q \leqslant 2^{1/q} \sigma^{-2/q} V(K)^{1-2/q} \|f\|_q \,, \quad f \in L^q(\mathbb{R}^n) \,.$$

If K_g is an integrable kernel on $\mathbb{R}^n$ satisfying (6.1.**H**)*, then*

$$\|U_{K_g}\|_{q \to q} \leqslant (2\pi \delta_{0,g})^{2/q} V(K_g)^{1-2/q} \,.$$

Proof. — Let $m = \widehat{K}$ and consider first $q = 2$. By Parseval (2.12.**P**) we have

$$\|\nabla K * f\|_2^2 = \Bigl\| \Bigl(\sum_{j=1}^n |\partial_j K * f|^2 \Bigr)^{1/2} \Bigr\|_2^2 = \int_{\mathbb{R}^n} \sum_{j=1}^n 4\pi^2 \xi_j^2 |m(\xi)|^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \,,$$

and $\sum_{j=1}^n 4\pi^2 \xi_j^2 |m(\xi)|^2 = 4\pi^2 |\xi|^2 |m(\xi)|^2 \leqslant 2\sigma^{-2}$ by (5.17.**B**) (or by (6.1.**H**), it is $\leqslant 4\pi^2 \delta_{0,g}^2$), hence $\|U_K\|_{2\to 2} \leqslant \sqrt{2}\,\sigma^{-1}$ (or $\leqslant 2\pi\delta_{0,g}$). If $q \in (2, +\infty)$, we write $1/q = (1-\theta)/2 + \theta/\infty$ with $\theta = 1 - 2/q$. We get that $\|U_K\|_{q\to q} \leqslant (\sqrt{2}\,\sigma^{-1})^{2/q} V(K)^{1-2/q}$ (or we get $\leqslant (2\pi\delta_{0,g})^{2/q} V(K)^{1-2/q}$) by Lemma 7.9 and interpolation (L^2, L^∞). □

End of the proof of Lemma 7.7. — We use Lemma 7.8, then apply Lemma 7.11 to K_g with $1/q = 1 - 1/p$ and obtain that

$$\|T_{m_g^\#}\|_{p\to p} \leqslant (2\pi)^{-1} \rho_p \|U_{K_g}\|_{q\to q} \leqslant (2\pi)^{1-2/p} \rho_p\, \delta_{0,g}^{2-2/p} V(K_g)^{-1+2/p}. \quad \square$$

7.5.1. Conclusion

We finish the proof of Proposition 7.2. We first run over half of the way in the following lemma, which we shall refer to again in Section 8.

Lemma 7.12. — *Let K_g be an integrable kernel on $\mathbb{R}^n$ satisfying (7.5.$\mathbf{H}_\infty$), and let $m_g^\#$ be defined by (7.19). Let $\alpha \in (0,1)$ and suppose that $1 < p_0 < p < 2$. There exists a constant $\kappa(p, p_0)$, independent of n, such that*

$$\|(\xi \cdot \nabla)^\alpha m_g(\xi)\|_{p\to p} \leqslant \|K_g\|_{L^1(\mathbb{R}^n)} + \frac{\kappa(p,p_0)}{1-\alpha} \bigl(\|K_g\|_{L^1(\mathbb{R}^n)} + \|m_g^\#\|_{p_0\to p_0} \bigr)^{1-\theta} \Delta_{k(\theta)}^\theta \,,$$

where $\theta \in (0,1)$ is defined by $1/p = (1-\theta)/p_0 + \theta/2$ and $k(\theta) = \lceil 1/\theta \rceil$.

Proof. — Lemma 7.3 gives

$$\|(\xi \cdot \nabla)^\alpha m_g(\xi)\|_{p\to p} \leqslant \|K_g\|_{L^1(\mathbb{R}^n)} + \|d_t^\alpha m_g(t\xi)\bigr|_{t=1}\|_{p\to p} \,.$$

Let $\varepsilon = 1 - \alpha > 0$. We apply complex interpolation to the Müller family (m_z^ε) in the strip $S = \{z \in \mathbb{C} : -\varepsilon \leqslant z \leqslant \nu\}$ of width $w := \nu + \varepsilon$. We bound $m_\alpha^\varepsilon(\xi) = d_t^\alpha m_g(t\xi)\bigr|_{t=1}$ on $L^p(\mathbb{R}^n)$, using L^{p_0} estimates of m_z^ε for $\operatorname{Re} z = -\varepsilon$ and L^2 estimates when $\operatorname{Re} z = \nu$. The value ν must satisfy $\alpha = (1-\theta)(-\varepsilon) + \theta\nu$, hence $\nu = 1/\theta - \varepsilon > 1 - \varepsilon$ and $w = 1/\theta$. It follows from Lemma 7.6 that

$$\|m_{-\varepsilon+\mathrm{i}\tau}^\varepsilon\|_{p_0\to p_0} \leqslant 2\varepsilon^{-1} \lambda_{p_0}\, \mathrm{e}^{\pi|\tau|} \bigl(\|K_g\|_{L^1(\mathbb{R}^n)} + \|m_g^\#\|_{p_0\to p_0} \bigr) \,.$$

By Lemma 7.4, each operator $m^{\varepsilon}_{\nu+\mathrm{i}\tau}$ is bounded by

$$\kappa_\nu \varepsilon^{-1} \Delta_\ell (1+\tau^2)^{\nu/2-1/4} \,\mathrm{e}^{\pi|\tau|/2}$$

on $L^2(\mathbb{R}^n)$, with $\kappa_\nu \leqslant \kappa_w = 4\Gamma(\max(w,2))(3/2)^{w-1/2}\,\mathrm{e}^{\pi/4} =: c_\theta$, a function of θ alone, and $\ell = \lceil \nu+\varepsilon \rceil = \lceil w \rceil$. If we check the admissible growth condition in S, we can rely on Corollary 3.4, Remark 3.6 and (3.22) in order to get a bound

$$\|m^{\varepsilon}_{\alpha}(\xi)\|_{p\to p} \leqslant \kappa(p,p_0)\varepsilon^{-1}(\|K_g\|_{L^1(\mathbb{R}^n)} + \|m^{\#}_g\|_{p_0\to p_0})^{1-\theta} \Delta^{\theta}_{k(\theta)}\,,$$

with $k(\theta) = \ell = \lceil 1/\theta \rceil$ and $\kappa(p,p_0) \leqslant (1+w/2)^{\theta(\nu-1/2)}\,\mathrm{e}^{\pi w/2}(2\lambda_{p_0})^{1-\theta} c^{\theta}_{\theta}$. Observing that $\theta\nu < 1$, $w > 1$ and $\lambda_{p_0} \geqslant 1$, we may simplify this bound as

$$\kappa(p,p_0) \leqslant \kappa w\, \mathrm{e}^{\pi w/2}\, \lambda_{p_0} \Gamma(\max(w,2))^{1/w} \leqslant \kappa w^2\, \mathrm{e}^{\pi w/2}\, \lambda_{p_0}\,. \tag{7.22}$$

We now verify that the holomorphic function $H(z) = \langle m^{\varepsilon}_z f_z, g_z\rangle$ has an admissible growth in S. Since the kernels are bounded functions of ξ by Lemma 7.4, all multipliers m^{ε}_z, $z \in S$, are L^2-bounded with a bound of the form $\kappa\,\mathrm{e}^{\kappa|\operatorname{Im} z|}$. If we restrict to functions f and g bounded with bounded support, we have by (3.24) that f_z, g_z are uniformly bounded in $L^2(\mathbb{R}^n)$, and we can conclude as in Section 7.3. □

End of the proof of Proposition 7.2. — Given $p \in (1,2)$ and $\alpha = 1-\varepsilon \in (0,1)$, we select $p_0 \in (1,p)$ and let $\theta \in (0,1)$ satisfy $1/p = (1-\theta)/p_0 + \theta/2$. Since $1 < p_0 < p < 2$, we have that $0 < \theta < 2(1-1/p) < 1$. It follows from Lemma 7.7 that

$$\|m^{\#}_g\|_{p_0\to p_0} \leqslant (2\pi)^{1-2/p_0} \rho_{p_0}\, \delta^{2-2/p_0}_{0,g} V(K_g)^{2/p_0-1}.$$

By Lemma 7.12, and because $\|K_g\|_{L^1(\mathbb{R}^n)} \leqslant 1$, $\rho_{p_0} \geqslant 1$ (see Remark 2.3), we get

$$\|(\xi\cdot\nabla)^{\alpha} m_g(\xi)\|_{p\to p} \leqslant 1 + \kappa(p,p_0)\varepsilon^{-1} \rho_{p_0} \Delta^{\theta}_{k(\theta)} \bigl(1 + \delta^{1-\theta-(2/p-1)}_{0,g} V(K_g)^{2/p-1}\bigr).$$

We still have a choice of $\theta \in (0, 2-2/p)$. If θ gets small, then the power of $\Delta_{k(\theta)}$ gets small, but the number $k(\theta)$ of constants $\delta_{j,g}$ involved increases to infinity. In the log-concave case, the estimate (5.18) indicates a growth of order $\Delta_{k,c} \sim k!$ yielding $\Delta^{\theta}_{k(\theta)} \sim 1/\theta$. Furthermore, the width $w = 1/\theta$ of the strip and the associated interpolation constants also tend to $+\infty$ in this case, and we should thus keep θ away from 0, as much as possible. If θ approaches its upper limit $2(1-1/p)$, then p_0 tends to 1 and the constants such as λ_{p_0}, ρ_{p_0} tend to infinity. Choosing $\theta = (4/3)(1-1/p)$ has the merit to provide the relatively simple bound

$$\|(\xi\cdot\nabla)^{\alpha} m_g(\xi)\|_{p\to p} \leqslant 1 + \kappa(\alpha,p) \Delta^{(4/3)(1-1/p)}_{k(p)} \bigl(1 + \delta^{(2/3)(1-1/p)}_{0,g} V(K_g)^{2/p-1}\bigr)\,, \tag{7.23}$$

with $\kappa(\alpha, p) = \kappa(p, p_0)\varepsilon^{-1}\rho_{p_0} \leqslant \kappa\varepsilon^{-1}w^2\,\mathrm{e}^{\pi w/2}\,\lambda_{p_0}\rho_{p_0}$ by (7.22), with $p_0 - 1 = (p-1)/(5-2p)$ and $k(p) = \lceil 1/\theta\rceil = \lceil 3p/(4p-4)\rceil$. □

Remark 7.13. — It is usual to have a factor $1/\Gamma(z)$ in fractional derivatives, which led us to seeing $\mathrm{e}^{\pi|\tau|/2}$ in many places, ending with $\mathrm{e}^{\pi w/2}$ in our estimate (7.22) of $\kappa(p, p_0)$, with $w = 1/\theta \sim q := p/(p-1)$ after the final choice of $\theta = 4/(3q)$ above. We could avoid this exponential though. Consider the modified Müller family

$$\widetilde{m}_z^\varepsilon(\xi) = \Gamma(1+\varepsilon+z)\,\frac{\Gamma(2\varepsilon+z)}{\Gamma(1+\varepsilon)}\,m_z^\varepsilon(\xi)\,, \quad \operatorname{Re} z \geqslant -\varepsilon,\ \ \xi \in \mathbb{R}^n,$$

which coincides with the former at $z = \alpha$ since $\varepsilon + \alpha = 1$ and $\Gamma(2) = 1$. For the L^{p_0} bound when $z = -\varepsilon + \mathrm{i}\tau$, we decompose $\widetilde{m}^\varepsilon_{-\varepsilon+\mathrm{i}\tau}(\xi)$ as

$$\frac{1}{\Gamma(1+\varepsilon)}\,\Bigl[\Gamma(1+\mathrm{i}\tau)(1+|\xi|)^{-\mathrm{i}\tau}\Bigr]\biggl[\frac{\Gamma(\varepsilon+\mathrm{i}\tau)}{\Gamma(\varepsilon-\mathrm{i}\tau)}\,(1+|\xi|)\int_1^2 (s-1)^{\varepsilon-\mathrm{i}\tau-1} m_g(s\xi)\,\mathrm{d}s\biggr].$$

Introducing $\Gamma(1+\mathrm{i}\tau)$ in the Laplace type multiplier (7.18), we obtain a new function $\widetilde{a}_\tau(t)$ bounded by 1, and $\Gamma(2\varepsilon+z)$ is used for the bound of $d_t^{-\varepsilon+\mathrm{i}\tau} m_g(t\xi)\big|_{t=1}$ because $|\Gamma(\varepsilon+\mathrm{i}\tau)| = |\Gamma(\varepsilon-\mathrm{i}\tau)|$. We get in this way for $\widetilde{m}_z^\varepsilon(\xi)$ a bound

$$\|\widetilde{m}_z^\varepsilon(\xi)\|_{p_0\to p_0} \leqslant 2\varepsilon^{-1}\lambda_{p_0}\,(\|K_g\|_{L^1(\mathbb{R}^n)} + \|m^\#\|_{p_0\to p_0})$$

that replaces Lemma 7.6 (we use again $1/\Gamma(1+\varepsilon) \leqslant 2$). The L^∞ bounds obtained in (7.16) when $z = k - \varepsilon + \mathrm{i}\tau$, $k > 0$, have now a largest factor of $\Delta_k\varepsilon^{-1}$ equal to $\Gamma(k+1+\mathrm{i}\tau)\Gamma(k+\varepsilon+\mathrm{i}\tau)/[\Gamma(1+\varepsilon)\Gamma(-k+1+\varepsilon-\mathrm{i}\tau)]$ (when the index j_1 in (7.16) is equal to $k-1$). The modulus of this factor is the same as that of

$$\frac{\Gamma(k+1+\mathrm{i}\tau)\Gamma(k+\varepsilon+\mathrm{i}\tau)}{\Gamma(1+\varepsilon)\Gamma(-k+1+\varepsilon+\mathrm{i}\tau)} = \frac{\Gamma(1+\mathrm{i}\tau)}{\Gamma(1+\varepsilon)}\biggl(\prod_{j=1}^{k}(j+\mathrm{i}\tau)\biggr)\biggl(\prod_{j=-k+1}^{k-1}(j+\varepsilon+\mathrm{i}\tau)\biggr).$$

This is a bounded function of τ according to (3.1), with a rough bound given by $2\sqrt{2\pi}\,(k+|\tau|)^{3k}\,\mathrm{e}^{-\pi|\tau|/2} \leqslant 6.2^k k^{3k}$ (use $x/\sinh x \leqslant (1+2x)\,\mathrm{e}^{-x}$ for $x \geqslant 0$). One need not be too careful here since this term will be raised to the power $\theta = 1/w \lesssim 1/k$. We use it as in (7.17) for two values k, $k+1$ such that $k \leqslant \nu_0 + \varepsilon \leqslant k+1 \leqslant \ell = \lceil w\rceil < w+1$. One has then for the L^2 bound of $\widetilde{m}^\varepsilon_{\nu+\mathrm{i}\tau}(\xi)$ an estimate by $\kappa 2^w w^{3(w+1)}\Delta_\ell\varepsilon^{-1}$. By interpolation we have

$$\|m_\alpha^\varepsilon(\xi)\|_{p\to p} \leqslant \kappa\varepsilon^{-1}\lambda_{p_0}^{1-\theta}(2^w w^{3(w+1)})^\theta(\|K_g\|_{L^1(\mathbb{R}^n)} + \|m_g^\#\|_{p_0\to p_0})^{1-\theta}\Delta_\ell^\theta\,.$$

We thus get for $\kappa(p, p_0)$ in (7.22) a new estimate $\kappa'(p, p_0) \leqslant \kappa w^3\lambda_{p_0}$, leading in (7.23) to $\kappa'(\alpha, p) \leqslant \kappa\varepsilon^{-1}q^3\lambda_{p_0}\rho_{p_0}$. The final choice in the proof of Proposition 7.2 gives $p_0 - 1 = (p-1)/(5-2p)$ of order $p-1$ as $p \to 1$, and since λ_{p_0}, ρ_{p_0} are $O\bigl((p_0-1)^{-1}\bigr)$ as $p_0 \to 1$ (see (2.20) and (2.24)), we end up with $\kappa'(\alpha, p) \leqslant \kappa\varepsilon^{-1}q^5$, a bound which is polynomial but has no reason to be accurate. After these modifications, we have for Proposition 7.2 when $1 < p \leqslant 2$ a new form

$$\begin{aligned} &\|(\xi \cdot \nabla)^\alpha m_g(\xi)\|_{p\to p} \\ &\qquad \leqslant 1 + \kappa \varepsilon^{-1} q^5 \Delta_{k(p)}^{(4/3)(1-1/p)} \bigl(1 + \delta_{0,g}^{(2/3)(1-1/p)} V(K_g)^{2/p-1}\bigr) , \quad (7.24) \end{aligned}$$

with $k(p) = \lceil 3p/(4p-4) \rceil$ and $q = p/(p-1)$.

Remark 7.14. — With the new information above, we go back to the proof of Theorem 7.1. One has chosen there $\varepsilon = 1 - \alpha = 3(p-1)/(4p+4)$, and $p_0 \in (1, p)$ such that $p_0 - 1 = 3(p-1)/(5-p)$. Both ε and $p_0 - 1$ behave as multiples of $p - 1$ when $p \to 1$. If we consider the Poisson kernel P as another K_g satisfying ($7.5.\mathbf{H}_\infty$), and with $V(P) \leqslant 2/\pi$ by (7.4), we can apply to it (7.24) for the value p_0 and obtain that $\|(\xi \cdot \nabla)^\alpha \widehat{P}(\xi)\|_{p_0 \to p_0} \leqslant \kappa \varepsilon^{-1} q^5$. Applying also (7.24) to m_g and p_0, we get for $m = m_g - \widehat{P}$ that

$$\|(\xi \cdot \nabla)^\alpha m(\xi)\|_{p_0\to p_0} \leqslant \kappa q^6 \Delta_{k(p_0)}^{(4/3)(1-1/p_0)} \delta_{0,g}^{(2/3)(1-1/p_0)} (1 + \lambda^{2/p_0-1}) =: B .$$

We have $\alpha - 1/p_0 = 3(p-1)/(4p+4)$ which again is of order $p - 1$. The constant κ_{α,p_0} from (6.25), seen in (7.7), behaves thus as $(p-1)^{-1/p_0} \simeq q$, so $C''_{p_0}(\lambda)$ is bounded by $\kappa q(2 + B)$. Also $r_0 - 1 \simeq (p + p_0 - 2)/2$ in (7.8) is of order $p - 1 \simeq 1/q$. In (7.8), the constants C_{r_0} and C'_p are of order q. Indeed, we can take $C_{r_0} = \mathrm{q}_{r_0}$ from (2.4), that was estimated by $r_0/(r_0 - 1)$ in (2.5), and C'_p can be the bound for the maximal function of the Poisson kernel, see ($1.31.P^*$). Also, $1 - \gamma = (p-1)/(2p)$, and with Lemma 6.19 we know that

$$\sum_{k\in\mathbb{Z}} a_{\alpha,k}^{(1-\gamma)p/2} \leqslant \kappa \sum_{k\in\mathbb{Z}} 2^{-(1-\alpha)(1-\gamma)p|k|/2} \leqslant \frac{\kappa'}{(1-\alpha)(1-\gamma)} \leqslant \kappa'' q^2 .$$

Finally, we obtain for Theorem 7.1 another bizarre polynomial estimate

$$\begin{aligned} \|\mathrm{M}_{K_g}\|_{p\to p} &\leqslant \|M_K\|_{p\to p} + O(q) \leqslant C_{r_0}^2 C''_{p_0}(\lambda) (\kappa'' q^2)^2 + O(q) \\ &\leqslant \kappa q^{13} \Delta_{\lceil q \rceil}^{1-1/p} \Delta_1^{1-1/p} (1 + \lambda^{2/p-1}) , \quad 1 < p \leqslant 2 . \end{aligned}$$

8. Bourgain's article on cubes

In this section, Q is a cube in dimension n, more precisely, the symmetric cube

$$Q = Q_n = \Bigl[-\frac{1}{2}, \frac{1}{2} \Bigr]^n$$

of volume 1 in $\mathbb{R}^n$. It is isotropic, but if we look for a multiple bQ normalized by variance, we would need that the half-side $a = b/2$ of bQ satisfy $\sigma_{bQ}^2 = 1$, where

$$\sigma_{bQ}^2 = \frac{1}{|bQ|} \int_{bQ} x_1^2 \,\mathrm{d}x = \frac{1}{2a} \int_{-a}^{a} s^2 \,\mathrm{d}s = \frac{1}{a} \int_0^a s^2 \,\mathrm{d}s = \frac{a^2}{3},$$

and where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. This gives $a = \sqrt{3}$ in every dimension n, but the cube $[-\sqrt{3}, \sqrt{3}]^n$ is not very pleasant to manipulate, and we shall rather follow Bourgain [13] and keep the volume 1 cube Q. With $a = 1/2$, the covariance for Q is given by $(12)^{-1} \mathrm{I}_n$. Since the variance $\sigma_Q^2 = 1/12$ is independent of the dimension, we shall have no problem with the estimates (5.17.**B**) or (5.19). The Fourier transform of the probability measure μ_Q is given by

$$m_Q(\xi) = \widehat{\mu_Q}(\xi) = \widehat{K_Q}(\xi) = \prod_{j=1}^n \frac{\sin(\pi \xi_j)}{\pi \xi_j}, \qquad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$

Bourgain observes that a decay better than the usual (5.17.**B**) for a Fourier transform m_C would allow to relax the limitation $p > 3/2$ of Theorem 6.2, and that this better decay is achieved by m_Q is most directions. He says that his proof proceeds therefore to diverse localizations in Fourier space.

Theorem 8.1 (Bourgain [13]). — *For every p in $(1, +\infty]$, there exists a constant κ_p such that $\|\mathrm{M}_{Q_n}\|_{p \to p} \leqslant \kappa_p$ for every integer $n \geqslant 1$.*

We shall approach the maximal function problem for the cube by summing expressions such as $K^R - K^{2R}$, with

$$K^R = K_Q * G_{(1/R)},$$

where G is a Gaussian probability kernel, $G_{(1/R)}$ its dilate (2.7), and where R takes the values $1, 2, \ldots, 2^j, \ldots$ with j being any integer $\geqslant 0$. This is a Littlewood–Paley-type decomposition, similar to what we have seen before. By Prékopa–Leindler, K^R is a log-concave probability density. We shall set $m^R = \widehat{K^R}$ in what follows.

We will call the Carbery–Müller artillery and obtain when $1 < r < 2$, for every $\delta > 0$ and $R = 2^j$ with $j \geqslant 0$, bounds of the form

$$\Big\| \sup_{1 \leqslant t \leqslant 2} |K_{(t)}^R * f| \Big\|_r \leqslant \kappa_{\delta, r} R^\delta \|f\|_r, \quad \text{where } K_{(t)}^R := (K^R)_{(t)}.$$

Why this may be a decisive step will be explained below. According to Carbery's Proposition 6.14 (2), this bound will be consequence of the $L^r(\mathbb{R}^n)$-boundedness of the multiplier $(\xi \cdot \nabla)^\alpha m^R(\xi)$ for a value of $\alpha \in (1/r, 1)$. Next, following Müller, it will be enough to estimate in $L^s(\mathbb{R}^n)$, with $1 < s < r$, the "crucial" multiplier $|\xi| m^R(\xi)$. This is what Bourgain does along several pages, in a series of reductions bringing in many tools that are specific to the product structure of the cube.

