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A Lagrangian Neighbourhood Theorem for shifted symplectic derived schemes ^(*)

DOMINIC JOYCE ⁽¹⁾ AND PAVEL SAFRONOV ⁽²⁾

ABSTRACT. — Pantev, Toën, Vaquié and Vezzosi [19] defined k -shifted symplectic derived schemes and stacks \mathbf{X} for $k \in \mathbb{Z}$, and Lagrangians $f : \mathbf{L} \rightarrow \mathbf{X}$ in them. They have important applications to Calabi–Yau geometry and quantization. Bussi, Brav and Joyce [7] and Bouaziz and Grojnowski [5] proved “Darboux Theorems” giving explicit Zariski or étale local models for k -shifted symplectic derived schemes \mathbf{X} for $k < 0$ presenting them as twisted shifted cotangent bundles.

We prove a “Lagrangian Neighbourhood Theorem” which gives explicit Zariski or étale local models for Lagrangians $f : \mathbf{L} \rightarrow \mathbf{X}$ in k -shifted symplectic derived schemes \mathbf{X} for $k < 0$, relative to the “Darboux form” local models of [7] for \mathbf{X} . That is, locally such Lagrangians can be presented as twisted shifted conormal bundles. We also give a partial result when $k = 0$.

We expect our results will have future applications to shifted Poisson geometry [12], and to defining “Fukaya categories” of complex or algebraic symplectic manifolds, and to the categorification of Donaldson–Thomas theory of Calabi–Yau 3-folds and “Cohomological Hall Algebras”.

RÉSUMÉ. — Pantev, Toën, Vaquié et Vezzosi [19] ont défini des schémas et des champs dérivés symplectiques k -décalés \mathbf{X} pour $k \in \mathbb{Z}$, et des Lagrangiens $f : \mathbf{L} \rightarrow \mathbf{X}$ en eux. Ils ont des applications importantes pour la géométrie Calabi–Yau et la quantification. Bussi, Brav et Joyce [7] et Bouaziz et Grojnowski [5] ont prouvé des « théorèmes de Darboux » donnant des modèles locaux précis Zariski ou étale pour les schémas dérivés symplectiques k -décalés \mathbf{X} pour $k < 0$, les présentant comme des fibrés cotangent décalés tordus.

Nous prouvons un « théorème de voisinage Lagrangien » donnant des modèles locaux précis Zariski ou étale pour les Lagrangiens $f : \mathbf{L} \rightarrow \mathbf{X}$ dans les schémas dérivés symplectiques k -décalés \mathbf{X} pour $k < 0$, par rapport à la « forme Darboux » de Bussi–Brav–Joyce pour \mathbf{X} . C’est-à-dire, localement, ces Lagrangiens peuvent être présentés sous forme de fibrés conormaux décalés tordus. Nous donnons aussi un résultat partiel lorsque $k = 0$.

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Article proposé par Bertrand Toën.

Nous espérons que nos résultats auront de futures applications à la géométrie de Poisson k -décalée de [12], à la définition de « catégories de Fukaya » de variétés symplectiques complexes ou algébriques, à la catégorification de la théorie de Donaldson–Thomas des variétés de Calabi–Yau de dimension 3, et au « Algèbres de Hall Cohomologiques ».

1. Introduction

Using Toën and Vezzosi’s theory of Derived Algebraic Geometry [23, 24, 25, 27, 26], Pantev, Toën, Vaquié and Vezzosi [19] defined k -shifted symplectic structures $\omega_{\mathbf{X}}$ on a derived scheme or stack \mathbf{X} , for $k \in \mathbb{Z}$. If \mathbf{X} is a derived scheme and $\omega_{\mathbf{X}}$ a 0-shifted symplectic structure, then $\mathbf{X} = X$ is a smooth classical scheme and $\omega_{\mathbf{X}} \in H^0(\Lambda^2 T^* X)$ a classical symplectic structure on X . They proved that if Y is a Calabi–Yau m -fold then derived moduli stacks \mathcal{M} of (complexes of) coherent sheaves on Y have natural $(2 - m)$ -shifted symplectic structures $\omega_{\mathcal{M}}$.

Pantev et al. [19] also defined *Lagrangians* $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in a k -shifted symplectic derived stack $(\mathbf{X}, \omega_{\mathbf{X}})$, and showed that fibre products $\mathbf{L} \times_{\mathbf{X}} \mathbf{M}$ of Lagrangians $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$, $\mathbf{g} : \mathbf{M} \rightarrow \mathbf{X}$ are $(k - 1)$ -shifted symplectic. Calaque [11] proved that if X is a Fano $(m + 1)$ -fold and $Y \subseteq X$ a smooth anticanonical divisor, so that Y is a Calabi–Yau m -fold, and \mathcal{L}, \mathcal{M} are derived moduli stacks of (complexes of) coherent sheaves on X, Y with derived restriction morphism $\mathbf{f} : \mathcal{L} \rightarrow \mathcal{M}$, then \mathcal{L} is Lagrangian in the $(2 - m)$ -shifted symplectic $(\mathcal{M}, \omega_{\mathcal{M}})$.

Recently, Calaque, Pantev, Toën, Vaquié and Vezzosi [12] have also developed a related theory of k -shifted Poisson structures $\pi_{\mathbf{X}}$ on a derived scheme or stack \mathbf{X} , for $k \in \mathbb{Z}$, and *coisotropics* $\mathbf{f} : \mathbf{C} \rightarrow \mathbf{X}$ in $(\mathbf{X}, \pi_{\mathbf{X}})$. They prove [12, Th. 3.2.4] that the spaces of k -shifted symplectic structures $\omega_{\mathbf{X}}$ and nondegenerate k -shifted Poisson structures $\pi_{\mathbf{X}}$ on \mathbf{X} are equivalent. Costello–Rozenblyum and Pridham [21] have also announced similar results.

For a symplectic manifold (X, ω) , the classical Darboux Theorem chooses local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on X with $\omega = \sum_{j=1}^n d_{\text{dR}} x_j d_{\text{dR}} y_j$. Bussi, Brav and Joyce [7, Th. 5.18] proved a “ k -shifted symplectic Darboux Theorem”, which for a k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ with $k < 0$ chooses a *cdga* A^\bullet , a Zariski open inclusion $\mathbf{i} : \mathbf{Spec} A^\bullet \hookrightarrow \mathbf{X}$, and coordinates $x_j^i, y_j^{k-i} \in A^\bullet$ with $\mathbf{i}^*(\omega_{\mathbf{X}}) \simeq (\omega^0, 0, 0, \dots)$ for $\omega^0 = \sum_{i,j} d_{\text{dR}} x_j^i d_{\text{dR}} y_j^{k-i}$. (Actually, all this only holds for $k \not\equiv 2 \pmod{4}$, and for $k \equiv 2 \pmod{4}$ there is a more complicated expression also involving coordinates $z_j^{k/2}$.)

This was the foundation for a series of papers [2, 3, 4, 9, 8, 6, 7, 10, 13] concerning generalizations of Donaldson–Thomas theory for Calabi–Yau 3- and 4-folds, involving perverse sheaves, motives, and new enumerative invariants. It can also be used as part of a proof that k -shifted symplectic derived schemes carry nondegenerate k -shifted Poisson structures, though this was not used in [12, 21].

Given a Lagrangian $L \hookrightarrow X$ in a symplectic manifold (X, ω) , the classical Lagrangian Neighbourhood Theorem describes L, X, ω in local coordinates.

The purpose of this paper is to prove a “ k -shifted symplectic Lagrangian Neighbourhood Theorem”, Theorem 3.7 below, which given a Lagrangian $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in a k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$, and a “Darboux form” local description $\mathbf{i} : \mathbf{Spec} A^\bullet \hookrightarrow \mathbf{X}$, $x_j^i, y_j^{k-i} \in A^\bullet$, $\omega^0 = \sum_{i,j} d_{\mathrm{dR}} x_j^i d_{\mathrm{dR}} y_j^{k-i}$ for $(\mathbf{X}, \omega_{\mathbf{X}})$ as in [7], chooses a cdga B^\bullet , coordinates $\tilde{x}_j^i, u_j^i, v_j^{k-1-i} \in B^\bullet$, a Zariski open inclusion $\mathbf{j} : \mathbf{Spec} B^\bullet \hookrightarrow \mathbf{X}$, and a cdga morphism $\alpha : A^\bullet \rightarrow B^\bullet$ with $\tilde{x}_j^i = \alpha(x_j^i)$ in a homotopy commutative diagram

$$\begin{array}{ccc} \mathbf{Spec} B^\bullet & \xrightarrow{\quad} & \mathbf{L} \\ \downarrow \mathbf{Spec} \alpha & \mathbf{j} & \mathbf{f} \downarrow \\ \mathbf{Spec} A^\bullet & \xrightarrow{\quad \mathbf{i} \quad} & \mathbf{X}, \end{array} \tag{1.1}$$

such that the pullback $\mathbf{j}^*(h_{\mathbf{L}})$ of the Lagrangian structure $h_{\mathbf{L}}$ on $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ to $\mathbf{Spec} \alpha : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ using (1.1) has $\mathbf{j}^*(h_{\mathbf{L}}) \simeq (h^0, 0, 0, \dots)$ with $h^0 = \sum_{i,j} d_{\mathrm{dR}} u_j^i d_{\mathrm{dR}} v_j^{k-1-i}$. (Actually, all this only holds for $k \not\equiv 3 \pmod{4}$, and for $k \equiv 3 \pmod{4}$ there is a more complicated expression also involving coordinates $w_j^{(k-1)/2}$.) Theorem 3.11 also gives a partial result for $k = 0$.

Bouaziz and Grojnowski [5] proved their own k -shifted symplectic Darboux Theorem independently of [7], showing that a k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$ is (at least for $k \not\equiv 2 \pmod{4}$) étale locally equivalent to a *twisted k -shifted cotangent bundle* $T_t^*[k]\mathbf{Y}$, where \mathbf{Y} is an affine derived \mathbb{K} -scheme, and $t \in \mathcal{O}_{\mathbf{Y}}^{k+1}$ with $dt = 0$ is used to “twist” the k -shifted cotangent bundle $T^*[k]\mathbf{Y}$. Remark 2.15 below relates their picture to that of [7].

In Remark 3.4 we interpret our “ k -shifted Lagrangian Neighbourhood Theorem” in the style of Bouaziz and Grojnowski [5], by saying that if $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ is Lagrangian in a k -shifted symplectic $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$, and \mathbf{X} is locally modelled on $T_t^*[k]\mathbf{Y}$, then $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ is (at least for $k \not\equiv 3 \pmod{4}$) locally modelled on the inclusion morphism $N_{u/t}^*[k](\mathbf{Z}/\mathbf{Y}) \rightarrow T_t^*[k]\mathbf{Y}$, where $N_{u/t}^*[k](\mathbf{Z}/\mathbf{Y})$ is the *twisted k -shifted conormal bundle* of a morphism of

affine derived \mathbb{K} -schemes $\mathbf{g} : \mathbf{Z} \rightarrow \mathbf{Y}$, and $u \in \mathcal{O}_{\mathbf{Z}}^k$ with $du = -\mathbf{g}^*(t)$ is used to “twist” $N^*[k](\mathbf{Z}/\mathbf{Y})$.

If the k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ is a point $(\mathbf{Spec} \mathbb{K}, 0)$ then Lagrangians $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ are just $(k - 1)$ -shifted symplectic derived \mathbb{K} -schemes $(\mathbf{L}, \omega_{\mathbf{L}})$. In this case, our Lagrangian Neighbourhood Theorem reduces to the Darboux Theorem of [7]. So the proof in Section 4 is a generalization of that in [7, §5.6], and runs parallel to [7] at several points.

Like the Darboux Theorem of [5, 7], our Lagrangian Neighbourhood Theorem should have important applications. For example, it gives local models for moduli schemes of coherent sheaves on Fano $(m + 1)$ -folds X with restriction morphisms to moduli schemes of coherent sheaves on a Calabi–Yau anticanonical hypersurface $Y \subset X$. We briefly discuss some conjectures which we hope our theorem will help to prove.

CONJECTURE 1.1. — *Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a (-1) -shifted symplectic derived \mathbb{C} -scheme with an “orientation”. Then Bussi, Brav, Dupont, Joyce, and Szendrői [6, Cor. 6.11] construct a natural perverse sheaf $\mathcal{P}_{\mathbf{X}, \omega_{\mathbf{X}}}^{\bullet}$ on $X = t_0(\mathbf{X})$, such that if $(\mathbf{X}, \omega_{\mathbf{X}})$ is locally modelled on a critical locus $\mathbf{Crit}(\Phi : U \rightarrow \mathbb{A}^1)$, then $\mathcal{P}_{\mathbf{X}, \omega_{\mathbf{X}}}^{\bullet}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}_{U, \Phi}^{\bullet}$.*

Suppose $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ is a Lagrangian, with an “orientation” relative to that of \mathbf{X} , and \mathbf{f} is proper. Then we can define a natural element $\lambda_{\mathbf{L}}$ in the hypercohomology $\mathbb{H}^{\dim \mathbf{L}}(\mathcal{P}_{\mathbf{X}, \omega_{\mathbf{X}}}^{\bullet})$. These $\lambda_{\mathbf{L}}$ satisfy certain composition laws for composition of Lagrangian correspondences.

The first author has an outline of a proof of Conjecture 1.1.

As suggested in [6, Rem. 6.15], we would like to define a “Fukaya category” $\mathcal{F}(S)$ of (derived) complex or algebraic Lagrangians $L \rightarrow S$ in a complex or algebraic symplectic manifold (S, ω) of dimension $2n$, such that if L, M are oriented Lagrangians in S then the morphisms $L \rightarrow M$ in $\mathcal{F}(S)$ are $\mathrm{Hom}^*(L, M) = \mathbb{H}^{*-n}(\mathcal{P}_{L, M}^{\bullet})$, where $\mathcal{P}_{L, M}^{\bullet}$ is the perverse sheaf on the -1 -shifted symplectic $\mathbf{X} = L \times_S M$ described above.

As in Ben-Bassat [2], if L, M, N are (derived) Lagrangians in (S, ω) , then $\mathbf{Y} = L \times_S M \times_S N \rightarrow (L \times_S M) \times (M \times_S N) \times (N \times_S L)$ is Lagrangian in -1 -shifted symplectic. The hypercohomology class $\lambda_{\mathbf{Y}}$ associated to this in Conjecture 1.1 is what we need to define composition $\mathrm{Hom}^*(M, N) \times \mathrm{Hom}^*(L, M) \rightarrow \mathrm{Hom}^*(L, N)$ of morphisms in the “Fukaya category” $\mathcal{F}(S)$. Amorim and Ben-Bassat [1] discuss this proposal and Conjecture 1.1 in detail.

A stacky version of Conjecture 1.1 is what we need to define multiplication in a “Cohomological Hall Algebra” associated to a Calabi–Yau 3-fold in the sense of Kontsevich–Soibelman [15], defined using the perverse sheaves on Calabi–Yau 3-fold moduli stacks constructed in Ben-Bassat, Bussi, Brav and Joyce [3].

CONJECTURE 1.2. — *Let U be a smooth \mathbb{K} -scheme and $\Phi : U \rightarrow \mathbb{A}^1$ a regular function. Then the derived critical locus $\mathbf{Crit}(\Phi)$ is a -1 -shifted symplectic derived \mathbb{K} -scheme. We can also define the \mathbb{Z}_2 -graded dg-category of matrix factorizations $\mathbf{MF}(U, \Phi)$, as in Preygel [20] for instance.*

Suppose $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{Crit}(\Phi)$ is a Lagrangian, with $\mathrm{vdim} \mathbf{L} - \dim U$ even, equipped with an “orientation” and a “spin structure”, and that \mathbf{f} is proper. Then we can define an object $\mu_{\mathbf{L}} \in \mathbf{MF}(U, \Phi)$ associated to \mathbf{L} . In this way we interpret $\mathbf{MF}(U, \Phi)$ as a kind of “Fukaya category” of the -1 -shifted symplectic derived \mathbb{K} -scheme $\mathbf{Crit}(\Phi)$.

This is connected to the programme of Kapustin and Rozansky [14] for associating a 2-category to a complex symplectic manifold, locally described using matrix factorization categories.

For each of the Conjectures 1.1–1.2, using our Lagrangian Neighbourhood Theorem we can write down local models on \mathbf{L} for the coisotropic structure, and for $\lambda_{\mathbf{L}}$ and $\mu_{\mathbf{L}}$. The problem is to glue these local models together globally.

We begin in Section 2 with background material on Derived Algebraic Geometry and Pantev–Toën–Vaquié–Vezzosi’s shifted symplectic geometry. Section 3 gives our main results. Theorem 3.2 in Section 3.1 shows that a morphism $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ of derived \mathbb{K} -schemes is locally modelled on $\mathbf{Spec} \alpha : \mathbf{Spec} A^\bullet \rightarrow \mathbf{Spec} B^\bullet$, where A^\bullet, B^\bullet are cdgas and $\alpha : B^\bullet \rightarrow A^\bullet$ a morphism, all in a particularly nice form.

Theorem 3.7 in Section 3.3 is our “Lagrangian Neighbourhood Theorem”, showing that Lagrangians $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in k -shifted symplectic derived \mathbb{K} -schemes $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$ are locally modelled on explicit “Lagrangian Darboux form” examples given in Examples 3.3 and 3.5 in Section 3.2. Theorem 3.11 in Section 3.4 also gives a partial result for $k = 0$. Section 4 proves Theorems 3.2 and 3.7.

Conventions. — Throughout \mathbb{K} will be an algebraically closed field with characteristic zero. All classical \mathbb{K} -schemes are assumed locally of finite type, and all derived \mathbb{K} -schemes \mathbf{X} are assumed to be locally finitely presented. Our sign conventions for cdgas, exterior forms, etc., follow Bussi, Brav and Joyce [7].

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2. Background material

We begin with some background material and notation needed later. Some references are Toën and Vezzosi [23, 24, 25, 26, 27] for Sections 2.1–2.2, and Pantev, Toën, Vezzosi and Vaquié [19] for Sections 2.3–2.4, and Brav, Bussi and Joyce [7] for Section 2.5. Throughout the paper, \mathbb{K} will be an algebraically closed field of characteristic zero.

2.1. Commutative differential graded algebras

DEFINITION 2.1. — Write $\mathbf{cdga}_{\mathbb{K}}$ for the category of commutative differential graded \mathbb{K} -algebras in nonpositive degrees, and $\mathbf{cdga}_{\mathbb{K}}^{\text{op}}$ for its opposite category. Objects of $\mathbf{cdga}_{\mathbb{K}}$ are of the form $\cdots \rightarrow A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^0$. Here A^k for $k = 0, -1, -2, \dots$ is the \mathbb{K} -vector space of degree k elements of A^\bullet , and we have a \mathbb{K} -bilinear, associative, supercommutative multiplication $\cdot : A^k \times A^l \rightarrow A^{k+l}$ for $k, l \leq 0$, an identity $1 \in A^0$, and differentials $d : A^k \rightarrow A^{k+1}$ for $k < 0$ satisfying

$$d(a \cdot b) = (da) \cdot b + (-1)^k a \cdot (db)$$

for all $a \in A^k, b \in A^l$. We write such objects as A^\bullet or (A^*, d) .

Here and throughout we will use the superscript “ $*$ ” to denote graded objects (e.g. graded algebras or vector spaces), where $*$ stands for an index in \mathbb{Z} , so that A^* means $(A^k, k \in \mathbb{Z})$. We will use the superscript “ \bullet ” to denote differential graded objects (e.g. differential graded algebras or complexes), so that A^\bullet means (A^*, d) , the graded object A^* together with the differential d .

Morphisms $\alpha : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ are \mathbb{K} -linear maps $\alpha^k : A^k \rightarrow B^k$ for all $k \leq 0$ commuting with all the structures on A^\bullet, B^\bullet . A morphism $\alpha : A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism if $H^k(\alpha) : H^k(A^\bullet) \rightarrow H^k(B^\bullet)$ is an isomorphism on cohomology groups for all $k \leq 0$.

Remark 2.2. — A fundamental principle of derived algebraic geometry is that $\mathbf{cdga}_{\mathbb{K}}$ is not really the right category to work in, but instead one wants to define a new category (or better, ∞ -category) by inverting (localizing) quasi-isomorphisms in $\mathbf{cdga}_{\mathbb{K}}$.

In fact $\mathbf{cdga}_{\mathbb{K}}$ has the additional structure of a simplicial model category, with weak equivalences quasi-isomorphisms, in which all objects are fibrant, and in which $\mathbf{cdgas} A^\bullet$ with A^* free as a commutative graded \mathbb{K} -algebra are cofibrant. The n -simplices of the mapping space between two $\mathbf{cdgas} A^\bullet$ and B^\bullet are given by morphisms $A^\bullet \rightarrow B^\bullet \otimes \Omega^\bullet(\Delta^n)$, where $\Omega^\bullet(\Delta^n)$ is the \mathbf{cdga} generated by elements s_i of degree 0 and t_i of degree 1 for $i = 0, \dots, n$ with the relations $\sum s_i = 1$ and $\sum t_i = 0$ and the differential $ds_i = t_i$. Note that $\Omega^\bullet(\Delta^n)$ are concentrated in positive degrees, and are not elements of $\mathbf{cdga}_{\mathbb{K}}$.

We will write $\mathbf{cdga}_{\mathbb{K}}^\infty$ for the associated ∞ -category, so that the homotopy category $\mathrm{Ho}(\mathbf{cdga}_{\mathbb{K}}^\infty)$ is the localized category $\mathbf{cdga}_{\mathbb{K}}[Q^{-1}]$ with quasi-isomorphisms inverted, an ordinary category. We will not go into any detail about model categories and ∞ -categories below, but here is some basic orientation on one issue relevant to this paper, for readers unfamiliar with these ideas. The objects of $\mathbf{cdga}_{\mathbb{K}}$, $\mathbf{cdga}_{\mathbb{K}}^\infty$, $\mathrm{Ho}(\mathbf{cdga}_{\mathbb{K}}^\infty)$ are the same. If A^\bullet, B^\bullet are objects, a morphism $\phi : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ is also a morphism in $\mathbf{cdga}_{\mathbb{K}}^\infty$ and $\mathrm{Ho}(\mathbf{cdga}_{\mathbb{K}}^\infty)$. However, a morphism $\phi^\infty : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}^\infty$ (or equivalently, in $\mathrm{Ho}(\mathbf{cdga}_{\mathbb{K}}^\infty)$) need not correspond to any morphism $\phi : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$, unless A^\bullet is cofibrant. If A^\bullet is cofibrant, the mapping space in $\mathbf{cdga}_{\mathbb{K}}^\infty$ is given by the mapping space in $\mathbf{cdga}_{\mathbb{K}}$.

Standard model $\mathbf{cdgas} A^\bullet$ are “nearly cofibrant”. They have the property that if $\phi^\infty : A^\bullet \rightarrow B^\bullet$ is a morphism in $\mathbf{cdga}_{\mathbb{K}}^\infty$ with A^\bullet standard model, such that $H^0(\phi^\infty) : H^0(A^\bullet) \rightarrow H^0(B^\bullet)$ can be lifted to a \mathbb{K} -algebra morphism $\phi^0 : A^0 \rightarrow B^0$, then ϕ^∞ can be lifted to $\phi : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$.

All this will be important because if $X \simeq \mathbf{Spec} A^\bullet$ and $Y \simeq \mathbf{Spec} B^\bullet$ are affine derived \mathbb{K} -schemes and $f : Y \rightarrow X$ is a morphism, then $f \simeq \mathbf{Spec} \phi^\infty$ for some morphism $\phi^\infty : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}^\infty$. For our Lagrangian Neighbourhood Theorem in Section 3.3, we want to lift ϕ^∞ to $\phi : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$.

DEFINITION 2.3. — *Let $A^\bullet \in \mathbf{cdga}_{\mathbb{K}}$, and write $D(\mathrm{mod} A)$ for the derived category of dg -modules over A^\bullet . Define a derivation of degree k from A^\bullet to an A^\bullet -module M^\bullet to be a \mathbb{K} -linear map $\delta : A^\bullet \rightarrow M^\bullet$ that is homogeneous of degree k with*

$$\delta(fg) = \delta(f)g + (-1)^{k \deg f} f\delta(g).$$

Just as for ordinary commutative algebras, there is a universal derivation into an A^\bullet -module of Kähler differentials $\Omega_{A^\bullet}^1$, which can be constructed as I/I^2 for $I = \mathrm{Ker}(m : A^\bullet \otimes A^\bullet \rightarrow A^\bullet)$. The universal derivation $\delta : A^\bullet \rightarrow \Omega_{A^\bullet}^1$ is $\delta(a) = a \otimes 1 - 1 \otimes a \in I/I^2$. One checks that δ is a universal degree 0 derivation, so that $\circ\delta : \mathrm{Hom}_{A^\bullet}(\Omega_{A^\bullet}^1, M^\bullet) \rightarrow \mathrm{Der}^\bullet(A, M^\bullet)$ is an isomorphism of dg -modules.

Note that $\Omega_{A^\bullet}^1 = ((\Omega_{A^\bullet}^1)^*, d)$ is canonical up to strict isomorphism, not just up to quasi-isomorphism of complexes, or up to equivalence in $D(\text{mod } A)$. Also, the underlying graded vector space $(\Omega_{A^\bullet}^1)^*$, as a module over the graded algebra A^* , depends only on A^* and not on the differential d in $A^\bullet = (A^*, d)$.

Similarly, given a morphism of cdgas $\Phi : A^\bullet \rightarrow B^\bullet$, we can define the relative Kähler differentials $\Omega_{B^\bullet/A^\bullet}^1$.

The cotangent complex \mathbb{L}_{A^\bullet} of A^\bullet is related to the Kähler differentials $\Omega_{A^\bullet}^1$, but is not quite the same. If $\Phi : A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism of cdgas over \mathbb{K} , then $\Phi_* : (\Omega_{A^\bullet}^1) \otimes_{A^\bullet} B^\bullet \rightarrow \Omega_{B^\bullet}^1$ may not be a quasi-isomorphism of B^\bullet -modules. So Kähler differentials are not well-behaved under localizing quasi-isomorphisms of cdgas, which is bad for doing derived algebraic geometry.

The cotangent complex \mathbb{L}_{A^\bullet} is a substitute for $\Omega_{A^\bullet}^1$ which is well-behaved under localizing quasi-isomorphisms. It is an object in $D(\text{mod } A)$, canonical up to equivalence. We can define it by replacing A^\bullet by a quasi-isomorphic, cofibrant cdga B^\bullet , and then setting $\mathbb{L}_{A^\bullet} = (\Omega_{B^\bullet}^1) \otimes_{B^\bullet} A^\bullet$. We will be interested in the p^{th} exterior power $\Lambda^p \mathbb{L}_{A^\bullet}$, and the dual $(\mathbb{L}_{A^\bullet})^\vee$, which is called the tangent complex, and written $\mathbb{T}_{A^\bullet} = (\mathbb{L}_{A^\bullet})^\vee$.

There is a de Rham differential $d_{\text{dR}} : \Lambda^p \mathbb{L}_{A^\bullet} \rightarrow \Lambda^{p+1} \mathbb{L}_{A^\bullet}$, a morphism of complexes, with $d_{\text{dR}}^2 = 0 : \Lambda^p \mathbb{L}_{A^\bullet} \rightarrow \Lambda^{p+2} \mathbb{L}_{A^\bullet}$. Note that each $\Lambda^p \mathbb{L}_{A^\bullet}$ is also a complex with its own internal differential $d : (\Lambda^p \mathbb{L}_{A^\bullet})^k \rightarrow (\Lambda^p \mathbb{L}_{A^\bullet})^{k+1}$, and d_{dR} being a morphism of complexes means that $d \circ d_{\text{dR}} = d_{\text{dR}} \circ d$.

Similarly, given a morphism of cdgas $\Phi : A^\bullet \rightarrow B^\bullet$, we can define the relative cotangent complex $\mathbb{L}_{B^\bullet/A^\bullet}$.

DEFINITION 2.4. — Following [7, Def. 2.9], we will call $A^\bullet \in \mathbf{cdga}_{\mathbb{K}}$ of standard form if A^0 is a smooth finitely generated \mathbb{K} -algebra, and the cotangent module $\Omega_{A^0}^1$ is a free A^0 -module of finite rank, and the graded \mathbb{K} -algebra A^* is freely generated over A^0 by finitely many generators, all in negative degrees.

More explicitly, as A^0 is a smooth \mathbb{K} -algebra, $U = \mathbf{Spec} A^0$ is a smooth \mathbb{K} -scheme. Suppose that U admits étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$. Then $\Omega_{A^0}^1 \cong A^0 \otimes_{\mathbb{K}} \langle d_{\text{dR}} x_1^0, \dots, d_{\text{dR}} x_{m_0}^0 \rangle_{\mathbb{K}}$ is a free A^0 -module of rank m_0 . Suppose we are given elements $x_1^i, \dots, x_{m_i}^i$ in A^i for $i = -1, -2, \dots, k$, such that $A^* = A^0[x_j^i : i = -1, \dots, k, j = 1, \dots, m_i]$ is the graded \mathbb{K} -algebra freely generated over A^0 by the generators x_j^i in degree $i < 0$. Then $A^\bullet = (A^*, d)$ is a standard form cdga. The differential d on A^* is determined uniquely by the elements $dx_j^i \in A^{i+1}$ for $i = -1, -2, \dots, k$ and $j = 1, \dots, m_i$.

The virtual dimension of A^\bullet is $\text{vdim } A^\bullet = \sum_{i=0}^d (-1)^i m_i \in \mathbb{Z}$.

Then the Kähler differentials $\Omega_{A^\bullet}^1$ are given as an A^* -module by

$$\Omega_{A^\bullet}^1 \cong A^* \otimes_{\mathbb{K}} \langle d_{\mathrm{dR}} x_j^i : i = 0, -1, \dots, k, j = 1, \dots, m_i \rangle_{\mathbb{K}}. \quad (2.1)$$

As in [7, §2.3], an important property of standard form cdgas A^\bullet is that they are sufficiently cofibrant that the Kähler differentials $\Omega_{A^\bullet}^1$ provide a model for the cotangent complex \mathbb{L}_{A^\bullet} , so we can take $\Omega_{A^\bullet}^1 = \mathbb{L}_{A^\bullet}$, without having to replace A^\bullet by an unknown cdga B^\bullet . Thus standard form cdgas are convenient for doing explicit computations with cotangent complexes.

We say that a standard form cdga A^\bullet is minimal at $p \in \mathbf{Spec} A^\bullet$ if all the differentials in the complex of \mathbb{K} -vector spaces $\Omega_{A^\bullet}^1|_p$ are zero. This means that $m_i = \dim H^i(\mathbb{L}_{A^\bullet}|_p)$ for $i = 0, -1, \dots, d$, and A^\bullet is defined using the minimum number of variables x_j^i in each degree $i = 0, -1, \dots$, compared to all other cdgas locally equivalent to A^\bullet near p .

2.2. Derived algebraic geometry and derived schemes

DEFINITION 2.5. — Write $\mathbf{dSt}_{\mathbb{K}}$ for the ∞ -category of derived \mathbb{K} -stacks (or D^- -stacks) defined by Toën and Vezzosi [27, Def. 2.2.2.14], [23, Def. 4.2]. Objects \mathbf{X} in $\mathbf{dSt}_{\mathbb{K}}$ are ∞ -functors

$$\mathbf{X} : \{\text{simplicial commutative } \mathbb{K}\text{-algebras}\} \longrightarrow \{\text{simplicial sets}\}$$

satisfying sheaf-type conditions. There is a spectrum functor

$$\mathbf{Spec} : (\mathbf{cdga}_{\mathbb{K}}^{\infty})^{\mathrm{op}} \longrightarrow \mathbf{dSt}_{\mathbb{K}}.$$

A derived \mathbb{K} -stack \mathbf{X} is called an affine derived \mathbb{K} -scheme if \mathbf{X} is equivalent in $\mathbf{dSt}_{\mathbb{K}}$ to $\mathbf{Spec} A^\bullet$ for some cdga A^\bullet over \mathbb{K} . As in [23, §4.2], a derived \mathbb{K} -stack \mathbf{X} is called a derived \mathbb{K} -scheme if it may be covered by Zariski open $\mathbf{Y} \subseteq \mathbf{X}$ with \mathbf{Y} an affine derived \mathbb{K} -scheme. Write $\mathbf{dSch}_{\mathbb{K}}$ for the full ∞ -subcategory of derived \mathbb{K} -schemes in $\mathbf{dSt}_{\mathbb{K}}$, and $\mathbf{dSch}_{\mathbb{K}}^{\mathrm{aff}} \subset \mathbf{dSch}_{\mathbb{K}}$ for the full ∞ -subcategory of affine derived \mathbb{K} -schemes. Then \mathbf{Spec} is an equivalence $(\mathbf{cdga}_{\mathbb{K}}^{\infty})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{dSch}_{\mathbb{K}}^{\mathrm{aff}}$.

We shall assume throughout this paper that all derived \mathbb{K} -schemes \mathbf{X} are locally finitely presented in the sense of Toën and Vezzosi [27, Def. 1.3.6.4].

With this assumption, derived schemes have a virtual dimension $\mathrm{vdim} \mathbf{X}$, which is a locally constant function $\mathrm{vdim} \mathbf{X} : \mathbf{X} \rightarrow \mathbb{Z}$. If $\mathbf{X} = \mathbf{Spec} A^\bullet$ for A^\bullet a standard form cdga then $\mathrm{vdim} \mathbf{X} = \mathrm{vdim} A^\bullet$, for $\mathrm{vdim} A^\bullet$ as in Definition 2.4.

There is a classical truncation functor $t_0 : \mathbf{dSch}_{\mathbb{K}} \rightarrow \mathbf{Sch}_{\mathbb{K}}$ taking a derived \mathbb{K} -scheme \mathbf{X} to the underlying classical \mathbb{K} -scheme $X = t_0(\mathbf{X})$. On

affine derived schemes $\mathbf{dSch}_{\mathbb{K}}^{\text{aff}}$ this maps $t_0 : \mathbf{Spec} A^\bullet \mapsto \mathbf{Spec} H^0(A^\bullet) = \mathbf{Spec}(A^0/d(A^{-1}))$.

Toën and Vezzosi show that a derived \mathbb{K} -scheme \mathbf{X} has a cotangent complex $\mathbb{L}_{\mathbf{X}}$ [27, §1.4], [23, §4.2.4–4.2.5] in a stable ∞ -category $L_{\text{qcoh}}(\mathbf{X})$ defined in [23, §3.1.7, §4.2.4]. We will be interested in the p^{th} exterior power $\Lambda^p \mathbb{L}_{\mathbf{X}}$, and the dual $(\mathbb{L}_{\mathbf{X}})^\vee$, which is called the tangent complex $\mathbb{T}_{\mathbf{X}}$.

By a point of a derived \mathbb{K} -scheme \mathbf{X} , written $x \in \mathbf{X}$, we will always mean that $x \in X(\mathbb{K})$ is a \mathbb{K} -point of the underlying classical \mathbb{K} -scheme $X = t_0(\mathbf{X})$.

When $\mathbf{X} = X$ is a classical scheme, the homotopy category of $L_{\text{qcoh}}(\mathbf{X})$ is the triangulated category $D_{\text{qcoh}}(X)$ of complexes of quasicoherent sheaves. These have the usual properties of (co)tangent complexes. For instance, if $f : \mathbf{X} \rightarrow \mathbf{Y}$ is a morphism in $\mathbf{dSch}_{\mathbb{K}}$ there is a distinguished triangle

$$f^*(\mathbb{L}_{\mathbf{Y}}) \xrightarrow{\mathbb{L}f} \mathbb{L}_{\mathbf{X}} \longrightarrow \mathbb{L}_{\mathbf{X}/\mathbf{Y}} \longrightarrow f^*(\mathbb{L}_{\mathbf{Y}})[1],$$

where $\mathbb{L}_{\mathbf{X}/\mathbf{Y}}$ is the relative cotangent complex of f .

Now suppose A^\bullet is a cdga over \mathbb{K} , and \mathbf{X} a derived \mathbb{K} -scheme with $\mathbf{X} \simeq \mathbf{Spec} A^\bullet$ in $\mathbf{dSch}_{\mathbb{K}}$. Then we have an equivalence of triangulated categories $\text{Ho}(L_{\text{qcoh}}(\mathbf{X})) \simeq D(\text{mod } A^\bullet)$, which identifies cotangent complexes $\mathbb{L}_{\mathbf{X}} \simeq \mathbb{L}_{A^\bullet}$. If also A^\bullet is of standard form then $\mathbb{L}_{A^\bullet} \simeq \Omega_{A^\bullet}^1$, so $\mathbb{L}_{\mathbf{X}} \simeq \Omega_{A^\bullet}^1$.

Bussi, Brav and Joyce [7, Th. 4.1] prove:

THEOREM 2.6. — *Suppose \mathbf{X} is a derived \mathbb{K} -scheme (as always, assumed locally finitely presented), and $x \in \mathbf{X}$. Then there exist a standard form cdga A^\bullet over \mathbb{K} which is minimal at $p \in \mathbf{Spec} A^\bullet$, in the sense of Definition 2.4, and a Zariski open inclusion $\mathbf{i} : \mathbf{Spec} A^\bullet \hookrightarrow \mathbf{X}$ with $\mathbf{i}(p) = x$.*

They also explain [7, Th. 4.2] how to compare two such standard form charts $\mathbf{Spec} A^\bullet \hookrightarrow \mathbf{X}$, $\mathbf{Spec} B^\bullet \hookrightarrow \mathbf{X}$ on their overlap in \mathbf{X} , using a third chart.

2.3. PTVV’s shifted symplectic geometry

Next we summarize parts of the theory of shifted symplectic geometry, as developed by Pantev, Toën, Vaquié, and Vezzosi in [19]. We explain them for derived \mathbb{K} -schemes \mathbf{X} , although Pantev et al. work more generally with derived stacks.

