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YVES CORNULIER AND ADRIEN LE BOUDEC

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Densely related groups ^(*)

YVES CORNULIER ⁽¹⁾ AND ADRIEN LE BOUDEC ⁽²⁾

ABSTRACT. — We study the class of densely related groups. These are finitely generated (or more generally, compactly generated locally compact) groups satisfying a strong negation of being finitely presented, in the sense that new relations appear at all scales. Here, new relations means relations that do not follow from relations of smaller size. Being densely related is a quasi-isometry invariant among finitely generated groups.

We check that a densely related group has none of its asymptotic cones simply connected. In particular a lacunary hyperbolic group cannot be densely related.

We prove that the Grigorchuk group is densely related. We also show that a finitely generated group that is (infinite locally finite)-by-cyclic and which satisfies a law must be densely related. Given a class \mathcal{C} of finitely generated groups, we consider the following dichotomy: every group in \mathcal{C} is either finitely presented or densely related. We show that this holds within the class of nilpotent-by-cyclic groups and the class of metabelian groups. In contrast, this dichotomy is no longer true for the class of 3-step solvable groups.

RÉSUMÉ. — On s'intéresse à la classe des groupes densément présentés. Il s'agit des groupes de type fini (ou plus généralement des groupes localement compacts compactement engendrés) dans lesquels de nouvelles relations apparaissent à intervalles réguliers. Une relation est dite nouvelle si elle n'est pas conséquence de relations de longueur plus petites. Pour un groupe de type fini, être densément présenté est un invariant de quasi-isométrie.

On vérifie qu'un groupe densément présenté ne peut pas avoir de cône asymptotique simplement connexe. En particulier un groupe lacunaire hyperbolique n'est jamais densément présenté.

On montre que le groupe de Grigorchuk est densément présenté. On prouve également que tout groupe de type fini (non virtuellement cyclique) qui est (localement fini)-par- \mathbb{Z} et qui satisfait une loi, est densément présenté. Étant donnée une classe \mathcal{C} de groupes de type fini, on considère l'alternative suivante: tout groupe dans \mathcal{C}

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⁽¹⁾ CNRS and Univ Lyon, Univ Claude Bernard Lyon 1, Institut Camille Jordan, 43 blvd du 11 novembre 1918, 69622 Villeurbanne, France — cornulier@math.univ-lyon1.fr

⁽²⁾ CNRS, Unité de Mathématiques Pures et Appliquées, ENS Lyon, France — adrien.le-boudec@ens-lyon.fr

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est soit finiment présenté, soit densément présenté. On montre que cette alternative est satisfaite par la classe des groupes nilpotents-par-cyclique et la classe des groupes métabéliens. A contrario, cette dichotomie n'est plus vraie pour les groupes résolubles de classe 3.

1. Introduction

If G is a group and S a generating subset, a *relation* in G is by definition an element of the kernel of the natural map $F_S \twoheadrightarrow G$. A relation $w \in F_S$ is said to be generated by a subset $R \subset F_S$ if w belongs to the normal subgroup of F_S generated by R . Equivalently, w can be written as a product of conjugates of elements of $R^{\pm 1}$. To every pair (G, S) , where G is a group and S a generating subset, we study the *relation range* $\mathcal{R}_S(G)$, which is the set of lengths of relations in G that are not generated by relations of smaller length. Here the length of an element of F_S refers to the word length associated to S . The set $\mathcal{R}_S(G)$ is finite precisely when the kernel of $F_S \twoheadrightarrow G$ is normally generated by elements of bounded length. When S is finite, this exactly means that the group G is finitely presented. The relation range was introduced by Bowditch in [9], but has not been explicitly considered since then, except in the small cancelation case [30].

The study of the relation range takes part in a program to find “measures” of the failure of finite presentability for finitely generated groups. Several approaches, of independent interest, have been carried out so far:

- In [5], several strengthenings of infinite presentability have been investigated, in terms of the study of the poset of normal subgroups contained in the kernel of epimorphisms from finitely presented groups to a given finitely generated group. However, these properties are not known to be quasi-isometry invariant, or are known not to be (see Remark 4.5).
- In [13], to one finitely generated group G , one associates the set of nonprincipal ultrafilters ω such that the asymptotic cone $\text{Cone}^\omega(G, (1/n))$ is simply connected. This is obviously a quasi-isometry invariant.

The study of the relation range is in spirit closer to the latter one, but it is not based on asymptotic cones (although we make a connection at some point) and therefore more of combinatorial flavor. The relation range has a distinct behavior than the aforementioned invariant from [13], and may detect properties not seen by this invariant (see Remark 3.6).

We stress out that the relation range of a finitely generated group is not defined in terms of a presentation of the group. Moreover in general there is no connection between the relation range and the set of lengths of relators in a given presentation of the group, even in the case of a minimal presentation (see Remark 3.7).

We should mention that there exist other notions turning out to be strengthenings of infinite presentability, such as the negation of being of homological type FP_2 over a given commutative ring. This is a quasi-isometry invariant; Alonso's proof from [1] in the case of the ring \mathbf{Z} carrying over the general case.

DEFINITION 1.1. — *A finitely generated group G is densely related if there exists $c > 0$ such that for all $n \geq 1$, we have $\mathcal{R}_S(G) \cap [n, cn] \neq \emptyset$. (This does not depend on the finite generating subset S , see Corollary 1.6.)*

These are groups in which new relations (that is, not generated by relations of smaller length) appear at all scales, in a sense precisely defined in the sequel. A typical example of a densely related group is the wreath product $\Gamma = \mathbf{Z} \wr \mathbf{Z}$, which admits the presentation $\langle t, x \mid [t^n x t^{-n}, x] = 1 \text{ for all } n \geq 1 \rangle$. By a result of Baumslag [3], this presentation is minimal. The relation $[t^n x t^{-n}, x]$ is not generated by relations of smaller length, and therefore $\mathcal{R}_{\{t,x\}}(\Gamma)$ contains the integer $4n + 4$ for every $n \geq 1$.

Recall that Gromov proved that if a finitely generated group G has all its asymptotic cones simply connected, then G is finitely presented and has a polynomially bounded Dehn function [21, §5.F]. It turns out that simple connectedness of *one* asymptotic cone of G already has consequences, namely that G cannot be densely related (see Proposition 3.16 for a more precise result):

THEOREM 1.2 (Corollary 3.17). — *Densely related finitely generated groups have no simply connected asymptotic cone.*

Any (non-trivial) wreath product is an example of a densely related group (Proposition 4.11). The following construction provides other kind of examples: Let G be a non-Hopfian group, and let $\varphi : G \rightarrow G$ be a surjective endomorphism of G with non-trivial kernel. We denote by K_φ the increasing union of the normal subgroups $\ker(\varphi^n)$, $n \geq 1$, and by $G_\varphi = G/K_\varphi$ the associated quotient.

PROPOSITION 1.3 (Corollary 4.20). — *Let G be a compactly generated locally compact group. Assume that φ is a continuous surjective and non-injective endomorphism of G such that K_φ is a closed subgroup of G . Then the group $G_\varphi = G/K_\varphi$ is densely related.*

Examples of groups covered by Proposition 1.3 are the semidirect products $\mathbf{Z}[1/pq] \rtimes \mathbf{Z}$, where the action is by multiplication by p/q , where $p, q \geq 2$ are coprime integers. See Section 4.3 for more examples.

Proposition 1.3 is a particular case of a more general construction investigated in Section 4.3. Let G be a group with a finite index normal subgroup H , which come with a finite set $\Phi = \{\varphi_i\}$ of surjective homomorphisms $\varphi_i : H \rightarrow G$. Under appropriate assumptions on G, H and Φ (see Section 4.3), this naturally defines an increasing sequence of normal subgroups $K_n \triangleleft G$, so that the groups G/K_n form a directed system. We show that the associated direct limit G/K_∞ is always densely related (Theorem 4.15). An example of group which may be obtained via this construction is the Grigorchuk group \mathfrak{G} introduced in [17], so that we obtain the following result.

THEOREM 1.4. — *The Grigorchuk group \mathfrak{G} is densely related. In particular no asymptotic cone of \mathfrak{G} is simply connected.*

Finitely generated groups that are not densely related are called *lacunary presented*. Of course this includes finitely presented groups, but here we are especially interested by infinitely presented ones.

The idea of considering sequences of relations of sparse lengths was used by Bowditch in order to provide a continuum of pairwise non-quasi-isometric infinitely presented small cancellation groups [9]. It was then shown by Thomas and Velickovic that this yields the first example of a finitely generated group having non-homeomorphic asymptotic cones (more precisely, one asymptotic cone that is a real tree, and one asymptotic cone that is not simply connected) [31]. This motivated the introduction of *lacunary hyperbolic groups* (groups with at least one asymptotic cone a real tree) by Olshanskii, Osin and Sapir [26]. They showed that this class of groups is actually very large. These are examples of groups that are lacunary presented, not all of which being finitely presented. However, the class of lacunary presented groups is much larger (see Remark 3.19).

Given $A, B \subset \mathbf{R}_+$, we write $A \preccurlyeq B$ if there exists $c > 0$ such that for every $a \in A$, there exists an element of B in the interval $[c^{-1}a, ca]$. We write $A \sim B$ when $A \preccurlyeq B$ and $B \preccurlyeq A$, and we say that A, B are at finite multiplicative Hausdorff distance. This defines an equivalence relation for subsets of \mathbf{R}_+ , which identifies all non-empty finite subsets.

The following result is a particular case of Theorem 2.3, which holds in the more general context of connected graphs. We refer to Section 2.3 for the relevant terminology.

THEOREM 1.5. — *Let G, S and G', S' be two pairs of groups and generating subsets. If the metric space (G, d_S) is a large-scale Lipschitz retract of $(G', d_{S'})$, then $\mathcal{R}_S(G) \preceq \mathcal{R}_{S'}(G')$.*

COROLLARY 1.6 (Bowditch). — *If the metric spaces (G, d_S) and $(G', d_{S'})$ are quasi-isometric, then $\mathcal{R}_S(G) \sim \mathcal{R}_{S'}(G')$.*

Corollary 1.6 implies in particular that if G is endowed with a (possibly discrete) locally compact topology, and if S is a compact generating subset, then the set $\mathcal{R}_S(G)$ does not depend, up to finite multiplicative Hausdorff distance, on the choice of S . We denote by $\mathcal{R}(G)$ the associated equivalence class, and we call $\mathcal{R}(G)$ the *relation range* of the group G . According to Corollary 1.6, the relation range $\mathcal{R}(G)$ is an invariant of the quasi-isometry class of G .

In this setting, we have the following two extreme situations:

- $\mathcal{R}(G)$ is finite. When G is a finitely generated group, the fact that $\mathcal{R}(G)$ is finite exactly means that G is finitely presented. More generally in the locally compact setting, the groups having a finite relation range are precisely the compactly presented groups. For an introduction to compactly presented groups, we refer the reader to the book [12].
- $\mathcal{R}(G) \sim \mathbf{N}$. By definition the groups G such that $\mathcal{R}(G) \sim \mathbf{N}$ are the densely related groups.