8.1. Holding on Müller and Carbery

Let us specify the preceding rough outline. The final objective is to bound in $L^p(\mathbb{R}^n)$ the maximal operator M_Q for p below the limit $3/2$ that is known so far, proving that

$$\Big\| \sup_{t>0} |(K_Q)_{(t)} * f| \Big\|_p \leqslant \kappa_p \, \|f\|_p \,, \quad 1 < p < 2, \;\; f \in L^p(\mathbb{R}^n) \,.$$

We fix a value $p \in (1, 2)$ in all that follows. In order to obtain the property $(A_{\mathbf{2}})$, needed for applying Carbery's Proposition 6.6, we must show that

$$\Big\| \sup_{1 \leqslant t \leqslant 2} |K_{(t)} * f| \Big\|_{p_2} \leqslant \kappa \|f\|_{p_2} \,, \quad 1 < p_2 < p < 2 \,, \tag{$\mathbf{A}$}$$

where $K = K_Q - P$ and P is the Poisson kernel (1.32). This is the only missing fact for lowering the limitation $p > 3/2$ down to $p > 1$, as explained in the proof of Müller's Theorem 7.1. For the Poisson side it is fine, it remains to work on K_Q. We introduce the Gaussian kernel $G = (\gamma_n)_{(\sqrt{2}/\sqrt{\pi})}$ on $\mathbb{R}^n$. The variance of G is equal to $2/\pi$, thus independent of n, and $\widehat{G}(\xi) = \mathrm{e}^{-4\pi|\xi|^2}$ for every $\xi \in \mathbb{R}^n$. With this normalization for G, we have by (7.3) that

$$V(G) = \sqrt{\pi/2}\, V(\gamma_n) = 1 \,. \tag{8.1}$$

We decompose the Dirac probability measure δ_0 at the origin, in the sense of distributions, by means of the simple telescopic series

$$\delta_0 = G_{(1)} + (G_{(1/2)} - G_{(1)}) + \cdots + (G_{(2^{-k-1})} - G_{(2^{-k})}) + \cdots$$

and we decompose K_Q accordingly, using the approximations $K^R = K_Q * G_{(1/R)}$, for $R = 2^j \geqslant 1$ and j nonnegative integer, under the form

$$K_Q = K^1 + (K^2 - K^1) + \cdots + (K^{2^{j+1}} - K^{2^j}) + \cdots \,.$$

By Prékopa–Leindler, each K^R is a log-concave symmetric probability density on $\mathbb{R}^n$. It is isotropic, with a variance σ_R^2 satisfying

$$12^{-1} < \sigma_R^2 = 12^{-1} + 2\pi^{-1} R^{-2} < 1 \,, \quad R \geqslant 1 \,. \tag{8.2}$$

We set $\mathrm{d}\mu^R(x) = K^R(x)\,\mathrm{d}x$, $m^R = \widehat{K^R} = \widehat{\mu^R}$. It follows from (5.19) that m^R satisfies $(7.5.\mathbf{H}_\infty)$ with constants independent of n. We get

$$\Big| \frac{\mathrm{d}^j}{\mathrm{d}t^j}\, m^R(t\theta) \Big| \leqslant \frac{\delta_{j,c}}{1 + \pi|t|/\sqrt{3}} \leqslant \frac{\delta_{j,c}}{1 + |t|} \,, \quad \theta \in S^{n-1} \,, \;\; t \in \mathbb{R} \,, \;\; j \geqslant 0 \,. \tag{8.3}$$

We shall obtain the desired estimate (**A**) for p_2 by interpolating between p_1 and 2, where $1 < p_1 < p_2 < p < 2$. As we have said previously, we will show that for every $\delta > 0$, we have for all $R = 2^j \geqslant 1$ that

$$\Big\| \sup_{1 \leqslant t \leqslant 2} |K^R_{(t)} * f| \Big\|_{p_1} \leqslant \kappa_\delta R^\delta \, \|f\|_{p_1} , \quad f \in L^{p_1}(\mathbb{R}^n) , \tag{B}$$

and on the other hand, we prove that for every $f \in L^2(\mathbb{R}^n)$ we have

$$\Big\| \sup_{t>0} |(K^R - K^{2R})_{(t)} * f| \Big\|_2 \leqslant \kappa R^{-1/2} \, \|f\|_2 , \quad \Big\| \sup_{t>0} |K^1_{(t)} * f| \Big\|_2 \leqslant \kappa \, \|f\|_2 . \tag{C}$$

The second inequality in (**C**) is the log-concave version of Bourgain's L^2 theorem, Theorem 5.2. One can obtain the first part of (**C**) by the $\Gamma_B(K)$ criterion, Lemma 5.14. We have indeed, uniformly in $\theta \in S^{n-1}$ and in the dimension n (observe that $\widehat{G}$ has a radial expression independent of n), that

$$|\widehat{G}(u\theta) - \widehat{G}(2u\theta)| \lesssim u^2 \wedge \mathrm{e}^{-4\pi u^2} \leqslant |u| \wedge |u|^{-1} \quad \text{and}$$
$$|\theta \cdot \nabla \widehat{G}(u\theta) - \theta \cdot \nabla \widehat{G}(2u\theta)| \lesssim |u| (1 \wedge \mathrm{e}^{-4\pi u^2}) \leqslant 1 \wedge |u|^{-1} , \qquad u \in \mathbb{R} .$$

We apply Lemma 6.18 with $K_1 = K_Q$, $K_2 = G$, replacing $2^{|k|}$ with R and obtaining

$$\sum_{j \in \mathbb{Z}} \Big(\alpha_j(K^R) + \sqrt{\alpha_j(K^R) \beta_j(K^R)} \Big) \lesssim R^{-1/2} .$$

If $\delta > 0$ is sufficiently small we deduce by interpolation between (**B**) and (**C**) that there exists $\delta_1 > 0$ such that

$$\Big\| \sup_{1 \leqslant t \leqslant 2} |(K^R_{(t)} - K^{2R}_{(t)}) * f| \Big\|_{p_2} \leqslant \kappa_\delta R^{-\delta_1} \, \|f\|_{p_2}$$

and we get Property (**A**) by summing on the values $R = 2^j$ for all integers $j \geqslant 0$. We fix thus a value $\delta_* = \delta_*(p, p_2, p_1) > 0$ of δ, sufficiently small for implying that $\delta_1 > 0$ whenever $0 < \delta \leqslant \delta_*$. Precisely, if $\lambda \in (0, 1)$ is such that

$$\frac{1}{p_2} = \frac{1 - \lambda}{p_1} + \frac{\lambda}{2} ,$$

we need to choose $\delta_* > 0$ so that $-\delta_1 = (1 - \lambda)\delta_* - \lambda/2 < 0$, i.e., we select a value $\delta_* = \delta_*(p, p_2, p_1)$ such that $0 < \delta_*(p, p_2, p_1) < (p_2 - p_1)/(2p_1 - p_2 p_1)$.

For obtaining (**B**) we shall use the conclusion (2) of Carbery's Proposition 6.14 and also apply Müller's analysis. We need to show that for some $\alpha \in (1/p_1, 1)$ and $0 < \delta \leqslant \delta_*$, we have

$$2 \|m^R\|_{p_1 \to p_1} + \|(\xi \cdot \nabla)^\alpha m^R(\xi)\|_{p_1 \to p_1} \leqslant \kappa R^\delta \tag{8.4}$$

for all $R = 2^j$, $j \in \mathbb{N}$. There is no problem for m^R, which corresponds to convolution with a probability density, and for the other term we shall apply

Lemma 7.12 with $1 < p_0 < p_1 < 2$. For technical reasons, the value p_0, close to 1, is chosen in a way that its conjugate q_0 is an integer of the form 2^ν, with ν integer > 0. If we can prove that for a fixed $\delta > 0$ and for every $R = 2^j$, we have

$$\big\| |\xi| \, m^R(\xi) \big\|_{p_0 \to p_0} \leqslant \kappa_\delta R^\delta , \tag{8.5}$$

it follows from Lemma 7.12 that $\|(\xi \cdot \nabla)^\alpha m^R(\xi)\|_{p_1 \to p_1} \leqslant \kappa'_\delta (1 + R^{\delta\beta}) \leqslant \kappa''_\delta R^\delta$ for some $\beta \in (0, 1)$, uniformly in the dimension n according to (8.3). The conclusion (8.4) is then obtained.

By exploiting the inequality (2.22) on Riesz transforms, Müller's plan went on with a reduction to estimating the expression $\big\| \nabla \mu^R * g \big\|_{q_0}$ when $g \in L^{q_0}(\mathbb{R}^n)$ and $1/q_0 + 1/p_0 = 1$. We must show that for every $R = 2^j$ we have

$$\big\| \nabla \mu^R * g \big\|_{q_0} \leqslant \kappa_{p_0, \delta} R^\delta \|g\|_{q_0} ,$$

yielding (8.5) by Lemma 7.8. We use (8.1) and (7.3), which give

$$V(K^R) = V(\mu_Q * G_{(1/R)}) \leqslant V(G_{(1/R)}) = R . \tag{8.6}$$

By Lemma 7.9, this bound for the mass of $\theta \cdot \nabla \mu^R$ when $\theta \in S^{n-1}$ implies that $\|\nabla \mu^R * g\|_{L^\infty(\mathbb{R}^n)} \leqslant R \|g\|_{L^\infty(\mathbb{R}^n)}$. Then, by interpolation with the L^2 case given by (**C**), we can find when $2 < q < +\infty$ a bound in $L^q(\mathbb{R}^n)$ of the form

$$\begin{aligned} \|\nabla(\mu^R - \mu^{2R}) * g\|_{L^q(\mathbb{R}^n)} &\leqslant \kappa (R^{-1/2})^{2/q} R^{1-2/q} \|g\|_{L^q(\mathbb{R}^n)} \\ &= \kappa R^{1-3/q} \|g\|_{L^q(\mathbb{R}^n)} . \end{aligned}$$

This interpolation (L^∞, L^2) does not give the desired bound R^δ in $L^{q_0}(\mathbb{R}^n)$, with δ small, when $q_0 \geqslant 3$. However, it does give the right ingredient for the Bourgain–Carbery Theorem 6.2 when $3/2 < p \leqslant 2$, since $1 - 3/q < 0$ in this case.

For going farther than Müller, one has to prove inequalities that allow one to work in $L^r(\mathbb{R}^n)$, $2 < r < +\infty$, instead of $L^\infty(\mathbb{R}^n)$. This is done with the help of certain analytic semi-groups (Section 8.2), as well as a *ad hoc* method a la Bourgain, which he says inspired from martingale techniques (Section 8.3). Theorem 8.1 will be obtained once we have the following proposition, which we can apply with a value $\delta \leqslant \delta_*(p, p_2, p_1)$. We then conclude by the preceding discussion.

Proposition 8.2. — *For every $\delta > 0$ and $q_0 = 2^\nu$, with ν an integer $\geqslant 1$, there exists a constant $\kappa(q_0, \delta)$ such that for every $n \geqslant 1$ and $R = 2^k$, $k = 0, 1, \ldots$, one has*

$$\|\nabla \mu^R * g\|_{L^{q_0}(\mathbb{R}^n)} \leqslant \kappa(q_0, \delta) R^\delta \|g\|_{L^{q_0}(\mathbb{R}^n)} , \quad g \in L^{q_0}(\mathbb{R}^n) .$$

We shall keep $\delta > 0$, $p_0 = q_0/(q_0-1)$ and $R = 2^{k_0}$ fixed in the rest of Section 8.

8.1.1. *A priori* estimate

The proof will play with an *a priori* estimate

$$\big\|\nabla \mu^R * g\big\|_{L^{q_0}(\mathbb{R}^n)} \leqslant B(q_0, R, n) \|g\|_{L^{q_0}(\mathbb{R}^n)}\,, \quad g \in L^{q_0}(\mathbb{R}^n)\,, \tag{8.7}$$

and will aim to find a relation of the form $B(q_0, R, n) \leqslant c(q_0, \delta) R^\delta + \varepsilon B(q_0, R, n)$ for some $\varepsilon < 1$ and for R larger than some R_1, for example with $\varepsilon = 1/2$, thus reaching the conclusion that $B(q_0, R, n) \leqslant 2c(q_0, \delta) R^\delta$ when $R > R_1$. We know that $B(q_0, R, n)$ is finite for every dimension n, for instance as a consequence of the trivial bound $\|\nabla \mu^R\|_1 \leqslant \|\nabla G^R\|_1 \leqslant \kappa \sqrt{n}\, R$.

We must notice that the *a priori* estimate in $\mathbb{R}^n$ yields the same estimate for the dimensions $\ell \leqslant n$, with a smaller or equal constant, precisely, we must know that $B(q_0, R, \ell) \leqslant B(q_0, R, n)$ when $1 \leqslant \ell \leqslant n$. Indeed, the forthcoming proof in dimension n will bring the question down to dimensions $\ell \leqslant n$, where we shall use the *a priori* bound by $B(q_0, R, \ell)$. For justifying the validity of the same bound when $\ell \leqslant n$, apply the case n to a function g of the form $g_1 \otimes \varphi$, namely

$$g(x_1, x_2) = g_1(x_1) \varphi(x_2)\,,$$

where x_1 is in $\mathbb{R}^\ell$, $g_1 \in L^{q_0}(\mathbb{R}^\ell)$, $x_2 \in \mathbb{R}^{n-\ell}$ and where φ is a fixed C^∞ function with compact support in $\mathbb{R}^{n-\ell}$, not identically zero. The indicator of the cube and the Gaussian density have a product structure, which allows us to write

$$K^R(x_1, x_2) = K_1^R(x_1) \psi(x_2)\,, \quad \mathrm{d}\mu^R(x_1, x_2) = \mathrm{d}\mu_1^R(x_1) \otimes (\psi(x_2)\, \mathrm{d}x_2)\,,$$

where K_1, K_1^R and $\mathrm{d}\mu_1^R(x_1) = K_1^R(x_1)\, \mathrm{d}x_1$ correspond to the cube in $\mathbb{R}^\ell$, and ψ is a probability density on $\mathbb{R}^{n-\ell}$ corresponding to the cube in $\mathbb{R}^{n-\ell}$. We also have

$$\mu^R * g = (\mu_1^R * g_1) \otimes (\psi * \varphi)\,.$$

The gradient of $\mu^R * g$ contains $(\nabla \mu_1^R * g_1) \otimes (\psi * \varphi)$ in its first ℓ coordinates, thus

$$\begin{aligned} \big\|\nabla \mu_1^R * g_1\big\|_{L^{q_0}(\mathbb{R}^\ell)} \big\|\psi * \varphi\big\|_{L^{q_0}(\mathbb{R}^{n-\ell})} &= \big\|(\nabla \mu_1^R * g_1) \otimes (\psi * \varphi)\big\|_{L^{q_0}(\mathbb{R}^n)} \\ &\leqslant \big\|\nabla \mu^R * g\big\|_{L^{q_0}(\mathbb{R}^n)} \leqslant B(q_0, R, n) \, \|g\|_{L^{q_0}(\mathbb{R}^n)} \\ &= B(q_0, R, n) \, \|g_1\|_{L^{q_0}(\mathbb{R}^\ell)} \|\varphi\|_{L^{q_0}(\mathbb{R}^{n-\ell})}\,. \end{aligned}$$

This yields $B(q_0, R, \ell) \leqslant B(q_0, R, n) \, \|\varphi\|_{L^{q_0}(\mathbb{R}^{n-\ell})} \big/ \big\|\psi * \varphi\big\|_{L^{q_0}(\mathbb{R}^{n-\ell})}$ and by spreading φ, replacing it with $\varphi_k : x \mapsto \varphi(x/k)$, $k \to +\infty$, one makes the quotient of norms tend to 1, thus proving that $B(q_0, R, \ell) \leqslant B(q_0, R, n)$.

8.2. First reduction

One applies a result of Pisier [62] about holomorphic semi-groups. If $\mathbf{T} = (T_j)_{j=1}^n$ is a family of bounded linear operators on $L^q(X, \Sigma, \mu)$, $1 \leqslant q \leqslant +\infty$, we introduce for every subset $J \subset N = \{1, \ldots, n\}$ the operators

$$\mathbf{T}^J = \prod_{j \in J} T_j \,, \quad \mathbf{T}^{\sim J} = \mathbf{T}^{N \setminus J} = \prod_{j \notin J} T_j \,,$$

and $\mathbf{T}^{\sim j}$ will be a short form for $\mathbf{T}^{\sim \{j\}}$, $1 \leqslant j \leqslant n$. We found the notation $\mathbf{T}^{\sim J}$ convenient, but it might be ambiguous, since it depends on the ambient set N.

Given commuting projectors $(E_j)_{j=1}^n$, one can consider the semi-group

$$T_t = \prod_{j=1}^n \bigl(E_j + \mathrm{e}^{-t}(I - E_j)\bigr), \quad t \geqslant 0 \,,$$

where I denotes the identity operator. If we set $z = \mathrm{e}^{-t}$ and expand the product, we can arrange it according to powers of z, displaying in this way homogeneous parts $z^k H_k$ of degree k. We see that

$$T_t = \sum_{k=0}^n z^k \Bigl(\sum_{|J|=k} \mathbf{E}^{\sim J} (\mathbf{I} - \mathbf{E})^J \Bigr) = \sum_{k=0}^n z^k H_k = \sum_{k=0}^n \mathrm{e}^{-kt} \, H_k \,.$$

Letting Σ_k denote the family of subsets $J \subset N$ of cardinality k, we have

$$H_k = \sum_{J \in \Sigma_k} \mathbf{E}^{\sim J} (\mathbf{I} - \mathbf{E})^J \,, \quad k = 0, \ldots, n \,, \quad \text{and} \quad \sum_{k=0}^n H_k = T_0 = I \,. \tag{8.8}$$

Proposition 8.3 (after Pisier [62]). — *Let $(E_j)_{j=1}^n$ be a family of commuting conditional expectation projectors on $L^q(X, \Sigma, \mu)$, $1 < q < +\infty$, and consider the semi-group*

$$P_t = \prod_{j=1}^n \bigl(\mathrm{e}^{-t} I + (1 - \mathrm{e}^{-t}) E_j\bigr) = \prod_{j=1}^n \bigl(E_j + \mathrm{e}^{-t}(I - E_j)\bigr), \quad t \geqslant 0 \,.$$

This semi-group is analytic on $L^q(X, \Sigma, \mu)$, $1 < q < +\infty$, with an extension $(P_z)_{z \in \Omega_{\varphi_q}}$ to a sector $\Omega_{\varphi_q} = \{z = r\,\mathrm{e}^{\mathrm{i}\theta} : r > 0,\ |\theta| < \varphi_q\}$ in $\mathbb{C}$, where $\varphi_q > 0$ depends on q only. The extension is bounded uniformly in q on every compact subset of Ω_{φ_q}. There exists $h_q \geqslant 1$ independent of n such that whenever $0 \leqslant k \leqslant n$, the homogeneous part H_k in (8.8) is bounded on $L^q(X, \Sigma, \mu)$ by $(h_q)^k$.

That $h_q \geqslant 1$ can be seen on any example $P_t f = E_1 f + \mathrm{e}^{-t}(f - E_1 f)$ with $n = 1$ and $E_1 \neq I$. Then H_1 is the projector $I - E_1 \neq 0$, hence

$h_q \geqslant \|H_1\|_{q\to q} \geqslant 1$. If $(E_{j,s})_{j=1}^n$, $s \in [0,1]$, is a family of such conditional expectations, where $E_{j,s}$ and $E_{k,t}$ commute for all $j \neq k$ and all $s, t \in [0,1]$, and if we set for example

$$U_j = \int_0^1 E_{j,s}\,\mathrm{d}s\,, \quad j = 1, \ldots, n\,,$$

then we see that

$$Q_t = \prod_{j=1}^n \bigl(\mathrm{e}^{-t}\, I + (1 - \mathrm{e}^{-t}) U_j\bigr) = \int_{[0,1]^n} P_{t,s_1,\ldots,s_n}\,\mathrm{d}s_1 \ldots \mathrm{d}s_n\,,$$

where each $P_{t,s_1,\ldots,s_n} = \prod_{j=1}^n \bigl(\mathrm{e}^{-t}\, I + (1-\mathrm{e}^{-t}) E_{j,s_j}\bigr)$ is of "Pisier type". Also, the corresponding homogeneous parts are of the form

$$\begin{aligned} \widetilde{H}_k &= \sum_{J\in\Sigma_k} \mathbf{U}^{\sim J} (\mathbf{I} - \mathbf{U})^J \\ &= \int_{[0,1]^n} \sum_{J\in\Sigma_k} \Bigl(\prod_{i\notin J} E_{i,s_i}\Bigr)\Bigl(\prod_{j\in J} (I - E_{j,s_j})\Bigr)\,\mathrm{d}s_1\mathrm{d}s_2 \ldots \mathrm{d}s_n \end{aligned}$$

that are averages of terms $H_k(s_1, \ldots, s_n)$ bounded by h_q^k according to Proposition 8.3. The result of Proposition 8.3 generalizes thus to families such as $(U_j)_{j=1}^n$.

We shall apply Proposition 8.3 to operators $(E_j)_{j=1}^n$ of conditional expectation on $L^q(\mathbb{R}^n)$, where each E_j is acting in the x_j variable and $1 \leqslant j \leqslant n$. For one variable and $s_0 \in \mathbb{R}$ fixed, we associate to a locally integrable function f on $\mathbb{R}$ its averages on length one intervals $I_r = [s_0 + r, s_0 + r + 1)$, $r \in \mathbb{Z}$, defining E_{s_0} by

$$(E_{s_0} f)(v) = \sum_{r\in\mathbb{Z}} \Bigl(\int_{I_r} f(s)\,\mathrm{d}s\Bigr) \mathbf{1}_{I_r}(v)\,, \quad v \in \mathbb{R}\,.$$

This operator is a conditional expectation, as considered in Remark 1.2. We define operators E_{j,s_0}, $j = 1, \ldots, n$, on $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by the analogous formula, acting on the x_j variable. When $j = 1$ for example, we let

$$(E_{1,s_0} f)(x_1, x_2, \ldots, x_n) = \sum_{r\in\mathbb{Z}} \Bigl(\int_{I_r} f(s, x_2, \ldots, x_n)\,\mathrm{d}s\Bigr) \mathbf{1}_{I_r}(x_1)\,.$$

Averaging on values of s_0, one can replace the E_js by convolution operators with probability densities χ on $\mathbb{R}$ of the form

$$\chi(x) = \int_{\mathbb{R}} \mathbf{1}_{[s,s+1]}(x)\,\mathrm{d}\nu(s)\,, \quad x \in \mathbb{R}\,, \tag{8.9}$$

where ν is a probability measure on the line. We see that $\chi(x) = F(x) - F(x-1)$, with $F(x) = \nu[(-\infty, x)]$ non-decreasing, $F(-\infty) = 0$ and

$F(+\infty) = 1$. One can also proceed to changes of scale. Summarizing, we have the lemma that follows.

LEMMA 8.4 (Bourgain [13], Lemma 5). — *Let χ be a compactly supported probability density on $\mathbb{R}$ of the form (8.9). Denote by T_j the convolution operator with $\chi_{(t_j)}$ in the x_j variable, $t_j > 0$, $j = 1, \ldots, n$. For $0 \leqslant k \leqslant n$, the norm of the operator*

$$H_k := \sum_{S \in \Sigma_k} \mathbf{T}^{\sim S} (\mathbf{I} - \mathbf{T})^S$$

on $L^q(\mathbb{R}^n)$ is bounded by h_q^k, with $1 < q < +\infty$ and h_q from Proposition 8.3.

In what follows, we denote by T_j, $j = 1, \ldots, n$, the convolution in the x_j variable on $L^q(\mathbb{R}^n)$ by $\eta_{(w_0)}(x_j)$, where $w_0 = R^{-\delta/2}$ will stay fixed and where

$$\eta(x) = (1-|x|)_+ = \int_{-1/2}^{1/2} \mathbf{1}_{[-1/2+s, 1/2+s]}(x) \, \mathrm{d}s = (\mathbf{1}_{[-1/2,1/2]} * \mathbf{1}_{[-1/2,1/2]})(x) \, .$$

Since η is a convolution square, $\widehat{\eta}$ is real and nonnegative. We have

$$\widehat{\eta}(t) = \Bigl(\frac{\sin(\pi t)}{\pi t} \Bigr)^2 , \quad \text{and} \quad \widehat{\eta}\,''(t) = -4\pi^2 \int_{\mathbb{R}} s^2 (1 - |s|)_+ \cos(2\pi s t) \, \mathrm{d}s$$

for every $t \in \mathbb{R}$, thus

$$|\widehat{\eta}\,''(t)| \leqslant 8\pi^2 \int_0^1 s^2 (1-s) \, \mathrm{d}s = \frac{8\pi^2}{12} < 8 \, .$$

By the Taylor formula we get

$$0 \leqslant 1 - \widehat{\eta}(t) \leqslant (4t^2) \wedge 1 \, . \tag{8.10}$$

For every subset $S \subset N := \{1, \ldots, n\}$ let us set

$$\Gamma^S = \mathbf{T}^{\sim S} (\mathbf{I} - \mathbf{T})^S \, . \tag{8.11}$$

The homogeneous parts (H_k) in $Q_t = \prod_{j=1}^n \bigl(\mathrm{e}^{-t} I + (1 - \mathrm{e}^{-t}) T_j \bigr)$ have the form

$$H_k = \sum_{S \in \Sigma_k} \Gamma^S , \quad 0 \leqslant k \leqslant n \, , \quad \text{and} \quad \sum_{k=0}^n H_k = I \, .$$

In particular, $H_0 = \Gamma^\emptyset = \mathbf{T}^N = \prod_{j=1}^n T_j$ has norm $\leqslant 1$ on every space $L^q(\mathbb{R}^n)$, for $1 \leqslant q \leqslant +\infty$, since H_0 is the convolution with the product probability density $\prod_{j=1}^n \eta_{(w_0)}(x_j)$. When $1 < q < +\infty$ and $1 \leqslant k \leqslant n$, we have $\|H_k\|_{q \to q} \leqslant h_q^k$ by Proposition 8.3. It is convenient to set $H_k = 0$ below when $k > n$.