Given a (locally finitely presented) derived \mathbb{K} -scheme \mathbf{X} and $p \geq 0, k \in \mathbb{Z}$, Pantev et al. [19] define complexes of k -shifted p -forms $\mathcal{A}_{\mathbb{K}}^p(\mathbf{X}, k)$ and k -shifted closed p -forms $\mathcal{A}_{\mathbb{K}}^{p,\text{cl}}(\mathbf{X}, k)$. These are defined first for affine derived

\mathbb{K} -schemes $\mathbf{Y} = \mathbf{Spec} A^\bullet$ for A^\bullet a cdga over \mathbb{K} , and shown to satisfy étale descent. Then for general \mathbf{X} , k -shifted (closed) p -forms are defined as a mapping stack; basically, a k -shifted (closed) p -form ω on \mathbf{X} is the functorial choice for all \mathbf{Y} , \mathbf{f} of a k -shifted (closed) p -form $\mathbf{f}^*(\omega)$ on \mathbf{Y} whenever $\mathbf{Y} = \mathbf{Spec} A^\bullet$ is affine and $\mathbf{f} : \mathbf{Y} \rightarrow \mathbf{X}$ is a morphism.

DEFINITION 2.7. — *Let $\mathbf{Y} \simeq \mathbf{Spec} A^\bullet$ be an affine derived \mathbb{K} -scheme, for A^\bullet a cdga over \mathbb{K} . A k -shifted p -form on \mathbf{Y} for $k \in \mathbb{Z}$ is an element $\omega_{A^\bullet}^0 \in (\Lambda^p \mathbb{L}_{A^\bullet})^k$ with $d\omega_{A^\bullet}^0 = 0$ in $(\Lambda^p \mathbb{L}_{A^\bullet})^{k+1}$, so that $\omega_{A^\bullet}^0$ defines a cohomology class $[\omega_{A^\bullet}^0] \in H^k(\Lambda^p \mathbb{L}_{A^\bullet})$. When $p = 2$, we call $\omega_{A^\bullet}^0$ nondegenerate if the induced morphism $\omega_{A^\bullet}^0 \cdot : \mathbb{T}_{A^\bullet} \rightarrow \mathbb{L}_{A^\bullet}[k]$ is a quasi-isomorphism.*

A k -shifted closed p -form on \mathbf{Y} is a sequence $\omega_{A^\bullet} = (\omega_{A^\bullet}^0, \omega_{A^\bullet}^1, \omega_{A^\bullet}^2, \dots)$ such that $\omega_{A^\bullet}^i \in (\Lambda^{p+i} \mathbb{L}_{A^\bullet})^{k-i}$ for $i \geq 0$, with $d\omega_{A^\bullet}^0 = 0$ and $d\omega_{A^\bullet}^{1+i} + d_{\mathrm{dR}}\omega_{A^\bullet}^i = 0$ in $(\Lambda^{p+i+1} \mathbb{L}_{A^\bullet})^{k-i}$ for all $i \geq 0$. Note that if $\omega_{A^\bullet} = (\omega_{A^\bullet}^0, \omega_{A^\bullet}^1, \dots)$ is a k -shifted closed p -form then $\omega_{A^\bullet}^0$ is a k -shifted p -form.

When $p = 2$, we call a k -shifted closed 2-form ω_{A^\bullet} a k -shifted symplectic form if the associated 2-form $\omega_{A^\bullet}^0$ is nondegenerate.

If \mathbf{X} is a general derived \mathbb{K} -scheme, then Pantev et al. [19, §1.2] define k -shifted 2-forms $\omega_{\mathbf{X}}^0$, which may be nondegenerate, and k -shifted closed 2-forms $\omega_{\mathbf{X}}$, which have an associated k -shifted 2-form $\omega_{\mathbf{X}}^0$, and where $\omega_{\mathbf{X}}$ is called a k -shifted symplectic form if $\omega_{\mathbf{X}}^0$ is nondegenerate. We will not go into the details of this definition for general \mathbf{X} .

The important thing for us is this: if $\mathbf{Y} \subseteq \mathbf{X}$ is a Zariski open affine derived \mathbb{K} -subscheme with $\mathbf{Y} \simeq \mathbf{Spec} A^\bullet$ then a k -shifted symplectic form $\omega_{\mathbf{X}}$ on \mathbf{X} induces a k -shifted symplectic form ω_{A^\bullet} on \mathbf{Y} in the sense above, where ω_{A^\bullet} is unique up to cohomology in the complex $(\prod_{i \geq 0} (\Lambda^{2+i} \mathbb{L}_{A^\bullet})^{*-i}, d + d_{\mathrm{dR}})$.

As in [19, §2.1], in the stacky case, an important source of examples of shifted symplectic derived stacks are Calabi–Yau moduli stacks:

THEOREM 2.8. — *Suppose Y is a Calabi–Yau m -fold over \mathbb{K} , and \mathcal{M} the derived moduli stack of complexes of coherent sheaves on Y . Then \mathcal{M} has a natural $(2 - m)$ -shifted symplectic form $\omega_{\mathcal{M}}$.*

2.4. Lagrangians in shifted symplectic derived schemes

Following Pantev et al. [19, §2.2], we define:

DEFINITION 2.9. — *Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a k -shifted symplectic derived \mathbb{K} -scheme, and $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ a morphism of derived \mathbb{K} -schemes. An isotropic structure on \mathbf{f} is a homotopy $h_{\mathbf{L}}$ from 0 to $\mathbf{f}^*(\omega_{\mathbf{X}})$ in the complex $\mathcal{A}_{\mathbb{K}}^{2, \mathrm{cl}}(\mathbf{L}, k)$,*

regarded as a simplicial set. Truncating to the first term $\mathcal{A}_{\mathbb{K}}^{2,\text{cl}}(\mathbf{L}, k) \rightarrow \mathcal{A}_{\mathbb{K}}^2(\mathbf{L}, k)$ gives a homotopy $h_{\mathbf{L}}^0$ from 0 to $\mathbf{f}^*(\omega_{\mathbf{X}}^0)$ in $\mathcal{A}_{\mathbb{K}}^2(\mathbf{L}, k)$.

This induces a 2-commutative diagram in $L_{\text{qcoh}}(\mathbf{L})$:

$$\begin{array}{ccc}
 \mathbb{T}_{\mathbf{L}} & \xrightarrow{\hspace{10em}} & 0 \\
 \downarrow \mathbb{T}_{\mathbf{f}} & \begin{array}{c} \xrightarrow{\mathbf{f}^*(\omega_{\mathbf{X}}^0) \cdot} \\ \simeq \end{array} & \begin{array}{c} \xrightarrow{h_{\mathbf{L}}^0 \cdot \Downarrow} \\ \mathbf{f}^*(\mathbb{L}_{\mathbf{X}}[k]) \end{array} & \xrightarrow{\mathbb{L}_{\mathbf{f}}[k]} & \mathbb{L}_{\mathbf{L}}[k] \\
 \mathbf{f}^*(\mathbb{T}_{\mathbf{X}}) & & & & \downarrow & \\
 & & & & & \mathbb{L}_{\mathbf{L}}[k].
 \end{array} \tag{2.2}$$

We say that $h_{\mathbf{L}}^0$ is nondegenerate if (2.2) is homotopy Cartesian (equivalently, homotopy co-Cartesian), and then we say that \mathbf{L} (with its morphism $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ and isotropic structure $h_{\mathbf{L}}$) is Lagrangian in $(\mathbf{X}, \omega_{\mathbf{X}})$.

An alternative way to explain the nondegeneracy of $h_{\mathbf{L}}^0$ is to note that it induces a natural morphism $\chi : \mathbb{T}_{\mathbf{L}/\mathbf{X}} \rightarrow \mathbb{L}_{\mathbf{L}}[k-1]$ via the diagram

$$\begin{array}{ccc}
 \mathbb{T}_{\mathbf{L}} & \xrightarrow{\hspace{10em}} & 0 \\
 \downarrow \mathbb{T}_{\mathbf{f}} & \begin{array}{c} \xrightarrow{\mathbf{f}^*(\omega_{\mathbf{X}}^0) \cdot} \\ \simeq \end{array} & \begin{array}{c} \xrightarrow{h_{\mathbf{L}}^0 \cdot \Downarrow} \\ \mathbf{f}^*(\mathbb{L}_{\mathbf{X}}[k]) \end{array} & \xrightarrow{\mathbb{L}_{\mathbf{f}}[k]} & \mathbb{L}_{\mathbf{L}}[k] \\
 \downarrow & & & \nearrow & \\
 \mathbb{T}_{\mathbf{L}/\mathbf{X}}[1], & \xrightarrow{\hspace{10em}} & \mathbb{L}_{\mathbf{L}}[k] & & \\
 & \chi[1] & & &
 \end{array} \tag{2.3}$$

and $h_{\mathbf{L}}^0$ is nondegenerate if $\chi : \mathbb{T}_{\mathbf{L}/\mathbf{X}} \rightarrow \mathbb{L}_{\mathbf{L}}[k-1]$ is a quasi-isomorphism.

Now suppose that $\mathbf{X} \simeq \mathbf{Spec} A^{\bullet}$ and $\mathbf{L} \simeq \mathbf{Spec} B^{\bullet}$ are affine, and \mathbf{f} is induced by a morphism $\alpha : A^{\bullet} \rightarrow B^{\bullet}$ in $\mathbf{cdga}_{\mathbb{K}}$, and $\omega_{\mathbf{X}}$ lifts to $\omega_{A^{\bullet}} = (\omega_{A^{\bullet}}^0, \omega_{A^{\bullet}}^1, \omega_{A^{\bullet}}^2, \dots)$ in $(\prod_{i \geq 0} (\Lambda^{2+i} \mathbb{L}_{A^{\bullet}}[k])^{*-i}, d + d_{\text{dR}})$ as in Definition 2.7. Then we can write $h_{\mathbf{L}}$ as a sequence (h^0, h^1, h^2, \dots) with $h^i \in (\Lambda^{2+i} \mathbb{L}_{B^{\bullet}})^{1+k-i}$ for $i = 0, 1, \dots$, where $h_{\mathbf{L}}$ an isotropic structure is equivalent to the equations

$$\alpha_*(\omega_{A^{\bullet}}^0) = dh^0, \quad \alpha_*(\omega_{A^{\bullet}}^i) = dh^i + d_{\text{dR}} h^{i-1}, \quad i = 1, 2, \dots \tag{2.4}$$

Remark 2.10. — Let us discuss virtual dimensions of shifted symplectic derived \mathbb{K} -schemes and their Lagrangians. If $(\mathbf{X}, \omega_{\mathbf{X}})$ is a k -shifted symplectic derived \mathbb{K} -scheme, it is easy to show (e.g. using the ‘‘Darboux Theorem’’ in Section 2.5) that

- (i) If $k \equiv 0 \pmod{4}$ then $\text{vdim } \mathbf{X}$ is even in \mathbb{Z} .
- (ii) If $k \equiv 1 \pmod{4}$ then $\text{vdim } \mathbf{X} = 0$.
- (iii) If $k \equiv 2 \pmod{4}$ then $\text{vdim } \mathbf{X}$ can take any value in \mathbb{Z} .
- (iv) If $k \equiv 3 \pmod{4}$ then $\text{vdim } \mathbf{X} = 0$.

Now suppose $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$, $h_{\mathbf{L}}$ is Lagrangian in $(\mathbf{X}, \omega_{\mathbf{X}})$. Then we find that:

- (i') If $k \equiv 0 \pmod{4}$ then $\mathrm{vdim} \mathbf{L} = \frac{1}{2} \mathrm{vdim} \mathbf{X}$.
- (ii') If $k \equiv 1 \pmod{4}$ then $\mathrm{vdim} \mathbf{L}$ can take any even value in \mathbb{Z} .
- (iii') If $k \equiv 2 \pmod{4}$ then $\mathrm{vdim} \mathbf{X}$ must be even (at least near the image of \mathbf{L} in \mathbf{X}), and $\mathrm{vdim} \mathbf{L} = \frac{1}{2} \mathrm{vdim} \mathbf{X}$.
- (iv') If $k \equiv 3 \pmod{4}$ then $\mathrm{vdim} \mathbf{L}$ can take any value in \mathbb{Z} .

So if $k \equiv 2 \pmod{4}$ and $\mathrm{vdim} \mathbf{X}$ is odd then no Lagrangians exist in $(\mathbf{X}, \omega_{\mathbf{X}})$.

Example 2.11. — Take $\mathbf{X} = \mathbf{Spec} \mathbb{K}$ to be the point $*$, regarded as a k -shifted symplectic derived \mathbb{K} -scheme with symplectic form $\omega_{\mathbf{X}} = 0$. Then Lagrangians \mathbf{L} in $(*, 0)$ are equivalent to $(k - 1)$ -shifted symplectic derived \mathbb{K} -schemes $(\mathbf{L}, \omega_{\mathbf{L}})$.

Pantev et al. [19, Th. 2.10] prove:

THEOREM 2.12. — *Suppose $(\mathbf{X}, \omega_{\mathbf{X}})$ is a k -shifted symplectic derived \mathbb{K} -scheme, and $\mathbf{f}_1 : \mathbf{L}_1 \rightarrow \mathbf{X}$ and $\mathbf{f}_2 : \mathbf{L}_2 \rightarrow \mathbf{X}$ are Lagrangians in $(\mathbf{X}, \omega_{\mathbf{X}})$. Then the fibre product $\mathbf{L}_1 \times_{\mathbf{f}_1, \mathbf{X}, \mathbf{f}_2} \mathbf{L}_2$ in $\mathbf{dSch}_{\mathbb{K}}$ has a natural $(k - 1)$ -shifted symplectic structure.*

In the stacky case, Calaque [11, §3.2] extends Theorem 2.8:

THEOREM 2.13. — *Suppose X is a Fano $(m + 1)$ -fold over \mathbb{K} , and $Y \subseteq X$ is a smooth anticanonical divisor, so that Y is a Calabi–Yau m -fold. Write \mathcal{L}, \mathcal{M} for the derived moduli stacks of complexes of coherent sheaves on X, Y , and $\mathbf{f} : \mathcal{L} \rightarrow \mathcal{M}$ for the morphism of derived restriction from X to Y . Theorem 2.8 gives a $(2 - m)$ -shifted symplectic structure $\omega_{\mathcal{M}}$ on \mathcal{M} . Then there is a natural isotropic structure $h_{\mathcal{L}}$ on $\mathbf{f} : \mathcal{L} \rightarrow \mathcal{M}$ making \mathcal{L} into a Lagrangian in $(\mathcal{M}, \omega_{\mathcal{M}})$.*

2.5. A shifted symplectic “Darboux Theorem”

Bussi, Brav and Joyce [7] prove “Darboux Theorems” for k -shifted symplectic derived \mathbb{K} -schemes $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$, which give explicit Zariski or étale local models for $(\mathbf{X}, \omega_{\mathbf{X}})$. We will explain their main result in Theorem 2.18 below. First, in Examples 2.14 and 2.16 we define families of explicit “Darboux form” k -shifted symplectic cdgas $A^{\bullet, \omega}$ for $k < 0$.

Example 2.14. — Let $k = -1, -2, \dots$, and set $d = \lceil (k + 1)/2 \rceil$, so that $d = k/2$ if k is even (giving $k = 2d$), and $d = (k + 1)/2$ if k is odd (giving $k = 2d - 1$). Following [7, Examples 5.8 & 5.9], we will define a simple class of standard form cdgas $A^{\bullet} = (A^*, d)$ equipped with explicit k -shifted symplectic forms $\omega = (\omega^0, 0, 0, \dots)$, which we will call of *Darboux form*.

Fix nonnegative integers $m_0, m_{-1}, m_{-2}, \dots, m_d$. Choose a smooth \mathbb{K} -algebra A^0 of dimension m_0 . Localizing A^0 if necessary, we may assume that there exist $x_1^0, \dots, x_{m_0}^0 \in A^0$ such that $d_{\mathrm{dR}}x_1^0, \dots, d_{\mathrm{dR}}x_{m_0}^0$ form a basis of $\Omega_{A^0}^1$ over A^0 . Geometrically, $U = \mathbf{Spec} A^0$ is a smooth \mathbb{K} -scheme of dimension m_0 , and $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$ are global étale coordinates on U .

Define A^* as a commutative graded \mathbb{K} -algebra to be the free graded algebra over A^0 generated by variables

$$\begin{aligned} x_1^i, \dots, x_{m_i}^i & \quad \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\ y_1^{k-i}, \dots, y_{m_i}^{k-i} & \quad \text{in degree } k - i \text{ for } i = 0, -1, \dots, d. \end{aligned} \quad (2.5)$$

So the upper index i in x_j^i, y_j^i always indicates the degree. The variables come in pairs x_j^i, y_j^{k-i} , with total degree k . We will define the differential d in the cdga $A^\bullet = (A^*, d)$ later.

As in Section 2.1, the spaces $(\Lambda^p \Omega_{A^\bullet}^1)^k$ and the de Rham differential d_{dR} upon them depend only on the commutative graded algebra A^* , not on the (not yet defined) differential d . Note that $\Omega_{A^\bullet}^1$ is the free A^* -module with basis $d_{\mathrm{dR}}x_j^i, d_{\mathrm{dR}}y_j^{k-i}$ for $i = 0, -1, \dots, d$ and $j = 1, \dots, m_i$. Define an element

$$\omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\mathrm{dR}}x_j^i d_{\mathrm{dR}}y_j^{k-i} \quad \text{in } (\Lambda^2 \Omega_{A^\bullet}^1)^k. \quad (2.6)$$

Clearly $d_{\mathrm{dR}}\omega^0 = 0$ in $(\Lambda^3 \Omega_{A^\bullet}^1)^k$.

Now choose a superpotential Φ in A^{k+1} , called the *Hamiltonian*, which we require to satisfy the *classical master equation*

$$\sum_{i=-1}^d \sum_{j=1}^{m_i} \frac{\partial \Phi}{\partial x_j^i} \frac{\partial \Phi}{\partial y_j^{k-i}} = 0 \quad \text{in } A^{k+2}. \quad (2.7)$$

Define the differential d on A^* by $d = 0$ on A^0 , and

$$dx_j^i = (-1)^{(i+1)(k+1)} \frac{\partial \Phi}{\partial y_j^{k-i}}, \quad dy_j^{k-i} = \frac{\partial \Phi}{\partial x_j^i}, \quad i = 0, \dots, d, \quad j = 1, \dots, m_i. \quad (2.8)$$

Equation (2.7) implies that $d \circ d = 0$.

Then $A^\bullet = (A^*, d)$ is a standard form cdga, as in Definition 2.4, with

$$\mathrm{vdim} A^\bullet = \begin{cases} 2 \sum_{i=0}^d (-1)^i m_i, & k \text{ even,} \\ 0, & k \text{ odd,} \end{cases}$$

so that $\mathrm{vdim} A^\bullet$ is always even (compare Remark 2.10). Also $d\omega^0 = d_{\mathrm{dR}}\omega^0 = 0$, and $\omega := (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A^\bullet$, as in [7, §5.3]. Define $\phi \in (\Omega_{A^\bullet}^1)^k$ by

$$\phi = \sum_{i=0}^d \sum_{j=1}^{m_i} [i x_j^i d_{\mathrm{dR}} y_j^{k-i} + (-1)^{(i+1)(k+1)} (k-i) y_j^{k-i} d_{\mathrm{dR}} x_j^i]. \quad (2.9)$$

Then we have

$$d\Phi = 0 \in A^{k+2}, \quad d_{\mathrm{dR}}\Phi + d\phi = 0 \in (\Omega_{A^\bullet}^1)^{k+1}, \quad d_{\mathrm{dR}}\phi = k\omega^0 \in (\Lambda^2 \Omega_{A^\bullet}^1)^k. \quad (2.10)$$

We say that A^\bullet, ω are in *Darboux form*.

In the first case $k = -1$, as in [7, Prop. 5.7(b)] we impose an additional condition on Φ . In this case $\Phi : U \rightarrow \mathbb{A}^1$ is a regular function, and $\mathbf{X} = \mathbf{Crit}(\Phi)$ is the derived critical locus of Φ , so $X = t_0(\mathbf{X}) = \mathrm{Crit}(\Phi)$ is the classical critical locus of Φ . The restriction $\Phi|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \rightarrow \mathbb{A}^1$ of Φ to the reduced \mathbb{K} -subscheme X^{red} of X is locally constant. By adding a constant to Φ (which does not change \mathbf{X}) and localizing, we may assume that $\Phi|_{X^{\mathrm{red}}} = 0$.

Remark 2.15. — Continue in the situation of Example 2.14. The following notation was not defined in [7], but will be important in Sections 3–4. Define A_+^\bullet to be the sub-cdga of A^\bullet generated (either as a cdga or equivalently as a graded algebra) by A^0 and the variables x_j^i for $i = -1, -2, \dots, d$ and $j = 1, \dots, m_i$. Then A_+^\bullet is of standard form with $\mathrm{vdim} A_+^\bullet = \sum_{i=0}^d (-1)^i m_i$. Write $\iota : A_+^\bullet \hookrightarrow A^\bullet$ for the inclusion morphism, which is a submersion. Then we have a fibre sequence

$$\mathbb{L}_{A_+^\bullet} \otimes_{A_+^\bullet} A^\bullet \xrightarrow{\mathbb{L}_\iota} \mathbb{L}_{A^\bullet} \longrightarrow \mathbb{L}_{A^\bullet/A_+^\bullet}.$$

Taking $\mathbb{L}_{A^\bullet} = \Omega_{A^\bullet}^1$, $\mathbb{L}_{A_+^\bullet} = \Omega_{A_+^\bullet}^1$, and $\mathbb{L}_{A^\bullet/A_+^\bullet} = \Omega_{A^\bullet/A_+^\bullet}^1$ as A^\bullet, A_+^\bullet are of standard form and ι is a submersion, as in (2.1) we have

$$\begin{aligned} \mathbb{L}_{A^\bullet} &\cong A^* \otimes_{\mathbb{K}} \langle d_{\mathrm{dR}} x_j^i, d_{\mathrm{dR}} y_j^{k-i} : i=0, -1, \dots, d, j=1, \dots, m_i \rangle_{\mathbb{K}}, \\ \mathbb{L}_{A_+^\bullet} \otimes_{A_+^\bullet} A^\bullet &\cong A^* \otimes_{\mathbb{K}} \langle d_{\mathrm{dR}} x_j^i : i=0, -1, \dots, d, j=1, \dots, m_i \rangle_{\mathbb{K}}, \\ \mathbb{L}_{A^\bullet/A_+^\bullet} &\cong A^* \otimes_{\mathbb{K}} \langle d_{\mathrm{dR}} y_j^{k-i} : i=0, -1, \dots, d, j=1, \dots, m_i \rangle_{\mathbb{K}}. \end{aligned}$$

We will also find it helpful to decompose $\Phi \in A^{k+1}$ into components. Observe that as $\deg(y_j^{k-i}) \leq k-d$ and $2(k-d) < k+1$, for degree reasons Φ can be at most linear in the variables y_j^{k-i} , so we may write

$$\Phi = \Phi_+ + \sum_{i=-1}^d \sum_{j=1}^{m_i} \Phi_j^{i+1} y_j^{k-i}, \quad (2.11)$$

where $\Phi_+ \in A_+^{k+1}$ and $\Phi_j^{i+1} \in A_+^{i+1}$ for all i, j do not involve the variables y_j^i . Then equation (2.7) is equivalent to the equations

$$\sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \Phi_j^{i+1} \frac{\partial \Phi_+}{\partial x_j^i} = 0 \quad \text{in } A_+^{k+2}, \quad (2.12)$$

$$\sum_{i=-1}^{i'+1} \sum_{j=1}^{m_i} (-1)^{i+1} \Phi_j^{i+1} \frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} = 0 \quad \text{in } A_+^{i'+2}, \quad (2.13)$$

$$i' = -1, \dots, d, \quad j' = 1, \dots, m_{i'},$$

and (2.8) is equivalent to the equations for $i = 0, \dots, d$ and $j = 1, \dots, m_i$:

$$dx_j^i = (-1)^{i+1} \Phi_j^{i+1}, \quad dy_j^{k-i} = \frac{\partial \Phi_+}{\partial x_j^i} + \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} \frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} y_{j'}^{k-i'}. \quad (2.14)$$

Define

$$\phi_+ = - \sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{(i+1)(k+1)} y_j^{k-i} d_{\text{dR}} x_j^i \quad \text{in } (\Omega_{A^\bullet}^1)^k. \quad (2.15)$$

Then as for (2.10), calculation shows that

$$d\Phi_+ = 0, \quad d_{\text{dR}} \Phi_+ + d\phi_+ = 0, \quad \text{and} \quad d_{\text{dR}} \phi_+ = -\omega^0. \quad (2.16)$$

A nice interpretation of $\iota : A_+^\bullet \rightarrow A^\bullet$, which we will not actually use, is that $\mathbf{Spec} \iota : \mathbf{Spec} A^\bullet \rightarrow \mathbf{Spec} A_+^\bullet$ is a *Lagrangian fibration* of $(\mathbf{Spec} A^\bullet, \omega)$.

We can also use this example to explain the relation between the ‘‘Darboux Theorems’’ of Bussi, Brav and Joyce [7], and Bouaziz and Grojnowski [5]. Bouaziz and Grojnowski show that any k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$ with $k \not\equiv 2 \pmod{4}$ is étale locally equivalent to a *twisted k -shifted cotangent bundle* $T_t^*[k]\mathbf{Y}$, where \mathbf{Y} is an affine derived \mathbb{K} -scheme, and $t \in \mathcal{O}_{\mathbf{Y}}^{k+1}$ with $dt = 0$ is used to ‘‘twist’’ the k -shifted cotangent bundle $T^*[k]\mathbf{Y}$.

To make the two pictures correspond, we should identify \mathbf{Y} with $\mathbf{Spec} A_+^\bullet$, and $\Phi_+ \in A_+^{k+1}$ with $t \in \mathcal{O}_{\mathbf{Y}}^{k+1}$. The data Φ_j^{i+1} in Φ in (2.11) is used to define the differential d in $A_+^\bullet = (A_+^*, d)$, via $dx_j^i = (-1)^{i+1} \Phi_j^{i+1}$ in (2.14). The classical master equation (2.7) reduces to (2.12)–(2.13), where (2.12) means that $d\Phi_+ = 0$, and (2.13) means that $d \circ d = 0$ in $A_+^\bullet = (A_+^*, d)$, necessary for A_+^\bullet to be a cdga and \mathbf{Y} a derived scheme.

Remark 3.4 will explain how our ‘‘Lagrangian Neighbourhood Theorem’’ relates to Bouaziz and Grojnowski’s ‘‘twisted cotangent bundle’’ picture.

Bussi, Brav and Joyce [7, Examples 5.10 & 5.12] also give two variations on Example 2.14 when $k \equiv 2 \pmod{4}$:

Example 2.16. — Let $k = -2, -6, -10, \dots$ be negative with $k \equiv 2 \pmod{4}$, and set $d = k/2$, so that d is negative and odd. Fix nonnegative integers $m_0, m_{-1}, m_{-2}, \dots, m_d$. Choose $A^0, x_1^0, \dots, x_{m_0}^0$ and U as in Example 2.14.

Modifying (2.5), define A^* as a commutative graded \mathbb{K} -algebra to be the free graded algebra over A^0 generated by variables

$$\begin{aligned} x_1^i, \dots, x_{m_i}^i & \quad \text{in degree } i \text{ for } i = -1, -2, \dots, d+1, \text{ and} \\ z_1^d, \dots, z_{m_d}^d & \quad \text{in degree } d, \text{ and} \\ y_1^{k-i}, \dots, y_{m_i}^{k-i} & \quad \text{in degree } k-i \text{ for } i = 0, -1, \dots, d+1. \end{aligned}$$

Let q_1, \dots, q_{m_d} be invertible elements of A^0 , and generalizing (2.6) define

$$\omega^0 = \sum_{i=0}^{d+1} \sum_{j=1}^{m_i} d_{\text{dR}} x_j^i d_{\text{dR}} y_j^{k-i} + \sum_{j=1}^{m_d} d_{\text{dR}} (q_j z_j^d) d_{\text{dR}} z_j^d \quad \text{in } (\Lambda^2 \Omega_{A^\bullet}^1)^k. \quad (2.17)$$

Choose a Hamiltonian Φ in A^{k+1} , which as in (2.7) we require to satisfy the *classical master equation*

$$\sum_{i=-1}^{d+1} \sum_{j=1}^{m_i} \frac{\partial \Phi}{\partial x_j^i} \frac{\partial \Phi}{\partial y_j^{k-i}} + \frac{1}{4} \sum_{j=1}^{m_d} \frac{1}{q_j} \left(\frac{\partial \Phi}{\partial z_j^d} \right)^2 = 0 \quad \text{in } A^{k+2}. \quad (2.18)$$

As for (2.8), define the differential d on A^\bullet by $d = 0$ on A^0 , and

$$\begin{aligned} dx_j^0 &= 0, \quad dy_j^k = \frac{\partial \Phi}{\partial x_j^0} - \sum_{j'=1}^{m_d} \frac{z_{j'}^d}{2q_{j'}} \frac{\partial q_{j'}}{\partial x_j^0} \frac{\partial \Phi}{\partial z_{j'}^d}, \quad j = 1, \dots, m_0, \\ dx_j^i &= (-1)^{i+1} \frac{\partial \Phi}{\partial y_j^{k-i}}, \quad dy_j^{k-i} = \frac{\partial \Phi}{\partial x_j^i}, \quad i = -1, \dots, d+1, \quad j = 1, \dots, m_i, \\ & \quad \text{and} \quad dz_j^d = \frac{1}{2q_j} \frac{\partial \Phi}{\partial z_j^d}, \quad j = 1, \dots, m_d. \end{aligned} \quad (2.19)$$

Then $A^\bullet = (A^*, d)$ is of standard form, with $\text{vdim } A^\bullet = 2 \sum_{i=0}^{d+1} (-1)^i m_i - m_d$. Also $d\omega^0 = d_{\text{dR}} \omega^0 = 0$, and $\omega := (\omega^0, 0, 0, \dots)$ is a k -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec } A^\bullet$, as in [7, §5.3]. Defining $\phi \in (\Omega_{A^\bullet}^1)^k$ by

$$\begin{aligned} \phi &= \sum_{i=0}^{d+1} \sum_{j=1}^{m_i} [i x_j^i d_{\text{dR}} y_j^{k-i} + (-1)^{i+1} (k-i) y_j^{k-i} d_{\text{dR}} x_j^i] \\ & \quad + k \sum_{j=1}^{m_d} q_j z_j^d d_{\text{dR}} z_j^d, \end{aligned} \quad (2.20)$$

as in (2.9), then (2.10) holds. We say that A^\bullet, ω are in *weak Darboux form*.

If all the above holds with $q_j = 1$ for $j = 1, \dots, m_d$, we say that A^\bullet, ω are in *strong Darboux form*.

Remark 2.17. — Actually, when $k \equiv 2 \pmod{4}$, Bussi, Brav and Joyce [7, §5.3] did not define “Darboux form” A^\bullet, ω as in Example 2.14 involving only variables x_j^i, y_j^{k-i} , but instead only defined “weak Darboux form” and “strong Darboux form” as in Example 2.16, involving variables x_j^i, y_j^{k-i}, z_j^d .

We can relate Example 2.16 to Example 2.14 with $k \equiv 2 \pmod{4}$. Let A^\bullet, ω be in strong Darboux form as in Example 2.16, so that we have variables x_j^i, y_j^{k-i}, z_j^d and $q_1 = \dots = q_{m_d} = 1$, and suppose m_d is even (equivalently, suppose $\text{vdim } A^\bullet$ is even). Then we may change variables from $z_1^d, \dots, z_{m_d}^d$ to $x_1^d, \dots, x_{m_d/2}^d, y_1^d, \dots, y_{m_d/2}^d$, where

$$x_j^d = z_{2j-1}^d + \sqrt{-1}z_{2j}^d, \quad y_j^d = z_{2j-1}^d - \sqrt{-1}z_{2j}^d, \quad j = 1, \dots, m_d/2,$$

and replace m_d by $m_d/2$, and then the “strong Darboux form” of Example 2.16 is equivalent to the “Darboux form” of Example 2.14. Here $\sqrt{-1} \in \mathbb{K}$ as \mathbb{K} is algebraically closed.

As in Remark 2.10, if $(\mathbf{X}, \omega_{\mathbf{X}}^\bullet)$ is k -shifted symplectic with $k \equiv 2 \pmod{4}$ and $\text{vdim } \mathbf{X}$ is odd, then no Lagrangians exist in $(\mathbf{X}, \omega_{\mathbf{X}})$. Because of this, in this paper we are happy to use the local form of Example 2.14 when $k \equiv 2 \pmod{4}$, which only works when $\text{vdim } \mathbf{X}$ is even, rather than that of Example 2.16, which works for all $\text{vdim } \mathbf{X}$.

Here is the main result of Bussi, Brav and Joyce [7, Th. 5.18]. They state only (i)–(iii), part (iv) is deduced from (iii) as in Remark 2.17. The reason we need \mathbf{i} to be étale in (iii) (and hence (iv)) is that to reduce from weak Darboux form to strong Darboux form in Example 2.16, it is necessary to take square roots of the functions q_1, \dots, q_{m_d} , and this is only possible étale locally rather than Zariski locally.

THEOREM 2.18. — *Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a k -shifted symplectic derived \mathbb{K} -scheme for $k < 0$, and $x \in \mathbf{X}$. Then there exists a standard form $\text{cdga } A^\bullet$ over \mathbb{K} which is minimal at $p \in \mathbf{Spec } A^\bullet$, a k -shifted symplectic form ω on $\mathbf{Spec } A^\bullet$, and a morphism $\mathbf{i} : \mathbf{Spec } A^\bullet \rightarrow \mathbf{X}$ with $\mathbf{i}(p) = x$ and $\mathbf{i}^*(\omega_{\mathbf{X}}) \sim \omega$, such that:*

- (i) *If $k \equiv 0, 1$ or $3 \pmod{4}$, then \mathbf{i} is a Zariski open inclusion, and A^\bullet, ω are in Darboux form, as in Example 2.14.*
- (ii) *If $k \equiv 2 \pmod{4}$, then \mathbf{i} is a Zariski open inclusion, and A^\bullet, ω are in weak Darboux form, as in Example 2.16.*

- (iii) *Alternatively, if $k \equiv 2 \pmod{4}$, we may instead take \mathbf{i} to be étale, and A^\bullet, ω to be in strong Darboux form, as in Example 2.16.*
- (iv) *Alternatively, if $k \equiv 2 \pmod{4}$, and $\dim \mathbf{X}$ is even near x , we may take \mathbf{i} to be étale, and A^\bullet, ω to be in Darboux form, as in Example 2.14.*

Following [7, Examples 5.15 & 5.16], we explain Examples 2.14 and 2.16 in more detail in the first cases $k = -1$ and $k = -2$.

Example 2.19. — Choose a smooth \mathbb{K} -algebra A^0 of dimension m_0 and elements $x_1^0, \dots, x_{m_0}^0 \in A^0$ such that $d_{\mathrm{dR}}x_1^0, \dots, d_{\mathrm{dR}}x_{m_0}^0$ form a basis of $\Omega_{A^0}^1$ over A^0 . Choose an arbitrary Hamiltonian $\Phi \in A^0$.

Example 2.14 with $k = -1$ defines $A^\bullet = A^0[y_1^{-1}, \dots, y_{m_0}^{-1}]$, where $y_1^{-1}, \dots, y_{m_0}^{-1}$ are variables of degree -1 , with differential

$$dx_j^0 = 0, \quad dy_j^{-1} = \frac{\partial \Phi}{\partial x_j^0}, \quad j = 1, \dots, m_0,$$

and -1 -shifted 2-form

$$\omega^0 = d_{\mathrm{dR}}x_1^0 d_{\mathrm{dR}}y_1^{-1} + \dots + d_{\mathrm{dR}}x_{m_0}^0 d_{\mathrm{dR}}y_{m_0}^{-1}.$$

Then $\omega = (\omega^0, 0, 0, \dots)$ is a -1 -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A^\bullet$. We have $H^0(A^\bullet) = A^0 / (\frac{\partial \Phi}{\partial x_1^0}, \dots, \frac{\partial \Phi}{\partial x_{m_0}^0}) = A^0 / (d_{\mathrm{dR}}\Phi)$.

Geometrically, $U = \mathbf{Spec} A^0$ is a smooth classical \mathbb{K} -scheme with étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, and $\Phi : U \rightarrow \mathbb{A}^1$ is regular, and \mathbf{X} is the derived critical locus of Φ , with $X = t_0(\mathbf{X})$ the classical critical locus of Φ .

Thus, *the important geometric data in writing a -1 -shifted symplectic derived \mathbb{K} -scheme (\mathbf{X}, ω) in Darboux form, is a smooth affine \mathbb{K} -scheme U and a regular function $\Phi : U \rightarrow \mathbb{A}^1$, such that $\mathbf{X} \simeq \mathbf{Crit}(\Phi)$.* The remaining data is a choice of étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, but this is not very interesting geometrically.

Example 2.20. — Choose a smooth \mathbb{K} -algebra A^0 of dimension m_0 and elements $x_1^0, \dots, x_{m_0}^0$ in A^0 such that $d_{\mathrm{dR}}x_1^0, \dots, d_{\mathrm{dR}}x_{m_0}^0$ form a basis of $\Omega_{A^0}^1$ over A^0 . Fix $m_{-1} \geq 0$, and as a graded algebra set $A^* = A^0[y_1^{-2}, \dots, y_{m_0}^{-2}, z_1^{-1}, \dots, z_{m_{-1}}^{-1}]$, where y_j^{-2} has degree -2 and z_j^{-1} degree -1 , as in Example 2.16 with $k = -2$.

