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Instantons and framed sheaves on Kähler Deligne–Mumford stacks ^(*)

PHILIPPE EYSSIDIEUX ⁽¹⁾ AND FRANCESCO SALA ⁽²⁾

ABSTRACT. — We provide stacky generalizations of classical gauge-theoretic results inspired by Donaldson, the Uhlenbeck–Yau theorem and variants due to Bando and his collaborators. Moreover, we show an application of this machinery in the study of ALE spaces.

RÉSUMÉ. — Nous donnons une généralisation champêtre de résultats classiques de théorie de jauge, comme la caractérisation par Donaldson des instantons sur \mathbb{R}^4 en termes algébro-géométriques, le théorème de Uhlenbeck–Yau et diverses variantes dues à Bando et ses collaborateurs. Nous appliquons cette machinerie à la classification des instantons sur certains espaces ALE.

1. Introduction

It is a general principle (or a *metatheorem*), which was explained to us by C. Simpson, that the basic results of Differential Geometry (Real or Complex) obtained through the analysis of Geometric Partial Differential Equations extend to *Deligne–Mumford stacks*⁽¹⁾. Doing the proof of such a result for stacks amounts to reading the proof in the manifold case while checking that the extension to stacks of the constructions used in the proof is possible and sufficiently well documented in the literature. In this article, we enforce this principle and state stacky generalizations of classical gauge-theoretic results inspired by Donaldson, the Uhlenbeck–Yau theorem [45] and variants due to Bando and his collaborators.

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(1) Complex differential geometry on *Artin stacks* is much more delicate but may have more applications — however we will not pursue this direction here.

Let us first briefly discuss the history of the result we want to generalize. In [16] Donaldson proved that there is a one-to-one correspondence between $U(r)$ -instantons of charge n on \mathbb{R}^4 (modulo gauge equivalence) and *framed vector bundles* of rank r and second Chern class n on the complex projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup l_\infty$. A framed vector bundle is a pair (E, ϕ_E) , where E is a vector bundle on \mathbb{P}^2 of rank r and $c_2(E) = n$ and $\phi_E: E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus r}$ a trivialization along the line l_∞ . Donaldson's correspondence was generalized to instantons on the blowup $\widetilde{\mathbb{C}^2}$ of \mathbb{C}^2 at n points (for $n = 1$ by King [25] and for $n \geq 2$ by Buchdahl [9]). Then in [2], Bando generalized Donaldson's characterization of instantons on \mathbb{R}^4 to general Kähler manifolds. In all these examples, in order to state a correspondence between instantons and framed vector bundles on a surface one needs first to endow the noncompact surface with a Kähler structure, compactify it by adding a smooth projective curve, and consider framed vector bundles on the corresponding smooth projective surface. Our main result is a stacky generalizations of Bando's instanton theorem:

THEOREM. — *Let (\mathcal{X}, ω_0) be a n -dimensional ($n \geq 2$) smooth connected compact Kähler Deligne–Mumford stack and let $\mathcal{D} \subset \mathcal{X}$ be a non-empty smooth divisor such that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is positive on \mathcal{D} . Let ω_c be a Kähler metric with cone-like singularities on $X^\circ := \mathcal{X} \setminus \mathcal{D}$. Then a holomorphic vector bundle \mathcal{E} on \mathcal{X} is such that $\mathcal{E}|_{\mathcal{D}}$ can be endowed with a flat unitary connection ∇ iff there exists an Hermite–Einstein asymptotically flat vector bundle (\mathcal{E}', h') on (X°, ω_c) having trivial holonomy at infinity which induces ∇ and \mathcal{E} is an extension of \mathcal{E}' .*

Our policy with respect to writing-up stacky generalizations of gauge-theoretic results on manifolds will be to concentrate on purely stack-theoretic issues, which amounts to setting up the definitions correctly, and briefly indicate how the deeper gauge-theoretic or analytical aspects of the proof go through. One may perhaps regret that no comprehensive reference on global analytic methods on stacks in the style of [30] is available. This is a symptom of a cultural gap between differential geometers and specialists of non-linear analysis on manifolds, who are reticent towards stacks and seem to be content with Thurston's theory of orbifolds in spite of its lack of flexibility, and algebraic geometers who have embraced stacky methods but are seldom interested in non-linear analysis. To help bridge this cultural gap, we believe it is timely and useful to point out which statements do generalize and which don't and what kind of applications to differential geometry on classical non-stacky objects can be obtained by stacky methods. We also believe it is useful to set up the basic definitions for Riemannian Geometry and Gauge theory for stacks and state the basic theorems since this is absent from the literature to the best of our knowledge.

The initial motivation of this work was a question by the second author whether one could generalize the Donaldson instanton/framed vector bundles correspondance to obtain a purely algebro-geometric description of instantons on ALE spaces. The proposed description of these classical non-stacky objects was inherently stack-theoretical and the full justification could only follow from an extension of gauge theory to stacks. We will limit our discussion to ALE spaces of type A_{k-1} , where $k \geq 2$ is an integer. Bando’s theorem [2] could only apply if one could compactify an ALE space X° of type A_{k-1} by adding a smooth divisor D which has positive normal line bundle so that one obtains a compact Kähler manifold $X := X^\circ \cup D$. If such compactification X exists, the smooth curve D would be connected (since X° has only one end), would have genus zero (because the fundamental group of this end is finite) hence the holonomy at infinity would be the trivial representation of the fundamental group of the end. But this fundamental group is non trivial. The way we propose to circumvent this obstacle is to consider orbifold compactifications of X° . Let us fix $X^\circ = X_k$, where X_k is the minimal resolution of the A_{k-1} toric singularity of $\mathbb{C}^2/\mathbb{Z}_k$. In [7] a compactification \mathcal{X}_k of X_k is constructed, which turns out to be a projective toric orbifold. Moreover $\mathcal{X}_k \setminus X_k$ is a smooth effective Cartier divisor \mathcal{D}_∞ with positive normal bundle.

Our main theorem applies to yield:

THEOREM. — *There is a one-to-one correspondence between holomorphic vector bundles on \mathcal{X}_k , which are isomorphic along \mathcal{D}_∞ to a fixed vector bundle \mathcal{F}_∞ endowed with a flat unitary connection ∇ , and vector bundles on X_k endowed with an Hermite–Einstein metric such that the curvature is square integrable and the holonomy at infinity is given by the holonomy of ∇ .*

Here, X_k is endowed with one of Kronheimer’s hyperkähler metrics [28]. Instanton bundles on gravitational instantons have been studied in depth by [29, 35] who gave an ADHM type classification theorem and described the hyperkähler structure of their moduli space. On the other hand, the analogue of Donaldson’s theorem which is the main objective of this article was not considered there.

Let us describe the organisation of this article. In Sections 2 and 3, we set up the definitions of the basic ingredients of the differential geometry of Deligne–Mumford stacks with an emphasis on the Kähler case. In Sections 4 and 5, we extend to Deligne–Mumford stacks the Uhlenbeck–Yau and Bando theorems. In Section 6, we recall the construction of the orbifold compactification of X_k and conclude with the proof of the above theorem (Theorem 6.11).

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2. Topological, differentiable and smooth analytic stacks

In this section we briefly describe topological, differentiable and smooth analytic stacks. Our main references are [3, 6, 40]. We assume that the reader is familiar with the notions of *category fibered in groupoids* and of *stack* (cf. [30, §2 and 3]).

2.1. Topological stacks

Let \mathbf{Top} be the category of topological spaces. We fix a final object $*$, the point, in the category \mathbf{Top} of topological spaces. We endow \mathbf{Top} with the usual Grothendieck topology (covers are simply topological open covers); so we can talk about (the 2-category of) stacks over \mathbf{Top} : it is closed under fibre products and by Yoneda's lemma the category of topological spaces embeds as a full subcategory of this 2-category of stacks.

We say a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of stacks over \mathbf{Top} is *representable*, if for any map $U \rightarrow \mathcal{X}$ from a topological space U to \mathcal{X} , the fiber product $V := U \times_{\mathcal{X}} \mathcal{Y}$ is equivalent to a topological space.

Several properties \mathbf{P} of maps of topological spaces are stable under base change (cf. [40, §4.1]). For example: *to be open maps, epimorphisms, surjective maps, embeddings, closed embeddings, open embeddings, local homeomorphisms, covering maps, maps with finite fibers, maps with discrete fibers*. We say a representable morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of stacks over \mathbf{Top} satisfies a property \mathbf{P} if for any map $U \rightarrow \mathcal{X}$ from a topological space U to \mathcal{X} , the base extension $V \rightarrow U$ of f satisfies \mathbf{P} .

DEFINITION 2.1. — *A pre-Deligne–Mumford topological stack is a stack \mathcal{X} for which there exists an epimorphism $u: U \rightarrow \mathcal{X}$ from a topological space U , such that u is representable by local homeomorphisms. We call the pair*

(U, u) an atlas of \mathcal{X} . A pre-Deligne–Mumford topological stack \mathcal{X} is called a Deligne–Mumford topological stack if U is Hausdorff for some atlas (U, u) and the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by proper maps with closed discrete fibers.

One can define a site $\mathcal{S}(\mathcal{X})$ of a Deligne–Mumford topological stack \mathcal{X} in the following way. The objects of the underlying category of $\mathcal{S}(\mathcal{X})$ are the atlases (U, u) of \mathcal{X} , the arrows are the morphisms $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ of two atlases where $\varphi: U \rightarrow V$ is a local homeomorphism of topological spaces and $\alpha: u \xrightarrow{\sim} v \circ \varphi$ is a 2-isomorphism. The topology on this category is the one induced by the pre-topology, where the *covering families* are of the following form: for an atlas (U, u) , we denote by $\text{Cov}(U, u)$ the set of families of morphisms $\{(\varphi_i, \alpha_i): (U_i, u_i) \rightarrow (U, u)\}_{i \in I}$ such that the map

$$\bigsqcup_{i \in I} \varphi_i: \bigsqcup_{i \in I} U_i \rightarrow U \tag{2.1}$$

is open and surjective.

Remark 2.2. — The previous definition of pre-Deligne–Mumford topological stacks comes from [6, §3.1]. In [40, §7], Noohi gives a more general definition of *pre-topological stacks*. In [40], our pre-Deligne–Mumford topological stacks are called *weak Deligne–Mumford topological stacks* (cf. [40, Def. 14.3]). On the other hand our definition of Deligne–Mumford topological stacks is more restrictive than the one given in [6]: indeed, in [6, Def. 3.1] the authors do not require existence of a Hausdorff atlas and require only that the diagonal is representable by closed maps with discrete finite fibers. As the authors point out in [6, §3.1] closedness is a property invariant under base extension only via local homeomorphism. On the other hand, we impose properness instead of closeness because we need a property invariant under *any* base extension.