It is natural to ask which subsets of \mathbf{N} are, up to multiplicative Hausdorff distance, the relation range of a finitely generated group. The easy answer is: all of them. Indeed we can start from the above presentation of $\mathbf{Z} \wr \mathbf{Z}$ and choose an arbitrary subset of relators. Namely, given any subset $I \subset \mathbf{N} \setminus \{0\}$, defining $\Gamma_I = \langle t, x \mid [t^n x t^{-n}, x] = 1 \text{ for all } n \in I \rangle$, the relation range of $(\Gamma_I, \{t, x\})$ is equal to $\{0\} \cup (4I + 4)$, and every non-empty subset of \mathbf{N} is at finite Hausdorff distance to a subset of this form. As a consequence, the relation range distinguishes continuum many quasi-isometry classes of groups, a fact that was established by Bowditch in [9] (using small cancellation rather than the previous presentation). Bowditch's strategy, combined with work of Olshanskii–Osin–Sapir [26], actually shows that every subset $I \subset \mathbf{N}$ such that $I \approx \mathbf{N}$ is the relation range of a finitely generated lacunary hyperbolic group (see Section 4.1).

Still, it remains natural to ask the same question within more restricted classes of groups. First recall that if \mathcal{P}, \mathcal{Q} are properties, a group is said to be \mathcal{P} -by- \mathcal{Q} when there is a normal subgroup with \mathcal{P} whose associated quotient has \mathcal{Q} .

PROPOSITION 1.7. — *Every subset of \mathbf{N} is (equivalent to) the relation range of a (3-nilpotent locally finite)-by-abelian finitely generated group.*

Note in particular that infinitely presented solvable groups can be lacunary presented. In contrast, non-elementary lacunary hyperbolic groups are never solvable. Indeed, more generally they cannot satisfy a law [16, Cor. 6.13]. Recall that a group G satisfies a law if there exists a non-trivial group word $w(x_1, \dots, x_k)$ in a free group such that $w(g_1, \dots, g_k) = 1$ for all $g_1, \dots, g_k \in G$.

Now given a certain class of groups, we want to study the behavior of the relation range within this class. Proposition 1.7 says that the class of amenable (and even solvable) groups is not yet sufficiently restrictive, whereby the necessity to focus on smaller classes of groups. The following result provides classes of finitely generated groups for which the behavior of the relation range is completely understood.

THEOREM 1.8. — *Let G be a finitely generated group. Assume that one of the following holds:*

- (1) *G is metabelian, or more generally center-by-metabelian;*
- (2) *G is nilpotent-by-cyclic.*

Then G is either finitely presented or densely related.

In combination with Theorem 1.2, this implies the following:

COROLLARY 1.9. — *Let G be as in Theorem 1.8. If G admits one simply connected asymptotic cone, then G is finitely presented.*

Recall that it follows from Bieri–Strebel theorem that a (non virtually cyclic) finitely generated group G that is (locally finite)-by-cyclic is infinitely presented [7]. By a construction due to Olshanskii, Osin and Sapir [26, §3.5], (locally finite)-by-cyclic groups are not always densely related. Moreover, although we will not go into this direction here, the flexibility of their construction suggests that the relation range of (locally finite)-by-cyclic groups may be arbitrary. The following result shows that, under the additional assumption that the group satisfies a law, the relation range is forced to be as large as possible.

THEOREM 1.10. — *Let $G = N \rtimes \mathbf{Z}$ be a finitely generated group, where N is an infinite locally finite subgroup. If G satisfies a law, then G is densely related. In particular G has no simply connected asymptotic cone.*

The existence of a law in a finitely generated group or in some large enough subgroup was already known to have consequences at the level of

asymptotic cones. For instance if G satisfies a law then no asymptotic cone of G can be tree-graded in the sense of Drutu and Sapir [16, Cor. 6.13]. Also if G has a subgroup H of relative exponential growth that satisfies a law, then no asymptotic cone of G can be real tree [26, Th. 3.18(c)], [23, Th. 1.4]. The idea is that the existence of a law prevents ultraproducts from containing free subgroups, and this has consequences on their possible transitive isometric group actions. The approach of Theorem 1.10 is different in the sense that the existence of a law is interpreted directly at the level of the group (rather than ultraproducts) in order to produce appropriate relations.

A common feature of the proofs of Theorem 1.8 and Theorem 1.10 is the study of the relation range of finitely generated groups with a given homomorphism onto the group \mathbf{Z} . Recall that given $\pi : G \twoheadrightarrow \mathbf{Z}$, we say that the action of \mathbf{Z} *contracts into a finitely generated subgroup* of G if there is a decomposition of G as an ascending HNN-extension over a finitely generated group whose associated homomorphism onto \mathbf{Z} is equal to π .

THEOREM 1.11. — *Let $G = M \rtimes \mathbf{Z}$ be a finitely generated group satisfying a law. Then either the action of \mathbf{Z} contracts into a finitely generated subgroup of M , or the group G is densely related.*

Finally we end this introduction with the following problem. While the class of finitely generated linear groups (that is, isomorphic to a subgroup of $\mathrm{GL}_n(K)$ for some n and some field K) contains both finitely presented groups and densely related groups (e.g. the wreath product $\mathbf{Z} \wr \mathbf{Z}$), we do not know any other behavior of the relation range within this class of groups.

QUESTION 1.12. — *Is it true that every finitely generated linear group is either finitely presented or densely related?*

We point out that a positive answer to this question would imply the non-existence of infinitely presented linear lacunary hyperbolic groups, and the existence of such groups was asked by Olshanskii, Osin and Sapir in [26]. We refer to Section 3.4 for details.

Outline

This article is organized as follows. Section 2 introduces the analogue of the relation range for graphs, and contains the proof of Theorem 1.5. In Section 3 we establish preliminary results on the relation range and make the connection with asymptotic cones (Theorem 1.2). Finally the study of the relation range for specific classes of groups is carried out in Section 4, which contains the proofs of all other results stated in the introduction.

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2. Relation range of graphs

Following [9], this section introduces the analogue of the relation range in the setting of connected graphs.

2.1. Equivalence relation on subsets of \mathbf{R}_+

Given two subsets A, B of \mathbf{R}_+ , we write $A \preccurlyeq B$ if there exists $c > 0$ such that there is an element of B in the interval $[c^{-1}a, ca]$ for every $a \in A$. The relation $A \sim B$ defined by $A \preccurlyeq B$ and $B \preccurlyeq A$ is an equivalence relation on the set of subsets of \mathbf{R}_+ . Equivalently, we have $A \sim B$ if and only if A, B are at finite multiplicative Hausdorff distance, i.e. $\log(A)$ and $\log(B)$ are at finite Hausdorff distance. The class of a subset A will be denoted $[A]$. We leave as an exercise the verification that the operation $[A] \cup [B] = [A \cup B]$ is well defined, and that the relation defined by $[A] \subset [B]$ if there exist $A' \sim A$ and $B' \sim B$ such that $A' \subset B'$, is a partial order on the set of equivalence classes.

2.2. Definition for graphs

Let X be a graph. A path α of length $n \geq 0$ in X is a sequence of vertices x_0, \dots, x_n such that x_i and x_{i+1} are adjacent for every $i = 0, \dots, n-1$. We say that x_0 and x_n are respectively the initial point and the endpoint of α . We say that α is a loop if $x_0 = x_n$.

Assume now that X is a connected graph, and choose a base point $x_0 \in X$. For every $n \geq 0$, we consider the subgroup $\pi_1^{(n)}(X, x_0)$ of $\pi_1(X, x_0)$ generated by loops of the form $p^{-1} \cdot \alpha \cdot p$, where p is a path with initial point x_0 , and α is a loop (based at the endpoint of p) of length at most n . The subgroup $\pi_1^{(n)}(X, x_0)$ is normal in $\pi_1(X, x_0)$, and clearly $\pi_1^{(n)}(X, x_0)$ is a subgroup

of $\pi_1^{(m)}(X, x_0)$ for $m \geq n$. Note that $\pi_1(X, x_0)$ is the increasing union of the subgroups $\pi_1^{(n)}(X, x_0)$. We take the convention that $\pi_1^{(-1)}(X, x_0)$ is the trivial subgroup.

DEFINITION 2.1. — We denote by $\Phi(X)$ the set of integers $n \geq 0$ such that $\pi_1^{(n)}(X, x_0)$ properly contains $\pi_1^{(n-1)}(X, x_0)$. This does not depend on x_0 .

For simplicity we will use the same notation for $\Phi(X)$ and its \sim -class.

Remark 2.2. — For every integer $k \geq 1$, the set $\Phi(X)$ is at finite multiplicative Hausdorff distance to the set of integers n such that $\pi_1^{(kn)}(X, x_0)$ properly contains $\pi_1^{(n-1)}(X, x_0)$.

2.3. Quasi-isometry invariance

In this paragraph we study the behavior of $\Phi(X)$ under large scale Lipschitz retracts of connected graphs.

A map $f : X \rightarrow Y$ between two metric spaces is (C, C') -LS-Lipschitz (where LS stands for Large-Scale) if

$$d(f(x), f(x')) \leq \max(Cd(x, x'), C') \text{ for all } x, x' \in X;$$

and f is *LS-Lipschitz* if it is (C, C') -LS-Lipschitz for some constants C, C' . The space X is an *LS-Lipschitz retract* of Y if there are LS-Lipschitz maps $X \rightarrow Y \rightarrow X$ whose composite map is at bounded distance to the identity map of X .

THEOREM 2.3. — Let X, Y be connected graphs. Assume that X is an *LS-Lipschitz retract* of Y . Then $[\Phi(X)] \subset [\Phi(Y)]$. In particular, if X and Y are quasi-isometric then $[\Phi(X)] = [\Phi(Y)]$.

Let us proceed to prove the theorem. It will follow from the more quantitative Lemma 2.5.

If X is a metric space and $c \geq 0$, let $R_c(X)$ be the Rips complex: this is the simplicial complex whose set of vertices is X and there is a n -simplex between any $n + 1$ points pairwise at distance $\leq c$. (We only consider the Rips complex as a topological space, when endowed with the usual inductive limit topology.)

If $f : X \rightarrow Y$ maps any two points at distance $\leq c$ to points at distance $\leq c'$, then it induces a map $\bar{f} : R_c(X) \rightarrow R_{c'}(Y)$, defined to be equal to f on vertices and extended to be affine on simplices; then \bar{f} is continuous and $f \mapsto \bar{f}$ is functorial. In particular:

- if f is (C, C') -LS-Lipschitz then for all $c \geq C'/C$, f induces a map $\bar{f} : R_c(X) \rightarrow R_{cC}(Y)$;
- if f has distance $\leq k$ to the identity, i.e. $d(f(x), x) \leq k$ for all $x \in X$, then $d(f(x), f(x')) \leq d(x, x') + 2k$, so this defines \bar{f} from $R_c(X)$ to $R_{c+2k}(X)$ for all $c \geq 0$.

