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GENKI HOSONO AND TAKAYUKI KOIKE

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On metrics with minimal singularities of line bundles whose stable base loci admit holomorphic tubular neighborhoods ^(*)

GENKI HOSONO ⁽¹⁾ AND TAKAYUKI KOIKE ⁽²⁾

ABSTRACT. — We investigate the minimal singularities of metrics on a big line bundle L over a projective manifold when the stable base locus Y of L is a submanifold of codimension $r \geq 1$. Under some assumptions on the normal bundle and a neighborhood of Y , we give a explicit description of the minimal singularity of metrics on L . We apply this result to study a higher (co-)dimensional analogue of Zariski's example, in which the line bundle L is not semi-ample, however it is nef and big.

RÉSUMÉ. — Nous étudions les singularités minimales des métriques d'un fibre en droites L sur une variété projective lorsque le locus de base stable Y de L est une sous-variété de codimension $r \geq 1$. Sous certaines hypothèses sur le fibre normal et le voisinage de Y , nous donnons une description explicite de la singularité minimale des métriques de L . Nous appliquons ce résultat pour étudier un analogue (co-dimensionnel) plus élevé de l'exemple de Zariski, dans lequel le fibre en droites L n'est pas semi-ample, mais il est nef et gros.

1. Introduction

The purpose of this paper is to investigate *metrics with minimal singularities* on a big line bundle L on a projective manifold X . Metrics with minimal singularities have been introduced in [6, Definition 1.4] as a weak analytic analogue of the so-called Zariski decomposition. There exists a metric with minimal singularities uniquely up to certain equivalence of singularities

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(1) Mathematical Institute, Tohoku University, 6-3, Aramaki Aza-Aoba, Aoba-ku, Sendai, 980-8578 (Japan) — genki.hosono@gmail.com

(2) Graduate School of Science, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku Osaka, 558-8585 (Japan) — tkoike@sci.osaka-cu.ac.jp

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when L is pseudo-effective [6, Theorem 1.5]. Indeed, the *equilibrium metric* h_e of any C^∞ Hermitian metric h on L has minimal singularities (see Example 2.3).

On a higher-dimensional variety, a line bundle does not necessarily admit the Zariski decomposition. Nakayama constructed an example of a line bundle which admits no Zariski decomposition even after any modification [15, IV, §2.6]. Nakayama's example is constructed as the relative tautological bundle on certain projective space bundle over an abelian variety. Boucksom [1] posed a decomposition called *divisorial Zariski decomposition*, in which the negative part of a big line bundle L is identified with the divisorial (i.e. one-codimensional) part of the singularities of a metric with minimal singularities on L . From this point of view, it is important for a study of the Zariski decomposition to investigate the higher-codimensional part of the singularities of metrics with minimal singularities in detail. In [10], the second author explicitly described the metrics with minimal singularities for Nakayama's example we mentioned above.

We also investigate the case where the line bundle L is nef (and big) and thus L has no negative part in the sense of Zariski decompositions. In this case, our main interest is in the semi-positivity of the line bundle, i.e. whether L admits a C^∞ metric with semi-positive curvature or not. In [11], the second author studied the metrics with minimal singularities on a line bundle called Zariski's example, which is known to be nef and big, but not semi-ample. As a result, it was shown that Zariski's example admits a C^∞ Hermitian metric with semi-positive curvature.

In this paper, we investigate the metrics of L with minimal singularities for more general cases than both [10] and [11]. Our main result has the following application:

THEOREM 1.1. — *Take two general quadric surfaces Q_1 and Q_2 in \mathbb{P}^3 and fix general N points p_1, p_2, \dots, p_N in $Q_1 \cap Q_2$ ($N \geq 12$). Denote by $\pi: X := \text{Bl}_{\{p_1, p_2, \dots, p_N\}} \mathbb{P}^3 \rightarrow \mathbb{P}^3$ the blow-up of \mathbb{P}^3 at these N points, and by D_1 and D_2 the strict transform of Q_1 and Q_2 , respectively. Define a line bundle L by $L := \pi^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_X(D_1)$. Then, the local weight function $\varphi_{\min, L}$ of a metric with minimal singularities $h_{\min, L}$ of L (i.e. $\varphi_{\min, L}$ is a locally defined function such that $h_{\min, L} = e^{-\varphi_{\min, L}}$) is written as*

$$\varphi_{\min, L}(z, y) = \frac{N-12}{N-8} \cdot \log(|z_1|^2 + |z_2|^2) + O(1)$$

on a neighborhood of every point of $Y := D_1 \cap D_2$, where y is a coordinate of Y and $z = (z_1, z_2)$ is a system of local defining functions of Y . We have that $\varphi_{\min, L}$ is locally bounded on $X \setminus Y$.

When $N = 12$, the line bundle L in this theorem is nef and big, but not semi-ample. Hence it can be regarded as a higher-dimensional analogue of Zariski's example. In this case, we will show that L admits a C^∞ Hermitian metric with semi-positive curvature (see Section 6.2 for detail), which can be regarded as a two-codimensional analogue of [11, Theorem 1.1].

In what follows, L denotes a big line bundle on a projective manifold X . We study the metric with minimal singularities when (X, L) satisfies the following condition.

CONDITION 1.2.

- (i) *The stable base locus $Y = \mathbf{B}(L)$ of L is a smooth (i.e. non-singular) compact subvariety of codimension $r \geq 1$,*
- (ii) *the normal bundle $N_{Y/X}$ of Y admits a direct sum decomposition $N_{Y/X} = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ into r negative line bundles.*

Nakayama's example satisfies Condition 1.2. For the pair (X, L) with Condition 1.2, we define the following convex set:

$$\square_L = \left\{ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \left| \begin{array}{l} |\alpha| \leq 1, \text{ and } c_1(L|_Y) + \sum_{\lambda=1}^r \alpha_\lambda c_1(N_\lambda^{-1}) \\ \text{is pseudo-effective} \end{array} \right. \right\},$$

where $|\alpha| := \sum_{\lambda=1}^r \alpha_\lambda$. In [10], a metric with minimal singularities on Nakayama's example was described explicitly by using \square_L on a neighborhood of Y . It is easily observed that a metric with minimal singularities is locally bounded on the complement $X \setminus Y$ of Y (see Example 2.4). Hence our interest is in the behavior of metrics with minimal singularities near Y . We always take a system of local defining functions $z = (z_1, z_2, \dots, z_r)$ of Y so that, for each λ , the subbundle of $N_{Y/X}^{-1}$ generated by dz_λ corresponds to the direct component N_λ^{-1} .

Our main result is stated as follows:

THEOREM 1.3. — *Let X be a projective manifold, L be a big line bundle on X , and $Y = \mathbf{B}(L)$ be the stable base locus of L . Assume that Y admits a holomorphic tubular neighborhood (see below) and (X, L, Y) satisfies Condition 1.2. Assume also that $L|_Y \otimes N_\lambda^{-1}$ and $K_Y^{-1} \otimes N_\lambda^{-1}$ are positive for every $\lambda = 1, 2, \dots, r$. Take C^∞ Hermitian metrics $h_{L|_Y}$ on $L|_Y$ and h_{N_λ} on N_λ satisfying $\Theta_{h_{L|_Y} \otimes h_{N_\lambda}^{-1}} > 0$ for every λ . Then the local weight function $\varphi_{\min, L}$ of a metric with minimal singularities $h_{\min, L}$ is written as*

$$\varphi_{\min, L}(z, y) = \log \max_{\alpha \in \square_L} \left(\prod_{\lambda=1}^r |z_\lambda|^{2\alpha_\lambda} \right) \cdot e^{(\varphi_\alpha)_e(y)} + O(1)$$

on a neighborhood of any given point of Y , where we are formally regarding 0^0 as 1, y is a coordinate of Y , $z = (z_1, z_2, \dots, z_r)$ is a system of local defining functions of Y as above, and $(\varphi_\alpha)_e$ is the local weight function of the equilibrium metric of $h_{L|_Y} \otimes h_{N_1}^{-\alpha_1} \otimes h_{N_2}^{-\alpha_2} \cdots \otimes h_{N_r}^{-\alpha_r}$ (see §2 for the notion of the “metric” $h_{L|_Y} \otimes h_{N_1}^{-\alpha_1} \otimes h_{N_2}^{-\alpha_2} \cdots \otimes h_{N_r}^{-\alpha_r}$ for real α_λ ’s).