Choose invertible functions $q_1, \dots, q_{m_{-1}}$ in A^0 . Define

$$\begin{aligned} \omega^0 &= d_{\mathrm{dR}}x_1^0 d_{\mathrm{dR}}y_1^{-2} + \dots + d_{\mathrm{dR}}x_{m_0}^0 d_{\mathrm{dR}}y_{m_0}^{-2} \\ &\quad + d_{\mathrm{dR}}(q_1 z_1^{-1}) d_{\mathrm{dR}}z_1^{-1} + \dots + d_{\mathrm{dR}}(q_{m_{-1}} z_{m_{-1}}^{-1}) d_{\mathrm{dR}}z_{m_{-1}}^{-1} \end{aligned}$$

in $(\Lambda^2 \Omega_{A^\bullet}^1)^{-2}$, as in (2.17). A general element Φ in A^{-1} may be written

$$\Phi = z_1^{-1} s_1 + \cdots + z_{m-1}^{-1} s_{m-1},$$

for $s_1, \dots, s_{m-1} \in A^0$. Then the classical master equation (2.18) reduces to

$$\frac{(s_1)^2}{q_1} + \cdots + \frac{(s_{m-1})^2}{q_{m-1}} = 0 \quad \text{in } A^0. \quad (2.21)$$

By (2.19), the differential d on A^\bullet is given by

$$dx_i^0 = 0, \quad dz_j^{-1} = \frac{s_j}{2q_j}, \quad dy_i^{-2} = \sum_{j=1}^{m-1} z_j^{-1} \left(\frac{\partial s_j}{\partial x_i^0} - \frac{s_j}{2q_j} \frac{\partial q_j}{\partial x_i^0} \right),$$

and $d \circ dy_i^{-2} = 0$ follows from applying $\frac{1}{4} \frac{\partial}{\partial x_i^0}$ to (2.21). We have

$$H^0(A^\bullet) = A^0 / (s_1/2q_1, \dots, s_{m-1}/2q_{m-1}) = A^0 / (s_1, \dots, s_{m-1}),$$

as q_1, \dots, q_{m-1} are invertible.

Geometrically, we have a smooth classical \mathbb{K} -scheme $U = \mathbf{Spec} A^0$ with étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, a trivial vector bundle $E \rightarrow U$ with fibre \mathbb{K}^{m-1} , a nondegenerate quadratic form Q on E given by $Q(e_1, \dots, e_{m-1}) = \frac{1}{q_1} e_1^2 + \cdots + \frac{1}{q_{m-1}} e_{m-1}^2$ for all regular functions $e_1, \dots, e_{m-1} : U \rightarrow \mathbb{A}^1$, and a section $s = (s_1, \dots, s_{m-1})$ in $H^0(E)$ with $Q(s, s) = 0$ by (2.21). The underlying classical \mathbb{K} -scheme $X = t_0(\mathbf{X}) = \mathbf{Spec} H^0(A^\bullet)$ is the \mathbb{K} -subscheme $s^{-1}(0)$ in U .

Thus, *the important geometric data in writing a -2 -shifted symplectic derived \mathbb{K} -scheme (\mathbf{X}, ω) in weak Darboux form, is a smooth affine \mathbb{K} -scheme U , a vector bundle $E \rightarrow U$, a nondegenerate quadratic form Q on E , and a section $s \in H^0(E)$ with $Q(s, s) = 0$, such that $X = t_0(\mathbf{X}) \cong s^{-1}(0) \subseteq U$. The remaining data is a choice of étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$ and a trivialization $E \cong U \times \mathbb{A}^{m-1}$, but these are not very interesting geometrically.*

3. The main results

3.1. A local standard form for derived scheme morphisms

As in Theorem 2.6, our favourite local model for a derived scheme \mathbf{X} near a point $x \in \mathbf{X}$ is $\mathbf{Spec} A^\bullet \hookrightarrow \mathbf{X}$ for A^\bullet a standard form cdga, and we can take $\mathbf{Spec} A^\bullet$ minimal at x . We used this in the Darboux Theorem, Theorem 2.18.

We will need a favourite local model for a morphism $\mathbf{f} : \mathbf{Y} \rightarrow \mathbf{X}$ in $\mathbf{dSch}_{\mathbb{K}}$ near $y \in \mathbf{Y}$ with $\mathbf{f}(y) = x \in \mathbf{X}$. For this we will use a homotopy commutative diagram (3.1) below, where $\alpha : A^\bullet \rightarrow B^\bullet$ is a *submersion* in $\mathbf{cdga}_{\mathbb{K}}$, following Borisov and Joyce [4, §2.1], which we take to be *minimal* at $j^{-1}(y) \in \mathbf{Spec} B^\bullet$.

DEFINITION 3.1. — *A morphism $\alpha : A^\bullet \rightarrow B^\bullet$ of standard form cdgas will be called a submersion if the corresponding morphism $\alpha_* : (\Omega_{A^\bullet}^1) \otimes_{A^\bullet} B^\bullet \rightarrow \Omega_{B^\bullet}^1$ is injective in every degree. (By analogy, a smooth map of manifolds $f : X \rightarrow Y$ is a submersion if $(df)^* : f^*(T^*Y) \rightarrow T^*X$ is injective.)*

If $\alpha : A^\bullet \rightarrow B^\bullet$ is a submersion of standard form cdgas then the relative Kähler differentials $\Omega_{B^\bullet/A^\bullet}^1$ are a model for the relative cotangent complex $\mathbb{L}_{B^\bullet/A^\bullet}$, so we can take $\Omega_{B^\bullet/A^\bullet}^1 = \mathbb{L}_{B^\bullet/A^\bullet}$. Thus submersions are a convenient class of morphisms for doing explicit computations with cotangent complexes.

In a similar way to Definition 2.4, we say that a submersion $\alpha : A^\bullet \rightarrow B^\bullet$ is *minimal* at $q \in \mathbf{Spec} B^\bullet$ if all the differentials in the complex of \mathbb{K} -vector spaces $\Omega_{B^\bullet/A^\bullet}^1|_q$ are zero. This means that regarding A^\bullet as fixed, B^\bullet is defined using the minimum number of variables in each degree $i = 0, -1, \dots$, compared to all other cdgas locally equivalent to B^\bullet near q with submersions to A^\bullet .

Here is a relative analogue of Theorem 2.6, which will be proved in Section 4.1.

THEOREM 3.2. — *Let $\mathbf{f} : \mathbf{Y} \rightarrow \mathbf{X}$ be a morphism in $\mathbf{dSch}_{\mathbb{K}}$, and $y \in \mathbf{Y}$ with $\mathbf{f}(y) = x \in \mathbf{X}$. Suppose A^\bullet is a standard form cdga over \mathbb{K} , and $p \in \mathbf{Spec} A^\bullet$, and $\mathbf{i} : \mathbf{Spec} A^\bullet \hookrightarrow \mathbf{X}$ is a Zariski open inclusion with $\mathbf{f}(p) = x$. This is possible by Theorem 2.8. We do not assume A^\bullet is minimal at p .*

Then there exists a standard form cdga B^\bullet over \mathbb{K} , a point $q \in \mathbf{Spec} A^\bullet$, a submersion $\alpha : A^\bullet \rightarrow B^\bullet$ minimal at q with $\mathbf{Spec} \alpha(q) = p$, and a Zariski open inclusion $\mathbf{j} : \mathbf{Spec} B^\bullet \hookrightarrow \mathbf{Y}$ with $\mathbf{j}(q) = y$ in a homotopy commutative diagram

$$\begin{array}{ccc}
 \mathbf{Spec} B^\bullet & \hookrightarrow & \mathbf{Y} \\
 \downarrow \mathbf{Spec} \alpha & \begin{array}{c} \xrightarrow{j} \\ \searrow f \end{array} & \downarrow f \\
 \mathbf{Spec} A^\bullet & \hookrightarrow & \mathbf{X} \\
 & \begin{array}{c} \xrightarrow{i} \\ \nearrow \end{array} &
 \end{array} \tag{3.1}$$

If instead \mathbf{i} is étale rather than a Zariski open inclusion, then \mathbf{j} is étale.

3.2. k -shifted “Lagrangian Darboux form” local models for $k < 0$

In Examples 3.3 and 3.5 we will explain our local models for Lagrangians in k -shifted symplectic derived \mathbb{K} -schemes, which we call “Lagrangian Darboux form”, for the cases $k < 0$ with $k \not\equiv 3 \pmod{4}$, and $k < 0$ with $k \equiv 3 \pmod{4}$, respectively. They are analogues of the “Darboux form” Examples 2.14 and 2.16 in Section 2.5, and work over a target in “Darboux form”. Theorem 3.7 in Section 3.3 will show that Lagrangians $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in a k -shifted symplectic \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$ are (Zariski or étale) locally modelled on one of Examples 3.3 and 3.5.

The next example is rather long.

Example 3.3. — Let $k < 0$ with $k \not\equiv 3 \pmod{4}$, suppose A^\bullet, ω are in k -shifted Darboux form, as in Example 2.14, and use the notation of Remark 2.15. These define a standard form cdga A^\bullet over \mathbb{K} , a sub-cdga $A_+^\bullet \subseteq A^\bullet$, coordinates x_j^i in $A_+^i \subseteq A^i$ and y_j^{k-i} in A^{k-i} for $i = 0, -1, \dots, d = [(k+1)/2]$ and $j = 1, \dots, m_i$, and a k -shifted 2-form $\omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} x_j^i d_{\text{dR}} y_j^{k-i}$. They also define $\Phi \in A^{k+1}$ satisfying (2.7), which determines the differential d in A^\bullet by (2.8), and $\phi \in (\Omega_{A^\bullet}^1)^k$ satisfying $d\Phi = 0$, $d_{\text{dR}}\Phi + d\phi = 0$ and $d_{\text{dR}}\phi = k\omega^0$. As in (2.11) we write $\Phi = \Phi_+ + \sum_{i=-1}^d \sum_{j=1}^{m_i} \Phi_j^{i+1} y_j^{k-i}$, where $\Phi_+ \in A_+^{k+1}$ and $\Phi_j^{i+1} \in A_+^{i+1}$ for all i, j do not involve the y_j^i , and we define $\phi_+ \in (\Omega_{A^\bullet}^1)^k$ by (2.15).

Write $e = [k/2]$, so that if k is even then $e = d$ and $k = 2e = 2d$, and if k is odd then $e = d - 1$ and $k = 2e + 1 = 2d - 1$. Choose nonnegative integers n_0, n_{-1}, \dots, n_e . Choose a smooth \mathbb{K} -algebra B^0 of dimension $m_0 + n_0$, and a smooth morphism $\alpha^0 : A^0 \rightarrow B^0$. Localizing B^0 if necessary, we may assume there exist $u_1^0, \dots, u_{n_0}^0 \in B^0$ such that $d_{\text{dR}}\tilde{x}_1^0, \dots, d_{\text{dR}}\tilde{x}_{m_0}^0, d_{\text{dR}}u_1^0, \dots, d_{\text{dR}}u_{n_0}^0$ form a basis of $\Omega_{B^0}^1$ over B^0 , where we write $\tilde{x}_j^0 = \alpha^0(x_j^0) \in B^0$.

Define B^* as a commutative graded \mathbb{K} -algebra to be the free graded algebra over B^0 generated by variables

$$\begin{aligned} \tilde{x}_1^i, \dots, \tilde{x}_{m_i}^i & \quad \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\ u_1^i, \dots, u_{n_i}^i & \quad \text{in degree } i \text{ for } i = -1, -2, \dots, e, \text{ and} \\ v_1^{k-1-i}, \dots, v_{n_i}^{k-1-i} & \quad \text{in degree } k-1-i \text{ for } i = 0, -1, \dots, e. \end{aligned} \tag{3.2}$$

So the upper index i in $\tilde{x}_j^i, u_j^i, v_j^i$ always indicates the degree.

Define a morphism $\alpha_+ : A_+^* \rightarrow B^*$ of commutative graded \mathbb{K} -algebras by $\alpha_+^0 = \alpha^0$ in degree 0 and

$$\alpha_+(x_j^i) = \tilde{x}_j^i, \quad i = -1, -2, \dots, d, \quad j = 1, \dots, m_i. \tag{3.3}$$

This is well-defined as A_+^* is freely generated over A^0 by the x_j^i .

Now choose a superpotential Ψ in B^k , which we require to satisfy

$$\sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + \alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial \Psi}{\partial \tilde{x}_j^i} = 0 \quad (3.4)$$

in B^{k+1} . Extend α_+ to a morphism $\alpha : A^* \rightarrow B^*$ by $\alpha|_{A_+^*} = \alpha_+$ and

$$\alpha(y_j^{k-i}) = (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i}, \quad i = 0, -1, \dots, d, \quad j = 1, \dots, m_i. \quad (3.5)$$

This is well-defined as A^* is freely generated over A_+^* by the y_j^{k-i} . Then from (2.11) and (3.5) we see that (3.4) may be rewritten

$$\sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + \alpha(\Phi) = 0.$$

Define the differential d in the cdga $B^\bullet = (B^*, d)$ by $d = 0$ on B^0 , and

$$\begin{aligned} d\tilde{x}_j^i &= (-1)^{i+1} \alpha_+(\Phi_j^{i+1}), \quad i = -1, -2, \dots, d, \quad j = 1, \dots, m_i, \\ du_j^i &= (-1)^{(i+1)k} \frac{\partial \Psi}{\partial v_j^{k-1-i}}, \quad dv_j^{k-1-i} = \frac{\partial \Psi}{\partial u_j^i}, \quad i = 0, -1, \dots, e, \quad j = 1, \dots, n_i. \end{aligned} \quad (3.6)$$

To prove that $d \circ d = 0$, note that for $i' = -1, \dots, d$ and $j' = 1, \dots, m_{i'}$ we have

$$\begin{aligned} d \circ d\tilde{x}_{j'}^{i'} &= (-1)^{i'+1} \sum_{i=-1}^d \sum_{j=1}^{m_i} d\tilde{x}_j^i \cdot \frac{\partial}{\partial \tilde{x}_j^i} [\alpha_+(\Phi_{j'}^{i'+1})] \\ &= (-1)^{i'+1} \alpha_+ \left[\sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \Phi_j^{i+1} \cdot \frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right] = 0, \end{aligned} \quad (3.7)$$

where in the first step we use that $\partial/\partial u_{j'}^{i'}$, $\partial/\partial v_{j'}^{k-1-i'}$ are zero on $\alpha_+(\Phi_j^{i+1})$ as this is a function of the \tilde{x}_j^i only, in the second (3.3) and the first line of (3.6), and in the third the second line of (2.12).

For $i' = -1, \dots, e$ and $j' = 1, \dots, n_{i'}$ we have

$$\begin{aligned}
 d \circ du_{j'}^{i'} &= (-1)^{(i'+1)k} \left[\sum_{i=-1}^d \sum_{j=1}^{m_i} d\tilde{x}_j^i \cdot \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial v_j^{k-1-i'}} \right. \\
 &\quad \left. + \sum_{i=-1}^e \sum_{j=1}^{n_i} du_j^i \cdot \frac{\partial^2 \Psi}{\partial u_j^i \partial v_j^{k-1-i'}} + \sum_{i=-1}^e \sum_{j=1}^{n_i} dv_j^{k-1-i} \cdot \frac{\partial^2 \Psi}{\partial v_j^{k-1-i} \partial v_j^{k-1-i'}} \right] \\
 &= (-1)^{(i'+1)k} \left[\sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \cdot \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial v_j^{k-1-i'}} \right. \\
 &\quad + \sum_{i=-1}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \frac{\partial \Psi}{\partial v_j^{k-1-i}} \cdot \frac{\partial^2 \Psi}{\partial u_j^i \partial v_j^{k-1-i'}} \\
 &\quad \left. + \sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \cdot \frac{\partial^2 \Psi}{\partial v_j^{k-1-i} \partial v_j^{k-1-i'}} \right] = 0, \tag{3.8}
 \end{aligned}$$

where in the first and second steps we use (3.6), and in the third we apply $\partial/\partial v_j^{k-1-i'}$ to (3.4), noting that $\partial/\partial v_j^{k-1-i'}$ is zero on $\alpha_+(\Phi_+)$, $\alpha_+(\Phi_j^{i+1})$ as these are functions of the \tilde{x}_j^i only, and dealing with signs appropriately. Similarly, for $i' = 0, \dots, e$ and $j' = 1, \dots, n_{i'}$ we have

$$\begin{aligned}
 d \circ dv_{j'}^{k-1-i'} &= \sum_{i=-1}^d \sum_{j=1}^{m_i} d\tilde{x}_j^i \cdot \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial u_{j'}^{i'}} \\
 &\quad + \sum_{i=-1}^e \sum_{j=1}^{n_i} du_j^i \cdot \frac{\partial^2 \Psi}{\partial u_j^i \partial u_{j'}^{i'}} + \sum_{i=-1}^e \sum_{j=1}^{n_i} dv_j^{k-1-i} \cdot \frac{\partial^2 \Psi}{\partial v_j^{k-1-i} \partial u_{j'}^{i'}} \\
 &= \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \cdot \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial u_{j'}^{i'}} \\
 &\quad + \sum_{i=-1}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \frac{\partial \Psi}{\partial v_j^{k-1-i}} \cdot \frac{\partial^2 \Psi}{\partial u_j^i \partial u_{j'}^{i'}} + \sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \cdot \frac{\partial^2 \Psi}{\partial v_j^{k-1-i} \partial u_{j'}^{i'}} = 0, \tag{3.9}
 \end{aligned}$$

where in the last step we apply $\partial/\partial u_{j'}^{i'}$ to (3.4).

This proves that $d \circ d = 0$, so B^\bullet is a standard form cdga over \mathbb{K} . Also

$$d \circ \alpha(x_{j'}^{i'}) = d\tilde{x}_{j'}^{i'} = (-1)^{i'+1} \alpha_+(\Phi_{j'}^{i'+1}) = \alpha[(-1)^{i'+1} \Phi_{j'}^{i'+1}] = \alpha \circ dx_{j'}^{i'}, \tag{3.10}$$

using equations (2.14), (3.3) and (3.6), and

$$\begin{aligned}
 d \circ \alpha(y_{j'}^{k-i'}) &= (-1)^{i'+1} d \frac{\partial \Psi}{\partial \tilde{x}_{j'}^{i'}} = (-1)^{i'+1} \left[\sum_{i=-1}^d \sum_{j=1}^{m_i} d\tilde{x}_j^i \cdot \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial \tilde{x}_{j'}^{i'}} \right. \\
 &\quad \left. + \sum_{i=-1}^e \sum_{j=1}^{n_i} du_j^i \cdot \frac{\partial^2 \Psi}{\partial u_j^i \partial \tilde{x}_{j'}^{i'}} + \sum_{i=-1}^e \sum_{j=1}^{n_i} dv_j^{k-1-i} \cdot \frac{\partial^2 \Psi}{\partial v_j^{k-1-i} \partial \tilde{x}_{j'}^{i'}} \right] \\
 &= (-1)^{i'+1} \left[\sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \cdot \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial \tilde{x}_{j'}^{i'}} \right. \\
 &\quad \left. + \sum_{i=-1}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \frac{\partial \Psi}{\partial v_j^{k-1-i}} \cdot \frac{\partial^2 \Psi}{\partial u_j^i \partial \tilde{x}_{j'}^{i'}} + \sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \cdot \frac{\partial^2 \Psi}{\partial v_j^{k-1-i} \partial \tilde{x}_{j'}^{i'}} \right] \\
 &= \frac{\partial}{\partial \tilde{x}_{j'}^{i'}} \alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} \frac{\partial}{\partial \tilde{x}_{j'}^{i'}} [\alpha_+(\Phi_j^{i+1})] \cdot (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} = \alpha \circ d(y_{j'}^{k-i'}),
 \end{aligned} \tag{3.11}$$

where in the first step we use (3.5), in the third (3.6), in the fourth we apply $\partial/\partial \tilde{x}_{j'}^{i'}$ to (3.4) and deal with signs, and in the fifth we use the second equation of (2.14) and (3.5). Equations (3.10)–(3.11) imply that $d \circ \alpha = \alpha \circ d : A^* \rightarrow B^*$, so $\alpha : A^\bullet \rightarrow B^\bullet$ and hence $\alpha_+ : A_+^\bullet \rightarrow B^\bullet$ are morphisms in $\mathbf{cdga}_{\mathbb{K}}$. Note that α_+ is a submersion, in the sense of Definition 3.1.

Following (2.6), define $h^0 \in (\Lambda^2 \Omega_{B^\bullet}^1)^{k-1}$ by

$$h^0 = \sum_{i=0}^e \sum_{j=1}^{n_i} d_{\mathrm{dR}} u_j^i d_{\mathrm{dR}} v_j^{k-1-i}. \tag{3.12}$$

Then $d_{\mathrm{dR}} h^0 = 0$, and

$$\begin{aligned}
 dh^0 &= \sum_{i=0}^e \sum_{j=1}^{n_i} [(d \circ d_{\mathrm{dR}} u_j^i) d_{\mathrm{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (d \circ d_{\mathrm{dR}} v_j^{k-1-i}) d_{\mathrm{dR}} u_j^i] \\
 &= - \sum_{i=0}^e \sum_{j=1}^{n_i} [(d_{\mathrm{dR}} \circ du_j^i) d_{\mathrm{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (d_{\mathrm{dR}} \circ dv_j^{k-1-i}) d_{\mathrm{dR}} u_j^i] \\
 &= - \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \left[d_{\mathrm{dR}} \left(\frac{\partial \Psi}{\partial v_j^{k-1-i}} \right) d_{\mathrm{dR}} v_j^{k-1-i} + d_{\mathrm{dR}} \left(\frac{\partial \Psi}{\partial u_j^i} \right) d_{\mathrm{dR}} u_j^i \right] \\
 &= d_{\mathrm{dR}} \left[- \sum_{i=0}^e \sum_{j=1}^{n_i} \left[d_{\mathrm{dR}} v_j^{k-1-i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + d_{\mathrm{dR}} u_j^i \frac{\partial \Psi}{\partial u_j^i} \right] \right] \\
 &= d_{\mathrm{dR}} \left[-d_{\mathrm{dR}} \Psi + \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\mathrm{dR}} \tilde{x}_j^i \frac{\partial \Psi}{\partial \tilde{x}_j^i} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\mathrm{dR}} \left(d_{\mathrm{dR}}(\alpha(x_j^i)) \cdot (-1)^{i+1} \alpha(y_j^{k-i}) \right) \\
 &= \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\mathrm{dR}}(\alpha(x_j^i)) d_{\mathrm{dR}}(\alpha(y_j^{k-i})) = \alpha_* \left[\sum_{i=0}^d \sum_{j=1}^{m_i} d_{\mathrm{dR}} x_j^i d_{\mathrm{dR}} y_j^{k-i} \right] = \alpha_*(\omega^0),
 \end{aligned} \tag{3.13}$$

using (3.12) in the first step, $d \circ d_{\mathrm{dR}} + d_{\mathrm{dR}} \circ d = 0$ in the second, (3.6) in the third, $d_{\mathrm{dR}} \circ d_{\mathrm{dR}} = 0$ in the fifth, (3.3) and (3.5) in the sixth, $\alpha_* \circ d_{\mathrm{dR}} = d_{\mathrm{dR}} \circ \alpha_*$ in the eighth, and (2.6) in the ninth.

Definition 2.9 and equation (3.13) now imply that $h := (h^0, 0, 0, \dots)$ is an isotropic structure for $\mathbf{Spec} \alpha : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ and the k -shifted symplectic structure $\omega = (\omega^0, 0, 0, \dots)$ on $\mathbf{Spec} A^\bullet$. We will now prove that this isotropic structure is nondegenerate, so that $\mathbf{Spec} B^\bullet$ is Lagrangian in $(\mathbf{Spec} A^\bullet, \omega)$. To do this, we have to show that the morphism $\chi : \mathbb{T}_{B^\bullet/A^\bullet} \rightarrow \mathbb{L}_{A^\bullet}[k-1]$ of B^\bullet -modules defined in (2.3) is a quasi-isomorphism.

It is enough to apply $-\otimes_{B^\bullet} H^0(B^\bullet)$, and show the corresponding morphism of complexes of $H^0(B^\bullet)$ -modules is an isomorphism. The analogue of (2.3) is

$$\begin{array}{ccc}
 (\Omega_{B^\bullet}^1)^\vee \otimes_{B^\bullet} H^0(B^\bullet) & \longrightarrow & 0 \\
 \downarrow (\Omega_\alpha^1)^\vee & \searrow^{h_L^0 \cdot \Downarrow} & \downarrow \\
 (\Omega_{A^\bullet}^1)^\vee \otimes_{A^\bullet} H^0(B^\bullet) & \xrightarrow{\omega^0} \Omega_{A^\bullet}^1[k] \otimes_{A^\bullet} H^0(B^\bullet) \xrightarrow{\Omega_\alpha^1[k]} \Omega_{B^\bullet}^1[k] \otimes_{B^\bullet} H^0(B^\bullet) & \\
 \downarrow & \searrow^{\chi[1]} & \\
 \mathbb{T}_{B^\bullet/A^\bullet}[1] \otimes_{B^\bullet} H^0(B^\bullet) & &
 \end{array} \tag{3.14}$$

Here we have used $\Omega_{A^\bullet}^1, \Omega_{B^\bullet}^1, (\Omega_{A^\bullet}^1)^\vee, (\Omega_{B^\bullet}^1)^\vee$ as models for $\mathbb{L}_{A^\bullet}, \mathbb{L}_{B^\bullet}, \mathbb{T}_{A^\bullet}, \mathbb{T}_{B^\bullet}$, since A^\bullet, B^\bullet are standard form cdgas. As a model for $\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet)$ we will use the cone of $(\Omega_\alpha^1)^\vee$ in (3.14), so that

$$\begin{aligned}
 &((\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet))^*, d) \\
 &= \left(((\Omega_{B^\bullet}^1)^\vee \otimes_{B^\bullet} H^0(B^\bullet))^* \oplus ((\Omega_{A^\bullet}^1)^\vee \otimes_{A^\bullet} H^0(B^\bullet))^{*-1}, \left(\begin{array}{cc} d_{B^\bullet} & 0 \\ (\Omega_\alpha^1)^\vee & d_{A^\bullet} \end{array} \right) \right).
 \end{aligned}$$

As $H^0(B^\bullet)$ -modules, the i^{th} graded pieces of $\Omega_{A^\bullet}^1 \otimes_{A^\bullet} H^0(B^\bullet), \Omega_{B^\bullet}^1 \otimes_{B^\bullet} H^0(B^\bullet), (\Omega_{A^\bullet}^1)^\vee \otimes_{A^\bullet} H^0(B^\bullet)$, and $(\Omega_{B^\bullet}^1)^\vee \otimes_{B^\bullet} H^0(B^\bullet)$ are

$$\begin{aligned}
 (\Omega_{A^\bullet}^1 \otimes_{A^\bullet} H^0(B^\bullet))^i &= \langle d_{\mathrm{dR}} x_j^i, j = 1, \dots, m_i, \\
 &\quad d_{\mathrm{dR}} y_j^i, j = 1, \dots, m_{k-i} \rangle_{H^0(B^\bullet)},
 \end{aligned} \tag{3.15}$$

$$\begin{aligned}
 (\Omega_{B^\bullet}^1 \otimes_{B^\bullet} H^0(B^\bullet))^i &= \langle d_{\mathrm{dR}} \tilde{x}_j^i, j = 1, \dots, m_i, \\
 &\quad d_{\mathrm{dR}} u_j^i, j = 1, \dots, n_i, d_{\mathrm{dR}} v_j^i, j = 1, \dots, n_{k-1-i} \rangle_{H^0(B^\bullet)},
 \end{aligned} \tag{3.16}$$

$$((\Omega_{A^\bullet}^1)^\vee \otimes_{A^\bullet} H^0(B^\bullet))^i = \left\langle \frac{\partial}{\partial x_j^{-i}}, j = 1, \dots, m_{-i}, \right. \\ \left. \frac{\partial}{\partial y_j^{-i}}, j = 1, \dots, m_{i-k} \right\rangle_{H^0(B^\bullet)}, \quad (3.17)$$

$$((\Omega_{B^\bullet}^1)^\vee \otimes_{B^\bullet} H^0(B^\bullet))^i = \left\langle \frac{\partial}{\partial \tilde{x}_j^{-i}}, j = 1, \dots, m_{-i}, \right. \\ \left. \frac{\partial}{\partial u_j^{-i}}, j = 1, \dots, n_{-i}, \frac{\partial}{\partial v_j^{-i}}, j = 1, \dots, n_{i+1-k} \right\rangle_{H^0(B^\bullet)}, \quad (3.18)$$

where $\langle \dots \rangle_{H^0(B^\bullet)}$ denotes the free $H^0(B^\bullet)$ -module with basis “...”.

The next diagram shows $\chi : \mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet) \rightarrow \Omega_{B^\bullet}^1[k-1] \otimes_{B^\bullet} H^0(B^\bullet)$ in degrees $i, i+1$, together with d in both complexes.

$$\begin{array}{ccc} (\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet))^i & & (\Omega_{B^\bullet}^1[k-1] \otimes_{B^\bullet} H^0(B^\bullet))^i \\ = \left\langle \frac{\partial}{\partial \tilde{x}_j^{-i}}, \forall j \right\rangle_{H^0(B^\bullet)} & & = \left\langle d_{\text{dR}} \tilde{x}_j^{k-1+i}, \forall j \right\rangle_{H^0(B^\bullet)} \\ \oplus \left\langle \frac{\partial}{\partial u_j^{-i}}, \frac{\partial}{\partial v_j^{-i}}, \forall j \right\rangle_{H^0(B^\bullet)} & \xrightarrow{\chi^i = \begin{pmatrix} 0 & 0 & * & \omega^0 \\ 0 & h^0 & * & 0 \end{pmatrix}} & \oplus \left\langle d_{\text{dR}} u_j^{k-1+i}, d_{\text{dR}} v_j^{k-1+i}, \forall j \right\rangle_{H^0(B^\bullet)} \\ \oplus \left\langle \frac{\partial}{\partial x_j^{1-i}}, \forall j \right\rangle_{H^0(B^\bullet)} & & \\ \oplus \left\langle \frac{\partial}{\partial y_j^{1-i}}, \forall j \right\rangle_{H^0(B^\bullet)} & & \\ \downarrow d = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ \alpha^* & 0 & * & 0 \\ * & * & * & * \end{pmatrix} & & \downarrow d = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \\ (\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet))^{i+1} & & (\Omega_{B^\bullet}^1[k-1] \otimes_{B^\bullet} H^0(B^\bullet))^{i+1} \\ = \left\langle \frac{\partial}{\partial \tilde{x}_j^{i-1}}, \forall j \right\rangle_{H^0(B^\bullet)} & & = \left\langle d_{\text{dR}} \tilde{x}_j^{k+i}, \forall j \right\rangle_{H^0(B^\bullet)} \\ \oplus \left\langle \frac{\partial}{\partial u_j^{i-1}}, \frac{\partial}{\partial v_j^{i-1}}, \forall j \right\rangle_{H^0(B^\bullet)} & \xrightarrow{\chi^{i+1} = \begin{pmatrix} 0 & 0 & * & \omega^0 \\ 0 & h^0 & * & 0 \end{pmatrix}} & \oplus \left\langle d_{\text{dR}} u_j^{k+i}, d_{\text{dR}} v_j^{k+i}, \forall j \right\rangle_{H^0(B^\bullet)} \\ \oplus \left\langle \frac{\partial}{\partial x_j^i}, \forall j \right\rangle_{H^0(B^\bullet)} & & \\ \oplus \left\langle \frac{\partial}{\partial y_j^i}, \forall j \right\rangle_{H^0(B^\bullet)} & & \end{array} \quad (3.19)$$

We have divided $\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet)$ into the direct sum of four pieces, and $\Omega_{B^\bullet}^1[k-1] \otimes_{B^\bullet} H^0(B^\bullet)$ into two. The morphisms d, χ^i, χ^{i+1} are written in matrix form, where “*” denotes some morphism. In the left hand d , the “ α^* ” maps $\frac{\partial}{\partial \tilde{x}_j^{-i}} \mapsto \frac{\partial}{\partial x_j^{-i}}$, up to sign. In χ^i , ω^0 maps $\frac{\partial}{\partial y_j^{1-i}} \mapsto d_{\text{dR}} \tilde{x}_j^{k-1+i}$, and h^0 maps $\frac{\partial}{\partial u_j^{-i}} \mapsto d_{\text{dR}} v_j^{k-1+i}$ and $\frac{\partial}{\partial v_j^{-i}} \mapsto d_{\text{dR}} u_j^{k-1+i}$, all up to sign, and similarly for χ^{i+1} . The important thing is that these α^*, ω^0, h^0 in d, χ^i, χ^{i+1} are all isomorphisms of $H^0(B^\bullet)$ -modules.

Now consider the graded $H^0(B^\bullet)$ -submodule

$$\begin{aligned} C^* &:= \{0\} \oplus \left\langle \frac{\partial}{\partial u_j^{-*}}, \frac{\partial}{\partial v_j^{-*}}, \forall j \right\rangle_{H^0(B^\bullet)} \oplus \{0\} \oplus \left\langle \frac{\partial}{\partial y_j^{1-*}}, \forall j \right\rangle_{H^0(B^\bullet)} \\ &\subseteq (\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet))^*. \end{aligned}$$

The form of the left hand “d” in (3.19) implies that C^* is closed under d, so $C^\bullet = (C^*, d)$ is a subcomplex of $\mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet)$. The isomorphism “ α^* ” plus two other zeroes in the left hand “d” in (3.19) imply that the inclusion $\text{inc} : C^\bullet \hookrightarrow \mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet)$ is a quasi-isomorphism. And the isomorphisms “ h^0 ”, “ ω^0 ” plus two other zeroes in χ^i, χ^{i+1} in (3.19) imply that $\chi|_{C^\bullet}$ is a strict isomorphism of complexes. Thus we have a commutative diagram

$$\begin{array}{ccc} C^\bullet & \xrightarrow{\chi|_{C^\bullet}} & \\ \simeq \downarrow \text{inc} & \cong & \\ \mathbb{T}_{B^\bullet/A^\bullet} \otimes_{B^\bullet} H^0(B^\bullet) & \xrightarrow{\chi} & \Omega_{B^\bullet}^1[k-1] \otimes_{B^\bullet} H^0(B^\bullet), \end{array} \quad (3.20)$$

so χ is a quasi-isomorphism. Therefore the isotropic structure h is nondegenerate, and $\mathbf{Spec} B^\bullet$ is Lagrangian in $(\mathbf{Spec} A^\bullet, \omega)$. We say that $A^\bullet, \omega, B^\bullet, \alpha, h$ are in *Lagrangian Darboux form*.

Following (2.9), define $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$ by

$$\psi = \sum_{i=0}^e \sum_{j=1}^{n_i} [i u_j^i d_{\text{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (k-1-i) v_j^{k-1-i} d_{\text{dR}} u_j^i]. \quad (3.21)$$

As a relative version of (2.10), we will prove that

$$d\Psi = -\alpha(\Phi + \Phi_+) \quad \text{in } B^{k+1}, \quad (3.22)$$

$$d_{\text{dR}} \Psi + d\psi = -\alpha_*(\phi + \phi_+) \quad \text{in } (\Omega_{B^\bullet}^1)^k, \text{ and} \quad (3.23)$$

$$d_{\text{dR}} \psi = (k-1)h^0 \quad \text{in } (\Lambda^2 \Omega_{B^\bullet}^1)^{k-1}. \quad (3.24)$$

Note that $dh^0 = \alpha_*(\omega^0)$ in (3.13) also follows from

$$\begin{aligned} (k-1)dh^0 &= d \circ d_{\text{dR}} \psi = -d_{\text{dR}} \circ d\psi = -d_{\text{dR}} [d_{\text{dR}} \Psi + d\psi] \\ &= d_{\text{dR}} \circ \alpha_*(\phi + \phi_+) \\ &= \alpha_*(d_{\text{dR}} \phi + d_{\text{dR}} \phi_+) = \alpha_*(k\omega^0 - \omega^0) = (k-1)\alpha_*(\omega^0), \end{aligned}$$

using equations (2.10), (2.16), (3.23), and (3.24).

