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On the Nodal set of a second Dirichlet eigenfunction in a doubly connected domain ^(*)

ROLA KIWAN ⁽¹⁾

ABSTRACT. — We investigate the geometry of the nodal set of a second eigenfunction of the Dirichlet Laplacian in a doubly connected Euclidean plane domain of the form $\Omega = D \setminus \bar{B}$ and obtain results of Payne’s type. For instance, we prove that when D and B are symmetric and convex with respect to a line, then the nodal set cannot enclose B . Moreover, if Ω has a second axis of symmetry, then the nodal line intersects both ∂B and ∂D .

We also use these results in the optimization of the second eigenvalue for the problem of optimal placement of B within D .

RÉSUMÉ. — Ce papier étudie la géométrie de l’ensemble nodal de la seconde fonction propre du laplacien avec conditions de Dirichlet dans un domaine doublement connexe de forme $\Omega = D \setminus \bar{B}$. Les résultats obtenus sont utilisés dans un problème d’optimisation de la seconde valeur propre.

1. Introduction and main results

Consider the fixed membrane eigenvalue problem in a bounded domain (open and connected) $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We denote by $\{\lambda_k(\Omega)\}_{k \geq 1}$ the nondecreasing unbounded sequence of eigenvalues. The first eigenvalue is always simple and the corresponding eigenfunction does not vanish in Ω . All the higher order eigenfunctions must change

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sign inside Ω and, consequently, have nontrivial nodal sets. Recall that the nodal set of an eigenfunction u is defined as the closure of the zero set of u ,

$$\mathcal{N}(u) = \overline{\{x \in \Omega; u(x) = 0\}}.$$

The connected components of $\Omega \setminus \mathcal{N}(u)$ are called “nodal domains”.

According to the Courant nodal domain theorem, a second eigenfunction admits exactly two nodal domains.

The geometry of $\mathcal{N}(u)$ has been extensively investigated since the work of Payne [12, 13] who conjectured that, for a plane domain, the nodal line $\mathcal{N}(u)$ of a second eigenfunction should intercept the boundary of Ω at exactly two distinct points (see Yau [16] for a higher dimensional version).

This conjecture was proved under various symmetry and partial convexity conditions (Payne [13], Lin [10], Pütter [14], Damascelli [2], Yang and Guo [15] etc.) and by Jerison [8] for long thin convex domains. The validity of the conjecture for all convex plane domains has been proved first by Melas [11] under an additional hypothesis of smoothness of the boundary, and, then, by Alessandrini [1] in a more general setting.

Nevertheless, Payne’s conjecture is not valid for all bounded domains. Indeed, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [7] constructed an example of a non-simply connected plane domain with the property that the nodal line of a second eigenfunction is a closed curve in Ω . Fournais [5] constructed a similar example in higher dimensions, and Kennedy [9] constructed an example of a second eigenfunction, whose nodal set is closed, in a domain which is homeomorphic to the Euclidean ball in \mathbb{R}^n .

Motivated by the “optimal placement” problem for eigenvalues (in the spirit of [3, 4] in collaboration with El Soufi), this paper considers doubly connected domains of the form $\Omega := D \setminus \overline{B}$, where D and B are simply connected domains in \mathbb{R}^2 with piecewise C^2 boundary such that $\overline{B} \subset D$. The main goal is to investigate the geometry of the nodal set of a second eigenfunction of Ω . As examples and counter-examples show this is a difficult problem. We will answer this question under additional conditions of symmetry and partial convexity similar to those considered by Payne [13], Lin [10], Pütter [14].

