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Topological properties of eigenfunctions of Riemannian surfaces

SUGATA MONDAL ⁽¹⁾

Dedicated to Jean-Pierre Otal on his 60-th birthday

ABSTRACT. — We provide a short survey of the results [37] of B. Sevennec, [28] of J-P. Otal, [29] of J-P. Otal and E. Rosas, [25], [26] of the author and [2], [3] of the author with his collaborators W. Ballmann and H. Matthiesen. The motivation is to give the reader a general idea how, in these (relatively) recent works, topological arguments were used to prove delicate results in the spectral geometry of surfaces.

RÉSUMÉ. — Nous examinons les résultats [37] de B. Sevennec, [28] de J-P. Otal, [29] de J-P. Otal et E. Rosas, [25], [26] de l'auteur et [2], [3] de l'auteur avec ses collaborateurs W. Ballmann et H. Matthiesen. Notre motivation est de donner au lecteur une idée générale de la façon dont, dans ces travaux (relativement) récents, des arguments topologiques ont été utilisés pour prouver des résultats délicats sur la géométrie spectrale des surfaces.

1. Introduction

A surface of finite topological type is a two dimensional manifold that is diffeomorphic to a closed surface from which finitely many distinct points and finitely many open disks with disjoint closures are removed. Although many of the results discussed in this article are true for (or can be extended to) non-orientable surfaces, here we shall consider orientable surfaces only.

Let S be a surface of finite topological type equipped with a smooth and complete Riemannian metric. Let Δ denote the Laplace operator corresponding to this metric that acts on the space of smooth functions on S . The spectrum $\sigma(\Delta)$ of Δ consists of discrete and essential parts (see Section 2). This article would focus mostly on the discrete part which for compact surfaces

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covers the whole spectrum. From now on we refer to $\sigma(\Delta)$ as the spectrum of S . Another aspect of $\sigma(\Delta)$ that will appear in our discussion (see Section 2 and Section 5) is the *generalized* and L^2 *eigenfunctions* corresponding to the points in $\sigma(\Delta)$. More precisely, part of our discussion (Section 5) will be around a conjecture, due to Otal–Rosas from [29], concerning *cuspidal* eigenvalues of Δ .

The asymptotic properties of the spectrum are somewhat global in nature (e.g. Weyl’s law, distribution of measures of the set of zeros, called the *nodal sets*, of eigenfunctions [14], quantum ergodicity [42], etc.) where topological tools are rarely helpful. On the contrary, for low eigenvalues, at least for surfaces, topological tools are extremely useful and we plan to explain this aspect in this article. We begin by considering the question of the multiplicity of a point in $\sigma(\Delta)$.

1.1. Multiplicity of eigenvalues

Any point λ in the discrete spectrum is called an (L^2) *eigenvalue* of S . A non-zero function $\varphi \in C^2(S) \cap L^2(S)$ that satisfies:

$$\Delta\varphi = \lambda \cdot \varphi \tag{1.1}$$

is called an (L^2) λ -*eigenfunction*. In [10] Cheng proved a structure theorem for the nodal set of any local solution of (1.1), see Theorem 3.1. This result proved to be particularly important in the study of the multiplicity of eigenvalues. The first among these was obtained by Cheng in the same paper [10]. There he showed that the multiplicity of the i -th eigenvalue (after an ascending arrangement of the eigenvalues) of a closed Riemannian surface of genus g is at most $(2g + 1 + i)(2g + 2 + i)/2$. Cheng’s arguments were later refined by Besson in [4] to replace the last bound, which is quadratic in g , by a bound that is linear in g .

The first eigenvalue is somewhat more important than others, see [37]. In [27] Nadirashvili extended the methods of Besson even further and showed that for any closed surface S the multiplicity of its first eigenvalue is at most $5 - 2 \cdot \chi(S)$, where $\chi(S)$ denotes the Euler characteristic of S . Relatively recently, in [37], Sévenec took these topological arguments to a higher level. There he first proved a Borsuk–Ulam type theorem and then used that to improve the latter bound to $5 - \chi(S)$.

THEOREM 1.1 (Sévenec [37]). — *Let S be a smooth closed surface equipped with a smooth Riemannian metric and $\chi(S) < 0$. Then the multiplicity of the first eigenvalue of S is at most $5 - \chi(S)$.*

Subsequently, in [28], Otal adopted the arguments of Sévenec for surfaces equipped with real analytic metrics of negative curvature to improve the bound to $-\chi(S) - 1$ when the first eigenvalue is a *small eigenvalue* (see below). In fact, Otal's bound is valid for any small eigenvalue be it the first one or not.

Remark 1.2. — It is worth mentioning that these bounds on the multiplicity are by no means evident and are false in dimensions at least three, see [13]. Even in dimension two, Colbois–Verdière [12] showed that there are hyperbolic surfaces of finite area for which the multiplicity of the first eigenvalue is as large as the square root of the genus of the surface.

Remark 1.3. — Although multiplicities do occur they are rare in the appropriate sense. The result [40] of Uhlenbeck shows that a generic metric on any surface (with proper boundary condition) has no multiple eigenvalue!

1.2. Small eigenvalues

Let \tilde{S} denote the universal cover of S equipped with the lifted metric. Let $\lambda_0(\tilde{S})$ denote the smallest number in the spectrum of Δ on \tilde{S} , called the *bottom of the spectrum* of \tilde{S} . The variational way of characterizing this number is

$$\lambda_0(\tilde{S}) = \inf_{\phi \in C_c^\infty(\tilde{S})} \frac{\int_{\tilde{S}} |\nabla \phi|^2}{\int_{\tilde{S}} \phi^2} \quad (1.2)$$

where $C_c^\infty(\tilde{S})$ denotes the space of smooth functions with compact support in \tilde{S} . Any eigenvalue of S in $[0, \lambda_0(\tilde{S})]$ is called a *small eigenvalue* of S .

Examples 1.4.

- (i) For S with finite area $\lambda_0(S) = 0$.
- (ii) For the Euclidean plane \mathbb{R}^2 equipped with the Euclidean metric, which is the universal cover for all flat tori, $\lambda_0(\mathbb{R}^2) = 0$.
- (iii) For the hyperbolic plane \mathbb{H}^2 equipped with the Poincaré metric, which is the universal cover for all hyperbolic surfaces, $\lambda_0(\mathbb{H}^2) = 1/4$.

Historically, the first appearance of small eigenvalues was (probably) in [15] where Huber mentioned these eigenvalues in connection with his study of distribution of lengths of closed geodesics on hyperbolic surfaces. Later, Selberg's 1/4 conjecture contributed to their fame. Initial results on small eigenvalues date back to McKean [23] and Randol [33]. Buser, in [7], gave an elegant construction of hyperbolic surfaces with maximal number of small eigenvalues. He also had initial contributions to the question of bounding

the number of small eigenvalues in terms of the Euler characteristic of the surface. Later, in [34] and [35], Schmutz extended some of Buser's results. From their results Schmutz, in [35], and Buser, in [8], conjectured that for any closed hyperbolic surface S of genus g the number of eigenvalues of $S < 1/4$ is at most $2g - 2 = -\chi(S)$. Refining the method developed in [28], in [29], Otal and Rosas proved an extended version of this conjecture.