A complex submanifold $Y \subset X$ is said to have a *holomorphic tubular neighborhood* if there exist a neighborhood V of Y in X , a neighborhood \tilde{V} of the zero section in $N_{Y/X}$ and a biholomorphism $i: V \rightarrow \tilde{V}$ such that $i|_Y$ coincides with the natural isomorphism. Note that the description of the singularity of $\varphi_{\min, L}$ as in Theorem 1.3 does not depend on the choice of the coordinates (up to $O(1)$). This theorem is a generalization of the main result of [10]. Moreover it is also a generalization of [11] in higher codimensional cases. In this theorem, Condition 1.2 and the condition that Y admits a holomorphic tubular neighborhood are essential and can not be dropped (see Section 6.3).

When Y is an abelian variety (as in Nakayama’s example), we have a sufficient condition for the existence of a holomorphic tubular neighborhood of Y by Grauert’s theory on a neighborhood of an exceptional subvariety [8] (see Section 5). As a result, we have the following theorem:

THEOREM 1.4. — *Let X be a projective manifold, let L be a big line bundle on X , and let $Y = \mathbf{B}(L)$ be the stable base locus of L . Assume Condition 1.2. Assume also that Y is an abelian variety, $L|_Y \otimes N_\lambda^{-1}$ is positive for every $\lambda = 1, 2, \dots, r$, and that $N_\lambda \cong N_\mu$ for every λ and μ . Then the local weight function $\varphi_{\min, L}$ of a metric with minimal singularities $h_{\min, L}$ is written as*

$$\varphi_{\min, L}(z, y) = \log \max_{\alpha \in \square_L} \prod_{\lambda=1}^r |z_\lambda|^{2\alpha_\lambda} + O(1)$$

on a neighborhood of any given point of Y , where y is a coordinate of Y and $z = (z_1, z_2, \dots, z_r)$ is a system of local defining functions of Y as above.

When $L|_Y$ is pseudo-effective, it is natural to ask whether $(h_{\min, L})|_Y$ is a metric with minimal singularities on $L|_Y$. For (X, L, Y) in Theorem 1.3, it follows by definition that the convex set \square_L includes the origin 0 if $L|_Y$ is pseudo-effective. In this case, it is directly deduced from Theorem 1.3 that $h_{\min, L}|_Y \leq (h_{L|_Y})_e \cdot e^{O(1)}$, which means that $h_{\min, L}|_Y$ has minimal singularities. Therefore we have the following:

COROLLARY 1.5. — *Let X, L , and Y be those in Theorem 1.3. Assume that $L|_Y$ is pseudo-effective. Then $h_{\min, L}|_Y$ is a singular Hermitian metric of $L|_Y$ with minimal singularities.*

The proof of Theorem 1.3 is based on the arguments in [11]. We first study a special case where X is a projective space bundle over Y and L is the relative tautological bundle. After that, we apply the exact description of metrics with minimal singularities for this special case to the study of general (X, L, Y) by using, what we call, the maximum construction technique (here we use the assumption of a holomorphic tubular neighborhood).

The organization of the paper is as follows. In Section 2, we introduce fundamental notation and recall some facts on projective space bundles and singular Hermitian metrics. In Section 3, we show the main result in the special case where X is the total space of a projective space bundle. In Section 4, we prove Theorem 1.3 in general. In Section 5, we give a sufficient condition for the existence of a holomorphic tubular neighborhood by using Grauert's theory. Here we also show Theorem 1.4. In Section 6, we give several examples.

2. Preliminaries

2.1. Notations on projective space bundles

Let Y be a compact complex manifold. Let M_1, M_2, \dots, M_r and M_{r+1} be holomorphic line bundles on Y . Let $\{U_j\}_j$ be an open cover of Y . Assume that every U_j is sufficiently small so that $M_\lambda|_{U_j}$ is trivial for every λ and j . Then there exist local holomorphic trivializations given by sections $s_{j,\lambda} \in H^0(U_j, M_\lambda)$. Denote by E the vector bundle $M_1 \oplus M_2 \oplus \dots \oplus M_{r+1}$. Then $(s_{j,\lambda})_\lambda$ forms a holomorphic frame of E on U_j . Let $(\xi_{j,\lambda})_\lambda$ be the dual frame of $(s_{j,\lambda})_\lambda$.

We fix the notation on $\mathbb{P}(E)$ as follows. Let us denote by $\mathbb{P}(E)$ the projective space bundle of *hyperplanes* of E over Y , i.e. $\mathbb{P}(E) := \bigcup_y (E_y^* \setminus 0) / \mathbb{C}^*$. We will denote the bundle of *lines* by $\mathbf{P}(E)$ in this paper. Let π denote the natural projection $\mathbb{P}(E) \rightarrow Y$. We will use the notation $\mathbb{P}(E)|_{U_j}$ to denote $\pi^{-1}(U_j)$. By using homogeneous coordinates, $([x_{j,1} : x_{j,2} : \dots : x_{j,r+1}], y)$ denotes the point $[x_{j,1}\xi_{j,1} + x_{j,2}\xi_{j,2} + \dots + x_{j,r+1}\xi_{j,r+1}] \in \mathbb{P}(E)_y$ on $\mathbb{P}(E)|_{U_j}$. Here $\mathbb{P}(E)_y$ denotes the fiber $\pi^{-1}(y)$. Let $U_j^{(\lambda)}$ be an open set $\{([x_{j,1}\xi_{j,1} + x_{j,2}\xi_{j,2} + \dots + x_{j,r+1}\xi_{j,r+1}], y) \mid y \in U_j, x_{j,\lambda} \neq 0\}$ of $\mathbb{P}(E)$. Note that $\{U_j^{(\lambda)}\}_{j,\lambda}$ forms an open cover of $\mathbb{P}(E)$. The *tautological line bundle* $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is defined by setting its fiber on $([\xi], y)$ as $E_y / \text{Ker } \xi$, where ξ denotes an element of $E_y^* \setminus 0$. Let Γ_λ be the divisor of $\mathbb{P}(E)$ defined as $\mathbb{P}(M_1 \oplus M_2 \oplus \dots \widehat{M_\lambda} \oplus \dots \oplus M_{r+1})$. The following fact is obtained by a simple computation.

LEMMA 2.1.

- (1) $[\Gamma_\lambda] \otimes \pi^* M_\lambda = \mathcal{O}_{\mathbb{P}(E)}(1)$, where $[\Gamma_\lambda]$ denotes the line bundle defined by the divisor Γ_λ .
- (2) $N_{Y/X} = \bigoplus_{\lambda=1}^r \mathcal{O}_{\mathbb{P}(E)}(1)|_Y \otimes M_\lambda^{-1}$.

2.2. Singular Hermitian metrics

In this subsection, we review some properties of singular Hermitian metrics on line bundles.

DEFINITION 2.2. — *Let X be a (possibly non-compact) complex manifold and let L be a line bundle on X . A singular Hermitian metric h on L is defined as a metric of L with the form $\|s\|_h^2 = |s|^2 e^{-\phi}$ on U for each trivialization $L|_U \cong U \times \mathbb{C}$, where $\phi \in L^1_{\text{loc}}(U)$. In this situation, we will write as $h = e^{-\phi}$ and call ϕ as a local weight. Note that ϕ is a collection of a function defined on small open sets. The curvature of a singular Hermitian metric $h = e^{-\phi}$ is defined as a $(1,1)$ -current $\Theta_h = \sqrt{-1} \partial \bar{\partial} \phi$.*

A singular Hermitian metric $h = e^{-\phi}$ is *semi-positively curved* (or h admits *semi-positive curvature*) if its local weight ϕ is plurisubharmonic on the set where ϕ is defined. In this case, its curvature is non-negative as a $(1,1)$ -current.