For a stack \mathcal{X} , we denote by $\pi_0 \mathcal{X}$ the sheaf associated to the presheaf of sets on **Top** defined by $W \mapsto \{\text{isomorphism classes in } \mathcal{X}(W)\}$. To any Deligne–Mumford topological stack \mathcal{X} one can associate a topological space X , called the *coarse moduli space* of \mathcal{X} : as a set X is equal to $\pi_0 \mathcal{X}(\ast)$. For any open substack $\mathcal{X}' \subseteq \mathcal{X}$ (i.e. any representable open embedding $\mathcal{X}' \rightarrow \mathcal{X}$), we have a natural inclusion of coarse moduli spaces $X' \subseteq X$. These are defined to be the open sets of X .

DEFINITION 2.3. — *Let \mathcal{X} be a Deligne–Mumford topological stack. We say that \mathcal{X} is connected if it has no proper open–closed substack. We say that \mathcal{X} is compact if its coarse moduli space X is compact.*

The natural constructions of homotopy theory of topological spaces can be extended to Deligne–Mumford topological stacks. For instance, one can

define a reasonable notion of homotopy between maps, and this allows us to define the n -th homotopy group of a pointed topological stack (\mathcal{X}, x) as pointed homotopy classes of maps $[(S^n, *), (\mathcal{X}, x)]$ (cf. [40, §17]).

As explained in [3], there is a well defined singular (co)homology theory with integral coefficients for Deligne–Mumford topological stacks. The (co)homology theory with integral coefficients depends on the stacky structure, unlike the (co)homology theory with rational coefficients. Indeed, one has the following result.

PROPOSITION 2.4 ([3, Prop. 36]). — *Let \mathcal{X} be a Deligne–Mumford topological stack and X its coarse moduli space. Then the coarse moduli space morphism $\pi: \mathcal{X} \rightarrow X$ induces isomorphisms*

$$H^k(\mathcal{X}; \mathbb{Q}) \simeq H^k(X; \mathbb{Q}). \quad (2.2)$$

2.2. Analytic and differentiable stacks

Let **Comp** be the category of complex manifolds⁽²⁾, endowed with the usual Grothendieck topology (where covers are simply topological open covers). As in the case of topological stacks, we can construct the 2-category of stacks over **Comp**, and by Yoneda’s lemma this category contains the category of complex manifolds as a full subcategory.

We say a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of stacks over **Comp** is *representable by local biholomorphisms*, if for any map $U \rightarrow \mathcal{X}$ from a complex manifold U to \mathcal{X} , the fiber product $V := U \times_{\mathcal{X}} \mathcal{Y}$ is equivalent to a complex manifold, and the map $V \rightarrow U$ is a local biholomorphism.

DEFINITION 2.5. — *A stack \mathcal{X} over **Comp** is called a smooth pre-Deligne–Mumford analytic stack if there exists an epimorphism $u: U \rightarrow \mathcal{X}$ from a complex manifold U such that u is representable by local biholomorphisms. We call the pair (U, u) an atlas of \mathcal{X} .*

Remark 2.6. — Since a holomorphic map of complex manifolds is a local biholomorphism if and only if it is a local homeomorphism, the previous definition coincides with the one given in [6, §3.2].

We say a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of smooth pre-Deligne–Mumford analytic stacks is *representable*, if for any map $U \rightarrow \mathcal{X}$ from a complex manifold U to \mathcal{X} that is representable by local biholomorphisms, the fiber product $V := U \times_{\mathcal{X}} \mathcal{Y}$ is equivalent to a complex manifold. Let **P** be a property of morphisms of complex manifolds that is invariant under base change with

⁽²⁾ We are assuming that manifolds have a countable basis for their topology.

respect to local biholomorphisms. For example we can take \mathbf{P} =closedness, \mathbf{P} =to have finite fibers, \mathbf{P} =to have discrete fibers, \mathbf{P} =properness, \mathbf{P} =to be unramified, or \mathbf{P} =to be a covering space. Then, we say a representable morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$ of smooth pre-Deligne–Mumford analytic stacks is \mathbf{P} , if for any map $U \rightarrow \mathcal{X}$ from a complex manifold U to \mathcal{X} that is representable by local biholomorphisms, the base extension $V \rightarrow U$ is \mathbf{P} .

DEFINITION 2.7. — *A smooth pre-Deligne–Mumford analytic stack \mathcal{X} is called a smooth Deligne–Mumford analytic stack if, for some atlas (U, u) of \mathcal{X} , U is Hausdorff and the diagonal $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by proper unramified⁽³⁾ finite maps. An analytic orbifold is a smooth Deligne–Mumford analytic stack with generically trivial stabilizer.*

Also in this case, one can define a site $\mathcal{S}(\mathcal{X})$ in a way similar than before.

Remark 2.8. — Our definition of smooth Deligne–Mumford analytic stacks is more restrictive than the one given in [6]: in [6, Def. 3.3] the authors assume that the diagonal is representable only by closed maps with finite fibers. On the other hand, one can give a definition of Deligne–Mumford analytic stacks over the category **Analytic** of all analytic spaces (cf. [40]). As pointed out in [6, Rem. 3.4], the 2-category of smooth Deligne–Mumford analytic stacks defined above is equivalent to the sub 2-category of the 2-category of Deligne–Mumford analytic stacks of [40] consisting of smooth Deligne–Mumford analytic stacks. Since the Deligne–Mumford analytic stacks we are interested in are smooth, we preferred not to work over **Analytic** since this category may be less familiar to the reader.

By substituting in the previous definitions the term “local biholomorphism” with “local diffeomorphism”, a definition of smooth (pre-)Deligne–Mumford differentiable stacks over the category **Diff** of differentiable manifolds can be given and was actually introduced in [3]. Similarly, one can give the notion of *differentiable orbifolds*.

One can define the coarse moduli space X of a smooth Deligne–Mumford analytic (resp. differentiable) stack \mathcal{X} . It is an analytic space, but it may not in general be a smooth complex manifold.

PROPOSITION 2.9 ([3, 40]). — *Let \mathcal{X} be a Deligne–Mumford topological (resp. smooth differentiable, resp. smooth analytic) stack. Then there is a covering $\{\mathcal{U}_i\}$ of \mathcal{X} by open substacks such that each \mathcal{U}_i is a quotient stack $[Z/G]$, where Z is a Hausdorff topological space (resp. a Hausdorff differentiable manifold, resp. Hausdorff complex manifold), and G a finite group acting continuously (resp. differentiably, resp. analytically) on Z .*

(3) The attribute “unramified” means “injective differential” from the differential geometric point of view.

As explained in [3, Cor. 25], there is a well-defined de Rham cohomology theory for any smooth Deligne–Mumford differentiable stack \mathcal{X} .

DEFINITION 2.10. — *Let \mathcal{X} be a smooth Deligne–Mumford differentiable stack. We say that \mathcal{X} is oriented if for any atlas (U, u) of \mathcal{X} the associated groupoid*

$$U \times_{\mathcal{X}} U \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} U. \tag{2.3}$$

is oriented, i.e., the manifolds U and $U \times_{\mathcal{X}} U$ and the maps s and t are oriented in a compatible way. In addition, we say that \mathcal{X} is of finite type if U and $U \times_{\mathcal{X}} U$ have finite good covers compatible with s and t .

Let \mathcal{X} be a finite type smooth Deligne–Mumford differentiable stack. Then there is a well-defined theory of cohomology with compact supports for \mathcal{X} . In addition, there exists an *integration map*

$$\int_{\mathcal{X}} : H_c^{\dim(\mathcal{X})}(\mathcal{X}) \rightarrow \mathbb{R}, \tag{2.4}$$

such that the induced pairing $H_{dR}^k(\mathcal{X}) \otimes H_c^{\dim(\mathcal{X})-k}(\mathcal{X}) \rightarrow \mathbb{R}$ is perfect. If \mathcal{X} is compact, $H_{dR}^k(\mathcal{X}) \simeq H_c^k(\mathcal{X})$ for any k and there is a well defined map (2.4) which induces a perfect pairing.

2.3. Comparing algebraic, analytic, differentiable and topological stacks

In [32] a nice interpretation of differentiable stacks as Lie groupoids is thoroughly discussed, which helped us to understand how the different types of stacks are related.

Let \mathcal{X} be a smooth separated Deligne–Mumford algebraic stack of finite type over \mathbb{C} (cf. [30, §4]). Then by [24], there exists a *coarse moduli space* $\pi: \mathcal{X} \rightarrow X$. In general X is a separated *algebraic space* of finite type over \mathbb{C} .

Let **AlgDM** be the 2-category of smooth separated Deligne–Mumford algebraic stacks of finite type over \mathbb{C} . As explained in [6, §3.3] (see also [32]), it is equivalent to the weak 2-category of groupoids up to Morita equivalence

$$X_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} X_0 \tag{2.5}$$

in the category of separated schemes of finite type over \mathbb{C} , where X_0 is smooth, s, t are étale morphisms and

$$(s, t): X_1 \rightarrow X_0 \times X_0 \tag{2.6}$$

is a proper unramified quasi-compact morphism (cf. [30]). Denote by **AnDM** (resp. **DiffDM**, resp. **TopDM**) the 2-category of smooth Deligne–Mumford analytic stacks (resp. differentiable stacks, resp. topological stacks). The argument in [6, §3.3] (see also [41] for the differential case) proves that the 2-category of Deligne–Mumford topological stacks (resp. smooth Deligne–Mumford analytic stacks, resp. smooth Deligne–Mumford differentiable stacks) is equivalent to the weak 2-category of groupoids up to Morita equivalence

$$X_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} X_0 \tag{2.7}$$

in the category of topological spaces (resp. complex manifolds, resp. differentiable manifolds), where the maps s, t are local homeomorphisms (resp. local biholomorphisms, resp. local diffeomorphisms) and

$$(s, t): X_1 \rightarrow X_0 \times X_0 \tag{2.8}$$

is a proper map with closed discrete fibers (resp. proper unramified finite map in the analytic and differential settings). Then one can define natural functors

$$\mathbf{AlgDM} \xrightarrow{-\text{an}} \mathbf{AnDM} \xrightarrow{-\text{diff}} \mathbf{DiffDM} \xrightarrow{-\text{top}} \mathbf{TopDM}.$$

Moreover, these functors respect the coarse moduli space construction. We shall also denote by $\xrightarrow{-\text{top}}$, resp. $\xrightarrow{-\text{diff}}$ any composition of these functors ending with $\xrightarrow{-\text{top}}$, resp. $\xrightarrow{-\text{diff}}$.

Remark 2.11. — In the following, if \mathcal{X} is a smooth Deligne–Mumford algebraic (analytic, differentiable) stack, its fundamental group is by definition the fundamental group of its underlying Deligne–Mumford topological stack.