THEOREM 1.5 (Otal–Rosas). — *Let S be a surface of finite topological type equipped with a real analytic metric with negative Euler characteristic and hyperbolic ends. Then the number of small eigenvalues of S is at most $-\chi(S)$.*

Remark 1.6. — A metric on S is said to have hyperbolic ends if the punctures of S has neighborhoods, called ends (see Section 2), where the metric is hyperbolic (i.e. has constant curvature equal to -1). In the theorem above, and in the rest of the paper, the eigenvalues are always counted with multiplicity.

Recently, the author and his collaborators W. Ballmann and H. Matthiesen have been able to extend Theorem 1.5 to any complete Riemannian metric on surfaces of finite topological type (with or without boundary, with or without finite area); see [2], [3].

1.3. Small cuspidal eigenvalues

Let S be a non-compact hyperbolic surface of finite topological type with finite area and without boundary. Such surfaces have at least one cusp. The spectrum $\sigma(\Delta)$, for such surfaces, is more complicated compared to compact surfaces. In Section 2 we give a quick review of the background.

A (*generalized*) eigenfunction φ of S is called *cuspidal* if the topological closure of the nodal set Z_φ of φ contains all the punctures of S . We call an eigenvalue λ *cuspidal* if at least one λ -eigenfunction is cuspidal. Following the notion of multiplicity of an eigenvalue, for an eigenvalue λ we define the *cuspidal multiplicity* $m^c(\lambda)$ of λ to be the dimension of the space spanned by the cuspidal λ -eigenfunctions. In [28] Otal shows that the cuspidal multiplicity of any small eigenvalue is at most $-\chi(\bar{S}) - 1$, where \bar{S} denotes the topological closure of S . Recall that the topological closure of a surface of topological type (g, n) is a closed surface of (topological type $(g, 0)$ i.e. a closed surface of) genus g .

In [29], after proving Theorem 1.5, Otal–Rosas discussed the issue of the number of small cuspidal eigenvalues. They conjectured, motivated by the

main result of [28], that the number is not more than $-\chi(\bar{S}) - 1$. The reason why this is a hard problem lies in the details of the proof of Theorem 1.5. Vaguely speaking the reason is that a small cuspidal eigenfunction satisfies Lemma 3.7 but there is apparently no reason for an arbitrary finite linear combination of small cuspidal eigenfunctions to satisfy the same.

The study and results in [26] was motivated by the above conjecture. Since proving a general result seems hard, the main objective in [26] was to provide existence of surfaces that satisfy bounds similar to the conjectural one. The methods employed to show such existence are a combination of variational and topological ones. On the one hand, motivated by the results of L. Ji and S. Wolpert, it used how small cuspidal eigenfunctions vary when the underlying hyperbolic metric varies. On the other hand, it used Lemma 3.7 to discard certain behaviors (of cuspidal eigenfunctions) that are allowed in the variational approach.

Remark 1.7. — It is worth mentioning that the variational study of cuspidal eigenpairs is a ridiculously hard problem if the eigenvalue lies in $(1/4, \infty)$, see [32]. This is another reason why the variational methods could only yield the “existence” and not a general bound.

1.4. Bounds for the first eigenvalue

The last topic that we consider is that of *largeness* of the first non-zero eigenvalue λ_1 . This question is interesting when we restrict ourselves to some sort of normalization, see Remark 6.1. Here we consider curvature normalization i.e. we consider hyperbolic surfaces of finite area.

For a pair of positive integers (g, n) with $2g - 2 + n > 0$ we consider the *moduli space* $\mathcal{M}_{g,n}$ of all hyperbolic metrics of finite area on a surface of topological type (g, n) . The first eigenvalue λ_1 can be thought of as a function $\lambda_1 : \mathcal{M}_{g,n} \rightarrow \mathbb{R}$. We are interested in the quantity

$$\Lambda_1(g, n) = \sup_{S \in \mathcal{M}_{g,n}} \lambda_1(S) \tag{1.3}$$

and ask the question if $\Lambda_1(g, n) \geq 1/4$. The motivation to pose this question comes from the works of Atle Selberg in [37] where he first proves that for any congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$

$$\lambda_1(\mathbb{H}/\Gamma) \geq \frac{3}{16} \tag{1.4}$$

and later on conjectures that the lower bound is in fact $1/4$. In [16], Huxley proved this conjecture for congruence subgroups of level ≤ 6 . The best known bound, in the general case, is $\frac{975}{4096}$ due to Kim and Sarnak [21]. Our question

above was addressed in the literature, first by Burger–Buser–Dodziuk in [9] and later by Brooks–Makover in [6]. The ideas in [9] and [6], in the light of Kim–Sarnak bound from [21], provide the existence of surfaces of arbitrary large genus with $\lambda_1 \geq \frac{975}{4096} - \epsilon$ for any preassigned $\epsilon > 0$.

1.4.1. Genus two

The existence of genus two closed hyperbolic surfaces with $\lambda_1 > \frac{1}{4}$ was first shown in [18]. It is known that the *Bolza* surface has λ_1 approximately 3.8 (see [38] for more details). Hence $\mathcal{B}_2(\frac{1}{4}) = \{S \in \mathcal{M}_2 : \lambda_1(S) > \frac{1}{4}\}$ is a non-empty subset of \mathcal{M}_2 . In [25], using topological techniques, the author obtained the following.

THEOREM 1.8. — $\mathcal{B}_2(\frac{1}{4})$ is an unbounded set that disconnects \mathcal{M}_2 .

The methods in the proof of this theorem fail in higher genus, see Section 6.1.

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2. Preliminaries

We say that a surface S is of *finite topological type* if it is diffeomorphic to a closed surface with a finite number of distinct points and finite number of open disks with mutually disjoint closures are removed. The points removed are called *punctures* and the disks removed are called *holes*. Observe that the non-compactness (if any) of a surface of finite topological type is caused by the punctures. Each of these punctures has neighborhoods in S that are topologically punctured disks. Such neighborhoods are called the *ends* of the surface S .

For any integer $g \geq 0$ let S_g denote the closed surface of genus g . For a pair of positive integers (g, n) let $S_{g,n}$ denote the surface obtained from S_g by removing n distinct points. Finally, for a triple of positive integers (g, n, m) let $S_{g,n,m}$ denote the surface obtained from S_g by removing n distinct points and m disks with mutually disjoint closures.

Remark 2.1. — A surface S of finite type with $\chi(S) < 0$ admits decompositions into pairs of pants, that is, into building blocks P of the following type:

- (1) a sphere with three holes;
- (2) a sphere with two holes and one puncture;
- (3) a sphere with one hole and two punctures;
- (4) a sphere with three punctures.

Each of these building blocks has Euler characteristic -1 and circles as boundary components. Hence S is built of $-\chi(S)$ such blocks where a block P of type (4) occurs if and only if $S = P$. This collection of building blocks is said to form a *pants decomposition* of the surface.

A Riemannian metric on a surface is called a *hyperbolic metric* if it has constant curvature equal to -1 . A surface equipped with a hyperbolic metric will be called a *hyperbolic surface*. It is well known that a connected surface S of finite type admits a complete hyperbolic metric of finite area with closed geodesics as boundary circles if and only if $\chi(S) < 0$ [8]. That is, excluded are sphere, torus, disk and annulus. A metric on S is said to have *hyperbolic ends* if each puncture of S has (end) neighborhoods where the metric is hyperbolic.