DEFINITION 1.1. — *A domain $D \subset \mathbb{R}^2$ is said to be convex with respect to a line L of \mathbb{R}^2 if, for all $x \in L$, the intersection $D \cap \Delta(x)$ of D with the line $\Delta(x)$ passing through x perpendicularly to L , is either empty or a connected segment in \mathbb{R}^2 .*

DEFINITION 1.2. — *Let \mathcal{C} be a piecewise- C^1 simple closed curve. We say that \mathcal{C} encloses an open set $B \subset \mathbb{R}^2$ if B is contained in the compact region bounded by \mathcal{C} .*

We first prove the following general result :

THEOREM 1.3. — *Assume that both D and B are symmetric and convex with respect to the same line. Then the nodal set $\mathcal{N}(u)$ of a second eigenfunction in $\Omega = D \setminus \overline{B}$ does not enclose B .*

This theorem can also be extended to domains with several holes, i.e. multi-connected domains of the form $\Omega = D \setminus \overline{B}$ where $B = \bigcup_{i=1}^k B_i$. If D and the B_i , $1 \leq i \leq k$, are symmetric and convex with respect to the same line, then the nodal set of a second eigenfunction in Ω cannot enclose $\bigcup_{i=1}^k B_i$.

THEOREM 1.4. — *Assume that $\Omega = D \setminus \overline{B}$ is a plane domain which admits two distinct symmetry axes and that both D and B are convex with respect to one of these axes. Then the nodal line of a second eigenfunction in Ω is the union of two simple curves joining the exterior boundary ∂D to the interior boundary ∂B of Ω .*

We mention here that in the special case where D and B are both disks, the domain Ω is a circular annulus, and the nodal line will be simply the intersection of Ω with one diameter of D .

In view of applications to the maximization problem for the second eigenvalue of (1.1) for doubly connected domains, we consider now a plane domain $\Omega = D \setminus \overline{B}$ such that:

- (i) Ω has two perpendicular symmetry axes, say $\{x_1 = 0\}$ and $\{x_2 = 0\}$,
- (ii) in the region $\{x_1 > 0, x_2 > 0\}$, the distance from the origin to the exterior boundary ∂D (resp. the interior boundary ∂B) is a decreasing (resp. non decreasing) function of the argument θ .

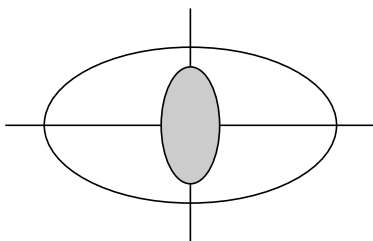


Figure 1.1. Example of a domain satisfying the conditions (i) and (ii) above

THEOREM 1.5. — *If $\Omega = D \setminus \overline{B}$ is a plane domain satisfying the conditions (i) and (ii) above, then the second eigenvalue $\lambda_2(\Omega)$ is simple and the corresponding eigenfunction u is antisymmetric with respect to the x_1 -variable, and $\mathcal{N}(u) = \overline{\Omega} \cap \{x_1 = 0\}$.*

In the paper [4] (in collaboration with A. El Soufi), we proved that, among all domains of \mathbb{R}^n bounded by two spheres of given radii, the second Dirichlet eigenvalue λ_2 is maximized when the spheres are concentric. The proof uses the fact that for a spherical shell, the nodal set of a second eigenfunction lies on a hyperplane of symmetry. Thanks to Theorem 1.5, it is possible to extend this optimization result, in the 2-dimensional case, to more general domains.

THEOREM 1.6. — *Let $\Omega = D \setminus \overline{B}$ be a domain satisfying the conditions (i) and (ii) above. For all $t \in \mathbb{R}$, let $B_t = B + te_2$ be the translate of B by the vector $(0, t)$. Then, for all $t \neq 0$ such that $\overline{B}_t \subset D$, one has*

$$\lambda_2(D \setminus \overline{B}_t) < \lambda_2(D \setminus \overline{B}).$$

We note here that for the first eigenvalue, the inequality

$$\lambda_1(D \setminus \overline{B}_t) < \lambda_1(D \setminus \overline{B}),$$

follows from the work of Harrell–Kröger–Kurata [6].

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2. Proof of Theorems 1.3 and 1.4

The idea of the proof of Theorem 1.3 is inspired by Payne [13]. We may assume, without loss of generality, that both D and B are convex and symmetric with respect to the axis $\{x_1 = 0\}$.