3. Differential geometry on smooth Deligne–Mumford analytic and differentiable stacks

In this section we shall sketch the generalization of some basic results of differential geometry.

3.1. Vector bundles on smooth Deligne–Mumford differentiable stacks

Let \mathcal{X} be a smooth Deligne–Mumford differentiable stack.

DEFINITION 3.1. — *A C^∞ complex (resp. real) vector bundle of rank r on \mathcal{X} is the following set of data:*

- *for any atlas (U, u) a C^∞ complex (resp. real) vector bundle $E_{U,u}$ of a rank r on U ,*
- *for any morphism $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ of two atlases, where $\varphi: U \rightarrow V$ is a local diffeomorphism of differentiable manifolds and $\alpha: u \xrightarrow{\sim} v \circ \varphi$ is a 2-isomorphism, an isomorphism of C^∞ complex (resp. real) vector bundles of rank r*

$$\vartheta_{\varphi, \alpha}: \varphi^*(E_{U,u}) \xrightarrow{\sim} E_{V,v}$$

such that for any composition $(U, u) \xrightarrow{(\varphi, \alpha)} (V, v) \xrightarrow{(\psi, \beta)} (W, w)$, we have

$$\varphi^* \vartheta_{\psi, \beta} \circ \vartheta_{\varphi, \alpha} = \vartheta_{\psi \circ \varphi, \varphi^* \beta \circ \alpha}.$$

Remark 3.2. — In [5] the authors give another definition of vector bundles on \mathcal{X} . Our definition is equivalent to their definition because of [5, Prop. 3.2].

There are several operations on C^∞ vector bundles on \mathcal{X} . We can indeed form the *trivial* vector bundle $V_{\mathcal{X}} := V \times \mathcal{X}$ given any \mathbb{C} (resp. \mathbb{R})-vector space V , resp. the complex conjugate \bar{E} or the dual E^\vee of a vector bundle E , resp. the direct sum $E \oplus F$, the tensor product $E \otimes F$ or the bundle of morphisms $\text{Hom}(E, F)$ of two vector bundles E and F , resp. the pull back f^*E of a complex vector bundle E on \mathcal{X} by a morphism $f: \mathcal{Y} \rightarrow \mathcal{X}$. Furthermore, one can define a morphism ψ of C^∞ vector bundles on \mathcal{X} from E to F as the data for any atlas (U, u) of a morphism $\psi_{U,u}: E_{U,u} \rightarrow F_{U,u}$ commuting with $\vartheta_{\varphi, \alpha}$. This enables to define $C^\infty(\mathcal{X}, E)$, the *space of smooth sections of E* , as the set of morphisms from $\mathbb{C}_{\mathcal{X}}$ to E .

A *connection* ∇ on E is the data of a connection $\nabla_{U,u}$ on $E_{U,u}$ for any atlas (U, u) which is compatible to the $\vartheta_{\varphi, \alpha}$. In a similar vein, one can define riemannian metrics, hermitian metrics, Levi-Civita connection, principal bundles and connections, curvature tensors, . . . The definitions are left to the reader, see however [17] for orbifolds, and [5]. The theory of Chern–Weil forms of connections and the fact that, on smooth differentiable Deligne–Mumford stacks, they compute the rational Chern classes defined as in [3] is established in [31].

In [5, §3.2], the *tangent bundle* $T\mathcal{X} \rightarrow \mathcal{X}$ of a smooth Deligne–Mumford differential stack \mathcal{X} is introduced as the smooth Deligne–Mumford differentiable stack realized as the quotient stack $[TX_1/TX_0]$ given by a groupoid $X_1 \rightrightarrows X_0$ associated with \mathcal{X} . As pointed out in loc. cit., the construction is functorial and it is independent of the choice of the groupoid presentation of \mathcal{X} . Moreover, $T\mathcal{X}$ is a C^∞ real vector bundle only if \mathcal{X} is a differentiable orbifold, otherwise it is a stack and the map $T\mathcal{X} \rightarrow \mathcal{X}$ may not even be representable. Moreover, in loc. cit. the authors introduce the *normal bundle* $\mathcal{N}_{\mathcal{D}|\mathcal{X}}$ of a smooth Deligne–Mumford differential substack \mathcal{D} in a smooth Deligne–Mumford differential stack \mathcal{X} . When \mathcal{X} is a differentiable orbifold, $\mathcal{N}_{\mathcal{D}|\mathcal{X}}$ can be realized as the quotient vector bundle $T\mathcal{X}_{|\mathcal{D}}/T\mathcal{D}$.

3.2. Holonomy

Let M be a differentiable manifold. Denote by Conn_M the groupoid whose objects are C^∞ complex vector bundles over M with a connection and whose arrows are *gauge equivalences* (vector bundle isomorphisms preserving the connection). Fix a C^∞ complex vector bundle of rank r with a connection (E, ∇) on a smooth connected Deligne–Mumford differentiable stack \mathcal{X} . Then, see [31]⁽⁴⁾, we can define the holonomy functor

$$\begin{aligned} \text{Hom}(M, \mathcal{X}) &\rightarrow \text{Conn}_M, \\ f &\mapsto (f^*E, f^*\nabla). \end{aligned}$$

When $M = I$ is an open interval of the real numbers, $x_0, x_1 \in I$ and $\delta: I \rightarrow \mathcal{X}$ is a *differentiable* path we can define the holonomy $h(\delta, x_0, x_1)$ along δ to be the natural linear transformation $\delta^*E_{x_0} \rightarrow \delta^*E_{x_1}$. If ∇ is flat then $h(\delta, x_0, x_1)$ depends only on the differentiable homotopy class of $f: (I, \{x_0, x_1\}) \rightarrow (\mathcal{X}, \{f(x_0), f(x_1)\})$.

It is easy to define differentiable paths $[0, 1] \rightarrow \mathcal{X}$ as *continuous* paths $[0, 1] \rightarrow \mathcal{X}^{\text{top}}$ extending differentially to a slightly larger interval. Now if $\gamma: [0, 1] \rightarrow \mathcal{X}^{\text{top}}$ is a continuous path we can approximate it in the C^0 -topology by differentiable paths. Since any homotopy of maps may be approximated by a differentiable homotopy in C^1 -topology and homotopy classes are open in C^1 -topology, we conclude as in the classical case that the holonomy of the flat connection ∇ on E gives rise to a representation of the *Poincaré groupoid* of \mathcal{X} hence to a group representation $\rho: \pi_1(\mathcal{X}, x) \rightarrow GL(r, \mathbb{C})$. If the flat connection in question preserves a hermitian metric h , then ρ takes values in the unitary group of (E_x, h_x) .

⁽⁴⁾ Note that our smooth differentiable Deligne–Mumford stacks are equivalent to *proper Lie groupoids* in the sense of [34].

Conversely, let $\tilde{\mathcal{X}}$ be a *universal cover*⁽⁵⁾ of \mathcal{X} and consider an atlas (U, u) of \mathcal{X} . Then the fibred product $(\tilde{U} := U \times_{\mathcal{X}} \tilde{\mathcal{X}}, \tilde{u})$ is an atlas of $\tilde{\mathcal{X}}$ and $\tilde{U} \rightarrow U$ is topological Galois covering space of U such that its deck transformations group is $\pi_1(\mathcal{X}, y)$, where y is point in \mathcal{X} . Let $\rho: \pi_1(\mathcal{X}, y) \rightarrow GL(r, \mathbb{C})$ be a linear representation. Then $\pi_1(\mathcal{X}, y)$ acts on $\tilde{U} \times \mathbb{C}^r$ by ρ on the second factor and induces after passing to quotient a C^∞ complex vector bundle of rank r with a flat connection $(E_{\rho_U}, \nabla_{\rho_U})$. This construction being functorial for morphisms of atlases, one obtains a C^∞ complex vector bundle on \mathcal{X} of rank r with a flat connection (E_ρ, ∇_ρ) having ρ as its holonomy. This construction is an equivalence of categories.

3.3. Metric geometry of Riemannian Deligne–Mumford stacks

DEFINITION 3.3. — *A Riemannian Deligne–Mumford stack is a pair (\mathcal{X}, g) , where \mathcal{X} is a smooth Deligne–Mumford differentiable stack and g a Riemannian metric on it.*

Every smooth Deligne–Mumford differentiable stack admits a Riemannian metric since we are assuming second countability. One can define geodesics on \mathcal{X} or more generally harmonic mappings from manifolds to (\mathcal{X}, g) in the usual way thanks to Proposition 2.9. The infimal length of a path between two points of \mathcal{X}^{top} is the Riemannian distance d between these two points and, since with our definitions the stacks are separated, this gives a distance function on \mathcal{X}^{top} inducing its topological structure (which is Hausdorff). The extension of the basic theory of geodesics is carefully carried out in [22].

For instance, the distance function permits to define the space of Lipschitz functions on \mathcal{X} which can be characterized by Rademacher’s Theorem as almost everywhere differentiable functions with g -bounded differential. The space of continuous functions on \mathcal{X} coincides with the space of continuous functions on \mathcal{X}^{top} . However the space of Lipschitz (or of C^∞) functions on \mathcal{X} does not depend only on the structure of the underlying topological stack \mathcal{X}^{top} , but it is a truly stacky invariant. For instance, if $G \subset GL(n, \mathbb{C})$ is a finite complex reflection group (e.g. $G = \{\pm 1\} \in \mathbb{C}^*$) the distance function attached to a G -invariant hermitian form is just Hölder continuous with respect to a Riemannian metric in the usual sense on $\mathbb{C}^n/G \simeq \mathbb{C}^n$. In the example, $d_{[\mathbb{C}_w/\{\pm 1\}]}(0, z) = |z|^{1/2}$ where $z = w^2$ is the natural coordinate on $\mathbb{C}_w = [\mathbb{C}_w/\{\pm 1\}]^{\text{top}}$.

⁽⁵⁾ The theory of covering stacks is developed in [40, §18].

3.4. Sheaves and vector bundles on smooth Deligne–Mumford analytic stacks

In this section we shall give definitions of (quasi)coherent analytic sheaves and vector bundles on a smooth Deligne–Mumford analytic stack which resembles the definition of (quasi)coherent sheaves given in [46] in the algebro-geometric setting.

Let \mathcal{X} be a smooth Deligne–Mumford analytic stack.