For equation (3.22), we have

$$\begin{aligned}
 d\Psi &= \sum_{i=-1}^d \sum_{j=1}^{m_i} d\tilde{x}_j^i \frac{\partial\Psi}{\partial\tilde{x}_j^i} + \sum_{i=-1}^e \sum_{j=1}^{n_i} du_j^i \frac{\partial\Psi}{\partial u_j^i} + \sum_{i=-1}^e \sum_{j=1}^{n_i} dv_j^{k-1-i} \frac{\partial\Psi}{\partial v_j^{k-1-i}} \\
 &= \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial\Psi}{\partial\tilde{x}_j^i} + \sum_{i=-1}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \frac{\partial\Psi}{\partial v_j^{k-1-i}} \frac{\partial\Psi}{\partial u_j^i} \\
 &\quad + \sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial\Psi}{\partial u_j^i} \frac{\partial\Psi}{\partial v_j^{k-1-i}} \tag{3.25} \\
 &= \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial\Psi}{\partial\tilde{x}_j^i} + 2 \sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial\Psi}{\partial u_j^i} \frac{\partial\Psi}{\partial v_j^{k-1-i}} \\
 &= \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial\Psi}{\partial\tilde{x}_j^i} \\
 &\quad - 2 \left[\alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial\Psi}{\partial\tilde{x}_j^i} \right] \\
 &= -\alpha(\Phi + \Phi_+),
 \end{aligned}$$

using (3.6) in the second step, (3.4) in the fourth, and (2.11) and (3.5) in the fifth. For equation (3.23), we have

$$\begin{aligned}
 d_{\text{dR}}\Psi + d\psi &= \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}}\tilde{x}_j^i \frac{\partial\Psi}{\partial\tilde{x}_j^i} + \sum_{i=0}^e \sum_{j=1}^{n_i} \left[d_{\text{dR}}u_j^i \frac{\partial\Psi}{\partial u_j^i} + d_{\text{dR}}v_j^{k-1-i} \frac{\partial\Psi}{\partial v_j^{k-1-i}} \right] \\
 &\quad + \sum_{i=0}^e \sum_{j=1}^{n_i} \left[i du_j^i d_{\text{dR}}v_j^{k-1-i} + (-1)^{(i+1)k} (k-1-i) dv_j^{k-1-i} d_{\text{dR}}u_j^i \right] \\
 &\quad + \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} \left[(-1)^{i'} i' u_{j'}^{i'} d \circ d_{\text{dR}}v_{j'}^{k-1-i'} - (-1)^{i'+k} (k-1-i') v_{j'}^{k-1-i'} d \circ d_{\text{dR}}u_{j'}^{i'} \right] \\
 &= \sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{(i+1)k} \frac{\partial\Psi}{\partial\tilde{x}_j^i} d_{\text{dR}}\tilde{x}_j^i \\
 &\quad + \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \left[(i+1) \frac{\partial\Psi}{\partial v_j^{k-1-i}} d_{\text{dR}}v_j^{k-1-i} + (k-i) \frac{\partial\Psi}{\partial u_j^i} d_{\text{dR}}u_j^i \right] \\
 &\quad - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} \left[(-1)^{i'} i' u_{j'}^{i'} d_{\text{dR}} \left[\frac{\partial\Psi}{\partial u_{j'}^{i'}} \right] - (-1)^{i'+k} (k-1-i') v_{j'}^{k-1-i'} d_{\text{dR}} \left[\frac{\partial\Psi}{\partial v_{j'}^{k-1-i'}} \right] \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=0}^d \sum_{j=1}^{m_i} \left[(-1)^{(i+1)k} \frac{\partial \Psi}{\partial \widetilde{x}_j^i} - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(i+1)k} i' u_{j'}^{i'} \frac{\partial^2 \Psi}{\partial u_{j'}^{i'} \partial \widetilde{x}_j^i} \right. \\
 &\quad \left. - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(i+1)k} (k-1-i') v_{j'}^{k-1-i'} \frac{\partial^2 \Psi}{\partial v_{j'}^{k-1-i'} \partial \widetilde{x}_j^i} \right] d_{\text{dR}} \widetilde{x}_j^i \\
 &+ \sum_{i=0}^e \sum_{j=1}^{n_i} \left[(-1)^{(i+1)k} (k-i) \frac{\partial \Psi}{\partial u_j^i} - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(i+1)k} i' u_{j'}^{i'} \frac{\partial^2 \Psi}{\partial u_{j'}^{i'} \partial u_j^i} \right. \\
 &\quad \left. - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(i+1)k} (k-1-i') v_{j'}^{k-1-i'} \frac{\partial^2 \Psi}{\partial v_{j'}^{k-1-i'} \partial u_j^i} \right] d_{\text{dR}} u_j^i \\
 &+ \sum_{i=0}^e \sum_{j=1}^{n_i} \left[(-1)^{(i+1)k} (i+1) \frac{\partial \Psi}{\partial v_j^{k-1-i}} - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(i+1)k} i' u_{j'}^{i'} \frac{\partial^2 \Psi}{\partial u_{j'}^{i'} \partial v_j^{k-1-i}} \right. \\
 &\quad \left. - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(i+1)k} (k-1-i') v_{j'}^{k-1-i'} \frac{\partial^2 \Psi}{\partial v_{j'}^{k-1-i'} \partial v_j^{k-1-i}} \right] d_{\text{dR}} v_j^{k-1-i} \\
 &= - \sum_{i=0}^d \sum_{j=1}^{m_i} \left[(-1)^{(i+1)k} (k-1-i) \frac{\partial \Psi}{\partial \widetilde{x}_j^i} - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i+1)k} i' \widetilde{x}_{j'}^{i'} \frac{\partial^2 \Psi}{\partial \widetilde{x}_{j'}^{i'} \partial \widetilde{x}_j^i} \right] d_{\text{dR}} \widetilde{x}_j^i \\
 &+ \sum_{i=0}^e \sum_{j=1}^{n_i} \sum_{i'=0}^e \sum_{j'=1}^{m_{i'}} (-1)^{(i+1)k} i' \widetilde{x}_{j'}^{i'} \left[\frac{\partial^2 \Psi}{\partial \widetilde{x}_{j'}^{i'} \partial u_j^i} d_{\text{dR}} u_j^i + \frac{\partial^2 \Psi}{\partial \widetilde{x}_{j'}^{i'} \partial v_j^{k-1-i}} d_{\text{dR}} v_j^{k-1-i} \right] \\
 &= - \sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{(i+1)(k+1)} (k-1-i) \alpha(y_j^{k-1-i}) d_{\text{dR}} [\alpha(x_j^i)] \\
 &\quad - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} i' \alpha(x_{j'}^{i'}) d_{\text{dR}} [\alpha(y_j^{k-i})] \\
 &= -\alpha_* \left[\sum_{i=0}^d \sum_{j=1}^{m_i} [(-1)^{(i+1)(k+1)} (k-1-i) y_j^{k-i} d_{\text{dR}} x_j^i + i x_j^i d_{\text{dR}} y_j^{k-i}] \right] \\
 &= -\alpha_*(\phi + \phi_+), \tag{3.26}
 \end{aligned}$$

using (3.21) in the first step, (3.6) and $d \circ d_{\text{dR}} + d_{\text{dR}} \circ d = 0$ in the second, in the fourth that

$$\begin{aligned}
 &\left[\sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} i' x_{j'}^{i'} \frac{\partial}{\partial \widetilde{x}_{j'}^{i'}} + \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} i' u_{j'}^{i'} \frac{\partial}{\partial u_{j'}^{i'}} + (k-1-i') v_{j'}^{k-1-i'} \frac{\partial}{\partial v_{j'}^{k-1-i'}} \right] \frac{\partial \Psi}{\partial \widetilde{x}_j^i} \\
 &= (k-i) \frac{\partial \Psi}{\partial \widetilde{x}_j^i},
 \end{aligned}$$

since $\frac{\partial \Psi}{\partial \tilde{x}_j^i}$ has degree $k - i$, and similar equations for $\frac{\partial \Psi}{\partial u_j^i}, \frac{\partial \Psi}{\partial v_j^{k-1-i}}$, equations (3.3) and (3.5) in the fifth, and (2.9) and (2.15) in the seventh. Equation (3.24) is immediate from (3.12) and (3.21).

Let us summarize our progress:

- Example 2.14 defined a cdga A^\bullet and $\omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} x_j^i d_{\text{dR}} y_j^{k-i}$ in $(\Lambda^2 \Omega_{A^\bullet}^1)^k$ such that $\omega = (\omega^0, 0, \dots)$ is k -shifted symplectic on $\mathbf{Spec} A^\bullet$, and $\Phi \in A^{k+1}$, $\phi \in (\Omega_{A^\bullet}^1)^k$ with $d\Phi = 0$, $d_{\text{dR}} \Phi + d\phi = 0$ and $d_{\text{dR}} \phi = k\omega^0$.
- Remark 2.15 defined a sub-cdga $A_+^\bullet \subseteq A^\bullet$ and $\Phi_+ \in A_+^{k+1}$, $\phi_+ \in (\Omega_{A_+^\bullet}^1)^k$ satisfying $d\Phi_+ = 0$, $d_{\text{dR}} \Phi_+ + d\phi_+ = 0$ and $d_{\text{dR}} \phi_+ = -\omega^0$.
- We define a cdga B^\bullet and a morphism $\alpha : A^\bullet \rightarrow B^\bullet$ such that $\alpha_+ := \alpha|_{A_+^\bullet} : A_+^\bullet \rightarrow B^\bullet$ is a submersion, a Lagrangian isotropic structure $h = (h^0, 0, 0, \dots)$ with $h^0 = \sum_{i=0}^e \sum_{j=1}^{n_i} d_{\text{dR}} u_j^i d_{\text{dR}} v_j^{k-1-i}$ for α , and $\Psi \in B^k$, $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$ with $d\Psi = -\alpha(\Phi + \Phi_+)$, $d_{\text{dR}} \Psi + d\psi = -\alpha_*(\phi + \phi_+)$ and $d_{\text{dR}} \psi = (k-1)h^0$. We say $A^\bullet, \omega, B^\bullet, \alpha, h$ are in *Lagrangian Darboux form*.

This finally concludes Example 3.3.

Remark 3.4. — Bouaziz and Grojnowski [5] proved their own k -shifted symplectic Darboux Theorem independently of [7], showing that any k -shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ for $k < 0$ with $k \not\equiv 2 \pmod{4}$ is étale locally equivalent to a *twisted k -shifted cotangent bundle* $T_t^*[k]\mathbf{Y}$, where \mathbf{Y} is an affine derived \mathbb{K} -scheme, and $t \in \mathcal{O}_{\mathbf{Y}}^{k+1}$ with $dt = 0$ is used to “twist” the k -shifted cotangent bundle $T^*[k]\mathbf{Y}$. Remark 2.15 related their picture to Theorem 2.18.

We can explain our k -shifted Lagrangian Neighbourhood Theorem 3.7(i) below in the style of Bouaziz and Grojnowski [5], by saying that a k -shifted Lagrangian $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ for $k \not\equiv 3 \pmod{4}$ is étale locally equivalent to the *twisted $(k-1)$ -shifted relative cotangent bundle* $T_{u/t}^*[k-1](\mathbf{Z}/\mathbf{Y}) \rightarrow T_t^*[k]\mathbf{Y}$ of a morphism of affine derived \mathbb{K} -schemes $\mathbf{g} : \mathbf{Z} \rightarrow \mathbf{Y}$, with “twisting” $u \in \mathcal{O}_{\mathbf{Z}}^k$ satisfying $du + \mathbf{g}^*(t) = 0$, or equivalently, to the *twisted k -shifted conormal bundle* $N_{u/t}^*[k](\mathbf{Z}/\mathbf{Y}) \rightarrow T_t^*[k]\mathbf{Y}$, since $N^*(\mathbf{Z}/\mathbf{Y}) = T^*[-1](\mathbf{Z}/\mathbf{Y})$.

Explicitly, in the situation of Example 3.3 and following Remark 2.15, define B_+^\bullet to be the sub-cdga of B^\bullet generated by B^0 and the variables \tilde{x}_j^i, u_j^i for all $i < 0$ and j . Then B^* is freely generated over B_+^* by the variables v_j^{k-1-i} for all i, j . Also α_+ maps $A_+^\bullet \rightarrow B_+^\bullet \subseteq B^\bullet$. For degree reasons Ψ can

be at most linear in the v_j^{k-1-i} , so as in (2.11) we may write

$$\Psi = \Psi_+ + \sum_{i=-1}^e \sum_{j=1}^{n_i} \Psi_j^{i+1} v_j^{k-1-i}, \quad (3.27)$$

where $\Psi_+ \in B_+^k$ and $\Psi_j^{i+1} \in B_+^{i+1}$ for all i, j do not involve the variables v_j^{k-1-i} .

Then as in (2.12)–(2.13), equation (3.4) is equivalent to the equations

$$\sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \Psi_j^{i+1} \frac{\partial \Psi_+}{\partial u_j^i} + \alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial \Psi_+}{\partial \tilde{x}_j^i} = 0, \quad (3.28)$$

$$\sum_{i=-1}^{i'+1} \sum_{j=1}^{m_i} (-1)^{i+1} \Psi_j^{i+1} \frac{\partial \Psi_{j'}^{i'+1}}{\partial u_j^i} + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial \Psi_{j'}^{i'+1}}{\partial \tilde{x}_j^i} = 0, \quad (3.29)$$

where (3.28) holds in B_+^{k+1} , and (3.29) holds in $B_+^{i'+2}$ for all $i' = -1, \dots, e$ and $j' = 1, \dots, n_{i'}$. Also equations (3.5) and (3.6) may be rewritten

$$\alpha(y_j^{k-i}) = (-1)^{i+1} \left[\frac{\partial \Psi_+}{\partial \tilde{x}_j^i} + \sum_{i'=i-1}^e \sum_{j'=1}^{n_{i'}} \frac{\partial \Psi_{j'}^{i'+1}}{\partial \tilde{x}_j^i} v_{j'}^{k-1-i'} \right], \quad (3.30)$$

$$\begin{aligned} d\tilde{x}_j^i &= (-1)^{i+1} \alpha_+(\Phi_j^{i+1}), & du_j^i &= (-1)^{i+1} \Psi_j^{i+1}, \\ dv_j^{k-1-i} &= \frac{\partial \Psi_+}{\partial u_j^i} + \sum_{i'=i-1}^e \sum_{j'=1}^{n_{i'}} \frac{\partial \Psi_{j'}^{i'+1}}{\partial u_j^i} v_{j'}^{k-1-i'}. \end{aligned} \quad (3.31)$$

From these we see that (3.28) is equivalent to

$$d\Psi_+ + \alpha_+(\Phi_+) = 0. \quad (3.32)$$

Now write $\mathbf{Y} = \mathbf{Spec} A_+^\bullet$ and $\mathbf{Z} = \mathbf{Spec} B_+^\bullet$, as affine derived \mathbb{K} -schemes, and $\mathbf{g} = \mathbf{Spec} \alpha_+ : \mathbf{Z} \rightarrow \mathbf{Y}$. As in Remark 2.15, we interpret $\mathbf{Spec} A^\bullet$ as a twisted k -shifted cotangent bundle $T_t^*[k]\mathbf{Y}$ with projection $\mathbf{Spec} \iota : T_t^*[k]\mathbf{Y} \rightarrow \mathbf{Y}$, where $\iota : A_+^\bullet \hookrightarrow A^\bullet$ is the inclusion, and the “twist” $t \in \mathcal{O}_{\mathbf{Y}}^{k+1}$ is $t = \Phi_+$. Similarly, we interpret $\mathbf{Spec} B^\bullet$ as a twisted k -shifted conormal bundle $N_{u/t}^*[k](\mathbf{Z}/\mathbf{Y})$ of $\mathbf{g} : \mathbf{Z} \rightarrow \mathbf{Y}$, with projection $\mathbf{Spec} j : N_{u/t}^*[k](\mathbf{Z}/\mathbf{Y}) \rightarrow \mathbf{Z}$, where $j : B_+^\bullet \hookrightarrow B^\bullet$ is the inclusion, and we interpret $\mathbf{Spec} \alpha$ as the twisted inclusion morphism $N_{u/t}^*[k](\mathbf{Z}/\mathbf{Y}) \rightarrow T_t^*[k]\mathbf{Y}$.

Here Ψ_+ is the “twist” u of the k -shifted conormal bundle $N^*[k](\mathbf{Z}/\mathbf{Y})$, and (3.32) is the compatibility condition $du + \mathbf{g}^*(t) = 0$ with the “twist” $t = \Phi_+$ of $T^*[k]\mathbf{Y}$. The data Ψ_j^{i+1} in Ψ in (3.27) defines the differential d in

$B_+^\bullet = (B_+^*, d)$, via $du_j^i = (-1)^{(i+1)k} \Psi_j^{i+1}$ in (3.31). Equation (3.29) means that $d \circ d = 0$ in $B_+^\bullet = (B_+^*, d)$, necessary for B_+^\bullet to be a cdga and \mathcal{Z} a derived scheme.

The next example, defining “weak Lagrangian Darboux form” and “strong Lagrangian Darboux form” when $k \equiv 3 \pmod{4}$, is related to Example 3.3 in the same way that Example 2.16 is related to Example 2.14 in Section 2.5.

Example 3.5. — Let $k < 0$ with $k \equiv 3 \pmod{4}$, suppose A^\bullet, ω are in k -shifted Darboux form, as in Example 2.14, and use the notation of Remark 2.15. These define a standard form cdga A^\bullet over \mathbb{K} , a sub-cdga $A_+^\bullet \subseteq A^\bullet$, coordinates x_j^i in $A_+^i \subseteq A^i$ and y_j^{k-i} in A^{k-i} for $i = 0, -1, \dots, d = [(k+1)/2]$ and $j = 1, \dots, m_i$, and a k -shifted 2-form $\omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} x_j^i d_{\text{dR}} y_j^{k-i}$. They also define $\Phi \in A^{k+1}$ satisfying (2.7), which determines the differential d in A^\bullet by (2.8), and $\phi \in (\Omega_{A^\bullet}^1)^k$ satisfying $d\Phi = 0$, $d_{\text{dR}}\Phi + d\phi = 0$ and $d_{\text{dR}}\phi = k\omega^0$. As in (2.11) we write $\Phi = \Phi_+ + \sum_{i=-1}^d \sum_{j=1}^{m_i} \Phi_j^{i+1} y_j^{k-i}$, where $\Phi_+ \in A_+^{k+1}$ and $\Phi_j^{i+1} \in A_+^{i+1}$ for all i, j do not involve the y_j^i , and we define $\phi_+ \in (\Omega_{A^\bullet}^1)^k$ as in (2.15).

Write $e = [k/2]$, so that $e = d-1$ and $k = 2e+1 = 2d-1$. Choose nonnegative integers n_0, n_{-1}, \dots, n_e . Choose a smooth \mathbb{K} -algebra B^0 of dimension $m_0 + n_0$, and a smooth morphism $\alpha^0 : A^0 \rightarrow B^0$. Localizing B^0 if necessary, we assume there exist $u_1^0, \dots, u_{n_0}^0 \in B^0$ such that $d_{\text{dR}} \tilde{x}_1^0, \dots, d_{\text{dR}} \tilde{x}_{m_0}^0, d_{\text{dR}} u_1^0, \dots, d_{\text{dR}} u_{n_0}^0$ form a basis of $\Omega_{B^0}^1$ over B^0 , where we write $\tilde{x}_j^0 = \alpha^0(x_j^0) \in B^0$.

As in (3.2), define B^* as a commutative graded \mathbb{K} -algebra to be the free graded algebra over B^0 generated by variables

$$\begin{array}{ll} \tilde{x}_1^i, \dots, \tilde{x}_{m_i}^i & \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\ u_1^i, \dots, u_{n_i}^i & \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\ w_1^e, \dots, w_{n_e}^e & \text{in degree } e, \text{ and} \\ v_1^{k-1-i}, \dots, v_{n_i}^{k-1-i} & \text{in degree } k-1-i \text{ for } i = 0, -1, \dots, d. \end{array}$$

So the upper index i in $\tilde{x}_j^i, u_j^i, v_j^i, w_j^i$ always indicates the degree.

As in (3.3), define a morphism $\alpha_+ : A_+^* \rightarrow B^*$ of commutative graded \mathbb{K} -algebras by $\alpha_+^0 = \alpha^0$ in degree 0 and

$$\alpha_+(x_j^i) = \tilde{x}_j^i, \quad i = -1, -2, \dots, d, \quad j = 1, \dots, m_i.$$

This is well-defined as A_+^* is freely generated over A^0 by the x_j^i .

Let q_1, \dots, q_{n_e} be invertible elements of B^0 . Choose a superpotential Ψ in B^k , which as in (2.18) and (3.4) we require to satisfy

$$\begin{aligned} \sum_{i=-1}^d \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + \frac{1}{4} \sum_{j=1}^{n_e} \frac{1}{q_j} \left(\frac{\partial \Psi}{\partial w_j^e} \right)^2 \\ + \alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial \Psi}{\partial \tilde{x}_j^i} = 0 \end{aligned} \quad (3.33)$$

in B^{k+1} . As in (3.5), extend α_+ to $\alpha : A^* \rightarrow B^*$ by $\alpha|_{A_+^*} = \alpha_+$ and

$$\begin{aligned} \alpha(y_j^{k-i}) &= (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i}, \quad i = -1, \dots, d, \quad j = 1, \dots, m_i, \\ \alpha(y_j^k) &= -\frac{\partial \Psi}{\partial \tilde{x}_j^0} + \sum_{j'=1}^{n_e} \frac{w_{j'}^e}{2q_{j'}} \frac{\partial q_{j'}}{\partial \tilde{x}_j^0} \frac{\partial \Psi}{\partial w_{j'}^e}, \quad j = 1, \dots, m_0. \end{aligned} \quad (3.34)$$

This is well-defined as A^* is freely generated over A_+^* by the y_j^{k-i} . Then from (2.11) and (3.34) we see that (3.33) may be rewritten

$$\sum_{i=-1}^d \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + \frac{1}{4} \sum_{j=1}^{n_e} \frac{1}{q_j} \left(\frac{\partial \Psi}{\partial w_j^e} \right)^2 + \alpha(\Phi) = 0.$$

As in (2.19) and (3.6), define the differential d in the cdga $B^\bullet = (B^*, d)$ by $d = 0$ on B^0 , and

$$\begin{aligned} d\tilde{x}_j^i &= (-1)^{i+1} \alpha_+(\Phi_j^{i+1}), & i = -1, -2, \dots, d, \quad j = 1, \dots, m_i, \\ du_j^i &= (-1)^{i+1} \frac{\partial \Psi}{\partial v_j^{k-1-i}}, & i = -1, -2, \dots, d, \quad j = 1, \dots, n_i, \\ dv_j^{k-1-i} &= \frac{\partial \Psi}{\partial u_j^i}, & i = -1, -2, \dots, d, \quad j = 1, \dots, n_i, \\ dv_j^{k-1} &= \frac{\partial \Psi}{\partial u_j^0} - \sum_{j'=1}^{n_e} \frac{w_{j'}^e}{2q_{j'}} \frac{\partial q_{j'}}{\partial u_j^0} \frac{\partial \Psi}{\partial w_{j'}^e}, & j = 1, \dots, n_0, \\ dw_j^e &= \frac{1}{2q_j} \frac{\partial \Psi}{\partial w_j^e}, & j = 1, \dots, n_e. \end{aligned} \quad (3.35)$$

We prove that $d \circ d = 0$ as in (3.7)–(3.9), applying $\partial/\partial u_{j'}^{i'}$, $\partial/\partial v_{j'}^{k-1-i'}$, $\partial/\partial w_{j'}^e$ to (3.33). Thus B^\bullet is a standard form cdga over \mathbb{K} .

As in (3.10)–(3.11) we can check that $d \circ \alpha(x_j^i) = \alpha \circ dx_j^i$ and $d \circ \alpha(y_j^{k-i}) = \alpha \circ dy_j^{k-i}$, so that $d \circ \alpha = \alpha \circ d$, and $\alpha : A^\bullet \rightarrow B^\bullet$ is a cdga morphism.

Following (2.17) and (3.12), define $h^0 \in (\Lambda^2 \Omega_{B^\bullet}^1)^{k-1}$ by

$$h^0 = \sum_{i=0}^d \sum_{j=1}^{n_i} d_{\mathrm{dR}} u_j^i d_{\mathrm{dR}} v_j^{k-1-i} + \sum_{j=1}^{n_e} d_{\mathrm{dR}} (q_j w_j^e) d_{\mathrm{dR}} w_j^e. \quad (3.36)$$

Then $d_{\mathrm{dR}} h^0 = 0$, and as in (3.13) we can show that $dh^0 = \alpha_*(\omega^0)$. Therefore $h := (h^0, 0, 0, \dots)$ is an isotropic structure for $\mathbf{Spec} \alpha : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ and the k -shifted symplectic structure $\omega = (\omega^0, 0, 0, \dots)$ on $\mathbf{Spec} A^\bullet$. Following the argument of (3.14)–(3.20) we can prove this isotropic structure is nondegenerate, so that $\mathbf{Spec} B^\bullet$ is Lagrangian in $(\mathbf{Spec} A^\bullet, \omega)$.

Following (2.20) and (3.21), define $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$ by

$$\begin{aligned} \psi = \sum_{i=0}^d \sum_{j=1}^{n_i} [i u_j^i d_{\mathrm{dR}} v_j^{k-1-i} + (-1)^{i+1} (k-1-i) v_j^{k-1-i} d_{\mathrm{dR}} u_j^i] \\ + (k-1) \sum_{j=1}^{n_e} q_j w_j^e d_{\mathrm{dR}} w_j^e. \end{aligned}$$

As in (3.25)–(3.26), we can show that equations (3.22)–(3.24) hold.

Following the notation of weak and strong Darboux form in Example 2.16, we say that $A^\bullet, \omega, B^\bullet, \alpha, h$ are in *weak Lagrangian Darboux form*. If all the above holds with $q_j = 1$ for $j = 1, \dots, n_e$, we say that $A^\bullet, \omega, B^\bullet, \alpha, h$ are in *strong Lagrangian Darboux form*. This concludes Example 3.5.

The next example, similar to Example 2.20, discusses Example 3.5 in more detail when $k = -1$.

Example 3.6. — Consider “weak Lagrangian Darboux form” in Example 3.5 when $k = -1$. Example 2.14 gives A^\bullet, ω , where A^0 is a smooth \mathbb{K} -algebra, elements $x_1^0, \dots, x_{m_0}^0 \in A^0$ such that $d_{\mathrm{dR}} x_1^0, \dots, d_{\mathrm{dR}} x_{m_0}^0$ form a basis of $\Omega_{A^0}^1$ over A^0 , and a Hamiltonian $\Phi \in A^0$. The classical master equation (2.7) is trivial in this case, so Φ is arbitrary. We have $A^\bullet = A^0[y_1^{-1}, \dots, y_{m_0}^{-1}]$, where $y_1^{-1}, \dots, y_{m_0}^{-1}$ have degree -1 , with differential

$$dx_j^0 = 0, \quad dy_j^{-1} = \frac{\partial \Phi}{\partial x_j^0}, \quad j = 1, \dots, m_0,$$

and -1 -shifted 2-form

$$\omega^0 = d_{\mathrm{dR}} x_1^0 d_{\mathrm{dR}} y_1^{-1} + \dots + d_{\mathrm{dR}} x_{m_0}^0 d_{\mathrm{dR}} y_{m_0}^{-1}.$$

Then $\omega = (\omega^0, 0, 0, \dots)$ is a -1 -shifted symplectic structure on $\mathbf{X} = \mathbf{Spec} A^\bullet$. Note that $H^0(A^\bullet) = A^0 / (\frac{\partial \Phi}{\partial x_1^0}, \dots, \frac{\partial \Phi}{\partial x_{m_0}^0}) = A^0 / (d_{\mathrm{dR}} \Phi)$.

Geometrically, $U = \mathbf{Spec} A^0$ is a smooth classical \mathbb{K} -scheme with étale coordinates $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$, and $\Phi : U \rightarrow \mathbb{A}^1$ is regular, and

$\mathbf{X} = \mathbf{Crit}(\Phi)$ is the derived critical locus of Φ , with $X = t_0(\mathbf{X})$ the classical critical locus $\mathbf{Crit}(\Phi)$. As in Example 2.14 and [7, Prop. 5.7(b)], the restriction $\Phi|_{X^{\text{red}}} : X^{\text{red}} \rightarrow \mathbb{A}^1$ of Φ to the reduced \mathbb{K} -subscheme X^{red} of X is locally constant. By adding a constant to Φ , we suppose that $\Phi|_{X^{\text{red}}} = 0$.

Example 3.5 now chooses a smooth \mathbb{K} -algebra B^0 , a smooth morphism $\alpha^0 : A^0 \rightarrow B^0$, and elements $u_1^0, \dots, u_{n_0}^0$ such that $d_{\text{dR}}\tilde{x}_1^0, \dots, d_{\text{dR}}\tilde{x}_{m_0}^0, d_{\text{dR}}u_1^0, \dots, d_{\text{dR}}u_{n_0}^0$ form a basis of $\Omega_{B^0}^1$ over B^0 , where $\tilde{x}_j^0 = \alpha^0(x_j^0)$. As a graded \mathbb{K} -algebra we have $B^* = B^0[v_1^{-2}, \dots, v_{n_0}^{-2}, w_1^{-1}, \dots, w_{n-1}^{-1}]$ for some $n-1 \geq 0$, with v_j^{-2} in degree -2 and w_j^{-1} in degree -1 .

We choose invertible elements q_1, \dots, q_{n-1} in B^0 , and a superpotential $\Psi \in B^{-1}$, which we write in the form

$$\Psi = s_1 w_1^{-1} + \dots + s_{n-1} w_{n-1}^{-1}$$

for $s_1, \dots, s_{n-1} \in B^0$. The p.d.e. (3.33) which Ψ must satisfy reduces to

$$\frac{1}{4} \sum_{j=1}^{n-1} \frac{(s_j)^2}{q_j} + \alpha^0(\Phi) = 0. \tag{3.37}$$

By (3.34), the morphism $\alpha : A^\bullet \rightarrow B^\bullet$ is determined by $\alpha|_{A^0} = \alpha^0$ and

$$\alpha(y_j^{-1}) = - \sum_{j'=1}^{n_e} \left[\frac{\partial s_{j'}}{\partial \tilde{x}_j^0} - \frac{s_{j'}}{2q_{j'}} \frac{\partial q_{j'}}{\partial \tilde{x}_j^0} \right] w_{j'}^{-1}, \quad j = 1, \dots, m_0.$$

By (3.35), the differential d in $B^\bullet = (B^*, d)$ is given by $d = 0$ on B^0 and

$$\begin{aligned} dv_j^{-2} &= \sum_{j'=1}^{n_e} \left[\frac{\partial s_{j'}}{\partial u_j^0} - \frac{s_{j'}}{2q_{j'}} \frac{\partial q_{j'}}{\partial u_j^0} \right] w_{j'}^{-1}, & j &= 1, \dots, n_0, \\ dw_j^{-1} &= \frac{s_j}{2q_j}, & j &= 1, \dots, n-1. \end{aligned}$$

Then $d \circ dv_j^{-2} = 0$ follows by applying $\frac{\partial}{\partial u_j^0}$ to (3.37). The Lagrangian structure is $h = (h^0, 0, 0, \dots)$, where $h^0 \in (\Lambda^2 \Omega_{B^\bullet}^1)^{-2}$ is given by

$$h^0 = \sum_{j=1}^{n_0} d_{\text{dR}}u_j^0 d_{\text{dR}}v_j^{-2} + \sum_{j=1}^{n-1} d_{\text{dR}}(q_j w_j^{-1}) d_{\text{dR}}w_j^{-1}.$$

Geometrically, we have a smooth classical \mathbb{K} -scheme $V = \mathbf{Spec} B^0$ with étale coordinates $(\tilde{x}_1^0, \dots, \tilde{x}_{m_0}^0, u_1^0, \dots, u_{n_0}^0) : V \rightarrow \mathbb{A}^{m_0+n_0}$, a smooth morphism $\pi = \mathbf{Spec} \alpha^0 : V \rightarrow U$ acting in coordinates by $\pi :$

$(\tilde{x}_1^0, \dots, \tilde{x}_{m_0}^0, u_1^0, \dots, u_{n_0}^0) \mapsto (\tilde{x}_1^0, \dots, \tilde{x}_{m_0}^0)$, a trivial vector bundle $E \rightarrow V$ with fibre \mathbb{K}^{n-1} , a nondegenerate quadratic form Q on E given by

$$Q(e_1, \dots, e_{n-1}) = \frac{(e_1)^2}{q_1} + \dots + \frac{(e_{n-1})^2}{q_{n-1}}$$

for all regular functions $e_1, \dots, e_{n-1} : V \rightarrow \mathbb{A}^1$, and a section $s = (s_1, \dots, s_{n-1})$ in $H^0(E)$ which by (3.37) satisfies

$$Q(s, s) + 4\pi^*(\Phi) = 0.$$

To summarize:

- The important geometric data in writing a -1 -shifted symplectic derived \mathbb{K} -scheme (\mathbf{X}, ω) in “Darboux form” is a smooth \mathbb{K} -scheme U and a regular function $\Phi : U \rightarrow \mathbb{A}^1$ with $\Phi|_{\text{Crit}(\Phi)^{\text{red}}} = 0$, and then \mathbf{X} is the derived critical locus $\mathbf{Crit}(\Phi)$.
- The important geometric data in writing a Lagrangian $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in (\mathbf{X}, ω) in “weak Lagrangian Darboux form” is a smooth \mathbb{K} -scheme V , a smooth morphism $\pi : V \rightarrow U$, a vector bundle $E \rightarrow V$, a nondegenerate quadratic form Q on E , and a section $s \in H^0(E)$ with $Q(s, s) + 4\pi^*(\Phi) = 0$. Then $t_0(\mathbf{L})$ is the \mathbb{K} -subscheme $s^{-1}(0)$ in V , and $t_0(\mathbf{f})$ is $\pi|_{s^{-1}(0)}$.

The remaining data is choices of étale coordinates $(x_1^0, \dots, x_{m_0}^0)$ on U and $(\tilde{x}_1^0, \dots, \tilde{x}_{m_0}^0, u_1^0, \dots, u_{n_0}^0)$ on V , and a trivialization $E \cong V \times \mathbb{A}^{n-1}$, but these are not very interesting geometrically.

3.3. A “ k -shifted Lagrangian Neighbourhood Theorem” for $k < 0$

Here is the main result of this paper, proved in Sections 4.2–4.7.

THEOREM 3.7. — *Suppose $(\mathbf{X}, \omega_{\mathbf{X}})$ is a k -shifted symplectic derived \mathbb{K} -scheme for $k < 0$, and $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ is a Lagrangian derived \mathbb{K} -scheme in $(\mathbf{X}, \omega_{\mathbf{X}})$, with isotropic structure $h_{\mathbf{L}} : 0 \xrightarrow{\sim} \mathbf{f}^*(\omega_{\mathbf{X}})$. Let $y \in \mathbf{L}$ with $\mathbf{f}(y) = x \in \mathbf{X}$.*

Suppose we are given a standard form cdga A^\bullet over \mathbb{K} , a k -shifted symplectic form ω on $\mathbf{Spec} A^\bullet$ with A^\bullet, ω in Darboux form (as in Example 2.14 and Remark 2.15, which also define a sub-cdga $A_+^\bullet \subseteq A^\bullet$), a point $p \in \mathbf{Spec} A^\bullet$, and a morphism $\mathbf{i} : \mathbf{Spec} A^\bullet \rightarrow \mathbf{X}$ which is either a Zariski open inclusion or étale, with $\mathbf{i}(p) = x$ and $\omega \sim \mathbf{i}^(\omega_{\mathbf{X}})$. We do not assume A^\bullet is minimal at p .*

(As an aside, we note that Theorem 2.18(i),(iv) guarantee such $A^\bullet, \omega, p, \mathbf{i}$ exist, where we may take A^\bullet to be minimal at p , and \mathbf{i} to be a Zariski open inclusion for $k \not\equiv 2 \pmod{4}$, and étale for $k \equiv 2 \pmod{4}$, since if $k \equiv 2 \pmod{4}$ then $\text{vdim } \mathbf{X} = 2 \text{vdim } \mathbf{L}$ is even near x .)

Then there exist a standard form $\text{cdga } B^\bullet$ over \mathbb{K} , a point $q \in \mathbf{Spec } B^\bullet$, a morphism $\alpha : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ with $\mathbf{Spec } \alpha(q) = p$ such that $\alpha_+ := \alpha|_{A_+^\bullet} : A_+^\bullet \rightarrow B^\bullet$ is a submersion minimal at q in the sense of Definition 3.1, a morphism $\mathbf{j} : \mathbf{Spec } B^\bullet \hookrightarrow \mathbf{L}$ which is either a Zariski open inclusion or étale, with $\mathbf{j}(q) = y$, in a homotopy commutative diagram

$$\begin{array}{ccc}
 \mathbf{Spec } B^\bullet & \xrightarrow{\quad \mathbf{j} \quad} & \mathbf{L} \\
 \downarrow \mathbf{Spec } \alpha & & \downarrow \mathbf{f} \\
 \mathbf{Spec } A^\bullet & \xrightarrow{\quad \mathbf{i} \quad} & \mathbf{X},
 \end{array} \tag{3.38}$$

and a Lagrangian structure $h : 0 \xrightarrow{\sim} \alpha_*(\omega)$ on $\mathbf{Spec } B^\bullet$ which is compatible with $h_{\mathbf{L}}$ in the sense that the following diagram homotopy commutes

$$\begin{array}{ccc}
 0 = \mathbf{j}^*(0) & \xrightarrow{\quad \mathbf{j}^*(h_{\mathbf{L}}) \quad} & \mathbf{j}^* \circ \mathbf{f}^*(\omega_{\mathbf{X}}) \\
 \downarrow h & & \sim \downarrow \\
 (\mathbf{Spec } \alpha)^*(\omega) = \alpha_*(\omega) & \xrightarrow{\quad \sim \quad} & (\mathbf{Spec } \alpha)^* \circ \mathbf{i}^*(\omega_{\mathbf{X}}),
 \end{array} \tag{3.39}$$

where the bottom equivalence comes from the homotopy $\omega \sim \mathbf{i}^*(\omega_{\mathbf{X}})$, and the right equivalence from the homotopy across (3.38). Furthermore:

- (i) If $k \not\equiv 3 \pmod{4}$ and \mathbf{i} is a Zariski open inclusion, then we may take \mathbf{j} to be a Zariski open inclusion, and $A^\bullet, \omega, B^\bullet, \alpha, h$ to be in Lagrangian Darboux form, as in Example 3.3.
If instead \mathbf{i} is étale, the same holds with \mathbf{j} étale.
- (ii) If $k \equiv 3 \pmod{4}$ and \mathbf{i} is a Zariski open inclusion, then we may take \mathbf{j} to be a Zariski open inclusion, and $A^\bullet, \omega, B^\bullet, \alpha, h$ to be in weak Lagrangian Darboux form, as in Example 3.5.
- (iii) If $k \equiv 3 \pmod{4}$ then we may take \mathbf{j} to be étale, and $A^\bullet, \omega, B^\bullet, \alpha, h$ to be in strong Lagrangian Darboux form, as in Example 3.5.

Remark 3.8. — Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a k -shifted symplectic derived \mathbb{K} -scheme for $k < 0$, and $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$, $h_{\mathbf{L}}$ a Lagrangian in $(\mathbf{X}, \omega_{\mathbf{X}})$, and $y \in \mathbf{L}$ with $\mathbf{f}(y) = x \in \mathbf{X}$. For clarity, we spell out what Theorems 2.18 and 3.7 together tell us about joint local models for \mathbf{X}, \mathbf{L} near x, y , for various $k = -1, -2, \dots$:

- (a) If $k < 0$ with $k \equiv 0$ or $1 \pmod{4}$, then Theorem 2.18 gives a “Darboux form” Zariski local model A^\bullet, ω for $(\mathbf{X}, \omega_{\mathbf{X}})$, and Theorem 3.7 gives a “Lagrangian Darboux form” Zariski local model B^\bullet, α, h for $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$.