Notation 2.1.

$$\Omega_i^+ = \{x = (x_1, x_2) \in \Omega; x_i > 0\}, \text{ for } i = 1, 2;$$

we denote by S_i (for $i = 1, 2$), the reflections with respect to $\{x_i = 0\}$:

$$S_1(x) = (-x_1, x_2) \quad \text{and} \quad S_2(x) = (x_1, -x_2).$$

A function is said to be symmetric (resp. antisymmetric) w.r.t x_i if $u \circ S_i = u$ (resp. $u \circ S_i = -u$).

LEMMA 2.2. — *If $\Omega = D \setminus \overline{B}$ has a second eigenfunction u whose nodal line $\mathcal{N}(u)$ encloses B then there exists a second eigenfunction \bar{u} which is symmetric w.r.t. x_1 , such that $\mathcal{N}(\bar{u})$ encloses B . Moreover $\mathcal{N}(\bar{u})$ can only touch the boundary (if touching) in $\{x_1 = 0\}$.*

Proof. — First notice that the lemma is true for a second eigenfunction u which is symmetric w.r.t. to x_1 (simply take $\bar{u} = u$). Let u be a second eigenfunction in $\Omega = D \setminus \overline{B}$ which is not symmetric w.r.t. x_1 such that $\mathcal{N}(u)$ encloses B . It is well known by Courant's nodal Theorem that $\mathcal{N}(u)$ cuts Ω in exactly two nodal domains. It follows that $\mathcal{N}(u) \cap \partial\Omega$ is either empty or contains at most one point of ∂B and one point of ∂D . This means that ∂B is the interior boundary of one nodal domain of u , while ∂D is the exterior boundary of the other nodal domain of u . Thus, we may assume that $u > 0$ in a neighborhood of $\partial D \setminus \mathcal{N}(u)$ in Ω and $u < 0$ in a neighborhood of $\partial B \setminus \mathcal{N}(u)$ in Ω .

Now since u is not symmetric w.r.t. x_1 , the function $\bar{u}(x) := \frac{1}{2}(u(x) + u(S_1(x)))$ is a second eigenfunction in Ω which is not identically zero. Moreover, \bar{u} is positive in a neighborhood of ∂D except possibly one point (in the case when $\mathcal{N}(u) \cap \partial D$ is non empty), negative in a neighborhood of ∂B except possibly at one point (in the case where $\mathcal{N}(u) \cap \partial B$ is non empty). It follows that the nodal line $\mathcal{N}(\bar{u})$ encloses B . The intersection of $\mathcal{N}(\bar{u})$ with $\partial\Omega$ is either empty or has at most one point of ∂B and one point of ∂D . These points of intersection, if existing, should lie on the axis $\{x_1 = 0\}$ by the symmetry of \bar{u} . \square

LEMMA 2.3. — *If Ω has a second eigenfunction u which is symmetric w.r.t. x_1 and such that $\mathcal{N}(u)$ encloses B , then we can find a domain V such that V and $\Omega_1^+ \setminus \overline{V}$ have non empty interiors and $v = \frac{\partial u}{\partial x_1}$ satisfies:*

$$\begin{cases} \Delta v + \lambda_2(\Omega)v = 0 & \text{in } V, \\ v = 0 & \text{on } \partial V, \\ v > 0 & \text{in } V. \end{cases}$$

Proof. — Since $\mathcal{N}(u)$ encloses B , we can assume that u is positive in a neighborhood of ∂D and negative in a neighborhood of ∂B .

Using the symmetry and the convexity of the domain, one can verify that, for all $x \in (\partial D \cap \{x_1 > 0\}) \setminus \mathcal{N}(u)$, and sufficiently small $t > 0$, the point $x_t = ((1 - t)x_1, x_2)$ belongs to Ω , with $u(x_t) \geq 0$. Thus, $\frac{\partial u}{\partial x_1}(x) \leq 0$.