LEMMA 2.4. — *If f has distance $\leq k$ to the identity then $\bar{f} : R_c(X) \rightarrow R_{c+2k}(X)$ is homotopic to \bar{i} , where i is the identity of X .*

Proof. — We write coordinates in an $(n + 1)$ simplex of vertices (x_0, \dots, x_n) as $\sum_i^* \lambda_i x_i$, where $\lambda_i \in [0, 1]$ and $\sum \lambda_i = 1$. Note that when two of the x_i are equal this is meaningful and lies in a smaller simplex. For $t \in [0, 1]$, we define $\gamma_t : \sum_i^* \lambda_i x_i \mapsto \sum_i^* (1 - t)\lambda_i x_i + \sum_i^* t\lambda_i f(x_i)$. We need to check this is meaningful. First, if (x_i) forms a simplex, then $d(x_i, f(x_j)) \leq d(x_i, x_j) + k$ and $d(f(x_i), f(x_j)) \leq d(x_i, x_j) + 2k$, hence the x_i and $f(x_j)$ together form a simplex in R_{c+2k} (possibly there are some equalities $x_i = f(x_j)$). We also need to check that the definition of γ_t matches between different simplices, and this is the case.

To check continuity, we first observe that by definition of the inductive limit topology, it is enough to check continuity on finite subcomplexes (i.e. on $F \times [0, 1]$ where F ranges over finite subcomplexes of $R_c(X)$). Since such a finite subcomplex maps into a finite subcomplex, this is equivalent to checking the continuity when the simplices are endowed with some standard metric, for instance the Euclidean metric with edges of length 1.

The formula being Lipschitz in all variables (λ_i) and t , the map $(t, x) \mapsto \gamma_t(x)$ is continuous and defines a homotopy between \bar{f} and \bar{i} (note that the x_i lying on a discrete set, there is no continuity issue at this level). \square

For two path-connected metric spaces Z_1, Z_2 , a continuous map $f : Z_1 \rightarrow Z_2$ is said to be π_1 -injective (respectively π_1 -surjective) if the induced map f_* at the level of fundamental groups is injective (respectively surjective).

Now we fix a base-point on X and we need the following condition on X :

there exists c_0 such that for all $c \geq c_0$,

$$\text{the inclusion map } R_{c_0}(X) \rightarrow R_c(X) \text{ is } \pi_1\text{-surjective. } (*)$$

This is ensured by some coarse geodesic assumptions such as: for every $c \geq c_0$, for any two points at distance $\leq c + 1$ there is a third point at distance $\leq c$ from both. This in particular holds with $c_0 = 1$ in a combinatorial connected graph.

We fix another metric space Y that also satisfies $(*)$ (with the same c_0). We also fix inverse quasi-isometries f, g between X and Y . We can assume

(increasing c_0 if necessary) that $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are (C, Cc_0) -LS-Lipschitz and (C', c_0) -LS-Lipschitz for some $C, C' \geq 1$. We assume all these maps are basepoint-preserving and that $g \circ f$ has distance $\leq c_0/2$ to the identity of X . (We use nothing on $f \circ g$, hence we just suppose that X is a QI-retract of Y .)

For a metric space Z , we denote by $\Psi(Z)$ is the set of r such that $R_r(Z) \rightarrow R_{2r}(Z)$ is not π_1 -injective. Note that when Z is a combinatorial connected graph, we have $\Psi(Z) \sim \Phi(Z)$, so that the following lemma implies Theorem 2.3.

LEMMA 2.5. — *There exists $r_0 > 0$ and $s \geq 1$ such that if $r \geq r_0$ and $R_r(X) \rightarrow R_{2r}(X)$ is not π_1 -injective, then $R_{Cs^{-1}r}(Y) \rightarrow R_{2Cr}(Y)$ is not π_1 -injective.*

In particular, up to finite Hausdorff multiplicative distance, we have the inclusion of $\Psi(X)$ into $\Psi(Y)$.

Proof. — Fix a positive number q satisfying $q \leq \min((C'C)^{-1}, 1/2)$. Fix $r > q^{-1}c_0$ such that $R_r(X) \rightarrow R_{2r}(X)$ is not π_1 -injective.

Assume by contradiction that $R_{Cqr}(Y) \rightarrow R_{2Cr}(Y)$ is π_1 -injective. Then we have the following diagram of continuous maps, where the horizontal arrows are standard inclusions and the top square commutes (for each vertical arrow, a simple inequality shows that it is well-defined).

$$\begin{array}{ccc}
 R_{qr}(X) & \longrightarrow & R_{2r}(X) \\
 \downarrow \bar{f} & & \downarrow \bar{f} \\
 R_{Cqr}(Y) & \longrightarrow & R_{2Cr}(Y) \\
 \downarrow \bar{g} & & \\
 R_{CC'qr}(X) & &
 \end{array}$$

The upper square induces a commutative square of maps between the fundamental groups. Pick a non-trivial element $\gamma \in \pi_1(R_r(X))$ having a trivial image in $\pi_1(R_{2r}(X))$. The assumption (*) implies that γ can be chosen to lie in $\pi_1(R_{qr}(X))$.

Composing (right and then down), we see that γ has a trivial image in $\pi_1(R_{Cqr}(Y))$. The injectivity of $R_{Cqr}(Y) \rightarrow R_{2Cr}(Y)$ then implies that $\bar{f}(\gamma) \in \pi_1(R_{Cqr}(Y))$ is trivial. Again composing by g , it follows that $\bar{g} \circ \bar{f}$ maps γ to a trivial element in $\pi_1(R_{C'Cqr}(X))$. On the other hand, by Lemma 2.4, $\bar{g} \circ \bar{f}$, as a map from $R_{qr}(X)$ to $R_{\max(C'Cqr, qr+c_0)}(X)$, is homotopic to the identity. Hence the identity maps γ to a trivial element in $\pi_1(R_{\max(C'Cqr, qr+c_0)}(X))$. The upper bound on q implies that

$\max(C'Cr, qr + c_0) \leq r$. Hence we obtain that the image of γ in $\pi_1(R_r(X))$ is trivial, which is a contradiction. \square

3. Relation range of a group

3.1. Definition

Let G be a group, and S a generating subset of G , so that we have a short exact sequence

$$1 \longrightarrow N \longrightarrow F_S \longrightarrow G \longrightarrow 1,$$

where F_S is the free group on S . For every $n \geq 0$, we let N_n be the normal subgroup of F_S generated as such by elements of N or word length at most n . By definition (N_n) is an increasing sequence of normal subgroups of F_S ascending to the subgroup N .

DEFINITION 3.1. — We denote by $\mathcal{R}_S(G)$ the set of integers $n \geq 0$ such that N_n properly contains N_{n-1} . We call $\mathcal{R}_S(G)$ the relation range of the group G with respect to the generating subset S .

DEFINITION 3.2. — A group G is said to be:

- (1) boundedly presented over S if $\mathcal{R}_S(G)$ is finite;
- (2) densely related over S if $\mathcal{R}_S(G) \sim \mathbf{N}$. Otherwise G is said to be lacunary presented over S .

Recall that the Cayley graph $\text{Cay}(G, S)$ of G with respect to S is the graph with G as set of vertices, and (x, y) is an edge if there is $s \in S^{\pm 1}$ such that $y = xs$. Since loops in $\text{Cay}(G, S)$ correspond to relations in G , we have the following.

LEMMA 3.3. — $\mathcal{R}_S(G) = \Phi(\text{Cay}(G, S))$.

Theorem 2.3 has the following consequence.

COROLLARY 3.4. — Let G be a compactly generated locally compact group, and S a compact generating subset. Then the \sim -class of $\mathcal{R}_S(G)$ does not depend on S , and is actually a quasi-isometry invariant of the group G .

More generally $[\mathcal{R}(H)] \subset [\mathcal{R}(G)]$ whenever H is a large-scale Lipschitz retract of G (e.g. a group retract).

Recall that a group retract of G is a subgroup $H \leq G$ such that there exists a homomorphism $G \rightarrow H$ whose restriction to H is the identity.

DEFINITION 3.5. — *If G is a compactly generated locally compact group, we call $\mathcal{R}(G) = [\mathcal{R}_S(G)]$ the relation range of G , which does not depend on the choice of compact generating subset S . We say that G is densely related if $\mathcal{R}(G) \sim \mathbf{N}$, and lacunary presented otherwise.*

Remark 3.6. — Following [13], if G is compactly generated locally compact group, let $\nu(G)$ be the set of non-principal ultrafilters on \mathbf{N} such that the asymptotic cone $\text{Cone}^\omega(G, (1/n))$ is simply connected. This is a quasi-isometry invariant, and its complement $\nu(G)^c$ is related in spirit to the relation range. However, it behaves differently: there exist finitely presented groups for which $\nu(G)^c$ can be non-empty, or even be the set of all non-principal ultrafilters. For instance, any lattice Γ in the 3-dimensional Lie group SOL has $\nu(\Gamma)$ empty (that is, $\nu(\Gamma)^c$ is the set of all ultrafilters), in spite of being finitely presented. Hence if we consider a family of groups (Λ_i) achieving all possible relation ranges (up to equivalence), then so does the family $(\Gamma \times \Lambda_i)$, but all these groups have ν empty and thus ν does not distinguish these groups.

Remark 3.7 (*On minimal presentations and relation range*). — Let G be a finitely generated group with a minimal presentation $G = \langle S \mid R \rangle$, meaning that no relator belongs to the normal subgroup generated by other relators. Then in general there is no relation between the relation range $\mathcal{R}_S(G)$ and the set $\mathcal{L}_R = \{|r|_S : r \in R\}$, in the sense that both inclusions $[\mathcal{R}_S(G)] \subset [\mathcal{L}_R]$ and $[\mathcal{L}_R] \subset [\mathcal{R}_S(G)]$ may fail:

- (1) Let H be a finitely generated densely related group having a minimal presentation $H = \langle S \mid (r_n) \rangle$, and form the group $G = H * \mathbf{Z}$. Since H is a retract of G , the group G remains densely related by Corollary 3.4. On the other hand G admits the presentation $G = \langle S, a \mid (a^{u_n} r_n a^{-u_n}) \rangle$ for arbitrary (u_n) , and this presentation remains minimal. In particular if (u_n) grows very fast, we have a densely related group with a minimal presentation such that \mathcal{L}_R is not \sim -equivalent to \mathbf{N} .
- (2) Consider the partial presentation of the lamplighter group

$$H = \left\langle t, x \mid [t^{(2n)!} x t^{-(2n)!}, x] = 1 \text{ for all } n \geq 1 \right\rangle.$$

Again form the group $G = H * \mathbf{Z}$, whose relation range is equivalent to $\{4(2n)!+4\} \sim \{(2n)!\}$ (Lemma 4.8). Denoting $r_n = [t^{(2n)!} x t^{-(2n)!}, x]$, consider the presentation $G = \langle t, x, a \mid a^{u_n} r_n a^{-u_n} = 1 \rangle$, where u_n is chosen so that the length of $a^{u_n} r_n a^{-u_n}$ is equal to $(2n+3)!$ for all $n \geq 1$. This presentation remains minimal, but the \sim -class of \mathcal{L}_R is not included inside the relation range of G .

3.2. Relation range of quotients

If N is a normal subgroup of a group G and S generates G , we denote by $\mathcal{X}_S(N, G)$ the set of integers n such that there is an element of N of length n outside the normal subgroup of G normally generated by elements of N of length at most $n - 1$.

PROPOSITION 3.8. — *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups. Assume that S is a generating subset of G such that G is boundedly presented over S . Then $\mathcal{R}_S(Q) = \mathcal{X}_S(N, G)$.*

Before giving the proof, let us derive the following consequence:

COROLLARY 3.9. — *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups. If G is finitely presented, then $\mathcal{R}(Q) \sim \mathcal{X}_S(N, G)$.*

Proposition 3.8 will follow from Lemmas 3.10 and 3.11.

