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Classification of Bagnera–de Franchis Varieties in Small Dimensions ^(*)

ANDREAS DEMLEITNER ⁽¹⁾

ABSTRACT. — A *Bagnera–de Franchis variety* $X = A/G$ is the quotient of an Abelian variety A by a free action of a finite cyclic group $G = \langle g \rangle \subset \text{Bihol}(A)$, which does not contain only translations. Constructing explicit polarizations and using a method introduced by F. Catanese, we classify split Bagnera–de Franchis varieties (i.e., there is a direct sum decomposition of A into G -invariant Abelian subvarieties, indexed by $k|\#G$, on which the linear part of g acts with eigenvalues of order k) up to complex conjugation in dimensions ≤ 4 .

RÉSUMÉ. — Une *variété de Bagnera–de Franchis* X est un quotient A/G d’une variété abélienne A par un groupe cyclique fini $G = \langle g \rangle \subset \text{Bihol}(A)$ agissant librement et ne consistant pas uniquement en des translations. Par une construction explicite des polarisations, et en suivant une méthode introduite par F. Catanese, nous donnons une classification exhaustive à conjugaison complexe près et au sens suivant des variétés de Bagnera–de Franchis scindées jusqu’en dimension 4 : A admet une décomposition en somme directe de sous-variétés abéliennes G -invariantes indexées par $k|\#G$ sur lesquelles la partie linéaire de g agit avec des valeurs propres d’ordre k .

1. Introduction

This work studies free group actions of finite groups G on Abelian varieties A and the corresponding quotients. Here, the group G is a group of affine transformations of A , but not a subgroup of the group of translations (else, the quotient would be again an Abelian variety). A quotient of an

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Abelian variety by such a group G is called a *generalized hyperelliptic variety*. More generally, one defines a *generalized hyperelliptic manifold* to be the quotient of a complex torus by a group G as above. In the case, where G is cyclic, we call the quotient a *Bagnera–de Franchis variety* (resp. a *Bagnera–de Franchis manifold*). The study of these dates back to the beginning of the 20th century, when Bagnera and de Franchis as well as Enriques and Severi published their seminal works [1] and [7], respectively. In the surface case, the classification result of Bagnera and de Franchis shows that there are no non-projective hyperelliptic manifolds. Since then, several authors have studied hyperelliptic manifolds, as well as related areas that contributed a lot to today’s understanding of this topic. To cite only a few: [2, 5, 8, 13]. In 2001, Lange ([10]) gave a method to classify BdF-varieties up to dimension 4, using heavily the tables of linear automorphisms of Abelian surfaces and threefolds (loc. cit), although he omitted some calculations in his work. It does not seem that this method can be used for the classification in dimension > 4 (because tables of linear automorphisms are, as far as we know, currently only known up to dimension 3). Instead, Catanese [4] introduced a method for the classification based on elementary linear algebra and number theory which will be explained and used in this paper for the classification in higher dimensions.

Let us explain how this work is organized. The first section mainly recalls some basic facts we will need and establishes some elementary results concerning combinatorics of automorphisms of complex tori. In Section 2.2 we introduce Bagnera–de Franchis varieties as quotients of an Abelian variety A by a free action of a finite cyclic group G which is not a subgroup of the group of translations and state a characterization for them: a BdF-variety $X = A/G$ splits as $A = (B_1 \times B_2)/(G \times T_r)$, where T_r is a finite group of translations and $B_i \subset A$ are Abelian subvarieties, such that suitable properties are satisfied (cf. Theorem 2.10). Here, G acts on B_1 by translation and linearly on B_2 . In Section 3 we follow [4] and introduce the *Hodge type* of a G -Hodge decomposition, an invariant attached to a faithful representation $G \rightarrow \mathrm{GL}(\Lambda)$, where Λ is a free Abelian group of even rank. Catanese’s method (loc. cit.) for the classification of BdF-varieties will be discussed in Section 3. We will assume here that the lattice Λ_2 of B_2 is a module over a direct sum of cyclotomic rings (in this case we call X *split*). This yields a decomposition of our Abelian variety B_2 into G -invariant Abelian subvarieties $B_{2,k}$, on which G acts with eigenvalues of order k . We go on classifying complex tori which admit a linear automorphism acting only with eigenvalues of order k to be able to list all possible decompositions for B_2 . In Sections 5 and 6, we put all pieces together (such that the conditions in the characterization of BdF-varieties are satisfied) and obtain the following classification result:

THEOREM 1.1. — *The following classification results hold.*

- (1) *There are no BdF-curves.*
- (2) *Families of split BdF-varieties X of dimension ≤ 4 are fully classified up to complex conjugation in Tables 5–7.*
- (3) *Families of complex tori of dimension ≤ 5 , which admit a linear automorphism of order $m := |G|$ whose eigenvalues are only primitive m -th roots of unity are fully classified up to complex conjugation in Sections 4.1 to 4.4.*
- (4) *Each family of complex tori of dimension ≤ 5 as in (3) contains an Abelian variety.*

Moreover, except possibly for the cases listed in Table 7, every family listed in (3) contains a principally polarized Abelian variety.

The one-dimensional case (1) is an easy consequence of the Riemann–Hurwitz formula, while the classification result for two-dimensional BdF-varieties is exactly the classification result of Bagnera–de Franchis, Enriques–Severi ([1] and [7]). The threefold case was treated by Lange ([10]). However, the result (2) in $\dim(X) = 4$ is new, as well as (3) and (4) are (as far as we know). The problem we face during the classification of BdF-varieties is that we do not know whether the classified families of complex tori in (3) really contain Abelian varieties. This question will be dealt with in the last section: we find explicit polarizations for these, which turn out to be principal in most cases. We also discuss how, and in which cases, projectivity is implied by the following result of T. Ekedahl, unpublished, but one can find a detailed proof in the preprint [6].

THEOREM 1.2. — *Let (T, G) be a rigid group action of a finite group G on a complex torus T . Then T/G is projective (or equivalently, $T = A$ is an Abelian variety).*

Notation. — We fix the following notation throughout the whole work. We will work over the field \mathbb{C} of complex numbers. By an *Abelian variety*, we will therefore mean a complex Abelian variety. The notion of a *ring* will always mean a commutative ring with unit element. The set of natural numbers \mathbb{N} will denote the set of all non-negative integers. The dual space of a vector space V is denoted V^\vee .

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2. Preliminaries

In this section, we recall some basic facts which we will need in the sequel. Let $T = V/\Lambda$ be a complex torus. It is well-known (cf. for instance [3]) that T is an Abelian variety if and only if there is an alternating \mathbb{Z} -bilinear form E on Λ such that the associated \mathbb{R} -bilinear form $H: V \times V \rightarrow \mathbb{R}$ given by $H(v, w) = E(\iota v, w) + \iota E(v, w)$ is Hermitian and positive definite. These conditions are publicly known as the two *Riemann Bilinear Relations*. The Riemann Bilinear Relations can also be expressed in the following way. The form E extends \mathbb{C} -linearly to a form E on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \bar{V}$. We have

$$E \in (V^{\vee} \otimes V^{\vee}) \oplus (V^{\vee} \otimes \bar{V}^{\vee}) \oplus (\bar{V}^{\vee} \otimes V^{\vee}) \oplus (\bar{V}^{\vee} \otimes \bar{V}^{\vee}).$$

Hence, E splits as a sum $E = E_1 + H_1 + H_2 + E_2$ (where E_1 is in the first direct summand, H_1 is in the second one, and so on). Now we have (cf. [9, p. 327]):

PROPOSITION 2.1. — *The following statements hold:*

- (1) *The first Riemann Bilinear Relation holds if and only if $E_1 = 0$.*
- (2) *The second Riemann Relation holds if and only if the first Riemann Relation holds and the form $H_1(-\iota z, \bar{w})$ is positive definite.*

2.1. Combinatorics of Automorphisms of Complex Tori

In this section we develop combinatorial and group-theoretical restrictions concerning automorphisms of complex tori. These results will be very useful to determine all possible families of BdF-varieties (cf. Sections 4 and 5).