Similarly, for all $y \in (\partial B \cap \{x_1 > 0\}) \setminus \mathcal{N}(u)$ and sufficiently small $t > 0$, the point $y_t = ((1 + t)y_1, y_2)$ belongs to Ω , with $u(y_t) \leq 0$. Thus, $\frac{\partial u}{\partial x_1}(y) \leq 0$.

Consider the function $v = \frac{\partial u}{\partial x_1}$ and let

$$V = \{x \in \Omega_1^+, v(x) > 0\}. \tag{2.1}$$

Notice that, as observed above, v is nonpositive on $(\partial\Omega \setminus \mathcal{N}(u)) \cap \{x_1 > 0\}$ and that $v(x) = 0$ for all $x \in \Omega \cap \{x_1 = 0\}$ by symmetry. It follows that v vanishes on ∂V .

In addition, neither V nor $\Omega_1^+ \setminus \bar{V}$ can be empty because v must change sign in the interior of Ω_1^+ as u vanishes on ∂B , ∂D , and $\mathcal{N}(u)$. Moreover, by continuity, V and $\Omega_1^+ \setminus \bar{V}$ have both non empty interiors.

Hence, the function v is a solution of the eigenvalue problem:

$$\begin{cases} \Delta v + \lambda_2(\Omega)v = 0 & \text{in } V, \\ v = 0 & \text{on } \partial V, \\ v > 0 & \text{in } V. \end{cases} \quad \square$$

Proof of Theorem 1.3. — Suppose by contradiction that there exists a second eigenfunction u such that $\mathcal{N}(u)$ encloses B . By Lemma 2.2 it is possible to assume that u is symmetric w.r.t. x_1 . Using now Lemma 2.3, there is a domain V such that V and $\Omega_1^+ \setminus V$ have non empty interior, and an eigenfunction v associated to $\lambda_2(\Omega)$ such that $v = 0$ on ∂V and $v > 0$ in V . Consequently, $\lambda_2(\Omega)$ is the first Dirichlet eigenvalue of the Laplacian in V , and so $\lambda_2(\Omega) = \lambda_1(V) > \lambda_1(\Omega_1^+)$ (using the domain monotonicity of the eigenvalues).

On the other hand we have $\lambda_2(\Omega) \leq \lambda_1(\Omega_1^+)$ since any first eigenfunction of Ω^+ extended antisymmetrically to Ω is an eigenfunction in Ω , which is orthogonal to the (symmetric) first eigenfunction in Ω .

We deduce that $\lambda_2(\Omega) = \lambda_1(V) > \lambda_1(\Omega_1^+) \geq \lambda_2(\Omega)$ which leads to a contradiction. \square

Proof of Theorem 1.4. — Let u be a second eigenfunction in $\Omega = D \setminus \bar{B}$ and denote by \bar{u} the symmetrization of u w.r.t. x_1 and x_2 , i.e.

$$\bar{u}(x_1, x_2) = \frac{1}{4} \left[u(x_1, x_2) + u(-x_1, x_2) + u(x_1, -x_2) + u(-x_1, -x_2) \right].$$

Suppose now *by contradiction* that $\mathcal{N}(u) \cap \partial D = \emptyset$. Then ∂D is contained in the boundary of one nodal domain of u . Hence u does not change sign in a neighborhood of ∂D and it follows that \bar{u} is not equal to zero, and that \bar{u} cannot change sign in a neighborhood of ∂D (for symmetry reasons). Thus $\mathcal{N}(\bar{u}) \cap \partial D = \emptyset$, and ∂D is contained in the boundary of one nodal domain of \bar{u} . Now we use the symmetry of \bar{u} , and the fact that \bar{u} can have only two nodal domains. This cannot be verified unless $\mathcal{N}(\bar{u})$ encloses B . But this result is in contradiction with Theorem 1.3. Therefore, $\mathcal{N}(u) \cap \partial D$ cannot be empty.