To every given function g in $L^q(\mathbb{R}^n)$, we shall apply a decomposition of the form $g = H_0 g + \cdots + H_{M-1} g + h$, and consider the corresponding expression

$$\nabla\mu^R * g = \nabla\mu^R * H_0 g + \cdots + \nabla\mu^R * H_{M-1} g + \nabla\mu^R * h\,, \tag{8.12}$$

where $M \geqslant 1$ will be chosen as a function of the already fixed p_0 and $\delta > 0$. We have to estimate in $L^{q_0}(\mathbb{R}^n)$ the successive terms in (8.12). The function h is considered as a small rest, the mapping $g \mapsto \nabla\mu^R * h$ will be handled in $L^2(\mathbb{R}^n)$ by a Fourier estimate, and in some $L^{q_1}(\mathbb{R}^n)$, $q_1 > q_0$, as a consequence of Proposition 8.3. We choose M large enough for deducing from $\|\nabla\mu^R * h\|_2 \leqslant \kappa R^{1-\delta M/2} \|g\|_2$ and $\|\nabla\mu^R * h\|_{q_1} \leqslant \kappa R \|g\|_{q_1}$ that one has by interpolation

$$\|\nabla\mu^R * h\|_{q_0} \leqslant \kappa(q_0, \delta)\, \|g\|_{q_0}\,, \tag{8.13}$$

which is just perfect in the direction of (8.7). Recall that μ_j^R denotes the jth partial derivative $\partial_j \mu^R = (\partial_j \mu_Q) * G^R$ of μ^R, so that $|\nabla\mu^R * h|^2 = \sum_{j=1}^n |\mu_j^R * h|^2$.

We factor the mapping $g \mapsto \nabla\mu^R * h$ into $U_{K^R} : h \mapsto \nabla\mu^R * h$ and $A : g \mapsto h$, i.e., $A = I - H_0 - \cdots - H_{M-1} = \sum_{k \geqslant M} H_k$. We look for estimates in L^2 and L^q, $q_0 < q < +\infty$. For U_{K^R} we use Lemma 7.11 and get by (8.2) and (8.6) that

$$\|U_{K^R}\|_{q \to q} \leqslant 2^{1/q} \sigma_R^{-2/q} V(K^R)^{1-2/q} \leqslant (24)^{1/q} R^{1-2/q} < 5\,R$$

since $q \geqslant 2$. On the other hand, by Lemma 8.4, the mapping $A : g \mapsto h$ is bounded in $L^q(\mathbb{R}^n)$ by $1 + \sum_{k=0}^{M-1} h_q^k \leqslant (M+1) h_q^{M-1}$. It follows that

$$\|\nabla\mu^R * h\|_q = \|U_{K^R} h\|_q \leqslant 5R \|h\|_q \leqslant 5R(M+1) h_q^{M-1} \|g\|_q\,. \tag{8.14}$$

This is also valid when $q = 2$, but the point is that we will then get a much better bound by factoring now $g \mapsto \nabla\mu^R * h$ as $U_{G^R} \circ B$, with $U_{G^R} : f \mapsto \nabla G^R * f$ and $B : g \mapsto \mu_Q * Ag = \mu_Q * h$. We begin by estimating

$$\|\mu_Q * h\|_2 = \Big\| \mu_Q * \Big(\sum_{k \geqslant M} H_k \Big) g \Big\|_2 = \Big\| \mu_Q * \Big(\sum_{|S| \geqslant M} \Gamma^S \Big) g \Big\|_2 .$$

One needs to control the $L^\infty(\mathbb{R}^n)$ norm of the function $\xi \mapsto L(\xi)$, where L is the multiplier associated to the mapping B. It is the aim of the next lemma. One sees that

$$L(\xi) := \Big(\prod_{j=1}^n \frac{\sin(\pi\xi_j)}{\pi\xi_j} \Big) \Big(\sum_{|S| \geqslant M} \prod_{j \notin S} \widehat{\eta}(w_0 \xi_j) \prod_{j \in S} \big(1 - \widehat{\eta}(w_0 \xi_j)\big) \Big) .$$

Lemma 8.5 (see [13, Equations (2.9), (2.11)]). — *For $0 \leqslant u \leqslant 1/4$ and every $\xi \in \mathbb{R}^n$, one has that*

$$\left| \Big(\prod_{j=1}^{n} \frac{\sin(\pi\xi_j)}{\pi\xi_j} \Big) \Big(\sum_{|S| \geqslant M} \prod_{j \notin S} \widehat{\eta}(u\xi_j) \prod_{j \in S} \big(1 - \widehat{\eta}(u\xi_j)\big) \Big) \right| \leqslant u^M .$$

Proof. — We know from (8.10) that $0 \leqslant \widehat{\eta}(t) \leqslant 1$ and $1 - \widehat{\eta}(t) \leqslant (4t^2) \wedge 1$. We introduce $v = 1/u \geqslant 4$ and begin by checking that for every $t \geqslant 0$, we have

$$X(t) := \left| \frac{\sin(\pi t)}{\pi t} \right| \big(1 + v[(4u^2t^2) \wedge 1]\big) \leqslant 1 .$$

Consider first the case $0 \leqslant t \leqslant 1/(2u)$. One has then $4u^2t^2 \leqslant 1$ and it follows that $1 + v[(4u^2t^2) \wedge 1] = 1 + 4ut^2$. If in addition $0 \leqslant t \leqslant 1$, then, for example by the Euler product formula (3.2.**E**), we have $\big|\sin(\pi t)/\pi t\big| \leqslant 1 - t^2$, and since $4u \leqslant 1$ by assumption, we get

$$X(t) \leqslant (1 - t^2)(1 + 4ut^2) \leqslant (1 - t^2)(1 + t^2) \leqslant 1 .$$

When $1 < t \leqslant 1/(2u)$, we have

$$\left| \frac{\sin(\pi t)}{\pi t} \right| \big(1 + 4ut^2\big) \leqslant \frac{1 + 4ut^2}{\pi t} = \frac{1}{\pi}\big(1/t + 4ut\big) \leqslant \frac{3}{\pi} < 1 .$$

In the second case, when $2ut > 1$, we can write

$$X(t) \leqslant \frac{1 + v}{\pi t} \leqslant \frac{2u(1 + v)}{\pi} \leqslant \frac{1/2 + 2}{\pi} < 1 .$$

Expanding the product $\prod_{j=1}^{n} X(\xi_j)$ and since X is even, one sees that

$$\begin{aligned} 1 \geqslant \prod_{j=1}^{n} X(\xi_j) &= \prod_{j=1}^{n} \left| \frac{\sin(\pi\xi_j)}{\pi\xi_j} \right| \big(1 + v[(4u^2\xi_j^2) \wedge 1]\big) \\ &\geqslant \prod_{j=1}^{n} \left| \frac{\sin(\pi\xi_j)}{\pi\xi_j} \right| \Big(\widehat{\eta}(u\xi_j) + v\big(1 - \widehat{\eta}(u\xi_j)\big) \Big) \\ &\geqslant v^M \left| \Big(\prod_{j=1}^{n} \frac{\sin(\pi\xi_j)}{\pi\xi_j} \Big) \Big(\sum_{|S| \geqslant M} \prod_{j \notin S} \widehat{\eta}(u\xi_j) \prod_{j \in S} \big(1 - \widehat{\eta}(u\xi_j)\big) \Big) \right| . \qquad \square \end{aligned}$$

By Lemma 7.11, we have that $\|U_{G^R}\|_{2\to 2} \leqslant \sqrt{\pi} R < 2R$, because the variance of G^R is $2\pi^{-1}R^{-2}$. Let us define R_0 by $R_0^{\delta/2} = 4$. If $R \geqslant R_0$, then $w_0 = R^{-\delta/2} \leqslant 1/4$, we obtain from Lemma 8.5 with $u = w_0$ the final control

$$\|\nabla\mu^R * h\|_2 = \|U_{G^R}(\mu_Q * h)\|_2 \leqslant 2R\|\mu_Q * h\|_2 \leqslant 2R R^{-\delta M/2}\|g\|_2 .$$

We use now (8.14) with for example $q = q_1 = 2q_0/p_0 > q_0$. Letting $\theta = 1/p_0$, we have $(1 - \theta)/2 + \theta/q_1 = 1/q_0$ and we see by interpolation for $g \mapsto \nabla\mu^R * h$

that

$$\|\nabla\mu^R * h\|_{q_0} \leqslant \bigl(2R^{-\delta M/2}\bigr)^{1/q_0} R\bigl(5(M+1)h_{q_1}^{M-1}\bigr)^{1/p_0} \|g\|_{q_0}\,.$$

We select $M = M(\delta) = \lceil 2q_0/\delta \rceil$, so that $\delta M/(2q_0) \geqslant 1$. When $R \geqslant R_0$ we get

$$\|\nabla\mu^R * h\|_{q_0} \leqslant \kappa_{q_0,\delta}\|g\|_{q_0} \quad \text{with} \quad \kappa_{q_0,\delta} \leqslant 5(2+2q_0/\delta)^{1/p_0} h_{2q_0/p_0}^{2q_0/(\delta p_0)}\,. \tag{8.15}$$

In what follows we assume that $R \geqslant R_0$, hence $R^\delta \geqslant 16$. In the conclusion section, we shall need the following bound for a Fourier transform.

Lemma 8.6. — *For every $r \in \mathbb{R}$, $\ell \geqslant 1$ and all $\xi = (\xi_1, \ldots, \xi_\ell) \in \mathbb{R}^\ell$, one has that*

$$(1 - \mathrm{e}^{-r^2|\xi|^2}) \prod_{j=1}^{\ell} \widehat{\eta}(\xi_j) \leqslant r^2\,.$$

Proof. — We observe first that

$$\widehat{\eta}(t) = \Bigl(\frac{\sin(\pi t)}{\pi t}\Bigr)^2 \leqslant \frac{1}{1+t^2}\,.$$

This is clear when $|t| \geqslant 1$ because $\widehat{\eta}(t) \leqslant (\pi t)^{-2}$ and $1 + t^2 < \pi^2 t^2$ in this case. When $|t| \leqslant 1$ we have $\widehat{\eta}(t) \leqslant |\sin(\pi t)|/|\pi t| \leqslant (1-t^2) \leqslant (1+t^2)^{-1}$ by (3.2.**E**). It suffices thus to bound for $x \in \mathbb{R}^\ell$ the expression

$$F(x) = (1 - \mathrm{e}^{-r^2|x|^2}) \prod_{j=1}^{\ell} \frac{1}{1+x_j^2} \geqslant 0\,.$$

The function F tends to 0 at infinity, we have at any maximum $\overline{x} \neq 0$ that

$$\frac{2r^2 \overline{x}_j\, \mathrm{e}^{-r^2|\overline{x}|^2}}{1 - \mathrm{e}^{-r^2|\overline{x}|^2}} = \frac{2\overline{x}_j}{1+\overline{x}_j^2}\,, \quad j = 1, \ldots, \ell\,.$$

The nonzero coordinates of $\overline{x}$ have the same square $\overline{x}_j^2 =: y > 0$, and if k denotes their cardinality, we have $0 < k \leqslant \ell$ and $|\overline{x}|^2 = ky$. It follows that

$$kr^2 y \leqslant \mathrm{e}^{kr^2 y} - 1 = r^2(1+y) \leqslant r^2(1+y)^k\,.$$

Finally, we have $F(\overline{x}) = (1 - \mathrm{e}^{-kr^2 y})(1+y)^{-k} \leqslant kr^2 y(1+y)^{-k} \leqslant r^2$. □

8.2.1. Decoupling

We have to analyze each of the expressions $\nabla\mu^R * H_k g$ in (8.12), for $0 \leqslant k < M$. When $1 \leqslant k < M$, we handle this by a decoupling argument that will allow us to essentially reduce to the cases where $k = 0, 1$, but in a dimension $\ell \leqslant n$. Before proceeding by a Bourgainian technique of "selectors", we split

$$\bigl|\nabla\mu^R * H_k g\bigr| = \Bigl(\sum_{j=1}^n |\mu_j^R * H_k g|^2\Bigr)^{1/2} = \Bigl(\sum_{j=1}^n \Bigl|\mu_j^R * \Bigl(\sum_{S\in\Sigma_k} \Gamma^S g\Bigr)\Bigr|^2\Bigr)^{1/2}$$

into two. For each j in $\{1,\dots,n\}$, let Σ_k^j and $\Sigma_k^{\sim j}$ denote respectively the family of subsets S of $\{1,\dots,n\}$ with cardinality $|S| = k$ containing j, resp. such that $j \notin S$. Then $\bigl|\nabla\mu^R * H_k g\bigr|$ is bounded by the sum of the two expressions

$$\mathbf{E}_k(R,n,g) := \Bigl(\sum_{j=1}^n \Bigl|\mu_j^R * \Bigl(\sum_{S\in\Sigma_k^{\sim j}} \Gamma^S g\Bigr)\Bigr|^2\Bigr)^{1/2} \tag{8.16a}$$

and

$$\mathbf{F}_k(R,n,g) := \Bigl(\sum_{j=1}^n \Bigl|\mu_j^R * \Bigl(\sum_{S\in\Sigma_k^{j}} \Gamma^S g\Bigr)\Bigr|^2\Bigr)^{1/2}. \tag{8.16b}$$

Assume that $1 \leqslant k < M = M(\delta)$. Let $(\gamma_i)_{1\leqslant i\leqslant n}$ be independent $\{0,1\}$-valued random variables with mean $1/(k+1)$ on some probability space $(\Omega,\mathcal{F},P)$. For each j in $\{1,\dots,n\}$ and $S \in \Sigma_k^{\sim j}$, let $\sigma_{S,j} = \gamma_j \prod_{i\in S}(1-\gamma_i)$. We have that

$$\mathrm{E}\,\sigma_{S,j} = \frac{1}{k+1}\Bigl(1 - \frac{1}{k+1}\Bigr)^k = \frac{k^k}{(k+1)^{k+1}} =: e_k\,, \quad j = 1,\dots,n\,,$$

and $e_k^{-1} \leqslant \mathrm{e}(k+1) \leqslant \mathrm{e}\,M$ because $\mathrm{e}^{1/k} > 1 + 1/k$. By convexity, we see that

$$\begin{aligned} e_k\,\mathbf{E}_k(R,n,g) &= \Bigl(\sum_{j=1}^n \Bigl|\mu_j^R * \Bigl(\sum_{S\in\Sigma_k^{\sim j}} \bigl[\mathrm{E}_\omega\,\sigma_{S,j}(\omega)\bigr]\,\Gamma^S g\Bigr)\Bigr|^2\Bigr)^{1/2} \\ &\leqslant \mathrm{E}_\omega\,\Bigl[\Bigl(\sum_{j=1}^n \Bigl|\mu_j^R * \Bigl(\sum_{S\in\Sigma_k^{\sim j}} \sigma_{S,j}(\omega)\,\Gamma^S g\Bigr)\Bigr|^2\Bigr)^{1/2}\Bigr]. \end{aligned}$$

Let $q \geqslant 1$ be given. It follows that for some $\omega_0 \in \Omega$, we have

$$\begin{aligned} &\bigl\|\mathbf{E}_k(R,n,g)\bigr\|_{L^q(\mathbb{R}^n)} \\ &\qquad \leqslant \mathrm{e}\,M \Bigl\|\Bigl(\sum_{j=1}^n \Bigl|\mu_j^R * \Bigl(\sum_{S\in\Sigma_k^{\sim j}} \sigma_{S,j}(\omega_0)\,\Gamma^S g\Bigr)\Bigr|^2\Bigr)^{1/2}\Bigr\|_{L^q(\mathbb{R}^n)}. \end{aligned} \tag{8.17}$$

Let $J_0 = \{j : \gamma_j(\omega_0) = 1\}$. Then $\sigma_{S,j}(\omega_0) = 0$ whenever S meets J_0 or $j \notin J_0$. The $L^q(\mathbb{R}^n)$ norm at the right-hand side of (8.17) is therefore the norm of

$$E(J_0, g) := \Bigl(\sum_{j \in J_0} \Bigl| \mu_j^R * \Bigl(\sum_{S \in \Sigma_k^{\sim J_0}} \Gamma^S g \Bigr) \Bigr|^2 \Bigr)^{1/2},$$

where $\Sigma_k^{\sim J_0}$ denotes the family of subsets S of $\{1, \ldots, n\}$ such that $|S| = k$ and that are disjoint from J_0. Let us introduce the operator

$$\mathbf{U} = \sum_{S \in \Sigma_k^{\sim J_0}} \mathbf{T}^{\sim (J_0 \cup S)} \, (\mathbf{I} - \mathbf{T})^S \quad \text{and the function} \quad \Psi = \mathbf{U}\, g$$

on $\mathbb{R}^n$. We see that $\mathbf{T}^{J_0} \mathbf{U} = \sum_{S \in \Sigma_k^{\sim J_0}} \Gamma^S$, and the operator $\mathbf{U}$ acts on the variables not in J_0 as does the kth homogeneous part H_k relative to $\mathbb{R}^{\{1,\ldots,n\} \setminus J_0}$. Consequently, applying Proposition 8.3 in the variables $\mathbf{x}^{\sim J_0} = (x_i)_{i \notin J_0}$, we get

$$\|\Psi_{\mathbf{x}^{J_0}}\|_{L^q(\mathbb{R}^{\sim J_0})} \leqslant h_q^k \, \|g_{\mathbf{x}^{J_0}}\|_{L^q(\mathbb{R}^{\sim J_0})} \tag{8.18}$$

for every fixed $\mathbf{x}^{J_0} = (x_i)_{i \in J_0}$, where $f_{\mathbf{x}^J}(\mathbf{x}^{\sim J}) := f(\mathbf{x}^J, \mathbf{x}^{\sim J}) = f(x)$, and we see that $E(J_0, g) = \bigl(\sum_{j \in J_0} \bigl| \mu_j^R * \mathbf{T}^{J_0} \Psi \bigr|^2 \bigr)^{1/2}$. Assume that there exists $b_0(q_0, R, n)$ such that for every subset J of $\{1, \ldots, n\}$ and $f \in L^{q_0}(\mathbb{R}^J)$ we have

$$\Bigl\| \Bigl(\sum_{j \in J} \Bigl| (\mu_{Q^J})_j^R * \Bigl(\prod_{i \in J} T_i \Bigr) f \Bigr|^2 \Bigr)^{1/2} \Bigr\|_{L^{q_0}(\mathbb{R}^J)} \leqslant b_0(q_0, R, n) \, \|f\|_{L^{q_0}(\mathbb{R}^J)}, \tag{8.19}$$

with μ_{Q^J} uniform on $Q^J := [-1/2, 1/2]^J$ in $\mathbb{R}^J$. It follows from (8.18), by integrating in the J_0 variables, that

$$\|E(J_0, g)\|_{L^{q_0}(\mathbb{R}^n)} \leqslant b_0(q_0, R, n) \, h_q^k \, \|g\|_{L^{q_0}(\mathbb{R}^n)}.$$

For $\mathbf{F}_k(R, n, g)$ we proceed similarly, writing each $S \in \Sigma_k^j$ as $S = \{j\} \cup S_1$, with $|S_1| = k - 1$, and using now $\sigma_{S_1, j} = \gamma_j \prod_{i \in S_1} (1 - \gamma_i)$ for which we have $\mathrm{E}\, \sigma_{S_1, j} = k^{k-1} (k+1)^{-k} \geqslant 1/(\mathrm{e}\, k) > 1/(\mathrm{e}\, M)$. We obtain for some $\omega_0 \in \Omega$ that

$$\bigl\| \mathbf{F}_k(R, n, g) \bigr\|_{L^q(\mathbb{R}^n)} \leqslant \mathrm{e}\, M \Bigl\| \Bigl(\sum_{j=1}^n \Bigl| \mu_j^R * \Bigl(\sum_{S \in \Sigma_k^j} \sigma_{S_1, j}(\omega_0) \, \Gamma^S g \Bigr) \Bigr|^2 \Bigr)^{1/2} \Bigr\|_{L^q(\mathbb{R}^n)}.$$

Considering again $J_0 = \{j : \gamma_j(\omega_0) = 1\}$, we get instead of $E(J_0, g)$ the expression

$$F(J_0, g) = \Bigl(\sum_{j \in J_0} \Bigl| \mu_j^R * \Bigl(\sum_{S_1 \in \Sigma_{k-1}^{\sim J_0}} \Gamma^{\{j\} \cup S_1} g \Bigr) \Bigr|^2 \Bigr)^{1/2}.$$

When $k = 1$, we have $S_1 = \emptyset$, $S = \{j\}$ and $\sigma_{S_1,j} = \gamma_j$, the argument remains correct but becomes "inactive". Let now $\Psi = \sum_{S_1 \in \Sigma_{k-1}^{\sim J_0}} \mathbf{T}^{\sim(J_0 \cup S_1)}(\mathbf{I}-\mathbf{T})^{S_1} g$, satisfying by Proposition 8.3 applied to $L^q(\mathbb{R}^{\sim J_0})$ the inequality

$$\|\Psi_{\mathbf{x}^{J_0}}\|_{L^q(\mathbb{R}^{\sim J_0})} \leqslant h_q^{k-1} \|g_{\mathbf{x}^{J_0}}\|_{L^q(\mathbb{R}^{\sim J_0})} .$$

For each $j \in J_0$, let $B_j = (I - T_j)\mathbf{T}^{J_0 \setminus \{j\}}$. Then $F(J_0, g) = \big(\sum_{j \in J_0} |\mu_j^R * B_j \Psi|^2\big)^{1/2}$. If there exists $b_1(q_0, R, n)$ such that for every subset J of $\{1, \dots, n\}$ and every function $f \in L^{q_0}(\mathbb{R}^J)$ we have an inequality

$$\Big\| \Big(\sum_{j \in J} |\mu_j^R * (I - T_j) \Big(\prod_{i \in J,\, i \neq j} T_i \Big) f|^2 \Big)^{1/2} \Big\|_{L^{q_0}(\mathbb{R}^J)} \leqslant b_1(q_0, R, n) \|f\|_{L^{q_0}(\mathbb{R}^J)} , \quad (8.20)$$

it implies that $F(J_0, g)$ may be bounded by $b_1(q_0, R, n)\, h_q^{k-1} \|g\|_{L^{q_0}(\mathbb{R}^n)}$ in $L^{q_0}(\mathbb{R}^n)$.