Let $T = V/\Lambda$ be a complex torus of dimension n . Let $\alpha \in \text{Aut}(T) := \text{Bihol}(T)$ and $\rho: \text{Aut}(T) \rightarrow \text{GL}(V)$ be the complex representation. This yields a representation $\rho': \text{Aut}(T) \rightarrow \text{GL}(\Lambda)$, called the rational representation. An easy observation is the following lemma (cf. [3]).

LEMMA 2.2. — *The representations $\rho' \otimes 1$ and $\rho \oplus \bar{\rho}$ are equivalent.*

Before we start discussing combinatorial restrictions on automorphisms of complex tori, let us first fix some notation. We define $\text{Aut}(T)$ to be the group of all (not necessarily linear) automorphisms of T and $\text{Aut}(T, 0)$ to be the subgroup of all linear automorphisms of T , and analogously $\text{End}(T, 0)$. For $\alpha \in \text{Aut}(T, 0)$, we define $\text{Fix}(\alpha)$ as the set consisting of all $a \in T$ which are fixed under α .

The following result can be found, together with its proof, in [2]; we give a different, more elementary proof here.

PROPOSITION 2.3. — *Let $\alpha \in \text{Aut}(T)$ be a linear automorphism of order $m = p^k$ for a prime p and assume that Λ is a free $\mathbb{Z}[\zeta_m]$ -module. Suppose furthermore that the eigenvalues of $\rho(\alpha)$ are all primitive m -th roots of unity. Then $\text{Fix}(\alpha) \cong (\mathbb{Z}/p\mathbb{Z})^{2 \cdot \dim(T)/\varphi(m)}$. Here, φ denotes the Euler totient function.*

Proof. — Denote by x the class of X in $\mathbb{Z}[\zeta_m] = \mathbb{Z}[X]/(\phi_m(X))$ and note that α acts as multiplication with x . We know that $v \in \text{Fix}(\alpha)$ iff $v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ and $(x-1) \cdot v \in \Lambda$. It is well known that the p^k -th cyclotomic polynomial is $\phi_p(x^{p^{k-1}}) = \sum_{i=0}^{p-1} x^{p^{k-1} \cdot i}$. Since $\Lambda \cong \mathbb{Z}[\zeta_m]^{2 \cdot \dim(A)/\varphi(m)}$, it suffices to show that $\{w \in \mathbb{Z}[\zeta_m] \otimes_{\mathbb{Z}} \mathbb{Q} \mid (x-1) \cdot w \in \mathbb{Z}[\zeta_m]\} \cong \mathbb{Z}/p\mathbb{Z}$. For this, we abbreviate $l = \varphi(p^k) = \deg(\phi_{p^k})$ and we write $w = a_0 + a_1x + \dots + a_{l-1}x^{l-1}$ with $a_i \in \mathbb{Q}$ for all i . Then a straightforward calculation shows the claim:

$$(x-1)w = -\sum_{i=0}^{l-1} a_i x^i + \sum_{i=1}^{l-1} a_{i-1} x^i - \sum_{i=0}^{p-2} a_{l-1} x^{p^{k-1} \cdot i},$$

where we have used the equality $x^l = -\sum_{i=0}^{p-2} x^{p^{k-1} \cdot i}$.

The condition is now that all the coefficients of powers of x must be integers, so the sum of the coefficients has to be an integer. Adding the coefficients, we find that $-p \cdot a_{l-1}$ has to be an integer. This shows the assertion, as all a_i are congruent to a_{l-1} modulo \mathbb{Z} . \square

Remark 2.4. — Let $T \cong_{\text{top.}} (S^1)^{2n}$ be a torus of real dimension $2n$. Then the preceding homeomorphism gives $T[m] \cong (\mathbb{Z}/m\mathbb{Z})^{2n}$.

Remark 2.5. — It follows from standard combinatorics that the number of elements of order exactly m in $(\mathbb{Z}/m\mathbb{Z})^r$ is precisely

$$\sum_{k=0}^{r-1} \varphi(m) \cdot (m - \varphi(m))^k \cdot m^{r-k-1}.$$

2.2. An Introduction to Bagnera–de Franchis Varieties

We start this section with a few preliminaries on generalized hyperelliptic varieties.

DEFINITION 2.6. — *A generalized hyperelliptic variety (resp. generalized hyperelliptic manifold) $X = A/G$ of dimension n is the quotient of an Abelian variety (resp. a complex torus) A by a finite group G of affine transformations of A , such that G acts freely and is not a subgroup of the group of translations. If G is cyclic, X is called a Bagnera–de Franchis variety (resp. Bagnera–de Franchis manifold); for short: BdF-variety (BdF-manifold).*

Remark 2.7. — Let $X = A/G$ be a generalized hyperelliptic variety. By virtue of our assumption that, in the above situation, G is not a subgroup of the group of translations, we can assume without loss of generality that G does not contain any translation: Let G_T be the subgroup of translations in G , which is a normal subgroup. Hence, we get an Abelian variety $A' = A/G_T$ and a group $G' = G/G_T$ without translations such that $A'/G' \cong A/G$.

Also note that there are no hyperelliptic varieties of dimension 1 in the above sense, since the Riemann–Hurwitz-formula holds.

Remark 2.8. — Let $X = A/G$ be a generalized hyperelliptic variety, and $A = V/\Lambda$. Write $\text{id} \neq g \in G$ as $g(x) = \alpha x + b$, where $\alpha \in \text{GL}(V)$ and $b \in V$. Then the property of g acting freely on A is equivalent to the fact that no pair of $(x, \lambda) \in V \times \Lambda$ solves the equation

$$(\alpha - \text{id})x = \lambda - b. \tag{2.1}$$

Hence, the freeness of the action of $g \in G$ implies that α has the eigenvalue 1.

DEFINITION 2.9. — *A Bdf-variety (resp. Bdf-manifold) $X = A/G$ is said to be of product type, if $A = B_1 \times B_2$ is a product of Abelian varieties (resp. complex tori) and $G \cong \mathbb{Z}/m\mathbb{Z}$ is generated by an automorphism g acting as $g(a_1, a_2) = (a_1 + b', \alpha' a_2)$, where $b' \in B_1$ is an element of order m and $\alpha' \in \text{Aut}(B_2)$ is a linear automorphism of order m , not admitting the eigenvalue 1.*

We have the following characterization of Bdf-varieties:

THEOREM 2.10. — *The following two statements are equivalent:*

- (1) $X = A/G$ is a Bdf-variety.
- (2) X is the quotient of a Bdf-variety $(B_1 \times B_2)/G$ of product type by a finite group T_r of translations such that the following conditions hold.
 - (a) T_r is the graph of an isomorphism $T_{r,1} \rightarrow T_{r,2}$, where $T_{r,i}$ is a finite group of translations of B_i .
 - (b) If $g \in G$ is given by $g(a_1, a_2) = (a_1 + b', \alpha' a_2)$, then

$$(\alpha' - \text{id})T_{r,2} = \{0\}.$$

- (c) If g is as above of order m , then $\langle b' \rangle \cap T_{r,1} = \{0\}$.

In particular, we may write $X = (B_1 \times B_2)/(G \times T_r)$.

Proof. — See for instance [4, Proposition 15]. □

3. Actions of Finite Groups on Complex Tori

We give the most important definitions for the classification of BdF-varieties.

DEFINITION 3.1. — *Let Λ be a free Abelian group of even rank and $G \rightarrow \mathrm{GL}(\Lambda)$ be a faithful representation of a finite group G . A G -Hodge decomposition is a decomposition*

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = H^{1,0} \oplus H^{0,1}, \quad \overline{H^{1,0}} = H^{0,1},$$

into G -invariant linear subspaces.

Splitting $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ using the canonical decomposition, we can write

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\chi} U_{\chi},$$

where the sum runs over all characters belonging to irreducible representations of G . Thus, $U_{\chi} = W_{\chi} \otimes M_{\chi}$. Here, W_{χ} is the corresponding irreducible representation and M_{χ} is a trivial representation. Write

$$V := H^{1,0} = \bigoplus_{\chi} V_{\chi}$$

with $V_{\chi} = W_{\chi} \otimes M_{\chi}^{1,0}$. We define

DEFINITION 3.2. — *The Hodge type of a G -Hodge decomposition is the collection of the dimensions $\nu(\chi) = \dim_{\mathbb{C}} M_{\chi}^{1,0}$. Here, χ runs over all non-real characters.*

Remark 3.3.