- (b) If $k < 0$ with $k \equiv 2 \pmod{4}$, then Theorem 2.18 gives a “Darboux form” local model A^\bullet, ω for $(\mathbf{X}, \omega_{\mathbf{X}})$ only in the étale topology, so Theorem 3.7 gives a corresponding “Lagrangian Darboux form” local model B^\bullet, α, h for $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ only in the étale topology.
- (c) If $k < 0$ with $k \equiv 3 \pmod{4}$, Theorem 2.18 gives a “Darboux form” Zariski local model A^\bullet, ω for $(\mathbf{X}, \omega_{\mathbf{X}})$. Then Theorem 3.7 gives a “weak Lagrangian Darboux form” Zariski local model B^\bullet, α, h , and also a “strong Lagrangian Darboux form” étale local model B^\bullet, α, h , for $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$.

Our theorems thus do not provide local models in the Zariski topology for Lagrangians in general k -shifted symplectic derived schemes when $k \equiv 2 \pmod{4}$. This is due to the laziness of the authors. One should find a “weak Lagrangian Darboux form” adapted to the “weak Darboux form” of Example 2.16.

3.4. The case $k = 0$

When $k = 0$, a 0-shifted symplectic derived \mathbb{K} -scheme $(\mathbf{X}, \omega_{\mathbf{X}})$ is simply a smooth classical \mathbb{K} -scheme X with a classical symplectic form ω_X . However, Lagrangians $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in them, in the sense of Section 2.4, need *not* be smooth classical Lagrangians; they can be truly derived objects, singular at the classical level, as Example 2.11 shows for Lagrangians in the point. So a “ k -shifted derived Lagrangian Neighbourhood Theorem” is still of interest when $k = 0$, and is more closely related to classical symplectic geometry than the $k < 0$ case.

If we try to extend Theorem 3.7 to the case $k = 0$, two things go wrong:

- (a) As is well known, although the classical Darboux Theorem holds for real C^∞ and complex symplectic manifolds, it is *false* for algebraic symplectic manifolds (symplectic schemes). So given a 0-shifted derived Lagrangian $\mathbf{f} : \mathbf{L} \rightarrow X$, h in a general classical symplectic scheme (X, ω_X) , we do not have “Darboux form” local models A^\bullet, ω for (X, ω_X) near $x \in X$.
- (b) Even if we assume that (X, ω_X) has a very nice local model near x (e.g. if $X = \mathbb{A}^{2n}$ and $\omega_X = \sum_{j=1}^n d_{\text{dR}}x_j d_{\text{dR}}y_j$), in the proof of Theorem 3.7 in Sections 4.2–4.7, Proposition 4.1 fails when $k = 0$, as there is an obstruction in $H_{\text{inf}}^1(H^0(B^\bullet))$ which need not vanish Zariski or étale locally.

The next two examples define notions of “Darboux form” for 0-shifted symplectic schemes, and “Lagrangian Darboux form” for Lagrangians in

them. Because of (a) and (b), they are not local models for general symplectic schemes and their Lagrangians, but at least they are local models for especially nice symplectic schemes and nice 0-shifted Lagrangians.

Example 3.9. — Suppose A_+^0 is a smooth \mathbb{K} -algebra of dimension m_0 , and $x_1^0, \dots, x_{m_0}^0 \in A_+^0$ such that $d_{\mathrm{dR}}x_1^0, \dots, d_{\mathrm{dR}}x_{m_0}^0$ form a basis of $\Omega_{A_+^0}^1$ over A_+^0 . Let $A^0 = A_+^0[y_1^0, \dots, y_{m_0}^0]$ be the \mathbb{K} -algebra freely generated over A_+^0 by variables $y_1^0, \dots, y_{m_0}^0$ in degree 0, and write $\iota : A_+^0 \hookrightarrow A^+$ for the inclusion. Regard A_+^0, A^0 as cdgas A_+^\bullet, A^\bullet concentrated in degree 0. Define

$$\omega^0 = \sum_{j=1}^{m_0} d_{\mathrm{dR}}x_j^0 d_{\mathrm{dR}}y_j^0 \quad \text{in } \Lambda^2 \Omega_{A^0}^1.$$

Then $\omega^0 \in (\Lambda^2 \Omega_{A^\bullet}^1)^0$ with $d\omega^0 = d_{\mathrm{dR}}\omega^0 = 0$, and $\omega := (\omega^0, 0, 0, \dots)$ is a 0-shifted symplectic structure on $\mathbf{Spec} A^\bullet$. We say that A^\bullet, ω (and $A_+^\bullet \subseteq A^\bullet$) are in *Darboux form*. Following (2.15) we also define

$$\phi_+ = \sum_{j=1}^{m_0} y_j^0 d_{\mathrm{dR}}x_j^0 \quad \text{in } \Lambda^1 \Omega_{A^0}^1,$$

and then $d_{\mathrm{dR}}\phi_+ = -\omega^0$ and $d\phi_+ = 0$.

Geometrically, $U = \mathbf{Spec} A_+^0$ is a smooth \mathbb{K} -scheme of dimension m_0 , and $(x_1^0, \dots, x_{m_0}^0) : U \rightarrow \mathbb{A}^{m_0}$ are étale coordinates on U , and $\mathbf{Spec} A^0 = T^*U$ is its cotangent bundle with projection $\pi = \mathbf{Spec} \iota : T^*U \rightarrow U$, and ω^0 is the canonical symplectic form on T^*U .

The next example, similar to Example 2.19, is basically Example 3.3 for $k=0$.

Example 3.10. — Use the notation of Example 3.9. Choose a smooth \mathbb{K} -algebra B^0 of dimension $m_0 + n_0$, and a smooth morphism $\alpha_+^0 : A_+^0 \rightarrow B^0$. Localizing B^0 if necessary, we may assume there exist $u_1^0, \dots, u_{n_0}^0 \in B^0$ such that $d_{\mathrm{dR}}\tilde{x}_1^0, \dots, d_{\mathrm{dR}}\tilde{x}_{m_0}^0, d_{\mathrm{dR}}u_1^0, \dots, d_{\mathrm{dR}}u_{n_0}^0$ form a basis of $\Omega_{B^0}^1$ over B^0 , where we write $\tilde{x}_j^0 = \alpha_+^0(x_j^0) \in B^0$. As in (3.2), define $B^* = B^0[v_1^{-1}, \dots, v_{n_0}^{-1}]$ to be the free graded algebra over B^0 generated by variables $v_1^{-1}, \dots, v_{n_0}^{-1}$ in degree -1 . Choose a superpotential Ψ in B^0 . The p.d.e. (3.4) is trivial in this case, so Ψ is arbitrary. As in (3.5), extend α_+ to $\alpha : A^\bullet \rightarrow B^\bullet$ by $\alpha|_{A_+^0} = \alpha_+$ and

$$\alpha(y_j^0) = -\frac{\partial \Psi}{\partial \tilde{x}_j^0}, \quad j = 1, \dots, m_0.$$

Define the differential d in the cdga $B^\bullet = (B^*, d)$ by $d = 0$ on B^0 , and

$$dv_j^{-1} = \frac{\partial \Psi}{\partial u_j^i}, \quad j = 1, \dots, n_0,$$

as in (3.6). Then $d \circ d = 0$ trivially. As in (3.12), define $h^0 \in (\Lambda^2 \Omega_{B^\bullet}^1)^{-1}$ by

$$h^0 = \sum_{j=1}^{n_0} d_{\mathrm{dR}} u_j^0 d_{\mathrm{dR}} v_j^{-1}.$$

Then $d_{\mathrm{dR}} h^0 = 0$, and (3.13) implies that $dh^0 = \alpha_*(\omega^0)$. Hence $h := (h^0, 0, 0, \dots)$ is an isotropic structure for $\mathbf{Spec} \alpha : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ and the 0-shifted symplectic structure $\omega = (\omega^0, 0, 0, \dots)$ on $\mathbf{Spec} A^\bullet$. Following (3.14)–(3.20) we prove that h is nondegenerate, so that $\mathbf{Spec} B^\bullet$ is Lagrangian in $(\mathbf{Spec} A^\bullet, \omega)$. We say that $A^\bullet, \omega, B^\bullet, \alpha, h$ are in *Lagrangian Darboux form*.

Following (3.21), define $\psi \in (\Omega_{B^\bullet}^1)^{-1}$ by

$$\psi = - \sum_{j=1}^{n_0} v_j^{-1} d_{\mathrm{dR}} u_j^0.$$

As for (3.23)–(3.24) we have

$$\begin{aligned} d_{\mathrm{dR}} \Psi + d\psi &= -\alpha_*(\phi_+) && \text{in } (\Omega_{B^\bullet}^1)^0, \text{ and} \\ d_{\mathrm{dR}} \psi &= -h^0 && \text{in } (\Lambda^2 \Omega_{B^\bullet}^1)^{-1}. \end{aligned}$$

Geometrically, $V = \mathbf{Spec} B^0$ is a smooth \mathbb{K} -scheme, $\pi := \mathbf{Spec} \alpha_+^0 : V \rightarrow U = \mathbf{Spec} A_+^0$ is a smooth morphism of \mathbb{K} -schemes, and $\Psi : V \rightarrow \mathbb{A}^1$ is a regular function. We should interpret $\mathbf{L} := \mathbf{Spec} B^\bullet$ as the *derived relative critical locus* $\mathbf{Crit}(\Psi/U)$ of $\Psi : V \rightarrow \mathbb{A}^1$ relative to $\pi : V \rightarrow U$. Heuristically, \mathbf{L} is the total space of a family of derived critical loci over the base U :

$$\mathbf{L} = \mathbf{Crit}(\Psi/U) \approx \coprod_{u \in U} \mathbf{Crit}(\Psi|_{V_u} : V_u \rightarrow \mathbb{A}^1),$$

where $V_u = \pi^{-1}(u)$ is the (smooth) fibre of $\pi : V \rightarrow U$ over $u \in U$.

The morphism $\mathbf{Spec} \alpha : \mathbf{L} \rightarrow T^*U$ can now be understood as follows. We have a commutative diagram of vector bundles on $\mathbf{L} \subseteq V$:

$$\begin{array}{ccccccc} & & & \mathcal{O}_{\mathbf{L}} & & & \\ & & \swarrow \lambda & \downarrow d_{\mathrm{dR}} \Psi|_{\mathbf{L}} & \searrow 0 & & \\ 0 & \longrightarrow & \pi^*(T^*U)|_{\mathbf{L}} & \longrightarrow & T^*V|_{\mathbf{L}} & \longrightarrow & T^*(V/U)|_{\mathbf{L}} \longrightarrow 0, \end{array}$$

with the bottom row exact. The section $d_{\mathrm{dR}} \Psi|_{\mathbf{L}}$ of $T^*V|_{\mathbf{L}}$ projects to 0 in $T^*(V/U)|_{\mathbf{L}}$, since \mathbf{L} is the derived zero locus of $d_{\mathrm{dR}} \Psi$ in $T^*(V/U)$. Hence by exactness $d_{\mathrm{dR}} \Psi|_{\mathbf{L}}$ lifts to a section λ of $\pi^*(T^*U)|_{\mathbf{L}}$, and $\mathbf{Spec} \alpha : \mathbf{L} \rightarrow T^*U$ can be interpreted as the graph of $-\lambda$.

We can now prove the following somewhat weak and unsatisfactory “0-shifted Lagrangian Neighbourhood Theorem”:

THEOREM 3.11. — *Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a 0-shifted symplectic derived \mathbb{K} -scheme, and $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ be a Lagrangian derived \mathbb{K} -scheme in $(\mathbf{X}, \omega_{\mathbf{X}})$, with isotropic structure $h_{\mathbf{L}} : 0 \xrightarrow{\sim} \mathbf{f}^*(\omega_{\mathbf{X}})$. Let $y \in \mathbf{L}$ with $\mathbf{f}(y) = x \in \mathbf{X}$.*

Suppose we are given a standard form cdga A^\bullet over \mathbb{K} , a 0-shifted symplectic form ω on $\mathbf{Spec} A^\bullet$ with A^\bullet, ω in Darboux form as in Example 3.9, which also defines $A^\bullet_+ \subseteq A^\bullet$, a point $p \in \mathbf{Spec} A^\bullet$, and a Zariski open inclusion $\mathbf{i} : \mathbf{Spec} A^\bullet \rightarrow \mathbf{X}$ with $\mathbf{i}(p) = x$ and $\omega \sim \mathbf{i}^(\omega_{\mathbf{X}})$.*

Then we can define an obstruction class $[\gamma]$ in

$$H^1_{\text{inf}}(t_0(\mathbf{L}))_y = \varinjlim_{y \in U \subseteq t_0(\mathbf{L})} H^1_{\text{inf}}(U), \tag{3.40}$$

where the direct limit is over Zariski open neighbourhoods U of y in the classical \mathbb{K} -scheme $t_0(\mathbf{L})$, and $H^1_{\text{inf}}(\cdot)$ is algebraic de Rham cohomology.

If this obstruction class $[\gamma]$ is zero then there exist a standard form cdga B^\bullet over \mathbb{K} , a point $q \in \mathbf{Spec} B^\bullet$, a morphism $\alpha : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ with $\mathbf{Spec} \alpha(q) = p$ such that $\alpha_+ := \alpha|_{A^\bullet_+} : A^\bullet_+ \rightarrow B^\bullet$ is a submersion minimal at q in the sense of Definition 3.1, a Zariski open inclusion $\mathbf{j} : \mathbf{Spec} B^\bullet \hookrightarrow \mathbf{L}$ with $\mathbf{j}(q) = y$ in a homotopy commutative diagram (3.38), and a Lagrangian structure $h : 0 \xrightarrow{\sim} \alpha_(\omega)$ on $\mathbf{Spec} B^\bullet$ for which (3.39) homotopy commutes, such that $A^\bullet, \omega, B^\bullet, \alpha, h$ are in Lagrangian Darboux form, as in Example 3.10.*

If instead \mathbf{i} is étale, the same holds with \mathbf{j} étale.

Proof. — We follow the proof of Theorem 3.7 for $k < 0$ even in Sections 4.2–4.5, setting $k = 0$ and $\Phi_+ = \Phi = \phi = 0$. The only place where taking $k = 0$ causes problems is in the proof of Proposition 4.1. Then in place of (4.9) we have that

$$\gamma = (-\alpha_*(\phi_+), -h^0, -h^1, \dots)$$

is a 0-shifted closed 1-form on $\mathbf{Spec} B^\bullet$, that is, a closed element of degree 0 in the complex $(\prod_{i \geq 0} (\Lambda^{i+1} \mathbb{L}_{B^\bullet})[i], d_{\text{dR}} + d)$. The analogue of (4.10) is

$$\begin{aligned} H^0 \left(\prod_{i \geq 0} (\Lambda^{i+1} \mathbb{L}_{B^\bullet})[i], d_{\text{dR}} + d \right) &\cong \text{HN}^0(B^\bullet)(1) \cong \text{HP}^0(B^\bullet)(1) \\ &\cong \text{HP}^0(H^0(B^\bullet))(1) \cong H^1_{\text{inf}}(H^0(B^\bullet)) \cong H^1_{\text{inf}}(U), \end{aligned}$$

where $U = t_0(\mathbf{j})(\mathbf{Spec} H^0(B^\bullet))$ is a Zariski open neighbourhood of y in $t_0(\mathbf{L})$.

The vanishing theorems used in Proposition 4.1 fail when $k = 0$, so we may have $H^1_{\text{inf}}(U) \neq 0$. If the cohomology class $[\gamma]$ of γ in $H^1_{\text{inf}}(U)$ is nonzero then Ξ, ψ in Proposition 4.1 do not exist, so we cannot continue the proof.

If there is some Zariski open neighbourhood V of q in $\mathbf{Spec} H^0(B^\bullet)$ such that $[\gamma]$ becomes zero when restricted to $H_{\text{inf}}^1(V)$, then by localizing B^\bullet we can make $[\gamma] = 0$, so Ξ, ψ in Proposition 4.1 do exist, and the rest of Sections 4.2–4.5 works without a hitch. The condition for there to exist some such V is that the image of $[\gamma]$ should be zero in the direct limit (3.40). This completes the proof. \square

Remark 3.12. — It seems likely that there exists a good theory of *Derived Complex Analytic Geometry*, a complex analytic version of Derived Algebraic Geometry, including *derived complex analytic spaces*, built using complex manifolds and holomorphic functions. Within this there should exist good notions of *k-shifted symplectic derived complex analytic space*, and *Lagrangians* in these. So far as the authors know, neither theory is yet available in the literature.

If such theories were constructed, the authors expect that the obvious complex analytic generalizations of the k -shifted symplectic Darboux Theorem 2.18 and Lagrangian Neighbourhood Theorem 3.7 will hold.

Observe that the two problems (a) and (b) in the case $k = 0$ above will not occur in the complex analytic case. For (a) the classical Darboux Theorem holds for complex symplectic manifolds, and for (b), the complex analytic analogue of (3.40) is zero, since H^i cohomology classes for $i > 0$ on a complex analytic space are zero locally in the complex analytic topology. So the complex analytic versions of Theorems 2.18 and 3.7 should also work when $k = 0$. The authors hope in future work to use these ideas to define a notion of “derived Lagrangian” in complex symplectic manifolds, generalizing the “d-critical loci” of Joyce [13].

3.5. k -shifted Poisson structures and coisotropics

Recently, Calaque, Pantev, Toën, Vaquié and Vezzosi [12] defined *k-shifted Poisson structures* $\pi_{\mathbf{X}}$ on a derived scheme or stack \mathbf{X} , for $k \in \mathbb{Z}$, and *coisotropics* $\mathbf{f} : \mathbf{C} \rightarrow \mathbf{X}$ in $(\mathbf{X}, \pi_{\mathbf{X}})$. They prove [12, Th. 3.2.4] that the spaces of k -shifted symplectic structures $\omega_{\mathbf{X}}$ and nondegenerate k -shifted Poisson structures $\pi_{\mathbf{X}}$ on \mathbf{X} are equivalent, and for fixed equivalent $\omega_{\mathbf{X}}, \pi_{\mathbf{X}}$ they conjecture [12, Conj. 3.4.5] that the spaces of Lagrangian structures on $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in $(\mathbf{X}, \omega_{\mathbf{X}})$ and nondegenerate coisotropic structures on $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ in $(\mathbf{X}, \pi_{\mathbf{X}})$ are equivalent. Recently this conjecture has been proved in [18, Th. 4.22].

The purpose of this section is to observe that for our “Darboux form” local models for k -shifted symplectic derived schemes in Section 2.5, we

can write down simple, explicit (strict) k -shifted Poisson structures, and for our “Lagrangian Darboux form” local models for Lagrangians in k -shifted symplectic derived schemes in Section 3.2, we can write down simple, explicit (strict) coisotropic structures.

DEFINITION 3.13. — *A \mathbb{P}_{k+1} -algebra is a cdga A^\bullet equipped with the data of a Lie bracket $\{\cdot, \cdot\} : A^\bullet \otimes A^\bullet \rightarrow A^\bullet[-k]$ satisfying the following equations:*

- (i) $\{f, g\} = -(-1)^{(|f|+k)(|g|+k)}\{g, f\}$,
- (ii) $\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|+k)(|g|+k)}\{g, \{f, h\}\}$,
- (iii) $d\{f, g\} = \{df, g\} + (-1)^{|f|+k}\{f, dg\}$,
- (iv) $\{f, gh\} = \{f, g\}h + (-1)^{|g|(|f|+k)}g\{f, h\}$,

for all elements $f, g, h \in A^\bullet$.

Note that if A^\bullet is a \mathbb{P}_{k+1} -algebra, then, forgetting the multiplication, $A^\bullet[k]$ is a dg Lie algebra.

DEFINITION 3.14. — *Let A^\bullet be a cdga. A k -shifted Poisson structure on A^\bullet is a \mathbb{P}_{k+1} -algebra \tilde{A}^\bullet equipped with a quasi-isomorphism of cdgas $\tilde{A}^\bullet \rightarrow A^\bullet$.*

We say that a k -shifted Poisson structure on A^\bullet is *strict* if $\tilde{A}^\bullet \rightarrow A^\bullet$ is an isomorphism; that is, A^\bullet itself is a \mathbb{P}_{k+1} -algebra.

Let $\mathbf{X} = \mathbf{Spec} A^\bullet$ be an affine derived scheme and assume that A is a cofibrant cdga. Then one can define the complex of k -shifted polyvector fields to be

$$\mathrm{Pol}(\mathbf{X}, k) = \mathrm{Hom}_{A^\bullet}(\mathrm{Sym}(\Omega_A^1[k+1]), A^\bullet).$$

This is a graded \mathbb{P}_{k+2} -algebra, where the grading comes from the symmetric algebra, and the Lie bracket is the Schouten bracket of polyvector fields that we denote by $[\cdot, \cdot]$. We write $\widehat{\mathrm{Pol}}(\mathbf{X}, k)$ for its completion with respect to the grading, and $\widehat{\mathrm{Pol}}^{\geq 2}(\mathbf{X}, k)$ for the part in degrees at least 2.

If A^\bullet is a \mathbb{P}_{k+1} -algebra, we get a bivector $\pi_{A^\bullet}^2 \in \mathrm{Pol}(\mathbf{X}, k)$ by the formula

$$\{f, g\} = (-1)^{|f|+k+1} \iota_{\pi_{A^\bullet}^2}(\mathrm{d}_{\mathrm{dR}}f \mathrm{d}_{\mathrm{dR}}g).$$

satisfying the equations

$$d\pi_{A^\bullet}^2 = 0, \quad [\pi_{A^\bullet}^2, \pi_{A^\bullet}^2] = 0,$$

i.e. $\pi_{A^\bullet}^2 \in \widehat{\mathrm{Pol}}^{\geq 2}(\mathbf{X}, k)$ is a Maurer–Cartan element. More generally, Melani [17, Th. 3.2] proves:

THEOREM 3.15 (Melani). — *Let $\mathbf{X} = \mathbf{Spec} A^\bullet$ be an affine derived scheme. Then the space of k -shifted Poisson structures on \mathbf{X} is equivalent to the space of Maurer–Cartan elements in $\widehat{\text{Pol}}^{\geq 2}(\mathbf{X}, k)$.*

A k -shifted Poisson structure $\pi_{\mathbf{X}}$ on \mathbf{X} defines a map $\pi_{\mathbf{X}}^2 : \mathbb{L}_{\mathbf{X}} \rightarrow \mathbb{T}_{\mathbf{X}}[-k]$. We say that $\pi_{\mathbf{X}}$ is *nondegenerate* if this map is a quasi-isomorphism. Calaque et al. [12, Th. 3.2.4] prove the following theorem (see also Pridham [21] for related results).

THEOREM 3.16 (Calaque–Pantev–Toën–Vaquié–Vezzosi). — *The space of nondegenerate k -shifted Poisson structures on a derived stack \mathbf{X} is equivalent to the space of k -shifted symplectic structures on \mathbf{X} .*

It is difficult in general to explicitly invert a k -shifted symplectic structure to obtain a k -shifted Poisson structure, but this can be easily done in the “Darboux form” models of Bussi, Brav and Joyce [7] from Section 2.5 where, in fact, we obtain *strict* k -shifted Poisson structures.

Example 3.17. — Let $k < 0$ and suppose A^\bullet, ω are in k -shifted Darboux form, as in Example 2.14. The differential of vector fields is given by

$$\begin{aligned} d \frac{\partial}{\partial x_j^i} &= \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i+i'} \frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \frac{\partial}{\partial x_{j'}^{i'}} + \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i+1} \frac{\partial^2 \Phi_+}{\partial x_j^i \partial x_{j'}^{i'}} \frac{\partial}{\partial y_{j'}^{k-i'}} \\ &\quad + \sum_{i', i''=0}^d \sum_{j'=1}^{m_{i'}} \sum_{j''=1}^{m_{i''}} (-1)^{i+1} \frac{\partial^2 \Phi_{j'}^{i'+1}}{\partial x_j^i \partial x_{j''}^{i''}} y_{j'}^{k-i'} \frac{\partial}{\partial y_{j''}^{k-i''}}, \\ d \frac{\partial}{\partial y_j^{k-i}} &= \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{(k+i)(i+i')+1} \frac{\partial \Phi_j^{i+1}}{\partial x_{j'}^{i'}} \frac{\partial}{\partial y_{j'}^{k-i'}}. \end{aligned}$$

The morphism $\omega^0 : \mathbb{T}_{A^\bullet} \rightarrow \mathbb{L}_{A^\bullet}[k]$ is a strict isomorphism given by

$$\frac{\partial}{\partial x_j^i} \mapsto d_{\text{dR}} y_j^{k-i}, \quad \frac{\partial}{\partial y_j^{k-i}} \mapsto (-1)^{(i+1)(k+1)} d_{\text{dR}} x_j^i.$$

Its inverse $\mathbb{L}_{A^\bullet} \rightarrow \mathbb{T}_{A^\bullet}[-k]$ is given by

$$d_{\text{dR}} y_j^{k-i} \mapsto (-1)^k \frac{\partial}{\partial x_j^i}, \quad d_{\text{dR}} x_j^i \mapsto (-1)^{i k + i + 1} \frac{\partial}{\partial y_j^{k-i}}.$$

This gives a degree $-k$ bivector

$$\pi_{A^\bullet}^2 = \sum_{i=0}^d \sum_{j=1}^{m_i} \frac{\partial}{\partial x_j^i} \frac{\partial}{\partial y_j^{k-i}}. \quad (3.41)$$

Its differential is

$$\begin{aligned}
 d\pi_{A^\bullet}^2 &= \sum_{i,i'=0}^d \sum_{j=1}^{m_i} \sum_{j'=1}^{m_{i'}} (-1)^{i+i'} \left[\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \frac{\partial}{\partial x_{j'}^{i'}} \frac{\partial}{\partial y_j^{k-i}} - \frac{\partial \Phi_j^{i+1}}{\partial x_{j'}^{i'}} \frac{\partial}{\partial x_j^i} \frac{\partial}{\partial y_{j'}^{k-i'}} \right] \\
 &\quad + \sum_{i,i'=0}^d \sum_{j=1}^{m_i} \sum_{j'=1}^{m_{i'}} (-1)^{i+1} \frac{\partial^2 \Phi_+}{\partial x_j^i \partial x_{j'}^{i'}} \frac{\partial}{\partial y_{j'}^{k-i'}} \frac{\partial}{\partial y_j^{k-i}} \\
 &\quad + \sum_{i,i',i''=0}^d \sum_{j=1}^{m_i} \sum_{j'=1}^{m_{i'}} \sum_{j''=1}^{m_{i''}} (-1)^{i+1} \frac{\partial^2 \Phi_{j'}^{i'+1}}{\partial x_j^i \partial x_{j'}^{i'}} y_{j''}^{k-i'} \frac{\partial}{\partial y_{j'}^{k-i'}} \frac{\partial}{\partial y_{j''}^{k-i}} = 0,
 \end{aligned}$$

where each line vanishes separately after exchanging the indices i and i' .

Clearly, $[\pi_{A^\bullet}^2, \pi_{A^\bullet}^2] = 0$ as the bivector has constant coefficients. Therefore, it defines a strict k -shifted Poisson structure on $\mathbf{X} = \mathbf{Spec} A^\bullet$.

The same formulae also work trivially when $k = 0$, defining A^\bullet, ω in 0-shifted Darboux form as in Example 3.9, with coordinates x_j^0, y_j^0 , with $\Phi_+ = \Phi_j^{i+1} = 0$ and $d \frac{\partial}{\partial x_j^i} = d \frac{\partial}{\partial y_j^{k-i}} = d\pi_{A^\bullet}^2 = 0$ for degree reasons, and $\pi_{A^\bullet}^2$ is a classical Poisson structure on $A^\bullet = A^0$.

Example 3.18. — Let $k < 0$ with $k \equiv 2 \pmod{4}$ and suppose A^\bullet, ω are in k -shifted weak Darboux form, as in Example 2.16. Example 3.17 generalizes to this case, where instead of (3.41) we have

$$\begin{aligned}
 \pi_{A^\bullet}^2 &= \sum_{i=0}^{d+1} \sum_{j=1}^{m_i} \frac{\partial}{\partial x_j^i} \frac{\partial}{\partial y_j^{k-i}} + \sum_{j=1}^{m_d} \frac{1}{4q_j} \frac{\partial}{\partial z_j^d} \frac{\partial}{\partial z_j^d} \\
 &\quad - \sum_{j=1}^{m_d} \sum_{j'=1}^{m_0} \frac{\partial q_j}{\partial x_{j'}^0} \frac{z_j^d}{2q_j} \frac{\partial}{\partial z_j^d} \frac{\partial}{\partial y_{j'}^k}. \quad (3.42)
 \end{aligned}$$

Then $d\pi_{A^\bullet}^2 = 0$ and $[\pi_{A^\bullet}^2, \pi_{A^\bullet}^2] = 0$, so $\pi_{A^\bullet}^2$ defines a strict k -shifted Poisson structure on $\mathbf{X} = \mathbf{Spec} A^\bullet$.

Combining Theorems 2.18 and 3.16 gives a k -shifted Poisson version of the k -shifted Darboux Theorem 3.7:

THEOREM 3.19. — *Let $(\mathbf{X}, \pi_{\mathbf{X}})$ be a nondegenerate k -shifted Poisson derived \mathbb{K} -scheme for $k < 0$, and $x \in \mathbf{X}$. Then $(\mathbf{X}, \pi_{\mathbf{X}})$ is Zariski locally modelled near x up to equivalence on a strict k -shifted Poisson affine derived \mathbb{K} -scheme $(\mathbf{Spec} A^\bullet, \pi_{A^\bullet}^2)$ in Example 3.17 if $k \not\equiv 2 \pmod{4}$, and in Example 3.18 if $k \equiv 2 \pmod{4}$. Also, when $k \equiv 2 \pmod{4}$, by instead taking $(\mathbf{X}, \pi_{\mathbf{X}})$ to be étale locally modelled on $(\mathbf{Spec} A^\bullet, \pi_{A^\bullet}^2)$ we may set $q_1 = \cdots = q_{m_d} = 1$ in Example 3.18.*

Now let us turn to k -shifted Lagrangians and coisotropics.

DEFINITION 3.20. — *A $\mathbb{P}_{[k+1,k]}$ -algebra is a triple of a \mathbb{P}_{k+1} -algebra A^\bullet , a \mathbb{P}_k -algebra B^\bullet and a morphism of \mathbb{P}_{k+1} -algebras*

$$A^\bullet \longrightarrow (\mathrm{Hom}_{B^\bullet}(\mathrm{Sym}(\Omega_{B^\bullet}^1[k]), B), \mathrm{d} + [\pi_{B^\bullet}^2, -]).$$

Note that if (A^\bullet, B^\bullet) is a $\mathbb{P}_{[k+1,k]}$ -algebra, the composite

$$A^\bullet \longrightarrow (\mathrm{Hom}_{B^\bullet}(\mathrm{Sym}(\Omega_{B^\bullet}^1[k]), B), \mathrm{d} + [\pi_{B^\bullet}^2, -]) \longrightarrow B^\bullet,$$

where the latter morphism is given by projection to the weight zero part, is a morphism of cdgas.

DEFINITION 3.21. — *Let $\alpha: A^\bullet \rightarrow B^\bullet$ be a morphism of cdgas. A k -shifted coisotropic structure on α is a $\mathbb{P}_{[k+1,k]}$ -algebra $(\tilde{A}^\bullet, \tilde{B}^\bullet)$ together with quasi-isomorphisms of cdgas $\tilde{A}^\bullet \rightarrow A^\bullet$ and $\tilde{B}^\bullet \rightarrow B^\bullet$ making the diagram of cdgas*

$$\begin{array}{ccc} \tilde{A}^\bullet & \longrightarrow & \tilde{B}^\bullet \\ \downarrow & & \downarrow \\ A^\bullet & \xrightarrow{\alpha} & B^\bullet \end{array}$$

commutative.

This definition in fact is equivalent to k -shifted coisotropic structures of [12] as shown in [22]. Moreover, one can define a nondegeneracy condition on a k -shifted coisotropic structure on $A^\bullet \rightarrow B^\bullet$ which in particular implies that the k -shifted Poisson structure on A^\bullet is nondegenerate. The following is [18, Th. 4.22].

THEOREM 3.22. — *The space of nondegenerate k -shifted Poisson structures on a morphism of derived stacks $\mathbf{L} \rightarrow \mathbf{X}$ is equivalent to the space of pairs of a k -shifted symplectic structure on \mathbf{X} and a Lagrangian structure on $\mathbf{L} \rightarrow \mathbf{X}$.*

As before, we say a k -shifted coisotropic structure is *strict* if the maps $\tilde{A}^\bullet \rightarrow A^\bullet$ and $\tilde{B}^\bullet \rightarrow B^\bullet$ are isomorphisms, so that (A^\bullet, B^\bullet) is a $\mathbb{P}_{[k+1,k]}$ -algebra. In general, it is difficult to construct a k -shifted coisotropic structure corresponding to a given k -shifted Lagrangian structure, but we will now explain how to perform this construction for our local models for Lagrangians which will moreover give strict coisotropic structures.

Example 3.23. — Let $k < 0$ with $k \not\equiv 3 \pmod{4}$, and consider data $A^\bullet, \omega, B^\bullet, \alpha, h$ in Lagrangian Darboux form as in Example 3.3, so that A^\bullet, ω is in Darboux form as in Example 2.14. Then Example 3.17 defines a strict k -shifted Poisson structure $\pi_{A^\bullet}^2$ on A^\bullet . We will define a strict k -shifted

coisotropic structure on $\alpha : A^\bullet \rightarrow B^\bullet$. We need to construct a $(k-1)$ -shifted Poisson structure π_{B^\bullet} on B^\bullet and provide a \mathbb{P}_{k+1} -morphism $\tilde{\alpha} : A^\bullet \rightarrow (\widehat{\text{Sym}}(\mathbb{T}_{B^\bullet}[-k]), d + [\pi_{B^\bullet}^2, -])$.

The morphism $\chi : \mathbb{T}_{B^\bullet/A^\bullet} \rightarrow \mathbb{L}_{B^\bullet}[k-1]$ is given by

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}_j^i} &\longmapsto 0, & \frac{\partial}{\partial u_j^i} &\longmapsto d_{\text{dR}} v_j^{k-1-i}, & \frac{\partial}{\partial v_j^{k-1-i}} &\longmapsto (-1)^{k(i+1)} d_{\text{dR}} u_j^i, \\ \frac{\partial}{\partial x_j^i} &\longmapsto (-1)^i d_{\text{dR}} \frac{\partial \Psi}{\partial \tilde{x}_j^i}, & \frac{\partial}{\partial y_j^{k-i}} &\longmapsto (-1)^{ik+k+i} d_{\text{dR}} \tilde{x}_j^i. \end{aligned}$$

We can find its one-sided inverse χ^{-1} so that the composite

$$\mathbb{L}_{B^\bullet} \xrightarrow{\chi^{-1}} \mathbb{T}_{B^\bullet/A^\bullet}[1-k] \longrightarrow \mathbb{T}_{B^\bullet}[1-k]$$

is given by

$$d_{\text{dR}} v_j^{k-1-i} \longmapsto (-1)^{k+1} \frac{\partial}{\partial u_j^i}, \quad d_{\text{dR}} u_j^i \longmapsto (-1)^{ik+1} \frac{\partial}{\partial v_j^{k-1-i}}, \quad d_{\text{dR}} \tilde{x}_j^i \longmapsto 0.$$

This gives a degree $1-k$ bivector

$$\pi_{B^\bullet}^2 = \sum_{i=0}^e \sum_{j=1}^{n_i} \frac{\partial}{\partial u_j^i} \frac{\partial}{\partial v_j^{k-1-i}}. \quad (3.43)$$

Clearly, $[\pi_{B^\bullet}^2, \pi_{B^\bullet}^2] = 0$ as the bivector has constant coefficients.

As a graded commutative algebra, A^* is freely generated over A_+^* by the variables y_j^{k-i} for $i = 0, -1, \dots, d$ and $j = 1, \dots, m_i$. We define the morphism $\tilde{\alpha}$ to be α on A_+^* and

$$\tilde{\alpha}(y_j^{k-i}) = (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} + (-1)^{i+1} \frac{\partial}{\partial \tilde{x}_j^i}. \quad (3.44)$$

Let us check that this $\tilde{\alpha}$ is compatible with the differential and the brackets:

$$d \circ \tilde{\alpha}(y_j^{k-i}) = (-1)^{i+1} \left(d \frac{\partial \Psi}{\partial \tilde{x}_j^i} + d \frac{\partial}{\partial \tilde{x}_j^i} + \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} \left[\frac{\partial}{\partial u_{j'}^{i'}} \frac{\partial}{\partial v_{j'}^{k-1-i'}}, \frac{\partial \Psi}{\partial \tilde{x}_j^i} \right] \right).$$

Equation (3.11) implies that

$$(-1)^{i+1} d \frac{\partial \Psi}{\partial \tilde{x}_j^i} = \alpha(dy_j^{k-i}) = \tilde{\alpha}(dy_j^{k-i}) - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i'+1} \alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) \frac{\partial}{\partial \tilde{x}_{j'}^{i'}}.$$

Therefore,

$$\begin{aligned}
 d \circ \tilde{\alpha}(y_j^{k-i}) &= \tilde{\alpha}(dy_j^{k-i}) - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i'+1} \alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) \frac{\partial}{\partial \tilde{x}_{j'}^{k-i'}} \\
 &+ \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i'+1} \alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) \frac{\partial}{\partial \tilde{x}_{j'}^{i'}} \\
 &+ \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{k(i'+1)} \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial v_{j'}^{k-1-i'}} \frac{\partial}{\partial u_{j'}^{i'}} + \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial u_{j'}^{i'}} \frac{\partial}{\partial v_{j'}^{k-1-i'}} \\
 &+ \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{(k+i)(k+i')+i+1} \frac{\partial^2 \Psi}{\partial v_{j'}^{k-1-i'} \partial \tilde{x}_j^i} \frac{\partial}{\partial u_{j'}^{i'}} \\
 &+ \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (-1)^{ii'+1} \frac{\partial^2 \Psi}{\partial u_{j'}^{i'} \partial \tilde{x}_j^i} \frac{\partial}{\partial v_{j'}^{k-1-i'}} = \tilde{\alpha}(dy_j^{k-i}), \tag{3.45}
 \end{aligned}$$

which shows that $\tilde{\alpha}$ commutes with the differential d .