In view of (8.19) and (8.20), all we need to do in order to control in $L^{q_0}(\mathbb{R}^n)$ the expressions $\nabla \mu^R * H_k g$, when $1 \leqslant k < M$, is to establish in all lower dimensions $\ell \leqslant n$ and for every function $f \in L^{q_0}(\mathbb{R}^\ell)$ the inequalities

$$\big\|\nabla \mu^R * H_0 f\big\|_{L^{q_0}(\mathbb{R}^\ell)} = \Big\| \Big(\sum_{j=1}^{\ell} |\mu_j^R * H_0 f|^2 \Big)^{1/2} \Big\|_{L^{q_0}(\mathbb{R}^\ell)} \leqslant b_0(q_0, R, n) \|f\|_{L^{q_0}(\mathbb{R}^\ell)} \quad (8.21)$$

and

$$\big\|\mathbf{F}(R, \ell, f)\big\|_{L^{q_0}(\mathbb{R}^\ell)} := \Big\| \Big(\sum_{j=1}^{\ell} |\mu_j^R * \Gamma^j f|^2 \Big)^{1/2} \Big\|_{L^{q_0}(\mathbb{R}^\ell)} \leqslant b_1(q_0, R, n) \|f\|_{L^{q_0}(\mathbb{R}^\ell)} \quad (8.22)$$

for suitable $b_0(q_0, R, n)$ and $b_1(q_0, R, n)$, with $\Gamma^j := \Gamma^{\{j\}} = (I - T_j)\, \mathbf{T}^{\{1,\dots,\ell\} \setminus \{j\}}$. Note that (8.21) controls the so far neglected term $k = 0$ in (8.12). From (8.13) and the preceding, this will permit us to estimate

$$\big\|\nabla \mu^R * g\big\|_{L^{q_0}(\mathbb{R}^n)} = \Big\| \Big(\sum_{j=1}^{n} |\mu_j^R * g|^2 \Big)^{1/2} \Big\|_{L^{q_0}(\mathbb{R}^n)} \leqslant C(q_0, R, n) \|g\|_{L^{q_0}(\mathbb{R}^n)} .$$

Recalling (8.15), (8.17) and that $M = M(\delta)$ depends on the fixed value $\delta > 0$, we have when $R \geqslant R_0$ that

$$C(q_0, R, n) \leqslant \kappa_{q_0,\delta} + \mathrm{e}\, M(\delta)^2\, h_{q_0}^{M(\delta)} \big(b_0(q_0, R, n) + b_1(q_0, R, n) \big) , \quad (8.23)$$

where the three terms correspond to the decompositions (8.12) and (8.16). By definition, it will follow that the *a priori* bound $B(q_0, R, n)$ is less than

$C(q_0, R, n)$. Bounds on $b_0(q_0, R, n)$ and $b_1(q_0, R, n)$ will be obtained below, and will use the other quantities $B(q_0, R, \ell) \leqslant B(q_0, R, n)$, with $\ell \leqslant n$. We shall get a relation

$$B(q_0, R, n) \leqslant c(q_0, \delta) R^{4\delta} + B(q_0, R, n)/2 \,, \quad n \geqslant 1 \,,$$

for R larger than some $R_1 \geqslant R_0$, and we shall be able to conclude.

8.3. Second reduction

Let $\tau > 0$ be given. We say that a nonnegative function f defined on $\mathbb{R}$ is *τ-stable with constant C* if whenever $|t| \leqslant \tau$, we have

$$f(s+t) \leqslant C f(s) \,, \quad s \in \mathbb{R} \,.$$

One sees that $C \geqslant 1$. Evident properties are to be observed about products, integrals, translations, convolutions... For example, if $f_1, \ldots, f_k$ are τ-stable with respective constants C_i, then clearly the product $f_1 \ldots f_k$ is τ-stable with constant $C_1 \ldots C_k$. If f is τ-stable with constant C and if $g \geqslant 0$, then for $|t| \leqslant \tau$ we have

$$\begin{aligned} (f * g)(s+t) &= \int_{\mathbb{R}} f(s+t-v) g(v) \,\mathrm{d}v \\ &\leqslant C \int_{\mathbb{R}} f(s-v) g(v) \,\mathrm{d}v = C (f * g)(s) \,, \end{aligned} \tag{8.24}$$

hence $f * g$ is also τ-stable with constant C. Suppose that f, g, h are non-negative on $\mathbb{R}$, and that f is τ-stable with constant C. If $|t| \leqslant \tau$ then

$$\begin{aligned} \int_{\mathbb{R}} f(s) g(s-t) h(t) \,\mathrm{d}s &\geqslant C^{-1} \int_{\mathbb{R}} f(s-t) g(s-t) h(t) \,\mathrm{d}s \\ &= C^{-1} h(t) \Bigl(\int_{\mathbb{R}} f(v) g(v) \,\mathrm{d}v \Bigr) \,, \end{aligned}$$

therefore

$$\int_{\mathbb{R}} f(s) (g * h)(s) \,\mathrm{d}s \geqslant C^{-1} \Bigl(\int_{|t| \leqslant \tau} h(t) \,\mathrm{d}t \Bigr) \Bigl(\int_{\mathbb{R}} f(v) g(v) \,\mathrm{d}v \Bigr) . \tag{8.25}$$

We shall now move to $\mathbb{R}^\ell$ with $\ell \geqslant 1$. Let Φ be a probability density on $\mathbb{R}$ that is τ-stable with constant C, for some $\tau > 0$. This implies that $\Phi(s) > 0$ for every $s \in \mathbb{R}$. Let us define $\beta \geqslant 1$ by

$$\beta^{-1} = \int_{|t| \leqslant \tau} \Phi(t) \,\mathrm{d}t \in (0, 1) \,. \tag{8.26}$$

We denote by Φ_j the operator on $L^q(\mathbb{R}^\ell)$ of convolution with Φ in the variable x_j, for each $j \in L = \{1, \ldots, \ell\}$. For instance, when $j = 1$ we let

$$(\Phi_1 f)(x_1, x_2, \ldots, x_\ell) = \int_{\mathbb{R}} f(x_1 - s, x_2, \ldots, x_\ell)\Phi(s)\,\mathrm{d}s\,.$$

For $j = 2, \ldots, \ell$ we let the transposition $\tau_j = (1\,j)$ act on $x = (x_1, \ldots, x_\ell)$ in $\mathbb{R}^\ell$ by $\tau_j\, x = (x_{\tau_j(i)})_{i=1}^\ell$ and on functions by $\tau_j(g) = g \circ \tau_j$. Letting $\tau_1 = I$, we have

$$\Phi_j f = \tau_j\big(\Phi_1(f \circ \tau_j)\big)\,, \quad j = 1, \ldots, n\,. \tag{8.27}$$

For every subset $J \subset L$ we set $\Phi^J = \prod_{k\in J} \Phi_k$, and $\Phi^{\sim j} = \Phi^{L\setminus\{j\}} = \prod_{k\neq j} \Phi_k$. We understand that $\Phi^\emptyset = I$. Each Φ^J is an operator acting on $L^q(\mathbb{R}^\ell)$ with norm equal to 1, when $1 \leqslant q \leqslant +\infty$. The next Bourgain's lemma is not too difficult, but the details are long and painful to write down precisely. We have chosen to break it into two parts, the first one containing the serious work.

Lemma 8.7 (a first part of Bourgain's [13, Lemma 7]). — *Let Φ be a probability density on $\mathbb{R}$ that is τ-stable with constant C, let $\beta \geqslant 1$ be defined by* (8.26)*. Let ℓ be an integer $\geqslant 1$, $L = \{1, \ldots, \ell\}$ and define Φ_j by* (8.27)*, for $j = 1, \ldots, \ell$. For all integers $q \geqslant 1$, for all nonnegative integrable functions $(f_j)_{j=1}^\ell$ on $\mathbb{R}^\ell$, one has*

$$\Big\|\sum_{j\in L} \Phi^{\sim j} f_j\Big\|_q \leqslant \beta C^{q-1} \Big\|\sum_{j\in L} \Phi^L f_j\Big\|_q + \sqrt{q-1}\,\Big\|\sum_{j\in L} \Phi^{\sim j} f_j^2\Big\|_{q/2}^{1/2}.$$

Proof. — The fundamental remark compares

$$\int_{\mathbb{R}} (\Phi_1 g_1)(s)(\Phi_1 g_2)(s) \ldots (\Phi_1 g_{k-1})(s) g_k(s)\,\mathrm{d}s$$

and

$$\int_{\mathbb{R}} (\Phi_1 g_1)(s)(\Phi_1 g_2)(s) \ldots (\Phi_1 g_{k-1})(s)(\Phi_1 g_k)(s)\,\mathrm{d}s\,,$$

when $k \geqslant 2$ and when the functions g_js are nonnegative on $\mathbb{R}$. We know by (8.24) that $\Phi_1 g = \Phi * g$ is τ-stable with constant C for every g nonnegative, so the product $f = (\Phi_1 g_1)(\Phi_1 g_2) \ldots (\Phi_1 g_{k-1})$ is τ-stable with constant C^{k-1}. Applying (8.25) and the definition of β with f, $g = g_k$, $h = \Phi$ and $g * h = \Phi_1 g_k$, we get

$$\begin{aligned}\int_{\mathbb{R}} &(\Phi_1 g_1)(s)(\Phi_1 g_2)(s) \ldots (\Phi_1 g_{k-1})(s) g_k(s)\,\mathrm{d}s \\ &\leqslant C^{k-1}\beta \int_{\mathbb{R}} (\Phi_1 g_1)(s)(\Phi_1 g_2)(s) \ldots (\Phi_1 g_{k-1})(s)(\Phi_1 g_k)(s)\,\mathrm{d}s\,. \end{aligned}\tag{8.28}$$

The case $q = 1$ of the lemma follows from $\beta \geqslant 1$ and $\int_{\mathbb{R}^\ell} g * f = \int_{\mathbb{R}^\ell} f$ for every probability density g. For the simplest non-trivial case, when $q = 2$, we write

$$\Bigl(\sum_{j \in L} \Phi^{\sim j} f_j \Bigr)^2 = \sum_{i \neq j} (\Phi^{\sim i} f_i)(\Phi^{\sim j} f_j) + \sum_{j \in L} (\Phi^{\sim j} f_j)^2 \,.$$

When $j \neq i$, the function $\Phi^{\sim i} f_i = \Phi^{L \setminus \{i\}} f_i = \Phi_j \Phi^{L \setminus \{i,j\}} f_i$ is of the form $\Phi_j g_1$, and letting $g_2 = \Phi^{\sim j} f_j$ we get by (8.28) for the x_j variable that

$$\begin{aligned} \int_{\mathbb{R}} (\Phi^{\sim i} f_i)(\Phi^{\sim j} f_j) \, \mathrm{d}x_j &= \int_{\mathbb{R}} (\Phi_j g_1) g_2 \, \mathrm{d}x_j \\ &\leqslant C\beta \int_{\mathbb{R}} (\Phi_j g_1)(\Phi_j g_2) \, \mathrm{d}x_j = C\beta \int_{\mathbb{R}} (\Phi^{\sim i} f_i)(\Phi^L f_j) \, \mathrm{d}x_j \end{aligned}$$

because $\Phi_j \Phi^{\sim j} = \Phi^L$. Integrating in the remaining variables, and since the functions are nonnegative, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^\ell} \sum_{i \neq j} (\Phi^{\sim i} f_i)(\Phi^{\sim j} f_j) \, \mathrm{d}x \leqslant C\beta \int_{\mathbb{R}^\ell} \sum_{i \neq j} (\Phi^{\sim i} f_i)(\Phi^L f_j) \, \mathrm{d}x \\ &\leqslant C\beta \int_{\mathbb{R}^\ell} \Bigl(\sum_{i \in L} \Phi^{\sim i} f_i \Bigr) \Bigl(\sum_{j \in L} \Phi^L f_j \Bigr) \mathrm{d}x \leqslant C\beta \Bigl\| \sum_{j \in L} \Phi^{\sim j} f_j \Bigr\|_2 \Bigl\| \sum_{j \in L} \Phi^L f_j \Bigr\|_2 . \end{aligned}$$

When $j = i$, we use $(\Phi^{\sim j} * g)^2 \leqslant \Phi^{\sim j} * g^2$ and get

$$\int_{\mathbb{R}^\ell} \sum_{j \in L} (\Phi^{\sim j} f_j)^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^\ell} \sum_{j \in L} \Phi^{\sim j} f_j^2 \, \mathrm{d}x =: B \,.$$

It follows that $E := \bigl\| \sum_{j \in L} \Phi^{\sim j} f_j \bigr\|_2$ satisfies an inequality $E^2 \leqslant AE + B$, where we let $A := C\beta \bigl\| \sum_{j \in L} \Phi^L f_j \bigr\|_2$. This yields $E \leqslant A + B^{1/2}$ and we have

$$\Bigl\| \sum_{j \in L} \Phi^{\sim j} f_j \Bigr\|_2 \leqslant C\beta \Bigl\| \sum_{j \in L} \Phi^L f_j \Bigr\|_2 + \Bigl\| \sum_{j \in L} \Phi^{\sim j} f_j^2 \Bigr\|_1^{1/2} .$$

This is Lemma 8.7 when $q = 2$. In general, when $q \geqslant 3$, we expand

$$\int_{\mathbb{R}^\ell} \Bigl(\sum_{j \in L} \Phi^{\sim j} f_j \Bigr)^q \mathrm{d}x = \sum_{j_1, j_2, \ldots, j_q \in L} \int_{\mathbb{R}^\ell} (\Phi^{\sim j_1} f_{j_1}) \ldots (\Phi^{\sim j_q} f_{j_q}) \, \mathrm{d}x \,. \quad (8.29)$$

Consider a multi-index $(j_1, j_2, \ldots, j_q) \in \{1, \ldots, \ell\}^q = L^q$ and suppose that j_q is not equal to any of $j_1, \ldots, j_{q-1}$. Then we can write $\Phi^{\sim j_k} f_{j_k} = \Phi_{j_q} g_k$ for each $k < q$, so as before, by (8.28) applied in the x_{j_q} variable, we get that

$$\int_{\mathbb{R}^\ell} (\Phi^{\sim j_1} f_{j_1}) \dots (\Phi^{\sim j_q} f_{j_q})\,\mathrm{d}x \leqslant C^{q-1}\beta \int_{\mathbb{R}^\ell} (\Phi^{\sim j_1} f_{j_1}) \dots (\Phi^{\sim j_{q-1}} f_{j_{q-1}})(\Phi^{L} f_{j_q})\,\mathrm{d}x\,.$$

Let us denote by $\sum_1$ the part of the summation at the right-hand side of (8.29) that is extended to all $j_1, \dots, j_q$ such that $j_q \notin \{j_1, \dots, j_{q-1}\}$. We obtain that

$$\begin{aligned}\sum\nolimits_1 \int_{\mathbb{R}^\ell} (\Phi^{\sim j_1} f_{j_1}) \dots (\Phi^{\sim j_q} f_{j_q})\,\mathrm{d}x &\leqslant C^{q-1}\beta \int_{\mathbb{R}^\ell} \Big(\sum_{j\in L} \Phi^{\sim j} f_j\Big)^{q-1} \Big(\sum_{j\in L} \Phi^{L} f_j\Big)\,\mathrm{d}x \\ &\leqslant C^{q-1}\beta \Big\|\sum_{j\in L} \Phi^{\sim j} f_j\Big\|_q^{q-1} \Big\|\sum_{j\in L} \Phi^{L} f_j\Big\|_q\,.\end{aligned}$$

The remaining sum $\sum_2$ is less than the sum of $q-1$ terms corresponding to which index j_k, $k = 1, \dots, q-1$ is equal to j_q. Each of these $q-1$ terms is similar to

$$\sum_{j_1, j_2, \dots, j_{q-1} \in L} \int_{\mathbb{R}^\ell} (\Phi^{\sim j_1} f_{j_1}) \dots (\Phi^{\sim j_{q-2}} f_{j_{q-2}})(\Phi^{\sim j_{q-1}} f_{j_{q-1}})^2\,\mathrm{d}x\,,$$

which is bounded by

$$\int_{\mathbb{R}^\ell} \Big(\sum_{j\in L} \Phi^{\sim j} f_j\Big)^{q-2} \Big(\sum_{j\in L} \Phi^{\sim j} f_j^2\Big) \leqslant \Big\|\sum_{j\in L} \Phi^{\sim j} f_j\Big\|_q^{q-2} \Big\|\sum_{j\in L} \Phi^{\sim j} f_j^2\Big\|_{q/2}\,.$$

We obtain for $E_q = \Big\|\sum_{j\in L} \Phi^{\sim j} f_j\Big\|_q^q$ a bound by $\Sigma_1 + \Sigma_2$ of the form

$$E_q \leqslant C^{q-1}\beta \Big\|\sum_{j\in L} \Phi^{L} f_j\Big\|_q E_q^{1-1/q} + (q-1)\Big\|\sum_{j\in L} \Phi^{\sim j} f_j^2\Big\|_{q/2} E_q^{1-2/q}\,,$$

which can be written also as

$$E_q^{2/q} \leqslant C^{q-1}\beta \Big\|\sum_{j\in L} \Phi^{L} f_j\Big\|_q E_q^{1/q} + (q-1)\Big\|\sum_{j\in L} \Phi^{\sim j} f_j^2\Big\|_{q/2}\,.$$

This implies as before that

$$\Big\|\sum_{j\in L} \Phi^{\sim j} f_j\Big\|_q = E_q^{1/q} \leqslant C^{q-1}\beta \Big\|\sum_{j\in L} \Phi^{L} f_j\Big\|_q + \sqrt{q-1}\,\Big\|\sum_{j\in L} \Phi^{\sim j} f_j^2\Big\|_{q/2}^{1/2}\,. \quad \square$$

Lemma 8.8 ([13, Lemma 7]). — *Let Φ be a probability density on $\mathbb{R}$ that is τ-stable with constant C, and let $\beta \geqslant 1$ be defined by* (8.26). *Let $\ell \geqslant 1$ be an integer, $L = \{1, \dots, \ell\}$ and define Φ_j by* (8.27)*, for $j = 1, \dots, \ell$. For every integer $\nu \geqslant 1$ and for all nonnegative integrable functions $(f_j)_{j=1}^{\ell}$ on $\mathbb{R}^\ell$, one has*

$$\kappa_\nu^{-1} \Big\| \sum_{j \in L} \Phi^{\sim j} f_j \Big\|_{2^\nu} \leqslant \sum_{k=0}^{\nu} \big\| \sum_{j \in L} \Phi^L f_j^{2^k} \big\|_{2^{\nu-k}}^{2^{-k}} \leqslant (\nu+1) \Big\| \sum_{j \in L} f_j \Big\|_{2^\nu}, \tag{8.30}$$

with $\kappa_\nu \leqslant \max(2^\nu, \beta C^{2^\nu})$. Each term $\big\| \sum_{j \in L} \Phi^L f_j^{2^k} \big\|_{2^{\nu-k}}^{2^{-k}}$, for $0 \leqslant k \leqslant \nu$, satisfies

$$\big\| \sum_{j \in L} \Phi^L f_j^{2^k} \big\|_{2^{\nu-k}}^{2^{-k}} = \Big\| \Big(\sum_{j \in L} \Phi^L f_j^{2^k} \Big)^{2^{-k}} \Big\|_{2^\nu} \leqslant \Big\| \sum_{j \in L} f_j \Big\|_{2^\nu}.$$

Proof. — We begin with the easy last sentence. For $r = 2^k$ and $k = 0, \dots, \nu$, we have

$$\Big\| \sum_{j \in L} \Phi^L f_j^r \Big\|_{2^{\nu-k}}^{1/r} \leqslant \Big\| \Phi^L \Big(\sum_{j \in L} f_j \Big)^r \Big\|_{2^{\nu-k}}^{1/r} \leqslant \Big\| \Big(\sum_{j \in L} f_j \Big)^r \Big\|_{2^{\nu-k}}^{1/r} = \Big\| \sum_{j \in L} f_j \Big\|_{2^\nu}.$$

The constant in the right-hand inequality of (8.30) is therefore bounded by $\nu + 1$.

We pass to the left-hand inequality. Let $q = 2^\nu$. By Lemma 8.7, we can reduce the case $q = 2^\nu$ to the case $q/2$. We proceed by induction, with a number of steps bounded by ν. Using $(a+b)^\alpha \leqslant a^\alpha + b^\alpha$ when $a, b \geqslant 0$ and $\alpha \in (0, 1]$, we obtain

$$\begin{aligned} \big\| \sum_{j \in L} \Phi^{\sim j} f_j \big\|_q &\leqslant \beta C^{q-1} \big\| \sum_{j \in L} \Phi^L f_j \big\|_q + \sqrt{q-1}\, \big\| \sum_{j \in L} \Phi^{\sim j} f_j^2 \big\|_{q/2}^{1/2} \\ &\leqslant \beta C^{2^\nu} \big\| \sum_{j \in L} \Phi^L f_j \big\|_{2^\nu} + 2^{\nu/2} \beta^{1/2} C^{2^{\nu-2}} \big\| \sum_{j \in L} \Phi^L f_j^2 \big\|_{2^{\nu-1}}^{1/2} \\ &\qquad + 2^{\nu/2} 2^{(\nu-1)/4} \big\| \sum_{j \in L} \Phi^{\sim j} f_j^4 \big\|_{2^{\nu-2}}^{1/4} \leqslant \dots \end{aligned}$$

and the successive factors in front of $\big\| \sum_{j \in L} \Phi^L f_j^{2^k} \big\|_{2^{\nu-k}}^{2^{-k}}$, for $0 \leqslant k \leqslant \nu$, have the form $q(\beta/q)^{2^{-k}} (C^q)^{4^{-k}} \leqslant q(\beta C^q/q)^{2^{-k}}$, leading to $\kappa_\nu \leqslant \max(q, \beta C^q)$. □

We can try to optimize the constant κ_ν in the following way. Suppose that the function $\ln \Phi$ is Lipschitz on $\mathbb{R}$ with constant λ. Then we see that

Φ is τ-stable with constant $C_\tau = \mathrm{e}^{\lambda\tau}$ for every $\tau > 0$, and

$$1 \geqslant \beta_\tau^{-1} := \int_{|t| \leqslant \tau} \Phi(t)\,\mathrm{d}t \geqslant 2\Phi(0) \int_0^\tau \mathrm{e}^{-\lambda t}\,\mathrm{d}t = 2\Phi(0) \frac{1 - \mathrm{e}^{-\lambda\tau}}{\lambda}\,.$$

Let $q = 2^\nu$ and select $\tau = 1/(\lambda q)$. Then $C_\tau \leqslant \mathrm{e}^{1/q}$ and

$$\beta_\tau C_\tau^q \leqslant \frac{\mathrm{e}\,\lambda}{2\Phi(0)(1 - \mathrm{e}^{-1/q})} \leqslant \frac{\mathrm{e}^2\,\lambda}{2\Phi(0)}\,q \leqslant \frac{4\lambda}{\Phi(0)}\,q\,.$$

Coming back to Lemma 8.8 and noticing that $\lambda \geqslant 2\Phi(0)$, we obtain

$$\kappa_\nu \leqslant \frac{4\lambda}{\Phi(0)}\,2^\nu\,. \tag{8.31}$$

We now introduce Bourgain's specific example φ of a function Φ, defined by

$$\forall\, s \in \mathbb{R}\,, \quad \varphi(s) = \frac{c}{1 + s^4}\,,$$

where $c = \sqrt{2}/\pi$ is chosen so that φ is a probability density. This value c is obtained by the residue theorem, which also gives the Fourier transform

$$\widehat{\varphi}(t) = \bigl(\cos(\pi\sqrt{2}|t|) + \sin(\pi\sqrt{2}|t|)\bigr)\,\mathrm{e}^{-\pi\sqrt{2}|t|}\,, \quad t \in \mathbb{R}\,.$$

Notice that $(\cos u + \sin u)\,\mathrm{e}^{-u} = \sqrt{2}\cos(u - \pi/4)\,\mathrm{e}^{-u} \geqslant \mathrm{e}^{-u^2}$ when $0 \leqslant u \leqslant \pi/2$, because $h(u) = \ln\bigl(\sqrt{2}\cos(u - \pi/4)\bigr) - u + u^2 \geqslant 0$ on this interval. Indeed, we have $h(0) = h'(0) = 0$ and $h''(u) = 1 - \tan(u - \pi/4)^2 \geqslant 0$ on $[0, \pi/2]$. It follows that

$$\widehat{\varphi}(t) \geqslant \mathrm{e}^{-2\pi^2 t^2} \quad \text{when} \quad \pi\sqrt{2}|t| \leqslant \frac{\pi}{2}\,,$$

in particular when $\pi|t| \leqslant 1$. We shall need later the estimate given by Lemma 8.9.