- (1) Note that for a non-real irreducible character χ , one has $\nu(\chi) + \nu(\bar{\chi}) = \dim_{\mathbb{C}} M_{\chi}$.
- (2) All G -Hodge decompositions of a fixed Hodge type are parametrized in an open set of a product of Grassmanians: for a real irreducible character χ , one simply chooses a $\frac{1}{2} \dim(M_{\chi})$ -dimensional subspace of M_{χ} , and for a non-real irreducible character, one chooses a $\nu(\chi)$ -dimensional subspace of M_{χ} . Then the condition is that $M_{\chi}^{1,0}$ and $M_{\bar{\chi}}^{0,1} = \overline{M_{\chi}^{1,0}}$ do not intersect.

Let $\rho: G \rightarrow \mathrm{GL}(V)$ be the linear representation which sends $g \in G$ to its linear part. Denote by G^{\vee} the group of characters of G , i.e., the group of group homomorphisms from G to \mathbb{C}^* . For simplicity, we write ρ_g instead of $\rho(g)$. Since G was assumed to be Abelian, all the irreducible representations of G have degree 1. The χ -eigenspace of G is denoted V_{χ} , i.e.,

$$V_{\chi} = \{v \in V \mid \rho_g(v) = \chi(g) \cdot v, \forall g \in G\}.$$

Thus, we can split $V = \bigoplus_{\chi} V_{\chi}$. Denote by M the set of all characters such that $V_{\chi} \neq \{0\}$. We then have $V = \bigoplus_{\chi \in M} V_{\chi}$. Of course, we want to apply these considerations to generalized hyperelliptic varieties. In particular, we will have G satisfying the following two properties:

- G acts freely.
- G contains no translations, i.e., ρ is faithful.

We have the following elementary result if G is even cyclic:

LEMMA 3.4. — *Let $\rho : G \rightarrow \mathrm{GL}(\Lambda)$ be a representation of a finite cyclic group $G = \langle g \rangle$, Λ a free Abelian group of rank $2n$ and $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\chi} U_{\chi}$ be the canonical decomposition. If two characters χ and χ' have the same order, the spaces U_{χ} and $U_{\chi'}$ have the same dimension.*

Proof. — We have $G \rightarrow \mathrm{GL}(\Lambda) \hookrightarrow \mathrm{GL}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})$. Denote by ρ^0 the representation $G \rightarrow \mathrm{GL}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})$. Let χ and χ' be two characters of the same order k . Since χ and χ' have order k and the characteristic polynomial f_g of ρ_g^0 has integral coefficients, a power of the k -th cyclotomic polynomial divides f_g . This yields that the multiplicities of $\chi(g)$ and $\chi'(g)$ as zeros of f_g are equal. We are now finished, because ρ_g^0 is diagonalizable. \square

4. Primary BdF-Varieties

In this section, we want to classify split BdF-varieties in small dimensions. We use Theorem 2.10, where we saw that a BdF-variety is the quotient of a BdF-variety of product type by a finite group of translations such that the conditions in the quoted theorem are satisfied. We follow [4] for the presentation of the method used for the classification.

DEFINITION 4.1.

- (1) *Assume that a cyclic group G generated by a linear automorphism g of order m acts on an Abelian variety (resp. a complex torus) A . Then (A, G) is called primary, if a generator g of G has only primitive m -th roots of unity as eigenvalues. By abuse of notation, we will call an Abelian variety (resp. a complex torus) A primary if the group G is clear from the context.*
- (2) *A BdF-variety (resp. BdF-manifold) $(B_1 \times B_2)/G$ of product type is said to be primary, if the Abelian variety B_2 (resp. the complex torus B_2) is primary.*

Let $X = (B_1 \times B_2)/G$ a BdF-variety of product type and write as usual $B_i = V_i/\Lambda_i$. In this situation, Λ_2 is a G -module, thus a module over the ring

$\mathbb{Z}[G] \cong \mathbb{Z}[X]/(X^m - 1)$. By the Chinese Remainder theorem, we find that $\mathbb{Z}[G]$ embeds into a direct sum of cyclotomic rings, i.e., we have

$$\mathbb{Z}[X]/(X^m - 1) \subset \bigoplus_{k|m} \mathbb{Z}[X]/(\phi_k(X)),$$

where $\phi_k(X)$ is the k -th cyclotomic polynomial.

Remark 4.2. — Fabrizio Catanese (private communication) pointed out that the inclusion above is not an isomorphism. In [4], he inadvertently claimed that exactly this was the case, while in fact, this is only true over \mathbb{Q} . Throughout this paper we only deal with the case where Λ_2 is as well a module over the direct sum $\bigoplus_{k|m} \mathbb{Z}[X]/(\phi_k(X))$. In the general case, one obtains tori which are isogenous to ones with splitting lattices as above (we shall deal with this question with more precision in a future paper).

We write $R := \mathbb{Z}[X]/(X^m - 1)$ and R_k for the cyclotomic ring $\mathbb{Z}[X]/(\phi_k(X))$. We assume that Λ_2 splits according to the order of the eigenvalues, $\Lambda_2 = \bigoplus_{k|m} \Lambda_{2,k}$ (note that $\Lambda_{2,k}$ is an R_k -module). The vector space V_2 splits accordingly, $V_2 = \bigoplus_{k|m} V_{2,k}$. We find that

$$B_2 = \bigoplus_{k|m} B_{2,k}, \tag{4.1}$$

where $B_{2,k}$ is a G -invariant Abelian subvariety of B_2 such that a generator $g \in G$ (as in Theorem 2.10) acts on $B_{2,k}$ with eigenvalues of order k .

DEFINITION 4.3. — *We call the Abelian variety B_2 split if it admits a splitting as in (4.1). A BdF variety $X = (B_1 \times B_2)/(G \times T)$ is called split if B_2 is split.*

Let X be a BdF-variety of dimension $n > 1$. We furthermore assume that $X = (B_1 \times B_2)/G$ is a primary BdF-variety. In this case, Λ_2 is a module over the ring $R_m = \mathbb{Z}[X]/(\phi_m(X))$, which is a Dedekind domain (see for example [14, Theorem 2.6]). In fact, Λ_2 is a projective R_m -module. We therefore get a splitting $\Lambda_2 = R_m^r \oplus I$ of Λ_2 into a free part and an ideal $I \subset R_m$ (see [11]). Indeed, Λ_2 is free if R_m is a PID, i.e., if the class number $h(R_m) := \#\text{Cl}(R_m)$ is equal to 1. We know that R_m is a free \mathbb{Z} -module of rank $\varphi(m)$ (where φ denotes the Euler totient function), hence we have

$$\Lambda_2 = R_m^r \cong \mathbb{Z}^{\varphi(m) \cdot r} \tag{4.2}$$

for all m satisfying $h(R_m) = 1$. As B_1 is an Abelian variety of positive dimension, we have the following important observation:

$$\varphi(m) \leq 2(n - 1). \tag{4.3}$$

DEFINITION 4.4. — *Let $T = V/\Lambda$ be a complex torus. If $\Lambda = R_m^k$, then T is called elementary.*

We have the following well-known table of values $\varphi(m)$:

$\varphi(m)$	m	$\varphi(m)$	m
1	1, 2	6	7, 9, 14, 18
2	3, 4, 6	8	15, 16, 20, 24, 30
4	5, 8, 10, 12	10	11, 22

According to the table on p. 353 of [14], we know in particular that $h(R_m) = 1$ for all values of $\varphi(m)$ listed above.

In the following sections, we will classify split BdF-varieties in small dimensions. To obtain a satisfying classification, one first determines all possibilities for B_2 . To give a complex structure to $(\Lambda_2 \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda_2$ it is sufficient and necessary to give a decomposition $\Lambda_2 \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \bar{V}$ (see [9, p. 326 and 327]). This amounts to determining all possible G -Hodge decompositions corresponding to Λ_2 .

Remark 4.5.