One can now see that $\mathcal{N}(u) \cap \partial D$ cannot have more than two points. Otherwise \bar{u} would have more than two nodal domains which contradicts Courant's Theorem.

Now, let us show that $\mathcal{N}(u) \cap \partial D$ has exactly two points. Again we will use a *proof by contradiction* similar to the previous argument. We assume that $\mathcal{N}(u) \cap \partial D$ is the unique simple point M . Then u does not change sign in a neighborhood of $\partial D \setminus \{M\}$ and we can assume that u is positive in this neighborhood. Hence \bar{u} is also positive in the same neighborhood of ∂D (or possibly in a neighborhood of $\partial D \setminus \{M \cup S_1(M) \cup S_2(M) \cup S_1(S_2(M))\}$). It follows that $\mathcal{N}(\bar{u})$ encloses B or $\Omega \setminus \mathcal{N}(\bar{u})$ has at least three connected components which is impossible.

Using the same arguments, we prove that $\mathcal{N}(u) \cap \partial B$ has exactly two points. We finally conclude that $\mathcal{N}(u)$ touches each boundary at exactly two points. \square

Remark 2.4. — Notice that the same proof cannot be extended to higher dimensions. One crucial point in the current proof is that any plane curve with two different axes of symmetry must enclose the origin. The same argument is not correct in \mathbb{R}^n , with $n > 2$. For instance, one can see that the embedded torus in \mathbb{R}^3 has infinitely many of symmetry planes and does not enclose the origin.

3. Doubly symmetric domains. Proof of Theorem 1.5

In this section, we consider domains $\Omega = D \setminus \bar{B}$ satisfying conditions (i) and (ii) of the introduction. We introduce the following parametrization:

$$\begin{aligned} D &= \{(r \cos \theta, r \sin \theta), \theta \in [-\pi, +\pi], 0 \leq r < f(\theta)\} \\ B &= \{(r \cos \theta, r \sin \theta), \theta \in [-\pi, +\pi], 0 \leq r < g(\theta)\} \\ \Omega &= \{(r \cos \theta, r \sin \theta); \theta \in [-\pi, +\pi], g(\theta) < r < f(\theta)\}. \end{aligned}$$

where f and g are two π -periodic and symmetric functions such that f is decreasing and g is non decreasing on $(0, \frac{\pi}{2})$.

Recall the notation, for $i = 1, 2$, $\Omega_i^+ = \{x = (x_1, x_2) \in \Omega; x_i > 0\}$.

LEMMA 3.1. — *For a domain Ω satisfying the above definition, we have*

$$\lambda_2(\Omega) < \lambda_1(\Omega_2^+). \tag{3.1}$$

In particular, there is no second eigenfunction of Ω that is antisymmetric with respect to the variable x_2 .

Proof. — Since Ω_2^+ is symmetric with respect to $\{x_1 = 0\}$, its first Dirichlet eigenfunction w satisfies $w(-x_1, x_2) = w(x_1, x_2)$ for all $x = (x_1, x_2)$ in Ω_2^+ . We extend it anti-symmetrically with respect to $\{x_2 = 0\}$ to the whole of Ω , and denote by w the resulting function. Hence

$$\begin{cases} \Delta w + \lambda_1(\Omega_2^+)w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ w = 0 & \text{on } \Omega \cap \{x_2 = 0\}. \end{cases}$$

Let $v = \frac{\partial w}{\partial \theta} = -x_2 \frac{\partial w}{\partial x_1} + x_1 \frac{\partial w}{\partial x_2}$. As w is antisymmetric w.r.t. x_2 , one can check that v is symmetric w.r.t. x_2 , non identically 0 and satisfies:

$$\begin{cases} \Delta v + \lambda_1(\Omega_2^+)v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Omega \cap \{x_1 = 0\}. \end{cases}$$

Moreover, since w vanishes on ∂B , then the gradient of w is orthogonal to the boundary ∂B , thus at every point $x = (g(\theta) \cos \theta, g(\theta) \sin \theta)$ of ∂B : $\nabla w = \frac{\partial w}{\partial \nu}(x) \cdot \nu(x)$ where ν is the outward unit normal to ∂B (i.e. pointing to the interior of Ω).