Finite area, for a hyperbolic surface S , is equivalent to the requirement that all the ends of S are cusps (see 2.3.2). However, one of the ends of S can also be a *funnel* instead (see 2.3.1), where the lengths of the bases (or the core geodesics) of the funnel may serve as additional parameters for the family of complete hyperbolic metrics.

2.1. Topology of embedded graphs and subsurfaces

A locally finite graph G on S consists of a pair (V, E) where V , called the set of *vertices* of G , is a discrete set of points of S and E , called the set of *edges* of G , is a locally finite collection (i.e. there are finitely many at each vertex) of mutually non-intersecting embedded arcs in S joining the points in V .

Using the Mayer–Vietoris sequence corresponding to the decomposition of S induced by G one can obtain the following equality

$$\chi(S) = \sum_i \chi(D_i) + \chi(G), \tag{2.1}$$

where D_i runs over the components of $S \setminus G$. Observe that each D_i is a smooth subsurface with piecewise smooth boundary. The formula (2.1) is called the *Euler–Poincaré formula*.

2.2. Spectrum of the Laplacian for smooth metrics on surfaces

Let S be a Riemannian surface, possibly not complete and possibly with non-empty boundary ∂S . For $k \geq 1$, integer, let $C^k(S)$ denote the space of C^k -functions on S , $C_c^k(S) \subseteq C^k(S)$ denote the space of C^k -functions on S with compact support. Denote by $L^2(S)$ the space of (equivalence classes of) square-integrable measurable functions on S . Let Δ denote the Laplace operator of S . For the following result, see [39, p. 85].

THEOREM 2.2. — *If S is complete, then the Laplacian Δ with domain*

$$\mathcal{D}_0 = \{\varphi \in C_c^\infty(S) \mid \varphi|_{\partial S} = 0\}$$

is essentially self-adjoint in $L^2(S)$.

In particular, the standard spectral theory for self-adjoint operators is applicable. Thus the spectrum $\sigma(\Delta)$ of Δ can be divided into two parts: *discrete* part and *essential* part. The essential part $\sigma_{\text{ess}}(\Delta)$ consists of all $\lambda \in \mathbb{R}$ such that $\Delta - \lambda \cdot I$ is not a Fredholm operator. The complement $\sigma_d(\Delta) = \sigma(\Delta) \setminus \sigma_{\text{ess}}(\Delta)$, the discrete spectrum of Δ , is a discrete subset of \mathbb{R} and consists of eigenvalues of Δ of finite multiplicity. In particular, every point in $\sigma_d(\Delta)$ comes with at least one (L^2) eigenfunction (see Section 1.1).

For S compact $\sigma_{\text{ess}}(\Delta)$ is empty and hence $\sigma(\Delta)$ is a discrete set of \mathbb{R} that diverges to ∞ . For S non-compact and hyperbolic with finite area $\sigma_{\text{ess}}(\Delta)$ comes with a “nice” description. The *Eisenstein series* corresponding to the cusps of S “spans” $\sigma_{\text{ess}}(\Delta)$ in this case; see [17] for details.

Another aspect in the spectral theory of hyperbolic surfaces of finite area is the “decomposition” of the space of L^2 functions into L^2 -eigenfunctions and *generalized* eigenfunction, see [17], [41]. For such a surface S , a function $\phi \in C^\infty(S)$ is called a generalized eigenfunction if there exists a $\lambda \in \mathbb{R}$ such that

- $\Delta\phi = \lambda \cdot \phi$ and
- ϕ has *at most polynomial growth* in the cusps of S ,

see [17], [41] for details.

The number λ is called a *generalized eigenvalue*, the pair (λ, ϕ) is called a *generalized eigen-pair* and ϕ is called a *generalized λ -eigenfunction*. A generalized eigenfunction ϕ , as above, is called an L^2 -eigenfunction if, further,

$\phi \in L^2(S)$. The λ and the pair (λ, ϕ) are, then, called an L^2 -eigenvalue and an L^2 -eigen-pair, respectively. This notion is slightly different from the last one (discrete and essential) because there could exist L^2 -eigenvalues embedded in the essential spectrum. In fact, they cover majority of the spectrum (in an appropriate sense) for congruence hyperbolic surfaces as shown by A. Selberg ([36]).

A generalized eigenfunction is called *cuspidal* if the topological closure of its nodal set contains all the punctures of the surface. It is known that cuspidal eigenfunctions are always L^2 -eigenfunctions. A generalized eigenvalue λ is called a *cuspidal eigenvalue* if at least one generalized λ -eigenfunction is cuspidal. It is known that the cuspidal eigenvalues of S , if exist, form a discrete set of \mathbb{R} without any accumulation point. We arrange them in the ascending order according to their cuspidal multiplicity and denote the i -th cuspidal eigenvalue of S by $\lambda_i^c(S)$.

2.3. The thick-thin decomposition of a hyperbolic surface

Let S be a hyperbolic surface without boundary. The *injectivity radius* of a point $p \in S$ is the radius of the largest geodesic disk that can be embedded in S with center p . For $\epsilon > 0$, the ϵ -thin part of S , $S^{(0,\epsilon)}$, is the set of points of S with injectivity radius $< \epsilon$. The complement of $S^{[\epsilon,\infty)} = S \setminus S^{(0,\epsilon)}$, the ϵ -thick part of S , is the set of points where the injectivity radius of S is $\geq \epsilon$.

2.3.1. Collars

Let γ be a simple closed geodesic on S . The collar theorem [20] says that γ has a *collar* neighborhood \mathcal{C}_γ in S that is diffeomorphic to the annulus $\mathbb{S}^1 \times [-w(\gamma), w(\gamma)]$ equipped with the hyperbolic metric $ds^2 = dr^2 + \ell_\gamma^2 \cosh^2 r d\theta^2$ where ℓ_γ is the length of γ and $w(\gamma) = \ell_\gamma \cosh(\sinh^{-1}(1/\sinh(\ell_\gamma/2)))$ is the *width* of \mathcal{C}_γ .

2.3.2. Cusps

Let S be a hyperbolic surface with at least one puncture. A *cusps* is an annular neighborhood of some puncture of S equipped with the hyperbolic metric $ds^2 = dr^2 + e^{-2r} d\theta^2$. Curves represented by $\{r = \text{constant}\}$ are called *horocycles*.

2.3.3. The decomposition

By Margulis lemma there exists a constant $\epsilon_0 > 0$, the Margulis constant, such that for all $\epsilon \leq \epsilon_0$, the thin part $S^{(0,\epsilon)}$ is a disjoint union of embedded collars, one for each geodesic of length less than 2ϵ , and of embedded cusps, one for each puncture.

2.4. Convergence of hyperbolic surfaces

Let $\mathcal{M}_{g,n}$ be the moduli space of hyperbolic surfaces of finite area and topological type (g, n) . It is known that $\mathcal{M}_{g,n}$ (is not compact and) can be compactified in such a way that in the compactification $\overline{\mathcal{M}}_{g,n}$ a sequence $(S_m) \in \mathcal{M}_{g,n}$ converges to $S_\infty \in \overline{\mathcal{M}}_{g,n}$ if and only if for any given $\epsilon > 0$ the ϵ -thick part $(S_m^{[\epsilon,\infty)})$ converges to $S_\infty^{[\epsilon,\infty)}$ in the Gromov–Hausdorff topology.