- (1) Note that, by the above inequality (4.3), the case $m = 2$ obviously occurs for all dimensions $n > 1$. The only automorphism of order 2 of an Abelian variety with finitely many fixed points is multiplication with -1 , so the case $m = 2$ can be easily classified. In what follows, this trivial case will be omitted during the calculations, but will be listed in the tables. (Note that the number of parameters for this case is given by the dimension of the Siegel upper half space.)
- (2) In the following sections, we classify families of complex tori admitting a faithfully acting linear automorphism of a certain order up to dimension 5 *up to complex conjugation*. Usually, we will drop the phrase “up to complex conjugation” in the statement of our results.
- (3) We refrain from listing the well-known table of elliptic curves admitting automorphisms of finite order.

4.1. The Surface Case

In this and the upcoming sections, we classify families of Abelian varieties admitting a linear automorphism of certain order acting faithfully. To abbreviate, we denote by $(*_n, m)$ the following condition to be satisfied by a torus T :

$(*_{n,m})$ $\dim(T) = n$, and T admits a linear automorphism of some order $m > 2$ having only primitive m -th roots of unity as eigenvalues

For now, we assume that T satisfies $(*_{2,m})$. We write $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \bar{V}$. Using inequality (4.3), we find that we only have two possibilities in equation (4.2):

- Case 1: $\varphi(m) = 2$ (hence $m \in \{3, 4, 6\}$) and $r = 2$, or
- Case 2: $\varphi(m) = 4$ (hence $m \in \{5, 8, 10, 12\}$) and $r = 1$.

In the first case, R_m has rank 2 as a \mathbb{Z} -module. We have the decomposition $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = W_{\chi} \oplus W_{\bar{\chi}}$ into 2-dimensional isotypical components. Recall that we have a decomposition $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \bar{V}$. As we have seen in Section 2, we also get a decomposition for V (i.e., we have $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V_{\chi} \oplus \bar{V}_{\chi} \oplus V_{\bar{\chi}} \oplus \bar{V}_{\bar{\chi}}$). We will always (i.e., in any dimension) distinguish between the cases where the following assumption holds (or does not hold).

Only pairwise non-conjugate characters appear in V . (4.4)

The result [2, Proposition 1.8(a)] together with (4.2) now give the following

PROPOSITION 4.6. — *If $T = V/\Lambda$ is elementary (of arbitrary dimension) such that the rank of Λ as an R_m -module is 1, then (4.4) holds.*

Note that the case where (4.4) holds is dealt with in [5]; according to the authors, the classification of two-dimensional tori which admit an automorphism of order m acting faithfully (such that (4.4) holds) is as follows. There are exactly

- $m = 3, 6$: one isomorphism class, namely $E_{\rho} \times E_{\rho}$ (E_{ρ} being the equianharmonic elliptic curve).
- $m = 4$: one isomorphism class, namely $E_l \times E_l$ (E_l being the harmonic elliptic curve).
- $m = 5, 10$: one isomorphism class S_{10} .
- $m = 8, 12$: two isomorphism classes S'_m, S''_m .

Moreover Catanese and Ciliberto prove:

PROPOSITION 4.7. — *Let T satisfy $(*_{n,m})$ and (4.4). Then the following hold.*

- (1) *If $m = 3, 6$, then $T \cong E_{\rho}^n$.*
- (2) *If $m = 4$, then $T \cong E_l^n$.*
- (3) *If $m = 5, 10$, then $T \cong S_{10}^{n/2}$.*
- (4) *If $m = 8, 12$, then there are $k, l \in \mathbb{N}$ such that $T \cong (S'_m)^k \times (S''_m)^l$.*

Now we analyze the case where (4.4) does not hold.

PROPOSITION 4.8. — *Assume that V splits into the two one-dimensional spaces V_χ and $V_{\bar{\chi}}$, i.e., (4.4) does not hold. Then there is a two-parameter family of tori T satisfying $(*_{2,m})$.*

We denote these families by S_4 for $m = 4$ and S_6 for $m \in \{3, 6\}$.

Proof. — It suffices to show that under the given conditions, one has a two-parameter family of complex tori modulo a group G with the desired properties. This is clear, as we have only the possibility $V = V_\chi \oplus V_{\bar{\chi}}$ (up to complex conjugation) to decompose V . But $(V_\chi, V_{\bar{\chi}})$ is contained in the open set $\{(W, W') \in \text{Gr}(1, 2) \times \text{Gr}(1, 2) \mid W \cap W' = \{0\}\}$, which is two-dimensional. \square

We sum up our results in the following

THEOREM 4.9. — *There are exactly 19 families (up to complex conjugation) of two-dimensional complex tori admitting a non-trivial linear automorphism of finite order m acting faithfully. These families are listed in Table 1.*

Table 1. By p , we denote the number of moduli of the corresponding family. The rest of the notation used is explained below the table.

m	T	$\text{Fix}(g)$	p	m	T	$\text{Fix}(g)$	p	m	T	$\text{Fix}(g)$	p
2	S	\mathbb{Z}_4^8	3	6	S_6	$\{0\}$	2	10	S_{10}	$\{0\}$	0
3	S_6	\mathbb{Z}_3^2	2	6	$E \times E_\rho$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	1	12	S'_{12}	$\{0\}$	0
3	$E_\rho \times E_\rho$	\mathbb{Z}_3^2	0	6	$E \times E_\rho$	\mathbb{Z}_2^2	1	12	S''_{12}	$\{0\}$	0
4	S_4	\mathbb{Z}_2^2	2	6	$E_\rho \times E_\rho$	\mathbb{Z}_3	0	12	$E_\rho \times E_\rho$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	0
4	$E_\iota \times E_\iota$	\mathbb{Z}_2^2	0	6	$E_\rho \times E_\rho$	$\{0\}$	0	12	$E_\iota \times E_\rho$	\mathbb{Z}_2	0
4	$E \times E_\iota$	\mathbb{Z}_2^3	1	8	S'_8	\mathbb{Z}_2^2	0	12	$E_\iota \times E_\rho$	\mathbb{Z}_2	0
5	S_{10}	\mathbb{Z}_5	0	8	S''_8	\mathbb{Z}_2^2	0				

The table is organized as follows. By E resp. S we denote an arbitrary elliptic curve resp. Abelian surface, while $S_4, S_6, S_{10}, S'_m, S''_m$ denote the (families of) Abelian surfaces introduced above. We decided to list the same torus T more than once if their entries in $\text{Fix}(g)$ are different. Finally, $\text{Fix}(g)$ denotes the fixed locus of a linear automorphism g of T of order m .

Remark 4.10. — The result of Theorem 4.9 was already achieved by Fujiki [8] with different methods, see also [12] for a study in the case where m is prime.

4.2. The Threefold Case

Let T satisfy $(*_{3,m})$. The only possibilities are:

- Case 1: $\varphi(m) = 2$ (hence $m \in \{3, 4, 6\}$) and $r = 3$, or
- Case 2: $\varphi(m) = 6$ (hence $m \in \{7, 9, 14, 18\}$) and $r = 1$.

We first deal with Case 1. In the same way as in the surface case there is the decomposition $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = W_{\chi} \oplus W_{\bar{\chi}} = V_{\chi} \oplus \overline{V_{\chi}} \oplus V_{\bar{\chi}} \oplus \overline{V_{\bar{\chi}}}$. We have the following proposition:

PROPOSITION 4.11. — *Let T satisfy $(*_{3,m})$ and $\varphi(m) = 2$. Then the following statements hold:*

- (1) *If (4.4) holds, then T is isomorphic to a product of elliptic curves. More precisely, if $m \in \{3, 6\}$, then $T = E_{\rho} \times E_{\rho} \times E_{\rho}$. If $m = 4$, then $T = E_{\iota} \times E_{\iota} \times E_{\iota}$.*
- (2) *If (4.4) does not hold, there is a four parameter family of complex tori admitting a linear automorphism of order m acting faithfully.*

We denote the respective families by A_m , $m \in \{4, 6\}$. We also use the notation A_6 for the family obtained for $m = 3$.

Proof. — The first assertion is clear. For the second assertion, note that the only possible Hodge type is $(1, 2)$, so the G -Hodge decompositions are parametrized by an open set in $\mathrm{Gr}(1, 3) \times \mathrm{Gr}(2, 3)$, which has dimension 4. \square

Case 2 is easy to analyze. Note that by Proposition 4.6 we can assume that the eigenvalues of a generator $g \in G$ are distinct and pairwise non-complex conjugate.