DEFINITION 3.4. — *A quasicoherent sheaf \mathcal{F} on \mathcal{X} is the following set of data:*

- for any atlas (U, u) a quasicoherent analytic sheaf $\mathcal{F}_{U,u}$ on U ,
- for any morphism $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ of two atlases, where $\varphi: U \rightarrow V$ is a local biholomorphism of complex manifolds and $\alpha: u \xrightarrow{\sim} v \circ \varphi$ is a 2-isomorphism, an isomorphism

$$\theta_{\varphi, \alpha}: \varphi^* \mathcal{F}_{U,u} \xrightarrow{\sim} \mathcal{F}_{V,v}$$

such that for any composition $(U, u) \xrightarrow{(\varphi, \alpha)} (V, v) \xrightarrow{(\psi, \beta)} (W, w)$, we have

$$\varphi^* \theta_{\psi, \beta} \circ \theta_{\varphi, \alpha} = \theta_{\psi \circ \varphi, \varphi^* \beta \circ \alpha}.$$

A coherent sheaf is a quasicoherent sheaf \mathcal{F} such that all $\mathcal{F}_{U,u}$ are coherent. A locally free sheaf is a coherent sheaf \mathcal{F} such that all $\mathcal{F}_{U,u}$ are locally free. A morphism $f: \mathcal{F} \rightarrow \mathcal{F}'$ is a collection of morphisms $f_{U,u}: \mathcal{F}_{U,u} \rightarrow \mathcal{F}'_{U,u}$ such that for any morphism $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ of two atlases we have $\varphi^*(f_{V,v}) \circ \theta_{\varphi, \alpha} = \theta'_{\varphi, \alpha} \circ f_{U,u}$.

Example 3.5. — The structure sheaf $\mathcal{O}_{\mathcal{X}}$ of \mathcal{X} is defined by $(\mathcal{O}_{\mathcal{X}})_{U,u} = \mathcal{O}_U$ for any atlas (U, u) , with the obvious isomorphisms. The sheaf of differential $\Omega^1_{\mathcal{X}}$ of \mathcal{X} is defined by $(\Omega^1_{\mathcal{X}})_{U,u} = \Omega^1_U$. Since for any morphism $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ of two atlases, the morphism φ is a local biholomorphism, there is a natural isomorphism $\Omega^1_U \simeq \varphi^*(\Omega^1_V)$.

A holomorphic vector bundle of rank r on \mathcal{X} is a C^∞ -complex vector bundle E over \mathcal{X} such that all $E_{U,u}$ are holomorphic vector bundles and all $\vartheta_{\varphi, \alpha}$ are isomorphisms of holomorphic vector bundles. As before in the differential setting, also in this setting one can perform the usual operations on vector bundles.

Let E be a rank r holomorphic vector bundle on \mathcal{X} . For any atlas (U, u) we denote by $\mathcal{E}_{U,u}$ the sheaf of holomorphic sections of the vector bundle

$E_{U,u}$. For any morphism $(\varphi, \alpha): (U, u) \rightarrow (V, v)$ of two atlases, the isomorphism $\vartheta_{\varphi, \alpha}$ induces an isomorphism

$$\theta_{\varphi, \alpha}: \varphi^* \mathcal{E}_{U,u} \xrightarrow{\sim} \mathcal{E}_{V,v}. \quad (3.1)$$

In this way, we define a coherent sheaf \mathcal{E} on \mathcal{X} , the *sheaf of sections of the vector bundle E* .

Remark 3.6. — The functor $-^{\text{an}}$ sends coherent sheaves to coherent analytic sheaves. The functor $-^{\text{diff}}$ sends holomorphic vector bundles to C^∞ complex vector bundles.

Let \mathcal{X} be a smooth Deligne–Mumford analytic stack. Then it is easy to see that there is a one-to-one correspondence between holomorphic vector bundles of rank r on \mathcal{X} and locally free sheaves of rank r on it. Assume moreover that \mathcal{X} is of the form $[Z/G]$, where Z is a complex manifold and G a complex Lie group acting on it. By the same argument as in [30, Example 12.4.6], the category of coherent sheaves on \mathcal{X} is equivalent to the category of G -equivariant coherent sheaves on Z . Similarly, the category of locally free sheaves on \mathcal{X} is equivalent to the category of G -equivariant locally free sheaves on Z (this follows from the descent result for locally free sheaves with respect to fppf morphisms, [4, Prop. A.11]). Since the category of G -equivariant locally free sheaves on Z is equivalent to the category of G -equivariant holomorphic vector bundles on Z (see, e.g., [11, §5.1]), the category of holomorphic vector bundles of rank r on \mathcal{X} is equivalent to the category of G -equivariant holomorphic vector bundles of rank r on Z .

In the following we shall call *hermitian bundle* a holomorphic vector bundle on \mathcal{X} with a hermitian metric on it. Then if \mathcal{X} is a smooth Deligne–Mumford analytic stack of the form $[Z/G]$, the category of hermitian bundles on \mathcal{X} is equivalent to the category of G -equivariant hermitian vector bundles on Z . Thus if E is a holomorphic vector bundle on a smooth Deligne–Mumford analytic stack, by Proposition 2.9 and a partition of unity argument on \mathcal{X}^{top} , one can always construct a hermitian metric on E .

The usual constructions of complex differential geometry (see [14, Chap. V]) carry over to smooth Deligne–Mumford analytic stacks⁽⁶⁾. A smooth Deligne–Mumford analytic stack carries a complexified tangent bundle $T_{\mathcal{X}}^{\mathbb{C}} = T_{\mathcal{X}}^{1,0} \oplus T_{\mathcal{X}}^{0,1}$ from which we may form the usual vector bundles $\Omega_{\mathcal{X}}^k$ of k -forms and $\Omega_{\mathcal{X}}^{p,q}$ of (p, q) -forms. A $(1, 1)$ form ω on \mathcal{X} is *positive* if so are its representatives $\omega_{U,u}$ for any atlas (U, u) of \mathcal{X} . A hermitian vector bundle (E, h) carries a canonical Chern connection ∇^h with a curvature form

⁽⁶⁾ One can mimic the arguments in [3]: one can define everything and the level of groupoid presentations of the stack and then prove that it is invariant under Morita equivalence.

$i\Theta(E, h) \in C^\infty(\mathcal{X}, \text{Herm}(E, h) \otimes \Omega_{\mathcal{X}}^{1,1})$. The curvature of a hermitian line bundle (L, h) is just a real $(1, 1)$ form on \mathcal{X} .

3.5. Kähler Deligne–Mumford analytic stacks

At a slightly deeper level, the De Rham, Dolbeault [14, Chap. IV, §6] and Hodge [14, Chap. VI, Thm. 3.16 and 3.17] isomorphism theorems are valid for smooth Deligne–Mumford analytic stacks. If we follow the proof of these basic results [14], we see that the only non obvious points are integration of top dimensional compactly supported forms on an oriented smooth Deligne–Mumford differentiable stack (so that we can define the L_2 norm on the space of sections of a metrized vector bundle on an oriented⁽⁷⁾ Riemannian Deligne–Mumford differentiable stack), Stokes’ theorem (so that we can perform integration by parts) and acyclicity of sheaves of (p, q) -forms (cf. Section 2.2 for the first two results, the latter descends from the analogous statement in the case of complex manifold simply by using a groupoid presentation of the stack). See also [44] the Grothendieck–Riemann–Roch theorem for proper morphisms of smooth Deligne Mumford algebraic stacks over the complex numbers.

Say a smooth Deligne–Mumford analytic stack is *Kähler* if it carries a closed positive $(1, 1)$ -form. This is equivalent to require that the coarse moduli space is Kähler. The Lefschetz package, the $\partial\bar{\partial}$ -lemma hold on compact Kähler Deligne–Mumford analytic stacks and so does the Kodaira–Akizuki–Nakano theorem. An application is an analytic version of Olsson–Matsuki’s proof of the Kawamata–Viehweg vanishing theorem in characteristic zero [33]. Yau’s solution of the Calabi conjecture [47] is also valid on stacks. For orbifolds, an early reference is [26], see also [18]. Although we will not use this, we note that the characterization of the Kähler cone extends to compact Kähler Deligne–Mumford analytic stacks.

THEOREM 3.7 ([15]). — *Let \mathcal{X} be a n -dimensional compact Kähler Deligne–Mumford analytic stack and let ω be Kähler form on \mathcal{X} . A class $\{\alpha\} \in H^{1,1}(\mathcal{X}, \mathbb{R})$ is represented by a Kähler form iff for every closed irreducible analytic substack $\mathcal{Z} \subset \mathcal{X}$ one has $\{\alpha\}^{\dim(\mathcal{Z})} \cdot [\mathcal{Z}] > 0$ and $\int_{\mathcal{X}} \{\alpha\}^{n-p} \{\omega\}^p > 0$ for all $1 \leq p \leq n - 1$.*

We will not need the full strength of this statement and will only sketch the proof of the consequence we shall use.

⁽⁷⁾ One needs to use densities for non-orientable stacks.

DEFINITION 3.8. — Let \mathcal{L} be a holomorphic line bundle on a smooth Deligne–Mumford analytic stack \mathcal{X} . Let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed substack of \mathcal{X} . We say \mathcal{L} is positive on \mathcal{Z} if there is an open analytic substack \mathcal{U} such that $\mathcal{Z} \subset \mathcal{U}$ and a hermitian metric h on \mathcal{L} such that $i\Theta(L|_{\mathcal{U}}, h|_{\mathcal{U}})$ is positive.

PROPOSITION 3.9. — Let \mathcal{L} be a holomorphic line bundle on a Deligne–Mumford analytic stack \mathcal{X} . Assume there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\mathcal{L}^{\otimes n} \simeq \pi^*(M)$ where $\pi: \mathcal{X} \rightarrow X$ is the natural map to the coarse moduli space X of \mathcal{X} and M is a holomorphic line bundle on X . Let $Z \subseteq X$ be a compact analytic subspace. Then \mathcal{L} is positive on $\mathcal{Z} := Z \times_X \mathcal{X}$ if and only if M is ample on Z (which is then a projective algebraic variety).

Proof. — The corresponding statement for analytic spaces was proved in Păun’s thesis, see [15, Prop. 3.3] for the required level of generality. By Proposition 2.9, it rests on gluing and regularization techniques for quasi-plurisubharmonic functions [13] that are local in nature and compatible with the action of a finite group. They are also insensitive to singularities. \square

Remark 3.10. — A positive line bundle in the above sense needs not be ample in the sense of [42] because this latter condition requires that the stack has cyclic isotropy groups and the isotropy representations of the line bundle are faithful. See [27] for related issues. One of the results that does not generalize to stacks is the Kodaira embedding theorem since a closed substack of the projective space is necessarily an algebraic variety. GAGA theorems do however generalize [23].

4. Hermite–Einstein metrics on compact Kähler Deligne–Mumford analytic stacks

4.1. Uhlenbeck–Yau theorem

Let \mathcal{X} be a n -dimensional connected compact smooth Kähler Deligne–Mumford analytic stack and let ω be Kähler form on \mathcal{X} .

Let \mathcal{E} be a coherent sheaf on \mathcal{X} . We call *rank* of \mathcal{E} the zero degree part of the Chern character of its K-theory class $[\mathcal{E}]$.