2.4.1. Convergence of functions

Let $S_m \rightarrow S_\infty$ in $\overline{\mathcal{M}}_{g,n}$. Fix an $\epsilon > 0$ and choose a base point $x_m \in S_m^{[\epsilon,\infty)}$ for each m . Now, for each $m \in \mathbb{N} \cup \{\infty\}$, fix a covering $\mathbb{H} \rightarrow S_m$ in such a way that the point $i \in \mathbb{H}$ is mapped to x_m . For a function f on S_m let \tilde{f} denote the lift of f under this covering. A sequence (f_m) of functions on (S_m) (i.e. f_m is a function on S_m) is said to *converges* to a function f_∞ on S_∞ if \tilde{f}_m converges to \tilde{f}_∞ , uniformly over compacta, for each choice of base points $x_m \in S_m^{[\epsilon,\infty)}$ and for each $\epsilon < \epsilon_0$, the Margulis constant.

3. Multiplicity of an eigenvalue

Let S be a surface equipped with a smooth Riemannian metric. For a smooth function φ on S the set of zeros $Z_\varphi = \{x \in S : \varphi(x) = 0\}$ of φ is called the *nodal set* of φ . In [10] S. Y. Cheng proved the following structure theorem for nodal sets of local solutions of the Laplace equation on Riemannian surfaces.

THEOREM 3.1 (Cheng). — *Let M be a smooth Riemannian surface. Then, for any solution of the equation $(\Delta + h(x))\varphi = 0$, $h \in C^\infty(M)$, the following are true:*

- (1) *The critical points on Z_φ are isolated.*

- (2) *When the nodal lines meet, they form an equiangular system.*
- (3) *The nodal lines consist of a number of C^2 -immersed one dimensional sub-manifolds. Therefore, when M is compact, they are a number of C^2 -immersed circles.*

Each connected component of $S \setminus Z_\varphi$ is called a *nodal domain* of φ . Using Theorem 3.1 and the *Courant nodal domain theorem*, in [10], Cheng obtained bounds on the multiplicity of the i -th eigenvalue λ_i in terms of i and the Euler characteristic of the surface.

THEOREM 3.2 (Cheng). — *Let S be a closed surface of genus g equipped with a smooth Riemannian metric. Then the multiplicity of λ_i is at most $(2g + 1 + i)(2g + 2 + i)/2$.*

These methods for bounding multiplicities of eigenvalues of Riemannian surfaces have proved to be fruitful in general, see [4], [27].

3.1. Sévenec’s Idea

The multiplicity of the first non-zero eigenvalue gained more interest than the others for reasons coming from other areas of mathematics (see [13], [37] and references there). In [37] B. Sévenec took a leap of thoughts to first prove a Borsuk–Ulam type theorem (see [37, Lemma 8]).

LEMMA 3.3 (Sévenec). — *Let $\cup_{i=1}^k \mathcal{P}_i = P^d$ be a decomposition of the d -dimensional real projective space into k subsets. Assume that the characteristic class α of the standard covering map $\pi: S^d \rightarrow P^d$ satisfies $(\alpha|_{\mathcal{P}_i})^{\ell_i} = 0$, for all $1 \leq i \leq k$. Then $d + 1 \leq \ell_1 + \dots + \ell_k$.*

He then used this result, in [37], to prove Theorem 1.1. The ideas in his approach proved to be fruitful in the works of Otal [28], Otal–Rosas [29] and in the works of the author with his collaborators [2] and [3]. Here we explain the work [28] that says the following.

THEOREM 3.4 (Otal). — *Let S be a hyperbolic surface of finite area. Then the multiplicity of any small eigenvalue of S is at most $-\chi(S) - 1$.*

Remark 3.5. — In [28] Otal also proved a version of the above theorem for small cuspidal eigenvalues. We discuss that result and related problems in Section 5.

3.2. Sévenec's arguments

By elliptic regularity, the eigenspace E_1 of λ_1 as in Theorem 1.1 is finite dimensional. Now consider some norm on E_1 (all are equivalent), and, with respect to this norm, consider the unit sphere S^d in E_1 , where $d+1 = \dim E_1$ is the multiplicity of λ_1 . The main reason behind Sévenec's investigation of the Borsuk–Ulam theorem was that each non-zero $\varphi \in E_1$ has exactly two nodal domains, $\Omega_\varphi^- = \{\varphi < 0\}$ and $\Omega_\varphi^+ = \{\varphi > 0\}$, which can be used to get a decomposition of S^d into the strata

$$\begin{aligned} \mathcal{S}_1 &= \{\varphi \in S^d \mid b_1(\Omega_\varphi^+) + b_1(\Omega_\varphi^-) \leq 1\}, \\ \mathcal{S}_j &= \{\varphi \in S^d \mid b_1(\Omega_\varphi^+) + b_1(\Omega_\varphi^-) = j\}, \quad 1 < j \leq b_1(S), \end{aligned}$$

where b_1 indicates the first Betti number. Clearly, each \mathcal{S}_j is invariant under the antipodal map of S^d . Discussions about the properties of this decomposition of P^d into the strata $\mathcal{P}_i = \pi(\mathcal{S}_i)$ covers a significant part of [37]. The main results are $\ell_1 = 4$ and $\ell_j = 1$ for $1 < j \leq b_1(S)$ ([37, Theorem 9]).

Remark 3.6. — For the conjectural upper bound on the multiplicity of λ_1 , motivated by [13], the reader should consult [37, p. 196].

3.3. Otal's adaptation to small eigenvalues

In [28], Otal adapted this whole line of thoughts to find bounds for the multiplicity of small eigenvalues on hyperbolic surfaces of finite area. Recall that for a hyperbolic surface an eigenvalue λ is called small if $\lambda \leq 1/4$. As explained in the introduction, the number $1/4$ is significant because it is the bottom of the spectrum of \mathbb{H}^2 [23]. It follows from domain monotonicity of λ_0 , the first Dirichlet eigenvalue, that for any bounded domain Ω in \mathbb{H}^2 one has the strict inequality $\lambda_0(\Omega) > 1/4$.

3.4. Otal's arguments

Observe that the eigenvalues considered now need not be the first (non-zero) eigenvalue. Hence Sévenec's ideas can not be applied directly. To remedy this Otal, in [28], starts with a key observation that provides a strong constraint on the topology of the nodal set (and nodal domains) for small eigenfunctions.

Following [28] we call an embedded locally finite graph G in a surface S to be *incompressible* if every loop in G that is homotopically trivial in S is already trivial in G .

LEMMA 3.7. — *Let S be a hyperbolic surface and φ a non-trivial eigenfunction with eigenvalue $\lambda \leq 1/4$. Then the graph Z_φ is incompressible. Moreover, any component of $S \setminus Z_\varphi$ has negative Euler characteristic.*

Proof (Sketch). — Observe that for any nodal domain Ω of φ we have $\lambda_0(\Omega) = \lambda$. This follows easily from the observation that $\varphi|_\Omega$ satisfies the eigenvalue equation on Ω and φ has constant sign on Ω . Now let D be a nodal domain of φ that is a disk. Then $\lambda_0(D) = \lambda$. On the other hand, the universal covering $\pi: \mathbb{H}^2 \rightarrow S$ is trivial over D and so we can lift D to a disk \tilde{D} in \mathbb{H}^2 . In particular, \tilde{D} is isometric to D and hence $\lambda_0(\tilde{D}) = \lambda_0(D) = \lambda \leq 1/4$. This is a contradiction because from the first paragraph of this subsection we know that $\lambda_0(\tilde{D}) > 1/4$.