PROPOSITION 4.12. — *There are exactly two isomorphism classes of tori satisfying $(*_{3,m})$ for any $m \in \{7, 9, 14, 18\}$. We denote them by A'_7 , A''_7 for $m \in \{7, 14\}$ and by A'_9 , A''_9 for $m \in \{9, 18\}$.*

Proof. — We only list one element of each orbit for $m = 7, 9$. For $m = 7$, there are exactly two orbits corresponding to the tuples of pairwise non-complex conjugate characters $(1, 2, 3)$, $(1, 2, 4)$. For $m = 9$, the two orbits correspond $(1, 2, 4)$ and $(1, 4, 7)$. \square

Now, our assumption (4.1) guarantees that we can give a list of all isomorphism classes of Abelian threefolds admitting an automorphism of finite order. Here, T denotes an arbitrary Abelian threefold. Again, we refrain from listing the same tori more than once unless their entries in the $\mathrm{Fix}(g)$ -column are different. The rest of the notation is as in Table 1.

THEOREM 4.13. — *There are exactly 56 families (up to complex conjugation) of three-dimensional complex tori admitting a non-trivial linear automorphism of finite order m acting faithfully. These families are listed in Table 2.*

Table 2. Again, p denotes the number of moduli.

m	T	$\text{Fix}(g)$	p	m	T	$\text{Fix}(g)$	p	m	T	$\text{Fix}(g)$	p
2	T	\mathbb{Z}_2^6	6	6	$E \times E_\rho \times E_\rho$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	1	12	$E \times S_{12}''$	$\{0\}$	1
3	A_6	\mathbb{Z}_3^3	4	6	$E \times E_\rho \times E_\rho$	\mathbb{Z}_2^2	1	12	$E_\rho \times S_{12}'$	\mathbb{Z}_3	0
3	$E_\rho \times S_6$	\mathbb{Z}_3^3	2	6	$E_\rho \times E_\rho \times E_\rho$	\mathbb{Z}_3^2	0	12	$E_\rho \times S_{12}'$	$\{0\}$	0
3	$E_\rho \times E_\rho \times E_\rho$	\mathbb{Z}_3^3	0	6	$E_\rho \times E_\rho \times E_\rho$	\mathbb{Z}_3	0	12	$E_\rho \times S_{12}'$	\mathbb{Z}_3	0
4	A_4	\mathbb{Z}_2^3	4	6	$E_\rho \times E_\rho \times E_\rho$	$\{0\}$	0	12	$E_\rho \times S_{12}'$	\mathbb{Z}_2	0
4	$E \times S_4$	\mathbb{Z}_2^3	3	7	A_7'	\mathbb{Z}_7	0	12	$E_\rho \times S_{12}'$	$\{0\}$	0
4	$E_\iota \times S$	\mathbb{Z}_3^3	3	7	A_7''	\mathbb{Z}_7	0	12	$E_\rho \times S_{12}'$	\mathbb{Z}_2	0
4	$E_\iota \times S_4$	\mathbb{Z}_2^3	2	8	$E \times S_8'$	\mathbb{Z}_2^3	1	12	$E_\iota \times S_{12}'$	\mathbb{Z}_2	0
4	$E \times E_\iota \times E_\iota$	\mathbb{Z}_2^3	1	8	$E \times S_8''$	\mathbb{Z}_2^3	1	12	$E \times E_\rho \times E_\iota$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	1
4	$E_\iota \times E_\iota \times E_\iota$	\mathbb{Z}_2^3	0	8	$E_\iota \times S_8'$	\mathbb{Z}_2^2	0	12	$E \times E_\iota \times E_\rho$	\mathbb{Z}_2^3	1
6	A_6	$\{0\}$	4	8	$E_\iota \times S_8''$	\mathbb{Z}_2^2	0	12	$E_\rho \times E_\rho \times E_\iota$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$	0
6	$E \times S_6$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$	3	9	A_9'	\mathbb{Z}_3	0	12	$E_\rho \times E_\iota \times E_\iota$	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	0
6	$E \times S_6$	\mathbb{Z}_2^2	3	9	A_9''	\mathbb{Z}_3	0	12	$E_\rho \times E_\iota \times E_\rho$	$\mathbb{Z}_3^2 \times \mathbb{Z}_2$	0
6	$E_\rho \times S_6$	\mathbb{Z}_3^2	2	10	$E \times S_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_5$	1	12	$E_\iota \times E_\rho \times E_\rho$	\mathbb{Z}_2	0
6	$E_\rho \times S_6$	\mathbb{Z}_3	2	12	$E_\rho \times S_4$	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	2	12	$E_\iota \times E_\iota \times E_\rho$	\mathbb{Z}_2	0
6	$E_\rho \times S_6$	$\{0\}$	2	12	$E_\rho \times S_4$	\mathbb{Z}_2^2	2	14	A_7'	$\{0\}$	0
6	$E_\rho \times S$	$\{0\}$	3	12	$E_\iota \times S_6$	$\mathbb{Z}_2 \times \mathbb{Z}_3^2$	2	14	A_7''	$\{0\}$	0
6	$E_\rho \times S$	$\mathbb{Z}_3 \times \mathbb{Z}_2^4$	3	12	$E_\iota \times S_6$	\mathbb{Z}_2	2	18	A_9'	$\{0\}$	0
6	$E \times E_\rho \times E_\rho$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$	1	12	$E \times S_{12}'$	$\{0\}$	1	18	A_9''	$\{0\}$	0

4.3. The Fourfold Case

Let T satisfy $(*_4, m)$. As in the previous sections, we determine all possibilities for the order m of G :

- Case 1: $\varphi(m) = 2$ (hence $m \in \{3, 4, 6\}$) and $r = 4$, or
- Case 2: $\varphi(m) = 4$ (hence $m \in \{5, 8, 10, 12\}$) and $r = 2$, or
- Case 3: $\varphi(m) = 8$ (hence $m \in \{15, 16, 20, 24, 30\}$) and $r = 1$.

Most of the proofs of this section are very similar to the proofs given in the previous sections; therefore, we will omit most of them. We start with the first case.

PROPOSITION 4.14. — *Let T satisfy $(*_4, m)$ such that $\varphi(m) = 2$. Then the following statements hold:*

- (1) *If (4.4) holds, then T is isomorphic to a product of elliptic curves. More precisely, if $m \in \{3, 6\}$, then $T = E_\rho^4$. If $m = 4$, then $T = E_\iota^4$.*
- (2) *If (4.4) does not hold, there are exactly two families of four-dimensional complex tori admitting an automorphism of order m , which have 6 resp. 8 parameters, respectively.*

We denote the respective families in (2) by $X_m^6, X_m^8, m \in \{4, 6\}$. We also use the notation X_6^6, X_6^8 for the families obtained for $m = 3$.

For Case 2, we have:

PROPOSITION 4.15. — *Let T satisfy $(*_{4,m})$ such that $\varphi(m) = 4$. Then the following statements hold:*

- (1) *If (4.4) holds, then T is isomorphic to one of the following complex tori: $(S'_k)^2, S'_k \times S''_k, (S''_k)^2$.*
- (2) *If (4.4) does not hold, there are exactly two families of complex tori of dimension 4 admitting an automorphism of order m , which have 4 resp. 6 parameters.*

We denote the respective families in (2) by $X_m^4, X_m^6, m \in \{8, 10, 12\}$. We also use the notation X_{10}^2, X_{10}^4 for the families obtained for $m = 5$.

Finally, the third case is analyzed. Again, we have that (4.4) holds by Proposition 4.6.