Let h_1 and h_2 be singular Hermitian metrics on L . We say that h_1 is *more singular* than h_2 when, for every relatively compact set U , there is a constant $C > 0$ such that the inequality $h_1 \geq Ch_2$ holds on U . In this case we write $h_1 \succsim_{\text{sing}} h_2$. We say that h_1 and h_2 have *equivalent singularities* (written $h_1 \sim_{\text{sing}} h_2$) when both $h_1 \lesssim_{\text{sing}} h_2$ and $h_1 \succsim_{\text{sing}} h_2$ hold. A semi-positively curved singular Hermitian metric h on L has *minimal singularities* if $h \lesssim_{\text{sing}} h'$ for any semi-positively curved singular Hermitian metric h' . When X is compact, $h \succsim_{\text{sing}} h'$ holds if and only if there exists a constant C such that $h_1 \geq Ch_2$ on X .

To investigate singular Hermitian metrics, it will be convenient to consider globally defined functions corresponding to their local weights. For this reason, we introduce the notion of θ -plurisubharmonic functions here. Let θ be a smooth real $(1,1)$ -form. We say that a function $u \in L^1_{\text{loc}}(X)$ is a *θ -plurisubharmonic function* when the inequality $\theta + i\partial\bar{\partial}u \geq 0$ holds as currents. We denote the set of θ -plurisubharmonic functions on X by $\text{PSH}(X, \theta)$.

Let L be a holomorphic line bundle on X . Fix a smooth Hermitian metric h_0 on L with curvature θ . Then there is a one-to-one correspondence between θ -plurisubharmonic functions u and semi-positively curved singular

Hermitian metrics $h_0 \cdot e^{-u}$ on L . We define θ -plurisubharmonic functions with minimal singularities similarly to the case of metrics. Namely, a θ -plurisubharmonic function u has *minimal singularities* (in $\text{PSH}(X, \theta)$) if, for every θ -plurisubharmonic function u' , there exists a (local) constant C such that $u \geq u' + C$ on each compact set. For an \mathbb{R} -line bundle L (i.e. a formal “line bundle” corresponding to an \mathbb{R} -divisor), a notion of singular Hermitian metric on L is well-defined formally in this sense.

Example 2.3. — Assume X is compact and L is *pseudo-effective*, i.e. L admits a semi-positively curved singular Hermitian metric. Fix a smooth metric h with curvature θ . Then, the function defined by

$$V_\theta := \sup\{v \in \text{PSH}(X, \theta) \mid v \leq 0\}$$

is θ -plurisubharmonic. It is easily observed that V_θ has minimal singularities in $\text{PSH}(X, \theta)$. The corresponding singular Hermitian metric $h \cdot e^{-V_\theta}$ is denoted by h_e , which is called the *equilibrium metric*.

Example 2.4. — Fix a smooth Hermitian metric h_0 on L . Let $f_1, f_2, \dots, f_N \in H^0(X, L)$ be global holomorphic sections of L . Then we define a singular Hermitian metric h by the formula

$$\|f\|_h^2 := \frac{\|f\|_{h_0}^2}{\sum_{j=1}^N \|f_j\|_{h_0}^2}.$$

In this manner, we obtain a semi-positively curved singular Hermitian metric h which is smooth on the Zariski open set $\bigcup_{j=1}^N \{f_j \neq 0\}$. If L is big, there exist a finite number of sections $f_1, \dots, f_N \in H^0(X, L^m)$ for sufficiently large m such that $\{f_1 = \dots = f_N = 0\} = \mathbf{B}(L)$ ([14, 2.1.21]). Here we denote the tensor product $L^{\otimes m}$ by L^m . Then, we have a singular Hermitian metric on L^m which is smooth on $X \setminus \mathbf{B}(L)$. By taking m -th root, we can define a singular Hermitian metric on L (we call it a *Bergman-type metric* on L obtained by f_1, \dots, f_N).

Example 2.5. — Let X be a compact complex manifold and let L be a line bundle. Fix a smooth volume form dV on X and a smooth metric $h = e^{-\phi}$ on L . Let θ be the curvature of h . We define a θ -plurisubharmonic function $V_{\phi, B}$ by

$$V_{\phi, B} := V_{h, B} := \sup \left\{ \frac{1}{m} \log |f|_{h^m}^2 \mid m \in \mathbb{Z}, f \in H^0(X, L^m), \int_X |f|_{h^m}^2 dV \leq 1 \right\}.$$

The corresponding singular Hermitian metric on L and its local weight are denoted by $h_B = e^{-\phi_B}$. By Proposition 2.6 below, h_B has minimal singularities when L is big.

We use this construction when L is a \mathbb{Q} -line bundle with a smooth metric $h = e^{-\phi}$, that is, for some integer $m > 0$, L^m is an ordinary line bundle and $h^m = e^{-m\phi}$ is a smooth metric on L^m . In this case, we take the smallest integer $m > 0$ such that L^m is a \mathbb{Z} -line bundle and define ϕ_B by $(1/m)(m\phi)_B$.

To compare the metrics h_e and h_B , we need the following proposition.

PROPOSITION 2.6 ([11, Lemma 2.10]). — *Let X be a projective manifold and let L be a big line bundle. Let $h = e^{-\phi}$ be a smooth Hermitian metric on L . Fix a smooth volume form dV on X . Then, there is a constant C such that the inequality*

$$V_{\phi,B} - C \leq V_\theta \leq V_{\phi,B}$$

holds.

Before starting the proof, we shall explain how we use Proposition 2.6 in Section 3. We shall apply it to a family of \mathbb{Q} -line bundles of the form

$$L^\alpha := L_1^{\alpha_1} \otimes L_2^{\alpha_2} \otimes \cdots \otimes L_{r+1}^{\alpha_{r+1}},$$

where L_λ are \mathbb{Z} -line bundles, $\alpha_\lambda \geq 0$ and $\alpha_1 + \cdots + \alpha_{r+1} = 1$. Let $e^{-\phi_\lambda}$ be a fixed smooth metric on L_λ and let m be the smallest positive integer such that $(L^\alpha)^m$ is a \mathbb{Z} -line bundle. Then, the local weight $m\phi_\alpha := m \sum_\lambda \alpha_\lambda \phi_\lambda$ defines a smooth metric on $(L^\alpha)^m$. The constant C in Proposition 2.6 depends only on C_1 , C_2 and $C(\phi)$, which will be defined in the proof below. We note that constants C_1 and C_2 are independent of the choice of line bundles, and $C(\phi)$ only depends on the differences $(\sup_{B_j''} - \inf_{B_j''})\phi$, where $\{B_j''\}_j$ is an open cover of X consisting of open balls. Thus there exists a constant C_3 depending on the metrics $e^{-\phi_\lambda}$ and independent of α , such that

$$V_{m\phi^\alpha,B} - \log(C_1 C_2) - mC_3 \leq V_{m\theta_\alpha} \leq V_{m\phi^\alpha,B}.$$

Dividing by m , we have that

$$V_{\phi^\alpha,B} - \frac{1}{m} \log(C_1 C_2) - C_3 \leq V_{\theta_\alpha} \leq V_{\phi^\alpha,B}.$$

In conclusion, there exists a constant C such that we have

$$V_{\phi^\alpha,B} - C \leq V_{\theta_\alpha} \leq V_{\phi^\alpha,B} \tag{2.1}$$

for every α such that $\alpha_\lambda \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_{r+1} = 1$.

Proof of Proposition 2.6. — First we prove the inequality $V_\theta \leq V_{\phi,B}$. Since L is big, there exists a singular Hermitian metric ψ_+ on L such that its curvature is a Kähler current, i.e. $\Theta_{\psi_+} \geq \epsilon\omega$ for some $\epsilon > 0$ and some Kähler form ω . Define a θ -plurisubharmonic function V_+ by $\psi_+ = \phi + V_+$. We may assume that $V_+ \leq 0$. Let $V_\ell := (1 - 1/\ell)V_\theta + (1/\ell)V_+$ and $\phi_\ell := \phi + V_\ell$.