LEMMA 3.10. — *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups, and let S be a generating subset of G . Then $\mathcal{X}_S(N, G) \subset \mathcal{R}_S(Q)$.*

Proof. — We denote by N_Q the kernel of $F_S \rightarrow Q$. Let $x \in N$ be an element of length $n \geq 1$ that is not in the normal closure in G of the set of elements of N of length at most $n - 1$. Let $w \in F_S$ of length n that represents x in G . Then w lies in N_Q , and we claim w does not belong to the normal closure in F_S of the set of elements of N_Q of length at most $n - 1$.

Argue by contradiction and assume that there is a decomposition $w = \prod \alpha_i r_i \alpha_i^{-1}$, where $r_i \in N_Q$ has length at most $n - 1$. If x_i (resp. a_i) is the element of G represented by r_i (resp. α_i), then by pushing the above decomposition in G we obtain $x = \prod a_i x_i a_i^{-1}$. Contradiction. \square

The following shows that the converse inclusion in the previous lemma also holds when the extension is boundedly presented.

LEMMA 3.11. — *Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups. Assume that S is a generating subset of G such that G is boundedly presented over S . Then $\mathcal{R}_S(Q) \subset \mathcal{X}_S(N, G)$.*

Proof. — Again we let N_Q (resp. N_G) be the kernel of $F_S \rightarrow Q$ (resp. $F_S \rightarrow G$). Let $r \geq 1$ such that N_G is normally generated by its elements of length at most r . Let $n \geq r + 1$ such that there is a word $w \in N_Q \subset F_S$ of length n , and such that w does not belong to the normal closure of elements of N_Q of length at most $n - 1$. We shall prove that n belongs to $\mathcal{X}_S(N, G)$.

We let x be the element represented by the word w in G . Then $x \in N$, and we claim that x does not belong to the subgroup normally generated

by elements of N of length at most $n - 1$. Indeed, assume that in G there is a decomposition $x = \prod a_i x_i a_i^{-1}$, where $a_i \in G$ and $x_i \in N$ has length at most $n - 1$. For every i , we let $w_i \in F_S$ representing x_i in G and such that $|w_i|_S \leq n - 1$, and we choose $\alpha_i \in F_S$ such that α_i represents a_i in G . Then $w(\prod \alpha_i w_i \alpha_i^{-1})^{-1} \in N_G$. Since N_G is normally generated by its elements of length at most r and $n - 1 \geq r$ by assumption, we deduce that w belongs to the normal closure of the set of elements $t \in N_Q$ such that $|t|_S \leq n - 1$. Contradiction. \square

3.3. Lacunary approximations

Given a locally compact group G , we call approximation of G a sequence of continuous surjective homomorphisms with discrete kernel $G_0 \rightarrow G_1 \rightarrow \dots$ and $G_i \rightarrow G$ such that the obvious diagrams commute, and such that the kernel N of $G_0 \rightarrow G$ is equal to the union of kernels N_i of $G_0 \rightarrow G_i$, and such that all G_i are compactly presented.

Given such an approximation, fix a compact generating subset of G_0 . For $i \geq 1$, let $\rho_i \in [1, +\infty]$ be the smallest length of an element in $N \setminus N_i$. Let s_i be the smallest number s such that N_i is normally generated by the intersection of N_i with the s -ball in G_0 ; then $s_i < \infty$ because G_i is compactly presented.

We say that the approximation is *lacunary* if $\rho_i/s_i \rightarrow +\infty$.

PROPOSITION 3.12. — *A compactly generated locally compact group is lacunary presented if and only if it admits a lacunary approximation. Moreover, in the discrete case, G_0 can be chosen to be free.*

Proof. — This is easy. Start from any compactly presented group G_0 such that G can be written as a quotient of G_0 with discrete kernel N (we can choose G_0 free in the discrete case); fix a compact generating subset of G_0 . First define H_n as the quotient of G_0 with the normal closure N_n of $N \cap B(n)$, where $B(n)$ is the n -ball in G_0 . Then $N = \bigcup N_n$ (ascending union).

Assume now that G is lacunary presented. Then for every integers $c, n_0 \geq 1$, we can find $n \geq n_0$ such that $[n, cn]$ contains no element of the relation range. Therefore, we can extract from (H_n) a subsequence satisfying the required conditions.

Conversely, the existence of a lacunary approximation implies that there is no new relation of size between s_i and ρ_i and since the ratio ρ_i/s_i tends to infinity, this implies that the relation range is not at finite multiplicative distance to \mathbf{N} . \square

Let us provide an application of lacunary approximations to the Hopfian property, extending the recent result by Coulon and Guirardel [14] that lacunary hyperbolic groups are Hopfian. Recall that a group is Hopfian if all its surjective endomorphisms are injective.

PROPOSITION 3.13. — *Let G be a finitely generated group with a lacunary approximation by Hopfian groups. Then G is Hopfian.*

Proof. — First, we can find an epimorphism from a free group G_{-1} to G_0 and then shift indices to assume that G_0 is free. Next, let f be a surjective endomorphism of G . Since G_0 is free, we can lift it to an endomorphism \hat{f} of G_0 ; then \hat{f} maps the 1-ball into the k -ball for some k . We can describe G_i as the quotient of G_0 by the normal closure of a certain subset R_i of the s_i -ball. Then $\hat{f}(R_i)$ belongs to the ks_i -ball and also belongs to the kernel of $G_0 \rightarrow G$. If i is large enough, say $i \geq i_0$, so that $\rho_i > ks_i$, it follows that $\hat{f}(R_i)$ belongs to the kernel of $G_0 \rightarrow G_i$. Thus for $i \geq i_0$, \hat{f} factors to an endomorphism f_i of G_i , still lifting f . Then since f is surjective, we can write generators as elements of the image in G , and lift this to G_i for, say $i \geq i_1 \geq i_0$. Thus f_i is a surjective endomorphism for all $i \geq i_1$. Since G_i is Hopfian, it follows that f_i is an automorphism for all $i \geq i_1$, and therefore f is an automorphism as well. \square

COROLLARY 3.14 (Coulon–Guirardel [14]). — *Finitely generated lacunary hyperbolic groups are Hopfian.*

Proof. — Sela proved that torsion-free hyperbolic groups are Hopfian [29], and this was more recently extended to arbitrary hyperbolic groups by Reinfeldt and Weidmann in the preprint [28]. Given that finitely generated lacunary hyperbolic groups admit lacunary approximations by hyperbolic groups [26, Th. 3.3], we conclude by Proposition 3.13. \square

Proposition 3.13 can also be applied beyond the lacunary hyperbolic case. For instance, those partial finite presentations of $\mathbf{Z} \wr \mathbf{Z}$ are easily seen to be residually finite (and hence Hopfian), so the lacunary presented groups obtained in this way are Hopfian groups.

3.4. Asymptotic cones of densely related groups

The following terminology and notation are essentially borrowed from [27]. We let $\mathbf{I}_2 = [0, 1] \times [0, 1]$ be the unit Euclidean square of dimension two, and denote by $\partial\mathbf{I}_2$ its boundary. A collection of squares D_1, \dots, D_k is defined to be a partition of \mathbf{I}_2 if $D_i \cap D_j = \partial D_i \cap \partial D_j$ whenever $i \neq j$, and if \mathbf{I}_2 is the union of the squares D_i .

If X is a geodesic metric space, a loop in X is by definition a continuous map $\alpha : \partial\mathbf{I}_2 \rightarrow X$, and we freely identify a loop with its image in X . A partition π of α is a continuous map extending α to $\partial D_1 \cup \dots \cup \partial D_k$, where D_1, \dots, D_k is a partition of \mathbf{I}_2 . We define the *mesh* of π as the maximal length of the paths $\pi(\partial D_i)$.

LEMMA 3.15. — *Let X be a geodesic metric space, and denote by $\mathcal{C} = \text{Cone}^\omega(X, (x_n), (s_n))$ an asymptotic cone of X . Then*

- (a) *any loop in \mathcal{C} is the ω -limit of a sequence of loops in X ;*
- (b) *if α is a loop in \mathcal{C} which is the ω -limit of a sequence of loops (α_n) in X , then any partition of α is the ω -limit of a sequence of partitions of α_n .*

Proof. — Statement (a) is proved in [22, Prop. 2.2] for paths rather than loops, but the proof can be easily adapted to realize any loop in \mathcal{C} as the ω -limit of a sequence of loops in X .

Statement (b) is obtained similarly, working in each square of the partition. □

Recall that by a theorem of Gromov [21, §5.F], if a finitely generated group G has all its asymptotic cones simply connected, then G is finitely presented (and has a polynomially bounded Dehn function). One cannot hope to obtain the same conclusion if we weaken the hypothesis by requiring that *one* asymptotic cone of G is simply connected, as for example any non-hyperbolic lacunary hyperbolic group is infinitely presented [26, Appendix]. Recall that a group is *lacunary hyperbolic* if it admits (at least) one asymptotic cone that is a real tree [26].

We next show that if G has one asymptotic cone that is simply connected, then the group G is lacunary presented. The proof follows the same strategy as the proof of the direct implication of Theorem 4.4 in [15], which says that simple connectedness of all asymptotic cones implies a certain division property for loops (see [15] for the relevant definition).

PROPOSITION 3.16. — *Let G be a compactly generated locally compact group, and (s_n) a sequence of positive numbers. Assume that the asymptotic cone $\text{Cone}^\omega(G, (s_n))$ is simply connected for some ultrafilter ω such that $\lim_\omega s_n = +\infty$. Then $\{s_n : n \in \mathbf{N}\} \not\subset \mathcal{R}(G)$.*

Proof. — Let S be a compact generating subset of G . We argue by contradiction, and assume that $\{s_n : n \in \mathbf{N}\} \subset \mathcal{R}(G)$. This implies that there exist a constant $c > 0$ and a sequence of relations $r_n \in F_S$ so that

$$c^{-1}s_n \leq |r_n|_S \leq cs_n,$$

and r_n is not generated by relations of smaller length. By construction, the sequence of loops $\alpha_n : \partial \mathbf{I}_2 \rightarrow \text{Cay}(G, S)$ parametrized proportionally to the length associated to r_n yields a loop $\alpha : \partial \mathbf{I}_2 \rightarrow \text{Cone}^\omega(G, (s_n))$. Since $\text{Cone}^\omega(G, (s_n))$ is supposed to be simply connected, the map $\alpha : \partial \mathbf{I}_2 \rightarrow \text{Cone}^\omega(G, (s_n))$ can be extended to a continuous function $\sigma : \mathbf{I}_2 \rightarrow \text{Cone}^\omega(G, (s_n))$. Now since \mathbf{I}_2 is compact, the map σ is uniformly continuous, and there exists $\eta > 0$ so that $d_\omega(\sigma(t), \sigma(u))$ is at most $c^{-1}/5$ as soon as the distance between t and u is at most η . Let us consider the partition of \mathbf{I}_2 given by the net

$$\{(a\eta, b\eta) : a, b \in \mathbf{Z}, 0 \leq a, b \leq 1/\eta\}.$$

Since the mesh of this partition is equal to 4η , the restriction of σ to this partition yields a partition of the loop α in $\text{Cone}^\omega(G, (s_n))$ of mesh at most $4c^{-1}/5$.