DEFINITION 4.1. — A holomorphic vector bundle \mathcal{E} on \mathcal{X} is ω -stable if every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ satisfies $\mu(\mathcal{F}) < \mu(\mathcal{E})$, where $\mu(\mathcal{F})$ is the ω -slope of \mathcal{F} defined as

$$\mu(\mathcal{F}) := \frac{\int_{\mathcal{X}} c_1(\mathcal{F}) \cdot \{\omega\}^{\dim(\mathcal{X})-1}}{\mathrm{rk}(\mathcal{F})}.$$

The proof of the celebrated Uhlenbeck–Yau theorem carries over to Kähler Deligne–Mumford analytic stacks.

THEOREM 4.2. — *Let \mathcal{X} be a n -dimensional smooth connected compact Kähler Deligne–Mumford analytic stack and let ω be Kähler form on \mathcal{X} . Let \mathcal{E} be a ω -stable holomorphic vector bundle on \mathcal{X} . Then it carries a smooth Hermite–Einstein metric h namely a hermitian metric whose curvature form $\Theta(\mathcal{E}, h)$ as in the end of Section 3.4 satisfies the equation:*

$$\Lambda i\Theta(\mathcal{E}, h) = 2\pi\mu(\mathcal{E})\text{Id}_{\mathcal{E}}$$

Λ being the contraction by the Kähler form ω .

Proof. — Let h_0 be a smooth hermitian metric on \mathcal{E} . As in [45] one first solves the elliptic pde $\Lambda i\Theta_{h_\epsilon} - \Lambda i\bar{\partial}\nabla h = -\epsilon \log h$ where h is a smooth selfadjoint endomorphism of (\mathcal{E}, h_0) for $\epsilon \in]0, 1]$ by a continuity method. Openness comes from the implicit function theorem in function spaces since \mathcal{E} is simple indeed (cf. [45, (2.7) p. S264]) and closedness by a priori estimates following from local calculations, integration by parts and the maximum principle. As explained above, this works on stacks. If the solution h_ϵ does not converge towards a Hermite–Einstein when $\epsilon \rightarrow 0$ then the arguments of [45, §4] rely only on such techniques and yield a weakly holomorphic subsheaf of \mathcal{E} . The main point of loc. cit. is a local statement to the effect that a weakly holomorphic subsheaf is actually a holomorphic subsheaf of \mathcal{E} which is destabilizing. \square

4.2. Comparison with Nironi’s stability condition

Let \mathcal{P} be a n -dimensional cyclic orbifold with a projective scheme as coarse moduli space, endowed with an ample and positive line bundle $\mathcal{O}_{\mathcal{P}}(1)$. We shall call the pair $(\mathcal{P}, \mathcal{O}_{\mathcal{P}}(1))$ a *polarized* projective cyclic orbifold.

Let $n \geq 1$ and a_0, \dots, a_n be coprime positive integers. Define the action of \mathbb{C}^* on \mathbb{C}^{n+1} by $\lambda \cdot (x_0, \dots, x_n) := (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$. Then the corresponding quotient stack $\mathcal{P}(a_0, \dots, a_n) := [\mathbb{C}^{n+1}/\mathbb{C}^*]$ is a n -dimensional weighted projective stack, which is an example of a polarized projective cyclic orbifold, where $\mathcal{O}_{\mathcal{P}(a_0, \dots, a_n)}(1)$ is its *tautological line bundle*.

In this section, we shall compare the differential-geometric definition of the Mumford–Takemoto stability condition given in the previous section with the algebro-geometric one due to Nironi [39]. The latter one depends on the *modified slope* and the *modified Hilbert polynomial*. Let \mathcal{G} be a *generating sheaf* for \mathcal{P} , i.e., a locally free sheaf on \mathcal{P} such that its isotropy representations contain every irreducible representations of the isotropy groups. Then one can define the modified Hilbert polynomial $P_{\mathcal{G}}(\mathcal{E}, m)$ of a coherent sheaf

\mathcal{E} on \mathcal{P} as the Poincaré polynomial of $\pi_*(\mathcal{E} \otimes \mathcal{G}^\vee)$ where π is the morphism from \mathcal{P} to its coarse moduli space; by using the n -th and $(n-1)$ -th coefficient of this polynomial, one can define the modified slope $\mu_{\mathcal{G}}(\mathcal{E})$ of \mathcal{E} (cf. [8, §2.3]). Since for some m , the locally free sheaf $\mathcal{G} := \bigoplus_{i=1}^m \mathcal{O}_{\mathcal{P}}(1)^{\otimes i}$ is a generating sheaf (cf. [27, §5.2]), one can prove that for a vector bundle \mathcal{E} on \mathcal{P} we have (cf. the proof of [39, Prop. 3.18])

$$\mu_{\mathcal{G}}(\mathcal{E}) = m^n \frac{1}{\mathrm{rk}(\mathcal{E})} \int_{\mathcal{P}} c_1(\mathcal{E}) \cdot c_1(\mathcal{O}_{\mathcal{P}}(1))^{n-1}. \quad (4.1)$$

In [42] the authors define a Kahler form ω induced by the curvature of a metric on $\mathcal{O}_{\mathcal{P}}(1)$. Then for any vector bundle \mathcal{E} on \mathcal{X} we have

$$\frac{1}{\mathrm{rk}(\mathcal{E})} \int_{\mathcal{P}} c_1(\mathcal{E}) \cdot \{\omega\}^{n-1} = \frac{1}{\mathrm{rk}(\mathcal{E})} \int_{\mathcal{P}} c_1(\mathcal{E}) \cdot c_1(\mathcal{O}_{\mathcal{P}}(1))^{n-1}. \quad (4.2)$$

Therefore the ω -stability condition is equivalent to the $\mu_{\mathcal{G}}$ -stability condition of Nironi.

5. Hermite–Einstein metrics on some noncompact Kähler Deligne–Mumford analytic stacks

5.1. Deleted neighborhoods of smooth divisors

Let (\mathcal{X}, g) be a Riemannian Deligne–Mumford stack. Given an atlas (U, u) of \mathcal{X} one can construct a groupoid action of

$$U \times_{\mathcal{X}} U \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} U \quad (5.1)$$

on the tangent space to U and the resulting quotient stack $\pi: \mathrm{Tot}(T\mathcal{X}) \rightarrow \mathcal{X}$ is a differentiable Deligne–Mumford stack over \mathcal{X} which is the total space of the tangent vector bundle $T\mathcal{X}$ to \mathcal{X} . One can generalize this construction to any (real) vector bundle E on \mathcal{X} .

The zero section of any (real) vector bundle E defines a substack $Z(E) \rightarrow \mathrm{Tot}(E)$ of its total space. Moreover, $\mathrm{Tot}(E) \setminus Z(E)$ is representable (i.e. it is a manifold) if and only if the isotropy action at all points of \mathcal{X} has no non zero fixed vectors. Given a metric on E one can construct similarly the unit sphere bundle $S(E)$ as a smooth substack of $\mathrm{Tot}(E) \setminus Z(E)$ which is a locally trivial sphere fibration over \mathcal{X} . Moreover, the long exact sequence of homotopy groups holds true [40]. In particular, $\mathrm{Tot}(T\mathcal{X}) \setminus \mathcal{X}$ is an honest manifold (i.e. it is representable) if and only if the tangent isotropy actions have no non zero fixed vectors. One also has an equivalence $\mathrm{Tot}(E) \setminus Z(E) \rightarrow S(E) \times \mathbb{R}_{>0}$ the second factor $\rho: \mathrm{Tot}(E) \rightarrow \mathbb{R}_{>0}$ being the norm function.

Given $i: \mathcal{S} \rightarrow \mathcal{X}$ a closed smooth substack, the normal bundle $\mathcal{N}_{\mathcal{S}|\mathcal{X}}$ over \mathcal{S} is introduced in Section 3.1. Then, using the exponential map in normal directions, one constructs an open differentiable substack $\mathcal{V} \subset \mathcal{X}$ containing \mathcal{S} as a closed substack such that the pair $(\mathcal{V}, \mathcal{S})$ is equivalent to $(\mathrm{Tot}(\mathcal{N}_{\mathcal{S}|\mathcal{X}}), \mathcal{S})$ in the sense that there is an equivalence of stacks $\mathcal{V} \rightarrow \mathrm{Tot}(\mathcal{N}_{\mathcal{S}|\mathcal{X}})$ which 2-commutes with the inclusions of \mathcal{S} .

If \mathcal{X} is now a smooth Deligne–Mumford analytic stack and $\mathcal{D} \subset \mathcal{X}$ be a smooth integral codimension one closed substack. Then $\mathcal{N}_{\mathcal{D}^{\mathrm{diff}}|\mathcal{X}^{\mathrm{diff}}}$ is the smooth bundle underlying $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ and there is an open differentiable substack $\mathcal{V} \subset \mathcal{X}^{\mathrm{diff}}$ containing $\mathcal{D}^{\mathrm{diff}}$ as a retract and a homotopy equivalence $\mathcal{U}_{\mathcal{D}} := S(\mathcal{N}_{\mathcal{D}^{\mathrm{diff}}|\mathcal{X}^{\mathrm{diff}}}) \rightarrow \mathcal{V} \setminus \mathcal{D}^{\mathrm{diff}}$ such that the resulting map $\mathcal{U}_{\mathcal{D}} \rightarrow \mathcal{D}^{\mathrm{diff}}$ is a locally trivial S^1 -fibration. In particular one has an exact sequence:

$$1 \rightarrow C_{\mathcal{D}} \rightarrow \pi_1(\mathcal{U}_{\mathcal{D}}) \rightarrow \pi_1(\mathcal{D}) \rightarrow 1, \quad (5.2)$$

where $C_{\mathcal{D}}$ is a normal cyclic subgroup whose order may take any non negative integer value or $+\infty$. If $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is ample on \mathcal{D} , then $\mathcal{U}_{\mathcal{D}}$ is a complex manifold.

5.2. Asymptotically flat metrics on deleted neighborhoods

Assume now (\mathcal{X}, ω_0) is Kähler Deligne–Mumford stack and $\mathcal{D} \subset \mathcal{X}$ is a smooth divisor. A riemannian metric g_c on $\mathcal{X} \setminus \mathcal{D}$ has cone-like singularities if there exists a metric ds^2 on $\mathcal{U}_{\mathcal{D}}$ such that the restriction of g_c to $\mathcal{V} \setminus \mathcal{D}^{\mathrm{diff}}$ is asymptotic to $dr^2 + r^2 ds^2$ in the sense made precise by [2, §1, Def. 1] on manifolds where r is the distance function to any given point (note that $r \sim \rho^{-1}$). The curvature of such metric g_c decays to zero when approaching \mathcal{D} .