Let us now consider the function $v(x) = \nabla w \cdot \tau(x)$, where $\tau = (-g(\theta) \sin \theta, g(\theta) \cos \theta)$ is the polar unit vector at the point x ; (notice that v can be seen as the derivative of w with respect to θ). One can see that:

$$\nu(x) = \frac{1}{\sqrt{g^2(\theta) + g'^2(\theta)}}(-g'(\theta) \sin \theta - g(\theta) \cos \theta, g'(\theta) \cos \theta - g(\theta) \sin \theta).$$

Hence

$$v(x) = -(g^2(\theta) + g'^2(\theta))^{-\frac{1}{2}} \frac{\partial w}{\partial \nu} g(\theta) g'(\theta).$$

Notice that $\frac{\partial w}{\partial \nu} \geq 0$ at each point of $\partial B \cap \{x_2 > 0\}$ by Hopf boundary point lemma, and that $g'(\theta) \geq 0$ in $[0, \frac{\pi}{2}]$. Thus, $v \leq 0$ on $\partial B \cap \{x_1 > 0\} \cap \{x_2 > 0\}$ and, by symmetry, $v \leq 0$ on $\partial B \cap \{x_1 > 0\}$. Using similar arguments, we can see that $v \leq 0$ on $\partial D \cap \{x_1 > 0\}$. Thus, $v \leq 0$ on $\partial\Omega \cap \{x_1 > 0\}$. One can predict the sign of v by noticing that any arc of circle starting from a point on $\partial\Omega \cap \{x_1 > 0\} \cap \{x_2 > 0\}$ and heading in the direction of increasing θ 's is pointing outward the domain, and looking at the sign of w .

Let us prove that v has to change sign in Ω_1^+ . To this end, take a positive number r such that $\inf f < r < \sup f$. Since f is decreasing on $(0, \frac{\pi}{2})$, the circular arc $\Gamma = \{(r \cos \theta, r \sin \theta), 0 < \theta < \frac{\pi}{2}\}$ must intersect ∂D for some value θ_1 , moreover $\Gamma_1 = \{(r \cos \theta, r \sin \theta), 0 < \theta < \theta_1\}$ is included in $\Omega_1^+ \cap \Omega_2^+$.

Now consider the function $h_r(\theta) = w(r \cos \theta, r \sin \theta)$. One can see that $h_r(0) = h_r(\theta_1) = 0$, we deduce that the derivative $h_r'(\theta)$ must vanish and

change sign at some $\theta_0 \in (0, \theta_1)$. Thus, v vanishes at $x_0 = (r \cos \theta_0, r \sin \theta_0) \in \Omega_1^+$, and it changes sign in Ω_1^+ .

Therefore $V := \{x \in \Omega_1^+, v(x) > 0\} \neq \emptyset$. Furthermore, since $v \leq 0$ on $\partial\Omega \cap \{x_1 > 0\}$, \bar{V} is included in the interior of Ω_1^+ , and then v vanishes on ∂V . As in the proof of Theorem 1.3, $\lambda_1(\Omega_2^+)$ is an eigenvalue of order at least two of the domain $V \cup S_1(V)$, where $S_1(V)$ is the image of V by the reflection with respect to $\{x_1 = 0\}$, and by monotonicity principle we get:

$$\lambda_2(\Omega) < \lambda_2(V \cup S_1(V)) \leq \lambda_1(\Omega_2^+),$$

which leads to (3.1).

Now, if there exists a second eigenfunction u in Ω which is antisymmetric in the x_2 -variable, then its nodal line lies on the x_2 -axis and the restriction of u to Ω_2^+ would be a first eigenfunction of Ω_2^+ which implies $\lambda_2(\Omega) = \lambda_1(\Omega_2^+)$ contradicting (3.1). \square

Proof of Theorem 1.5. — We use arguments similar to those of Pütter [14].