Lemma 8.9. — *For all $s \in \mathbb{R}$, ℓ integer $\geqslant 1$ and $\xi = (\xi_1, \ldots, \xi_\ell) \in \mathbb{R}^\ell$, one has that*

$$\Bigl(1 - \prod_{j=1}^{\ell} \widehat{\varphi}(s\xi_j)\Bigr) \prod_{j=1}^{\ell} \widehat{\eta}(\xi_j) \leqslant 2\pi^2 s^2\,.$$

Proof. — Suppose that $\pi|s\xi| \leqslant 1$. Then $\pi|s\xi_j| \leqslant 1$ and $\widehat{\varphi}(s\xi_j) \geqslant \mathrm{e}^{-2\pi^2 s^2 \xi_j^2}$ for each index $j = 1, \ldots, \ell$, thus by Lemma 8.6 we have

$$\Bigl(1 - \prod_{j=1}^{\ell} \widehat{\varphi}(s\xi_j)\Bigr) \prod_{j=1}^{\ell} \widehat{\eta}(\xi_j) \leqslant \bigl(1 - \mathrm{e}^{-2\pi^2 s^2 |\xi|^2}\bigr) \prod_{j=1}^{\ell} \widehat{\eta}(\xi_j) \leqslant 2\pi^2 s^2\,.$$

Otherwise, we have $\pi|s\xi| \geqslant 1$ and applying Lemma 8.6 with $r \to 0$ we get

$$\Bigl(1 - \prod_{j=1}^{\ell} \widehat{\varphi}(s\xi_j)\Bigr) \prod_{j=1}^{\ell} \widehat{\eta}(\xi_j) \leqslant 2 \prod_{j=1}^{\ell} \widehat{\eta}(\xi_j) \leqslant 2|\xi|^{-2} \leqslant 2\pi^2 s^2\,.$$

□

We know that φ is 1-stable, because $F(x) = \ln \varphi(x)$ is Lipschitz. Indeed, its derivative $F'(x) = -4x^3/(1+x^4)$ is bounded on the real line. To be precise, the second derivative F'' vanishes when $x^4 = 3$, which implies that $|F'(x)| \leqslant 3^{3/4}$ for every x. When $|t| \leqslant 1$, we have thus

$$\varphi(s+t) \leqslant \mathrm{e}^{3^{3/4}} \varphi(s)\,, \qquad s \in \mathbb{R}\,,$$

with $\mathrm{e}^{3^{3/4}} < 9,772 < 10$. This shows that φ is 1-stable with constant $\leqslant 10$. We shall need more than the 1-stability of the function φ, namely, we shall use the polynomial character of $1/\varphi$. When $|t| \geqslant 1$ and $u \in \mathbb{R}$, we have

$$1 + (u-t)^4 \leqslant 1 + 8(u^4 + t^4) \leqslant 8(1+t^4)(1+u^4) \leqslant 16t^4(1+u^4)\,, \quad (8.32)$$

implying in this case, and with $u = s+t$, that $\varphi(s+t) \leqslant 16t^4\varphi(s)$.

We introduce $w_1 = w_0^2 = R^{-\delta} < w_0$. The dilate $\varphi_{(w_1)}$ of φ is w_1-stable with constant 10 and we shall consider from now on that $\Phi = \varphi_{(w_1)}$. We denote as before by Φ_j the operator on $L^q(\mathbb{R}^\ell)$ of convolution with $\varphi_{(w_1)}$ in the variable x_j, where $j \in L = \{1, \ldots, \ell\}$. For every subset $J \subset L$ we define Φ^J as before, as well as $\Phi^{\sim j} = \Phi^{L\setminus\{j\}}$. For $|t| \geqslant w_1$ we have by (8.32) the inequality

$$\varphi_{(w_1)}(s+t) \leqslant 16(t/w_1)^4 \varphi_{(w_1)}(s) = 16R^{4\delta}t^4\varphi_{(w_1)}(s)\,, \quad s \in \mathbb{R}\,. \qquad (8.33)$$

Here is perhaps the crux of the matter. The boundary measures μ_j, partial derivatives of μ_Q, will be swallowed and disappear as if by magic. The cube Q here is the cube Q_ℓ in $\mathbb{R}^\ell$.

Lemma 8.10 ([13, Lemma 8]). — *Let ν be an integer $\geqslant 1$, let $q = 2^\nu$, and let $f_1, \ldots, f_\ell$ be functions in $L^q(\mathbb{R}^\ell)$. Let μ_j denote the partial derivative $\partial_j \mu_Q$ of the probability measure μ_Q, for $j = 1, \ldots, \ell$. With Φ_j defined as in* (8.27) *from $\Phi f = \varphi_{(w_1)} * f$ when $f \in L^q(\mathbb{R})$, one has that*

$$\Big\| \Big(\sum_{j=1}^{\ell} \big|\mu_j * \Phi^{\sim j} f_j\big|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^\ell)} \leqslant \kappa \sqrt{q \ln q}\, R^{4\delta} \Big\| \Big(\sum_{j=1}^{\ell} |f_j|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^\ell)}.$$

Proof. — Let us write $L = \{1, \ldots, \ell\}$ and $\mathbb{R}^L$ for $\mathbb{R}^\ell$. For each $j \in L$, let $Q^{\sim j}$ denote the cube $\prod_{i \neq j}[-1/2, 1/2]$ in $\mathbb{R}^{L\setminus\{j\}}$, let $\mathrm{d}\mathbf{x}^{\sim j}$ be the Lebesgue measure on $\mathbb{R}^{L\setminus\{j\}}$ and consider the probability measures τ_j, $K^{\sim j}$, $K^{\{j\}}$ on

$\mathbb{R}^\ell$ defined by

$$\tau_j = \frac{1}{2}\big(\delta_{1/2}(x_j) + \delta_{-1/2}(x_j)\big) \otimes \Big(\otimes_{i\neq j}\ \delta_0(x_i)\Big),$$

$$K^{\sim j} = \delta_0(x_j) \otimes \Big(\otimes_{i\neq j}\ \mathbf{1}_{[-1/2,1/2]}(x_i)\,\mathrm{d}x_i\Big) = \delta_0(x_j) \otimes \big(\mathbf{1}_{Q^{\sim j}}\,\mathrm{d}\mathbf{x}^{\sim j}\big),$$

$$K^{\{j\}} = \Big(\mathbf{1}_{[-1/2,1/2]}(x_j)\,\mathrm{d}x_j\Big) \otimes \Big(\otimes_{i\neq j}\ \delta_0(x_i)\Big).$$

When convenient, we shall identify a kernel K and the convolution operator with that kernel. Note that the signed measure $\mu_j = \partial_j \mathbf{1}_Q$ satisfies

$$|\partial_j \mathbf{1}_Q| = \big(\delta_{1/2}(x_j) + \delta_{-1/2}(x_j)\big) \otimes \big(\mathbf{1}_{Q^{\sim j}}\,\mathrm{d}\mathbf{x}^{\sim j}\big) = 2\tau_j * K^{\sim j}.$$

Using $|\mu * f|^p \leqslant \mu * |f|^p$ when μ is a probability measure and $p \geqslant 1$, we have

$$\sum_{j=1}^{\ell} |\partial_j \mathbf{1}_Q * \Phi^{\sim j} f_j|^2 \leqslant 4 \sum_{j=1}^{\ell} \Phi^{\sim j}(\tau_j * K^{\sim j} * |f_j|^2).$$

We evaluate the L^q norm applying Lemma 8.8, obtaining that

$$\Big\| \Big(\sum_{j=1}^{\ell} \big|\mu_j * \Phi^{\sim j} f_j\big|^2 \Big)^{1/2} \Big\|_q^2 \leqslant 4 \Big\| \sum_{j=1}^{\ell} \Phi^{\sim j}(\tau_j * K^{\sim j} * |f_j|^2) \Big\|_{q/2} \leqslant 4\kappa_{\nu-1} \sum_{k=0}^{\nu-1} (E_k)^{2^{-k}},$$

where the expressions E_k are given by

$$E_k := \Big\| \sum_{j=1}^{\ell} \Phi^L \big[\tau_j * K^{\sim j} * |f_j|^2\big]^{2^k} \Big\|_{q/2^{k+1}}, \quad 1 \leqslant 2^k \leqslant q/2 = 2^{\nu-1}.$$

Using again $|\mu * f|^p \leqslant \mu * |f|^p$ for $p \geqslant 1$, we get

$$E_k \leqslant F_k := \big\| \sum_{j=1}^{\ell} \Phi^L (\tau_j * K^{\sim j} * |f_j|^{2^{k+1}}) \big\|_{q/2^{k+1}}.$$

Next, observe that $\varphi_{(w_1)}(s+t) \leqslant w_1^{-4}\varphi_{(w_1)}(s) = R^{4\delta}\varphi_{(w_1)}(s)$ for $|t| \leqslant 1/2$. Indeed, when $w_1 \leqslant |t| \leqslant 1/2$ we have $\varphi_{(w_1)}(s+t) \leqslant 16(t/w_1)^4\varphi_{(w_1)}(s) \leqslant w_1^{-4}\varphi_{(w_1)}(s)$ by (8.33), and $\varphi_{(w_1)}(s+t) \leqslant 10\,\varphi_{(w_1)}(s) \leqslant R^{4\delta}\varphi_{(w_1)}(s)$ when $|t| \leqslant w_1$, because we assumed that $R^\delta \geqslant R_0^\delta = 16$. When μ is a probability measure supported on $[-1/2, 1/2]$, it follows that $\mu * \varphi_{(w_1)} \leqslant R^{4\delta}\varphi_{(w_1)}$ and $\varphi_{(w_1)} \leqslant R^{4\delta}\mu * \varphi_{(w_1)}$. We have therefore $\tau_j \Phi_j \leqslant R^{4\delta}\Phi_j$ and $\Phi_j \leqslant R^{4\delta}\Phi_j K^{\{j\}}$. For g nonnegative we obtain

$$\Phi_j \tau_j g \leqslant R^{4\delta}\Phi_j g \leqslant R^{8\delta}\Phi_j K^{\{j\}} g.$$

Consequently, observing that $K^{\{j\}} * K^{\sim j} = \mathbf{1}_Q(x)\,\mathrm{d}x$, we have

$$\Phi^L(\tau_j K^{\sim j} g) = \Phi^{\sim j}\Phi_j \tau_j K^{\sim j} g \leqslant R^{8\delta} \Phi^{\sim j}\Phi_j K^{\{j\}} K^{\sim j} g = R^{8\delta}\Phi^L * \mathbf{1}_Q * g\,,$$

and by the last assertion of Lemma 8.8, we obtain for $k = 0, \ldots, \nu - 1$ that

$$\begin{aligned} F_k &\leqslant R^{8\delta} \Big\| \Big(\sum_{j=1}^{\ell} \Phi^L |f_j|^{2^{k+1}} \Big) * \mathbf{1}_Q \Big\|_{q/2^{k+1}} \\ &\leqslant R^{8\delta} \Big\| \sum_{j=1}^{\ell} \Phi^L |f_j|^{2^{k+1}} \Big\|_{q/2^{k+1}} \leqslant R^{8\delta} \Big\| \big(\sum_{j=1}^{\ell} |f_j|^2 \big)^{1/2} \Big\|_q^{2^{k+1}} . \end{aligned}$$

Finally, assuming $\big\| \big(\sum_{j=1}^{\ell} |f_j|^2 \big)^{1/2} \big\|_q \leqslant 1$ we get

$$\Big\| \Big(\sum_{j=1}^{\ell} \big| \mu_j * \Phi^{\sim j} f_j \big|^2 \Big)^{1/2} \Big\|_q^2 \leqslant 4\kappa_{\nu-1} \sum_{k=0}^{\nu-1} (R^{8\delta})^{2^{-k}} \leqslant 4\nu\kappa_{\nu-1} R^{8\delta} .$$

Since $\ln \varphi$ is Lipschitz on $\mathbb{R}$, we can estimate κ_ν by (8.31) and conclude. □

Recalling that G^R is a probability density and $\mu_j^R = \mu_j * G^R$, we immediately deduce the result that we really need.

Lemma 8.11 ([13, Lemma 9]). — *Assume that $q = 2^\nu$, with $\nu \geqslant 1$ an integer. Let $f_1, \ldots, f_\ell$ be elements of $L^q(\mathbb{R}^\ell)$. With Φ_j as in Lemma 8.10, we have*

$$\Big\| \Big(\sum_{j=1}^{\ell} \big| \mu_j^R * \Phi^{\sim j} f_j \big|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^\ell)} \leqslant \kappa_q R^{4\delta} \Big\| \Big(\sum_{j=1}^{\ell} |f_j|^2 \Big)^{1/2} \Big\|_{L^q(\mathbb{R}^\ell)} .$$

8.4. Conclusion

It remains to estimate the two terms $\mathbf{E}(R, \ell, f) := |\nabla \mu^R * H_0 f|$ and $\mathbf{F}(R, \ell, f)$ defined in (8.22), for $f \in L^{q_0}(\mathbb{R}^\ell)$, $q_0 = 2^\nu$ and $\ell \leqslant n$. Each one will be cut into two pieces, one of order a power of R^δ and the second bounded by a "small" multiple of $B(q_0, R, n)$. Let us start with $\mathbf{E}(R, \ell, f)$, and cut $|\nabla \mu^R * H_0 f|$ into

$$\mathbf{E}'(R, \ell, f) := \big| \nabla \mu^R * G_{(w_1)} * H_0 f \big| , \ \mathbf{E}''(R, \ell, f) := \big| \nabla \mu^R * (\delta_0 - G_{(w_1)}) * H_0 f \big| .$$

We begin with $\mathbf{E}'(R, \ell, f)$. The mapping $f \mapsto \nabla \mu^R * G_{(w_1)} * H_0 f$, equal to $U_{\mu^R * G_{(w_1)}} \circ H_0$ is studied by applying Lemma 7.11 to the log-concave probability density $\mu^R * G_{(w_1)}$. Using (7.3) and (8.1), we see that $V(\mu^R * G_{(w_1)}) \leqslant V(G_{(w_1)}) = w_1^{-1} = R^\delta$. The variance of $\mu^R * G_{(w_1)} = \mu_Q * G_{(1/R)} *$

$G_{(w_1)}$ is larger than that of μ_Q, which is equal to $(12)^{-1}$. By Lemma 7.11 and $q_0 \geqslant 2$, we get that

$$\big\|\mathbf{E}'(R,\ell,f)\big\|_{q_0} \leqslant 24^{1/q_0}(R^\delta)^{1-2/q_0}\,\|H_0 f\|_{q_0} \leqslant 5R^\delta\,\|f\|_{q_0} .$$

We study now $\mathbf{E}''(R,\ell,f)$ with the *a priori* estimate that involves the constant $B(q_0,R,\ell)$. By the definition (8.7), one writes

$$\begin{aligned}\big\|\mathbf{E}''(R,\ell,f)\big\|_{q_0} &= \big\|\nabla\mu^R * (\delta_0 - G_{(w_1)}) * H_0 f\big\|_{q_0}\\ &\leqslant B(q_0,R,\ell)\,\|(\delta_0 - G_{(w_1)}) * H_0 f\|_{q_0} .\end{aligned}$$

We continue by interpolation (L^∞, L^2) for $f \mapsto (\delta_0 - G_{(w_1)}) * H_0 f$. In $L^\infty(\mathbb{R}^\ell)$ one has simply $\|(\delta_0 - G_{(w_1)}) * H_0\|_{\infty\to\infty} \leqslant 2$ by using the L^1 norm of the convolution kernel. Lemma 8.6 with $r = 2\sqrt{\pi}\,w_1/w_0$ gives for the Fourier transform a bound

$$(1 - \mathrm{e}^{-4\pi w_1^2|\xi|^2}) \prod_{j=1}^{\ell} \widehat{\eta}(w_0\xi_j) \leqslant 4\pi(w_1/w_0)^2 = 4\pi w_0^2 = 4\pi R^{-\delta} , \qquad \xi \in \mathbb{R}^\ell ,$$

implying $\|(\delta - G_{(w_1)}) * H_0\|_{2\to2} \leqslant 4\pi R^{-\delta}$. We get in this way that

$$\|(\delta_0 - G_{(w_1)}) * H_0\|_{q_0\to q_0} \leqslant 2^{1-2/q_0}(4\pi R^{-\delta})^{2/q_0} \leqslant 4\pi R^{-2\delta/q_0} ,$$

thus $\big\|\mathbf{E}''(R,\ell,f)\big\|_{q_0} \leqslant \kappa B(q_0,R,\ell) R^{-2\delta/q_0}\,\|f\|_{q_0}$ and we obtain

$$\big\|\mathbf{E}(R,\ell,f)\big\|_{q_0} \leqslant \kappa\big(R^\delta + B(q_0,R,\ell)R^{-2\delta/q_0}\big)\|f\|_{q_0} .$$

Now we consider $\mathbf{F}(R,\ell,f)$ and we cut it into

$$\mathbf{F}'(R,\ell,f) := \Big(\sum_{j=1}^{\ell} \big|\mu_j^R * \Gamma^j \Phi^{\sim j} f\big|^2\Big)^{1/2} ,$$

$$\mathbf{F}''(R,\ell,f) := \Big(\sum_{j=1}^{\ell} \big|\mu_j^R * \Gamma^j (I - \Phi^{\sim j}) f\big|^2\Big)^{1/2} .$$

By Lemma 8.11, we have that

$$\big\|\mathbf{F}'(R,\ell,f)\big\|_{q_0} \leqslant \kappa_{q_0} R^{4\delta} \bigg\|\Big(\sum_{j=1}^{\ell} \big|\Gamma^j f\big|^2\Big)^{1/2}\bigg\|_{q_0} .$$

Using Khinchin's (1.22.**K**) and (1.27) we reduce to $\big\|\sum_{j=1}^{\ell} \pm\Gamma^j f\big\|_{q_0}$, and dividing according to the sign $\pm$, we further reduce to $\big\|\sum_{j\in J_1} \Gamma^j f\big\|_{q_0}$ and $\big\|\sum_{j\notin J_1} \Gamma^j f\big\|_{q_0}$, where $J_1 \subset \{1,\dots,\ell\}$. The first sum corresponds to the operator H_1 relative to J_1, the second is the one for $\sim J_1 := \{1,\dots,\ell\} \setminus J_1$.

By Proposition 8.3 for the set J_1 of variables, writing $x = (\mathbf{x}^{J_1}, \mathbf{x}^{\sim J_1}) \in \mathbb{R}^\ell$, we have for $1 < q < +\infty$ that

$$\Bigl\| \Bigl(\sum_{j \in J_1} \Gamma^j f \Bigr)_{\mathbf{x}^{\sim J_1}} \Bigr\|_{L^q(\mathbb{R}^{J_1})} \leqslant h_q \bigl\| f_{\mathbf{x}^{\sim J_1}} \bigr\|_{L^q(\mathbb{R}^{J_1})}, \quad \mathbf{x}^{\sim J_1} \in \mathbb{R}^{\sim J_1},$$

and integrating in the variables in $\sim J_1$ we get with A_q from (1.22.$\mathbf{K}$) that

$$\Bigl\| \Bigl(\sum_{j=1}^{\ell} \bigl| \Gamma^j f \bigr|^2 \Bigr)^{1/2} \Bigr\|_q \leqslant 2 A_q^{-1} h_q \|f\|_q \tag{8.34}$$

for $1 < q < +\infty$. It follows that $\bigl\| \mathbf{F}'(R, \ell, f) \bigr\|_{q_0} \leqslant \kappa'_{q_0} R^{4\delta} \|f\|_{q_0}$.

For the second term $\mathbf{F}''(R, \ell, f)$ we first obtain an L^2 bound for the nonlinear operator $V : f \mapsto \bigl(\sum_{j=1}^{\ell} |\Gamma^j (I - \Phi^{\sim j}) f|^2 \bigr)^{1/2}$, by estimating the Fourier transform

$$\chi(\xi) := \sum_{j=1}^{\ell} \bigl(1 - \widehat{\eta}(w_0 \xi_j) \bigr)^2 \Bigl(\prod_{i \neq j} \widehat{\eta}(w_0 \xi_i) \Bigr)^2 \bigl(1 - \prod_{i \neq j} \widehat{\varphi}(w_1 \xi_i) \bigr)^2 \leqslant 4\pi^2 R^{-\delta}.$$

Indeed, we know that $0 \leqslant \widehat{\eta}(t) \leqslant 1$ and $-1 \leqslant \widehat{\varphi}(t) \leqslant 1$, therefore

$$\begin{aligned} &\sum_{j=1}^{\ell} \bigl(1 - \widehat{\eta}(w_0 \xi_j) \bigr)^2 \Bigl(\prod_{i \neq j} \widehat{\eta}(w_0 \xi_i) \Bigr) \bigl(1 - \prod_{i \neq j} \widehat{\varphi}(w_1 \xi_i) \bigr) \\ &\quad \leqslant 2 \sum_{j=1}^{\ell} \bigl(1 - \widehat{\eta}(w_0 \xi_j) \bigr) \prod_{i \neq j} \widehat{\eta}(w_0 \xi_i) \leqslant 2 \prod_{j=1}^{\ell} \bigl((1 - \widehat{\eta}(w_0 \xi_j)) + \widehat{\eta}(w_0 \xi_j) \bigr) = 2, \end{aligned}$$

and by Lemma 8.9 applied to $\mathbb{R}^{L \setminus \{j\}}$ with $s = w_1 / w_0 = w_0 = R^{-\delta/2}$, it follows that

$$\chi(\xi) \leqslant 2 \max_{1 \leqslant j \leqslant \ell} \Bigl(\prod_{i \neq j} \widehat{\eta}(w_0 \xi_i) \Bigr) \Bigl(1 - \prod_{i \neq j} \widehat{\varphi}(w_1 \xi_i) \Bigr) \leqslant 4\pi^2 R^{-\delta}, \qquad \xi \in \mathbb{R}^\ell.$$

We get $\|Vf\|_2^2 \leqslant 4\pi^2 R^{-\delta} \|f\|_2^2$ and $\|V\|_{2 \to 2} \leqslant 2\pi R^{-\delta/2}$. On the other hand, given functions $(g_j)_{j=1}^{\ell}$ and independent Bernoulli random variables $(\varepsilon_j)_{j=1}^{\ell}$, we have

$$\begin{aligned} 2 \Bigl(\sum_{j=1}^{\ell} |\mu_j^R * g_j|^2 \Bigr)^{1/2} &= 2 \Bigl(\sum_{j=1}^{\ell} |\mu_j^R * \varepsilon_j g_j|^2 \Bigr)^{1/2} \\ &\leqslant \Bigl(\sum_{j=1}^{\ell} \bigl| \mu_j^R * \bigl(\varepsilon_j g_j + \sum_{i \neq j} \varepsilon_i g_i \bigr) \bigr|^2 \Bigr)^{1/2} + \Bigl(\sum_{j=1}^{\ell} \bigl| \mu_j^R * \bigl(\varepsilon_j g_j - \sum_{i \neq j} \varepsilon_i g_i \bigr) \bigr|^2 \Bigr)^{1/2} \end{aligned}$$

hence with $g_j = \Gamma^j (I - \Phi^{\sim j}) f$ and $F_\varepsilon = \sum_{i=1}^{\ell} \varepsilon_i \Gamma^i (I - \Phi^{\sim i}) f$ we see that

$$\big\| \mathbf{F}''(R, \ell, f) \big\|_{q_0} \leqslant \mathrm{E}_\varepsilon \bigg\| \Big(\sum_{j=1}^{\ell} |\mu_j^R * F_\varepsilon|^2 \Big)^{1/2} \bigg\|_{q_0} = \mathrm{E}_\varepsilon \big\| \nabla \mu^R * F_\varepsilon \big\|_{q_0} =: D\,.$$

With Khinchin (1.27) and the *a priori* bound (8.7) we obtain

$$D \leqslant B(q_0, R, \ell)\, \mathrm{E}_\varepsilon\, \|F_\varepsilon\|_{q_0} \leqslant B_{q_0} B(q_0, R, \ell) \bigg\| \Big(\sum_{i=1}^{\ell} |\Gamma^i (I - \Phi^{\sim i}) f|^2 \Big)^{1/2} \bigg\|_{q_0}.$$

In $L^{q_1}(\mathbb{R}^\ell)$ with $q_1 = 2q_0 = 2^{\nu+1}$ we have by (8.34) and Lemma 8.8 that

$$\begin{aligned} &\bigg\| \Big(\sum_{j=1}^{\ell} |\Gamma^j (I - \Phi^{\sim j}) f|^2 \Big)^{1/2} \bigg\|_{q_1} \\ &\qquad\qquad \leqslant \bigg\| \Big(\sum_{j=1}^{\ell} |\Gamma^j f|^2 \Big)^{1/2} \bigg\|_{q_1} + \bigg\| \sum_{j=1}^{\ell} \Phi^{\sim j} \big(\Gamma^j f \big)^2 \bigg\|_{q_1/2}^{1/2} \leqslant \kappa_{q_0} \|f\|_{q_1}. \end{aligned}$$

Interpolating with the L^2 bound, and with κ_{q_0} changing from line to line, we get

$$\bigg\| \Big(\sum_{j=1}^{\ell} |\Gamma^j (I - \Phi^{\sim j}) f|^2 \Big)^{1/2} \bigg\|_{q_0} \leqslant \kappa_{q_0} R^{-\delta/(2q_0-2)} \|f\|_{q_0} \leqslant \kappa_{q_0} R^{-\delta/(2q_0)} \|f\|_{q_0}\,,$$

therefore $\big\| \mathbf{F}''(R, \ell, f) \big\|_{q_0} \leqslant \kappa_{q_0} B(q_0, R, \ell) R^{-\delta/(2q_0)} \|f\|_{q_0}$ and

$$\big\| \mathbf{F}(R, \ell, f) \big\|_{q_0} \leqslant \kappa_{q_0} \big(R^{4\delta} + B(q_0, R, \ell) R^{-\delta/(2q_0)} \big) \|f\|_{q_0}\,.$$

The estimates are proved in every dimension $\ell \leqslant n$, we have thus realized our objectives (8.21) and (8.22). Noticing that $R \geqslant 1$, we have consequently

$$b_0(q_0, R, n) + b_1(q_0, R, n) \leqslant \kappa_{q_0} \big(R^{4\delta} + B(q_0, R, n) R^{-\delta/(2q_0)} \big)\,.$$

At last, we put all parts of (8.23) together. We may assume that $4\delta < 1$. We use again $R \geqslant 1$ in order to absorb the constant bound from (8.15), thus obtaining

$$\big\| \nabla \mu^R * g \big\|_{q_0} \leqslant c(q_0, \delta) \big(R^{4\delta} + B(q_0, R, n) R^{-\delta/(2q_0)} \big) \|g\|_{q_0}\,,$$

for $g \in L^{q_0}(\mathbb{R}^n)$ and $R \geqslant R_0$. Since $B(q_0, R, n)$ is the maximum of $\big\| \nabla \mu^R * g \big\|_{q_0}$ for g of norm $\leqslant 1$ in $L^{q_0}(\mathbb{R}^n)$, we deduce that $B(q_0, R, n) \leqslant c(q_0, \delta) R^{4\delta} + B(q_0, R, n)/2$ for $R \geqslant R_1$, if $R_1 \geqslant R_0$ is such that $c(q_0, \delta) R_1^{-\delta/(2q_0)} \leqslant 1/2$, thus $B(q_0, R, n) \leqslant 2c(q_0, \delta) R^{4\delta}$ for $R \geqslant R_1$. The value of R_1 depends on δ and q_0 that are fixed. For $R \leqslant R_1$, we may estimate directly

$\|\nabla\mu^R * g\|_{q_0} \leqslant \kappa R\|g\|_{q_0} \leqslant \kappa R_1^{1-4\delta} R^{4\delta}\|g\|_{q_0}$ by Lemma 7.11. It follows finally that $B(q_0, R, n) \leqslant c'(q_0, \delta)\, R^{4\delta}$, and δ being arbitrarily small, we have proved Proposition 8.2.