Then the curvature of the metric $e^{-\phi_\ell}$ is a Kähler current. Now consider the following approximations:

$$V_{\phi, B, m} := \sup^* \left\{ \frac{1}{m} \log |f|_{m\phi}^2 \mid f \in H^0(X, L^m), \int_X |f|^2 e^{-m\phi} dV \leq 1 \right\}, \text{ and}$$

$$V_{\phi_\ell, B, m} := \sup^* \left\{ \frac{1}{m} \log |f|_{m\phi_\ell}^2 \mid f \in H^0(X, L^m), \int_X |f|^2 e^{-m\phi_\ell} dV \leq 1 \right\}.$$

Then we have that $V_\ell + V_{\phi_\ell, B, m} \leq V_{\phi, B, m}$. Applying Demailly's approximation theorem ([4, Theorem 13.21]) to ϕ_ℓ , we have that $V_{\phi_\ell, B, m} \geq V_\ell - C/m$, where C is independent of ℓ and m . Hence we have $V_\ell - \frac{C}{m} \leq V_{\phi, B, m} \leq V_{\phi, B}$. By letting $m \rightarrow \infty$, we obtain $V_\ell \leq V_{\phi, B}$. After that, letting $\ell \rightarrow \infty$, we have that $V_\theta \leq V_{\phi, B}$.

Next we prove the inequality $V_{\phi, B} - C \leq V_\theta$. Fix a collection of open coordinate balls $B'_j \subset B''_j \subset B_j$ such that $\{B'_j\}_j$ is an open cover of X and the radii of B'_j , B''_j and B_j are $1/2$, 1 and 2 respectively. Fix a local trivialization of L . Take $f \in H^0(X, L^m)$ with $\int_X |f|^2 e^{-m\phi} dV \leq 1$. Then, for every $p \in B'_j$, we have that

$$\begin{aligned} |f(p)|^2 &\leq \frac{1}{\pi^n (1/2)^{2n} / n!} \int_{|z-p| < 1/2} |f|^2 d\lambda \\ &\leq C_1 C_2 \cdot e^{m \sup_{B''_j} \phi} \int_{|z-p| < 1/2} |f|^2 e^{-m\phi} dV \\ &\leq C_1 C_2 \cdot e^{m \sup_{B''_j} \phi}. \end{aligned}$$

Here we write the constants as $C_1 := \frac{1}{\pi^n (1/2)^{2n} / n!}$ and $C_2 := \sup_{B''_j} d\lambda/dV$. It follows that $|f(p)|^2 e^{-m\phi(p)} \leq C_1 C_2 \cdot \exp(m(\sup_{B''_j} \phi - \phi(p))) \leq C_1 C_2 \cdot \exp(m(\sup_{B''_j} - \inf_{B''_j})\phi)$. Thus we have that

$$\frac{1}{m} \log |f(p)|_{m\phi}^2 \leq (\log C_1 C_2)/m + \left(\sup_{B''_j} - \inf_{B''_j} \right) \phi.$$

It follows that

$$V_{\phi, B, m}(p) \leq (\log C_1 C_2)/m + \left(\sup_{B''_j} - \inf_{B''_j} \right) \phi.$$

The right-hand side is estimated by using the constants C_1, C_2 and a constant $C(\phi)$ depends only on ϕ . Taking the supremum over m , we have that $V_{\phi, B}(p) \leq \log(C_1 C_2) + C(\phi)$. We denote this constant by C . Considering all B_j , we have $V_{\phi, B} - C \leq V_\theta$ for some constant C . \square

3. Projective bundles

3.1. Settings in the case of \mathbb{P}^r -bundle

Let Y be a projective manifold. Let M_1, M_2, \dots, M_r and M_{r+1} be line bundles. We assume that the first r line bundles M_1, \dots, M_r are ample (we do not assume the ampleness of M_{r+1}). Define a manifold X by $X := \mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})$ and a line bundle L on X by $L := \mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})}(1)$. Let us recall that $\mathbb{P}(E)$ denotes the projective space bundle of *hyperplanes* of E . Let π denote the natural projection $X \rightarrow Y$. We regard Y as a submanifold of X via the inclusion $\mathbb{P}(M_{r+1}) \subset \mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})$ induced by the projection $M_1 \oplus M_2 \oplus \dots \oplus M_{r+1} \rightarrow M_{r+1}$. Let h_λ ($1 \leq \lambda \leq r+1$) be smooth Hermitian metrics on M_λ and θ_λ be the curvature forms of h_λ . Here we assume that every h_λ ($1 \leq \lambda \leq r$) has a positive curvature, i.e. the curvature form θ_λ is a positive (1,1)-form for every $\lambda = 1, \dots, r$. Let us denote by $h_L = e^{-\varphi_L}$ the naturally induced metric on L from h_1, \dots, h_{r+1} by considering the Euler sequence. We denote by θ_L the curvature of h_L . Let \square_L be a convex set defined in §1 as follows:

$$\square_L = \left\{ \alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \left| \begin{array}{l} |\alpha| \leq 1, \text{ and } c_1(L|_Y) + \sum_{\lambda=1}^r \alpha_\lambda c_1(M_\lambda \otimes L|_Y^{-1}) \\ \text{is pseudo-effective} \end{array} \right. \right\},$$

where $|\alpha|$ denotes $\alpha_1 + \alpha_2 + \dots + \alpha_r$. Here we use the direct decomposition $N_{Y/X} = \bigoplus_{\lambda=1}^r (L \otimes \pi^* M_\lambda^{-1})|_Y$ (see Lemma 2.1).

The following theorem is the main result of this section.

THEOREM 3.1. — *Let Y, M_λ, X, L and h_λ be as above. For an r -tuple of non-negative real numbers $\alpha = (\alpha_1, \dots, \alpha_r)$ with $\alpha_1 + \alpha_2 + \dots + \alpha_r \leq 1$ and a real number $\alpha_{r+1} := 1 - \alpha_1 - \alpha_2 - \dots - \alpha_r$, define a function $u_\alpha(x)$ on X as follows:*

$$u_\alpha(x) := \alpha_1 \log |s_1|_{h_1}^2 + \alpha_2 \log |s_2|_{h_2}^2 + \dots + \alpha_{r+1} \log |s_{r+1}|_{h_{r+1}}^2 + \pi^* V_{\theta_\alpha}.$$

Here, s_λ denotes the canonical section of a divisor Γ_λ , \widehat{h}_λ denotes the metric on the line bundle $[\Gamma_\lambda]$ defined by $h_L/\pi^* h_\lambda$, and θ_α denotes the (1,1)-form $\sum_{\lambda=1}^{r+1} \alpha_\lambda \theta_\lambda$. Then,

- (i) u_α is a θ_L -plurisubharmonic function.
- (ii) For every fixed $x \in X$, there exists the maximum value of the function $u_\alpha(x)$ of $\alpha \in \square_L$.
- (iii) Define a function $\widehat{V}(x)$ on X by $\widehat{V}(x) := \max_{(\alpha_1, \dots, \alpha_r) \in \square_L} u_\alpha(x)$. Then \widehat{V} is upper semi-continuous and θ_L -plurisubharmonic.
- (iv) \widehat{V} is a θ_L -plurisubharmonic function with minimal singularities.

3.2. The relation between Theorem 3.1 and Theorem 1.2.

Before proving Theorem 3.1, we shall explain the relation between Theorem 3.1 and Theorem 1.3. We assume that X , Y and L are those in Theorem 1.3. We construct a “projective space bundle model” $(\tilde{X}, \tilde{Y}, \tilde{L})$, to which we apply Theorem 3.1. We define \tilde{X} by $\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)$, \tilde{Y} by $\mathbb{P}(\mathbb{I}_Y)$, and \tilde{L} by $\mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1)$. In Section 4, we use a trick called the *maximum construction* to get a metric with minimal singularities on L from that on \tilde{L} .

To check that $(\tilde{X}, \tilde{Y}, \tilde{L})$ satisfies the assumption of Theorem 3.1, we have to choose appropriate line bundles M_λ on \tilde{Y} as in the following lemma.

LEMMA 3.2. — *Let N_λ be line bundles as in Theorem 1.3. If one take $M_\lambda = L|_Y \otimes N_\lambda^{-1}$ for $\lambda = 1, \dots, r$ and $M_{r+1} = L|_Y$, one have that $\tilde{X} \cong \mathbb{P}(M_1 \oplus \dots \oplus M_{r+1})$ and $\tilde{L} = \mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})}(1)$.*

Proof. — We have that

$$\begin{aligned} \mathbb{P}(M_1 \oplus \dots \oplus M_{r+1}) &= \mathbb{P}(L|_Y \otimes (N_1^{-1} \oplus \dots \oplus N_r^{-1} \oplus \mathbb{I}_Y)) \\ &\cong \mathbb{P}(N_1^{-1} \oplus \dots \oplus N_r^{-1} \oplus \mathbb{I}_Y) \\ &= \mathbb{P}(N_{Y/X}^* \oplus \mathbb{I}_Y) = \tilde{X}. \end{aligned}$$

We also have that

$$\mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus \dots \oplus M_{r+1})}(1) = \mathcal{O}_{\mathbb{P}(N_1^{-1} \oplus N_2^{-1} \oplus \dots \oplus N_r^{-1} \oplus \mathbb{I}_Y)}(1) \otimes \pi^* M_{r+1} = \tilde{L}. \quad \square$$

Note that, under this choice, we have $N_{Y/X} = \bigoplus_{\lambda=1}^r (L \otimes \pi^* M_\lambda^{-1})|_Y$. To use the maximum construction argument in §4, we have to use the following lemma.