PROPOSITION 4.16. — *There are exactly four isomorphism classes of 4-dimensional complex tori satisfying $(*_{4,m})$ for each $m \in \{15, 16, 20, 30\}$ and exactly five isomorphism classes when $m = 24$.*

Proof. — We only list one element of each orbit for $m = 15, 16, 20, 24$: for $m = 15$, the orbits correspond to the 4-tuples of pairwise non-complex conjugate characters $(1, 2, 4, 7), (1, 2, 4, 8), (1, 2, 7, 11), (1, 4, 7, 13)$. For $m = 16$, the orbits correspond to $(1, 3, 5, 9), (1, 3, 5, 7), (1, 3, 9, 11), (1, 5, 9, 13)$, while for $m = 20$, they correspond to $(1, 3, 7, 9), (1, 3, 7, 11), (1, 3, 11, 13), (1, 9, 13, 17)$. Finally, for $m = 24$, the tuples $(1, 5, 7, 11), (1, 5, 7, 13), (1, 5, 13, 17), (1, 7, 13, 19), (1, 11, 17, 19)$ give the five orbits. \square

We denote representatives of the respective isomorphism classes for $m \in \{15, 30\}$ by $X_{30}^{(i)}, i \in \{1, 2, 3, 4\}$. For $m \in \{16, 20\}$, the representatives are denoted $X_m^{(i)}, i \in \{1, 2, 3, 4\}$. Finally, for $m = 24$, the representatives of the five isomorphism classes are denoted $X_{24}^{(j)}$, with $j \in \{1, 2, 3, 4, 5\}$.

We now could, in principle, give a list of all 4-dimensional complex tori admitting a linear automorphism of finite order which have finite fixed locus. This will be omitted, as it is very tedious, yet not very enlightening. In lieu thereof, we will only list those 4-dimensional complex tori which admit a linear automorphism of order m with eigenvalues of order m .

THEOREM 4.17. — *The table below contains 4-dimensional complex tori which admit a linear automorphism g of order m , whose eigenvalues are only primitive m -th roots of unity. Conversely, every such complex torus belongs (up to complex conjugation) to one of the families listed below.*

Table 3. Here, \tilde{T} denotes an arbitrary 4-dimensional complex torus. The remaining notation is as introduced above.

m	T	$\text{Fix}(g)$	p
2	\tilde{T}	\mathbb{Z}_2^8	10
3	E_ρ^4	\mathbb{Z}_3^4	0
3	$E_\rho \times E_\rho \times S_6$	\mathbb{Z}_3^4	2
3	$S_6 \times S_6$	\mathbb{Z}_3^4	4
3	$E_\rho \times A_6$	\mathbb{Z}_3^4	4
3	X_6^6	\mathbb{Z}_3^4	6
3	X_6^8	\mathbb{Z}_3^4	8
4	E_i^4	\mathbb{Z}_2^4	0
4	$E_i \times E_i \times S_4$	\mathbb{Z}_2^4	2
4	$S_4 \times S_4$	\mathbb{Z}_2^4	4
4	$E_i \times A_4$	\mathbb{Z}_2^4	4
4	X_4^6	\mathbb{Z}_2^4	6
4	X_4^8	\mathbb{Z}_2^4	8
5	$S_{10} \times S_{10}$	\mathbb{Z}_5^2	0

m	T	$\text{Fix}(g)$	p
5	X_{10}^2	\mathbb{Z}_5^2	6
5	X_{10}^4	\mathbb{Z}_5^2	6
6	E_ρ^4	$\{0\}$	0
6	$E_\rho \times E_\rho \times S_6$	$\{0\}$	2
6	$S_6 \times S_6$	$\{0\}$	4
6	$E_\rho \times A_6$	$\{0\}$	4
6	X_6^6	$\{0\}$	6
6	X_6^8	$\{0\}$	8
8	$S_8' \times S_8'$	\mathbb{Z}_2^2	0
8	$S_8' \times S_8''$	\mathbb{Z}_2^2	0
8	$S_8'' \times S_8''$	\mathbb{Z}_2^2	0
8	X_8^2	\mathbb{Z}_2^2	6
8	X_8^4	\mathbb{Z}_2^2	6
10	$S_{10} \times S_{10}$	$\{0\}$	0

m	T	$\text{Fix}(g)$	p
10	X_{10}^2	$\{0\}$	6
10	X_{10}^4	$\{0\}$	6
12	$S_{12}' \times S_{12}'$	$\{0\}$	0
12	$S_{12}' \times S_{12}''$	$\{0\}$	0
12	$S_{12}'' \times S_{12}''$	$\{0\}$	0
12	X_{12}^2	$\{0\}$	6
12	X_{12}^4	$\{0\}$	6
15	$X_{30}^{(i)}, i = 1, 2, 3, 4$	$\{0\}$	0
16	$X_{16}^{(i)}, i = 1, 2, 3, 4$	\mathbb{Z}_2	0
20	$X_{20}^{(i)}, i = 1, 2, 3, 4$	$\{0\}$	0
24	$X_{24}^{(i)}, i = 1, \dots, 5$	$\{0\}$	0
30	$X_{30}^{(i)}, i = 1, 2, 3, 4$	$\{0\}$	0

4.4. The Fivefold Case

Let T satisfy $(*_5, m)$. We have the following possibilities:

Case 1: $\varphi(m) = 2$ (hence $m \in \{3, 4, 6\}$) and $r = 5$, or

Case 2: $\varphi(m) = 10$ (hence $m \in \{11, 22\}$) and $r = 1$.

PROPOSITION 4.18. — *Let T satisfy $(*_5, m)$ such that $\varphi(m) = 2$. Then:*

- (1) *If (4.4) holds, then T is isomorphic to a product of elliptic curves.*
- (2) *If (4.4) does not hold, then there are exactly two families of such Abelian varieties, which have 8 resp. 12 parameters.*

The analysis of the second case yields

PROPOSITION 4.19. — *There are exactly four isomorphism classes of tori satisfying $(*_5, m)$ for $m \in \{11, 22\}$.*

Proof. — It suffices to deal with the case $m = 11$. The four orbits correspond to the 5-tuples of pairwise non-complex conjugate characters $(1, 2, 3, 4, 5)$, $(1, 2, 3, 4, 6)$, $(1, 2, 3, 5, 7)$, $(1, 3, 4, 5, 9)$. \square

Writing a table containing all families of 5-dimensional complex tori which admit a linear automorphism of order m acting faithfully is omitted, as the table would need to much space.

5. Classification of BdF-Varieties in Small Dimensions

In this section, we list all candidates for split BdF-varieties of dimensions ≤ 4 . It suffices to give a list of m , B_1 , B_2 and $T_{r,i}$ with our desired properties

(cf. Theorem 2.10). As we saw, B_2 has to admit an automorphism of order m : all possibilities for B_2 up to dimension 3 with corresponding m were listed in the last section. An arbitrary Abelian variety of a suitable dimension can be chosen for B_1 . To compute all possible isomorphism classes for $T_{r,i}$, we use the combinatorical restrictions, namely [2, Corollary 1.7, Proposition 1.8] (note that part (c) of 1.8 in loc.cit. equals Proposition 2.3 in this paper), Remarks 2.4 and 2.5. We abbreviate $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. The notation is explained before the tables.

Remark 5.1. — At this point, it is not clear that all the BdF-manifolds listed in the tables below are really projective, at least in dimension > 2 . We will deal with this question in the next section.

We start out with the surface case. We list all BdF-surfaces of product type together with all possibilities of $T_{r,i}$.

THEOREM 5.2. — *There are exactly seven families of BdF-surfaces. They are listed in Table 4.*

Table 4. Here, p denotes the number of moduli.

m	B_1	B_2	Possibilities for $T_{r,i}$	p	m	B_1	B_2	Possibilities for $T_{r,i}$	p
2	E	E'	$\{0\}, \mathbb{Z}_2$	2	4	E	E_ι	$\{0\}, \mathbb{Z}_2$	1
3	E	E_ρ	$\{0\}, \mathbb{Z}_3$	1	6	E	E_ρ	$\{0\}$	1

This is the classification result of Bagnera–de Franchis [1] and Enriques–Severi [7].