If $\mathcal{O}_{\mathcal{X}}(\mathcal{D})|_{\mathcal{D}}$ is positive, there exists an hermitian metric $h_{\mathcal{D}}$ on $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ such that its curvature form is positive definite on a neighbourhood of \mathcal{D} . As in [2, §1], we can construct a cone-like complete metric ω_c on $\mathcal{X} \setminus \mathcal{D}$ by the following formula:

$$\omega_c := \sqrt{-1} \partial \bar{\partial} \frac{1}{a} \exp(a \log \|\sigma_{\mathcal{D}}\|_{h_{\mathcal{D}}}^{-2}) + C\omega_0, \quad (5.3)$$

where $\sigma_{\mathcal{D}} \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))$ is the *tautological section* of \mathcal{D} , a is an arbitrary positive number and C is a big enough positive real number.

DEFINITION 5.1. — *A holomorphic hermitian vector bundle (\mathcal{E}, h) on $\mathcal{X} \setminus \mathcal{D}$ is asymptotically flat if $|\Theta(\mathcal{E}|_{(X^\circ, \omega_c)}, h)|_{g_c} = O(r^{-2-\epsilon})$ as $r \rightarrow \infty$ for some $\epsilon > 0$.*

Restricting the Chern connection of (\mathcal{E}, h) to $\{r = R\}$ for $R \gg 1$ we get a hermitian connection D_R on the pull-back of \mathcal{E} to $\mathcal{U}_{\mathcal{D}}$.

The asymptotic flatness condition gives immediately (cf. [2, §1, Def. 3]):

LEMMA 5.2. — *The holonomy of D_R along any path has a limit when $R \rightarrow \infty$ which is homotopy invariant.*

This defines a unitary representation of $\pi_1(\mathcal{U}_{\mathcal{D}})$ and we say (\mathcal{E}, h) has *trivial holonomy at infinity* if the representation is trivial when restricted to the group $C_{\mathcal{D}}$ defined by (5.2).

5.3. Bando's Instanton Theorem

We are now ready to state an analog of [2, Thm. 1]:

THEOREM 5.3. — *Let (\mathcal{X}, ω_0) be a n -dimensional ($n \geq 2$) smooth connected compact Kähler Deligne–Mumford stack and let $\mathcal{D} \subset \mathcal{X}$ be a smooth divisor such that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is positive on \mathcal{D} . Let ω_c be a Kähler metric with cone-like singularities on $X^\circ := \mathcal{X} \setminus \mathcal{D}$. Then a holomorphic vector bundle \mathcal{E} on \mathcal{X} is such that $\mathcal{E}|_{\mathcal{D}}$ can be endowed with a flat unitary connection ∇ iff there exists an Hermite–Einstein asymptotically flat vector bundle (\mathcal{E}', h') on (X°, ω_c) having trivial holonomy at infinity which induces ∇ and \mathcal{E} is an extension of \mathcal{E}' .*

Proof. — Lemma 5.2 gives one direction. Bando's proof of the much harder converse applies mutatis mutandis: extend the flat hermitian metric to a deleted neighborhood of \mathcal{D} by using a retraction, then extend it to \mathcal{X} and flow it to an asymptotically flat Hermite–Einstein metric. All these operations make sense on stacks as explained above. The a priori estimates needed in the argument also rely on techniques that carry out to stacks: integral estimates using Stokes' formula and the maximum principle. \square

COROLLARY 5.4. — *If furthermore X° is an ALE manifold then we can replace the curvature decay condition by $\int_{\mathcal{X}} |\Theta(\mathcal{E}', h')|^n < +\infty$; it is equivalent to $|\Theta(\mathcal{E}', h')| = O(r^{-(2n-\epsilon)})$ for any $\epsilon > 0$.*

Proof. — Bando's argument for [2, Cor. 2] being local extends. \square

6. An application: orbifold compactifications of ALE spaces of type A_{k-1}

In this section we describe an explicit application of Theorem 5.3.

6.1. Homotopy theory of spherical curves

In this section we shall recall some homotopical properties of spherical Deligne–Mumford curves from [6].

For any pair of integers $m, n \geq 1$, we call *weighted projective line of type (m, n)* the smooth Deligne–Mumford analytic stack $\mathcal{P}(m, n) := [\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^*]$, where the action of \mathbb{C}^* on $\mathbb{C}^2 \setminus \{0\}$ is given by $t \cdot (x, y) := (t^m x, t^n y)$ for any $t \in \mathbb{C}^*$ and $(x, y) \in \mathbb{C}^2 \setminus \{0\}$. Note that $\mathcal{P}(1, 1) \simeq \mathbb{P}^1$. The stack $\mathcal{P}(m, n)$ has at most two orbifold points and its coarse moduli space is \mathbb{P}^1 . Obviously, $\mathcal{P}(m, n) \simeq \mathcal{P}(n, m)$. Moreover, a weighted projective line is an orbifold if and only if m and n are relatively coprime. We call these *orbifold weighted projective lines*. The weighted projective line $\mathcal{P}(m, n)$ is a μ_d -gerbe over $\mathcal{P}(\frac{m}{d}, \frac{n}{d})$, where $d = \gcd(m, n)$. As explained in [6, §4.3], the fundamental group $\pi_1(\mathcal{P}(m, n))$ is trivial.

For any pair of integers $m, n \geq 1$, a *football of type (m, n)* is an orbifold $\mathcal{F}(m, n)$ whose coarse moduli space is \mathbb{P}^1 and has two orbifold points of order m and n at 0 and ∞ , respectively. More precisely $\mathcal{F}(m, n) = \mathbb{P}^1(\sqrt[m]{0}) \times_{\mathbb{P}^1} \mathbb{P}^1(\sqrt[n]{\infty})$ is a root stack in the sense of [10]. When m and n are relatively prime, $\mathcal{F}(m, n)$ is naturally isomorphic to $\mathcal{P}(m, n)$.

DEFINITION 6.1. — *Let \mathcal{D} be a one-dimensional smooth Deligne–Mumford analytic stack. We say that \mathcal{D} is a spherical Deligne–Mumford curve if its universal cover is $\mathcal{P}(m, n)$ for some positive integer numbers m, n .*

Remark 6.2. — Let \mathcal{D} be one-dimensional smooth Deligne–Mumford analytic stack. By [6, Prop. 4.6] there exists a one-dimensional analytic orbifold $\tilde{\mathcal{D}}$ and a finite group H such that \mathcal{D} is a H -gerbe over $\tilde{\mathcal{D}}$. By [6, Prop. 7.4] \mathcal{D} is spherical if and only if $\tilde{\mathcal{D}}$ is spherical.

PROPOSITION 6.3 ([6, Prop. 5.5]). — *Let \mathcal{D} be a spherical Deligne–Mumford curve which is an orbifold with two orbifold points. Then \mathcal{D} is isomorphic to a football $\mathcal{F}(m, n)$ of type (m, n) for some positive integers m, n . The fundamental group of \mathcal{D} is \mathbb{Z}_d , where $d = \gcd(m, n)$ and its universal cover is $\mathcal{P}(\frac{m}{d}, \frac{n}{d})$ on which \mathbb{Z}_d acts by rotations.*

PROPOSITION 6.4. — *Let \mathcal{D} be a one-dimensional smooth Deligne–Mumford analytic stack, which is an H -gerbe over a football $\mathcal{F}(m, n)$. Then we have the exact sequence*

$$1 \rightarrow \mathbb{Z}_d \rightarrow H \rightarrow \pi_1(\mathcal{D}) \rightarrow \pi_1(\mathcal{F}(m, n)) \rightarrow 1, \quad (6.1)$$

where $d = \gcd(m, n)$. Moreover, the universal cover of \mathcal{D} is isomorphic to $\mathcal{P}(m, n)$.

Proof. — By [6, Prop. 7.6(i)], the fundamental group of \mathcal{D} fits into the exact sequence

$$1 \rightarrow \ker(H \rightarrow \pi_1(\mathcal{D})) \rightarrow H \rightarrow \pi_1(\mathcal{D}) \rightarrow \pi_1(\mathcal{F}(m, n)) \rightarrow 1.$$

On the other hand, as explained in [6, §9], $\ker(H \rightarrow \pi_1(\mathcal{D})) \simeq \pi_2(\mathrm{PGL}(m, n))$ (for a definition of $\mathrm{PGL}(m, n)$, see [6, §8]). Finally, by [6, Prop. 8.5 and 8.6], we get

$$\pi_2(\mathrm{PGL}(m, n)) = \begin{cases} \mathbb{Z}_d & \text{if } m \neq n \text{ and } d = \gcd(m, n), \\ \mathbb{Z}_m & \text{if } m = n. \end{cases}$$

The second statement follows from [6, Prop. 7.6(ii)]. □

6.1.1. Spherical Deligne–Mumford curves with universal cover

$\mathcal{P}(d, d)$

Let \mathcal{D} be a one-dimensional smooth Deligne–Mumford analytic stack with universal cover $\mathcal{P}(d, d)$. By [6, Cor. 9.8], \mathcal{D} is a global quotient stack of the form $[\mathbb{C}^2 \setminus \{0\}/E]$ where E is a central extension of a discrete group Γ by \mathbb{C}^* and the action of E on $\mathbb{C}^2 \setminus \{0\}$ is given by a \mathbb{C} -representation $\rho: E \rightarrow \mathrm{GL}(2, \mathbb{C})$ of E , which fits into the commutative triangle

$$\begin{array}{ccc} & \mathbb{C}^* & \\ & \swarrow & \searrow (t^d, t^d) \\ E & \xrightarrow{\rho} & \mathrm{GL}(2, \mathbb{C}) \end{array} \tag{6.2}$$

As explained in [6, §9.2], the fundamental group of \mathcal{D} is Γ .

Remark 6.5. — The Picard group $\mathrm{Pic}(\mathcal{D})$ of \mathcal{D} consists of E -equivariant line bundles on $\mathbb{C}^2 \setminus \{0\}$, i.e., characters $E \rightarrow \mathbb{C}^*$ (cf. Section 3.4). Let $\rho: \Gamma \rightarrow \mathbb{C}^*$ be a one-dimensional representation of $\Gamma \simeq \pi_1(\mathcal{D})$. Then the composition of morphisms $E \rightarrow \Gamma \xrightarrow{\rho} \mathbb{C}^*$ defines a line bundle L_ρ on \mathcal{D} . Moreover, L_ρ is endowed by an hermitian metric and a unitary flat connection associated with ρ (cf. Section 3.2).

6.2. A-type ALE hyperkähler spaces

Recall that an hyperkähler structure on a Riemannian manifold (M, g) consists of three integrable almost complex structures (I, J, K) satisfying the quaternionic relation $IJ = -JI = K$. Then there exist three associated Kähler forms $\omega_I, \omega_J, \omega_K$.