LEMMA 3.3. — *In this situation, one have $SB(\tilde{L}) \subset \tilde{Y}$.*

Proof. — Recall that s_λ is the canonical section of the line bundle $[\Gamma_\lambda] = \pi^* M_\lambda^{-1} \otimes \tilde{L}$ (Lemma 2.1). For every global section $f \in H^0(Y, \pi^* M_\lambda^m)$ ($m \geq 0$), we have that $s_\lambda^m \otimes \pi^* f \in H^0(\tilde{X}, \tilde{L}^m)$. By the assumption of Theorem 1.3, the line bundle $L|_Y \otimes N_\lambda^{-1}$ is ample for every $\lambda = 1, \dots, r$. Therefore, for sufficiently large m , there exist global sections of \tilde{L} whose common zero is Γ_λ . Using this argument for every λ , we have that $\mathbf{B}(\tilde{L}) \subset \tilde{Y}$. \square

3.3. Proof of Theorem 3.1

3.3.1. The outline of the proof

We obtain (i) easily from the construction of u_α . We will prove (ii) and (iii) in Section 3.3.2. Now we explain the outline proof of (iv). Fix a Kähler form ω on Y .

Define functions $\widehat{V}^{\mathbb{Q}}$ and $\widehat{V}_B^{\mathbb{Q}}$ on X by

$$\widehat{V}^{\mathbb{Q}} := \sup_{\substack{\alpha=(\alpha_1, \dots, \alpha_r) \\ \alpha \in \square_L \cap \mathbb{Q}^r}}^* u_{\alpha}$$

and

$$\widehat{V}_B^{\mathbb{Q}} := \sup_{\substack{\alpha=(\alpha_1, \dots, \alpha_r) \\ \alpha \in \square_L \cap \mathbb{Q}^r}}^* \left[\sum_{\lambda=1}^{r+1} \alpha_{\lambda} \log |s_{\lambda}|_{h_{\lambda}}^2 + \pi^* V_{h_1^{\alpha_1} h_2^{\alpha_2} \dots h_{r+1}^{\alpha_{r+1}}, B} \right],$$

where $V_{h_1^{\alpha_1} h_2^{\alpha_2} \dots h_{r+1}^{\alpha_{r+1}}, B}$ is a function on Y defined as in Example 2.5 with respect to the volume form ω^n on Y . In Section 3.3.4, we prove that $V_{\theta_L, B} \leq \widehat{V}_B^{\mathbb{Q}} + C$ holds for some constant C , where $V_{\theta_L, B}$ is also defined as in Example 2.5 (we specify the volume form on X later in Section 3.3.3). Then we have that, for some $C' \geq 0$,

$$V_{\theta_L, B} \leq \widehat{V}_B^{\mathbb{Q}} + C \leq \widehat{V}^{\mathbb{Q}} + C' \leq \widehat{V} + C'.$$

Here, the second inequality follows from the equation (2.1) before the proof of Proposition 2.6.

3.3.2. Proof of Theorem 3.1(ii) and (iii)

In this subsection, we will show the upper semicontinuity of \widehat{V} . For simplicity of notation, we write V_{α} instead of $V_{\theta_{\alpha}} = \sup\{\psi \in PSH(Y, \theta_{\alpha}) \mid \psi \geq 0\}$. We will show the following proposition.

PROPOSITION 3.4. — *The function $F: \square_L \times Y \rightarrow \mathbb{R} \cup \{\infty\}: F(\alpha, y) := V_{\alpha}(y)$ is upper semi-continuous.*

From this proposition and compactness of \square_L , a standard argument shows (ii) and (iii) of Theorem 3.1. In this subsection, we write $\alpha \leq \beta$ when $\alpha_{\lambda} \leq \beta_{\lambda}$ for every λ . To prove Proposition 3.4, we need the following lemma.

LEMMA 3.5. — *Let α and β be points in \square_L .*

- (1) *If $\alpha \leq \beta$, $\frac{V_{\alpha}}{1-|\alpha|} \leq \frac{V_{\beta}}{1-|\beta|}$.*
- (2) *$\lim_{\beta \downarrow \alpha} \frac{V_{\beta}}{1-|\beta|} = \frac{V_{\alpha}}{1-|\alpha|}$, where $\lim_{\beta \downarrow \alpha}$ means the limit as β approaches to α under the condition $\alpha \leq \beta$.*

Proof.

(1). — Consider the local weights $\varphi_{\alpha} := \sum_{\lambda=1}^r \alpha_{\lambda} \varphi_{\lambda} + (1-|\alpha|)\varphi_{r+1}$. First, we use the equation

$$\varphi_{\beta} + \frac{1-|\beta|}{1-|\alpha|} \cdot V_{\alpha} = \frac{1-|\beta|}{1-|\alpha|} \cdot (\varphi_{\alpha} + V_{\alpha}) + \sum_{\lambda=1}^r \left(\beta_{\lambda} - \frac{1-|\beta|}{1-|\alpha|} \cdot \alpha_{\lambda} \right) \cdot \varphi_{\lambda}.$$

As the right-hand side is plurisubharmonic, we have that $\frac{1-|\beta|}{1-|\alpha|} \cdot V_\alpha$ is θ_β -plurisubharmonic. Since this function is non-positive, we obtain

$$\frac{1-|\beta|}{1-|\alpha|} \cdot V_\alpha \leq V_\beta$$

by the definition of V_β .

(2). — Take a sequence $\{\beta^{(\nu)}\}_{\nu=1}^\infty$, $\beta^{(\nu)} = (\beta_1^{(\nu)}, \beta_2^{(\nu)}, \dots, \beta_r^{(\nu)})$, with $\beta_\lambda^{(\nu)} \downarrow \alpha_\lambda$ for every $\lambda = 1, 2, \dots, r$. We shall prove

$$\lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|} = \frac{V_\alpha}{1-|\alpha|}.$$

By (1), the inequality $\lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|} = \frac{V_\alpha}{1-|\alpha|}$ holds. Hence it is sufficient to prove the converse inequality.

Let us consider the local weight

$$\frac{1}{1-|\beta^{(\nu)}|} \cdot (\varphi_{\beta^{(\nu)}} + V_{\beta^{(\nu)}}) = \varphi_{r+1} + \sum_{\lambda=1}^r \frac{\beta_\lambda^{(\nu)}}{1-|\beta^{(\nu)}|} \cdot \varphi_\lambda + \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|}.$$

By the right-hand side, this weight is clearly decreasing in ν . Moreover, by focusing on the left-hand side, we have that this weight is plurisubharmonic. Therefore the limit $\varphi_\alpha/(1-|\alpha|) + \lim_{\nu \rightarrow \infty} V_{\beta^{(\nu)}}/(1-|\beta^{(\nu)}|)$ is also plurisubharmonic. As the function $(1-|\alpha|) \cdot \lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|}$ is non-positive and θ_α -plurisubharmonic, we have that

$$(1-|\alpha|) \cdot \lim_{\nu \rightarrow \infty} \frac{V_{\beta^{(\nu)}}}{1-|\beta^{(\nu)}|} \leq V_\alpha. \quad \square$$

Proof of Proposition 3.4. — Fix a point $(\alpha^0, y^0) \in \square_L \times Y$. We shall prove the upper semicontinuity of F at (α^0, y^0) .

First, we treat the case where $|\alpha^0| = 1$. It is sufficient to prove

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} V_\alpha(y) \leq V_{\alpha^0}(y^0).$$

Since the forms $\theta_1, \theta_2, \dots, \theta_r$ are positive, θ_{α^0} is also positive. Thus $V_{\alpha^0}(y^0) = 0$ and upper semicontinuity is trivial in this case.