9. The Aldaz weak type result for cubes, and improvements

We work again in this section with the symmetric cube Q_n of volume 1 in $\mathbb{R}^n$, that is to say, with $Q_1 = [-1/2, 1/2]$ when $n = 1$ and $Q_n = (Q_1)^n$. We first present, following Aubrun [3], a rather soft argument proving the result of Aldaz [1] that the weak type $(1,1)$ constant $\kappa_{Q,n}$ associated to the cubes Q_n is not bounded when n tends to infinity. We shall indicate and comment the quantitative improvement obtained by Aubrun [3], who gave a lower bound $\kappa_{Q,n} \geqslant \kappa_\varepsilon (\log n)^{1-\varepsilon}$ for every $\varepsilon > 0$. We then give a version of the proof of Iakovlev and Strömberg [46] who considerably improved this lower bound, showing that $\kappa_{Q,n} \geqslant \kappa n^{1/4}$. All the arguments though are based on the same initial principle that we now recall.

We begin with a few simple reflections. If we want to contradict the uniform boundedness of the weak type $(1,1)$ constant $\kappa_{Q,n}$ we must, in view of Bourgain's Theorem 8.1, look for functions f_n on $\mathbb{R}^n$ that do not stay bounded, as $n \to \infty$, in any $L^p(\mathbb{R}^n)$ with $p > 1$. Also, we may easily obtain by mollifying techniques that the weak type inequality for L^1 functions, stating that

$$c\big|\{x \in \mathbb{R}^n : (\mathrm{M}_Q f)(x) > c\}\big| \leqslant \kappa_{Q,n} \|f\|_{L^1(\mathbb{R}^n)}\,, \quad c > 0,\ f \in L^1(\mathbb{R}^n)\,, \tag{9.1}$$

where we let $\mathrm{M}_Q = \mathrm{M}_{Q_n}$, extends to bounded nonnegative measures μ on $\mathbb{R}^n$: if for every $x \in \mathbb{R}^n$ we define $(\mathrm{M}_Q\, \mu)(x)$ to be the supremum over $r > 0$ of all quotients $\mu(x + rQ)/|x + rQ|$, then (9.1) extends with the same constant $\kappa_{Q,n}$ as

$$c\big|\{x \in \mathbb{R}^n : (\mathrm{M}_Q\, \mu)(x) > c\}\big| \leqslant \kappa_{Q,n}\, \mu(\mathbb{R}^n)\,, \quad c > 0\,.$$

These two remarks lead naturally to consider measures on $\mathbb{R}^n$ that are sums of Dirac measures, in order to contradict the boundedness of $\kappa_{Q,n}$ when $n \to \infty$.

Let $\mu_N = \sum_{j=1}^N (\delta_{j-1/2} + \delta_{-j+1/2})$ stand for an "approximation" of the Lebesgue measure λ on a large segment $S_N = [-N, N]$. The measure μ_N has a unit mass at the middle of each interval $(j, j+1)$, j integer and $-N \leqslant j < N$. Every interval $[u, u+h)$ contained in S_N, with length an integer $h > 0$, has the same measure h for μ_N or for λ. However, if I is a segment of length $1 + \alpha$, $0 < \alpha = 1 - \varepsilon < 1$, centered at $s = 0$ or at any $s = j$, integer with $|j| < N$, then I contains $j \pm 1/2$ and

$$\mu_N(I) = 2 \quad \text{but} \quad \lambda(I) = 1 + \alpha = 2 - \varepsilon < 2\,,$$

so that $(\mathrm{M}_Q\, \mu_N)(s) \geqslant \mu_N(I)/\lambda(I) = 2/(2-\varepsilon)$. The same observation is valid if s is not too far from an integer j in $(-N, N)$, precisely, if $|s-j| < \alpha/2$. If we pass to $\mathbb{R}^n$ and to the tensor product measure $\mu_N^{(n)} := \otimes^n \mu_N$, we obtain a huge magnification due to dimension, which reads as

$$\bigl(\mathrm{M}_Q\, \mu_N^{(n)}\bigr)(x) \geqslant \Bigl(\frac{2}{2-\varepsilon}\Bigr)^n$$

when all coordinates x_i, $i = 1, \ldots, n$, of the point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ belong to the subset C_α of $[-N, N]$ defined by

$$C_\alpha = \bigcup_{-N<j<N} (j - \alpha/2, j + \alpha/2)\,. \tag{9.2}$$

If $\ell = 2h+1 \geqslant 1$ is an odd integer, if $J = (-h-1/2-\alpha/2, h+1/2+\alpha/2) = (\ell+\alpha)Q$ and if $s+J$ is contained in S_N, we see in the same way, when $s \in C_\alpha$, that the segment $s + J$ contains $\ell + 1 = 2h + 2$ of the unit masses forming μ_N. Consequently, we have $(\mathrm{M}_Q\, \mu_N)(s) \geqslant (\ell + 1)/|s + J| = (\ell + 1)/(\ell + \alpha)$ and

$$(\mathrm{M}_Q\, \mu_N^{(n)})(x) \geqslant \Bigl(\frac{\ell+1}{\ell+1-\varepsilon}\Bigr)^n$$

when $x = (x_1, \ldots, x_n)$ has all coordinates x_i in C_α and $x + J^n \subset S_N^n = 2NQ_n$.

This case is much too particular, since the set of such points x represents only a tiny proportion α^n of the cube S_N^n. One has actually to consider that *some* coordinates x_i of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are in C_α, say $m \leqslant n$ of them. For the other coordinates x_i, observe that any interval of length $\ell+\alpha$ contained in S_N contains at least ℓ points of the support of μ_N. Assuming that $x + (\ell + \alpha)Q \subset S_N^n$, we get for this point x with m coordinates in C_α the lower bound

$$(\mathrm{M}_Q\, \mu_N^{(n)})(x) \geqslant \frac{\mu_N^{(n)}\bigl(x + (\ell+\alpha)Q\bigr)}{\bigl|x + (\ell+\alpha)Q\bigr|} \geqslant \Bigl(\frac{\ell+1}{\ell+\alpha}\Bigr)^m \Bigl(\frac{\ell}{\ell+\alpha}\Bigr)^{n-m}. \tag{9.3}$$

We want the cardinality m of the “good”, “centered” coordinates x_i to be as big as possible. Since they are chosen out of subsets of length α in unit intervals $(j - 1/2, j + 1/2)$, it is likely that the proportion of “good coordinates” among n coordinates be around α, with a plausible deviation of order $\sqrt{n}$ from the expected number αn. We shall thus think henceforth that $m = \alpha n + \delta\sqrt{n}$ for some $\delta > 0$.

We try to make the lower bound (9.3) as large as possible, by a suitable choice of ℓ. Setting $\beta = 1 - \alpha$, we rewrite the right-hand side of (9.3) under the form

$$\left(\frac{\ell+\alpha+\beta}{\ell+\alpha}\right)^m \left(\frac{\ell+\alpha-\alpha}{\ell+\alpha}\right)^{n-m} = \left(1+\frac{\beta}{\ell+\alpha}\right)^m \left(1-\frac{\alpha}{\ell+\alpha}\right)^{n-m}.$$

Considering now $y = (\ell+\alpha)^{-1}$ as a real parameter, we will study

$$V(y) := (1+\beta y)^m (1-\alpha y)^{n-m}, \quad -1/\beta \leqslant y \leqslant 1/\alpha,$$

and find the maximal value $V(\overline{y})$. Equivalently, we let f denote the fraction m/n of coordinates of x that are in C_α, and we maximize $v_{f,\alpha}(s) = V(s)^{1/n}$ defined by

$$v_{f,\alpha}(s) = (1+\beta s)^f (1-\alpha s)^{1-f}, \quad s \in [-1/\beta, 1/\alpha].$$

We have to remember though that the lower bound $V(y)$ for $\mathrm{M}_Q \mu_N^{(n)}(x)$ given in (9.3) is only valid when $1/y - \alpha$ is an odd integer ℓ. We shall replace $\overline{y}$ by a value $y = y_{\mathbb{N}} > 0$ close to $\overline{y}$, such that $1/y_{\mathbb{N}} - \alpha$ is an odd integer, thus obtaining that $\mathrm{M}_Q \mu_N^{(n)}(x) \geqslant V(y_{\mathbb{N}})$. We must ensure that the value of $V(y)$ does not decrease too much when moving from $\overline{y}$ to $y_{\mathbb{N}}$. We would like to have

$$V(y_{\mathbb{N}}) \geqslant \mathrm{e}^{-c}\, V(\overline{y}) \;\text{ or }\; v_{f,\alpha}(y_{\mathbb{N}}) \geqslant \mathrm{e}^{-c/n}\, v_{f,\alpha}(\overline{y}), \quad \text{for some } c \geqslant 0. \tag{9.4}$$

The maximal argument $\overline{y}$ is produced from f and a choice of $\alpha < f$. We shall say that the couple (f, α) is c-*allowable* if the above condition (9.4) is satisfied.

Lemma 9.1. — *Let $0 < \alpha < f < 1$, $\sigma_\alpha^2 = \alpha(1-\alpha)$ and let us define $\tau > 0$ by writing $f = \alpha + \sigma_\alpha \tau$. The function $v_{f,\alpha}$ reaches its maximum at*

$$\overline{y} = \overline{y}_{f,\alpha} = \frac{\tau}{\sigma_\alpha} = \frac{f-\alpha}{\sigma_\alpha^2} > 0. \tag{9.5}$$

If $0 < y, \overline{y} \leqslant 1/2$ then

$$\mathrm{e}^{(y-\overline{y})^2/2}\, v_{f,\alpha}(y) \geqslant v_{f,\alpha}(\overline{y}) = \left(\frac{f}{\alpha}\right)^f \left(\frac{1-f}{1-\alpha}\right)^{1-f}. \tag{9.6}$$

If $0 < \overline{y} \leqslant 1/4$ and $\overline{y}^4 \leqslant c/n$, then the couple (f, α) is c-allowable.

Proof. — Let $w(s) = \ln v_{f,\alpha}(s) = f \ln(1+\beta s) + (1-f)\ln(1-\alpha s)$. We have

$$w'(s) = \frac{\beta f}{1+\beta s} - \frac{\alpha(1-f)}{1-\alpha s}, \quad w''(s) = -\frac{\beta^2 f}{(1+\beta s)^2} - \frac{\alpha^2(1-f)}{(1-\alpha s)^2}.$$

The maximal argument $\overline{y}$ is found by solving $w'(\overline{y}) = 0$, yielding

$$\overline{y} = \frac{f-\alpha}{\sigma_\alpha^2}, \quad 1+\beta\overline{y} = \frac{f}{\alpha}, \quad 1-\alpha\overline{y} = \frac{1-f}{1-\alpha}.$$

This gives us the maximal value $v_{f,\alpha}(\overline{y})$ at the right-hand side of (9.6). Suppose now that we have $0 < y, \overline{y} \leqslant 1/2$. Using Taylor–Lagrange at $\overline{y}$, we get

$$w(y) - w(\overline{y}) = w''(\xi)\frac{(y-\overline{y})^2}{2}\,,$$

for some ξ between y and $\overline{y}$, hence $0 < \xi \leqslant 1/2$. We have $1 - \alpha\xi \geqslant 1/2$ and

$$-w''(\xi) \leqslant \beta^2 f + 4\alpha^2(1-f) \leqslant \beta + 4\alpha^2(1-\alpha) \leqslant \beta + \alpha = 1\,,$$

because $\alpha(1-\alpha) \leqslant 1/4$. This implies the left-hand side of (9.6).

Suppose that $0 < \overline{y} \leqslant 1/4$. Moving around $\overline{y}$, we can find $y_{\mathbb{N}} > 0$ satisfying

$$\frac{|y_{\mathbb{N}} - \overline{y}|}{y_{\mathbb{N}}\overline{y}} = \left|\frac{1}{\overline{y}} - \frac{1}{y_{\mathbb{N}}}\right| \leqslant 1$$

and such that $1/y_{\mathbb{N}} - \alpha$ is an odd integer. From $|y_{\mathbb{N}} - \overline{y}| \leqslant y_{\mathbb{N}}\overline{y}$ and $\overline{y} \leqslant 1/4$ follows that $y_{\mathbb{N}} \leqslant 4\overline{y}/3 \leqslant 1/3 < 1/2$. Also, $|y_{\mathbb{N}} - \overline{y}| \leqslant 4\overline{y}^2/3 < \sqrt{2}\,\overline{y}^2$. By (9.6), we deduce that $v_{f,\alpha}(y_{\mathbb{N}}) \geqslant \mathrm{e}^{-\overline{y}^4}\, v_{f,\alpha}(\overline{y})$ and the conclusion is reached. □

Given f and α such that $0 < \alpha < f < 1$, let us now examine the optimal value

$$E_{f,\alpha} := v_{f,\alpha}(\overline{y}) = (1+\beta\overline{y})^f(1-\alpha\overline{y})^{1-f} = \left(\frac{f}{\alpha}\right)^f\left(\frac{1-f}{1-\alpha}\right)^{1-f}. \tag{9.7}$$

Consider the function ϕ_α defined on $(0,1)$ by

$$\phi_\alpha(s) = s\ln\left(\frac{s}{\alpha}\right) + (1-s)\ln\left(\frac{1-s}{1-\alpha}\right), \quad s \in (0,1)\,. \tag{9.8}$$

We see that $\phi'_\alpha(s) = \ln(s/\alpha) - \ln\big((1-s)/(1-\alpha)\big)$, $\phi''_\alpha(s) = 1/s + 1/(1-s) = 1/\big(s(1-s)\big)$, and $\phi^{(3)}_\alpha(s) = -s^{-2} + (1-s)^{-2}$. Note that $\phi_\alpha(\alpha) = \phi'_\alpha(\alpha) = 0$, and that $\phi''_\alpha(\alpha) = \sigma_\alpha^{-2}$.

Lemma 9.2. — *If $0 < \alpha < f = \alpha + \sigma_\alpha\tau < 1$, the maximal value $v_{f,\alpha}(\overline{y})$ satisfies*

$$\ln v_{f,\alpha}(\overline{y}) = \phi_\alpha(f) \geqslant \frac{\tau^2}{2} - \frac{1-2\alpha}{\sigma_\alpha}\frac{\tau^3}{6}\,. \tag{9.9}$$

Proof. — By Taylor–Lagrange for ϕ_α at the point α, we have

$$\phi_\alpha(f) = \phi''_\alpha(\alpha)\frac{(f-\alpha)^2}{2} + \phi^{(3)}_\alpha(\xi)\frac{(f-\alpha)^3}{6} = \frac{\tau^2}{2} + \phi^{(3)}_\alpha(\xi)\frac{(\sigma_\alpha\tau)^3}{6}$$

for some $\xi \in (\alpha, f)$. Since $\phi^{(3)}_\alpha$ is increasing, we get that

$$\phi_\alpha(f) - \frac{\tau^2}{2} \geqslant \phi^{(3)}_\alpha(\alpha)\frac{(\sigma_\alpha\tau)^3}{6} = \frac{2\alpha-1}{\alpha^2(1-\alpha)^2}\frac{\sigma_\alpha^3\tau^3}{6} = \frac{2\alpha-1}{\sigma_\alpha}\frac{\tau^3}{6}\,. \qquad □$$

In all that follows, we see $\Omega_1 := [-N, N]$ as a probability space equipped with the uniform probability measure, denoted here by P_1, and we shall consider the cube $S_N^n = 2NQ_n$, equipped with the product measure $P = P_1^{\otimes n}$, also the uniform probability measure, as being our main probability space $(\Omega, \mathcal{F}, P)$. On this space, the random variables $(\mathbf{1}_{C_\alpha}(x_i))_{i=1}^n$, where $x = (x_1, \ldots, x_n) \in \Omega$, are independent and equal to 0 or 1 with respective probabilities $1 - \alpha$ and α. Their expectation is α and their variance is equal to $\sigma_\alpha^2 = \alpha(1-\alpha) \leqslant 1/4$. For every $\alpha \in (0,1)$, we introduce the centered and variance 1 Bernoulli variable $X_{1,\alpha}$ defined on Ω_1 by

$$X_{1,\alpha} = \frac{\mathbf{1}_{C_\alpha} - \alpha}{\sigma_\alpha} = \sqrt{\frac{1-\alpha}{\alpha}}\,\mathbf{1}_{C_\alpha} - \sqrt{\frac{\alpha}{1-\alpha}}\,\mathbf{1}_{\Omega_1 \setminus C_\alpha}\,, \tag{9.10}$$

and we let

$$X_{n,\alpha}(x) = \frac{\sum_{i=1}^n X_{1,\alpha}(x_i)}{\sqrt{n}} = \sum_{i=1}^n \frac{\mathbf{1}_{C_\alpha}(x_i) - \alpha}{\sigma_\alpha \sqrt{n}}\,, \qquad x = (x_1, \ldots, x_n) \in \Omega\,.$$

We also let $N_{n,\alpha}(x) = \sum_{i=1}^n \mathbf{1}_{C_\alpha}(x_i)$ denote the number of coordinates of x that are in C_α. We are ready for a first explicit estimate of the maximal function $\mathrm{M}_Q \mu_N^{(n)}$.

Lemma 9.3. — *Let $0 < \alpha < 1$ and $\sigma_\alpha^2 = \alpha(1-\alpha)$. Let $n \in \mathbb{N}^*$, $t > 0$ and $0 < \theta < 1$ be such that $\sqrt{n} \geqslant 2t\sigma_\alpha^{-2}(1-\theta)^{-1}$. We have $\mathrm{M}_Q\, \mu_N^{(n)} > \mathrm{e}^{\theta t^2/2}$ on the set*

$$A_{\alpha,t}^{(n)} = \Bigl\{ x \in 2(N - t^{-1}\sqrt{n})Q_n : N_{n,\alpha}(x) = \sum_{i=1}^n \mathbf{1}_{C_\alpha}(x_i) > \alpha n + t\sigma_\alpha \sqrt{n} \Bigr\}\,,$$

where C_α is defined at (9.2). When the dimension n is large, and assuming the size N large enough compared to n, it follows that

$$\frac{\bigl|\{\mathrm{M}_Q\, \mu_N^{(n)} > \mathrm{e}^{\theta t^2/2}\}\bigr|}{|S_N^n|} \geqslant \frac{|A_{\alpha,t}^{(n)}|}{|2NQ_n|} > \frac{1}{2}\,\gamma_1\bigl((t, +\infty)\bigr)\,.$$

Proof. — By the central limit theorem (see [32] for instance), we know that the distribution of $X_{n,\alpha}$ tends to the distribution of a $N(0,1)$ Gaussian random variable G when n tends to infinity. This yields

$$P\bigl(N_{n,\alpha} > \alpha n + t\sigma_\alpha \sqrt{n}\bigr) = P\bigl(X_{n,\alpha} > t\bigr) \xrightarrow[n]{} P(G > t) = \gamma_1\bigl((t, +\infty)\bigr)\,.$$

Let $A_{\alpha,t}^{(n,0)}$ be the set of points $x \in \Omega$ where $N_{n,\alpha}(x) > \alpha n + t\sigma_\alpha\sqrt{n}$. Fix $x \in A_{\alpha,t}^{(n,0)}$ and let $m = N_{n,\alpha}(x)$. We shall apply Lemma 9.1 with $f = m/n$ and $\tau = t/\sqrt{n}$. By assumption, the optimal argument $\overline{y}$ satisfies

$$\overline{y} = \frac{t}{\sigma_\alpha \sqrt{n}} \leqslant \frac{\sigma_\alpha(1-\theta)}{2} < 1/4\,.$$

At (9.3), we used a cube centered at x, with side length $\ell+\alpha$, ℓ an odd integer. We can choose $\ell+\alpha < 1/\overline{y}+2 < 2/\overline{y} < t^{-1}\sqrt{n}$. This cube must be contained in $\Omega = S_N^n$, so we have to give up a small part of $A_{\alpha,t}^{(n,0)}$, close to the boundary of Ω. We thus introduce the subset $A_{\alpha,t}^{(n)} = A_{\alpha,t}^{(n,0)} \cap 2(N - t^{-1}\sqrt{n})Q_n$. The difference $A_{\alpha,t}^{(n,0)} \setminus A_{\alpha,t}^{(n)}$ gets negligible when the side $2N$ of S_N^n tends to infinity since $(1 - t^{-1}\sqrt{n}/N)^n \to_N 1$, so the set $A_{\alpha,t}^{(n)}$ has essentially the same probability as $A_{\alpha,t}^{(n,0)}$ when $N = N(n) > \kappa(t)n^{3/2}$ is large enough. When n tends to infinity, the probability of $A_{\alpha,t}^{(n)}$ is therefore, say, larger than $\gamma_1((t,+\infty))/2$.