From a geometric point of view, the fact that r_n does not belong to the normal subgroup of F_S generated by relations of length at most $|r_n|_S - 1$ implies that the mesh of any partition of the loop α_n is at least $|r_n|_S$, so in particular at least $c^{-1}s_n$. Now we claim that this implies that the mesh of any partition of the loop α is at least c^{-1} . Indeed according to Lemma 3.15, any partition π of the loop α is the ω -limit of a sequence of partitions π_n of the loop α_n in $\text{Cay}(G, S)$. Being the limit over the ultrafilter ω of the mesh of π_n rescaled by s_n , the mesh of π is at least c^{-1} by the previous observation. This readily gives a contradiction with the previous paragraph. \square

COROLLARY 3.17. — *Let G be a densely related compactly generated group. Then none of the asymptotic cones of G are simply connected.*

Since a real tree is a simply connected metric space, Corollary 3.17 also admits the following consequence, which can also be derived from Proposition 3.12 using the combinatorial characterization of lacunary hyperbolic groups from [26] in the discrete case, and from [23] in the locally compact case.

COROLLARY 3.18. — *Any lacunary hyperbolic locally compact group is lacunary presented.*

Remark 3.19. — The class of lacunary presented groups is much larger than the class of lacunary hyperbolic groups. For instance, in the discrete case, it contains all finitely presented groups and is stable under direct powers and direct products with finitely presented groups (Corollary 4.9), which are essentially never lacunary hyperbolic (recall that being direct limits of hyperbolic groups, lacunary hyperbolic groups contain no copy of \mathbf{Z}^2).

Remark 3.20. — (See also Question 1.12) We do not know any example of a finitely generated linear group that is neither finitely presented nor densely

related. If it is true that every finitely generated infinitely presented linear group is densely related, then this implies according to Corollary 3.18 that every finitely generated lacunary hyperbolic group that is linear is actually a hyperbolic group (because a finitely presented lacunary hyperbolic group is hyperbolic [26, Appendix]). Whether this last assertion is true was asked in [26].

4. Specific classes of groups and examples

4.1. Classes of groups with various relation ranges

4.1.1. Lacunary hyperbolic groups

Recall from Corollary 3.18 that every finitely generated lacunary hyperbolic group is lacunary presented. The following proposition, the proof of which is a combination of works of Bowditch [9] and Olshanskii–Osin–Sapir [26], says that this is actually the only restriction on the relation range of a lacunary hyperbolic group.

PROPOSITION 4.1. — *For every subset I of \mathbf{N} with $I \approx \mathbf{N}$, there exists a lacunary hyperbolic group Γ with $\mathcal{R}(\Gamma) \sim I$.*

Proof. — Upon changing I into some $I' \sim I$, we may easily find a presentation $\Gamma = \langle a, b \mid (r_n)_n \rangle$ so that $\{|r_n| : n \in \mathbf{N}\} = I'$ and satisfying the $C'(1/7)$ small cancelation condition. This last property means that $\{r_n : n \in \mathbf{N}\}$ is a set of reduced words stable under taking cyclic conjugates, and such that for every $n \in \mathbf{N}$ and every $m \neq n$, the largest common prefix of r_n and r_m has length at most $(1/7)|r_n|$. The $C'(1/7)$ condition implies on the one hand that the relation range of the group Γ is given by the set of lengths of relators [9, Lem. 5], i.e. $\mathcal{R}(\Gamma) \sim I' \sim I$, and on the other hand that Γ is lacunary hyperbolic since $I' \approx \mathbf{N}$ [26, Prop. 3.12]. \square

4.1.2. Solvable groups

If G is a group and $\ell : G \rightarrow \mathbf{R}_+$ is a length function on G , we denote by $\mathcal{X}_\ell(G)$ the set of integers n such that there is $g \in G$ such that $\ell(g) = n$ and g does not belong to the subgroup of G generated by elements of length at most $n - 1$. Note that if Z is a central subgroup of a group G and S is a generating subset of G , then we have $\mathcal{X}_{\ell_S}(Z) = \mathcal{X}_S(Z, G)$ (see Section 3.2 for the definition of $\mathcal{X}_S(Z, G)$).

The proof of the following lemma is routine, and we omit it.

LEMMA 4.2. — *If ℓ_1, ℓ_2 are bi-Lipschitz equivalent length functions on G , then $\mathcal{X}_{\ell_1}(G) \sim \mathcal{X}_{\ell_2}(G)$.*

In the sequel we denote by R the ring $\mathbf{F}_p[t, t^{-1}, (t+1)^{-1}]$, and we consider Abels' group $A_4(R)$ consisting of upper triangular matrices of $\mathrm{GL}_4(R)$ whose first and last diagonal entries are equal to 1 (see [13]).

For $i \geq 0$, we denote by $z_i \in A_4(R)$ the element whose diagonal entries are equal to 1, and whose only non-zero off-diagonal entry has coordinate $(1, 4)$ and is equal to t^i . We note that every z_i lies in the center of $A_4(R)$. If $I \subset \mathbf{N}$, we let $Z_I = \langle z_i \rangle_{i \in I}$. The group Z_I is abelian and isomorphic to the group $\mathbf{F}_p^{(I)}$. Every $z \in Z_I$ can be uniquely represented as $\sum_{k \geq 0} a_k t^k$, where all but finitely many $a_k \in \mathbf{F}_p$ are equal to zero. We consider the length function on Z_I defined by $\ell(z) = \sup \{k : a_k \neq 0\}$.

LEMMA 4.3. — *We have $\mathcal{X}_\ell(Z_I) = I$.*

Proof. — We have $\ell(z_i) = i$ for every $i \in I$, and clearly z_i does not belong to the subgroup generated by the z_k for $k < i$. This shows $I \subset \mathcal{X}_\ell(Z_I)$, and the converse is clear since $\ell : Z_I \rightarrow \mathbf{R}_+$ take values in I . \square

Recall that Grigorchuk gave, using growth exponents, a continuum of pairwise non quasi-isometric finitely generated groups of intermediate growth [18]. In view of Corollary 3.4, the following proposition provides a continuum of quasi-isometry classes of finitely generated solvable groups, a fact that was established in [13] using asymptotic cones. This was also obtained in [10] using compression of embeddings into \mathbf{L}^p -spaces.

PROPOSITION 4.4. — *Let $I \subset \mathbf{N}$, and let Q_I be the quotient of the group $A_4(R)$ by its central subgroup Z_I . Then $\mathcal{R}(Q_I) \sim I$.*

Proof. — Let S be a finite generating subset of the group $A_4(R)$. Since the group $A_4(R)$ is finitely presented [13, Th. 5.1], we have $\mathcal{R}_S(Q_I) = \mathcal{X}_S(Z_I, G)$ according to Corollary 3.9. Now since Z_I is central in $A_4(R)$, it follows that $\mathcal{X}_S(Z_I, G) = \mathcal{X}_{\ell_S}(Z_I) = \mathcal{X}_\ell(Z_I)$, where the last equality follows from Lemma 4.2 (since the metrics ℓ and ℓ_S are bi-Lipschitz equivalent on Z_I , see for instance [13, §4.3]). Therefore $\mathcal{R}_S(Q_I) = \mathcal{X}_\ell(Z_I)$, and the conclusion then follows from Lemma 4.3. \square

Remark 4.5. — This completes a remark in the introduction. In [5], the class of extrinsic condensation group is studied: these are finitely generated groups G such that for every finitely presented group H and surjective homomorphism $p : H \rightarrow G$, the kernel of p contains uncountably many normal subgroups of H . This property is a strengthening of being infinitely presented but is not a quasi-isometry invariant, and is actually not even closed under taking finite index overgroups. For instance, if B is Abels' group (denoted A_4/Z in [5, Example 5.13]); here it is rather, for some prime p , the

quotient of $A_4(\mathbf{Z}[1/p])$ by its center $Z \simeq \mathbf{Z}[1/p]$, then $B \times B$ is an extrinsic condensation group but its overgroup of index 2 $B \wr \mathbf{Z}/2\mathbf{Z}$ is not.

4.2. Graph products and wreath products

We will make use of the following easy lemma.

LEMMA 4.6. — *Let G be a group, H a retract of G and $\varphi : G \rightarrow H$ be a homomorphism whose restriction to H is the identity. Assume that S_G is a generating subset of G , and write $S_H = \varphi(S_G)$. Then $\mathcal{R}_{S_H}(H) \subset \mathcal{R}_{S_G}(G)$.*

Proof. — Let $w \in F_{S_H}$ be a relation in the group H . If w admits a decomposition in F_{S_G} as a product of conjugates of elements of length $\leq n$, then by replacing each letter by its image by φ we obtain a decomposition of w in F_{S_H} as a product of conjugates of elements of length $\leq n$. It follows that if w is not generated by relations of smaller length in F_{S_H} , then the same holds in F_{S_G} , and the result is proved. \square

We derive the following consequence about group retracts, which also follows from Theorem 2.3.

COROLLARY 4.7. — *Let $G = N \rtimes Q$ be a semidirect product decomposition of locally compact groups, with G, Q compactly generated. Then $\mathcal{R}(Q) \subset \mathcal{R}(G)$. In particular, if G is lacunary presented then so is Q (or equivalently, if Q is densely related then so is G).*

Let $X = (V, E)$ be a graph, and let $(G_v)_{v \in V}$ be a family of groups indexed by the set of vertices of X . Recall that the graph product P associated to this data is the quotient of the free product of all G_v by the relations $[G_v, G_w] = 1$ whenever (v, w) is an edge of X .

LEMMA 4.8. — *Assume that S_v is a generating subset of G_v for every $v \in V$, and let S be the disjoint union of all S_v . Then*

$$\mathcal{R}_S(P) = \bigcup_{v \in V} \mathcal{R}_{S_v}(G_v).$$

Proof. — Each G_v is a retract of P , so by Lemma 4.6 we have $\mathcal{R}_{S_v}(G_v) \subset \mathcal{R}_S(P)$ for every $v \in V$. Now let w be a word in the elements of S of length ≥ 5 , such that w is a relation in P that is not generated by relations of smaller length. Remark that in P every relation that is not generated by relations of length at most 4 comes from relations in the G_v . Since moreover w is not generated by relations of smaller length w must be a relation in one of the G_v , which proves the converse inclusion. \square

COROLLARY 4.9. — *Let a compactly generated locally compact group G be a graph product of finitely many groups G_1, \dots, G_n . Then $\mathcal{R}(G) \sim \bigcup \mathcal{R}(G_i)$. In particular, if all G_i are lacunary presented then so is G . In particular again, the class of lacunary presented groups is closed under taking direct or free products with compactly presented locally compact groups, and under taking direct or free powers ($G \mapsto G^k$ or G^{*k}).*

Remark 4.10. — Similarly to what happen for lacunary hyperbolic groups [26, Example 3.16], the class of lacunary presented groups is not closed under direct or free products. For instance, consider the subset $A_1 = \bigcup_{n \geq 1} [(2n)!, (2n+1)!] (where intervals are understood to be within integers), and let A_2 be the complement of A_1 in \mathbf{N} . Then there exist finitely generated groups Γ_1, Γ_2 with $\mathcal{R}(\Gamma_i) \sim A_i$ (see the introduction, or Section 4.1.2). Then both Γ_1 and Γ_2 are lacunary presented, but using Corollary 4.9, $\Gamma_1 \times \Gamma_2$ and $\Gamma_1 * \Gamma_2$ are densely related.$

Recall that a standard wreath product $H \wr G$, with H non-trivial and G infinite, is never finitely presented [3]. The following strengthens this result by showing that such groups are actually densely related.