Next, we treat the case when $|\alpha^0| < 1$. In this case, there exists $\varepsilon > 0$ such that $\alpha^0 + \varepsilon := (\alpha_1^0 + \varepsilon, \alpha_2^0 + \varepsilon, \dots, \alpha_r + \varepsilon)$ lies in the interior of \square_L . Take a sufficiently small neighborhood $U_{\alpha^0, \varepsilon}$ in \square_L of α^0 such that every point α in $U_{\alpha^0, \varepsilon}$ satisfies $\alpha \leq \alpha^0 + \varepsilon$. It follows from Lemma 3.5(1) that

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1-|\alpha|} = \limsup_{U_{\alpha^0, \varepsilon} \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1-|\alpha|} \leq \limsup_{Y \ni y \rightarrow y^0} \frac{V_{\alpha^0 + \varepsilon}(y)}{1-|\alpha^0 + \varepsilon|}$$

holds. By the upper semicontinuity of $\frac{V_{\alpha^0 + \varepsilon}}{1-|\alpha^0 + \varepsilon|}$, we have

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1-|\alpha|} \leq \frac{V_{\alpha^0 + \varepsilon}(y^0)}{1-|\alpha^0 + \varepsilon|}.$$

Letting $\epsilon \rightarrow 0$, we have

$$\limsup_{\square_L \times Y \ni (\alpha, y) \rightarrow (\alpha^0, y^0)} \frac{V_\alpha(y)}{1 - |\alpha|} \leq \frac{V_{\alpha^0}(y^0)}{1 - |\alpha^0|},$$

which shows the upper semicontinuity of the function $F(\alpha, y)/(1 - |\alpha|)$ near (α^0, y^0) . By multiplying by the continuous function $(\alpha, y) \mapsto 1 - |\alpha|$, we have that F itself is also upper semicontinuous. \square

3.3.3. Integral formula

In the following, we consider the local coordinates on $U_j^{(r+1)}$ defined as in §2. Recall that we defined homogeneous fiber coordinates $[x_{j,1} : x_{j,2} : \cdots : x_{j,r+1}]$ on $U_j^{(r+1)}$. We define the fiber coordinate z_1, \dots, z_r on $U_j^{(r+1)}$ as $z_\lambda := x_{j,\lambda}/x_{j,r+1}$. Then, we have $s_\lambda = z_\lambda$ ($1 \leq \lambda \leq r$) and $s_{r+1} = 1$. We rewrite $\widehat{V}_B^{\mathbb{Q}}$ as follows:

$$\widehat{V}_B^{\mathbb{Q}} = \sup_{\substack{\ell_\lambda, m \in \mathbb{Z} \\ \ell/m \in \square_L}} \left[\frac{2\ell_1}{m} \log |z_1| + \cdots + \frac{2\ell_r}{m} \log |z_r| + (\varphi_{\ell/m})_B(y) \right],$$

where $\varphi_{\ell/m}$ is defined by $(\ell_1/m)\varphi_1 + (\ell_2/m)\varphi_2 + \cdots + (\ell_r/m)\varphi_r + (1 - (\ell_1 + \ell_2 + \cdots + \ell_r)/m)\varphi_{r+1}$ and ϕ_B denotes a local weight corresponding to a function $V_{\phi,B}$. Here we regard z_λ as a holomorphic function on $U_j^{(r+1)}$.

Define a volume form dV on X by setting

$$dV := \frac{\pi^* \omega^n}{n!} \wedge \frac{e^{\varphi_1 + \cdots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \cdots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_r \wedge d\bar{z}_r$$

on $U_j^{(r+1)}$. Here, ω is a fixed Kähler form on Y as in Section 3.3.1. A simple calculation shows that this form extends to a smooth volume form on X .

In the proof of Theorem 3.1, we need the following integral formula. Here we integrate a function $F = |z_1|^{2t_1} \cdots |z_r|^{2t_r} e^{-\varphi_L} = \frac{|z_1|^{2t_1} \cdots |z_r|^{2t_r}}{|z_1|^2 e^{\varphi_1} + \cdots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}}}$ by using the volume form dV .

LEMMA 3.6. —

$$\begin{aligned} \int_{(z_1, \dots, z_r) \in \mathbb{C}^r} \frac{|z_1|^{2t_1} \cdots |z_r|^{2t_r} \cdot e^{\varphi_1 + \cdots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \cdots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+2}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_r \wedge d\bar{z}_r \\ = (2\pi)^r \frac{\Gamma(1+t_1) \cdots \Gamma(1+t_r) \Gamma(2 - (t_1 + \cdots + t_r))}{\Gamma(r+2) e^{\varphi_t}}. \end{aligned}$$

Proof. — To make ideas clear, we first prove the case that $r = 2$. In this case, the equation we want to prove is as follows:

$$\begin{aligned} \int_{(z_1, z_2) \in \mathbb{C}^2} \frac{|z_1|^{2t_1} |z_2|^{2t_2} \cdot e^{\phi_1 + \phi_2 + \phi_3}}{(|z_1|^2 e^{\phi_1} + |z_2|^2 e^{\phi_2} + e^{\phi_3})^4} (\sqrt{-1})^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ = (2\pi)^2 \frac{\Gamma(1+t_1) \Gamma(1+t_2) \Gamma(2 - (t_1 + t_2))}{\Gamma(4) e^{\phi_t}}, \end{aligned}$$

where $\phi_t := t_1\phi_1 + t_2\phi_2 + (1 - (t_1 + t_2))\phi_3$. Write $z_1 = a_1e^{i\theta_1}$ and $z_2 = a_2e^{i\theta_2}$ and define A_λ by $A_\lambda := e^{\phi_\lambda}$ ($\lambda = 1, 2, 3$). Then we have

$$\begin{aligned} \int_{\mathbb{C}^2} \frac{|z_1|^{2t_1}|z_2|^{2t_2} \cdot A_1A_2A_3}{(|z_1|^2A_1 + |z_2|^2A_2 + A_3)^4} (\sqrt{-1})^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ = (2\pi)^2 2^2 \int_{\mathbb{R}_{>0}^2} \frac{a_1^{1+2t_1} a_2^{1+2t_2} A_1 A_2 A_3}{(a_1^2 A_1 + a_2^2 A_2 + A_3)^4} da_1 da_2. \end{aligned}$$

Next, we use the following polar coordinates (s, θ) :

$$\begin{aligned} s &:= \left(\frac{A_1}{A_3} a_1^2 + \frac{A_2}{A_3} a_2^2 \right)^{1/2} \in \mathbb{R}_{\geq 0}, \\ \theta &:= \operatorname{Arctan} \left(\frac{\sqrt{A_2} a_2}{\sqrt{A_1} a_1} \right) \in [0, \pi/2]. \end{aligned}$$

Note that these coordinates are written as $a_1 = \sqrt{A_3/A_1} \cdot s \cos \theta$ and $a_2 = \sqrt{A_3/A_2} \cdot s \sin \theta$. Then we have

$$\begin{aligned} \int_{\mathbb{R}_{>0}^2} \frac{a_1^{1+2t_1} a_2^{1+2t_2} A_1 A_2 A_3}{(a_1^2 A_1 + a_2^2 A_2 + A_3)^4} da_1 da_2 \\ = A_1^{-t_1} A_2^{-t_2} A_3^{t_1+t_2-1} \int_{s=0}^{\infty} \frac{s^{3+2t_1+2t_2}}{(s^2+1)^4} ds \int_{\theta=0}^{\pi/2} (\cos \theta)^{1+2t_1} (\sin \theta)^{1+2t_2} d\theta. \end{aligned}$$

We denote this value by I . We will calculate the integration in s and θ respectively.

First we consider the integration in s . To compute, we use the substitution $\sigma = s^2$. Then,

$$\int_{s=0}^{\infty} \frac{s^{3+2t_1+2t_2}}{(s^2+1)^4} ds = \frac{1}{2} \int_{\sigma=0}^{\infty} \frac{\sigma^{1+t_1+t_2}}{(\sigma+1)^4} d\sigma = \frac{1}{2} B(2+t_1+t_2, 2-t_1-t_2).$$

At the last equality, we use the formula [16, 5.12.3] for the beta function.