We first show that the couple (f,α) is c-allowable with $c = (1-\theta)t^2/4$. We know that $\overline{y} < 1/4$ and on the other hand, we have

$$\overline{y}^4 = \frac{t^4}{\sigma_\alpha^4 n^2} = \frac{c}{n}\,\frac{4t^2}{(1-\theta)\sigma_\alpha^4 n} < \frac{c}{n}\Bigl(\frac{2t}{(1-\theta)\sigma_\alpha^2\sqrt{n}}\Bigr)^2 \leqslant \frac{c}{n}\,.$$

It follows from Lemma 9.1 that $\mathrm{M}_Q\mu_N^{(n)}(x) \geqslant \mathrm{e}^{-(1-\theta)t^2/4}\,V(\overline{y})$ for every $x \in A_{\alpha,t}^{(n)}$. It remains to estimate the optimal value $V(\overline{y})$. For this we apply (9.9). It implies that $V(\overline{y}) \geqslant \mathrm{e}^{t^2/2}$ when $\alpha \geqslant 1/2$, and when $\alpha \leqslant 1/2$, we see that

$$\frac{1-2\alpha}{\sigma_\alpha}\,\frac{\tau^3}{6} = \frac{(1-2\alpha)\tau}{3\sigma_\alpha}\,\frac{\tau^2}{2} < \frac{t}{3\sigma_\alpha\sqrt{n}}\,\frac{\tau^2}{2} < \frac{\sigma_\alpha(1-\theta)}{6}\,\frac{\tau^2}{2} < \frac{(1-\theta)\tau^2}{4}\,,$$

so that $V(\overline{y}) \geqslant \mathrm{e}^{t^2/2-(1-\theta)t^2/4}$ and $\mathrm{M}_Q\mu_N^{(n)}(x) \geqslant \mathrm{e}^{t^2/2}\,\mathrm{e}^{-(1-\theta)t^2/2} = \mathrm{e}^{\theta t^2/2}$. □

Given $\alpha \in (0,1)$, we have identified a subset $A_{\alpha,t}^{(n)}$ of S_N^n where $\mathrm{M}_Q\mu_N^{(n)}$ is large. We shall have to use several values of α, and show that the union of the corresponding sets provides a fair amount of the total volume of S_N^n. We thus introduce $0 < \alpha_0 < \alpha_2 < \ldots < \alpha_K < 1$ and we will prove that the probability of the union of sets $(A_{\alpha_j,t}^{(n)})_{j=0}^K$ gets $> 1/4$, say, when K is large but fixed and when n tends to infinity. Rather than relying, as Aubrun does, on the *law of iterated logarithm*, we apply easy facts behind the proof of that “law”. In a simple qualitative approach, we shall analyze the Gaussian limit of the joint distribution of $(X_{n,\alpha_j})_{j=0}^K$, which is the distribution of a Gaussian vector $(G_j)_{j=0}^K$ whose covariance matrix C is the same as that of $(X_{n,\alpha_j})_{j=0}^K$. Letting $\sigma_j^2 = \alpha_j(1-\alpha_j)$, the entries of C are

$$C_{j,k} = \mathrm{E}(X_{1,\alpha_j}X_{1,\alpha_k}) = \sigma_j^{-1}\sigma_k^{-1}(\alpha_j \wedge \alpha_k - 2\alpha_j\alpha_k + \alpha_j\alpha_k)\,, \quad 0 \leqslant j,k \leqslant K\,.$$

Note that $C_{j,j} = 1$. Assuming $\alpha_j \leqslant \alpha_k$, that is to say, assuming $j \leqslant k$, we get

$$C_{j,k} = \sigma_j^{-1}\sigma_k^{-1}\alpha_j(1-\alpha_k) = \sqrt{\frac{\alpha_j}{1-\alpha_j}}\sqrt{\frac{1-\alpha_k}{\alpha_k}}\,.$$

We fix $v \in (0,1)$ and set $w = \sqrt{1-v^2}$. We define $\alpha_j = (1+v^{2j})^{-1}$, $j = 0, \ldots, K$, and obtain $C_{j,k} = v^{|k-j|}$. We can realize the distribution of $(G_j)_{j=0}^K$ by considering the larger Gaussian sequence indexed by $\mathbb{Z}$, which is defined by the sums of the series $G_j = w \sum_{i \leqslant j} v^{j-i} U_i$, for every $j \in \mathbb{Z}$, where the $(U_i)_{i \in \mathbb{Z}}$ are independent $N(0,1)$ Gaussian variables. Indeed, if $j \leqslant k$ we have that

$$\mathrm{E}(G_j G_k) = (1-v^2) \sum_{i \leqslant j} v^{j+k-2i} = v^{k-j} = v^{|k-j|}\,.$$

We see that $G_j - vG_{j-1} = wU_j$ and it follows that

$$\max_{1 \leqslant j \leqslant J} |U_j| = w^{-1} \max_{1 \leqslant j \leqslant J} |G_j - vG_{j-1}| \leqslant w^{-1}(1+v) \max_{0 \leqslant j \leqslant J} |G_j|\,. \tag{9.11}$$

We now recall an extremely classical estimate.

Lemma 9.4. — *Let $J \geqslant 21$ be an integer and set*

$$s_J := \sqrt{2 \ln J - \ln(16\pi \ln J)}\,.$$

If $U_1, \ldots, U_J$ are independent $N(0,1)$ Gaussian variables, one has that

$$P\big(\max_{1 \leqslant j \leqslant J} U_j > s_J\big) > 1/2\,.$$

Proof. — We have for $s > 0$ that

$$\int_s^{+\infty} \mathrm{d}\gamma_1(s) > \frac{s}{\sqrt{2\pi}(1+s^2)}\, \mathrm{e}^{-s^2/2}\,, \tag{9.12}$$

consequence of

$$\mathrm{e}^{-s^2/2}/s = \int_s^{+\infty} (1+u^{-2})\, \mathrm{e}^{-u^2/2}\, \mathrm{d}u < (1+s^{-2}) \int_s^{+\infty} \mathrm{e}^{-u^2/2}\, \mathrm{d}u\,.$$

When $J \geqslant 21$, one has $\mathrm{e}^{-1} J^2 > 16\pi \ln J > 1$, hence $1 < s_J < \sqrt{2 \ln J}$. Therefore, we see by (9.12) for each $j = 1, \ldots, J$ that

$$P\big(U_j > s_J\big) \geqslant \frac{s_J}{\sqrt{2\pi}(1+s_J^2)} \frac{\sqrt{16\pi \ln J}}{J} \geqslant \frac{2s_J^2}{(1+s_J^2)J} \geqslant \frac{1}{J}\,.$$

It follows that

$$P\big(\max_{1 \leqslant j \leqslant J} U_j \leqslant s_J\big) \leqslant \Big(1 - \frac{1}{J}\Big)^J < \mathrm{e}^{-1} < \frac{1}{2}\,. \qquad \square$$

Theorem 9.5 (Aldaz [1]). — *The weak type $(1,1)$ constant $\kappa_{Q,n}$ in (9.1) does not stay bounded when the dimension n tends to infinity.*

Proof. — Given an arbitrary $t > 1$, we let $t_1 := tw^{-1}(1+v) > t$ and choose an integer $K \geqslant 21$ such that $s_K > t_1$. Applying Lemma 9.4, we obtain that the event $\big\{\max_{1 \leqslant j \leqslant K} |U_j| > t_1\big\}$ has probability $> 1/2$, and by (9.11), it follows that the event $\{\max_{0 \leqslant j \leqslant K} |G_j| > t\}$ also has probability $> 1/2$.

We see that $\sup |G_j|$ is the maximum of $\sup G_j$ and $\sup(-G_j)$ that have the same distribution, hence $\{\max_{0\leqslant j\leqslant K} G_j > t\}$ has probability $> 1/4$. Consequently, given any $t > 1$, we obtain by the central limit theorem that the union of sets $A^{(n)}_{\alpha_j,t}$, for $j = 0, \ldots, K$, has a probability close to that of $\{\max_{0\leqslant j\leqslant K} G_j > t\}$, hence $> 1/4$ when n is large. By Lemma 9.3, given $\theta \in (0, 1)$ and if $\sqrt{n}(1-\theta)\sigma_K^2 > 2t$, the maximal function $\mathrm{M}_Q\mu_N^{(n)}$ is larger than $\mathrm{e}^{\theta t^2/2}$ on the union $\bigcup_{j=0}^K A^{(n)}_{\alpha_j,t}$, i.e., on a subset of $\Omega = S_N^n$ having probability $> 1/4$, hence $\kappa_{Q,n} \geqslant \mathrm{e}^{\theta t^2/2}/4$ when n is large enough. □

Aubrun [3] gives a lower bound $\kappa_{Q,n} \geqslant \kappa_\varepsilon(\ln n)^{1-\varepsilon}$ for every $\varepsilon > 0$ by making quantitative the proof above. He applies to this end results proved years before (by Bretagnolle–Massart [14] in 1989 and previously, by Komlós–Major–Tusnády [51] in 1975) on the approximation of Brownian bridges, when $n \to +\infty$ and with explicit bounds, by binomial processes

$$Z_t^{(n)} = \sum_{i=1}^n \frac{\mathbf{1}_{\{Y_i\leqslant t\}} - t}{\sqrt{n}}, \qquad t \in [0, 1],$$

where the $(Y_i)_{i=1}^n$ are independent and uniform on $[0, 1]$. One can see that the distribution of the process $(Z_t^{(n)})_{t\in(0,1)}$ is equal to that of $(\sigma_t X_{n,t})_{t\in(0,1)}$.

Iakovlev and Strömberg [46] begin with the same observations, in particular introducing the measure $\mu_N^{(n)}$, using the fundamental estimate (9.3) and, in a less apparent manner, the value $\mathrm{e}^{\theta t^2/2}$ from Lemma 9.3. But instead of working in a probabilistic setting, they proceed to a finer combinatorial analysis. Contrary to Aubrun, they do not use values α close to 1, nor close to 0. In our exposition of their arguments, we shall work towards simplicity rather than optimality.

Let us digress a little with some comments on the Gaussian process viewpoint, and express in terms of stochastic maximal function the lower bound for $\mathrm{M}_Q\mu_N^{(n)}$ given in (9.3). Let $x \in \Omega$ and $m = N_{n,\alpha}(x)$, $\sigma_\alpha^2 = \alpha(1-\alpha)$ and write $m = \alpha n + \sigma_\alpha t\sqrt{n}$. Notice that $t = (m - \alpha n)/(\sigma_\alpha\sqrt{n}) = X_{n,\alpha}(x)$. We let f be the fraction m/n, and rewrite the preceding formula for m as $f = \alpha + \sigma_\alpha\tau$, with $\tau = t/\sqrt{n}$. We know the optimal argument $\overline{y}$ for $V(y)$, given in (9.5) by

$$\overline{y} = \frac{t}{\sigma_\alpha\sqrt{n}} = \frac{\tau}{\sigma_\alpha}, \quad \text{and} \quad \frac{\ln V(\overline{y})}{n} = f\ln\left(\frac{f}{\alpha}\right) + (1-f)\ln\left(\frac{1-f}{1-\alpha}\right).$$

By Lemma 9.2 we have $\ln E_{f,\alpha} = \phi_\alpha(f) \geqslant \tau^2/2 = t^2/(2n)$ if $\tau > 0$ and $\alpha \geqslant 1/2$. Let $1/2 \leqslant \alpha \leqslant 3/4$ and assume that $0 < t = X_{n,\alpha}(x) \leqslant n^{1/4}/2$.

We see then that $\overline{y} = t/(\sigma_\alpha \sqrt{n}) < 2n^{-1/4}/\sqrt{3}$, thus $n\overline{y}^4 \leqslant 16/9$, $\overline{y} \leqslant 1/4$ for $n > 455$ and by Lemma 9.1 we are then in the allowable case with $c \leqslant 16/9$. This yields

$$\mathrm{M}_Q \mu_N^{(n)}(x) \geqslant \kappa^{-1} E_{f,\alpha}^n \geqslant \kappa^{-1} \exp\left(\frac{t^2}{2}\right), \qquad \text{with } \kappa < \mathrm{e}^{16/9} < 6,\ n > 455\,. \tag{9.13}$$

Let us define a maximal function $X^*(x) = \sup_{1/2 \leqslant \alpha \leqslant 3/4} X_{n,\alpha}^{(1)}(x)$, where $X_{n,\alpha}^{(1)}(x) = X_{n,\alpha}(x)$ when $0 \leqslant 2X_{n,\alpha}(x) \leqslant n^{1/4}$ and $X_{n,\alpha}^{(1)}(x) = 0$ otherwise. We get

$$6\,\mathrm{M}_Q \mu_N^{(n)}(x) \geqslant \exp\left(\frac{X^*(x)^2}{2}\right)$$

and the weak type $(1,1)$ constant $\kappa_{Q,n}$ must satisfy the condition

$$P\big(\{X^* > s\}\big) \leqslant P\big(\{\mathrm{M}_Q \mu_N^{(n)} > \mathrm{e}^{s^2/2}/6\}\big) \leqslant 6\,\kappa_{Q,n}\,\mathrm{e}^{-s^2/2}, \quad s > 0\,.$$

This explains how delicate the question can be. Indeed, given a subgaussian process $(Y_t)_{t \in T}$ satisfying tail estimates of the form $P(Y > s) \leqslant \kappa\,\mathrm{e}^{-s^2/(2d^2)}$ for every $s > 0$, for each difference $Y = Y_{t_2} - Y_{t_1}$ and with $d = d(t_1, t_2) = \|Y_{t_1} - Y_{t_2}\|_2$, the well known chaining technique of Dudley [28] does not allow one to prove for the maximal process $\sup_{t \in T} Y_t$ a subgaussian inequality with the same bounding function $\mathrm{e}^{-s^2/2}$, but rather with $\mathrm{e}^{-Cs^2/2}$ for some $C < 1$, which is inoperative here.

Theorem 9.6 (Iakovlev and Strömberg [46]). — *One has that*

$$\kappa_{Q,n} \geqslant \kappa n^{1/4}\,.$$

Rather than exploiting the exponential asymptotics (9.13) of $E_{f,\alpha}^n$, we shall observe some more nice features of the expression $E_{f,\alpha}$ defined in (9.7), where $f = m/n = \alpha + t\sigma_\alpha/\sqrt{n} = \alpha + \sigma_\alpha \tau$. We replace the value $\mathrm{e}^{\theta t^2/2}$ seen in Lemma 9.3 by a fixed large value $V > 1$ and we try to keep the (conditional on allowability) lower bound $E_{f,\alpha}^n$ for $\mathrm{M}_Q \mu_N^{(n)}$ constantly equal to V. Equivalently, we keep

$$E_{f,\alpha} = \mathrm{e}^{\phi_\alpha(f)} = V^{1/n} > 1 \tag{9.14}$$

for all values of f (or of m) that will be handled. The possibility of finding α satisfying (9.14) comes from the fact that for every given $f \in (0,1)$, the function

$$\psi_f : s \mapsto \left(\frac{f}{s}\right)^f \left(\frac{1-f}{1-s}\right)^{1-f} = \mathrm{e}^{\phi_s(f)}, \quad s \in (0,1)\,, \tag{9.15}$$

is convex on $(0,1)$ (actually, log-convex), tends to infinity at 0 and at 1, and assumes its minimal value $\psi_f(f) = 1$ at $s = f$. Consequently, there are exactly two values $\alpha_0 < f < \alpha_1$ of $\alpha \in (0,1)$ solving (9.14), we shall consider the smallest one and set $\alpha(f) = \alpha_0$. Notice that $(\ln \psi_f)'(s) = -f/s + (1-f)/(1-s)$ vanishes at $s = f$, and

$$(\ln \psi_f)''(s) = \frac{f}{s^2} + \frac{1-f}{(1-s)^2} > f + (1-f) = 1\,. \tag{9.16}$$

We have therefore for every $s \in (0,1)$ that

$$\ln \psi_f(s) \geqslant (s-f)^2/2\,, \qquad \text{thus} \;\; (f - \alpha(f))^2/2 \leqslant \ln \psi_f(\alpha(f)) = (\ln V)/n\,. \tag{9.17}$$

From now on, we fix two values $0 < f_* < f^* \leqslant 1/2$, independent of the dimension n. For every integer m in the range $[f_* n, f^* n]$, we shall consider the set

$$F_m = \{x \in \Omega : N_{n,\alpha(f)}(x) = m\}\,, \quad \text{with} \;\; f = m/n\,.$$

Let us write $\alpha = \alpha(f)$ for brevity. We have that $E_{f,\alpha} = V^{1/n}$ and if we assume c-allowability for (f,α) we get $\mathrm{M}_Q \mu_N^{(n)}(x) \geqslant \mathrm{e}^{-c}\, V$ for every $x \in F_m$, by (9.4). The probability of F_m is $\alpha^m (1-\alpha)^{n-m} \binom{n}{m}$ and we see that

$$\begin{aligned} V P(F_m) &= \Bigl(\frac{f}{\alpha}\Bigr)^m \Bigl(\frac{1-f}{1-\alpha}\Bigr)^{n-m} \alpha^m (1-\alpha)^{n-m} \binom{n}{m} \\ &= \frac{m^m (n-m)^{n-m}}{n^n} \binom{n}{m}\,. \end{aligned}$$

Stirling's formula in the form $\mathrm{e}^{-1/(12p)}\, p! \leqslant p^p\, \mathrm{e}^{-p} \sqrt{2\pi p} \leqslant p!$ (see [66]) gives

$$\mathrm{e}^{-1/(12n)}\, V P(F_m) \leqslant \frac{\sqrt{n}}{\sqrt{2\pi m(n-m)}} \leqslant \mathrm{e}^{n/(12m(n-m))}\, V P(F_m)\,. \tag{9.18}$$

With $s_* = \sqrt{f_*(1-f_*)}$ and $s^* = \sqrt{f^*(1-f^*)}$, it follows that

$$V P(F_m) \geqslant \frac{\mathrm{e}^{-1/(12 f(1-f) n)}}{\sqrt{2\pi f(1-f) n}} \geqslant \frac{\mathrm{e}^{-1/(12 s_*^2 n)}}{s^* \sqrt{2\pi}}\, \frac{1}{\sqrt{n}}\,. \tag{9.19}$$

If the sets F_m were disjoint (and the couples $(f, \alpha(f))$ c-allowable) we would get immediately, by summing on m between $f_* n$ and $f^* n$, a lower bound of

$$\kappa_{Q,n} \geqslant \mathrm{e}^{-c}\, V P\bigl(\{\mathrm{M}_Q \mu_N^{(n)} \geqslant \mathrm{e}^{-c}\, V\}\bigr) \;\; \text{by} \;\; \kappa \left[\mathrm{e}^{-c} (f^* - f_*)/(s^* \sqrt{2\pi})\right] \sqrt{n}\,,$$

but this disjointness property is clearly not true. We shall specify a suitable large V such that the probability of the intersection of two events F_{m_1} and F_{m_2} will be small compared to the probability of F_{m_1}, when $m_1 < m_2$ are not too close. We shall find a subset $M \subset [f_* n, f^* n]$, as large as possible, consisting of "well spaced" values m_j giving rise to c-allowable couples. The final estimate has the form

$$\kappa_{Q,n} \geqslant \mathrm{e}^{-c} V P\Big(\bigcup_{m \in M} F_m\Big), \tag{9.20}$$

where the probability of the union will be larger than half of the sum of probabilities. The seemingly harmless allowability restriction that $y^{-1} - \alpha = \sigma_\alpha/\tau - \alpha$ must be an odd integer ℓ will actually cause a heavy loss at the end.

We fix $\varepsilon \in (0, f_*]$ and introduce $\eta := \sqrt{1 - \varepsilon/f_*}$. We define the "big" value V as $V = \mathrm{e}^{\varepsilon^2 n/2}$. By (9.17), we have that

$$0 < f - \alpha(f) \leqslant \varepsilon\,. \tag{9.21}$$

Lemma 9.7. — *Suppose that* $0 < \alpha < \xi \leqslant f \leqslant \alpha + \varepsilon$ *and* $f_* \leqslant f \leqslant 1/2$. *One has*

$$\eta^2 \leqslant \frac{\alpha}{\xi} < \frac{\alpha(1-\alpha)}{\xi(1-\xi)} < 1\,, \quad \textit{in particular} \;\; \eta\sigma_f = \eta\sqrt{f(1-f)} < \sigma_\alpha\,. \tag{9.22}$$

Assuming $V = \mathrm{e}^{\varepsilon^2 n/2}$, $\alpha = \alpha(f)$ *and writing* $\sigma_\alpha \tau = f - \alpha$, *one has that*

$$\eta\,\tau \leqslant \varepsilon \leqslant \tau\,. \tag{9.23}$$

Proof. — We see that $\alpha(1-\alpha) < \xi(1-\xi)$ because $0 < \alpha < \xi \leqslant 1/2$. Next, we get

$$\frac{\alpha(1-\alpha)}{\xi(1-\xi)} > \frac{\alpha}{\xi} \geqslant \frac{f-\varepsilon}{f} \geqslant 1 - \frac{\varepsilon}{f_*} = \eta^2\,.$$

By Taylor–Lagrange at α for the function ϕ_α defined in (9.8), we have

$$\phi_\alpha(f) = \phi''_\alpha(\xi_0)\frac{(f-\alpha)^2}{2} = \frac{\sigma_\alpha^2}{\xi_0(1-\xi_0)}\,\frac{\tau^2}{2} = \frac{\alpha(1-\alpha)}{\xi_0(1-\xi_0)}\,\frac{\tau^2}{2}$$

for some $\xi_0 \in (\alpha, f)$, and $\phi_\alpha(f) = \phi_{\alpha(f)}(f) = (\ln V)/n = \varepsilon^2/2$ by assumption. The inequalities in (9.23) follow then from (9.21) and (9.22). □

We have to understand how the values $\alpha(f)$ are distributed when f varies in $[f_*, f^*]$. To this end, we estimate the derivative $\alpha'(f)$.

Lemma 9.8. — *Let* $0 < \varepsilon \leqslant f_*$ *and* $V = \mathrm{e}^{\varepsilon^2 n/2}$. *The mapping* $(0,1) \ni f \mapsto \alpha(f)$ *implicitly defined at* (9.14) *is increasing, and when* $f \in [f_*, f^*]$ *we have that*

$$\eta^2 < \alpha'(f) < 1\,.$$

Proof. — We express the derivative $\alpha'(f)$ by differentiating with respect to f the equality $\phi_{\alpha(f)}(f) = (\ln V)/n$. Writing ϕ_α for $\phi_{\alpha(f)}$, we obtain

$$\phi'_\alpha(f) + \Big(\frac{\partial}{\partial\alpha}\,\phi_\alpha(f)\Big)\alpha'(f) = \phi'_\alpha(f) - \frac{f-\alpha}{\alpha(1-\alpha)}\,\alpha'(f) = 0\,.$$

By Taylor–Lagrange at α for $s \mapsto \phi'_\alpha(s)$, there is $\xi \in (\alpha, f)$ such that

$$\phi''_\alpha(\xi)(f-\alpha) = \phi'_\alpha(f) = \frac{f-\alpha}{\alpha(1-\alpha)}\,\alpha'(f), \quad \text{hence} \quad \alpha'(f) = \frac{\alpha(1-\alpha)}{\xi(1-\xi)} > 0$$

because $\phi''_\alpha(\xi) = \sigma_\xi^{-2}$. We have that $\alpha < \xi < f \leqslant \alpha + \varepsilon$ by (9.21), and when we further assume $f_* \leqslant f \leqslant f^* \leqslant 1/2$ the conclusion follows by (9.22). □

We need to study the intersections $F_{m_1} \cap F_{m_2}$, when $m_1, m_2 \in [f_* n, f^* n]$.

Lemma 9.9. — *Suppose that $f_* n \leqslant m_1 < m_2 \leqslant f^* n$. One has that*

$$\mathrm{e}^{-1/(6n)}\, P\big(F_{m_1} \cap F_{m_2}\big)/P(F_{m_1}) < \lambda\, \mathrm{e}^{-\delta^2\varepsilon^2(m_2-m_1)/2} \,/ \sqrt{2\pi(m_2-m_1)}\,,$$

with $\delta = \eta^3 s_*/(1-f_*)$ and $\lambda = \sqrt{1-f_*}/\sqrt{1-f^*}$.