Let $\Gamma \subset \mathrm{SL}(2)$ be a finite group. An ALE space (of order 4) is a smooth complex surface X , diffeomorphic to the minimal resolution of \mathbb{C}^2/Γ , endowed with a hyperkähler metric for which there exist a compact subset $K \subset X$ and a diffeomorphism $X \setminus K \xrightarrow{\sim} \mathbb{C}^2 \setminus B_r(0)/\Gamma$ under which the metric is approximated by the standard Euclidean metric on \mathbb{C}^2/Γ ; it is written in the Euclidean coordinates (x_i) as

$$g_{ij} = \delta_{ij} + a_{ij}, \tag{6.3}$$

with $\partial^p a_{ij} = O(r^{-4-p})$, $p \geq 0$, where $r^2 = \sum_{i=1}^4 x_i^2$ and ∂ denotes the differentiation with respect to the coordinates x_i see [38, §1.(iii)]. Here $B_r(0)$ is the ball of radius r centered at 0.

ALE spaces are classified by Kronheimer in [28]. Let us recall his result only in the A-type case.

Let $k \geq 2$ be an integer and denote by μ_k the group of k -th roots of unity in \mathbb{C} . A choice of a primitive k -th root of unity ω defines an isomorphism of groups $\mu_k \simeq \mathbb{Z}_k$. We define an action of $\mu_k \simeq \mathbb{Z}_k$ on \mathbb{C}^2 as

$$\omega \cdot (z_1, z_2) := (\omega z_1, \omega^{-1} z_2). \tag{6.4}$$

The action of \mathbb{Z}_k on \mathbb{C}^2 is free outside the origin 0. The singularity of the quotient space $\mathbb{C}^2/\mathbb{Z}_k$ at the origin is called a *rational double point* or *Du Val singularity* (cf. [12, Def. 10.4.10]). Let $\varphi_k: X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ be the minimal resolution of singularities.

PROPOSITION 6.6 ([28]). — *Let $\alpha_I, \alpha_J, \alpha_K \in H^2(X_k; \mathbb{R})$ be three cohomology classes satisfying the non-degeneracy condition*

$$(*) \text{ for each } \Sigma \in H_2(X; \mathbb{Z}) \text{ with } \Sigma \cdot \Sigma = -2, \text{ there exists } A \in \{I, J, K\} \text{ with } \alpha_A(\Sigma) \neq 0.$$

Then there exist a Riemannian metric g and a hyperkähler structure (I, J, K) on X_k such that the cohomology classes of the Kähler forms $\{\omega_A\}$ are the given α_A for $A \in \{I, J, K\}$ and the metric is an A-type ALE metric of order 4. Conversely, all A-type ALE hyperkähler four-manifolds of order 4 are obtained in the above manner and their isometry classes are uniquely determined by the cohomology classes $\alpha_I, \alpha_J, \alpha_K$.

The *McKay correspondence* states that there exists a one-to-one correspondence between the set of irreducible representations of \mathbb{Z}_k and the irreducible components D_1, D_2, \dots, D_{k-1} of the exceptional divisor $\varphi_k^{-1}(0)$ (cf. [12, Cor. 10.3.11]). Moreover, the intersection matrix $(D_i \cdot D_j)_{ij}$ is the negative of the Cartan matrix associated with the A_{k-1} -root system. In this way, the set $\{D_1, \dots, D_{k-1}\}$ can be identified with the set of simple roots of the A_{k-1} -root system. On the other hand, the set of homology classes

$\{[D_1], \dots, [D_k]\}$ gives a basis of $H_2(X_k; \mathbb{Z})$; hence there is an isomorphism between $H_2(X_k; \mathbb{Z})$ and the A_{k-1} -root system.

Gonzalez-Sprinberg and Verdier [21] give a geometrical explanation of the McKay correspondence as follows. Let M_i be the reflexive $\mathcal{O}_{\mathbb{C}^2/\mathbb{Z}_k}$ -module defined by a nontrivial irreducible representation of \mathbb{Z}_k ; set $R_i := \varphi_k^* M_i / \text{Tors}$. Then R_i is a locally free sheaf on X_k and the set $\{c_1(R_1), \dots, c_1(R_{k-1})\}$ is the dual basis of $\{[D_1], \dots, [D_k]\}$. Kronheimer and Nakajima extend this construction to any A-type ALE hyperkähler space (see [29, Appendix]). We call the locally free sheaves R_i the *tautological bundles* of X_k .

In [29, 35] moduli spaces of $U(r)$ -instantons on A-type ALE hyperkähler spaces are studied. In the first paper, Nakajima compactifies, as a V-manifold (in the way described in [28, §2]), an A-type ALE hyperkähler space by adding a point and studies $U(2)$ -instantons on the ALE space in terms of the corresponding one in the orbifold compactification. In the second paper, Kronheimer and Nakajima provide a characterization of $U(r)$ -instantons in terms of linear data satisfying some closed and open conditions (in a way similar to the one followed by Atiyah, Drinfel'd, Hitchin and Manin [1] to study $SU(r)$ -instantons on \mathbb{R}^4). By using such description, in [36] Nakajima completes metrically the moduli spaces of $U(r)$ -instantons: the resulting space (called nowadays *Nakajima quiver variety*) is an hyperkähler variety which can be obtained as an *Hamiltonian reduction* of the cotangent space of the representation space of the affine Dynkin diagram $A_{k-1}^{(1)}$ (see for example [43, 20] for more details).

Remark 6.7. — Note that also the moduli space of $SU(r)$ -instantons on \mathbb{R}^4 can be completed metrically and the resulting space is a Nakajima quiver variety associated with the Jordan quiver, i.e., the quiver with one vertex and one loop. There exists a description of this quiver variety as moduli space of *framed sheaves* on \mathbb{P}^2 , i.e., torsion-free sheaves on \mathbb{P}^2 trivial along a line at infinity (cf. [37, Chap. 2]). Unfortunately, an analogous sheaf-theoretic description for Nakajima quiver varieties associated with the affine A-type Dynkin diagram is still not completely available. As we shall explain below, in the next section we will describe a construction which can be seen as a first building block in order to provide such sheaf-theoretic description for certain Nakajima quiver varieties.

6.3. Orbifold compactification

In this section we describe a construction in [7] of a compactification of the minimal resolution of the A_{k-1} toric singularity of $\mathbb{C}^2/\mathbb{Z}_k$, which turns out to be a projective toric orbifold.

Normal compactification

Let $k \geq 2$ be an integer and denote by μ_k the group of k -th roots of unity in \mathbb{C} . A choice of a primitive k -th root of unity ω defines an isomorphism of groups $\mu_k \simeq \mathbb{Z}_k$. We define an action of $\mu_k \simeq \mathbb{Z}_k$ on \mathbb{C}^2 as

$$\omega \cdot (z_1, z_2) := (\omega z_1, \omega^{-1} z_2). \tag{6.5}$$

The quotient $\mathbb{C}^2/\mathbb{Z}_k$ is a normal toric affine surface. To describe its fan we need to introduce some notation. Let $N \simeq \mathbb{Z}^2$ be the lattice of 1-parameter subgroups of the torus $T := \mathbb{C}^* \times \mathbb{C}^*$. Fix a \mathbb{Z} -basis $\{e_1, e_2\}$ of N and define the vector $v_i := ie_1 + (1 - i)e_2 \in N$ for any integer $i \geq 0$. Then the fan of $\mathbb{C}^2/\mathbb{Z}_k$ consists of the two-dimensional cone $\sigma := \text{Cone}(v_0, v_k) \subset N_{\mathbb{Q}}$ and its subcones. The origin is the unique singular point of $\mathbb{C}^2/\mathbb{Z}_k$, and is a particular case of the so-called *rational double point* or *Du Val singularity* (see [12, Def. 10.4.10]).

By [12, Example 10.1.9 and Cor. 10.4.9], the minimal resolution of singularities of $\mathbb{C}^2/\mathbb{Z}_k$ is the smooth toric surface $\varphi_k: X_k \rightarrow \mathbb{C}^2/\mathbb{Z}_k$ defined by the fan $\Sigma_k \subset N_{\mathbb{Q}}$, where

$$\Sigma_k(0) := \{0\}, \tag{6.6}$$

$$\Sigma_k(1) := \{\rho_i := \text{Cone}(v_i) \mid i = 0, 1, 2, \dots, k\}, \tag{6.7}$$

$$\Sigma_k(2) := \{\sigma_i := \text{Cone}(v_{i-1}, v_i) \mid i = 1, 2, \dots, k\}. \tag{6.8}$$

Note that the vectors v_i are the minimal generators of the rays ρ_i for $i = 0, 1, \dots, k$.

Let us consider the vector $b_{\infty} := -v_0 - v_k = -ke_1 + (k - 2)e_2$ in N . Denote by ρ_{∞} the ray $\text{Cone}(b_{\infty}) \subset N_{\mathbb{R}}$ and by v_{∞} its minimal generator. For even k , $v_{\infty} = \frac{1}{2}b_{\infty}$; for odd k , $v_{\infty} = b_{\infty}$. Let $\sigma_{\infty, k}$ and $\sigma_{\infty, 0}$ be the two-dimensional cones $\text{Cone}(v_k, v_{\infty}) \subset N_{\mathbb{R}}$ and $\text{Cone}(v_0, v_{\infty}) \subset N_{\mathbb{R}}$ respectively. Let \bar{X}_k be the normal projective toric surface defined by the fan $\bar{\Sigma}_k \subset N_{\mathbb{R}}$:

$$\bar{\Sigma}_k(0) := \{0\} = \Sigma_k(0), \tag{6.9}$$

$$\bar{\Sigma}_k(1) := \{\rho_i \mid i = 0, 1, 2, \dots, k\} \cup \{\rho_{\infty}\} = \Sigma_k(1) \cup \{\rho_{\infty}\}, \tag{6.10}$$

$$\bar{\Sigma}_k(2) := \{\sigma_i \mid i = 1, 2, \dots, k\} \cup \{\sigma_{\infty, k}, \sigma_{\infty, 0}\} = \Sigma_k(2) \cup \{\sigma_{\infty, k}, \sigma_{\infty, 0}\}. \tag{6.11}$$

First note that $i: X_k \hookrightarrow \bar{X}_k$ as an open dense subset. We denote by D_{∞} the T -invariant divisor associated to the ray ρ_{∞} .

Canonical orbifold

Let \tilde{k} be k for odd k , otherwise $k/2$. Let $\pi_k^{\text{can}}: \mathcal{X}_k^{\text{can}} \rightarrow \bar{X}_k$ be the so-called *canonical toric orbifold* over \bar{X}_k with torus T . It is the unique (up to isomorphism) smooth two-dimensional separated toric⁽⁸⁾ Deligne–Mumford algebraic stack of finite type over \mathbb{C} such that the locus over which π_k^{can} is not an isomorphism has a nonpositive dimension.