PROPOSITION 4.11. — *Let G, H be finitely generated groups such that H is non-trivial and G is infinite. Then the standard wreath product $H \wr G$ is densely related.*

Proof. — Let S be the union of finite generating subsets S_G and S_H of respectively G and H . For every $n \geq 0$, let us consider the graph structure X_n on G defined by putting an edge between g_1 and $g_2 \neq g_1$ if and only if $d_{S_G}(g_1, g_2) \leq n$. We denote by P_n the graph product associated to X_n for which all groups are equal to H . Since the action of G on itself preserves the graph structure X_n , we can consider the semidirect product $\Gamma_n = P_n \rtimes G$. By construction we have a surjective homomorphism $\Gamma_n \twoheadrightarrow \Gamma_{n+1}$ for every $n \geq 0$. Since (X_n) converges to the complete graph on G , the direct limit of the sequence (Γ_n) is exactly the wreath product $H \wr G$.

Fix a non-trivial $s \in S_H$. For every $n \geq 1$, we choose some element $g_n \in G$ whose length with respect to S_G is exactly n (such an element always exists because S_G is finite and G is infinite). Choose a word v_n in the elements of S_G of length n representing g_n in the group G , and write $w_n = [v_n s v_n^{-1}, s]$. By construction w_n is a relation of length $4n + 4$ in $H \wr G$. Since there is no edge in X_{n-1} between 1_G and g_n , the elements $g_n s g_n^{-1}$ and s generate their free product in P_{n-1} [11, Lem. 2.3(2)]. In particular the word w_n is not trivial in Γ_{n-1} . Now by remarking that every relation in $H \wr G$ of length at most $4n$ is a relation in Γ_{n-1} , we obtain that w_n cannot be generated by relations of length $\leq 4n$. Therefore $\mathcal{R}_S(H \wr G)$ contains an element between $4n$ and $4n + 4$, and the proof is complete. \square

4.3. Iterations of endomorphisms

Following [20], we consider the following situation. Let G be a group with a finite index normal subgroup H , and denote by $X = \{x_1, \dots, x_r\}$ a system of coset representatives, so that $G = x_1H \cup \dots \cup x_rH$. Assume we are given a set $\Phi = \{\varphi_i\}$ of surjective homomorphisms

$$\varphi_i : H \rightarrow G, \text{ for } i = 1, \dots, r,$$

with the following properties:

- (1) $\bigcap_{i=1}^r \ker(\varphi_i) \neq 1$.
- (2) for every $i, j \in \{1, \dots, r\}$, there exists $k \in \{1, \dots, r\}$ such that

$$\varphi_i \circ \mu_{x_j} = \varphi_k,$$

where $\mu_{x_j} : H \rightarrow H$ is the automorphism of H induced by the conjugation by x_j .

- (3) The image of $H \rightarrow G^r, h \mapsto (\varphi_1(h), \dots, \varphi_r(h))$, contains the diagonal $\Delta(G)$ in G^r .

Remark 4.12. — It would actually be enough for our purpose that the image of $H \rightarrow G^r$ contains the diagonal $\Delta(H)$ in $H^r \leq G^r$.

From now on we fix G, H and Φ satisfying (1), (2) and (3). We define inductively a sequence of subgroups by

$$K_0 = 1, \text{ and } K_{n+1} = \bigcap_{i=1}^r \varphi_i^{-1}(K_n)$$

for all $n \geq 0$.

LEMMA 4.13. — $K_n \leq H$ is a normal subgroup of G , and K_n is properly contained in K_{n+1} for all $n \geq 0$.

Proof. — That K_n is a subgroup of H and $K_n \leq K_{n+1}$ follow by an easy induction. Let us check that K_n is normal in G . We also proceed by induction, the case $n = 0$ being trivial. Assume that K_n is normal in G , and choose $h_{n+1} \in K_{n+1}$ and $g \in G$. Given $i \in \{1, \dots, r\}$, we want to see that $\varphi_i(gh_{n+1}g^{-1}) \in K_n$. Let j and $h \in H$ be such that $g = x_jh$. We have:

$$\varphi_i(gh_{n+1}g^{-1}) = \varphi_i \circ \mu_{x_j}(hh_{n+1}h^{-1}) = \varphi_k(hh_{n+1}h^{-1}),$$

where the integer k is provided by (2). Therefore we have

$$\varphi_i(gh_{n+1}g^{-1}) = \varphi_k(h)\varphi_k(h_{n+1})\varphi_k(h)^{-1},$$

and $\varphi_k(h_{n+1}) \in K_n$ since $h_{n+1} \in K_{n+1}$. Since K_n is normal in G by assumption, we conclude that $\varphi_i(gh_{n+1}g^{-1})$ belongs to K_n .

To prove that the inclusion $K_n \subset K_{n+1}$ is proper for all n , assume by contradiction that there is $n \geq 1$ such that $K_n = K_{n+1}$, and take $g \in K_n$. According to (3) we may find $h \in H$ such that $\varphi_i(h) = g$ for every i , which implies that h belongs to K_{n+1} since $g \in K_n$. Therefore $h \in K_n$ by our assumption, and it follows that the element g belongs to K_{n-1} . So $K_{n-1} = K_n$. By repeating the argument we obtain that $K_0 = K_1$, which is a contradiction with (1). \square

We denote by

$$K_\Phi = \bigcup_{n \geq 0} K_n$$

the increasing union of the subgroups K_n , which is therefore a normal subgroup of G , and by $G_\Phi = G/K_\Phi$ be the associated quotient. In other words, we have a sequence of groups and surjective homomorphisms

$$G \xrightarrow{\pi_0} G/K_1 \xrightarrow{\pi_1} \cdots \longrightarrow G/K_n \xrightarrow{\pi_n} \cdots$$

whose direct limit is G_Φ .

In the sequel we assume moreover that G is a compactly generated locally compact group, that H is closed in G and that all $\varphi_i : H \rightarrow G$ are continuous. Note that the continuity of the φ_i implies that the K_n are closed subgroups of G , but it may happen that K_Φ is not closed (as the example $H = G = \mathbf{S}^1$ and $\varphi(z) = z^2$ shows, for which K_Φ is dense). From now on we will assume that K_Φ is also closed in G , and hence G_Φ is a locally compact group.

Remark 4.14. — If there is a neighbourhood of the identity U such that $U \cap \ker(\varphi_i) = 1$ and $\varphi_i(U) \subset U$ for every i , then K_Φ is a discrete subgroup of G .

Since K_Φ is the increasing union of its closed subgroups K_n , by the Baire category theorem the subgroups K_n must eventually be open in K_Φ . In particular for every $k > 0$, the intersection between K_Φ and S^k is compact in K_Φ , and therefore must be contained in some subgroup K_{n_k} . Since K_{n_k} is a normal subgroup of G , this shows that the normal subgroup of G generated as such by elements of K_Φ of length at most k is contained in K_{n_k} . Since all the inclusions $K_n \subset K_{n+1}$ are strict (Lemma 4.13), this shows that K_Φ is not compactly generated as a normal subgroup of G , and therefore the group G_Φ is not compactly presented by Lemma 3.10.

THEOREM 4.15. — *Let G be a compactly generated locally compact group, and assume that $\Phi = \{\varphi_i\}$ satisfy (1), (2) and (3). Then the group $G_\Phi = G/K_\Phi$ is densely related.*

We will use the following lemma, the proof of which is standard.

LEMMA 4.16. — *Let S be a compact generating subset of G . Then there exists $c > 0$ such that for every $g \in G$, there exists $h \in H$ such that $\varphi_i(h) = g$ for every i and $|h|_S \leq c|g|_S$.*

Proof. — The image of $\varphi : H \rightarrow G^r$, $h \mapsto (\varphi_1(h), \dots, \varphi_r(h))$, contains the diagonal in G^r thanks to (3). Let $\Sigma \subset H$ be a compact subset of H such that $\varphi(\Sigma) = S$ (where we identify S and its diagonal embedding), and let $c = \sup |\sigma|_S$, for $\sigma \in \Sigma$. Since Σ is compact, c is finite. Now let $g \in G$, and let $\ell = |g|_S$, so that there exist $s_1, \dots, s_\ell \in S$ such that $g = s_1 \cdots s_\ell$. For all k , let $\sigma_k \in \Sigma$ such that $\varphi(\sigma_k) = s_k$, and write $h = \sigma_1 \cdots \sigma_\ell$. Then h satisfies the conclusion, because each φ_i maps h to g , and $|h|_S \leq \sum |\sigma_i|_S \leq c\ell = c|g|_S$. \square

Proof of Theorem 4.15. — Let S be a compact generating subset of G , and $\pi : F_S \rightarrow G$ the canonical projection. For $k \geq 0$, let $N^{(k)} = \pi^{-1}(K_k)$. Note that $N^{(k)}$ is a normal subgroup of F_S in view of Lemma 4.13. The increasing union $N^{(\infty)}$ of the subgroups $N^{(k)}$ is nothing but the kernel of the natural map from F_S to the group G_Φ :

$$1 \longrightarrow N^{(\infty)} \longrightarrow F_S \longrightarrow G_\Phi \longrightarrow 1.$$

For every $k \geq 1$ we let $N_k^{(\infty)}$ be the normal subgroup of F_S generated as such by elements of $N^{(\infty)}$ of length at most k .

Fix an integer $k \geq 1$. Denote by n_k the smallest integer so that the set of elements of $N^{(\infty)}$ of word length at most k lies in $N^{(n_k)}$. The discussion before the proposition shows that such an integer n_k always exists. Note that since $N^{(n_k)}$ is a normal subgroup, we have $N_k^{(\infty)} \subset N^{(n_k)}$.

By definition of n_k , there must exist a word w in $N^{(\infty)} \setminus N^{(n_k-1)}$ of length at most k . Set $g = \pi(w)$. According to (3), there exists $h \in H$ such that $\varphi_i(h) = g$ for every i . Let $w' \in F_S$ such that $h = \pi(w')$. Then we claim that $w' \in N^{(\infty)}$. Indeed, for every i we have $\varphi_i(\pi(w')) = \varphi_i(h) = g = \pi(w)$, so if n is such that $w \in N^{(n)}$, then $w' \in N^{(n+1)}$. On the other hand the word w' does not belong to $N^{(n_k)}$, because otherwise we would have $w \in N^{(n_k-1)}$. A fortiori w' does not belong to $N_k^{(\infty)}$ since $N_k^{(\infty)} \subset N^{(n_k)}$. This implies that the relation range $\mathcal{R}_S(G_\Phi)$ contains an element between k and $|w'|_S$.