To conclude that this proves the first part of the lemma one observes that if a loop is trivial in S then by Schoefflies theorem it bounds a disk. Clearly the above also shows that no component of $S \setminus Z_\varphi$ can be a disk. The remaining assertion i.e. no component of $S \setminus Z_\varphi$ is an annulus, can be proved following similar arguments. One extra ingredient one needs is that any annulus in S can be lifted to a cyclic subcover \hat{S} of S and that by a result of Brooks [5] the bottom of the spectrum of \hat{S} , $\lambda_0(\hat{S}) = 1/4$. \square

Proof (Sketch) of Theorem 3.4. — The basic strategy is very similar to [37]. Consider the eigenspace E_λ where λ is a small eigenvalue. This is finite dimensional. The idea is to decompose the unit sphere S_λ in E_λ (with respect to some norm) into $-\chi(S) - 1$ many strata using the topology of $S \setminus Z_\varphi$. More precisely, the i -th stratum is given by

$$\mathcal{S}_i = \{\varphi \in S_\lambda : \chi(S \setminus Z_\varphi) = -i\}. \tag{3.1}$$

By Lemma 3.7 and the Euler–Poincaré formula (2.1) it follows that for any small eigenfunction φ

$$\chi(S) \leq \chi(S \setminus Z_\varphi) \leq -2. \tag{3.2}$$

In particular, $\mathcal{S}_i = \emptyset$ for $i \neq 2, \dots, -\chi(S)$. Hence the above stratification consists of at most $-\chi(S) - 1$ strata. From the definition it clear that \mathcal{S}_i is invariant under the antipodal map (see Section 3.1). Hence to conclude the theorem one needs to prove that the restriction of the covering $\pi: S_\lambda \rightarrow \mathbb{P}_\lambda$ to each stratum is trivial, where \mathbb{P}_λ denotes the projective space in E_λ .

The argument for this part relies on the following fact from topology. Let U, V be two disjoint sub-surfaces $U, V \subset S$ with piecewise smooth boundary. If at least one of U and V has negative Euler characteristic then there is no isotopy of S that interchanges U and V .

For $\varphi \in S_\lambda$, in the same line as [37], consider the decomposition $S \setminus Z_\varphi$ according to the sign of φ , i.e. $S \setminus Z_\varphi = C_\varphi^+ \cup C_\varphi^-$, where $\varphi < 0$ (resp. $\varphi > 0$)

on $C^+(\varphi)$ (resp. $C^-(\varphi)$). It is easy to see that each C_φ^\pm is a sub-surface of S with piecewise smooth boundary and the singularity of the boundary of C_φ^\pm is prescribed by Theorem 3.1.

Observe that for any $\psi \in S_\lambda$ sufficiently close to φ in C^0 , $\chi(C_\varphi^\pm) \geq \chi(C_\psi^\pm)$. If we further assume that $\varphi, \psi \in \mathcal{S}_i$ then the last two inequalities are actually equalities. In particular, there is an isotopy of S that sends $\chi(C_\varphi^\pm)$ to $\chi(C_\psi^\pm)$. Hence, by our earlier observation on the existence of such isotopies, the connected component of \mathcal{S}_i that contains φ can not contain $-\varphi$. This proves the triviality of the covering. \square

4. Number of small eigenvalues

In [24], McKean stated erroneously that closed hyperbolic surfaces do not carry any non-zero small eigenvalue. In [33], B. Randol showed for the first time that there are hyperbolic surfaces with non-zero small eigenvalues. Later Buser (see [8]) gave a geometric construction of such hyperbolic surfaces. His construction used pants decompositions of these surfaces (see Remark 2.1) and the surfaces he constructed are built from pairs of pants whose boundary geodesics are short (length). His method yield the following.

THEOREM 4.1. — *For any $\epsilon > 0$, there exists a closed hyperbolic surfaces of genus g with $2g - 3$ non-zero eigenvalues $< \epsilon$.*

Motivated by this, he later studied the question of bounding the number of small eigenvalues of closed hyperbolic surfaces of genus g and showed that the $4g - 2$ -th non-zero eigenvalue $\lambda_{4g-2} > 1/4$ [7]. This was extended to $\lambda_{4g-3} > 1/4$ by Schmutz [34]. Later, in [35] Schmutz, showed that $\lambda_2 \geq 1/4$ for any hyperbolic surface of genus 2. This last result motivated Schmutz to conjecture (in [35]) that a closed hyperbolic surface S can not have more than $-\chi(S)$ many eigenvalues $< 1/4$. Later in [8] Buser also made the same conjecture.

Observe that Theorem 3.4 already implies this conjecture if one knows that forgetting multiplicity there is exactly one small eigenvalue. Of course there can be more than one small eigenvalues, forgetting multiplicity, and so one needs to do some more work to prove the conjecture. An extended version of the conjecture was proved, following very similar lines as in the proof of Theorem 3.4, by Otal and Rosas in [29], see Theorem 1.5.

Sketch of proof of Theorem 1.5. — Although the line of approach is very similar to those explained in Section 3.1 and Section 3.3, there are several

new difficulties that appear. Now one needs to consider the vector space E spanned by the finitely many eigenspaces E_λ of S with $\lambda \leq \lambda_0(\tilde{S})$.

To extend the ideas in Section 3.3 one needs an extension of Lemma 3.7. Since the functions that we are considering now are linear combinations of eigenfunctions, Theorem 3.1 is no longer available. However, since the underlying Riemannian metric of S is real analytic, its eigenfunctions are real analytic functions and, therefore, also any (finite) linear combination of them. Hence (by [22], as explained in Proposition 3 of [29]), the nodal set of any such linear combination has the structure of a locally finite graph.

A next and more serious difficulty in extending the ideas from Section 3.3 is that Lemma 3.7 may no longer be true for the nodal sets of arbitrary linear combinations of small eigenfunctions.

For example, the nodal set Z_φ of φ may have components that are not incompressible. (Note also that E contains the constant functions so that the nodal set of $\varphi \in E$ may be empty.) To take care of this, just delete all those components of Z_φ that are contained in a topological disk to obtain the modified graph, $G_\varphi \subseteq Z_\varphi$. Now G_φ may still not be incompressible in S ; however, the components of $S \setminus G_\varphi$ are.

LEMMA 4.2. — *For any $\varphi \in E$, at least one component of $S \setminus G_\varphi$ has negative Euler characteristic.*

Proof. — Let $\varphi \in E$. Then the Rayleigh quotient $R(\varphi)$ of φ is at most $\lambda_0(\tilde{S})$, by the definition of E . On the other hand, if a component of $S \setminus G_\varphi$ is a disk or an annulus, then the Rayleigh quotient $R(\varphi|_C)$ of φ restricted to any such component C is strictly bigger than $\lambda_0(\tilde{S})$, by the argument in the first paragraph of Section 3.3 for disks and the argument at the end of the proof of Lemma 3.7 for annuli. If now all the components of $S \setminus G_\varphi$ are disks or annuli then the Rayleigh quotient of φ on all of S would be strictly bigger than $\lambda_0(\tilde{S})$, a contradiction. \square

We let Y_φ be the union of all components of $S \setminus G_\varphi$ with negative Euler characteristic. Then $\chi(Y_\varphi) < 0$ thanks to Lemma 4.2 above. We also have $\chi(S) \leq \chi(Y_\varphi)$ by (2.1) and the incompressibility of the components of $S \setminus G_\varphi$.