THEOREM 5.3. — *Every three-dimensional BdF-variety belongs to a family which is listed in Table 5.*

Table 5.

m	B_2	Possibilities for $T_{r,i}$	m	B_2	Possibilities for $T_{r,i}$
2	E	$\{0\}, \mathbb{Z}_2, \mathbb{Z}_2^2$	6	E_ρ	$\{0\}$
2	S	$\{0\}, \mathbb{Z}_2$	6	S_6	$\{0\}$
3	E_ρ	$\{0\}, \mathbb{Z}_3$	6	$E \times E_\rho$	$\{0\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2^3, \mathbb{Z}_6, \mathbb{Z}_2^2 \times \mathbb{Z}_3$
3	S_6	$\{0\}, \mathbb{Z}_3$	6	$E_\rho \times E_\rho$	$\{0\}, \mathbb{Z}_3$
3	$E_\rho \times E_\rho$	$\{0\}, \mathbb{Z}_3$	8	S'_8	$\{0\}, \mathbb{Z}_2$
4	E_ι	$\{0\}, \mathbb{Z}_2$	8	S''_8	$\{0\}, \mathbb{Z}_2$
4	S_4	$\{0\}, \mathbb{Z}_2, \mathbb{Z}_2^2$	10	S_{10}	$\{0\}$
4	$E_\iota \times E_\iota$	$\{0\}, \mathbb{Z}_2, \mathbb{Z}_2^2$	12	S'_{12}	$\{0\}$
4	$E \times E_\iota$	$\{0\}, \mathbb{Z}_2, \mathbb{Z}_2^2$	12	S''_{12}	$\{0\}$
5	S_{10}	$\{0\}, \mathbb{Z}_5$	12	$E_\iota \times E_\rho$	$\{0\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$

THEOREM 5.4. — *Every fourdimensional BdF-variety belongs to a family which is listed in Table 6.*

We leave it to the interested reader to write down all the possible isomorphism classes for $T_{r,i}$ in Table 6

Table 6.

m	B_2	m	B_2	m	B_2	m	B_2
2	E	4	$E \times E_l \times E_l$	8	$E \times S'_8$	12	$E_l \times S'_{12}$
2	S	4	$E_l \times E_l \times E_l$	8	$E \times S''_8$	12	$E_l \times S''_{12}$
2	X	5	S_{10}	8	$E_l \times S'_8$	12	$E \times E_\rho \times E_l$
3	E_ρ	6	E_ρ	8	$E_l \times S''_8$	12	$E_\rho \times E_l \times E_l$
3	S_6	6	S_6	9	A'_9	12	$E_\rho \times E_\rho \times E_l$
3	$E_\rho \times E_\rho$	6	$E \times E_\rho$	9	A''_9	14	A'_7
3	A_6	6	$E_\rho \times E_\rho$	10	S_{10}	14	A''_7
3	$E_\rho \times S_6$	6	A_6	10	$E \times S_{10}$	15	$E_\rho \times S_{10}$
3	$E_\rho \times E_\rho \times E_\rho$	6	$E \times S_6$	12	S'_{12}	18	A'_9
4	E_l	6	$E_\rho \times S_6$	12	S''_{12}	18	A''_9
4	S_4	6	$E_\rho \times S$	12	$E_l \times E_\rho$	20	$E_l \times S_{10}$
4	$E_l \times E_l$	6	$E \times E_\rho \times E_\rho$	12	$E_\rho \times S_4$	24	$E_\rho \times S'_8$
4	$E \times E_l$	6	$E_\rho \times E_\rho \times E_\rho$	12	$E_l \times S_6$	24	$E_\rho \times S''_8$
4	A_4	7	A'_7	12	$E \times S'_{12}$	30	$E_\rho \times S_{10}$
4	$E_l \times S$	7	A''_7	12	$E \times S''_{12}$		
4	$E_l \times S_4$	8	S'_8	12	$E_\rho \times S'_{12}$		
4	$E \times S_4$	8	S''_8	12	$E_\rho \times S''_{12}$		

Remark 5.5. — One could, in theory, write tables with all candidates for BdF-varieties up to dimension 11 using the presented method; in fact, the first case where Λ_2 is not a free R_m -module is $m = 23$, meaning $\dim(B_2) = 11$.

6. Projectivity of BdF-manifolds

In the last section, we listed all candidates for BdF-varieties. However, the answer to the following question still remains.

QUESTION 6.1. — *When do the families of BdF-manifolds listed in the tables above contain BdF-varieties?*

In the sequel, we will prove that each family of BdF-manifolds contains a projective member using different methods. The first method is a theorem of T. Ekedahl, unpublished at the moment, which briefly can be stated as “rigid group actions on tori are projective”, and whose strategy of proof was

sketched to F. Catanese at an Oberwolfach conference, see the preprint [6] by F. Catanese and the author. In the subsequent section, we find explicit forms for the polarization. These methods overlap somehow. Nevertheless we think that it is of interest to present both methods.

6.1. Rigid Group Actions on Tori

We state the result by T. Ekedahl we have mentioned before.

THEOREM 6.2 (Ekedahl). — *Let (T, G) be a rigid group action of a finite group G on a complex torus T . Then T (or, equivalently, T/G) is projective.*

Writing $T = V/\Lambda$, the rigidity of a pair (T, G) amounts to requiring that each character χ of G appears in at most one of $V^{1,0}$ and $V^{0,1}$, so we can easily determine which complex tori in the above tables are projective by Ekedahl’s Theorem:

COROLLARY 6.3. — *The complex tori admitting a linear automorphism of order $m > 2$ acting faithfully for which assumption (4.4) holds are projective. In particular, they give rise to BdF-varieties.*

6.2. Explicit Forms for the Polarization

In the following we find directly an explicit elementary form of the polarization, showing that each family of BdF-manifolds contains a projective member. Moreover, we determine the type of the polarization whenever possible.

LEMMA 6.4. — *The following statements hold: There is no non-degenerate G -invariant alternating bilinear form on $R = \mathbb{Z}[X]/(X^m - 1)$ for each $m > 2$. But there is such a form E on $\mathbb{Z}[X]$ satisfying*

$$\ker(E) = \begin{cases} (X^{m-1} + X^{m-3} \cdots + 1), & m \text{ odd} \\ (X^{m-2} + X^{m-4} + \cdots + 1), & m \text{ even.} \end{cases}$$

Proof. — We only prove the lemma in the case where m is odd, the other case is similar. Let E be a form on E as in the statement of the lemma. Then we have, by G -invariance,

$$E(X^i, X^j) = \begin{cases} 0, & i = j, \\ E(1, X^{j-i}), & j > i, \end{cases}$$

and defining the rest by \mathbb{Z} -bilinearity and alternation. Setting $\lambda_i = E(1, X^i)$ for $1 < i < m$, we find

$$0 = E(X^i, X^m - 1) = E(X^i, X^m) - E(X^i, 1) = \lambda_{m-i} + \lambda_i. \quad (6.1)$$

Hence, E is uniquely determined by $\lambda_1, \dots, \lambda_{\lfloor m/2 \rfloor}$, which are not all equal to 0. By slight abuse of notation, we use the letter E not only for the alternating form, but also for its matrix $E = (E(X^i, X^j))_{ij}$. One sees that $X^{m-1} + \dots + 1$ is always in the kernel of E , independent of the choice of the λ_i . Setting $\lambda_1 := 1$, $\lambda_{m-1} := -1$ and $\lambda_i := 0$ for $i \in \{2, \dots, m-2\}$, one obtains a desired form as stated in the lemma. \square

COROLLARY 6.5. — *The alternating form E of the previous lemma induces a non-degenerate G -invariant bilinear form on the \mathbb{Z} -module $R_k = \mathbb{Z}[X]/(\phi_k(X))$ for a divisor k of $n+1 = m$ for each $n > 1$. More precisely, E can be uniquely written as a sum of non-degenerate G -invariant bilinear forms E_k on R_k for k dividing $n+1$.*

Proof. — Write $R' = R_k \oplus R'_k = \mathbb{Z}[X]/(\phi_k(X)) \oplus \mathbb{Z}[X]/(\psi_k(X))$. It suffices to show that $E(R_k, R'_k) = 0$. Note that R_k and R'_k decompose in direct sums of character-eigenspaces; first, let $r \in R_k, r' \in R'_k$ such that r, r' belong to different character-eigenspaces. Hence the condition that E is G -invariant yields

$$g \cdot E(r, r') = \chi(g)\overline{\psi}(g)E(r, r') = E(r, r'),$$

with different characters $\chi \neq \psi$. Hence $E(r, r') = 0$. If $r \in R_k, r' \in R'_k$ are general elements (i.e., a sum of elements belonging to the respective character eigenspaces), we get $E(r, r') = 0$ by bilinearity. \square

Remark 6.6.