Next, we consider the integration in θ . By the formula [16, 5.12.2], we have

$$\int_{\theta=0}^{\pi/2} (\cos \theta)^{1+2t_1} (\sin \theta)^{1+2t_2} d\theta = \frac{1}{2} B(1+t_1, 1+t_2).$$

By [16, 5.12.1], we have

$$(2\pi)^2 2^2 I = (2\pi)^2 A_1^{-t_1} A_2^{-t_2} A_3^{t_1+t_2-1} \frac{\Gamma(2-t_1-t_2)\Gamma(1+t_1)\Gamma(1+t_2)}{\Gamma(4)}.$$

The proof in the case that $r = 2$ is complete.

Now we will prove the theorem in the general case. Since the proof is almost the same, we only explain the essential points. We use the coordinate change $z_\lambda = a_\lambda e^{i\theta_\lambda}$ and get the expression of a_1, \dots, a_r . Then we use the r -dimensional

polar coordinate

$$\begin{aligned}
 a_1 &= \sqrt{A_{r+1}/A_1} \cdot s \cos \theta_1 \\
 a_2 &= \sqrt{A_{r+1}/A_2} \cdot s \sin \theta_1 \cos \theta_2 \\
 a_3 &= \sqrt{A_{r+1}/A_3} \cdot s \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
 &\vdots \\
 a_{r-1} &= \sqrt{A_{r+1}/A_{r-1}} \cdot s \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{r-2} \cos \theta_{r-1} \\
 a_r &= \sqrt{A_{r+1}/A_r} \cdot s \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{r-2} \sin \theta_{r-1},
 \end{aligned}$$

where $A_\lambda = e^{\varphi_\lambda}$. The determinant of the Jacobian matrix is written as

$$\sqrt{\frac{A_{r+1}}{A_1} \dots \frac{A_{r+1}}{A_r}} s^{r-1} (\sin \theta_1)^{r-2} (\sin \theta_2)^{r-3} \dots (\sin \theta_{r-2})^1.$$

Finally we use formulae of the beta function to deduce the conclusion. \square

3.3.4. Proof of Theorem 3.1 (iv)

As we described in §3.3.1, we shall prove $V_{\theta_L, B} \leq \widehat{V}_B^{\mathbb{Q}} + C$. Fix a global section $F \in H^0(X, L^m)$ with

$$\int_X |F|_{h_L^m}^2 dV = 1.$$

We will decompose F into orthogonal components using the following claim.

CLAIM 3.7. — *The direct decomposition of $H^0(X, L^m)$ induced by the isomorphisms $H^0(X, L^m) = H^0(X, \mathcal{O}_X(m)) = H^0(Y, S^m(M_1 \otimes \dots \otimes M_{r+1})) \cong \bigoplus_{\ell_1 + \dots + \ell_{r+1} = m} H^0(Y, M_1^{\ell_1} \otimes \dots \otimes M_{r+1}^{\ell_{r+1}})$ is the orthogonal decomposition with respect to the L^2 -norm defined by Hermitian metric h_L^m of L^m and the volume form dV . Here, $S^m E$ denotes the m -th symmetric tensor of E .*

Proof. — By the decomposition above, we have injective morphisms $H^0(Y, M_1^{\ell_1} \otimes \dots \otimes M_{r+1}^{\ell_{r+1}}) \rightarrow H^0(X, L^m)$ for every $(r+1)$ -tuple of non-negative integers $\ell = (\ell_1, \dots, \ell_{r+1})$ with $\ell_1 + \dots + \ell_{r+1} = m$. In the following, $M_1^{\ell_1} \otimes \dots \otimes M_{r+1}^{\ell_{r+1}}$ will be denoted by M^ℓ . This morphism maps $f_\ell \in H^0(Y, M^\ell)$ to $s_1^{\ell_1} \cdot s_2^{\ell_2} \dots s_{r+1}^{\ell_{r+1}} \pi^* f_\ell$.

We will prove that, for any $\ell = (\ell_1, \dots, \ell_{r+1})$ and $\ell' = (\ell'_1, \dots, \ell'_{r+1})$ with $\ell \neq \ell'$, two sections $\beta := s_1^{\ell_1} \cdot s_2^{\ell_2} \dots s_{r+1}^{\ell_{r+1}} \pi^* f_\ell$ and $\beta' := s_1^{\ell'_1} \cdot s_2^{\ell'_2} \dots s_{r+1}^{\ell'_{r+1}} \pi^* f_{\ell'}$ are orthogonal. We regard β and β' as holomorphic functions via the local trivialization.

Then, by the equations $s_\lambda = z_\lambda$ ($\lambda = 1, 2, \dots, r$) and $s_{r+1} = 1$, it follows that

$$\begin{aligned} \int_X \langle \beta, \beta' \rangle_{h_L^m} dV &= \int_X (z_1^{\ell_1} \cdot z_2^{\ell_2} \dots z_r^{\ell_r} \pi^* f_\ell) \cdot \overline{(z_1^{\ell'_1} \cdot z_2^{\ell'_2} \dots z_r^{\ell'_r} \pi^* f_{\ell'})} e^{-m\varphi_L} dV \\ &= \int_{y \in Y} \left[\int_{z \in \pi^{-1}(y)} F(z) (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_r \wedge d\bar{z}_r \right] \frac{\omega^n}{n!}, \end{aligned}$$

where

$$F(z) = \frac{(z_1^{\ell_1} \dots z_r^{\ell_r} \pi^* f_\ell) \overline{(z_1^{\ell'_1} \dots z_r^{\ell'_r} \pi^* f_{\ell'})} e^{-m\varphi_L} e^{\varphi_1 + \dots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \dots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}}.$$

Write $z_\lambda = s_\lambda r^{i\theta_\lambda}$. Considering integration in θ_λ 's, we have that it becomes 0 if $\ell \neq \ell'$. Therefore, two sections are orthogonal for different ℓ and ℓ' . \square

Let us decompose $F \in H^0(X, L^m)$ into the sum of the components $\beta_\ell = s_1^{\ell_1} \cdot s_2^{\ell_2} \dots s_{r+1}^{\ell_{r+1}} \cdot \pi^* f_\ell$, where $f_\ell \in H^0(Y, M_1^{\ell_1} \otimes \dots \otimes M_{r+1}^{\ell_{r+1}})$, according to the orthogonal decomposition obtained in the claim. By the orthogonality, we have

$$\int_X |\beta_\ell|_{h_L^m}^2 dV \leq \int_X |F|_{h_L^m}^2 dV (= 1).$$

Next we estimate the norm of f_ℓ .

CLAIM 3.8. — *There exist constants C_1 and C_2 independent of m such that*

$$\int_{y \in Y} |f_\ell(y)|^2 e^{-m(\varphi_\ell/m)} \frac{\omega^n}{n!} \leq C_1 C_2^m \int_X |\beta_\ell|^2 e^{-m\varphi_L} dV,$$

where φ_ℓ/m stands for $\frac{\ell_1}{m}\varphi_1 + \frac{\ell_2}{m}\varphi_2 + \dots + \frac{\ell_r}{m}\varphi_r + (1 - \frac{\ell_1 + \dots + \ell_r}{m})\varphi_{r+1}$ as in Section 3.3.3.