As a global quotient stack, $\mathcal{X}_k^{\text{can}}$ is isomorphic to $[Z_{\bar{\Sigma}_k}/(\mathbb{C}^*)^k]$, where $Z_{\bar{\Sigma}_k}$ is the union over all cones $\sigma \in \bar{\Sigma}_k$ of the open subsets

$$Z_\sigma := \{x \in \mathbb{C}^{k+2} \mid x_i \neq 0 \text{ if } \rho_i \notin \sigma\} \subset \mathbb{C}^{k+2}, \quad (6.12)$$

and the action of $(\mathbb{C}^*)^k$ is given by

$$\begin{aligned} & (t_1, \dots, t_k) \cdot (z_1, \dots, z_{k+2}) \\ &= \begin{cases} \left(\prod_{i=1}^{k-1} t_i^i t_k^{2-k} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^k z_2, t_1 z_3, \dots, t_k z_{k+2} \right), & k \text{ odd}; \\ \left(\prod_{i=1}^{k-1} t_i^i t_k^{1-\tilde{k}} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{\tilde{k}} z_2, t_1 z_3, \dots, t_k z_{k+2} \right), & k \text{ even}, \end{cases} \end{aligned}$$

for $(t_1, \dots, t_k) \in (\mathbb{C}^*)^k$ and $(z_1, \dots, z_{k+2}) \in Z_{\bar{\Sigma}_k}$.

The effective Cartier divisor $\tilde{\mathcal{D}}_\infty := (\pi_k^{\text{can}})^{-1}(D_\infty)_{\text{red}}$ is a toric orbifold with torus \mathbb{C}^* . By [7, Prop. 3.10] $\tilde{\mathcal{D}}_\infty$ is isomorphic as a quotient stack to $[\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^* \times \mu_{\tilde{k}}]$, where the action is given by $(t, \omega) \cdot (z_1, z_2) = (t\omega z_1, t z_2)$ for $(t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}}$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$. Moreover, $D_\infty \simeq \mathbb{P}^1$ is the coarse moduli space of $\tilde{\mathcal{D}}_\infty$ and the line bundle $\mathcal{O}_{\mathcal{X}_k^{\text{can}}}(\tilde{\mathcal{D}}_\infty)$ is ample (cf. [7, Rem. 3.18]).

Remark 6.8. — As explained in [7, §3.3], $\tilde{\mathcal{D}}_\infty$ is an orbifold of type (\tilde{k}, \tilde{k}) with two orbifold points \tilde{p}_0 and \tilde{p}_∞ , which are $(\pi_k^{\text{can}})^{-1}(0)_{\text{red}}$ and $(\pi_k^{\text{can}})^{-1}(\infty)_{\text{red}}$ respectively, where $0, \infty \in D_\infty$ are the two torus fixed points. By Proposition 6.3, the fundamental group of $\tilde{\mathcal{D}}_\infty$ is $\mathbb{Z}_{\tilde{k}}$. Moreover, its universal cover is $\mathcal{P}(\frac{\tilde{k}}{k}, \frac{\tilde{k}}{k}) = \mathcal{P}(1, 1) \simeq \mathbb{P}^1$.

⁽⁸⁾ Following [19], a *Deligne–Mumford torus* \mathcal{T} is a product $T \times \mathcal{B}G$ where T is an ordinary torus and G a finite abelian group. A *smooth toric Deligne–Mumford stack* is a smooth separated Deligne–Mumford algebraic stack \mathcal{X} of finite type over \mathbb{C} , with as a coarse moduli space $\pi: \mathcal{X} \rightarrow X$ a scheme, together with an open immersion of a Deligne–Mumford torus $\iota: \mathcal{T} \hookrightarrow \mathcal{X}$ with dense image such that the action of \mathcal{T} on itself extends to an action $a: \mathcal{T} \times \mathcal{X} \rightarrow \mathcal{X}$.

Root stack over the canonical orbifold

Let $\phi_k: \mathcal{X}_k \rightarrow \mathcal{X}_k^{\text{can}}$ be the stack obtained from $\mathcal{X}_k^{\text{can}}$ by performing a k -root construction⁽⁹⁾ along the divisor $\tilde{\mathcal{D}}_\infty$. As explained in [7, §3.4], \mathcal{X}_k is a two-dimensional toric orbifold with torus T and with coarse moduli space $\pi_k := \pi_k^{\text{can}} \circ \phi_k: \mathcal{X}_k \rightarrow \bar{X}_k$. As a global quotient stack, \mathcal{X}_k is isomorphic to $[Z_{\bar{\Sigma}_k}/(\mathbb{C}^*)^k]$, where the action of $(\mathbb{C}^*)^k$ on $Z_{\bar{\Sigma}_k}$ is

$$(t_1, \dots, t_k) \cdot (z_1, \dots, z_{k+2}) = \begin{cases} \left(\prod_{i=1}^{k-1} t_i^i t_k^{2k-k^2} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{k^2} z_2, t_1 z_3, \dots, t_k z_k \right) & \text{for odd } k, \\ \left(\prod_{i=1}^{k-1} t_i^i t_k^{k-k\bar{k}} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{k\bar{k}} z_2, t_1 z_3, \dots, t_k z_k \right) & \text{for even } k. \end{cases}$$

for $(t_1, \dots, t_k) \in (\mathbb{C}^*)^k$ and $(z_1, \dots, z_{k+2}) \in Z_{\bar{\Sigma}_k}$.

Let \mathcal{D}_∞ be the effective Cartier divisor $\pi_k^{-1}(D_\infty)_{\text{red}}$; it is a smooth toric Deligne–Mumford stack with Deligne–Mumford torus $\mathcal{T} \simeq \mathbb{C}^* \times \mathcal{B}\mu_k$. By [7, Prop. 3.30] \mathcal{D}_∞ is isomorphic as a global quotient stack to

$$\left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mu_k} \right], \quad (6.13)$$

where the action is given by

$$(t, \omega) \cdot (z_1, z_2) = \begin{cases} (t^{\bar{k}} \omega z_1, t^{\bar{k}} \omega^{-1} z_2) & \text{for even } k, \\ (t^k \omega^{\frac{k+1}{2}} z_1, t^k \omega^{\frac{k-1}{2}} z_2) & \text{for odd } k, \end{cases} \quad (6.14)$$

for $(t, \omega) \in \mathbb{C}^* \times \mu_k$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$. Moreover, \mathcal{D}_∞ is a μ_k -gerbe over $\tilde{\mathcal{D}}_\infty$.

Under this description of \mathcal{D}_∞ , the restriction of the line bundle $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ on \mathcal{D}_∞ is associated with the character of $\mathbb{C}^* \times \mu_k$ given by the projection to the first factor (cf. [7, Lem. 3.35]). So it is easy to see that $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ is positive on \mathcal{D}_∞ (see also [8, §6.1]). Moreover, (cf. [7, §3.4])

$$\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)|_{\mathcal{D}_\infty}^{\otimes \bar{k}k} \simeq \pi_k^* \mathcal{O}_{\bar{X}_k}(\tilde{k}D_\infty)|_{D_\infty} \simeq \pi_k|_{\mathcal{D}_\infty}^* \mathcal{O}_{D_\infty}(k/\bar{k}). \quad (6.15)$$

So by Proposition 3.9 the line bundle $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ is positive on \mathcal{D}_∞ .

Remark 6.9. — Since $\mathcal{D}_\infty^{\text{top}}$ is a μ_k -gerbe over $\tilde{\mathcal{D}}_\infty^{\text{top}}$, its universal cover is $\mathcal{P}(\tilde{k}, \bar{k})$ by Proposition 6.4. The group $\mathbb{C}^* \times \mu_k$ fits into a triangle as (6.2) where the homomorphism $\rho: \mathbb{C}^* \times \mu_k \rightarrow \text{GL}(2, \mathbb{C})$ is given by the action (6.14) and the homomorphism $\mathbb{C}^* \rightarrow \mathbb{C}^* \times \mu_k$ is simply $t \mapsto (t, 1)$. Thus by the arguments in Section 6.1.1, we obtain that $\pi_1(\mathcal{D}_\infty^{\text{top}}) \simeq \mathbb{Z}_k$.

⁽⁹⁾ For the theory of root stacks we refer to [10].

Remark 6.10. — It is easily seen that $\mathcal{U}_{\mathcal{D}_\infty}$ and $\mathcal{U}_{\tilde{\mathcal{D}}_\infty}$ are actually diffeomorphic manifolds. Their homotopy exact sequences fit into the diagram:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & C_{\mathcal{D}_\infty} & \longrightarrow & \pi_1(\mathcal{U}_{\mathcal{D}_\infty}) & \longrightarrow & \pi_1(\mathcal{D}_\infty) \longrightarrow 1 \\
 & & \downarrow & & \downarrow \text{id} & & \downarrow \\
 1 & \longrightarrow & C_{\tilde{\mathcal{D}}_\infty} & \longrightarrow & \pi_1(\mathcal{U}_{\tilde{\mathcal{D}}_\infty}) & \longrightarrow & \pi_1(\tilde{\mathcal{D}}_\infty) \longrightarrow 1
 \end{array} \tag{6.16}$$

where the rightmost vertical arrow is a surjection and the leftmost is an injection. Since $\pi_1(\mathcal{D}_\infty^{\text{top}}) \simeq \mathbb{Z}_k$ and $\pi_1(\tilde{\mathcal{D}}_\infty^{\text{top}}) \simeq \mathbb{Z}_{\bar{k}}$, then $C_{\tilde{\mathcal{D}}_\infty}/C_{\mathcal{D}_\infty} = \mathbb{Z}_k/\bar{k}$.

By Theorem 5.3 and Corollary 5.4, whose hypotheses we just checked since \mathcal{X}_k is a smooth Deligne Mumford stack, \mathcal{D}_∞ is a smooth divisor such that $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ is positive on \mathcal{D}_∞ we obtain the following ⁽¹⁰⁾.

THEOREM 6.11. — *Given a holomorphic vector bundle \mathcal{E} on \mathcal{X}_k , its restriction $\mathcal{E}|_{\mathcal{D}_\infty}$ can be endowed with a flat unitary connection ∇ iff there exists a Hermite–Einstein vector bundle $(\mathcal{E}' = \mathcal{E}|_{X_k}, h')$ on (X_k, ω_k) , where ω_k is an ALE metric on X_k , such that $\int_{X_k} |\Theta(\mathcal{E}', h')|^2 < +\infty$. Then, (\mathcal{E}', h') has at infinity the holonomy given by the holonomy of ∇ .*

Remark 6.12. — The original result of Bando permits only to treat the case where $k = 2$ and the holonomy is trivial on $C_{\tilde{\mathcal{D}}_\infty}/C_{\mathcal{D}_\infty} = \mathbb{Z}_2$.

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