The Poisson bracket on A^\bullet is given by

$$\{x_j^i, x_{j'}^{i'}\} = 0, \quad \{y_j^i, y_{j'}^{i'}\} = 0, \quad \{y_j^i, x_{j'}^{i'}\} = (-1)^{i+1} \delta_{jj'} \delta^{ii'}. \tag{3.46}$$

Under $\tilde{\alpha}$ these are sent to

$$\begin{aligned}
 [\tilde{\alpha}(x_j^i), \tilde{\alpha}(x_{j'}^{i'})] &= [\tilde{x}_j^i, \tilde{x}_{j'}^{i'}] = 0, \\
 [\tilde{\alpha}(y_j^i), \tilde{\alpha}(y_{j'}^{i'})] &= (-1)^{i+i'} \left[\frac{\partial \Psi}{\partial \tilde{x}_j^i} + \frac{\partial}{\partial \tilde{x}_j^i}, \frac{\partial \Psi}{\partial \tilde{x}_{j'}^{i'}} + \frac{\partial}{\partial \tilde{x}_{j'}^{i'}} \right] \\
 &= (-1)^{i+i'} \frac{\partial^2 \Psi}{\partial \tilde{x}_j^i \partial \tilde{x}_{j'}^{i'}} + (-1)^{(i+1)(i'+1)} \frac{\partial^2 \Psi}{\partial \tilde{x}_{j'}^{i'} \partial \tilde{x}_j^i} = 0, \\
 [\tilde{\alpha}(y_j^i), \tilde{\alpha}(x_{j'}^{i'})] &= (-1)^{i+1} \left[\frac{\partial \Psi}{\partial \tilde{x}_j^i} + \frac{\partial}{\partial \tilde{x}_j^i}, \tilde{x}_{j'}^{i'} \right] = (-1)^{i+1} \delta_{jj'} \delta^{ii'} = \tilde{\alpha}(\{y_j^i, x_{j'}^{i'}\}).
 \end{aligned} \tag{3.47}$$

Therefore, $\tilde{\alpha}$ is a morphism of \mathbb{P}_{k+1} -algebras, and so (3.43) and (3.44) define a (strict) coisotropic structure on $\alpha : A^\bullet \rightarrow B^\bullet$.

The same formulae also work when $k = 0$, defining A^\bullet, ω in 0-shifted Darboux form as in Example 3.9, with $\Phi_j^{i+1} = 0$, and $A^\bullet, \omega, B^\bullet, \alpha, h$ in Lagrangian Darboux form as in Example 3.10.

Example 3.24. — Now let $k < 0$ with $k \equiv 3 \pmod{4}$, and consider data $A^\bullet, \omega, B^\bullet, \alpha, h$ in weak Lagrangian Darboux form as in Example 3.5, so that A^\bullet, ω is in Darboux form as in Example 2.14, and we have coordinates $\tilde{x}_j^i, u_j^i, v_j^{k-1-i}, w_j^e$ in B^\bullet , with the w_j^e associated to invertible $q_1, \dots, q_{n_e} \in B^0$, where $A^\bullet, \omega, B^\bullet, \alpha, h$ are in strong Lagrangian Darboux form if $q_1 = \dots = q_{n_e} = 1$.

All of Example 3.23 generalizes to this case, so we just give the definitions, leaving most verifications to the reader. As in (3.42)–(3.43), the bivector $\pi_{B^\bullet}^2$ is

$$\pi_{B^\bullet}^2 = \sum_{i=0}^d \sum_{j=1}^{n_i} \frac{\partial}{\partial u_j^i} \frac{\partial}{\partial v_j^{k-1-i}} + \sum_{j=1}^{n_e} \frac{1}{4q_j} \frac{\partial}{\partial w_j^e} \frac{\partial}{\partial w_j^e} - \sum_{j=1}^{n_e} \sum_{j'=1}^{n_0} \frac{\partial q_j}{\partial u_{j'}^0} \frac{w_j^e}{2q_j} \frac{\partial}{\partial w_j^e} \frac{\partial}{\partial v_{j'}^{k-1}}.$$

As this does not have constant coefficients, we check that $[\pi_{B^\bullet}^2, \pi_{B^\bullet}^2] = 0$ by

$$\begin{aligned} [\pi_{B^\bullet}^2, \pi_{B^\bullet}^2] &= \sum_{j=1}^{n_0} \sum_{j'=1}^{n_e} \left[\frac{\partial q_{j'}}{\partial u_j^0} \frac{1}{4q_{j'}^2} \frac{\partial}{\partial v_j^{k-1}} \frac{\partial}{\partial w_{j'}^e} \frac{\partial}{\partial w_{j'}^e} - \frac{\partial q_j}{\partial u_{j'}^0} \frac{1}{4q_j^2} \frac{\partial}{\partial v_{j'}^{k-1}} \frac{\partial}{\partial w_j^e} \frac{\partial}{\partial w_j^e} \right] \\ &\quad - \sum_{j,j'=1}^{n_0} \sum_{j''=1}^{n_e} \frac{\partial^2 q_{j'}}{\partial u_j^0 \partial u_{j''}^0} \frac{w_{j'}^e}{2q_{j'}} \frac{\partial}{\partial w_{j''}^e} \frac{\partial}{\partial v_{j'}^{k-1}} \frac{\partial}{\partial v_{j''}^{k-1}} = 0, \end{aligned}$$

which vanishes as the $\frac{\partial}{\partial w_j^e}$ are symmetric under multiplication, as e is odd, and the $\frac{\partial}{\partial v_j^{k-1}}$ are antisymmetric, as $k-1$ is even, and $\frac{\partial^2 q_{j'}}{\partial u_j^0 \partial u_{j''}^0}$ is symmetric in j, j' .

The \mathbb{P}_{k+1} -morphism $\tilde{\alpha} : A^\bullet \rightarrow (\widehat{\text{Sym}}(\mathbb{T}_{B^\bullet}[-k]), d + [\pi_{B^\bullet}^2, \cdot])$ is given by $\tilde{\alpha}|_{A_+^*} = \alpha|_{A_+^*}$ and, generalizing (4.43),

$$\begin{aligned} \tilde{\alpha}(y_j^{k-i}) &= \alpha(y_j^{k-i}) + (-1)^{i+1} \frac{\partial}{\partial \tilde{x}_j^i}, \quad i = -1, \dots, d, \quad j = 1, \dots, m_i, \\ \tilde{\alpha}(y_j^k) &= \alpha(y_j^k) - \frac{\partial}{\partial \tilde{x}_j^0} + \sum_{j'=1}^{n_e} \frac{1}{2q_{j'}} \frac{\partial q_{j'}}{\partial \tilde{x}_j^0} w_{j'}^e \frac{\partial}{\partial w_{j'}^e}, \quad j = 1, \dots, m_0. \end{aligned}$$

We can show that $d \circ \tilde{\alpha} = \tilde{\alpha} \circ d$ as in (3.45), and that $\tilde{\alpha}$ preserves $\{\cdot, \cdot\}, [\cdot, \cdot]$ as in (3.46)–(3.47). Thus $\pi_{B^\bullet}^2, \tilde{\alpha}$ define a (strict) coisotropic structure on $\alpha : A^\bullet \rightarrow B^\bullet$.

Examples 3.23 and 3.24 show that all of our (ordinary/weak/strong) “Lagrangian Darboux form” local models in k -shifted symplectic derived \mathbb{K} -schemes for $k \leq 0$ can be promoted to explicit (strict) coisotropic structures in k -shifted Poisson derived \mathbb{K} -schemes.

Theorems 3.7 and 3.22 imply a “Coisotropic Neighbourhood Theorem”, saying that a nondegenerate coisotropic $f : C \rightarrow X$ in a nondegenerate k -shifted Poisson derived \mathbb{K} -scheme X for $k < 0$ is Zariski or étale locally modelled on $\mathbf{Spec} \alpha : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ in $\mathbf{Spec} A^\bullet$ in Examples 3.23 or 3.24, in a similar way to Theorem 3.19.

4. Proofs of the main results

Sections 4.1 and 4.2–4.7 will prove Theorems 3.2 and 3.7, respectively.

4.1. Proof of Theorem 3.2

The proof is modelled on that of Theorem 2.6 in [7, §4.1]. As in Theorem 3.2, let $f : Y \rightarrow X$ be a morphism of derived \mathbb{K} -schemes, A^\bullet be a standard form cdga over \mathbb{K} , $i : \mathbf{Spec} A^\bullet \hookrightarrow X$ be a Zariski open inclusion (or étale morphism), and $y \in Y$, $p \in \mathbf{Spec} H^0(A^\bullet)$ with $f(y) = i(p) = x \in X$.

First, consider a homotopy pullback diagram in $\mathbf{dSch}_{\mathbb{K}}$:

$$\begin{array}{ccc} Z & \xrightarrow{h} & \mathbf{Spec} A^\bullet \\ \downarrow g & & \downarrow i \\ Y & \xrightarrow{f} & X, \end{array}$$

where $Z = Y \times_X \mathbf{Spec} A^\bullet$. The map $g : Z \rightarrow Y$ is a Zariski open immersion (or étale if i is étale). Also there is a unique point $z \in Z$ with $g(z) = y$ and $h(z) = p$. Let $k : \mathbf{Spec} C^\bullet \hookrightarrow Z$ be an affine Zariski neighbourhood of z for some finitely presented cdga C^\bullet , so that $r \in \mathbf{Spec} C^\bullet$ with $k(r) = z$.

Recall the distinction between the ordinary category $\mathbf{cdga}_{\mathbb{K}}$ and the ∞ -category $\mathbf{cdga}_{\mathbb{K}}^\infty$ of cdgas over \mathbb{K} , discussed in Remark 2.2. The morphism $h \circ k : \mathbf{Spec} C^\bullet \rightarrow \mathbf{Spec} A^\bullet$ is equivalent to $\mathbf{Spec} \gamma^\infty$ for some morphism $\gamma^\infty : A^\bullet \rightarrow C^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}^\infty$, unique up to equivalence. Later (after modifying C^\bullet) we will show that γ^∞ descends to a morphism γ in $\mathbf{cdga}_{\mathbb{K}}$.

Possibly after localizing C^\bullet , we will inductively construct a standard form cdga B^\bullet with a quasi-isomorphism $\beta : B^\bullet \xrightarrow{\sim} C^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$, such that $h \circ k \circ (\mathbf{Spec} \beta)^{-1} : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ is equivalent to $\mathbf{Spec} \alpha$, for $\alpha : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ a submersion of cdgas minimal at $q = \mathbf{Spec} \beta(r)$, with $(\mathbf{Spec} \beta)^{-1}$ a quasi-inverse for $\mathbf{Spec} \beta$. Then setting $j = g \circ k \circ (\mathbf{Spec} \beta)^{-1}$, so that $\mathbf{Spec} \alpha(q) = p$, $j(q) = y$, we have a homotopy commutative diagram

$$\begin{array}{ccccc} \mathbf{Spec} B^\bullet & & & & \\ & \xrightarrow{(\mathbf{Spec} \beta)^{-1}} & & \xrightarrow{\mathbf{Spec} \alpha} & \\ & \searrow \mathbf{Spec} \beta & \mathbf{Spec} C^\bullet & \xrightarrow{\mathbf{Spec} \gamma} & \mathbf{Spec} A^\bullet \quad (4.1) \\ & & \downarrow k & \downarrow h & \downarrow i \\ & & Y & \xrightarrow{f} & X, \\ & \searrow j & & & \end{array}$$

which gives (3.1). Also j is a Zariski open inclusion (or étale if i is étale), as $g, k, (\mathbf{Spec} \beta)^{-1}$ are, so this will prove Theorem 3.2.

As $\mathbf{Spec} C^\bullet$ is affine, we can choose an embedding $\mathbf{Spec} H^0(C^\bullet) \hookrightarrow \mathbb{A}^N$ for $N \gg 0$. We also have a composition of morphisms

$$\mathbf{Spec} H^0(C^\bullet) \xrightarrow{t_0(k)} t_0(\mathbf{Z}) \xrightarrow{t_0(h)} \mathbf{Spec} H^0(A^\bullet) \hookrightarrow \mathbf{Spec} A^0 =: U,$$

so the direct product is an embedding $e : \mathbf{Spec} H^0(C^\bullet) \hookrightarrow \mathbb{A}^N \times U$. Choose a smooth, affine, locally closed \mathbb{K} -subscheme V in $\mathbb{A}^N \times U$, such that V contains an open neighbourhood of $e(r)$ in $e[\mathbf{Spec} H^0(C^\bullet)]$ as a closed \mathbb{K} -subscheme, and the projection $V \rightarrow U$ is smooth, and $\dim V$ is minimal under these conditions. Localizing V if necessary, we can assume T^*V is a trivial vector bundle.

As V is affine we have $V = \mathbf{Spec} B^0$ for a smooth \mathbb{K} -algebra B^0 , with $\Omega_{B^0}^1$ a free B^0 -module. The (smooth) projection $V \rightarrow U$ is $\mathbf{Spec} \alpha^0$ for $\alpha^0 : B^0 \rightarrow A^0$ a smooth morphism of \mathbb{K} -algebras. Since V contains an open neighbourhood of $e(p)$ in $e[\mathbf{Spec} H^0(C^\bullet)]$, localizing C^\bullet if necessary we can suppose $e[\mathbf{Spec} H^0(C^\bullet)]$ is a closed \mathbb{K} -subscheme in V . Then the closed embedding $\mathbf{Spec} H^0(C^\bullet) \hookrightarrow V$ is $\mathbf{Spec} \beta'$ for some $\beta' : B^0 \rightarrow H^0(C^\bullet)$.

Since C^\bullet is the homotopy limit of its Postnikov tower $\cdots \rightarrow \tau_{\geq -1} C^\bullet \rightarrow \tau_{\geq 0} C^\bullet \simeq H^0(C^\bullet)$ in which each map is a square-zero extension of cdgas [16, Prop. 7.1.3.19], and as B^0 is smooth and hence maps out of it can be lifted along square-zero extensions, after replacing C^\bullet by an equivalent cdga we can lift $\beta' : B^0 \rightarrow H^0(C^\bullet)$ along the canonical map $C^\bullet \rightarrow H^0(C^\bullet)$ to obtain a map $\beta^0 : B^0 \rightarrow C^0 \subseteq C^\bullet$.

Set $\gamma^0 = \beta^0 \circ \alpha^0 : A^0 \rightarrow C^\bullet$, as a morphism in $\mathbf{cdga}_{\mathbb{K}}$. Then we have a homotopy commutative diagram in $\mathbf{cdga}_{\mathbb{K}}^\infty$, with $\alpha^0, \beta^0, \gamma^0$ morphisms in $\mathbf{cdga}_{\mathbb{K}}$:

$$\begin{array}{ccccc} & & \gamma^0 & & \\ & & \curvearrowright & & \\ A^0 & \xrightarrow{\alpha^0} & B^0 & \xrightarrow{\beta^0} & C^0 \\ & \downarrow & & & \downarrow \\ A^\bullet & \xrightarrow{\gamma^\infty} & & & C^\bullet \end{array} \quad (4.2)$$

Since A^\bullet is cofibrant over A^0 , and $A^0 \rightarrow C^\bullet$ is represented by a morphism γ^0 in $\mathbf{cdga}_{\mathbb{K}}$, up to equivalence γ^∞ descends to a morphism $\gamma : A^\bullet \rightarrow C^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$.

As A^\bullet is a standard form cdga, it is freely generated over A^0 by finitely many generators $x_1^i, \dots, x_{m_i}^i$ in degree i for $i = -1, -2, \dots$, where $m_i = 0$ for $i \ll 0$. Write $A^\bullet(k)$ for $k = 0, -1, \dots$ for the sub-cdga of A^\bullet generated over A^0 by the generators x_j^i in degrees $i = -1, -2, \dots, k$ only, so that $A^\bullet(0) \subseteq A^\bullet(-1) \subseteq A^\bullet(-2) \subseteq \cdots$, and $A^\bullet(0) = A^0$, and $A^\bullet(k) = A^\bullet$ for $k \ll 0$.

Next we inductively construct a sequence of standard form cdgas $B^\bullet(0) \subseteq B^\bullet(-1) \subseteq B^\bullet(-2) \subseteq \dots$, and submersions $\alpha(k) : A^\bullet(k) \rightarrow B^\bullet(k)$ in $\mathbf{cdga}_{\mathbb{K}}$, and morphisms $\beta(k) : B^\bullet(k) \rightarrow C^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$, such that $\alpha(k-1)|_{A^\bullet(k)} = \alpha(k)$, $\beta(k-1)|_{B^\bullet(k)} = \beta(k)$, and the following diagram commutes in $\mathbf{cdga}_{\mathbb{K}}$:

$$\begin{array}{ccccc}
 A^\bullet(k) & \hookrightarrow & A^\bullet(k-1) & \hookrightarrow & A^\bullet \\
 \downarrow \alpha(k) & & \downarrow \alpha(k-1) & & \downarrow \gamma \\
 B^\bullet(k) & \xrightarrow{\hspace{2cm}} & B^\bullet(k-1) & \xrightarrow{\beta(k-1)} & C^\bullet, \\
 & \searrow \beta(k) & & & \\
 & & & &
 \end{array} \tag{4.3}$$

and $\mathbb{L}_{C^\bullet/B^\bullet(k)}$ is concentrated in degrees $(-\infty, k-1]$, and $B^*(k-1)$ is freely generated over $B^*(k)$ by finitely many generators in degree $k-1$, where this number of generators is minimal such that the previous conditions hold near r .

For the first step, set $A^\bullet(0) = A^0$ and $B^\bullet(0) = B^0$, regarded as cdgas concentrated in degree 0, and $\alpha(0) = \alpha^0 : A^\bullet(0) \rightarrow B^\bullet(0)$, which is a submersion, and $\beta(0) = \beta^0 : B^\bullet(0) \rightarrow C^\bullet$. Then (4.2) implies that the outer rectangle of (4.3) commutes for $k = 0$.

For the inductive step, suppose that for some $k \leq 0$ we have chosen $B^\bullet(0), B^\bullet(-1), \dots, B^\bullet(k)$ and $\alpha(0), \alpha(-1), \dots, \alpha(k)$ and $\beta(0), \beta(-1), \dots, \beta(k)$ with the desired properties. Now $H^{k-1}(\mathbb{L}_{C^\bullet/B^\bullet(k)}|_r)$ is spanned by elements $(d_{\mathrm{dR}}y, d_{\mathrm{dR}}z)$ for $y \in C^{k-1}$ and $z \in B^k(k)$ with $dy = \beta(k)(z) \in C^k$ and $dz = 0 \in B^{k+1}(k)$. We have generators x_j^{k-1} of A^\bullet in degree $k-1$ with $dx_j^{k-1} \in A^\bullet(k) \subseteq A^\bullet$, so that $y = \gamma(x_j^{k-1})$, $z = \alpha(k)(dx_j^{k-1})$ satisfy $dy = \beta(k)(z)$ and $dz = 0$, and $(d_{\mathrm{dR}}\gamma(x_j^{k-1}), d_{\mathrm{dR}}\alpha(k)(dx_j^{k-1}))$ gives an element of $H^{k-1}(\mathbb{L}_{C^\bullet/B^\bullet(k)}|_r)$. Choose a minimal number of additional pairs $(y_1^{k-1}, z_1^k), \dots, (y_{n_{k-1}}^{k-1}, z_{n_{k-1}}^k)$ in $C^{k-1} \times B^k(k)$ with $dy_j^{k-1} = \beta(k)(z_j^k)$ and $dz_j^k = 0$, such that $H^{k-1}(\mathbb{L}_{C^\bullet/B^\bullet(k)}|_r)$ is spanned by $(d_{\mathrm{dR}}\gamma(x_j^{k-1}), d_{\mathrm{dR}}\alpha(k)(dx_j^{k-1}))$ for $j = 1, \dots, m_{k-1}$ and $(d_{\mathrm{dR}}y_j^{k-1}, d_{\mathrm{dR}}z_j^k)$ for $j = 1, \dots, n_{k-1}$.

Define $B^*(k-1)$ to be the commutative graded algebra freely generated over $B^*(k)$ by generators $\tilde{x}_1^{k-1}, \dots, \tilde{x}_{m_{k-1}}^{k-1}, \tilde{y}_1^{k-1}, \dots, \tilde{y}_{n_{k-1}}^{k-1}$ in degree $k-1$. Define the differential d in $B^\bullet(k-1) = (B^*(k-1), d)$ by $d|_{B^*(k)} = d_{B^*(k)}$, and $d(\tilde{x}_j^{k-1}) = \alpha(k)(dx_j^{k-1})$, $d\tilde{y}_j^{k-1} = z_j^k$ for all j . Then $d \circ d = 0$ as $d \circ \alpha(k)(dx_j^{k-1}) = 0$ and $dz_j^k = 0$ in $B^\bullet(k)$. Define $\alpha(k-1) : A^*(k-1) \rightarrow B^*(k-1)$ to be the unique graded algebra morphism with $\alpha(k-1)|_{A^*(k-1)} = \alpha(k)$ and $\alpha(k-1)(x_j^{k-1}) = \tilde{x}_j^{k-1}$ for $j = 1, \dots, m_{k-1}$. Then $\alpha(k-1)$ is a cdga morphism as $d\alpha(k-1)(x_j^{k-1}) = d\tilde{x}_j^{k-1} = \alpha(k)(dx_j^{k-1}) = \alpha(k-1)(dx_j^{k-1})$.

Define $\beta(k-1) : B^*(k-1) \rightarrow C^*$ to be the unique graded algebra morphism with $\beta(k-1)|_{B^*(k)} = \beta(k)$ and $\beta(k-1)(\tilde{x}_j^{k-1}) = \gamma(x_j^{k-1})$ and $\beta(k-1)(\tilde{y}_j^{k-1}) = y_j^{k-1}$ for all j . Then $\beta(k-1)$ is a cdga morphism $B^\bullet(k-1) \rightarrow C^\bullet$ as

$$\begin{aligned} d\beta(k-1)(\tilde{x}_j^{k-1}) &= d\gamma(x_j^{k-1}) = \gamma(dx_j^{k-1}) = \beta(k) \circ \alpha(k)(dx_j^{k-1}) \\ &= \beta(k-1) \circ d(\tilde{x}_j^{k-1}), \\ d\beta(k-1)(\tilde{y}_j^{k-1}) &= dy_j^{k-1} = \beta(k)(z_j^k) = \beta(k-1) \circ d(\tilde{y}_j^{k-1}). \end{aligned}$$

Also (4.3) commutes as $\beta(k-1) \circ \alpha(k-1)(x_j^{k-1}) = \gamma(x_j^{k-1})$.

As $\mathbb{L}_{C^\bullet/B^\bullet(k)}$ is concentrated in degrees $(-\infty, k-1]$, and the new generators of $B^\bullet(k-1)$ span $H^{k-1}(\mathbb{L}_{C^\bullet/B^\bullet(k)}|_r)$, we see that $\mathbb{L}_{C^\bullet/B^\bullet(k-1)}$ is concentrated in degrees $(-\infty, k-2]$ near $r \in \mathbf{Spec} C^\bullet$. Thus, localizing C^\bullet and $B^\bullet(0), \dots, B^\bullet(k-1)$, we can suppose that $\mathbb{L}_{C^\bullet/B^\bullet(k-1)}$ is concentrated in degrees $(-\infty, k-2]$. This completes the inductive step. Hence, by induction we have defined standard form cdgas $B^\bullet(0) \subseteq B^\bullet(-1) \subseteq B^\bullet(-2) \subseteq \dots$, submersions $\alpha(k) : A^\bullet(k) \rightarrow B^\bullet(k)$, and morphisms $\beta(k) : B^\bullet(k) \rightarrow C^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ for $k = 0, -1, -2, \dots$

Since A^\bullet is of standard form and C^\bullet is finitely presented, for $k \ll 0$ A^\bullet has no generators in degrees $< k$, and \mathbb{L}_{C^\bullet} is concentrated in degrees $[k, 0]$. Then we add no further generators in degrees $< k$, so that $A^\bullet(k) = A^\bullet(k-1) = \dots = A^\bullet$ and $B^\bullet(k) = B^\bullet(k-1) = \dots$. Set $B^\bullet = B^\bullet(k)$, $\alpha = \alpha(k)$ and $\beta = \beta(k)$ for such k . Then β is a quasi-isomorphism, since $\beta(i)$ is a quasi-isomorphism in degrees $> i$ and $\beta = \beta(k) = \beta(k-1) = \dots$, and $\gamma = \beta \circ \alpha$, so (4.1) commutes up to homotopy. Also $\alpha : A^\bullet \rightarrow B^\bullet$ is a submersion, as $\alpha(k) : A^\bullet(k) \rightarrow B^\bullet(k)$ is, and the minimality conditions we imposed on $\dim V$ and the number of (y_j^{k-1}, z_j^k) imply that α is minimal at q . This completes the proof of Theorem 3.2.

4.2. Beginning the proof of Theorem 3.7 for $k < 0$

Sections 4.2–4.7 will prove our main result, Theorem 3.7. This section begins by showing that when $k < 0$ we can choose $B^\bullet, q, \alpha, \mathbf{i}, \mathbf{j}$ and $h = (h^0, 0, 0, \dots)$ as in the first part of Theorem 3.7, and $\Xi \in B^k, \psi \in (\Omega_{B^\bullet}^1)^{k-1}$ with $d\Xi = -\alpha(\Phi + \Phi_+)$, $d_{\mathrm{dR}}\Xi + d\psi = -\alpha_*(\phi + \phi_+)$ and $h^0 = \frac{1}{k-1}d_{\mathrm{dR}}\psi$.

Sections 4.3–4.5 continue the proof for even $k < 0$. In Section 4.3 we choose coordinates $\tilde{x}_j^i, u_j^i, v_j^{k-1-i} \in B^\bullet$ with $h^0 = \sum_{i,j} d_{\mathrm{dR}} a_j^{k-1-i} d_{\mathrm{dR}} \tilde{x}_j^i + d_{\mathrm{dR}} u_j^i d_{\mathrm{dR}} v_j^{k-1-i}$ for functions $a_j^{k-1-i} \in B^{k-1-i}$. Section 4.4 defines $\Psi = \Xi - \sum_{i,j} i a_j^{k-1-i} d\tilde{x}_j^i$, and computes the p.d.e. satisfied by Ψ , and expressions for d in $B^\bullet = (B^*, d)$, and for $\alpha : A^\bullet \rightarrow B^\bullet$.

Section 4.5 explains how to replace α, h by an equivalent morphism $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$ and Lagrangian structure $\hat{h} = (\hat{h}^0, 0, 0, \dots)$, such that for $\hat{\alpha}, \hat{h}$ the functions a_j^{k-1-i} are zero, giving $\hat{h}^0 = d_{\mathrm{dR}} u_j^i d_{\mathrm{dR}} v_j^{k-1-i}$. This will complete the proof of Theorem 3.7 for even $k < 0$. Sections 4.6–4.7 discuss how to modify Sections 4.3–4.5 to prove Theorem 3.7 for the cases $k < 0$ with $k \equiv 1 \pmod{4}$, and $k < 0$ with $k \equiv 3 \pmod{4}$, respectively.

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a k -shifted symplectic derived \mathbb{K} -scheme for $k < 0$, and $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{X}$ be a Lagrangian derived \mathbb{K} -scheme in $(\mathbf{X}, \omega_{\mathbf{X}})$, with isotropic structure $h_{\mathbf{L}} : 0 \xrightarrow{\sim} \mathbf{f}^*(\omega_{\mathbf{X}})$. Let $y \in \mathbf{L}$ with $\mathbf{f}(y) = x \in \mathbf{X}$. Suppose A^\bullet is a standard form cdga over \mathbb{K} , ω is a k -shifted symplectic form on $\mathbf{Spec} A^\bullet$ with A^\bullet, ω in Darboux form, $p \in \mathbf{Spec} H^0(A^\bullet)$, and $i : \mathbf{Spec} A^\bullet \rightarrow \mathbf{X}$ is either a Zariski open inclusion or étale, with $i(p) = x$ and $\omega \sim i^*(\omega_{\mathbf{X}})$.

As in Example 2.14, we have coordinates x_j^i, y_j^{k-i} in A^\bullet for $i = 0, -1, \dots, d$, $j = 1, \dots, m_d$, and $\omega = (\omega^0, 0, 0, \dots)$ with $\omega^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\mathrm{dR}} x_j^i d_{\mathrm{dR}} y_j^{k-i}$ in $(\Lambda^2 \Omega_{A^\bullet}^1)^k$, and $\Phi \in A^{k+1}$, $\phi \in (\Omega_{A^\bullet}^1)^k$ with $d\Phi = 0$, $d_{\mathrm{dR}} \Phi + d\phi = 0$ and $d_{\mathrm{dR}} \phi = k\omega^0$. We also use the notation of Remark 2.15, which defined a sub-cdga $\iota : A_+^\bullet \hookrightarrow A^\bullet$ containing the x_j^i but not the y_j^{k-i} , and $\Phi_+ \in A_+^{k+1}$, $\phi_+ \in (\Omega_{A^\bullet}^1)^k$ satisfying $d\Phi_+ = 0$, $d_{\mathrm{dR}} \Phi_+ + d\phi_+ = 0$ and $d_{\mathrm{dR}} \phi_+ = -\omega^0$.

Form a homotopy commutative diagram

$$\begin{array}{ccccc}
 \mathbf{Z} & \xrightarrow{\quad h \quad} & \mathbf{Spec} A^\bullet & \xrightarrow{\quad \mathbf{Spec} \iota \quad} & \mathbf{Spec} A_+^\bullet \\
 \downarrow g & & \downarrow i & & \\
 \mathbf{L} & \xrightarrow{\quad f \quad} & \mathbf{X} & &
 \end{array} \tag{4.4}$$

with $\mathbf{Z} = \mathbf{L} \times_{\mathbf{f}, \mathbf{X}, i} \mathbf{Spec} A^\bullet$. Then i a Zariski open inclusion, or étale, implies that g is a Zariski open inclusion, or étale, respectively. There is a unique point $z \in \mathbf{Z}$ with $g(z) = y \in \mathbf{L}$ and $h(z) = p \in \mathbf{Spec} A^\bullet$.

Apply Theorem 3.2 with $\mathbf{Spec} \iota \circ h : \mathbf{Z} \rightarrow \mathbf{Spec} A_+^\bullet$, $\mathrm{id} : \mathbf{Spec} A_+^\bullet \rightarrow \mathbf{Spec} A_+^\bullet$, $z \in \mathbf{Z}$, $p \in \mathbf{Spec} A_+^\bullet$ in place of $\mathbf{f} : \mathbf{Y} \rightarrow \mathbf{X}$, $i : \mathbf{Spec} A^\bullet \rightarrow \mathbf{X}$, $y \in \mathbf{Y}$, $p \in \mathbf{Spec} A^\bullet$. This gives a standard form cdga B^\bullet , a point $q \in \mathbf{Spec} B^\bullet$, a submersion $\alpha_+ : A_+^\bullet \rightarrow B^\bullet$ minimal at q with $\mathbf{Spec} \alpha_+(q) = p$, and a Zariski open inclusion $e : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Z}$ with $e(q) = z$ such that the

following diagram is homotopy commutative

$$\begin{array}{ccc}
 \mathbf{Spec} B^\bullet & \xrightarrow{\mathbf{Spec} \alpha_+} & \mathbf{Spec} A_+^\bullet \\
 \downarrow e & & \text{id} \downarrow \\
 Z & \xrightarrow{\mathbf{Spec} \iota \circ h} & \mathbf{Spec} A_+^\bullet.
 \end{array} \tag{4.5}$$

The morphism $h \circ e : \mathbf{Spec} B^\bullet \rightarrow \mathbf{Spec} A^\bullet$ is equivalent to $\mathbf{Spec} \alpha^\infty$ for some morphism $\alpha^\infty : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}^\infty$, where $\alpha^\infty \circ \iota \simeq \alpha_+$. Since A^\bullet is freely generated over A_+^\bullet and so cofibrant over A_+^\bullet , up to equivalence α^∞ descends to a morphism $\alpha : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ with $\alpha \circ \iota = \alpha_+$. Thus, combining (4.4)–(4.5) gives a homotopy commutative diagram

$$\begin{array}{ccccc}
 \mathbf{Spec} B^\bullet & & & & \\
 \downarrow e & \searrow \mathbf{Spec} \alpha & & \searrow \mathbf{Spec} \alpha_+ & \\
 j \downarrow Z & \xrightarrow{h} & \mathbf{Spec} A^\bullet & \xrightarrow{\mathbf{Spec} \iota} & \mathbf{Spec} A_+^\bullet \\
 \downarrow g & & \downarrow i & & \\
 L & \xrightarrow{f} & X, & &
 \end{array} \tag{4.6}$$

where we write $j = g \circ e$, so that j is a Zariski open inclusion, or étale, if i is a Zariski open inclusion, or étale, respectively. This proves the first part of Theorem 3.7: we have constructed a standard form $\mathbf{cdga} B^\bullet$, a point $q \in \mathbf{Spec} B^\bullet$, a morphism $\alpha : A^\bullet \rightarrow B^\bullet$ in $\mathbf{cdga}_{\mathbb{K}}$ with $\mathbf{Spec} \alpha(q) = p$ such that $\alpha_+ = \alpha \circ \iota : A_+^\bullet \rightarrow B^\bullet$ is a submersion minimal at q , and a morphism $j : \mathbf{Spec} B^\bullet \hookrightarrow L$ which is either a Zariski open inclusion or étale with $j(q) = y$, such that (3.38) homotopy commutes by (4.6).

Next we discuss the isotropic structure. As we have the homotopy commutative diagram (3.38) with i, j étale, and $\omega \sim i^*(\omega_X)$, Definition 2.9 implies that the Lagrangian structure h_L for $f : L \rightarrow (X, \omega_X)$ lifts to a Lagrangian structure h for $\mathbf{Spec} \alpha : \mathbf{Spec} B^\bullet \rightarrow (\mathbf{Spec} A^\bullet, \omega)$, where $h = (h^0, h^1, h^2, \dots)$ with $h^i \in (\Lambda^{2+i} \mathbb{L}_{B^\bullet})^{k-1-i}$, which by (2.4) satisfy

$$dh^0 = \alpha_*(\omega^0) \quad \text{and} \quad d_{\text{dR}} h^i + dh^{i+1} = 0 \quad \text{for } i = 0, 1, \dots \tag{4.7}$$

First, we need a vanishing result for the isotropic structure. Note that “Lagrangian Darboux form” in Example 3.3 involves $\Psi \in B^k$ and $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$, and (4.8) is (3.22)–(3.23) with Ξ in place of Ψ , and $\tilde{h}^0 = \frac{1}{k-1} d_{\text{dR}} \psi$ is (3.24).

PROPOSITION 4.1. — *There exists $\Xi \in B^k$ and $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$ satisfying*

$$d\Xi = -\alpha(\Phi + \Phi_+) \quad \text{and} \quad d_{\text{dR}} \Xi + d\psi = -\alpha_*(\phi + \phi_+), \tag{4.8}$$

such that the isotropic structure $h = (h^0, h^1, \dots)$ is homotopic to $(\tilde{h}^0, 0, 0, \dots)$ where $\tilde{h}^0 = \frac{1}{k-1} d_{\text{dR}} \psi$.

Proof. — Combining equations (2.10), (2.16) and (4.7) gives the equations

$$\begin{aligned} d[-\alpha(\Phi) - \alpha(\Phi_+)] &= 0 \quad \text{in } B^{k+2}, \\ d_{\text{dR}}[-\alpha(\Phi) - \alpha(\Phi_+)] + d[-\alpha_*(\phi) - \alpha_*(\phi_+)] &= 0 \quad \text{in } (\Omega_{B^\bullet}^1)^{k+1}, \\ d_{\text{dR}}[-\alpha_*(\phi) - \alpha_*(\phi_+)] + d[(k-1)h^0] &= 0 \quad \text{in } (\Lambda^2\Omega_{B^\bullet}^1)^k, \\ d_{\text{dR}}[(k-1)h^i] + d[(k-1)h^{i+1}] &= 0 \quad \text{in } (\Lambda^{3+i}\Omega_{B^\bullet}^1)^{k-1-i}, \quad i \geq 0. \end{aligned}$$

Therefore

$$\gamma = (-\alpha(\Phi) - \alpha(\Phi_+), -\alpha_*(\phi) - \alpha_*(\phi_+), (k-1)h^0, (k-1)h^1, \dots) \quad (4.9)$$

is a $(k+1)$ -shifted closed 0-form on $\mathbf{Spec} B^\bullet$, that is, γ is a closed element of degree $k+1$ in the complex $(\prod_{i \geq 0} (\Lambda^i \mathbb{L}_{B^\bullet})[i], d_{\text{dR}} + d)$.

As in Bussi, Brav and Joyce [7, §5.2] we have isomorphisms

$$\begin{aligned} H^{k+1} \left(\prod_{i \geq 0} (\Lambda^i \mathbb{L}_{B^\bullet})[i], d_{\text{dR}} + d \right) &\cong \text{HN}^{k+1}(B^\bullet)(0) \cong \text{HP}^{k+1}(B^\bullet)(0) \\ &\cong \text{HP}^{k+1}(H^0(B^\bullet))(0) \cong H_{\text{inf}}^{k+1}(H^0(B^\bullet)), \quad (4.10) \end{aligned}$$

where $\text{HN}^*(\cdot)$ is negative cyclic homology, $\text{HP}^*(\cdot)$ is periodic cyclic homology, and $H_{\text{inf}}^*(\cdot)$ is algebraic de Rham cohomology.