Let us first notice that any second eigenfunction u of Ω is symmetric w.r.t. x_2 , otherwise the function $u(x) - u(S_2(x))$ will be a non-zero antisymmetric eigenfunction, which is in contradiction with Lemma 3.1.

Furthermore, u must be antisymmetric w.r.t. x_1 , otherwise the function $u(x) + u(S_1(x))$ will be a non-zero eigenfunction that is symmetric in two directions, and this cannot occur taking in consideration Courant Nodal Theorem and Theorem 1.4.

It follows that: $\mathcal{N}(u) = \bar{\Omega} \cap \{x_1 = 0\}$. To see that $\lambda_2(\Omega)$ is simple, we only need to notice that $\lambda_2(\Omega)$ is a first eigenvalue of Ω_1^+ , and that for any second eigenfunction u of Ω , Ω_1^+ is a nodal domain of u . \square

Remark 3.2. — Notice that a similar result can be shown if we assume that f is non increasing and g is increasing on $(0, \frac{\pi}{2})$. After a rotation, we recover the condition (ii) above. The nodal line in this case is given by $\mathcal{N}(u) = \bar{\Omega} \cap \{x_2 = 0\}$.

4. Application to the optimal placement problem for the second Dirichlet eigenvalue. Proof of Theorem 1.6

Let us first introduce $\Omega(t) = D \setminus \bar{B}(t)$ the domain obtained after removing the domain $B(t)$ from the domain D , where $B(t) = B + te_1$ be the translate of B by t on the x_1 -axis. It is obvious that $\Omega(t)$ is invariant by S_2 (reflection with respect to the line $x_2 = 0$).

The space $H = W_0^{1,2}(\Omega(t))$ splits naturally into two invariant subspaces $H = H_2^+ \oplus H_2^-$, where $H_2^\pm = \{u \in H ; u \circ S_2 = \pm u\}$.

We denote by $\{\lambda_i^\pm(t)\}_{i \geq 1}$ the spectra of the restriction of the operator Δ on each subspace. The spectrum $\lambda_i(\Omega(t))$ of Δ is equal to the re-ordered union of $\{\lambda_i^+(t)\}_{i \geq 1}$ and $\{\lambda_i^-(t)\}_{i \geq 1}$. One can check easily that

$$\lambda_1(\Omega(t)) = \lambda_1^+(t) < \lambda_1^-(t)$$

(the first eigenvalue is simple and the corresponding eigenfunction is symmetric) and that

$$\lambda_2(\Omega(t)) = \inf\{\lambda_1^-(t), \lambda_2^+(t)\}. \tag{4.1}$$

Proof of Theorem 1.6. — Notice first that, according to Theorem 1.5, a second eigenfunction of $\Omega(0)$ belongs to H_2^- , which gives, with (4.1),

$$\lambda_2(\Omega(0)) = \lambda_1^-(0). \tag{4.2}$$

On the other hand, thanks to the monotonicity property (ii) satisfied by ∂D (f is decreasing), for all $t \in [0, R]$ with $R := \min f - \max g$, the reflection of $\Omega(t) \cap \{x_2 > t\}$ with respect to $x_2 = t$ is a subset of $\Omega(t) \cap \{x_2 < t\}$. This enables us to apply the same arguments as in the proof of Theorem 1 of [4] and show that $\lambda_1^-(t)$ is a strictly decreasing function of t on $(0, R)$. Thus, for all $t \in (0, R)$,

$$\lambda_1^-(t) < \lambda_1^-(0),$$

which gives, using (4.1) and (4.2),

$$\lambda_2(\Omega(t)) \leq \lambda_1^-(t) < \lambda_1^-(0) = \lambda_2(\Omega(0)).$$

This inequality ends the proof. □

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