By definition, each component C of $S \setminus G_\varphi$ is a union of a nodal domain Ω of φ with a finite number of disks in S enclosed by Ω . We say that C is positive or negative if φ is positive or negative on Ω and let Y_φ^+ and Y_φ^- be the union of the positive and negative components of Y_φ , respectively. Then Y_φ is the disjoint union of Y_φ^+ and Y_φ^- .

One final modification is necessary for these Y_φ^\pm . Namely, if a component of $S \setminus Y_\varphi^+$ or $S \setminus Y_\varphi^-$ is an annulus, then attach that annulus to its neighbour components in Y_φ^+ or Y_φ^- , respectively, to obtain new subsurfaces

$X_\varphi^+ \supseteq Y_\varphi^+$ and $X_\varphi^- \supseteq Y_\varphi^-$. Note that $\chi(X_\varphi^\pm) = \chi(Y_\varphi^\pm)$ so that, in particular, applying (2.1) we have

$$\chi(S) \leq \chi(X_\varphi^+) + \chi(X_\varphi^-) < 0,$$

by what we said above.

Now we are ready to follow the line of approaches in Section 3.1 and Section 3.3. As before one considers the unit sphere S^d in E and the projective space P^d of E , where $\dim E = d + 1$. The strata of S^d as in Lemma 3.3 are now

$$\mathcal{S}_i = \{\phi \in \mathbb{S} : \chi(X_\varphi^+) + \chi(X_\varphi^-) = -i\}.$$

In order to show the triviality of the restriction of the covering $\pi: S^d \rightarrow P^d$ to $\mathcal{S}_i \rightarrow \mathcal{P}_i = \pi(\mathcal{S}_i)$, one argues that the isotopy type of the triples $(S, X_\varphi^+, X_\varphi^-)$ does not change under small perturbation of φ as long as the perturbation lies in the same stratum. The proof of this last fact follows a similar line as the one in the last part of the (sketch of the) proof of Theorem 3.4. \square

After proving Theorem 1.5, in [29], Otal and Rosas, posed the question if Theorem 1.5 can be extended to “all” smooth metrics. The author and his collaborators, in [2], [3], have answered this question affirmatively as in the following theorem.

THEOREM 4.3 (Ballmann–Matthiesen–Mondal). — *Let S be a smooth Riemannian surface of finite topological type and $\chi(S) < 0$. Then the number of small eigenvalues of S is at most $-\chi(S)$.*

Remark 4.4. — Observe that by the density of analytic metrics among smooth metrics (on a fixed surface) a weaker form of Theorem 4.3 follows directly from Theorem 1.5. For example, it implies that the $2g-2$ -th non-zero eigenvalue $\lambda_{2g-2}(S) \geq \lambda_0(\tilde{S})$ for any closed surface S of genus g equipped with a smooth Riemannian metric.

4.1. Main issues in the extension Theorem 4.3

There are two main issues in the above extension and, in fact, they were resolved in two different papers (mentioned above). The first issue is that the metric is no longer real analytic. The nodal sets of eigenfunctions are still locally finite graphs by [10] but that of (arbitrary) finite linear combinations of small eigenfunctions, a priori, has no reason to be a graph.

To tackle this issue we consider *approximate* nodal domains

$$Z_\epsilon(\varphi) = \{x \in S : |\varphi|(x) < \epsilon\} \tag{4.1}$$

of such a linear combination φ where $\pm\epsilon$ are regular values of φ . Clearly $Z_\epsilon(\varphi)$ is a sub-surface of S with smooth boundary. In [2] we follow the guidelines of [29] with nodal sets replaced by approximate nodal sets. Of course there are several issues to take care of since one can not expect something like Lemma 4.2 to be true for any regular value $\pm\epsilon$ and furthermore approximate nodal domains change with the regular value.

The second issue arises when we treat non-compact surfaces. Observe that we have no assumption on the area of the surface or on the behavior of the metric near the ends of the surface. This is particularly important because in [29] the fact that the metric has hyperbolic ends plays an important role. More precisely, for those metrics separation of variables (for the Laplacian) provides a precise description of the nodal sets of (linear combinations of) eigenfunctions near the ends.

To tackle this issue we consider “nice” exhaustions $(K_i)_{i=1}^\infty$ of S by compact sets such that $S \setminus K_i$ consist of the ends of S . Then we first study how the parts of the approximate nodal sets $Z_\epsilon(\varphi)$ inside K_i change as $\epsilon \rightarrow 0$ and then study how they evolve as $i \rightarrow \infty$. The main line of approach in [29] still works but due to this “double approximation” the arguments become fairly involved.

The details of the above results can, of course, be found in [2] and in [3]. We avoid giving complete details of the arguments here mainly because the author and his collaborators have written a recent survey [1] precisely on this topic.

5. Small cuspidal eigenvalues

We first complete the arguments for Otal’s bound for the cuspidal multiplicity of small cuspidal eigenvalues in [28]. We begin by recalling the result.

THEOREM 5.1 (Otal). — *Let S be a non-compact hyperbolic surface with finite area. Then the cuspidal multiplicity of any small cuspidal eigenvalue is at most $-\chi(\bar{S}) - 1$.*

Complement to the proof of Theorem 3.4. — For any small cuspidal eigenfunction ϕ its nodal set $\mathcal{Z}(\phi)$ contains all the punctures of S . Therefore, the number of strata appearing in the proof of Theorem 3.4 via the Euler characteristic of $S \setminus \mathcal{Z}(\phi)$ varies between -2 and $\chi(\bar{S})$. \square

CONJECTURE 5.2 (Otal–Rosas). — *Let S be a non-compact hyperbolic surface of finite area. Then the number of small cuspidal eigenvalues of S is at most $-\chi(\bar{S}) - 1$.*

5.1. Where does the argument fail

In [29] Lemma 3.7 is replaced by Lemma 4.2. This lemma is a bit too general to conclude that the punctures of S would not enter in the counting of cuspidal eigenvalues. Figure 5.1 is a possible situation that can not be ruled out by Lemma 4.2. It seems that, to argue along the lines of Otal–Rosas, one needs finer properties of nodal sets of cuspidal eigenfunctions.

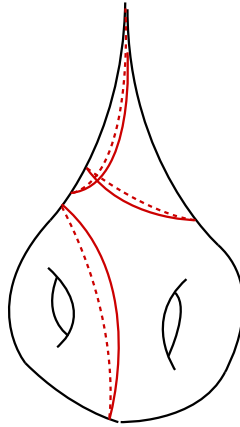


Figure 5.1. Nodal set of a finite linear combination of cuspidal eigenfunctions

5.2. Some existence results

Although the conjecture seems likely to be true in general, it is hard to find examples where one can “check” if it is the case. This is a general problem in this area, of course. The following results, obtained in [26], are motivated by the question of finding these examples. For $N \in \mathbb{N}$ consider the following subsets of $\mathcal{M}_{g,n}$

$$\mathcal{C}_{g,n}^{\frac{1}{4}}(N) = \{S \in \mathcal{M}_{g,n} : \lambda_N^c(S) > \frac{1}{4}\}. \quad (5.1)$$

Here λ_N^c denote the N -th cuspidal eigenvalue of S . The existence results are summarized as:

THEOREM 5.3 (Mondal). — $\mathcal{C}_{g,n}^{\frac{1}{4}}(2g - 2)$ contains a neighborhood of $\cup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ in $\overline{\mathcal{M}}_{g,n}$. Moreover, $\mathcal{C}_{g,n}^{\frac{1}{4}}(2g - 1)$ contains a neighborhood of $\mathcal{M}_{0,n+1} \cup \mathcal{M}_{g,1}$ in $\overline{\mathcal{M}}_{g,n}$.