- (1) Recall that, if $\Lambda = \mathbb{Z}[X]/(X^n + \dots + 1)$ (for even n and $m = n+1$), we have decompositions

$$\Lambda \otimes \mathbb{C} = V \oplus \overline{V} = \bigoplus_{\text{id} \neq \chi \in G^\vee} W_\chi = \bigoplus_{j < \frac{m}{2}} W_{\chi_j} \oplus W_{\overline{\chi_j}}$$

and $V = \bigoplus_\chi V_\chi$, such that, for each χ , either $W_\chi = V_\chi$ or $W_{\overline{\chi}} = V_{\overline{\chi}}$. We fix the indices such that χ_j corresponds to the eigenvalue ϵ^j (where $\epsilon = \exp(\frac{2\pi i}{m})$). One can do the same if $\Lambda = \mathbb{Z}[X]/(X^{n-1} + X^{n-3} + \dots + 1)$ in the case where n is odd. Unless otherwise stated, we choose the complex structure such that $W_{\chi_j} = V_{\chi_j}$ for every $j < \frac{m}{2}$.

- (2) From the above discussion it follows that a general alternating form E as in Lemma 6.4 takes the following shape

$$E_{ij} = \begin{cases} 0, & i = j, \\ \lambda_{j-i}, & j > i, \\ -\lambda_{i-j}, & j < i. \end{cases} \quad (6.2)$$

LEMMA 6.7. — *For each $n > 1$, there is a non-degenerate G -invariant alternating bilinear form E on $R' = \mathbb{Z}[X]/(X^n + \dots + 1)$ resp. $R' = \mathbb{Z}[X]/(X^n + X^{n-2} + \dots + 1)$ (depending on the parity of n), such that the associated form $H: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}$ defined by $H(v, w) = E(\iota \cdot v, w) + \iota E(v, w)$ and \mathbb{R} -bilinear extension is Hermitian and positive definite.*

Proof. — Let $m = n + 1$. We only prove the lemma in the case where n is even. Write $R = \mathbb{Z} \oplus R'$, where R is as in Lemma 6.4 and let E be given as in 6.2. It remains to show that the Riemann Bilinear Relations are satisfied. The first Riemann Bilinear Relation is satisfied if and only if $E(V, V) = 0$. Due to the previous corollary, it suffices to check this condition on eigenvectors: it is well-known that V_{χ_j} (for $j < \frac{m}{2}$) is generated by the element $v_j = \sum_{i=0}^n \epsilon^{-ji} X^i \in R$. Then one computes:

$$\begin{aligned} E(v_h, v_k) &= \sum_{j=1}^n \sum_{i=0}^n \lambda_j ((X^i)^\vee \otimes (X^{i+j})^\vee - (X^{i+j})^\vee \otimes (X^i)^\vee) (v_h, v_k) \\ &= \sum_{j=1}^n \lambda_j (\epsilon^{-kj} - \epsilon^{-hj}) \sum_{i=0}^n \epsilon^{-hi-ki}. \end{aligned}$$

The last sum is 0 unless $k + h \equiv 0 \pmod{m}$. But if $k + h \equiv 0 \pmod{m}$, we find that v_k and $v_{-k} = \overline{v_k}$ do not both belong to V . Restricting the form to R' , this shows that the first Riemann Bilinear Relation is satisfied, independently of the choice of the λ_i .

Now consider the special form E obtained by setting $\lambda_1 = 1$, $\lambda_j = 0$ for every other j , and restricting to R' . By abuse of notation, we denote the images of the v_k in R' again by v_k . We prove the second Riemann Bilinear Relation for this form: we have to check that $E(-\iota v_k, \overline{v_k}) > 0$ for every $k < \frac{m}{2}$. A simple calculation gives

$$E(-\iota v_k, \overline{v_k}) = 2n \cdot \sin\left(\frac{2\pi k}{m}\right) > 0. \quad \square$$

Summarizing our results, we have the following:

THEOREM 6.8. — *Let $T = V/\Lambda$ be a complex torus and G a cyclic group of order $m \geq 2$ of order m acting freely on T such that Λ is a cyclotomic submodule of $\mathbb{Z}[X]/(X^{m-1} + \dots + 1)$ or $\mathbb{Z}[X]/(X^{m-2} + X^{m-4} + \dots + 1)$ (depending on the parity of m). Assume furthermore that V splits as in Remark 6.6(1). Then there is a non-degenerate G -invariant alternating bilinear form E on Λ whose associated Hermitian form is positive definite, such that the polarization on T is principal.*

Proof. — The matrix of E as chosen in the proofs above has determinant 1. □

Remark 6.9. — Indeed, we have shown in this subsection that all of the following families of BdF-varieties $A = (B_1 \times B_2)/(T \times G)$ contain a projective member:

- The families where (4.4) is not satisfied (i.e., the non-rigid cases): In this case, a product of lower-dimensional Abelian varieties for which (4.4) holds is always in the same family as B_2 .
- The families where the complex structure is chosen as in Remark 6.6(1) (or the one conjugate to it).

We briefly treat the exceptional cases which we not dealt with in the previous discussion, i.e., the ones where the complex structure is different from the one in Remark 6.6. Note that all these cases are rigid, hence projective by Ekedahl’s Theorem 6.2; nevertheless, it is also of interest to give an explicit form of a polarization for these cases. One checks computationally that these values of λ_i give rise to a positive definite form on $\mathbb{Z}[X]/(X^{m-1} + \dots + 1)$ resp. $\mathbb{Z}[X]/(X^{m-2} + X^{m-4} + \dots + 1)$.

Table 7. Here we used the shorthand notation $a^k = \overbrace{(a, \dots, a)}^{k \text{ times}}$ for an integer a .

Case	Orbit	$(\lambda_1, \dots, \lambda_{\lfloor m/2 \rfloor})$	Case	Orbit	$(\lambda_1, \dots, \lambda_{\lfloor m/2 \rfloor})$
S''_8	(1, 5)	(-1, 1, 0)	X^2_{20}	(1, 3, 7, 11)	$((-1)^2, 0, -1, 0, -1, 0, -1, 0^2)$
S''_{12}	(1, 7)	(-1, 0, 1^2, -1)	X^3_{20}	(1, 3, 11, 13)	$((-1)^2, 0^8)$
A''_7	(1, 2, 4)	(0, -1, 1)	X^4_{20}	(1, 9, 13, 17)	$(1^2, 0, (-1)^2, 0^2, (-1)^2, 0)$
A''_9	(1, 2, 7)	$(0, (-1)^2, 0)$	X^2_{24}	(1, 5, 7, 13)	$(1^3, 0, (-1)^3, 0^2, 1, 0, -1, 0)$
X^2_{15}	(1, 2, 4, 8)	$((-1)^4, 1, 0, 1)$	X^3_{24}	(1, 5, 13, 17)	$((-1)^{12}, 0^2)$
X^3_{15}	(1, 2, 7, 11)	$((-1)^3, 0^2, (-1)^2)$	X^4_{24}	(1, 7, 13, 19)	$((-1)^5, 1^2, -1, 0^2, -1, 0)$
X^4_{15}	(1, 4, 7, 13)	$(-1, 1^2, (-1)^2, 1, -1)$	X^5_{24}	(1, 11, 17, 19)	$((-1)^3, 0, -1, 1^3, -1, 0^3)$
X^2_{16}	(1, 3, 5, 9)	$(0^2, 1, -1, 0, -1, 0^2)$	X^2_{11}	(1, 2, 3, 4, 6)	$((-1)^2, 1, 0, 1)$
X^3_{16}	(1, 3, 9, 11)	$(0, (-1)^7, 0)$	X^3_{11}	(1, 2, 3, 5, 7)	$((-1)^3, 1, -1)$
X^4_{16}	(1, 5, 9, 13)	$(0, -1, 1, (-1)^2, 1, -1, 0^2)$	X^4_{11}	(1, 3, 4, 5, 9)	$(1, 0^2, 1^2)$

Hence we have proved:

COROLLARY 6.10. — *Let $X = (B_1 \times B_2)/(G \times T)$ be a BdF-manifold. Then the family containing X contains a BdF-variety.*

This concludes the proof of Theorem 1.1.

Remark 6.11. — Quite interesting is also the datum of explicit equations for the given variety, or one isogenous to it. This problem requires a lot of time and effort and is therefore not dealt with in this paper. See for instance [5] for results in this direction.

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