Now we may plainly choose $w' \in F_S$ such that $|w'|_S = |h|_S$, and according to Lemma 4.16 we may also choose $h \in H$ such that $|h|_S \leq c|g|_S$ for some constant c depending only on S . But $g = \pi(w)$, so the word length of g with respect to S is at most $|w|_S$. Since $|w|_S \leq k$ by definition of w , it follows that $|w'|_S \leq ck$. Therefore $\mathcal{R}_S(G_\Phi) \cap [c, ck] \neq \emptyset$, and the statement is proved. \square

4.3.1. The Grigorchuk group

We consider the Grigorchuk group \mathfrak{G} introduced in [17].

THEOREM 4.17. — *The Grigorchuk group \mathfrak{G} is densely related.*

Proof. — Let $G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle$ be the free product $C_2 * (C_2 \times C_2)$, and H be its subgroup of index two which projects trivially on the first C_2 . H is generated by b, c, d and their conjugates by a . The assignments

$$\varphi_0 = (b \mapsto a, c \mapsto a, d \mapsto 1, aba \mapsto c, aca \mapsto c, ada \mapsto b)$$

and

$$\varphi_1 = \varphi_0 \circ \mu_a = (b \mapsto c, c \mapsto d, d \mapsto b, aba \mapsto a, aca \mapsto a, ada \mapsto 1)$$

extend to homomorphisms from H onto G .

It is well known that the group \mathfrak{G} is generated by four elements a, b, c, d satisfying the relations defining G , so that we have a surjective map $\pi : G \rightarrow \mathfrak{G}$. The image of H is the subgroup \mathfrak{H} of \mathfrak{G} stabilizing the first level of the rooted tree on which \mathfrak{G} acts. Moreover there exist morphisms $\psi_0, \psi_1 : \mathfrak{H} \rightarrow \mathfrak{G}$ such that the diagrams

$$\begin{array}{ccc} H & \xrightarrow{\varphi_i} & G \\ \downarrow \pi & & \downarrow \pi \\ \mathfrak{H} & \xrightarrow{\psi_i} & \mathfrak{G} \end{array}$$

commute ([19, §2]).

We shall check that the assumptions of Theorem 4.15 are satisfied. It is a simple verification that $\Phi = \{\varphi_0, \varphi_1\}$ satisfies (1), and (2) is clear since $\varphi_1 = \varphi_0 \circ \mu_a$ by definition. Now to see that (3) is satisfied, by the commutativity of the above diagrams it is enough to see that the image of $\psi = (\psi_0, \psi_1) : \mathfrak{H} \rightarrow \mathfrak{G} \times \mathfrak{G}$ contains the diagonal embedding of \mathfrak{G} . This later fact follows for instance from Proposition 1 in [19]. Therefore by applying Theorem 4.15, we deduce that the group G_Φ is densely related. The statement follows, since the group G_Φ is precisely the group \mathfrak{G} [20, Th. 4] (see also [19]). \square

Combined with Corollary 3.17, Theorem 4.17 implies the following result.

COROLLARY 4.18. — *No asymptotic cone of the Grigorchuk group is simply connected.*

We shall point out that an explicit infinite presentation of the group \mathfrak{G} has been given by Lysionok in [24], and this presentation has been shown to be minimal by Grigorchuk [19]. However we stress out that in general

the relation range cannot be seen on the lengths of relators even for a minimal presentation, see Remark 3.7. The presentation of the group \mathfrak{G} from [24] is an example of a finite endomorphic presentation in the sense of Bartholdi [2]. These are presentations which are obtained by successive iterations of (finitely many) endomorphisms of the free group to a finite initial subset of relators. However the proof of Theorem 4.15 does not seem to apply directly to every group with a finite endomorphic presentation (even in the case when the endomorphisms of the free group induce endomorphisms of the group associated to the endomorphic presentation). We do not know whether every such group is either finitely presented or densely related.

The group \mathfrak{G} is an example of group that is generated by a finite state automaton. As a preliminary version of this article was circulating, the following conjecture has been communicated to us by L. Bartholdi.

CONJECTURE 4.19 (Bartholdi). — *If a group G is generated by a finite state automaton, then G is either finitely presented or densely related.*

4.3.2. Non-Hopfian groups

A particular case of the construction carried out in Section 4.3 is when G is a non-Hopfian group, $H = G$ and $\Phi = \{\varphi\}$, where $\varphi : G \rightarrow G$ is a surjective, non-injective endomorphism of G .

COROLLARY 4.20. — *Let G be a compactly generated locally compact group, and assume that $\varphi : G \rightarrow G$ is a continuous, surjective, non-injective endomorphism of G such that K_φ is closed in G . Then the group $G_\varphi = G/K_\varphi$ is densely related.*

Proof. — The assumption (1) is satisfied because φ is non-injective. There is nothing to check for (2) in this case, and (3) also holds here because φ is surjective. The conclusion therefore follows from Theorem 4.15. \square

Here are examples of groups that can be obtained as direct limits of non-Hopfian groups by iterating a non-injective endomorphism. It follows from Corollary 4.20 that all the groups appearing in the following examples are densely related.

Example 4.21. — Let $r \geq 1$, and let $m, n \geq 2$ be two coprime integers. Let us consider the group $C(m, n, r)$ defined by the presentation

$$C(m, n, r) = \langle x, t \mid t^r x^m t^{-r} = x^n, [xt^{-1}, x] = \dots = [t^{r-1} x t^{-(r-1)}, x] = 1 \rangle.$$

This family of groups appeared in [4] as a generalization of Baumslag–Solitar groups $BS(m, n)$, which correspond to the case $r = 1$. When $m, n \geq 2$ are

chosen relatively prime, the endomorphism given by $\varphi(x) = x^m$ and $\varphi(t) = t$ is well defined, surjective but non-injective. In this case the limit group $C(m, n, r)_\varphi$ is the metabelian group $\mathbf{Z}[1/mn]^r \rtimes_M \mathbf{Z}$, where the action is defined by the $r \times r$ companion matrix

$$M = \begin{pmatrix} 0 & \cdots & 0 & n/m \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

Let us now recall the construction from [25] of non-Hopfian HNN-extensions. Let A be a group and let $\mu, \nu : A \rightarrow A$ be two injective and non-surjective endomorphisms. Assume that

- (a) $\mu \circ \nu = \nu \circ \mu$;
- (b) $\langle \mu(A), \nu(A) \rangle = A$;
- (c) $\exists \alpha \notin \mu(A), \beta \notin \nu(A); [\nu(\alpha), \mu(\beta)] = 1$;

and consider the HNN-extension of A associated to the subgroups $\mu(A)$ and $\nu(A)$

$$G = \langle A, t \mid t\mu(a)t^{-1} = \nu(a) \quad \forall a \in A \rangle.$$

The following was proved in [25, Lem. 1].

LEMMA 4.22. — *The endomorphism $\varphi : G \rightarrow G$ defined by $\varphi(t) = t$ and $\varphi(a) = \mu(a)$ for every $a \in A$, is surjective but non-injective.*

Example 4.23. — Let $\mathbf{K}_i, i = 1, 2$ be ultrametric local fields. Let \mathcal{O}_i be the unique maximal compact subring in \mathbf{K}_i . Let π_i be an element in \mathbf{K}_i with $0 < |\pi_i| < 1$. Let us take $A = \mathcal{O}_1 \times \mathcal{O}_2$ and $\mu(x, y) = (x, \pi_2 y)$ and $\nu(x, y) = (\pi_1 x, y)$.

The HNN-extension G defined above inherits a locally compact topology for which the inclusion of A in G is continuous and open. Since $\varphi(A) \subset A$ and φ is injective on A , by Remark 4.14 the subgroup K_φ is a discrete subgroup of G . A simple computation shows that the group G_φ is the group $(\mathbf{K}_1 \times \mathbf{K}_2) \rtimes \mathbf{Z}$, where the action is the multiplication by (π_1, π_2^{-1}) .

Example 4.24. — Discrete analogues of the construction of Example 4.23 may be carried out, for example with $A = \mathbf{Z}^2$ and $\mu(x, y) = (x, ny), \nu(x, y) = (nx, y)$ for some integer $n \geq 2$. Here G_φ is the metabelian group $\mathbf{Z}[1/n]^2 \rtimes \mathbf{Z}$, where the action is the multiplication by (n, n^{-1}) .

Interesting examples also arise with A non-abelian. Let for instance $A = H_3(\mathbf{Z})$ be the Heisenberg group, and μ, ν defined by

$$\begin{aligned} \mu &: \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & pz \\ & 1 & py \\ & & 1 \end{pmatrix}, \\ \nu &: \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & pz \\ & 1 & y \\ & & 1 \end{pmatrix}, \end{aligned}$$

where p is a prime. Then one can check that the group G_φ is the group

$$A_3(\mathbf{Z}[1/p]) = \begin{pmatrix} 1 & \mathbf{Z}[1/p] & \mathbf{Z}[1/p] \\ & p^{\mathbf{Z}} & \mathbf{Z}[1/p] \\ & & 1 \end{pmatrix}.$$

4.4. Finitely generated groups with a homomorphism to \mathbf{Z}

4.4.1. Preliminaries

We say that an automorphism $\alpha : G \rightarrow G$ contracts into a subgroup H if the sequence of subgroups $(\alpha^{-n}(H))_{n \geq 0}$ is increasing and ascends to G . If we have a homomorphism $\pi : G \rightarrow \mathbf{Z}$, we say that the action of \mathbf{Z} contracts into a finitely generated subgroup of G if there is $t \in G$ such that $\pi(t)$ is a generator of \mathbf{Z} and the conjugation by t contracts into a finitely generated subgroup of G . This is equivalent to saying that π splits ascendingly over a finitely generated subgroup, i.e. that there is a decomposition of G as an ascending HNN-extension whose associated homomorphism $G \rightarrow \mathbf{Z}$ is equal to π .

We now introduce some notation. Consider a finitely generated group $G = M \rtimes \mathbf{Z}$, and let $S = \{m_1, \dots, m_\ell\} \subset M$ and t a generator for \mathbf{Z} such that $S \cup \{t\}$ generates G . In the free group F freely generated by the m_i and t , we write $m_i^{[k]} = t^k m_i t^{-k}$. Let R be the kernel of $F \rightarrow G$, and R_n the normal subgroup of F generated by the intersection of R and the n -ball.

For $u \leq v$ let $M_{[u,v]}$ be the subgroup of M generated by $\bigcup_{u \leq k \leq v} t^k S t^{-k}$. For every $n \geq 1$, let G_n be the HNN-extension of $M_{[0,n]}$ along the isomorphism $M_{[0,n-1]} \rightarrow M_{[1,n]}$ given by conjugation by t . The inclusions $M_{[0,n]} \rightarrow M_{[0,n+1]}$ induce surjective homomorphisms $G_n \rightarrow G_{n+1}$. Let R'_n be the kernel of the natural map $F \rightarrow G_n$.

LEMMA 4.25. — *We have $R_{2n+3} \subset R'_n$.*

Note that typically, we have elements of the form $[m, t^{n+1}mt^{-n-1}]$ in R_{2n+4} that are not in R'_n , so the lemma is optimal.