If $k < -1$ then $H_{\text{inf}}^{k+1}(H^0(B^\bullet)) = 0$. If $k = -1$ then $H_{\text{inf}}^0(H^0(B^\bullet))$ is isomorphic to the locally constant functions $(\mathbf{Spec} H^0(B^\bullet))^{\text{red}} \rightarrow \mathbb{A}^1$. This identifies the cohomology class of γ with $-\alpha_*(\Phi)|_{(\mathbf{Spec} H^0(A^\bullet))^{\text{red}}}$. But as in Example 2.14, when $k = -1$ we impose the additional condition that $\Phi|_{(\mathbf{Spec} H^0(A^\bullet))^{\text{red}}} = 0$. Hence γ is exact in $(\prod_{i \geq 0} (\Lambda^i \mathbb{L}_{B^\bullet})[i], d_{\text{dR}} + d)$ for all $k < 0$. So we may write

$$\gamma = (d_{\text{dR}} + d)(\Xi, \psi, (k-1)\delta^0, (k-1)\delta^1, (k-1)\delta^2, \dots), \quad (4.11)$$

where $\Xi \in B^k$, $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$ and $\delta^i \in (\Lambda^{2+i}\Omega_{B^\bullet}^1)^{k-2-i}$ for $i = 0, 1, \dots$. Set $\tilde{h}^0 = \frac{1}{k-1} d_{\text{dR}}\psi$. Then combining (4.9)–(4.11) proves (4.8) and the equations

$$d\delta^0 = h^0 - \tilde{h}^0, \quad d_{\text{dR}}\delta^i + d\delta^{i+1} = h^{i+1}, \quad i = 0, 1, \dots$$

Thus $(\delta^0, \delta^1, \dots)$ is a homotopy from $(\tilde{h}^0, 0, 0, \dots)$ to $h = (h^0, h^1, \dots)$ in the complex $(\prod_{i \geq 0} (\Lambda^i \mathbb{L}_{B^\bullet})[i], d_{\text{dR}} + d)$. This completes the proof. \square

We replace $h = (h^0, h^1, \dots)$ by $(\tilde{h}^0, 0, 0, \dots)$, so from now on we suppose that $h = (h^0, 0, 0, \dots)$ with $h^0 = \frac{1}{k-1} d_{\text{dR}}\psi$. We continue the proof in the cases k even, $k \equiv 1 \pmod{4}$, and $k \equiv 3 \pmod{4}$, in Section 4.3, Section 4.6 and Section 4.7, respectively.

4.3. Choosing coordinates $\tilde{x}_j^i, u_j^i, v_j^{k-1-i}$ on B^\bullet for even $k < 0$

We carry on from Section 4.2, but now also assume that k is even. In the notation of Example 3.3 we have $d = [(k+1)/2]$, $e = [k/2]$, so that $e = d$ and $k = 2d = 2e$. Supposing k is even simplifies the proof as there is no “middle degree” $(k-1)/2$ in B^\bullet , which would require special treatment.

Sections 4.3–4.5 will complete the proof of Theorem 3.7 for even $k < 0$. To save work in Sections 4.6–4.7, in Sections 4.3–4.5 we give the correct signs in formulae for general k , so we include factors such as $(-1)^{(i+1)k}$ in (4.27) although k is even.

From Section 4.2, we have a submersion of standard form cdgas $\alpha_+ : A_+^\bullet \rightarrow B^\bullet$ minimal at $q \in \mathbf{Spec} B^\bullet$, and coordinates $x_j^i \in A_+^i$ for $i = 0, -1, \dots, d$ and $j = 1, \dots, m_i$, where $(x_1^0, \dots, x_{m_0}^0)$ are étale coordinates on $U = \mathbf{Spec} A^0$, and A_+^* is freely generated over A^0 by the x_j^i in degree i for $i < 0$. As $\alpha_+^0 : A^0 \rightarrow B^0$ is smooth, localizing B^\bullet if necessary, we may assume there exist $u_1^0, \dots, u_{n_0}^0 \in B^0$ such that $d_{\mathrm{dR}} \tilde{x}_1^0, \dots, d_{\mathrm{dR}} \tilde{x}_{m_0}^0, d_{\mathrm{dR}} u_1^0, \dots, d_{\mathrm{dR}} u_{n_0}^0$ form a basis of $\Omega_{B^0}^1$ over B^0 , where we write $\tilde{x}_j^0 = \alpha_+^0(x_j^0) \in B^0$. Geometrically, $(x_1^0, \dots, x_{m_0}^0)$ are étale coordinates on $U := \mathbf{Spec} A^0$, and $(\tilde{x}_1^0, \dots, \tilde{x}_{m_0}^0, u_1^0, \dots, u_{n_0}^0)$ are étale coordinates on $V := \mathbf{Spec} B^0$.

Since $\alpha_+ : A_+^\bullet \rightarrow B^\bullet$ is a submersion of standard form cdgas and A_+^* is freely generated over A^0 by the x_j^i for $i = -1, -2, \dots, d$ and $j = 1, \dots, m_i$, we see that B^* is freely generated over B^0 by the $\tilde{x}_j^i = \alpha_+(x_j^i)$ for $i = -1, -2, \dots, d$ and $j = 1, \dots, m_i$ plus some additional variables, which lie in degrees $-1, -2, \dots, k-1$ since \mathbb{L}_{B^\bullet} is concentrated in degrees $[k-1, 0]$ and α_+ is minimal at q . Thus, as in (3.2), we take B^* to be freely generated over B^0 by

$$\begin{aligned} \tilde{x}_1^i, \dots, \tilde{x}_{m_i}^i & \quad \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\ u_1^i, \dots, u_{n_i}^i & \quad \text{in degree } i \text{ for } i = -1, -2, \dots, e, \text{ and} \\ v_1^{k-1-i}, \dots, v_{n_i}^{k-1-i} & \quad \text{in degree } k-1-i \text{ for } i = 0, -1, \dots, e, \end{aligned} \tag{4.12}$$

where $\tilde{x}_j^i = \alpha_+(x_j^i)$, and later we will show that $n_i' = n_i$. Note that by (2.8) and (2.14) this implies that in $B^\bullet = (B^*, d)$ we have

$$d\tilde{x}_j^i = \alpha_+(dx_j^i) = (-1)^{(i+1)(k+1)} \alpha \left(\frac{\partial \Phi}{\partial y_j^{k-i}} \right) = (-1)^{i+1} \alpha_+(\Phi_j^{i+1}). \tag{4.13}$$

Since $\psi \in (\mathbb{L}_B^\bullet)^{k-1}$, we may write

$$\begin{aligned} \frac{1}{k-1}\psi &= \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} b_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} u_j^i \\ &\quad + \sum_{i=0}^e \sum_{j=1}^{n'_i} c_j^i \mathrm{d}_{\mathrm{dR}} v_j^{k-1-i}, \end{aligned} \quad (4.14)$$

where $a_j^i, b_j^i, c_j^i \in B^i$. For degree reasons, the c_j^i depend on B^0 and the $\tilde{x}_{j'}^{i'}, u_{j'}^{i'}$, but do not involve the $v_j^{k-1-i'}$. By leaving h^0 unchanged but replacing Ξ, ψ by

$$\begin{aligned} \tilde{\Xi} &= \Xi - (k-1) \mathrm{d} \left[\sum_{i=0}^e \sum_{j=1}^{n'_i} (-1)^i c_j^i v_j^{k-1-i} \right], \\ \tilde{\psi} &= \psi - (k-1) \mathrm{d}_{\mathrm{dR}} \left[\sum_{i=0}^e \sum_{j=1}^{n'_i} (-1)^i c_j^i v_j^{k-1-i} \right], \end{aligned} \quad (4.15)$$

we may assume that $c_j^i = 0$ for all i, j , as $\mathrm{d}_{\mathrm{dR}} c_j^i$ involves no terms in $\mathrm{d}_{\mathrm{dR}} v_j^{k-1-i'}$. Thus we have

$$\frac{1}{k-1}\psi = \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} b_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} u_j^i, \quad (4.16)$$

so that $h^0 = \frac{1}{k-1} \mathrm{d}_{\mathrm{dR}} \psi$ yields

$$h^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} \mathrm{d}_{\mathrm{dR}} a_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} \mathrm{d}_{\mathrm{dR}} b_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} u_j^i. \quad (4.17)$$

The next part of the argument follows (3.14)–(3.20) in Example 3.3. By Definition 2.9, h being a nondegenerate isotropic structure means that a certain morphism $\chi : \mathbb{T}_{B^\bullet/A^\bullet} \rightarrow \Omega_{B^\bullet}^1[k-1]$ is a quasi-isomorphism of B^\bullet -modules, so $\chi|_q : \mathbb{T}_{B^\bullet/A^\bullet}|_q \rightarrow \Omega_{B^\bullet}^1[k-1]|_q$ is also a quasi-isomorphism of complexes of \mathbb{K} -vector spaces. As for (3.15)–(3.18), as \mathbb{K} -vector spaces, the i^{th} graded pieces of $\Omega_{A^\bullet}^1|_p, \Omega_{B^\bullet}^1|_q, (\Omega_{A^\bullet}^1)^\vee|_p$, and $(\Omega_{A^\bullet}^1)^\vee|_p$ are

$$(\Omega_{A^\bullet}^1|_p)^i = \langle \mathrm{d}_{\mathrm{dR}} x_j^i, j = 1, \dots, m_i, \mathrm{d}_{\mathrm{dR}} y_j^i, j = 1, \dots, m_{k-i} \rangle_{\mathbb{K}}, \quad (4.18)$$

$$\begin{aligned} (\Omega_{B^\bullet}^1|_q)^i &= \langle \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i, j = 1, \dots, m_i, \mathrm{d}_{\mathrm{dR}} u_j^i, j = 1, \dots, n_i, \\ &\quad \mathrm{d}_{\mathrm{dR}} v_j^i, j = 1, \dots, n'_{k-1-i} \rangle_{\mathbb{K}}, \end{aligned} \quad (4.19)$$

$$((\Omega_{A^\bullet}^1)^\vee|_p)^i = \left\langle \frac{\partial}{\partial x_j^{-i}}, j = 1, \dots, m_{-i}, \frac{\partial}{\partial y_j^{-i}}, j = 1, \dots, m_{i-k} \right\rangle_{\mathbb{K}}, \quad (4.20)$$

$$\begin{aligned}
 ((\Omega_{A^\bullet}^1)^\vee|_p)^i &= \left\langle \frac{\partial}{\partial \tilde{x}_j^{-i}}, j = 1, \dots, m_{-i}, \frac{\partial}{\partial u_j^{-i}}, j = 1, \dots, n_{-i}, \right. \\
 &\quad \left. \frac{\partial}{\partial v_j^{-i}}, j = 1, \dots, n'_{i+1-k} \right\rangle_{\mathbb{K}}.
 \end{aligned} \tag{4.21}$$

As for (3.19), the next diagram shows $\chi|_q : \mathbb{T}_{B^\bullet/A^\bullet}|_q \rightarrow \Omega_{B^\bullet}^1[k-1]|_q$ in degrees $i, i+1$, together with d in both complexes:

$$\begin{array}{ccc}
 \begin{array}{l}
 (\mathbb{T}_{B^\bullet/A^\bullet}|_q)^i \\
 = \left\langle \frac{\partial}{\partial \tilde{x}_j^{-i}}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle \frac{\partial}{\partial u_j^{-i}}, \frac{\partial}{\partial v_j^{-i}}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle \frac{\partial}{\partial x_j^{1-i}}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle \frac{\partial}{\partial y_j^{1-i}}, \forall j \right\rangle_{\mathbb{K}}
 \end{array} & \xrightarrow{\chi|_q = \begin{pmatrix} h^0 & h^0 & * & \omega^0 \\ h^0 & h^0 & * & 0 \end{pmatrix}} & \begin{array}{l}
 (\Omega_{B^\bullet}^1[k-1]|_q)^i \\
 = \left\langle d_{\text{dR}} \tilde{x}_j^{k-1+i}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle d_{\text{dR}} u_j^{k-1+i}, d_{\text{dR}} v_j^{k-1+i}, \forall j \right\rangle_{\mathbb{K}}
 \end{array} \\
 \downarrow d = \begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ \alpha^* & 0 & * & 0 \\ * & * & * & * \end{pmatrix} & & \downarrow d = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \\
 \begin{array}{l}
 (\mathbb{T}_{B^\bullet/A^\bullet}|_q)^{i+1} \\
 = \left\langle \frac{\partial}{\partial \tilde{x}_j^{-i-1}}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle \frac{\partial}{\partial u_j^{-i-1}}, \frac{\partial}{\partial v_j^{-i-1}}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle \frac{\partial}{\partial x_j^{-i}}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle \frac{\partial}{\partial y_j^{-i}}, \forall j \right\rangle_{\mathbb{K}}
 \end{array} & \xrightarrow{\chi|_q^{i+1} = \begin{pmatrix} h^0 & h^0 & * & \omega^0 \\ h^0 & h^0 & * & 0 \end{pmatrix}} & \begin{array}{l}
 (\Omega_{B^\bullet}^1[k-1]|_q)^{i+1} \\
 = \left\langle d_{\text{dR}} \tilde{x}_j^{k+i}, \forall j \right\rangle_{\mathbb{K}} \\
 \oplus \left\langle d_{\text{dR}} u_j^{k+i}, d_{\text{dR}} v_j^{k+i}, \forall j \right\rangle_{\mathbb{K}}
 \end{array}
 \end{array} \tag{4.22}$$

Here in the left hand column of (4.22), the component of “ d ” mapping $\left\langle \frac{\partial}{\partial u_j^{-i}}, \frac{\partial}{\partial v_j^{-i}} \right\rangle_{\mathbb{K}} \rightarrow \left\langle \frac{\partial}{\partial u_j^{-i-1}}, \frac{\partial}{\partial v_j^{-i-1}} \right\rangle_{\mathbb{K}}$ is zero. This is because $\alpha_+ : A_+^\bullet \rightarrow B^\bullet$ is a submersion minimal at q , so in the additional variables u_j^i, v_j^{k-1-i} in B^\bullet the differential vanishes at q . Similarly, in the right hand column, the component of “ d ” mapping $\left\langle d_{\text{dR}} u_j^{k-1+i}, d_{\text{dR}} v_j^{k-1+i} \right\rangle_{\mathbb{K}} \rightarrow \left\langle d_{\text{dR}} u_j^{k+i}, d_{\text{dR}} v_j^{k+i}, \forall j \right\rangle_{\mathbb{K}}$ is zero.

The argument in Example 3.3 involving the subcomplex C^\bullet works again in this case. Thus in (4.22) we may replace the right hand column by the

subcomplex C^\bullet , giving a simpler diagram in which the rows are a quasi-isomorphism:

$$\begin{array}{ccc}
 \begin{array}{c} \langle \frac{\partial}{\partial u_j^{-i}}, \frac{\partial}{\partial v_j^{-i}}, \forall j \rangle_{\mathbb{K}} \\ \oplus \langle \frac{\partial}{\partial y_j^{1-i}}, \forall j \rangle_{\mathbb{K}} \end{array} & \xrightarrow{\chi|_q = \begin{pmatrix} h^0 & \omega^0 \\ h^0 & 0 \end{pmatrix}} & \begin{array}{c} \langle d_{\text{dR}} \tilde{x}_j^{k-1+i}, \forall j \rangle_{\mathbb{K}} \\ \oplus \langle d_{\text{dR}} u_j^{k-1+i}, d_{\text{dR}} v_j^{k-1+i}, \forall j \rangle_{\mathbb{K}} \end{array} \\
 \downarrow d = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} & & \downarrow d = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \\
 \begin{array}{c} \langle \frac{\partial}{\partial u_j^{-i-1}}, \frac{\partial}{\partial v_j^{-i-1}}, \forall j \rangle_{\mathbb{K}} \\ \oplus \langle \frac{\partial}{\partial y_j^{-i}}, \forall j \rangle_{\mathbb{K}} \end{array} & \xrightarrow{\chi|_q^{i+1} = \begin{pmatrix} h^0 & \omega^0 \\ h^0 & 0 \end{pmatrix}} & \begin{array}{c} \langle d_{\text{dR}} \tilde{x}_j^{k+i}, \forall j \rangle_{\mathbb{K}} \\ \oplus \langle d_{\text{dR}} u_j^{k+i}, d_{\text{dR}} v_j^{k+i}, \forall j \rangle_{\mathbb{K}} \end{array}
 \end{array} \quad (4.23)$$

In the top row of (4.23), the morphism $\omega^0 : \langle \frac{\partial}{\partial y_j^{1-i}} \rangle_{\mathbb{K}} \rightarrow \langle d_{\text{dR}} \tilde{x}_j^{k-1+i} \rangle_{\mathbb{K}}$ is an isomorphism. Using this, we see that $\chi|_q$ is a quasi-isomorphism if and only if the morphism $h^0 : \langle \frac{\partial}{\partial u_j^{-i}}, \frac{\partial}{\partial v_j^{-i}} \rangle_{\mathbb{K}} \rightarrow \langle d_{\text{dR}} u_j^{k-1+i}, d_{\text{dR}} v_j^{k-1+i} \rangle_{\mathbb{K}}$ in $\chi|_q^i$ is an isomorphism for all $i \in \mathbb{Z}$.

From (4.17), $h^0 : \langle \frac{\partial}{\partial u_j^{-i}}, \frac{\partial}{\partial v_j^{-i}} \rangle_{\mathbb{K}} \rightarrow \langle d_{\text{dR}} u_j^{k-1+i}, d_{\text{dR}} v_j^{k-1+i} \rangle_{\mathbb{K}}$ acts by

$$\begin{aligned}
 h^0 : \frac{\partial}{\partial u_j^{-i}} &\mapsto (-1)^{k(i+1)} \sum_{j'=1}^{n'_i} \frac{\partial b_j^{k-1+i}}{\partial v_{j'}^{k-1+i}} \Big|_q d_{\text{dR}} v_{j'}^{k-1+i}, \\
 h^0 : \frac{\partial}{\partial v_j^{-i}} &\mapsto \sum_{j'=1}^{n_{k-1+i}} \frac{\partial b_{j'}^{-i}}{\partial v_j^{-i}} \Big|_q d_{\text{dR}} u_{j'}^{k-1+i}.
 \end{aligned}$$

Therefore we see that h nondegenerate at q is equivalent to $n'_i = n_i$ and the following being an invertible matrix over \mathbb{K} for all $i = 0, -1, \dots, e$:

$$\left(\frac{\partial b_j^{k-1-i}}{\partial v_{j'}^{k-1-i}} \Big|_q \right)_{j, j'=1}^{n_i}. \quad (4.24)$$

Set $\hat{v}_j^{k-1-i} = (-1)^{(i+1)k} b_j^{k-i-i}$ for all $i = 0, -1, \dots, e$ and $j = 1, \dots, n_i$. The invertibility of (4.24) implies that near $q \in V$ we can invert this to write the $v_{j'}^{k-1-i'}$ as linear functions of the \hat{v}_j^{k-1-i} with coefficients in $B^0[\tilde{x}_{j'}^i, u_{j'}^i]$. Localizing B^\bullet we can suppose this invertibility holds globally on $V = \mathbf{Spec} B^0$, so that $\tilde{x}_j^i, u_j^i, \hat{v}_j^i$ is an alternative set of coordinates for B^\bullet . Thus, replacing the v_j^i by the \hat{v}_j^i , by (4.16)–(4.17) we can take ψ and h^0

to be of the form

$$\psi = (k-1) \left[\sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} v_j^{k-1-i} d_{\text{dR}} u_j^i \right], \quad (4.25)$$

$$h^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} d_{\text{dR}} u_j^i d_{\text{dR}} v_j^{k-1-i}. \quad (4.26)$$

Leaving h^0 unchanged, but by replacing Ξ, ψ by

$$\begin{aligned} \tilde{\Xi} &= \Xi + d \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{k-i} i a_j^{k-1-i} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^i i u_j^i v_j^{k-1-i} \right], \\ \tilde{\psi} &= \psi + d_{\text{dR}} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{k-i} i a_j^{k-1-i} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^i i u_j^i v_j^{k-1-i} \right], \end{aligned}$$

we have

$$\begin{aligned} \psi &= \sum_{i=0}^d \sum_{j=1}^{m_i} [(k-1-i) a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + (-1)^{(i+1)k} i \tilde{x}_j^i d_{\text{dR}} a_j^{k-1-i}] \\ &\quad + \sum_{i=0}^e \sum_{j=1}^{n_i} [i u_j^i d_{\text{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (k-1-i) v_j^{k-1-i} d_{\text{dR}} u_j^i]. \quad (4.27) \end{aligned}$$

4.4. Determining the equations for even $k < 0$

We continue in the situation of Section 4.3. Using (4.13), define $\Psi \in B^k$ by

$$\Psi = \Xi - \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} d \tilde{x}_j^i = \Xi - \sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{i+1} a_j^{k-1-i} \alpha_+(\Phi_j^{i+1}). \quad (4.28)$$

Writing things in terms of Ψ rather than Ξ will make many of the following formulae simpler. This section will compute the p.d.e. satisfied by Ψ , and expressions for d in $B^\bullet = (B^*, d)$, and for $\alpha : A^\bullet \rightarrow B^\bullet$.

By (4.8) we have $d_{\mathrm{dR}}\Xi + d\psi = -\alpha_*(\phi + \phi_+)$. Expanding this equation using (2.9), (2.15), (4.27), (4.28), and $\tilde{x}_j^i = \alpha(x_j^i)$ yields

$$\begin{aligned}
 & \sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{(i+1)k} \frac{\partial \Psi}{\partial \tilde{x}_j^i} d_{\mathrm{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \left[\frac{\partial \Psi}{\partial u_j^i} d_{\mathrm{dR}} u_j^i + \frac{\partial \Psi}{\partial v_j^{k-1-i}} d_{\mathrm{dR}} v_j^{k-1-i} \right] \\
 & + \sum_{i=0}^d \sum_{j=1}^{m_i} \left[(k-1-i) da_j^{k-1-i} d_{\mathrm{dR}} \tilde{x}_j^i + (-1)^{k-i} (k-2-i) a_j^{k-1-i} d_{\mathrm{dR}} \circ d\tilde{x}_j^i \right. \\
 & \quad \left. + (-1)^{(i+1)k} (i+1) d\tilde{x}_j^i d_{\mathrm{dR}} a_j^{k-1-i} + (-1)^{(i+1)(k+1)} i \tilde{x}_j^i d_{\mathrm{dR}} \circ da_j^{k-1-i} \right] \\
 & + \sum_{i=0}^e \sum_{j=1}^{n_i} \left[i (du_j^i d_{\mathrm{dR}} v_j^{k-1-i} - (-1)^i u_j^i d_{\mathrm{dR}} \circ dv_j^{k-1-i}) \right. \\
 & \quad \left. + (-1)^{(i+1)k} (k-1-i) (dv_j^{k-1-i} d_{\mathrm{dR}} u_j^i - (-1)^{k-1-i} v_j^{k-1-i} d_{\mathrm{dR}} \circ du_j^i) \right] \\
 & = - \sum_{i=0}^d \sum_{j=1}^{m_i} \left[i \tilde{x}_j^i d_{\mathrm{dR}} \circ \alpha(y_j^{k-i}) + (-1)^{(i+1)(k+1)} (k-1-i) \alpha(y_j^{k-i}) d_{\mathrm{dR}} \tilde{x}_j^i \right]. \quad (4.29)
 \end{aligned}$$

Rewriting terms of the form $d_{\mathrm{dR}} \circ d(\dots)$ using equations such as

$$\begin{aligned}
 d_{\mathrm{dR}} \circ du_j^i &= \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i+1)(i'+1)} \frac{\partial (du_j^i)}{\partial \tilde{x}_{j'}^{i'}} d_{\mathrm{dR}} \tilde{x}_{j'}^{i'} \\
 & + \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} \left[(-1)^{(i+1)(i'+1)} \frac{\partial (du_j^i)}{\partial u_{j'}^{i'}} d_{\mathrm{dR}} u_{j'}^{i'} + (-1)^{(i+1)(k-i')} \frac{\partial (du_j^i)}{\partial v_{j'}^{k-1-i'}} d_{\mathrm{dR}} v_{j'}^{k-1-i'} \right],
 \end{aligned}$$

and noting that $\frac{\partial}{\partial u_j^i} (d\tilde{x}_{j'}^{i'}) = \frac{\partial}{\partial v_j^{k-1-i}} (d\tilde{x}_{j'}^{i'}) = 0$ as $d\tilde{x}_j^i = \alpha(dx_j^i)$ for dx_j^i in A_+^\bullet , we can express (4.29) solely in terms of multiples of $d_{\mathrm{dR}} \tilde{x}_j^i$, $d_{\mathrm{dR}} u_j^i$ and $d_{\mathrm{dR}} v_j^{k-1-i}$. Then taking coefficients of these, multiplying by $(-1)^{(i+1)k}$, and rearranging, gives three equations:

$$\begin{aligned}
 & (-1)^i (k-1) \alpha(y_j^{k-i}) - (-1)^{(i+1)k} (k-1) da_j^{k-1-i} \\
 & + (k-1) \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i(k-1-i')} a_{j'}^{k-1-i'} \frac{\partial (d\tilde{x}_{j'}^{i'})}{\partial \tilde{x}_j^i} \\
 & = \frac{\partial}{\partial \tilde{x}_j^i} \left[\Psi + \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} \left[(-1)^{i'} i' \tilde{x}_{j'}^{i'} \alpha(y_{j'}^{k-i'}) + (i'+1) a_{j'}^{k-1-i'} d\tilde{x}_{j'}^{i'} \right. \right. \\
 & \quad \left. \left. + (-1)^{k-1-i'} i' da_{j'}^{k-1-i'} \tilde{x}_{j'}^{i'} \right] \right. \\
 & \quad \left. - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (i' u_{j'}^{i'} dv_{j'}^{k-1-i'} + (-1)^{(i'+1)k} (k-1-i') v_{j'}^{k-1-i'} du_{j'}^{i'}) \right], \quad (4.30)
 \end{aligned}$$

$$\begin{aligned}
 & - (k-1)dv_j^{k-1-i} \\
 & = \frac{\partial}{\partial u_j^i} \left[\Psi + \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} [(-1)^{i'} i' \tilde{x}_{j'}^{i'} \alpha(y_{j'}^{k-i'}) + (i'+1)a_{j'}^{k-1-i'} d\tilde{x}_{j'}^{i'} \right. \\
 & \quad \left. + (-1)^{k-1-i'} i' da_{j'}^{k-1-i'} \tilde{x}_{j'}^{i'} \right] \\
 & \quad - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (i' u_{j'}^{i'} dv_{j'}^{k-1-i'} + (-1)^{(i'+1)k} (k-1-i') v_{j'}^{k-1-i'} du_{j'}^{i'}) \Big], \quad (4.31)
 \end{aligned}$$

$$\begin{aligned}
 & - (-1)^{(i+1)k} (k-1) du_j^i \\
 & = \frac{\partial}{\partial v_j^{k-1-i}} \left[\Psi + \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} [(-1)^{i'} i' \tilde{x}_{j'}^{i'} \alpha(y_{j'}^{k-i'}) + (i'+1)a_{j'}^{k-1-i'} d\tilde{x}_{j'}^{i'} \right. \\
 & \quad \left. + (-1)^{k-1-i'} i' da_{j'}^{k-1-i'} \tilde{x}_{j'}^{i'} \right] \\
 & \quad - \sum_{i'=0}^e \sum_{j'=1}^{n_{i'}} (i' u_{j'}^{i'} dv_{j'}^{k-1-i'} + (-1)^{(i'+1)k} (k-1-i') v_{j'}^{k-1-i'} du_{j'}^{i'}) \Big], \quad (4.32)
 \end{aligned}$$

where (4.30) holds for all $i = 0, \dots, d$ and $j = 1, \dots, m_i$, and (4.31)–(4.32) hold for all $i = 0, \dots, e$ and $j = 1, \dots, n_i$.

Writing F for the function $[\dots]$ on the r.h.s. of (4.30)–(4.32), we have

$$kF = \sum_{i=0}^d \sum_{j=1}^{m_i} \tilde{x}_j^i \frac{\partial F}{\partial \tilde{x}_j^i} + \sum_{i=0}^e \sum_{j=1}^{n_i} \left[i u_j^i \frac{\partial F}{\partial u_j^i} + (k-1-i) v_j^{k-1-i} \frac{\partial F}{\partial v_j^{k-1-i}} \right],$$

since F has degree k . Thus, multiplying (4.30) by $i\tilde{x}_j^i$, and (4.31) by $i u_j^i$, and (4.32) by $(k-1-i)v_j^{k-1-i}$, and summing all three over all i, j , yields

$$\begin{aligned}
 & k \left[\Psi + \sum_{i=0}^d \sum_{j=1}^{m_i} [(-1)^i i \tilde{x}_j^i \alpha(y_j^{k-i}) + (i+1)a_j^{k-1-i} d\tilde{x}_j^i \right. \\
 & \quad \left. + (-1)^{k-1-i} i d(a_j^{k-1-i}) \tilde{x}_j^i \right] \\
 & \quad - \sum_{i=0}^e \sum_{j=1}^{n_i} (i u_j^i dv_j^{k-1-i} + (-1)^{(i+1)k} (k-1-i) v_j^{k-1-i} du_j^i) \Big] \\
 & = (k-1) \sum_{i=0}^d \sum_{j=1}^{m_i} [(-1)^i i \tilde{x}_j^i \alpha(y_j^{k-i}) + (i+1)a_j^{k-1-i} d\tilde{x}_j^i \\
 & \quad + (-1)^{k-1-i} i d(a_j^{k-1-i}) \tilde{x}_j^i] \\
 & \quad - (k-1) \sum_{i=0}^e \sum_{j=1}^{n_i} [i u_j^i dv_j^{k-1-i} + (-1)^{(i+1)k} (k-1-i) v_j^{k-1-i} du_j^i]. \quad (4.33)
 \end{aligned}$$

Here the second term on the r.h.s. of (4.33) comes from the third term on the l.h.s. of (4.30) using $\sum_{i,j} i \tilde{x}_j^i \frac{\partial(d\tilde{x}_{j'}^{i'})}{\partial \tilde{x}_j^i} = (i'+1)d\tilde{x}_{j'}^{i'}$, as $\frac{\partial(d\tilde{x}_{j'}^{i'})}{\partial u_j^i} = \frac{\partial(d\tilde{x}_{j'}^{i'})}{\partial v_j^{k-1-i}} = 0$.

Rearranging (4.33) shows that $F = -(k-1)\Psi$. Substituting this into (4.30)–(4.32), dividing by $1-k$, rearranging, and using (4.13) in (4.34) yields

$$\begin{aligned} \alpha(y_j^{k-i}) &= (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} - (-1)^{(i+1)(k+1)} da_j^{k-1-i} \\ &\quad - \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i+(i'+1)k} \alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) a_{j'}^{k-1-i'}, \end{aligned} \quad (4.34)$$

$$dv_j^{k-1-i} = \frac{\partial \Psi}{\partial u_j^i}, \quad (4.35)$$

$$du_j^i = (-1)^{(i+1)k} \frac{\partial \Psi}{\partial v_j^{k-1-i}}. \quad (4.36)$$

Using equations (2.14), (4.13), (4.28), (4.35) and (4.36) we see that

$$\begin{aligned} d\Xi &= \sum_{i=-1}^d \sum_{j=1}^{m_i} \left(d\tilde{x}_j^i \frac{\partial \Psi}{\partial \tilde{x}_j^i} + da_j^{k-1-i} d\tilde{x}_j^i \right) \\ &\quad + \sum_{i=-1}^e \sum_{j=1}^{n_i} \left(du_j^i \frac{\partial \Psi}{\partial u_j^i} + dv_j^{k-1-i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} \right) \\ &= \sum_{i=-1}^d \sum_{j=1}^{m_i} \alpha_+(\Phi_j^{i+1}) \left((-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} + (-1)^{(i+1)(k+1)} da_j^{k-1-i} \right) \\ &\quad + 2 \sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}}. \end{aligned} \quad (4.37)$$

Also we have

$$\begin{aligned} \alpha(\Phi + \Phi_+) &= 2\alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} \alpha_+(\Phi_j^{i+1}) \alpha(y_j^{k-i}) \\ &= 2\alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} \alpha_+(\Phi_j^{i+1}) \left[(-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} - (-1)^{(i+1)(k+1)} da_j^{k-1-i} \right]. \end{aligned} \quad (4.38)$$

Here in the first step we use (2.11), and in the second we use (4.34) to substitute for $\alpha(y_j^{k-i})$, and note that the terms coming from the second line of (4.34) are zero by (2.13). Since $d\Xi = -\alpha(\Phi + \Phi_+)$ by (4.8), equations (4.37)–(4.38) give

$$\sum_{i=-1}^e \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + \alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial \Psi}{\partial \tilde{x}_j^i} = 0, \quad (4.39)$$

which is equation (3.4) in Example 3.3.

4.5. Improving $\alpha : A^\bullet \rightarrow B^\bullet$ and completing the proof for even $k < 0$

We continue in the situation of Sections 4.3–4.4. Observe that the expressions (4.13), (4.35), (4.36) for d in $B^\bullet = (B^*, d)$, and the p.d.e. (4.39), all depend on the coordinates $\tilde{x}_j^i, u_j^i, v_j^{k-1-i}$ and functions $\Psi, \alpha_+(\Phi_+), \alpha_+(\Phi_j^{i+1})$ in B^\bullet , but are independent of the a_j^{k-i-1} , although the expressions (4.34) for $\alpha : A^\bullet \rightarrow B^\bullet$, and (4.26), (4.27) for $h^0 \in (\Lambda^2 \Omega_{B^\bullet}^1)^{k-1}$ and $\psi \in (\Omega_{B^\bullet}^1)^{k-1}$ do depend on the a_j^{k-1-i} .

Now $\alpha : A^\bullet \rightarrow B^\bullet$ was chosen in Section 4.2 to make (4.6) homotopy commute, and thus α is unique only up to homotopy in $\mathbf{cdga}_{\mathbb{K}}^\infty$, subject to the condition $\alpha \circ \iota = \alpha_+$. We will now show that keeping $B^\bullet, \alpha_+, \tilde{x}_j^i, u_j^i, v_j^{k-1-i}, \Psi$ fixed, we can replace α by a homotopic morphism $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$ with $\hat{\alpha} \circ \iota = \alpha_+$, and h^0, ψ by homotopic $\hat{h}^0, \hat{\psi}$, so as to make the a_j^{k-1-i} zero.

Define a morphism of graded \mathbb{K} -algebras $\hat{\alpha} : A^* \rightarrow B^*$ by $\hat{\alpha}|_{A_+^*} = \alpha_+$ and

$$\hat{\alpha}(y_j^{k-i}) = (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} \quad (4.40)$$

for $i = 0, -1, \dots, d$ and $j = 1, \dots, m_i$, as in (3.5), which would be (4.34) if we had $a_j^{k-1-i} = 0$ for all i, j . This is well-defined as A^* is freely generated over A_+^* by the y_j^{k-i} . The proof in equation (3.11) in Example 3.3 shows that $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$ is a cdga morphism. Define $\hat{h}^0 \in (\Lambda^2 \Omega_{B^\bullet}^1)^{k-1}, \hat{\psi} \in (\Omega_{B^\bullet}^1)^{k-1}$ by

$$\hat{h}^0 = \sum_{i=0}^e \sum_{j=1}^{n_i} d_{\mathrm{dR}} u_j^i d_{\mathrm{dR}} v_j^{k-1-i}, \quad (4.41)$$

$$\hat{\psi} = \sum_{i=0}^e \sum_{j=1}^{n_i} [i u_j^i d_{\mathrm{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (k-1-i) v_j^{k-1-i} d_{\mathrm{dR}} u_j^i], \quad (4.42)$$

as in (3.12) and (3.21), which would be h^0, ψ in (4.26)–(4.27) if we had $a_j^{k-1-i} = 0$ for all i, j . The proofs of (3.22)–(3.24) in Example 3.3 show that

$$d\Psi = -\hat{\alpha}(\Phi + \Phi_+), \quad d_{\mathrm{dR}} \Psi + d\hat{\psi} = -\hat{\alpha}_*(\phi + \phi_+), \quad (k-1)\hat{h}^0 = d_{\mathrm{dR}} \hat{\psi},$$

as in Proposition 4.1 with $\Psi, \hat{\psi}, \hat{h}^0$ in place of Ξ, ψ, \tilde{h}^0 . Also Example 3.3 shows that \hat{h}^0 is a Lagrangian structure for $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$ and ω .

PROPOSITION 4.2. — *In the situation above, $\alpha, \hat{\alpha} : A^\bullet \rightarrow B^\bullet$ are homotopic cdga morphisms, and under this homotopy, the Lagrangian structures $(\hat{h}^0, 0, \dots)$ for $\alpha : A^\bullet \rightarrow B^\bullet, \omega$ and $(\hat{h}^0, 0, \dots)$ for $\hat{\alpha} : A^\bullet \rightarrow B^\bullet, \omega$ are also homotopic.*

Proof. — By definition, a homotopy H from $\hat{\alpha}$ to α is a cdga morphism

$$H : A^\bullet \longrightarrow B^\bullet \otimes_{\mathbb{K}} \mathbb{K}[s, t],$$

where $\mathbb{K}[s, t]$ is the cdga over \mathbb{K} in *nonnegative* degrees which as a graded \mathbb{K} -algebra is freely generated by variables s in degree 0 and t in degree 1 with differential given by $ds = t, dt = 0$, such that the following commutes:

$$\begin{array}{ccccc}
 & & A^\bullet & & \\
 & \hat{\alpha} \curvearrowright & \downarrow H & \curvearrowleft \alpha & \\
 B^\bullet & \xleftarrow{s=0, t=0} & B^\bullet \otimes_{\mathbb{K}} \mathbb{K}[s, t] & \xrightarrow{s=1, t=0} & B^\bullet,
 \end{array} \tag{4.43}$$

where the bottom morphisms are evaluation at $s = t = 0$ and at $s = 1, t = 0$. These are 1-simplices in the simplicial model category $\mathbf{cdga}_{\mathbb{K}}$, as explained in Remark 2.2.

Note that as in Section 2.1, all cdgas C^\bullet considered so far have been concentrated in nonpositive degrees, so that $C^i = 0$ for $i > 0$. Here, however, $\mathbb{K}[s, t]$ lives in degrees 0, 1 (since $t^2 = 0$), and $B^\bullet \otimes_{\mathbb{K}} \mathbb{K}[s, t]$ in degrees 1, 0, $-1, -2, \dots$. So $\mathbf{Spec} \mathbb{K}[s, t], \mathbf{Spec}(B^\bullet \otimes_{\mathbb{K}} \mathbb{K}[s, t])$ *do not exist* as derived schemes in the usual sense. Nonetheless, this is a good definition of homotopy of cdga morphisms.