Proof. — Let $f_j = m_j/n$, $f_* \leqslant f_j \leqslant f^*$, and $\alpha_j = \alpha(f_j)$, for $j = 1, 2$. By Lemma 9.8, we have that $\alpha_1 < \alpha_2$ since $f_1 < f_2$. Let J be an arbitrary subset of $\{1, \ldots, n\}$ satisfying $|J| = m_1$, and let $A(J)$ be the subset of $\Omega = S_N^n$ defined by

$$A = A(J) = \big\{x = (x_1, \ldots, x_n) \in \Omega : J = \{i : x_i \in C_{\alpha_1}\}\big\}\,.$$

One has thus $N_{n,\alpha_1}(x) = m_1$ when $x \in A$. The conditional probability p_A that $N_{n,\alpha_2}(x) = m_2$ knowing that $x \in A$ is equal to the probability that $m_A := m_2 - m_1$ of the remaining $n_A := n - m_1 = (1-f_1)n \geqslant n/2$ coordinates of x (those coordinates that are in $\Omega_1 \setminus C_{\alpha_1}$) fall in $C_{\alpha_2} \setminus C_{\alpha_1}$. This is given by the binomial distribution corresponding to n_A and to $\alpha_A = (\alpha_2 - \alpha_1)/(1-\alpha_1)$, and we know therefore that

$$\begin{aligned} p_A := \frac{P\big(\{N_{n,\alpha_2} = m_2\} \cap A\big)}{P(A)} &= P\big(\{N_{n_A,\alpha_A} = m_A\}\big) \\ &= \alpha_A^{m_A} (1-\alpha_A)^{n_A - m_A} \binom{n_A}{m_A}. \end{aligned}$$

Let $f_A = m_A/n_A = (f_2 - f_1)/(1-f_1)$. Since $\alpha'(f) < 1$ on $[f_*, f^*]$, we get

$$\begin{aligned} f_A &= \frac{f_2 - f_1}{1-f_1} = \frac{f_2 - f_1}{1-\alpha_1}\left(1 + \frac{f_1 - \alpha_1}{1-f_1}\right) \\ &> \frac{\alpha_2 - \alpha_1}{1-\alpha_1} + \frac{(f_1-\alpha_1)(f_2-f_1)}{(1-\alpha_1)(1-f_1)} = \alpha_A + \frac{f_1 - \alpha_1}{(1-\alpha_1)(1-f_1)}\,(f_2 - f_1)\,. \end{aligned}$$

Let $f_1 - \alpha_1 = \sigma_{\alpha_1}\tau_1$. We have $\tau_1 \geqslant \varepsilon$ by (9.23), $\sigma_{f_1} > \sigma_{\alpha_1} > \eta\sigma_{f_1} \geqslant \eta s_*$ by (9.21) and (9.22), and $f_* \leqslant f_1 \leqslant 1/2$. By the leftmost inequality in (9.22), we obtain

$$\frac{1}{1-\alpha_1} > \frac{\alpha_1}{f_1(1-f_1)} \geqslant \frac{\eta^2}{1-f_1} \geqslant \frac{\eta^2}{1-f_*}\,,$$

therefore

$$f_A - \alpha_A > \frac{\eta^3 s_* \varepsilon}{(1-f_*)(1-f_1)}\,(f_2 - f_1) = \frac{\delta\varepsilon}{1-f_1}\,(f_2 - f_1)\,. \tag{9.24}$$

Recalling the function ψ_f from (9.15), we see that

$$p_A = \psi_{f_A}(\alpha_A)^{-n_A} f_A^{m_A} (1-f_A)^{n_A-m_A} \binom{n_A}{m_A}.$$

Applying Stirling as before in (9.18), and because we have that $n_A/(n_A - m_A) = (1-f_1)/(1-f_2) \leqslant (1-f_*)/(1-f^*)$, we obtain

$$\begin{aligned} e^{-1/(12n_A)}\, p_A &< \psi_{f_A}(\alpha_A)^{-n_A} \sqrt{\frac{n_A}{2\pi m_A(n_A - m_A)}} \\ &\leqslant \psi_{f_A}(\alpha_A)^{-n_A} \sqrt{\frac{1-f_*}{2\pi(1-f^*)m_A}}\,. \end{aligned}$$

For some $\xi \in (\alpha_A, f_A)$, and since $(\ln\psi_{f_A})''(\xi) > f_A/\xi^2 > 1/f_A$ by (9.16), we get

$$\ln\psi_{f_A}(\alpha_A) = (\ln\psi_{f_A})''(\xi)\, \frac{(f_A-\alpha_A)^2}{2} > \frac{(f_A-\alpha_A)^2}{2f_A}.$$

Consequently, we can write

$$p_A < e^{1/(12n_A)} \exp\Bigl(-n_A\frac{(f_A-\alpha_A)^2}{2f_A}\Bigr)\frac{\lambda}{\sqrt{2\pi m_A}}\,, \quad \text{with} \quad \lambda = \frac{\sqrt{1-f_*}}{\sqrt{1-f^*}}\,.$$

We see that $n_A/f_A = n_A^2/(m_2-m_1)$. By (9.24) we have

$$\frac{n_A}{f_A}(f_A-\alpha_A)^2 \geqslant \frac{n^2(1-f_1)^2}{m_2-m_1}\,\frac{\delta^2\varepsilon^2(f_2-f_1)^2}{(1-f_1)^2} = \delta^2\varepsilon^2(m_2-m_1).$$

Using also $n < 2n_A$ and the definition of p_A, we obtain for $A = A(J)$ that

$$\begin{aligned} &P\bigl(A(J)\cap\{N_{n,\alpha_2} = m_2\}\bigr) \\ &\qquad < \Bigl(e^{1/(6n)}\,\lambda\, e^{-\delta^2\varepsilon^2(m_2-m_1)/2}\big/\sqrt{2\pi(m_2-m_1)}\Bigr)\, P(A(J))\,. \end{aligned}$$

Summing on all subsets J of $\{1,\dots,n\}$ with $|J| = m_1$, and because $\bigcup_{|J|=m_1} A(J)$ is equal to $\{N_{n,\alpha_1} = m_1\} = F_{m_1}$, we get

$$P\bigl(F_{m_1}\cap F_{m_2}\bigr) < \Bigl(e^{1/(6n)}\,\lambda\, e^{-\delta^2\varepsilon^2(m_2-m_1)/2}\big/\sqrt{2\pi(m_2-m_1)}\Bigr)\, P(F_{m_1}). \ \square$$

End of proof of Theorem 9.6. — Let H be a sufficiently large integer, and let us now define $M = \{jH : j\in\mathbb{N}\}\cap[f_*n, f^*n]$ to be the set of multiples of H located in the segment $[f_*n, f^*n]$. We fix $m_1 \in M$ and let $m_2 > m_1$ be any other element of M. Then $m_2 - m_1 = kH$ with k integer $\geqslant 1$. Summing on $k \geqslant 1$ we see that

$$\sum_{k=1}^{+\infty}\frac{e^{-\delta^2\varepsilon^2 kH/2}}{\sqrt{kH}} < \int_0^{+\infty} e^{-\delta^2\varepsilon^2 Hs/2}\,\frac{ds}{\sqrt{Hs}} = \frac{\sqrt{2}\,\Gamma(1/2)}{\varepsilon H\delta} = \frac{\sqrt{2\pi}}{\varepsilon H\delta}\,.$$

By Lemma 9.9, we get $\sum_{m_2 \in M,\, m_2 > m_1} P\big(F_{m_1} \cap F_{m_2}\big) < P(F_{m_1})/2$ when εH is larger than $2\lambda\, \mathrm{e}^{1/(6n)}/\delta$. It follows then that at least one half of the set F_{m_1} is not covered by the other sets F_{m_2} for $m_2 > m_1$ and $m_2 \in M$, therefore $P(\bigcup_{m \in M,\, m \geqslant m_1} F_m) \geqslant P(\bigcup_{m \in M,\, m > m_1} F_m) + P(F_{m_1})/2$ for $m_1 \in M$. The probability of $\bigcup_{m \in M} F_m$ is thus at least equal to half of the sum of probabilities. By (9.19) and (9.20) we get

$$\kappa_{Q,n} \geqslant \mathrm{e}^{-c}\, V P\Big(\bigcup_{m \in M} F_m \Big) \geqslant \frac{\mathrm{e}^{-c}}{2} \sum_{m \in M} V P(F_m) \geqslant \frac{\mathrm{e}^{-c}}{2} \, \frac{\mathrm{e}^{-1/(12 s_*^2 n)}}{s^* \sqrt{2\pi}} \, \frac{|M|}{\sqrt{n}} \,. \tag{9.25}$$

So far we could hope for a lower bound of order $\sqrt{n}$ for the weak type constant. But we have to comply with the allowability restriction, and we must estimate the number of couples $(f, \alpha(f))$ that are c-allowable. We let

$$\varepsilon = \frac{s_* n^{-1/4}}{1 + s_* n^{-1/4}/f_*}, \quad \text{so that} \quad \frac{\varepsilon}{\eta^2} = \frac{\varepsilon}{1 - \varepsilon/f_*} = s_* n^{-1/4}$$

and $\varepsilon < f_*$. We choose a spacing $H \sim n^{1/4}$. For every $m \in M$, for $f = m/n$, $\alpha = \alpha(f)$ and $f = \alpha + \sigma_\alpha \tau$ we have by (9.5), (9.22) and (9.23) that

$$\overline{y} = \frac{\tau}{\sigma_\alpha} \leqslant \frac{\varepsilon}{\eta} \, \frac{1}{\eta \sigma_f} \leqslant \frac{\varepsilon}{\eta^2 s_*} = n^{-1/4} \,.$$

For $n > 256$ we see that $\overline{y} < 1/4$ and $\overline{y}^4 < 1/n$, thus $(f, \alpha(f))$ is allowable with constant $c = 1$ according to Lemma 9.1. We choose the spacing integer H such that $H > 2\lambda\, \mathrm{e}^{1/(6n)}/(\delta\varepsilon)$. Since $\varepsilon = \eta^2 s_* n^{-1/4}$, we arrive to the condition

$$H > (2\lambda/\delta\eta^2 s_*)\, \mathrm{e}^{1/(6n)}\, n^{1/4} \,.$$

We obtain a set $M \subset [f_* n, f^* n]$ of multiples of H with cardinality at least equal to $\lfloor (f^* n - f_* n)/H \rfloor > \big[\eta^2 \delta s_* (f^* - f_*)/(2\lambda)\big]\, \mathrm{e}^{-1/(6n)}\, n^{3/4} - 1$ where each element m produces a 1-allowable couple $(f, \alpha(f))$. By (9.25), we get that

$$\kappa_{Q,n} \geqslant \frac{1}{2\,\mathrm{e}} \, \frac{\mathrm{e}^{-1/(12 s_*^2 n)}}{s^* \sqrt{2\pi}} \, \frac{|M|}{\sqrt{n}} \geqslant \frac{\eta^2 \delta s_* (f^* - f_*)}{4\,\mathrm{e} \sqrt{2\pi} \lambda s^*} \, n^{1/4} - O(n^{-1/2}) \,.$$

Our version of the Iakovlev–Strömberg proof is not optimal, we shall however try to figure out a numerical value for the constant that we get in front of $n^{1/4}$. We have for n large that $\varepsilon = o(1)$, thus $\eta \simeq 1$. Let us introduce

$$z := \frac{\delta s_* (f^* - f_*)}{\eta^3 \lambda s^*} = \frac{s_*^2}{1 - f_*} \, \frac{f^* - f_*}{\sqrt{f^*(1 - f_*)}} = \frac{f_*(f^* - f_*)}{\sqrt{f^*(1 - f_*)}} \,.$$

This expression increases with f^*, so we set $f^* = 1/2$, the maximal possibility. Then, the resulting value of z is maximal for $f_* = 3/4 - \sqrt{11/48} \sim 0.271$, yielding $z > 0.102$. When n is large, we have

$$\kappa_{Q,n} > \frac{z}{4\,\mathrm{e}\sqrt{2\pi}}\, n^{1/4} - o(n^{1/4}) > 0.0037\, n^{1/4} > \frac{n^{1/4}}{271}\,.$$

Notice that we have set the constant value V as $V = V_n \sim \mathrm{e}^{\kappa\sqrt{n}}$ in dimension n. The corresponding sequence of values $t_n = \sqrt{2\ln V_n} \sim n^{1/4}$ for the "test sets" $\{X_{n,\alpha} > t_n\}$ is "invisible" to the Gaussian limit argument of Theorem 9.5. □

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In the index and notation list, 1, 1.2, 1.2.3 refer respectively to Section 1, subsection 1.1 or subsubsection 1.2.3. We also localize by equation number, or statement number, as in eq. (1.1), Prop. 1.2, or proximity (before, after) to one of those, as in bf. Lem. 3.3 for example.

Notation

$\mathbf{1}_A$	indicator function of the set A	1, p. 9
A_p, B_p	constants in Khinchin's inequalities	eq. (1.22.**K**), p. 26
$(A_\mathbf{0})\ldots(A_\mathbf{3})$	assumptions for Carbery's Proposition 6.6	bf. Prop. 6.6, p. 96
$A^{(n)}_{\alpha,t}$	set where the maximal function $\mathrm{M}_Q\mu^{(n)}_N$ is large	Lem. 9.3, p. 180
$a \wedge b$, $a \vee b$	minimum, maximum of two real numbers a and b	
$\mathcal{B}_Y$	Borel σ-field of a topological space Y	
$(B_t)_{t \geqslant 0}$	Brownian motion in $\mathbb{R}^n$	1.4.1, p. 22
B_τ	Brownian value $\omega \mapsto B_{\tau(\omega)}(\omega)$ at a stopping time τ	1.4.3, p. 31
$B(q_0, R, n)$	*a priori* bound in Bourgain's cube proof	8.1.1, p. 154
bar μ	barycenter of a probability measure μ on $\mathbb{R}^n$	1.4.1, p. 21
C_p, C'_p, C''_p	constants for Carbery's Proposition 6.6	eq. $(A_\mathbf{0})$–$(A_\mathbf{2})$, p. 96
$C_\alpha(m)$	Carbery's constant for a Fourier multiplier m	Prop. 6.14, p. 111
C_α	a subset of $[-N, N]$ in proof of Aldaz–Aubrun weak type theorem	eq. (9.2), p. 177
c_p	Burkholder–Gundy constant, $1 < p < +\infty$	Th. 1.6, p. 25
$D^z h$, $D^\alpha h$	fractional derivative of h	eq. (6.7), p. 101 & (6.10), p. 102
$D^\alpha_t h(\lambda t)$	fractional derivative of $t \mapsto h(\lambda t)$	eq. (6.7), p. 101 & (6.10), p. 102
$D^\alpha_t h(\lambda t)\Big\vert_{t=t_0}$	fractional derivative evaluated at t_0	af. eq. (6.8), p. 101
$(d_k)^N_{k=0}$	martingale difference sequence	1.4.2, p. 25
$d^z h$, $d^\alpha h$	another fractional derivative of h	eq. (7.9), p. 131
$\mathrm{E}\, f$	expectation of the random variable f	1, p. 9
$\mathrm{E}(f\vert\mathcal{G})$	conditional expectation of f on the σ-field $\mathcal{G}$	1, p. 9
$(\mathbf{e}_j)^n_{j=1}$	standard unit vector basis of $\mathbb{R}^n$	
$\mathcal{F}f$, $\widehat{f}$, $\widehat{\mu}$	Fourier transform of a function f, of a measure μ	2, p. 35
F_m	set of $x \in \mathbb{R}^n$ with $N_{n,\alpha(f)}(x) = m$ in Section 9	af. eq. (9.17), p. 185
$\Vert f\Vert_p$, $\Vert f\Vert_{L^p}$	norm of a function f in L^p	Th. 0.1, p. 3
f^*	uncentered classical maximal function of f	Intro., p. 3
$f_{\#}\mu$	pushforward image of the finite measure μ by the mapping f	p. 9
f^*_r, f^*_ℓ	right, left maximal function of a function f on $\mathbb{R}$	eq. (5.5), p. 79
f_*, f^*	lower, upper bound for $f = m/n$, proof of Iakovlev–Strömberg	af. eq. (9.17), p. 185
$G_s f$	Gaussian semi-group acting on f	eq. (1.19), p. 23
G	Gaussian kernel in Bourgain's cube proof, $\widehat{G}(\xi) = \mathrm{e}^{-4\pi\vert\xi\vert^2}$	af. eq. (**A**), p. 151
g_p	absolute p-th moment of the Gaussian measure γ_1 on $\mathbb{R}$	eq. (1.18), p. 22
$g^\vee$	inverse Fourier transform of a function g on $\mathbb{R}^n$	2, p. 36
$g(f)$, $g_k(f)$	Littlewood–Paley functions for f on $\mathbb{R}^n$, $k \geqslant 1$	2.1, p. 36
$g_{(\lambda)}$, $g_{[\lambda]}$	dilates of a function g on $\mathbb{R}^n$	eq. (2.7), p. 40
Hf, $H_\mathbb{T}f$	Hilbert transform of f on $\mathbb{R}$ or $\mathbb{T}$	2.3, p. 44
H_k	homogeneous parts in Pisier's semi-group theorem	bf. eq. (8.8), p. 155
$\Vert h\Vert_{L^2_\alpha}$	Carbery's multiplier norm of a function h on $(0, +\infty)$	eq. (6.21), p. 109
h_q	L^q bound for H_1 in Pisier's semi-group theorem	Prop. 8.3, p. 155
I	identity operator on $L^q(\mathbb{R}^n)$	8.2, p. 155
I_n	identity matrix of size $n \times n$	
$I^w f$, $I^\beta f$	fractional integration	eq. (6.9), p. 102
$i^w f$, $i^\beta f$	another fractional integration	7.1, p. 131
J_ν	Bessel function of order ν	
$\mathcal{K}(\mathbb{R}^n)$	space of compactly supported continuous functions on $\mathbb{R}^n$	
K_C	uniform probability density on a convex set C	5.1, p. 74
K_{lc}	a symmetric log-concave probability density on $\mathbb{R}^n$	Prop. 5.10, p. 86
K_g	a probability density or kernel on $\mathbb{R}^n$ satisfying (6.1.**H**)	bf. eq. (6.1.**H**), p. 93
K^R	Bourgain's kernels for the cube	8, p. 150
$L^p(\mathbb{R}^n)$	Lebesgue spaces, $1 \leqslant p \leqslant +\infty$	
$L(C)$	isotropy constant of the convex set C	eq. (5.2), p. 76
$\mathrm{M}f$	classical Hardy–Littlewood maximal function of f	eq. (0.1), p. 2
$\mathrm{M}_C f$	maximal function of f associated to C	eq. (0.3.**M**), p. 5

M_N^*	maximal function of a martingale $(M_k)_{k=0}^N$	1.1, p. 10
$\mathcal{M}f$	radial maximal function of f	4.1, p. 63
$\mathrm{M}_K f$, $M_K f$	maximal function of f associated to a kernel K	3.3, p. 62 & 5.1, p. 75
$\mathrm{M}_C^{(d)} f$	dyadic maximal function of f associated to C	6, p. 92
$\|m\|_{p\to p}$	norm on L^p of a multiplier m	2.2, p. 41
m_σ	Fourier transform of the uniform probability measure σ on S^{n-1}	4.2, p. 66
m^*	$m^*(\xi) = \xi \cdot \nabla m(\xi)$, for a multiplier m on $\mathbb{R}^n$	p. 69
m_C, $m_C(\xi)$	Fourier transform of K_C	5.1, p. 74
m_{lc}, m_g	Fourier transform of K_{lc}, of K_g	
m_z^ε	Müller's holomorphic family of multipliers	eq. (7.12), p. 132
$m^\#$	Müller's "crucial" multiplier $m^\#(\xi) = \|\xi\| m(\xi)$	eq. (7.19), p. 142
m^R	Bourgain's cube multiplier, Fourier transform of K^R	8, p. 150
$\mathbb{N}^*$	set of integers $n > 0$	
$N(0, \mathrm{I}_n)$	centered Gaussian distribution with covariance matrix I_n	1.4.1, p. 22
$N_{n,\alpha}(x)$	number of coordinates of $x \in \mathbb{R}^n$ that are in C_α	bf. Lem. 9.3, p. 180
$\mathcal{O}(n)$	orthogonal group	
P_t, $P_t f$	Poisson measure, Poisson semi-group acting on f	1.5, p. 33
$P_t^{(n)}$	Poisson kernel on $\mathbb{R}^n$	eq. (1.32), p. 35
p^*	$p^* := \max(p, p/(p-1))$, in Burkholder's unconditional constant $p^* - 1$	p. 30
Q_μ	covariance quadratic form of a measure μ	1.4.1, p. 21
$Q(C)$	Müller's constant for a convex set C	7, p. 127
Q, Q_n	symmetric cube of volume one in $\mathbb{R}^n$	8, p. 149
q_p	Littlewood–Paley constant, for $1 < p < +\infty$	eq. (2.4), p. 37
$q(C)$	modified Müller's constant for a symmetric convex set C	eq. (7.1), p. 127
$R_j f$	Riesz transforms of f in $\mathbb{R}^n$, $1 \leqslant j \leqslant n$	bf. eq. (2.21), p. 45
$\mathcal{R}f$	vector form of the Riesz transform in $\mathbb{R}^n$	2.3, p. 46
R_0	$R_0^{\delta/2} = 4$, in Bourgain's cube proof	p. 159
$\|S\|$	measure of a set S	
$\|S\|_n$	Lebesgue's n-dimensional measure of a set S in $\mathbb{R}^n$	
S_N	square function of a martingale	1.4.2, p. 25
$\mathcal{S}(\mathbb{R}^n)$	Schwartz function space on $\mathbb{R}^n$	1.5, p. 33
S^{n-1}	unit sphere in $\mathbb{R}^n$	
$S_p(\tau)$	one-side moments of a log-concave probability density on $[\tau, +\infty)$	Lem. 5.3, p. 79
S_N, S_N^n	sets $[-N, N]$, $[-N, N]^n$ in Section 9	af. eq. (9.1), p. 176
s_{n-1}	Lebesgue measure of the unit sphere in $\mathbb{R}^n$	eq. (1.34), p. 35
s^+	positive part of a real number s, $s^+ = \max(s, 0)$	
$\lfloor s \rfloor$, $\lceil s \rceil$	floor, ceiling of s, integers such that $s - 1 \leqslant \lfloor s \rfloor \leqslant s \leqslant \lceil s \rceil < s + 1$	
s_*, s^*	$s_* = \sqrt{f_*(1-f_*)}$, $s^* = \sqrt{f^*(1-f^*)}$, in Iakovlev–Strömberg	af. eq. (9.18), p. 185
$\mathbb{T}$	unit circle in $\mathbb{R}^2$ or $\mathbb{C}$	
$\|T\|_{p\to p}$	norm of an operator $T : L^p \to L^p$	
T_m	linear operator associated to the multiplier m on $\mathbb{R}^n$	bf. eq. (2.9), p. 40
$T_{j,v}$, T_j, T^*	operators for Carbery's maximal theorem	6.1, p. 94
$\mathbf{T}^J$	product $\prod_{j\in J} T_j$ of linear operators $(T_j)_{j\in J}$	8.2, p. 155
tC	dilate by $t > 0$ of the convex set C	
U_K	operator $f \mapsto \nabla K * f$	eq. (7.20), p. 144
$u(x, t)$	harmonic extension of $f(x)$, $x \in \mathbb{R}^n$, to the upper half-space in $\mathbb{R}^{n+1}$	1.5, p. 33
$V(K)$	directional variation of a kernel K	eq. (7.2), p. 128
V	fixed large value V in Iakovlev–Strömberg	af. Th. 9.6, p. 184
w_0	$w_0 = R^{\delta/2}$ in Bourgain's cube proof	af. Lem. 8.4, p. 157
w_1	$w_1 = w_0^2 = R^\delta$ in Bourgain's cube proof	af. eq. (8.32), p. 170
$X_{1,\alpha}$, $X_{n,\alpha}$	Bernoulli, binomial variable in Aldaz–Aubrun	eq. (9.10) & bf. Lem. 9.3, p. 180
$\|x\|$	norm of a vector x, usually Euclidean norm on $\mathbb{R}^n$	
$\|x\|_C$	norm of a vector x relative to a symmetric convex set C	1.4.3, p. 32