Sketch of proof. — Let (S_m) be a sequence of finite area hyperbolic surfaces of type (g, n) each having a cuspidal eigen-pair (λ_m, ϕ_m) . Using the notion of convergence in the moduli space $\mathcal{M}_{g,n}$ (see Section 2.4) one can extract a converging subsequence $S_m \rightarrow S_\infty \in \overline{\mathcal{M}}_{g,n}$ and along this convergence of surfaces study the behavior of the eigen-pair (λ_m, ϕ_m) in the sense of Section 2.4.1.

This problem was investigated in the works [19] of Lizhen Ji and [41] of Scott Wolpert. Since Ji's work focuses only on closed hyperbolic surfaces it is not directly applicable for us. Wolpert's work covers non-compact hyperbolic surfaces of finite area and cuspidal eigen-pairs on them. However, Wolpert only treats sequence of eigen-pairs with the property that the eigenvalue limits in the range $(1/4, \infty)$. Since we need a result on small cuspidal eigen-pairs, Wolpert's result is not directly applicable either. So a version of these results had to be proven for small cuspidal eigen-pairs. Thanks to their works [19], [41], this turned out to be relatively straight forward.

THEOREM 5.4 (Mondal). — *Let $S_m \rightarrow S_\infty$ in $\overline{\mathcal{M}}_{g,n}$. Let (λ_m, ϕ_m) be a normalized (L^2 -norm of $\phi_m = 1$) small cuspidal eigenpair of S_m . Assume that λ_m converges to λ_∞ . Then one of the following holds:*

- (1) *There exist constants $\epsilon, \delta > 0$ such that $\limsup \|\phi_m\|_{S_m^{[\epsilon, \infty)}} \geq \delta$. Then, up to extracting a subsequence, (ϕ_m) converges to a λ_∞ -eigenfunction ϕ_∞ of S_∞ .*
- (2) *For each $\epsilon > 0$, $\limsup \|\phi_m\|_{S_m^{[\epsilon, \infty)}} = 0$. Then $S_\infty \in \partial\mathcal{M}_{g,n}$ and $\lambda_\infty = \frac{1}{4}$. Moreover, there exist constants $K_m \rightarrow \infty$ such that, up to extracting a subsequence, $(K_m\phi_m)$ converges to a linear combination of Eisenstein series and (possibly) a cuspidal λ_∞ -eigenfunction of S_∞ .*

With this result in hand we argue by contradiction. If the first assertion of Theorem 5.3 is false then we can find a sequence of surfaces (S_m) in $\mathcal{M}_{g,n}$ such that $S_m \rightarrow S_\infty \in \cup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ in $\overline{\mathcal{M}}_{g,n}$ and $\lambda_{2g-2}^c(S_m) \leq 1/4$. Observe that S_∞ has exactly $n+1$ components of which exactly n are thrice punctured spheres. Observe also that each of these n thrice punctured sphere components of S_∞ contains an *old cusp* i.e. cusps of S_∞ which are limits of cusps of S_m . The construction from the proof of [8, Theorem 8.1.3] implies that, for m large, S_m has at least n eigenvalues that converge to zero as $m \rightarrow \infty$.

We first make the observation that each eigenfunction corresponding to these first n eigenvalues are residual (not cuspidal) for m sufficiently large. To see this we again argue by contradiction and assume that one of the corresponding eigenfunctions ϕ_m is cuspidal. Then by Theorem 5.4, ϕ_m converges

uniformly over compacta to a function ϕ_∞ and ϕ_∞ is an eigenfunction for the eigenvalue 0. So ϕ_∞ is constant on each component of S_∞ . On those components of $S_\infty^{(\epsilon, \infty)}$ that contains an old cusp ϕ_∞ is necessarily zero because ϕ_m being cuspidal the average of ϕ_m over any horocycle (see Section 2.3.2) is zero. On the component of S_∞ of type $(g-1, 2)$ (that does not contain an old cusp) ϕ is again zero because the mean of ϕ_∞ over S_∞ is equal to the mean of ϕ_m over S_m which is zero. Therefore, ϕ_∞ is the zero function which is a contradiction by Theorem 5.4.

Going back to the original claim we observe that if $\lambda_{2g-2}^c(S_m) \leq \frac{1}{4}$ then each S_m has at least $2g-2+n$ non-zero small eigenvalues. This is a contradiction to [29, Theorem 2].

The proof of the second part is more involved. We again argue by contradiction and assume that we have a sequence of hyperbolic surfaces (S_m) in $\mathcal{M}_{g,n}$ such that $S_m \rightarrow S_\infty \in \mathcal{M}_{0,n+1} \cup \mathcal{M}_{g,1}$ and $\lambda_{2g-1}^c(S_m) \leq 1/4$. For each $1 \leq i \leq 2g-1$ and $m \geq 1$ choose a small cuspidal eigenpair (λ_m^i, ϕ_m^i) of S_m with the following properties:

- (1) $\{\phi_m^i\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S_m)$,
- (2) λ_m^i is the i -th eigenvalue of S_m .

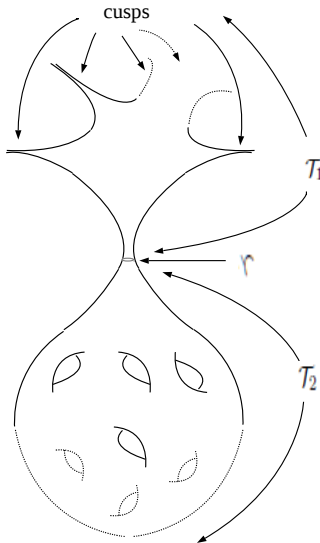


Figure 5.2. Pinching

For $1 \leq i \leq 2g - 1$ let λ_∞^i denote the limit of (λ_m^i) as $m \rightarrow \infty$. By Theorem 5.4 the sequence (ϕ_m^i) can exhibit two types of behavior:

- (1) the sequence (ϕ_m^i) converges to a λ_∞^i -eigenfunction ϕ_∞^i on S_∞ or
- (2) (λ_m^i, ϕ_m^i) satisfies condition (2) in Theorem 5.4 for some i .

A crucial observation is that in the second situation some portion of the L^2 norm of ϕ_m^i gets concentrated around a geodesic (γ in the above figure) that (gets pinched along (S_m) and) separates S_m into two components one of type $(0, n + 1)$ and the other of type $(g, 1)$; see Figure 5.2. This concentration does not allow the nodal set of ϕ_m^i to cross the separating geodesic. This contradicts Lemma 3.7 via a topological argument, see [26, Lemma 5.17]. Thus each ϕ_m^i limits to a non-zero function ϕ_∞^i on S_∞ . Using essentially the same argument one further obtains that each of these limit functions is cuspidal. Since no L^2 norm is lost (of any ϕ_m^i) the limit functions $\{\phi_\infty^1, \dots, \phi_\infty^{2g-1}\}$ are linearly independent.