Proof. — Consider an element w in the $(2n+3)$ -ball of F that is a relation in G . The number of occurrences of t and its inverse is even. First suppose this is $2n + 2$: then up to cyclic conjugation, this element has the form $t^{2n+2}m$ and this is a contradiction. Hence it is at most $2n$. We can view the occurrences of t as a loop in \mathbf{Z} of length $\leq 2n$; hence its diameter in \mathbf{Z} is at most n . Therefore, after conjugating w (and possibly increasing its length), we can suppose that this path in \mathbf{Z} has range in $[0, n]$; this means that $w \in R'_n$. \square

4.4.2. Abelian-by-cyclic groups

In this paragraph we prove that a finitely generated abelian-by-cyclic group that is not finitely presented must be densely related. Although a more general result will be obtained later, here we include an elementary proof for this class of groups.

LEMMA 4.26. — *Assume that the action of \mathbf{Z} does not contract into any finitely generated subgroup of M . Then for every $n \geq 1$ there exist i, j such that $m_i \notin M_{[1, n]}$ and $m_j^{[n]} \notin M_{[0, n-1]}$.*

Proof. — Argue by contradiction and assume that for all i , we have $m_i \in M_{[1, n]}$. This implies that $M_{[1, n]} = M_{[0, n]}$, and hence G is an ascending HNN-extension over a finitely generated subgroup of M , a contradiction.

Similarly, if $m_j^{[n]} \in M_{[0, n-1]}$ for all j , then $M_{[0, n-1]} = M_{[0, n]}$ and again G is an ascending HNN-extension, resulting to a similar contradiction. \square

PROPOSITION 4.27. — *Assume that the action of \mathbf{Z} does not contract into any finitely generated subgroup of M . Then for every $n \geq 1$ there exist i, j such that $[m_i, m_j^{[n+1]}]$ is not contained in R'_n .*

Proof. — Let i, j be as in the conclusion of Lemma 4.26. Suppose that $[m_i, m_j^{[n+1]}]$ is a relation in G_n . We rewrite it as the equality in G_n

$$m_i t m_j^{[n]} t^{-1} m_i^{-1} t (m_j^{[n]})^{-1} t^{-1} = 1.$$

Once we view $m_i^{[n]}$ as an element of $M_{[0, n]}$, the Britton lemma implies that this expression is not reduced (with respect to the given HNN decomposition of G_n). This means that either $m_i \in M_{[1, n]}$, or $m_j^{[n]} \in M_{[0, n-1]}$, and both cases contradict the definition of i and j .

Hence $[m_i, m_j^{[n+1]}]$ is not a relation in G_n , which exactly means that $[m_i, m_j^{[n+1]}]$ does not belong to R'_n . \square

COROLLARY 4.28. — *Let $G = M \rtimes \mathbf{Z}$ be an abelian-by-cyclic finitely generated group. Then G is either finitely presented or densely related.*

Proof. — We assume that G is not finitely presented, and we show that G is densely related. The action of \mathbf{Z} cannot contract into a finitely generated subgroup of M , because otherwise G would be an ascending HNN-extension over a finitely presented subgroup because M is abelian. Therefore G would be finitely presented, which is a contradiction. So we may apply Proposition 4.27, which, combined with Lemma 4.25, says that the relation range of G with respect to $S \cup \{t\}$ contains an element of the interval $[2n + 4, 4n + 8]$ for every $n \geq 1$. This implies the statement. \square

4.4.3. Groups satisfying a law

In this paragraph we deal with the larger class of groups G satisfying a law. Recall that this means that there exists a non-trivial reduced word $w(x_1, \dots, x_k)$ in some letters x_1, \dots, x_k such that $w(g_1, \dots, g_k) = 1$ for all $g_1, \dots, g_k \in G$. Note that if we have $G = M \rtimes \mathbf{Z}$, then G satisfies a law if and only if M satisfies a law.

We follow a similar approach as in the previous paragraph, except that $[m_i, m_j^{[n+1]}]$ need not be a relation anymore. We need the following simple lemma.

LEMMA 4.29. — *Consider a group amalgam $A *_C B$, and $a \in A \setminus C$, $b \in B \setminus C$. Then for every $k \in \mathbf{Z} \setminus \{0\}$, we have $(ab)^k \notin A \cup B$.*

If moreover we have, for all $k \in \mathbf{Z} \setminus \{0\}$, $a^k \notin C$ and $b^k \notin C$, then a, b freely generate a free subgroup.

Proof. — For $k \geq 1$ and $b_0 \in B$, we can write $(ab)^k b_0^{-1} = abab \dots a(bb_0^{-1})$, which is a reduced word, hence it represents a non-trivial element by the Britton lemma. So $(ab)^k \notin B$, and the same proof shows that $(ab)^k \notin A$.

Under the stronger assumption, it first follows that a, b are not torsion. Hence if they satisfy a non-trivial relation, it can be chosen of the form $a^{m_1} b^{n_1} \dots a^{m_r} b^{n_r}$ with $r \geq 1$ and m_i, n_i nonzero integers. The stronger assumption again implies that this word is reduced. \square

LEMMA 4.30. — *Let $n \geq 1$ such that there exist i, j so that $m_i \notin M_{[1, n]}$ and $m_j^{[n]} \notin M_{[0, n-1]}$. Then*

$$a = m_i^{[-2]} m_j^{[n-1]} \quad \text{and} \quad b = m_i m_j^{[n+1]}$$

freely generate a free subgroup in G_n .

Proof. — G_n being a HNN-extension, it is a semidirect product $N \rtimes \mathbf{Z}$ with N being a 2-sided iterated amalgam

$$\dots M_{[-2,n-2]} *_{M_{[-1,n-2]}} M_{[-1,n-1]} *_{M_{[0,n-1]}} M_{[0,n]} *_{M_{[1,n]}} M_{[1,n+1]} *_{M_{[2,n+1]}} \dots,$$

which thus is the amalgam of two 1-sided iterated amalgams:

$$\begin{aligned} & (\dots M_{[-2,n-2]} *_{M_{[-1,n-2]}} M_{[-1,n-1]}) \\ & *_{M_{[0,n-1]}} (M_{[0,n]} *_{M_{[1,n]}} M_{[1,n+1]} *_{M_{[2,n+1]}} \dots) =: A *_C B. \end{aligned}$$

Here $a = m_i^{[-2]} m_j^{[n-1]}$ belongs to $M_{[-2,n-2]} *_{M_{[-1,n-2]}} M_{[-1,n-1]}$. Note that by assumption $m_i^{[-2]} \in M_{[-2,n-2]} \setminus M_{[-1,n-2]}$ and $m_j^{[n-1]} \in M_{[-1,n-1]} \setminus M_{[-1,n-2]}$, so it follows from the first item in Lemma 4.29 that for all $k \in \mathbf{Z} \setminus \{0\}$, we have $a^k \notin M_{[-2,n-2]} \cup M_{[-1,n-1]} = M_{[-2,n-1]}$.

The same argument applied to $b = m_i m_j^{[n+1]} \in M_{[0,n]} *_{M_{[1,n]}} M_{[1,n+1]}$ shows that $b^k \notin M_{[0,n+1]}$ for all $k \in \mathbf{Z} \setminus \{0\}$. In particular neither a^k nor b^k belongs to $M_{[0,n-1]}$, and the second item in Lemma 4.29 implies that a, b freely generate a free subgroup. \square

The following is the main result of this paragraph.

PROPOSITION 4.31. — *Consider a finitely generated group $G = M \rtimes \mathbf{Z}$ satisfying a law. Then either the action of \mathbf{Z} contracts into a finitely generated subgroup of M , or the group G is densely related.*

Proof. — Let $w(x_1, \dots, x_k)$ be a reduced word in some letters x_1, \dots, x_k such that $w(g_1, \dots, g_k) = 1$ for every $g_1, \dots, g_k \in G$, and let ℓ be the length of w .

Assume that the action of \mathbf{Z} does not contract into any finitely generated subgroup of M . Lemmas 4.26 and 4.30 ensure the existence, for every $n \geq 1$, of elements $a_n, b_n \in F$, both of length $2n + 4$, whose images in G_n freely generate a free subgroup. For simplicity we still denote a_n and b_n the corresponding elements of G_n . Then one easily check that $a_n b_n, a_n^2 b_n^2, \dots, a_n^k b_n^k$ freely generate a free subgroup of rank k in G_n . In particular the word $w(a_n b_n, \dots, a_n^k b_n^k)$ does not belong to R'_n , and by Lemma 4.25 it does not belong to R_{2n+3} either. But now by definition of w , the word $w(a_n b_n, \dots, a_n^k b_n^k)$ belongs to R . It has length at most $2k\ell(2n + 4)$, so the relation range of G with respect to $S \cup \{t\}$ must contain an element in the interval $[2n + 4, 2k\ell(2n + 4)]$. \square

We derive from Proposition 4.31 the following consequence. Recall that a group is called coherent if every finitely generated subgroup is finitely presented. Examples of coherent groups are locally nilpotent groups, or locally finite groups.

COROLLARY 4.32. — *Let $G = M \rtimes \mathbf{Z}$ be a finitely generated group, where M is coherent. If G satisfies a law, then G is either finitely presented or densely related.*

Proof. — Assume that G is not densely related. Then by Proposition 4.31 the action of \mathbf{Z} contracts into a finitely generated subgroup of M , and G is an ascending HNN-extension over a finitely presented group because M is coherent. This clearly implies that G is finitely presented. \square

COROLLARY 4.33. — *Every nilpotent-by-cyclic finitely generated group is either finitely presented or densely related.*

Remark 4.34. — We shall point out that Corollary 4.33 extends neither to nilpotent-by-abelian groups according to Proposition 4.4, nor to (locally nilpotent)-by-cyclic groups in view of the examples from [26, §3.5].

4.4.4. Metabelian groups

The following important theorem, the proof of which is a combination of results of Bieri–Strebel [7, 8] and Bieri–Groves [6], characterizes infinitely presented groups among finitely generated metabelian groups. This result was explained and used in [5] to determine the finitely generated metabelian groups that are of extrinsic condensation (see [5] for this terminology).

THEOREM 4.35 (Bieri–Groves–Strebel). — *Let G be a finitely generated metabelian group. Then G admits a homomorphism onto \mathbf{Z} that does not contract into any finitely generated subgroup if and only if G is not finitely presented.*

The following is the main result of this paragraph.

THEOREM 4.36. — *Let G be a finitely generated metabelian group. Then G is either finitely presented or densely related.*

Proof. — If G is not finitely presented, then Theorem 4.35 ensures the existence of a homomorphism from G onto \mathbf{Z} that does not contract into any finitely generated subgroup of G . Now since G is metabelian, we may plainly apply Proposition 4.31, from which the conclusion follows. \square

The dichotomy appearing in Theorem 4.36 actually extends to the larger class of center-by-metabelian finitely generated groups.

COROLLARY 4.37. — *Let G be a center-by-metabelian finitely generated group. Then G is either finitely presented or densely related.*

Proof. — Let Z be a central subgroup of G such that $Q = G/Z$ is metabelian. If Q is finitely presented then Z has to be finitely generated by Lemma 3.10, and therefore finitely presented since Z is abelian. So G lies in an extension of finitely presented groups, and is therefore finitely presented.

Now assume that Q is not finitely presented. Since Q is metabelian, by Theorem 4.35 there is a homomorphism $Q \rightarrow \mathbf{Z}$ that does not contract into any finitely generated subgroup of Q . This homomorphism may clearly be lifted to a homomorphism $G \rightarrow \mathbf{Z}$ with the same property, and since G is solvable, Proposition 4.31 implies that G is densely related. \square

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