Heuristically, we can pretend “ $\mathbf{Spec} \mathbb{K}[s, t] \cong [0, 1]$ ” is an interval, so that “ $\mathbf{Spec}(B^\bullet \otimes_{\mathbb{K}} \mathbb{K}[s, t]) \cong (\mathbf{Spec} B^\bullet) \times [0, 1]$ ”, and (4.43) corresponds to a diagram

$$\begin{array}{ccccc}
 & & \mathbf{Spec} A^\bullet & & \\
 & \mathbf{Spec} \hat{\alpha} \curvearrowright & \uparrow \text{“Spec } H \text{”} & \curvearrowleft \mathbf{Spec} \alpha & \\
 \mathbf{Spec} B^\bullet & \xrightarrow{\text{“id} \times 0 \text{”}} & \text{“}(\mathbf{Spec} B^\bullet) \times [0, 1]\text{”} & \xleftarrow{\text{“id} \times 1 \text{”}} & \mathbf{Spec} B^\bullet,
 \end{array} \tag{4.44}$$

so that “ $\mathbf{Spec} H$ ” is a homotopy from $\mathbf{Spec} \hat{\alpha}$ to $\mathbf{Spec} \alpha$ in the usual sense in topology. But we do not claim that (4.44) actually makes sense in $\mathbf{dSch}_{\mathbb{K}}$.

Define $H : A^* \longrightarrow B^* \otimes_{\mathbb{K}} \mathbb{K}[s, t]$ to be the morphism of graded \mathbb{K} -algebras given by $H|_{A_{\pm}^{\bullet}} = \alpha_{\pm} \otimes 1$ and

$$H(y_j^{k-i}) = (-1)^{i+1} \frac{\partial \Psi}{\partial \tilde{x}_j^i} - (-1)^{(i+1)(k+1)} s d a_j^{k-1-i} - (-1)^{(i+1)(k+1)} t \alpha_j^{k-1-i} \\ - \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{i+(i'+1)k} s \alpha_{+} \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) a_{j'}^{k-1-i'} \quad (4.45)$$

for $i = 0, -1, \dots, d$ and $j = 1, \dots, m_i$. This is well-defined as A^* is freely generated over A_{+}^* by the y_j^{k-i} . To see that H is a cdga morphism, note that $H|_{A_{\pm}^{\bullet}} : A_{\pm}^{\bullet} \rightarrow B^{\bullet} \otimes_{\mathbb{K}} \mathbb{K}[s, t]$ is a cdga morphism, and

$$d \circ H(y_j^{k-i}) = (-1)^{i+1} d \left[\frac{\partial \Psi}{\partial \tilde{x}_j^i} + \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i'+1)k} s \alpha_{+} \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) a_{j'}^{k-1-i'} \right] \\ = (-1)^{i+1} \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} d \tilde{x}_{j'}^{i'} \frac{\partial^2 \Psi}{\partial \tilde{x}_{j'}^{i'} \partial \tilde{x}_j^i} \\ + (-1)^{i+1} \sum_{i'=-1}^e \sum_{j'=1}^{n_{i'}} \left[d u_{j'}^{i'} \frac{\partial^2 \Psi}{\partial u_{j'}^{i'} \partial \tilde{x}_j^i} + d v_{j'}^{k-1-i'} \frac{\partial^2 \Psi}{\partial v_{j'}^{k-1-i'} \partial \tilde{x}_j^i} \right] \\ - \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} \sum_{i''=-1}^d \sum_{j''=1}^{m_{i''}} (-1)^{i+(i'+1)k} s d \tilde{x}_{j''}^{i''} \alpha_{+} \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_{j''}^{i''} \partial x_j^i} \right) a_{j'}^{k-1-i'} \\ - \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i'+1)(k+1)} s \alpha_{+} \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) d a_{j'}^{k-1-i'} \\ - \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{i+(i'+1)k} t \alpha_{+} \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) a_{j'}^{k-1-i'} \\ = - \frac{\partial}{\partial \tilde{x}_j^i} \left[\sum_{i'=-1}^e \sum_{j'=1}^{n_{i'}} \frac{\partial \Psi}{\partial u_{j'}^{i'}} \frac{\partial \Psi}{\partial v_{j'}^{k-1-i'}} + \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{i'+1} \alpha_{+} (\Phi_{j'}^{i'+1}) \frac{\partial \Psi}{\partial \tilde{x}_{j'}^{i'}} \right] \\ + \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{i'+1} \alpha_{+} \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) \frac{\partial \Psi}{\partial \tilde{x}_{j'}^{i'}} \\ + \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} \sum_{i''=-1}^d \sum_{j''=1}^{m_{i''}} (-1)^{i+i'+(i'+1)k} s \alpha_{+} (\Phi_{j'}^{i'+1}) \alpha_{+} \left(\frac{\partial \Phi_{j''}^{i''+1}}{\partial x_{j'}^{i'} \partial x_{j''}^{i''}} \right) a_{j'}^{k-1-i'}$$

$$\begin{aligned}
 & - \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i'+1)(k+1)} s\alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) da_{j'}^{k-1-i'} \\
 & - \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} (-1)^{i+(i'+1)k} t\alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) a_{j'}^{k-1-i'}, \tag{4.46}
 \end{aligned}$$

using (4.45) and $d[sda_j^{k-1-i} + ta_j^{k-1-i}] = 0$ in the first step, and (4.13), (4.35) and (4.36) in the third. Also equations (2.14) and (4.45) imply that

$$\begin{aligned}
 H \circ dy_j^{k-i} &= \frac{\partial}{\partial \tilde{x}_j^i} [\alpha_+(\Phi_+)] + \sum_{i'=i-1}^d \sum_{j'=1}^{m_{i'}} \alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial x_j^i} \right) \left[(-1)^{i'+1} \frac{\partial \Psi}{\partial \tilde{x}_{j'}^{i'}} \right. \\
 & \quad - (-1)^{(i'+1)(k+1)} sda_{j'}^{k-1-i'} - (-1)^{(i'+1)(k+1)} ta_{j'}^{k-1-i'} \\
 & \quad \left. - \sum_{i''=0}^d \sum_{j''=1}^{m_{i''}} (-1)^{i'+k(i'+1)} s\alpha_+ \left(\frac{\partial \Phi_{j''}^{i''+1}}{\partial \tilde{x}_{j''}^{i''}} \right) a_{j''}^{k-1-i'} \right]. \tag{4.47}
 \end{aligned}$$

Combining (4.46), (4.47) and adding $\frac{\partial}{\partial \tilde{x}_j^i}$ applied to (4.39) yields

$$\begin{aligned}
 & d \circ H(y_j^{k-i}) - H \circ dy_j^{k-i} \tag{4.48} \\
 &= \sum_{i'=-1}^d \sum_{j'=1}^{m_{i'}} \sum_{i''=-1}^d \sum_{j''=1}^{m_{i''}} (-1)^{i+i'+(i'+1)k} s\alpha_+(\Phi_{j'}^{i'+1}) \alpha_+ \left(\frac{\partial \Phi_{j''}^{i''+1}}{\partial x_j^{i'}, \partial x_j^{i''}} \right) a_{j''}^{k-1-i'} \\
 & \quad + \sum_{i'=i-1}^d \sum_{j'=1}^{m_{i'}} \sum_{i''=0}^d \sum_{j''=1}^{m_{i''}} (-1)^{i'+k(i'+1)} s\alpha_+ \left(\frac{\partial \Phi_{j''}^{i''+1}}{\partial x_j^{i''}} \right) \alpha_+ \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial \tilde{x}_{j'}^{i'}} \right) a_{j'}^{k-1-i'}.
 \end{aligned}$$

But by applying $\frac{\partial}{\partial \tilde{x}_j^i}$ to (2.13) for i', j' , we see that the r.h.s. of (4.48) is zero. Hence $d \circ H(y_j^{k-i}) = H \circ dy_j^{k-i}$, and H is a cdga morphism. From (4.34), (4.40) and (4.45) we see that restricting H to $s = t = 0$ gives $\hat{\alpha}$, and restricting H to $s = 1, t = 0$ gives α . Thus (4.43) commutes, and H is a homotopy of cdga morphisms from $\hat{\alpha}$ to α , as we have to prove.

Next we must show that the Lagrangian structures $(\hat{h}^0, 0, \dots)$ for $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$, ω and $(h^0, 0, \dots)$ for $\alpha : A^\bullet \rightarrow B^\bullet$, ω are homotopic, over the homotopy H from $\hat{\alpha}$ to α . It is enough to find $\dot{h}^0 \in (\Lambda^2 \Omega_{B^\bullet}^1 \otimes_{\mathbb{K}} \mathbb{K}[s, t])^{k-1}$ satisfying

$$\begin{aligned}
 \dot{h}^0|_{s=0, t=0} &= \hat{h}^0, & \dot{h}^0|_{s=1, t=0} &= h^0, \\
 d_{\text{dR}} \dot{h}^0 &= 0, & d\dot{h}^0 &= H_*(\omega^0), \tag{4.49}
 \end{aligned}$$

since then $(\hat{h}^0, 0, \dots)$ is a homotopy from $(\hat{h}^0, 0, \dots)$ to $(h^0, 0, \dots)$ over H . Set

$$\hat{h}^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}}(sa_j^{k-1-i}) d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} d_{\text{dR}} u_j^i d_{\text{dR}} v_j^{k-1-i}. \quad (4.50)$$

The first two equations of (4.49) follow from (4.26) and (4.41), and the third is also immediate. For the fourth equation, we have

$$\begin{aligned} d\hat{h}^0 &= \sum_{i=0}^d \sum_{j=1}^{m_i} [(d \circ d_{\text{dR}}(sa_j^{k-1-i})) d_{\text{dR}} \tilde{x}_j^i + (-1)^{(i+1)k} (d \circ d_{\text{dR}} \tilde{x}_j^i) d_{\text{dR}}(sa_j^{k-1-i})] \\ &\quad + \sum_{i=0}^e \sum_{j=1}^{n_i} [(d \circ d_{\text{dR}} u_j^i) d_{\text{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (d \circ d_{\text{dR}} v_j^{k-1-i}) d_{\text{dR}} u_j^i] \\ &= - \sum_{i=0}^d \sum_{j=1}^{m_i} [(d_{\text{dR}} \circ d(sa_j^{k-1-i})) d_{\text{dR}} \tilde{x}_j^i + (-1)^{(i+1)k} (d_{\text{dR}} \circ d \tilde{x}_j^i) d_{\text{dR}}(sa_j^{k-1-i})] \\ &\quad - \sum_{i=0}^e \sum_{j=1}^{n_i} [(d_{\text{dR}} \circ d u_j^i) d_{\text{dR}} v_j^{k-1-i} + (-1)^{(i+1)k} (d_{\text{dR}} \circ d v_j^{k-1-i}) d_{\text{dR}} u_j^i] \\ &= - \sum_{i=0}^d \sum_{j=1}^{m_i} \left[(-1)^{(i+1)(k+1)} d_{\text{dR}} \tilde{x}_j^i (d_{\text{dR}} \circ d(sa_j^{k-1-i})) \right. \\ &\quad \left. + (-1)^{(i+1)k} \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{i'+1} d_{\text{dR}} \tilde{x}_{j'}^{i'} H \left(\frac{\partial \Phi_j^{i'+1}}{\partial x_{j'}^{i'}} \right) d_{\text{dR}}(sa_j^{k-1-i}) \right] \\ &\quad - \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^{(i+1)k} \left[d_{\text{dR}} \left(\frac{\partial \Psi}{\partial v_j^{k-1-i}} \right) d_{\text{dR}} v_j^{k-1-i} + d_{\text{dR}} \left(\frac{\partial \Psi}{\partial u_j^i} \right) d_{\text{dR}} u_j^i \right] \\ &= d_{\text{dR}} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} \left[-(-1)^{(i+1)k} d_{\text{dR}} \tilde{x}_j^i [sda_j^{k-1-i} + ta_j^{k-1-i}] \right. \right. \\ &\quad \left. \left. + \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i'+1)k} d_{\text{dR}} \tilde{x}_{j'}^{i'} sH \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial \tilde{x}_{j'}^{i'}} \right) a_{j'}^{k-1-i'} \right] \right. \\ &\quad \left. - \sum_{i=0}^e \sum_{j=1}^{n_i} \left[d_{\text{dR}} v_j^{k-1-i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + d_{\text{dR}} u_j^i \frac{\partial \Psi}{\partial u_j^i} \right] \right] \end{aligned}$$

$$\begin{aligned}
&= \mathrm{d}_{\mathrm{dR}} \left[-\mathrm{d}_{\mathrm{dR}} \Psi + \sum_{i=0}^d \sum_{j=1}^{m_i} \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i \left[\frac{\partial \Psi}{\partial x_j^i} - (-1)^{(i+1)k} s \mathrm{d} a_j^{k-1-i} \right. \right. \\
&\quad \left. \left. - (-1)^{(i+1)k} t a_j^{k-1-i} + \sum_{i'=0}^d \sum_{j'=1}^{m_{i'}} (-1)^{(i'+1)k} s H \left(\frac{\partial \Phi_{j'}^{i'+1}}{\partial \tilde{x}_j^i} \right) a_{j'}^{k-1-i'} \right] \right] \\
&= \sum_{i=0}^d \sum_{j=1}^{m_i} \mathrm{d}_{\mathrm{dR}} \left(\mathrm{d}_{\mathrm{dR}} (H(x_j^i)) \cdot (-1)^{i+1} H(y_j^{k-i}) \right) \\
&= \sum_{i=0}^d \sum_{j=1}^{m_i} \mathrm{d}_{\mathrm{dR}} (H(x_j^i)) \mathrm{d}_{\mathrm{dR}} (H(y_j^{k-i})) \\
&= H_* \left[\sum_{i=0}^d \sum_{j=1}^{m_i} \mathrm{d}_{\mathrm{dR}} x_j^i \mathrm{d}_{\mathrm{dR}} y_j^{k-i} \right] = H_*(\omega^0),
\end{aligned}$$

using (4.50) in the first step, equations (2.14), (4.35) and (4.36) and $H|_{A_+^\bullet} = \alpha_+ \otimes 1$ in the third, $ds = t$ in the fourth, $H|_{A_+^\bullet} = \alpha_+ \otimes 1$ and (4.45) in the sixth, and (2.6) in the ninth. This completes the proof of Proposition 4.2. \square

Proposition 4.2 shows that we may replace α, h^0 by $\hat{\alpha}, \hat{h}^0$. We have now proved all the assumptions of Example 3.3 in the case $k < 0$ with k even. Equation (3.3) was the definition of \tilde{x}_j^i . The classical master equation (3.4) is (4.39). The definition (3.5) of $\alpha(y_j^{k-i})$ is (4.40), and the definition (3.6) of d in $B^\bullet = (B^*, d)$ is equations (4.13), (4.35) and (4.36). The definitions (3.12) and (3.21) of h^0 and ψ are equations (4.41) and (4.42). Thus, $A^\bullet, \omega, B^\bullet, \alpha, h$ are in Lagrangian Darboux form. This proves Theorem 3.7(i) for $k < 0$ with k even.

4.6. Modifications to the proof for $k < 0$ with $k \equiv 1 \pmod{4}$

Next we explain how to modify Sections 4.2–4.5 to prove Theorem 3.7 when $k < 0$ with $k \equiv 1 \pmod{4}$. In the notation of Example 3.3 we have $d = [(k+1)/2]$, $e = [k/2]$, so that $e = d - 1$ and $k = 2d - 1 = 2e + 1$, with d, k odd and e even.

First follow Section 4.2 without change. Then follow Section 4.3 with the following modifications. In place of (4.12), we take B^* to be freely generated

over B^0 by

$$\begin{aligned}
 \tilde{x}_1^i, \dots, \tilde{x}_{m_i}^i & && \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\
 u_1^i, \dots, u_{n_i}^i & && \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\
 w_1^e, \dots, w_{p_e}^e & && \text{in degree } e, \text{ and} \\
 v_1^{k-1-i}, \dots, v_{n_i'}^{k-1-i} & && \text{in degree } k-1-i \text{ for } i = 0, -1, \dots, d,
 \end{aligned} \tag{4.51}$$

where $\tilde{x}_j^i = \alpha_+(x_j^i)$. In place of (4.14), we write

$$\begin{aligned}
 \frac{1}{k-1}\psi &= \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} b_j^{k-1-i} d_{\text{dR}} u_j^i \\
 &+ \sum_{j=1}^{p_e} \sum_{j'=1}^{p_e} c_{j'j}^0 w_{j'}^e d_{\text{dR}} w_j^e + \sum_{j=1}^{p_e} d_j^e d_{\text{dR}} w_j^e + \sum_{i=0}^d \sum_{j=1}^{n_i} e_j^i d_{\text{dR}} v_j^{k-1-i}, \tag{4.52}
 \end{aligned}$$

where $a_j^i, b_j^i, c_{j'j}^i, d_j^i, e_j^i \in B^i$, and d_j^e includes no terms in $w_{j'}^e$, as these are written separately in the $c_{j'j}^0$ terms.

As in (4.15), by leaving h^0 unchanged but replacing Ξ, ψ by

$$\begin{aligned}
 \tilde{\Xi} &= \Xi - (k-1)d \left[\frac{1}{2} \sum_{j=1}^{p_e} \sum_{j'=1}^{p_e} c_{j'j}^0 w_{j'}^e w_j^e + \sum_{j=1}^{p_e} d_j^e w_j^e + \sum_{i=0}^d \sum_{j=1}^{n_i} (-1)^i e_j^i v_j^{k-1-i} \right], \\
 \tilde{\psi} &= \psi - (k-1)d_{\text{dR}} \left[\frac{1}{2} \sum_{j=1}^{p_e} \sum_{j'=1}^{p_e} c_{j'j}^0 w_{j'}^e w_j^e + \sum_{j=1}^{p_e} d_j^e w_j^e + \sum_{i=0}^d \sum_{j=1}^{n_i} (-1)^i e_j^i v_j^{k-1-i} \right],
 \end{aligned}$$

we may assume that $c_{j'j}^0 = -c_{jj'}^0$ for all j, j' and $d_j^e = e_j^i = 0$ for all i, j , as $d_{\text{dR}} d_j^e, d_{\text{dR}} e_j^i$ involve no terms in $d_{\text{dR}} w_{j'}^e, d_{\text{dR}} v_{j'}^{k-1-i}$. Here the minus sign in $c_{j'j}^0 = -c_{jj'}^0$ occurs as e is even. Thus the analogue of (4.17) is

$$\begin{aligned}
 h^0 &= \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} d_{\text{dR}} b_j^{k-1-i} d_{\text{dR}} u_j^i \\
 &+ \sum_{j=1}^{p_e} \sum_{j'=1}^{p_e} d_{\text{dR}} (c_{j'j}^0 w_{j'}^e) d_{\text{dR}} w_j^e. \tag{4.53}
 \end{aligned}$$

The argument of (4.18)–(4.24) now shows that h nondegenerate at q is equivalent to $n_i = n_i'$ for $i = 0, -1, \dots, d$, and the following being an invertible matrix over \mathbb{K} for all $i = 0, -1, \dots, d$:

$$\left(\frac{\partial b_j^{k-1-i}}{\partial v_{j'}^{k-1-i}} \Big|_q \right)_{j, j'=1}^{n_i}, \tag{4.54}$$

and for the case $i = e$, the following being an invertible matrix over \mathbb{K} :

$$(c_{j'j}^0|_q)_{j,j'=1}^{p_e} \quad (4.55)$$

As (4.55) is an invertible and antisymmetric, p_e is even, so we write $p_e = 2n_e$.

Regard $(c_{j'j}^0)_{j,j'=1}^{p_e}$ as an antisymmetric form on the trivial vector bundle $\mathbb{K}^{2n_e} \times V \rightarrow V$ over $V = \mathbf{Spec} B^0$, which is nondegenerate at $q \in V$, and hence near q . Since nondegenerate antisymmetric forms can be standardized Zariski locally by a change of basis, after localizing B^\bullet we can choose an invertible change of variables of the $w_1^e, \dots, w_{2n_e}^e$, such that w.r.t. the new w_j^e we have

$$c_{j'j}^0 = \begin{cases} \frac{1}{2}, & j' = 1, \dots, n_e, \quad j = j' + n_e, \\ -\frac{1}{2}, & j = 1, \dots, n_e, \quad j' = j + n_e, \\ 0, & \text{otherwise.} \end{cases}$$

As in Section 4.3, set $\hat{v}_j^{k-1-i} = (-1)^{i+1} v_j^{k-1-i}$ for all $i = 0, -1, \dots, d$ and $j = 1, \dots, n_i$. Since (4.54) is invertible, localizing B^\bullet we can suppose $\tilde{x}_j^i, u_j^i, \hat{v}_j^i, w_j^e$ is an alternative set of coordinates for B^\bullet . Also define $u_j^e = w_j^e$ and $v_j^e = w_{j+n_e}^e$ for $j = 1, \dots, n_e$. Then, modifying (4.51), B^* is freely generated over B^0 by

$$\begin{array}{ll} \tilde{x}_1^i, \dots, \tilde{x}_{m_i}^i & \text{in degree } i \text{ for } i = -1, -2, \dots, d, \text{ and} \\ u_1^i, \dots, u_{n_i}^i & \text{in degree } i \text{ for } i = -1, -2, \dots, e, \text{ and} \\ v_1^{k-1-i}, \dots, v_{n_i'}^{k-1-i} & \text{in degree } k-1-i \text{ for } i = 0, -1, \dots, e. \end{array}$$

Also, from (4.52) we have

$$\begin{aligned} \frac{1}{k-1} \psi = \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} (-1)^{i+1} v_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} u_j^i \\ + \sum_{j=1}^{n_e} \left[\frac{1}{2} u_j^e \mathrm{d}_{\mathrm{dR}} v_j^e - \frac{1}{2} v_j^e \mathrm{d}_{\mathrm{dR}} u_j^e \right]. \end{aligned} \quad (4.56)$$

Leaving h^0 unchanged but replacing Ξ, ψ by

$$\tilde{\Xi} = \Xi - (k-1) \mathrm{d} \left[\frac{1}{2} \sum_{j=1}^{n_e} u_j^e v_j^e \right], \quad \tilde{\psi} = \psi - (k-1) \mathrm{d}_{\mathrm{dR}} \left[\frac{1}{2} \sum_{j=1}^{n_e} u_j^e v_j^e \right],$$

equation (4.56) becomes

$$\frac{1}{k-1} \psi = \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} (-1)^{i+1} v_j^{k-1-i} \mathrm{d}_{\mathrm{dR}} u_j^i,$$

which agrees with (4.25), so $h^0 = \frac{1}{k-1} d_{\text{dR}} \psi$ yields

$$h^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^e \sum_{j=1}^{n_i} d_{\text{dR}} u_j^i d_{\text{dR}} v_j^{k-1-i},$$

as in (4.26). The rest of the proof, from (4.27) to the end of Section 4.5, works without further changes. This proves Theorem 3.7(i) when $k < 0$ with $k \equiv 1 \pmod{4}$, and so completes the proof of Theorem 3.7(i).

4.7. Modifications to the proof for $k < 0$ with $k \equiv 3 \pmod{4}$

We now explain how to modify Sections 4.2–4.6 to prove Theorem 3.7 when $k < 0$ with $k \equiv 3 \pmod{4}$, that is, to prove Theorem 3.7(ii),(iii). In the notation of Example 3.3 we have $d = \lfloor (k+1)/2 \rfloor$, $e = \lfloor k/2 \rfloor$, so that $e = d - 1$ and $k = 2d - 1 = 2e + 1$, with d even and e, k odd.

The first part of the proof follows that for $k \equiv 1 \pmod{4}$ in Section 4.6, as far as equation (4.55). The only difference is that just before (4.53) we have $c_{j'j}^0 = c_{jj'}^0$, since e is odd rather than even. So $(c_{j'j}^0)_{j,j'=1}^{p_e}$ is now a symmetric matrix of functions on V , rather than an antisymmetric matrix. Write $n_e = p_e$.

Now in general, nondegenerate quadratic forms cannot be trivialized Zariski locally, but they can at least be diagonalized. That is, in general we cannot find a (Zariski local) change of variables of the $w_1^e, \dots, w_{n_e}^e$ to make $(c_{j'j}^0)_{j,j'=1}^{n_e}$ the identity matrix, but we can change variables to make $(c_{j'j}^0)_{j,j'=1}^{n_e}$ a diagonal matrix. Thus, after localizing B^\bullet if necessary and changing variables w_j^e , we can suppose there are invertible elements q_1, \dots, q_{n_e} in B^0 such that $c_{jj}^0 = q_j$ and $c_{j'j}^0 = 0$ if $j' \neq j$. Also replacing v_j^{k-1-i} by $\hat{v}_j^{k-1-i} = (-1)^{i+1} b_j^{k-i-i}$ for all $i = 0, -1, \dots, d$ and $j = 1, \dots, n_i$, we now have the analogue of (4.25):

$$\begin{aligned} \frac{1}{k-1} \psi &= \sum_{i=0}^d \sum_{j=1}^{m_i} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} (-1)^{i+1} v_j^{k-1-i} d_{\text{dR}} u_j^i \\ &\quad + \sum_{j=1}^{n_e} q_j w_j^e d_{\text{dR}} w_j^e, \end{aligned}$$

so that $h^0 = \frac{1}{k-1} d_{\text{dR}} \psi$ yields the analogue of (4.26):

$$h^0 = \sum_{i=0}^d \sum_{j=1}^{m_i} d_{\text{dR}} a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} d_{\text{dR}} u_j^i d_{\text{dR}} v_j^{k-1-i} + \sum_{j=1}^{n_e} d_{\text{dR}} (q_j w_j^e) d_{\text{dR}} w_j^e.$$

Leaving h^0 unchanged, but by replacing Ξ, ψ by

$$\begin{aligned} \tilde{\Xi} &= \Xi + d \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{k-i} i a_j^{k-1-i} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} (-1)^i i u_j^i v_j^{k-1-i} \right], \\ \tilde{\psi} &= \psi + d_{\text{dR}} \left[\sum_{i=0}^d \sum_{j=1}^{m_i} (-1)^{k-i} i a_j^{k-1-i} \tilde{x}_j^i + \sum_{i=0}^d \sum_{j=1}^{n_i} (-1)^i i u_j^i v_j^{k-1-i} \right], \end{aligned}$$

we have the analogue of (4.27):

$$\begin{aligned} \psi &= \sum_{i=0}^d \sum_{j=1}^{m_i} [(k-1-i) a_j^{k-1-i} d_{\text{dR}} \tilde{x}_j^i + (-1)^{i+1} i \tilde{x}_j^i d_{\text{dR}} a_j^{k-1-i}] \\ &\quad + \sum_{i=0}^d \sum_{j=1}^{n_i} [i u_j^i d_{\text{dR}} v_j^{k-1-i} + (-1)^{i+1} (k-1-i) v_j^{k-1-i} d_{\text{dR}} u_j^i] \\ &\quad + \sum_{j=1}^{n_e} d_{\text{dR}} (q_j w_j^e) d_{\text{dR}} w_j^e. \end{aligned}$$

We now follow Section 4.4 adding extra terms involving the q_j, w_j^e , which we leave as an exercise. The definition (4.28) of Ψ is unchanged. For the final results, equation (4.34) is unchanged, the analogues of (4.35)–(4.36) are

$$\begin{aligned} du_j^i &= (-1)^{i+1} \frac{\partial \Psi}{\partial v_j^{k-1-i}}, & i = 0, -1, \dots, d, \quad j = 1, \dots, n_i, \\ dv_j^{k-1-i} &= \frac{\partial \Psi}{\partial u_j^i}, & i = -1, -2, \dots, d, \quad j = 1, \dots, n_i, \\ dv_j^{k-1} &= \frac{\partial \Psi}{\partial u_j^0} - \sum_{j'=1}^{n_e} \frac{w_{j'}^e}{2q_{j'}} \frac{\partial q_{j'}}{\partial u_j^0} \frac{\partial \Psi}{\partial w_{j'}^e}, & j = 1, \dots, n_0, \\ dw_j^e &= \frac{1}{2q_j} \frac{\partial \Psi}{\partial w_j^e}, & j = 1, \dots, n_e, \end{aligned}$$

as in (3.35), and as in (3.33), equation (4.39) is replaced by

$$\begin{aligned} \sum_{i=-1}^d \sum_{j=1}^{n_i} \frac{\partial \Psi}{\partial u_j^i} \frac{\partial \Psi}{\partial v_j^{k-1-i}} + \frac{1}{4} \sum_{j=1}^{n_e} \frac{1}{q_j} \left(\frac{\partial \Psi}{\partial w_j^e} \right)^2 \\ + \alpha_+(\Phi_+) + \sum_{i=-1}^d \sum_{j=1}^{m_i} (-1)^{i+1} \alpha_+(\Phi_j^{i+1}) \frac{\partial \Psi}{\partial \tilde{x}_j^i} = 0. \end{aligned}$$

Next we follow Section 4.5 adding extra terms involving the q_j, w_j^e , to show that we can define an alternative cdga morphism $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$ by $\hat{\alpha}|_{A_+^\bullet} = \alpha_+$ and $\hat{\alpha}(y_j^{k-i})$ as in (3.34), and a Lagrangian structure $(\hat{h}^0, 0, \dots)$ for $\hat{\alpha} : A^\bullet \rightarrow B^\bullet$, ω with \hat{h}^0 as in (3.36), such that $\alpha, \hat{\alpha} : A^\bullet \rightarrow B^\bullet$ are homotopic cdga morphisms, and under this homotopy, the Lagrangian structures $(h^0, 0, \dots)$ for α, ω and $(\hat{h}^0, 0, \dots)$ for $\hat{\alpha}, \omega$ are also homotopic. Replacing α, h^0 by $\hat{\alpha}, \hat{h}^0$, we can set $a_j^{k-1-i} = 0$ for all i, j . Then $A^\bullet, \omega, B^\bullet, \alpha, h$ are in weak Lagrangian Darboux form, in the sense of Example 3.5. This completes the proof of Theorem 3.7(ii).

All the proof so far has worked with B^\bullet, α, i, j in the homotopy commutative diagram (4.6) with $Z \simeq \mathbf{L} \times_{f, \mathbf{X}, i} \mathbf{Spec} A^\bullet$ and $e : \mathbf{Spec} B^\bullet \hookrightarrow Z$ a Zariski open inclusion, so that j is a Zariski open inclusion (or étale) if i is a Zariski open inclusion (or étale), as required by Theorem 3.7(i),(ii). For Theorem 3.7(iii) we allow j to be étale even if i is a Zariski open inclusion, so $e : \mathbf{Spec} B^\bullet \hookrightarrow Z$ in (4.6) can be étale, and we can use étale local operations to construct B^\bullet, α, j .

In the proof above we have invertible elements q_1, \dots, q_{n_e} in B^0 . Write $\check{B}^\bullet = B^\bullet[q_1^{1/2}, \dots, q_{n_e}^{1/2}]$ for the cdga obtained by adjoining square roots of q_1, \dots, q_{n_e} to B^\bullet . The inclusion $j : B^\bullet \hookrightarrow \check{B}^\bullet$ is an étale cover of degree 2^{n_e} . Let $\check{q} \in \mathbf{Spec} \check{B}^\bullet$ be one of the 2^{n_e} preimages of $q \in \mathbf{Spec} B^\bullet$. Write $\check{\alpha}_+ = j \circ \alpha_+ : A_+^\bullet \rightarrow \check{B}^\bullet$, $\check{\alpha} = j \circ \alpha : A^\bullet \rightarrow \check{B}^\bullet$, $\check{e} = e \circ \mathbf{Spec} j : \mathbf{Spec} \check{B}^\bullet \rightarrow Z$, and $\check{i} = i \circ \mathbf{Spec} j : \mathbf{Spec} \check{B}^\bullet \rightarrow \mathbf{L}$. In the proof in Section 4.2, replace $B^\bullet, q, \alpha_+, \alpha, e, i$ by $\check{B}^\bullet, \check{q}, \check{\alpha}_+, \check{\alpha}, \check{e}, \check{i}$, respectively.

The new features are that $e : \mathbf{Spec} B^\bullet \hookrightarrow Z$ in (4.6) is now étale rather than a Zariski open inclusion, and the invertible functions $q_1, \dots, q_{n_e} \in B^0$ above now have square roots $q_j^{1/2}$ in B^0 . Thus, in the first part of Section 4.7, we may change variables from $w_1^e, \dots, w_{n_e}^e$ to $\check{w}_1^e, \dots, \check{w}_{n_e}^e$, where $\check{w}_j^e = q_j^{1/2} w_j^e$, while leaving the other variables $\tilde{x}_j^i, u_j^i, v_j^{k-1-i}$ unchanged. This has the effect of setting $q_j = 1$ for all $j = 1, \dots, n_e$, so at the end of the argument above, $A^\bullet, \omega, B^\bullet, \alpha, h$ are in strong Lagrangian Darboux form, in the sense of Example 3.5, at the cost of working étale locally rather than

Zariski locally. This proves Theorem 3.7 (iii), and finally completes the proof of Theorem 3.7.

Bibliography

- [1] L. AMORIM & O. BEN-BASSAT, “Perversely categorified Lagrangian correspondences”, *Adv. Theor. Math. Phys.* **21** (2017), no. 2, p. 289-381.
- [2] O. BEN-BASSAT, “Multiple derived Lagrangian intersections”, in *Stacks and categories in geometry, topology, and algebra*, Contemporary Mathematics, vol. 643, American Mathematical Society, 2015, p. 119-126.
- [3] O. BEN-BASSAT, C. BRAV, V. BUSSI & D. JOYCE, “A ‘Darboux theorem’ for shifted symplectic structures on derived Artin stacks, with applications”, *Geom. Topol.* **19** (2015), no. 3, p. 1287-1359.
- [4] D. BORISOV & D. JOYCE, “Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds”, *Geom. Topol.* **21** (2017), no. 6, p. 3231-3311.
- [5] E. BOUAZIZ & I. GROJNOWSKI, “A d -shifted Darboux theorem”, <https://arxiv.org/abs/1309.2197>, 2013.
- [6] C. BRAV, V. BUSSI, D. DUPONT, D. JOYCE & B. SZENDRŐI, “Symmetries and stabilization for sheaves of vanishing cycles”, *J. Singul.* **11** (2015), p. 85-151, with an appendix by Jörg Schürmann.
- [7] C. BRAV, V. BUSSI & D. JOYCE, “A Darboux theorem for schemes with shifted symplectic structure”, *J. Am. Math. Soc.* **32** (2019), no. 2, p. 399-443.
- [8] V. BUSSI, “Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles”, <https://arxiv.org/abs/1404.1329>, 2014.
- [9] ———, “Generalized Donaldson–Thomas theory over fields $\mathbb{K} \neq \mathbb{C}$ ”, <https://arxiv.org/abs/1403.2403>, 2014.
- [10] V. BUSSI, D. JOYCE & S. MEINHARDT, “On motivic vanishing cycles of critical loci”, *J. Algebr. Geom.* **28** (2019), no. 3, p. 405-438.
- [11] D. CALAQUE, “Lagrangian structures on mapping stacks and semi-classical TFTs”, in *Stacks and categories in geometry, topology, and algebra*, Contemporary Mathematics, vol. 643, American Mathematical Society, 2015, p. 1-23.
- [12] D. CALAQUE, T. PANTEV, B. TOËN, M. VAQUIÉ & G. VEZZOSI, “Shifted Poisson structures and deformation quantization”, *J. Topol.* **10** (2017), no. 2, p. 483-584.
- [13] D. JOYCE, “A classical model for derived critical loci”, *J. Differ. Geom.* **101** (2015), no. 2, p. 289-367.
- [14] A. KAPUSTIN & L. ROZANSKY, “Three-dimensional topological field theory and symplectic algebraic geometry II”, *Commun. Number Theory Phys.* **4** (2010), no. 3, p. 463-549.
- [15] M. KONTSEVICH & Y. SOIBELMAN, “Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants”, *Commun. Number Theory Phys.* **5** (2011), no. 2, p. 231-352.
- [16] J. LURIE, “Higher algebra”, prepublication book draft, <http://www.math.harvard.edu/~lurie/papers/HA.pdf>, 2011.
- [17] V. MELANI, “Poisson bivectors and Poisson brackets on affine derived stacks”, *Adv. Math.* **288** (2016), p. 1097-1120.
- [18] V. MELANI & P. SAFRONOV, “Derived coisotropic structures II: stacks and quantization”, *Sel. Math., New Ser.* **24** (2018), no. 4, p. 3119-3173.

- [19] T. PANTEV, B. TOËN, M. VAQUIÉ & G. VEZZOSI, “Shifted symplectic structures”, *Publ. Math., Inst. Hautes Étud. Sci.* **117** (2013), p. 271-328.
- [20] A. PREYGEL, “Thom–Sebastiani and duality for matrix factorizations”, <https://arxiv.org/abs/1101.5834>, 2011.
- [21] J. P. PRIDHAM, “Shifted Poisson and symplectic structures on derived N -stacks”, *J. Topol.* **10** (2017), no. 1, p. 178-210.
- [22] P. SAFRONOV, “Braces and Poisson additivity”, *Compos. Math.* **154** (2018), no. 8, p. 1698-1745.
- [23] B. TOËN, “Higher and derived stacks: a global overview”, in *Algebraic geometry—Seattle 2005. Part 1*, Proceedings of Symposia in Pure Mathematics, vol. 80, American Mathematical Society, 2009, p. 435-487.
- [24] ———, “Derived algebraic geometry”, *EMS Surv. Math. Sci.* **1** (2014), no. 2, p. 153-240.
- [25] ———, “Derived algebraic geometry and deformation quantization”, in *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, Kyung Moon Sa, 2014, p. 769-792.
- [26] B. TOËN & G. VEZZOSI, “From HAG to DAG: derived moduli stacks”, in *Axiomatic, enriched and motivic homotopy theory*, NATO Science Series II: Mathematics, Physics and Chemistry, vol. 131, Kluwer Academic Publishers, 2004, p. 173-216.
- [27] ———, “Homotopical algebraic geometry. II. Geometric stacks and applications”, *Mem. Am. Math. Soc.* **193** (2008), no. 902.