Finally we count the number of small eigenvalues of S_∞ using [29] to conclude that at least one of ϕ_∞^i is nonzero on the component of S_∞ of type $(0, n + 1)$. This is a contradiction to [28, Proposition 2]. \square

5.3. An open question

Along the line of approach of Theorem 5.3 towards Conjecture 5.2 one important set that needs further understanding is the topological boundary $\partial\mathcal{C}_{g,n}^{\frac{1}{4}}(2g - 2)$ of $\mathcal{C}_{g,n}^{\frac{1}{4}}(2g - 2)$. One natural candidate for this set is

$$\mathcal{D}_{g,n}^\mu(2g - 2) = \{S \in \mathcal{M}_{g,n} : \lambda_{2g-2}^c(S) = \mu\},$$

for $\mu = 1/4$. So we ask if this indeed is the case.

QUESTION 5.5. — Is $\partial\mathcal{C}_{g,n}^{\frac{1}{4}}(2g - 2) = \mathcal{D}_{g,n}^{\frac{1}{4}}(2g - 2)$?

If the answer to the above question is yes, then there “maybe” a possible approach to prove the conjecture in the “generic” sense. The idea is that, due to a conjecture of Philips–Sarnak [32], the spaces $\mathcal{D}_{g,n}^\mu(N)$ are believed to be non-separating if $\mu > 1/4$. Although the conjecture does not include $\mathcal{D}_{g,n}^{1/4}(2g - 2)$ explicitly, there is some hope in this case as well, see [31].

6. Bounds for the First eigenvalue

Let S be a hyperbolic surface of finite area. For any such S the number 0 is always an (L^2) eigenvalue with the constant function as the corresponding

eigenfunction. The next entry in the spectrum λ_1 , called the first eigenvalue, is of importance to geometers for different reasons. We discussed the multiplicity issue of this eigenvalue in Section 3 and here we discuss how large this value could be for hyperbolic surfaces of finite area. Of course, there are geometric inequalities e.g. Cheeger’s inequality, that provide such bounds in general.

Remark 6.1. — This question is not that interesting in the complete generality of arbitrary smooth Riemannian metrics. This is because one can rescale a metric to get λ_1 as large as one wishes. So basically some sort of normalization is necessary. See [30] and reference therein for a related problem with area normalization.

The famous 1/4-conjecture of A. Selberg (see Introduction) probably originated the developments towards this question. The particular question of our interest is if, for a given topological type (g, n) , there are finite area hyperbolic surfaces of type (g, n) with $\lambda_1 \geq 1/4$. In genus two (i.e. $(g, n) = (2, 0)$), it is known that the *Bolza* surface has λ_1 approximately 3.8 (see [18] and [38] for more details).

For surfaces of higher genus the question is still open. In [9] Burger–Buser–Dodziuk and later in [6] Brooks–Makover used the results (on λ_1) for arithmetic hyperbolic surfaces to construct closed hyperbolic surfaces with large λ_1 . Since Selberg 1/4 is still not known to be true, this construction do not (and probably can not) provide examples of surfaces with $\lambda_1 \geq 1/4$.

The result that we prove in this section (is from [25] and) is for genus two hyperbolic surfaces. The motivation being that since the methods are “topological” in nature, they maybe helpful (with other analytic tools) to solve the problem in general. Recall that \mathcal{M}_g denotes the moduli space of closed hyperbolic surfaces of genus g . We consider the subset

$$\mathcal{B}_g(\tfrac{1}{4}) = \{S \in \mathcal{M}_g : \lambda_1(S) > \tfrac{1}{4}\}.$$

The question of existence of surfaces without any small eigenvalues translates to the following.

QUESTION 6.2. — *Is $\mathcal{B}_g(\frac{1}{4})$ non-empty for $g \geq 2$?*

From the continuity of λ_1 , as a function $\mathcal{M}_g \rightarrow \mathbb{R}$, it is clear that $\mathcal{B}_g(\frac{1}{4})$ is open. From [18] we have $\mathcal{B}_2(\frac{1}{4}) \neq \emptyset$. The main result we prove here is a description of how large this set is.

THEOREM 6.3 (Mondal). — *$\mathcal{B}_2(\frac{1}{4})$ is unbounded and disconnects \mathcal{M}_2 .*

Sketch of Proof. — Observe that any closed hyperbolic surface of genus two can have at most one eigenvalue $\leq 1/4$. So, for $S \in \mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$ its $\lambda_1(S)$ -eigenfunction φ_S is defined uniquely up to a constant multiple. Applying

Euler–Poincaré formula (2.1) for the graph Z_{φ_S} we get

$$\chi(S) = \chi(S \setminus Z_{\varphi_S}) + \chi(Z_{\varphi_S}). \quad (6.1)$$

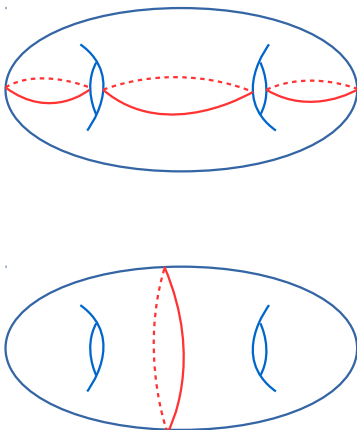


Figure 6.1. Possible decompositions of a genus two hyperbolic surface

Observe that $\chi(S) = -2$ and by Lemma 3.7 each component of $S \setminus Z_{\varphi_S}$ has negative Euler characteristic. Since $S \setminus Z_{\varphi_S}$ has exactly two components (by Courant nodal domain theorem) it follows that $\chi(Z_{\varphi_S}) = 0$. This, in particular, implies that Z_{φ_S} is a union of simple closed curves that divide S into exactly two pieces. Up to isotopy, there are precisely two possible configurations for Z_{φ_S} , see Figure 6.1.

Now let $\mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$ be connected and hence path connected. Let S_1, S_2 be two surfaces in $\mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$. Consider a path α that joins S_1 and S_2 . Since λ_1 has multiplicity one for all surfaces in $\mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$ we can consider a branch of $\lambda_1(\alpha(t))$ -eigenfunctions $\phi_{\alpha(t)}$ along $\alpha(t)$. This means that the one parameter family of functions $\phi_{\alpha(t)}$ varies continuously with respect to t . Thus using the above description of nodal sets it is possible to deduce that $Z(\phi_{S_1})$ and $Z(\phi_{S_2})$ are isotopic. This shows that the isotopy type of Z_{φ_S} is constant for $S \in \mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$.

To conclude the theorem it suffices to construct two surfaces $M, N \in \mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$ such that Z_{φ_M} and Z_{φ_N} are not isotopic. Existence of such surfaces ([25, Proposition 3.1]) can be shown via a pinching argument and the main result of [11]. \square

6.1. What goes wrong in higher genus

In genus three and higher, the method fails drastically. The first and the biggest obstacle is the multiplicity of λ_1 . More precisely, in higher genus λ_1 may have multiplicity ≥ 2 even if λ_1 is small. The second problem is that the topology (e.g. $-\chi(S)$) increases and hence the nodal sets of the first eigenfunctions need not be “simple” anymore.

To tackle the first problem one may consider *branches* of eigenvalues (or eigen-pairs) instead of true eigenvalues (or eigen-pairs). This approach was taken in [25] to prove some results about the branches *starting as* λ_1 ; see [25] for details.

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