Proof. — Let $z^\ell := z_1^{\ell_1} \cdot z_2^{\ell_2} \dots z_r^{\ell_r}$. Then we have $z^\ell \pi^* f_\ell = \beta_\ell$ under the trivialization. We estimate the right-hand side from below. We have that

$$\begin{aligned} \int_X |\beta_\ell|^2 e^{-m\varphi_L} dV &= \int_X |z^\ell f_\ell(x)|^2 e^{-m\varphi_L} dV \\ &= \int_{y \in Y} |f_\ell(x)|^2 \left[\int_{z \in \pi^{-1}(y)} |z^\ell|^2 e^{-m\varphi_L} dP \right] \frac{\omega^n}{n!}, \end{aligned}$$

where dP is the measure on the fiber $\pi^{-1}(y)$ defined as

$$dP := \frac{e^{\varphi_1 + \dots + \varphi_{r+1}}}{(|z_1|^2 e^{\varphi_1} + \dots + |z_r|^2 e^{\varphi_r} + e^{\varphi_{r+1}})^{r+1}} (\sqrt{-1})^r dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_r \wedge d\bar{z}_r.$$

By Hölder's inequality, we have that

$$\int_{z \in \pi^{-1}(y)} |z^\ell|^{2/m} \cdot e^{-\varphi_L} dP \leq \left[\int_{\pi^{-1}(y)} |z^\ell|^2 \cdot e^{-m\varphi_L} dP \right]^{\frac{1}{m}} \cdot \left[\int_{\pi^{-1}(y)} 1 \cdot dP \right]^{\frac{m-1}{m}}.$$

A straightforward computation similar to the proof of Lemma 3.6 shows that the value of the integral $\int_{\pi^{-1}(y)} dP$ in the right-hand side is a constant independent

of y , which we will denote by I . By Lemma 3.6, the integral in the left-hand side is equal to

$$(2\pi)^r \frac{\Gamma\left(1 + \frac{\ell_1}{m}\right) \cdots \Gamma\left(1 + \frac{\ell_r}{m}\right) \Gamma\left(2 - \left(\frac{\ell_1 + \cdots + \ell_r}{m}\right)\right)}{\Gamma(r+2)e^{\varphi_\ell}}.$$

As $\Gamma(t)$ is bounded for $1 \leq t \leq 2$ from below, there exists a positive constant $C > 0$ such that $\Gamma(t) \geq C$ for $1 \leq t \leq 2$. Combining these estimates, we have that

$$(2\pi)^r \frac{C^{r+1}e^{-\varphi_\ell}}{\Gamma(r+2)} \leq \left[\int_{z \in \pi^{-1}(y)} |z^\ell|^2 e^{-m\varphi_L} dP \right]^{\frac{1}{m}} \cdot I^{\frac{m-1}{m}}.$$

Thus we obtain

$$\begin{aligned} \int_X |\beta_\ell|^2 e^{-m\varphi_L} dV &= \int_{y \in Y} |f_\ell(y)|^2 \left[\int_{z \in \pi^{-1}(y)} |z^\ell|^2 e^{-m\varphi_L} dP \right] \frac{\omega^n}{n!} \\ &\geq \int_{y \in Y} |f_\ell(y)|^2 I^{-(m-1)} \left[(2\pi)^r \frac{C^{r+1}e^{-\varphi_\ell}}{\Gamma(r+2)} \right]^m \frac{\omega^n}{n!} \\ &= \frac{(2\pi)^{rm} C^{rm} I^{-(m-1)}}{\Gamma(r+2)^m} \int_{y \in Y} |f_\ell(y)|^2 e^{-m\varphi_\ell} \frac{\omega^n}{n!}. \quad \square \end{aligned}$$

By the previous estimates and the definition of the Bergman-type metric, we have

$$\frac{1}{m} \log |f_\ell|^2 \leq (\varphi_\ell)_B + \frac{\log C_1 + m \log C_2}{m}.$$

To prove the desired inequality $V_{\theta_L, B} \leq \widehat{V}_B^{\mathbb{Q}} + C$, we will estimate the norm of the section $F = \sum_\ell z^\ell \pi^* f_\ell$ from above. Assume $\ell = (\ell_1, \dots, \ell_r)$ satisfies $\ell/m \in \square_L$. By the argument as in the proof of [11, Proposition 2.5(2)], we have

$$\frac{2}{m} \log |F| \leq \max_\ell \left[\frac{2\ell_1}{m} \log |z_1| + \cdots + \frac{2\ell_r}{m} \log |z_r| + (\phi_\ell/m)_B \right] + \frac{1}{m} \log C_1 + \log C_2.$$

Let $C := \log C_1 + \log C_2$. Then the right-hand side is estimated by $\widehat{V}_B^{\mathbb{Q}} + C$ by definition. Since m is arbitrary, the supremum of the left-hand side over m and F is $V_{\theta_L, B}$, and the proof is complete.

4. Proof of the main results

In this section, we prove Theorem 1.4 and Theorem 1.3. Let X, Y , and L be those as in Theorem 1.3 (in particular, we assume that Y admits a holomorphic tubular neighborhood). The idea of the proof is based on [11]: we first construct a new ‘‘projective space bundle model’’ $(\widetilde{X}, \widetilde{Y}, \widetilde{L})$ from (X, Y, L) , and construct a metric of L with minimal singularities by using the metric on \widetilde{L} as in Section 3. See also Section 3.2 for the relation between the models (X, Y, L) and $(\widetilde{X}, \widetilde{Y}, \widetilde{L})$.

4.1. The projective space bundle model $(\tilde{X}, \tilde{Y}, \tilde{L})$

Let X, Y , and L be as in Theorem 1.3. Denote by \tilde{X} the total space of the projective space bundle $\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)$, by \tilde{Y} the subvariety $\mathbb{P}(\mathbb{I}_Y)$, and by \tilde{L} the line bundle $\mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1) \otimes \pi^* L|_Y$, where $\pi: \tilde{X} \rightarrow Y$ is the natural projection. Note that $\tilde{X} = \mathbf{P}(\mathbb{I}_Y \oplus N_{Y/X})$ and $\tilde{Y} = \mathbf{P}(\mathbb{I}_Y)$. It is easily observed that \tilde{X} is a compactification of the normal bundle $N_{Y/X}$, and from this point of view, \tilde{Y} is regarded as a zero section of $N_{Y/X}$. Therefore, by the assumption on the existence of a holomorphic tubular neighborhood, there exists a neighborhood V of Y in X, \tilde{V} of \tilde{Y} in \tilde{X} and a biholomorphic map $i: \tilde{V} \cong V$ with $i|_{\tilde{V}} = \pi|_{\tilde{V}}$.

PROPOSITION 4.1. — *If one choose V sufficiently small, one have that $i^* L \cong \tilde{L}$.*

The proof is based on [11, §3]. First we prove the following proposition as a higher codimensional analogue of [11, Proposition 3.1]:

PROPOSITION 4.2. — *By shrinking V suitably, one have the following:*

- (1) *(a version of Rossi's theorem) The natural map $H^1(V, \mathcal{O}_V/I_Y^n)$ is injective for some $n \geq 1$, where I_Y the defining ideal sheaf of $Y \subset V$.*
- (2) *If $H^1(Y, S^\ell N_{Y/X}^*)$ vanishes for every $\ell \geq 1$, then the groups $\text{Pic}^0(V)$ and $\text{Pic}^0(Y)$ are isomorphic.*

Proof. — See the proof of [11, Proposition 3.1] (We intrinsically use Rossi's theorem [17, Theorem 3]). Here we remark that, by Lemma 4.3 below, we may assume that V is a strongly pseudoconvex domain which has Y as a maximal compact set). \square

LEMMA 4.3. — *There exists a strongly pseudoconvex holomorphic tubular neighborhood V of Y which has Y as a maximal compact analytic set.*

Proof. — As Y admits a holomorphic tubular neighborhood, it is sufficient to show the lemma by assuming $X = \tilde{X}$. Take a C^∞ Hermitian metric $h_{N_\lambda^{-1}}$ on N_λ^{-1} with positive curvature. Taking a local coordinate y of Y and pulling it back by π , we regard $(z, y) = (z_1, z_2, \dots, z_r, y)$ as a local coordinates system of X , where z_λ is a fiber coordinate of N_λ . By considering the sublevel set of the function $\Phi: N_{Y/X} \rightarrow \mathbb{R}$ defined by

$$\Phi(z, y) := \sum_{\lambda=1}^r |z_\lambda|^2 e^{\varphi_\lambda(y)},$$

where φ_λ is the local weight of $h_{N_\lambda^{-1}}$, the lemma follows. \square

Proof of Proposition 4.1. — First note that

$$K_Y^{-1} \otimes S^\ell N_{Y/X}^* = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^r, |\alpha|=\ell} K_Y^{-1} \otimes \left(\bigotimes_{\lambda=1}^r N_\lambda^{-\alpha_\lambda} \right)$$

holds. As $K_Y^{-1} \otimes N_\lambda^{-1}$ and N_μ are positive for each $\lambda, \mu = 1, 2, \dots, r$, it follows from Kodaira's vanishing theorem that $H^1(Y, S^\ell N_{Y/X}^*)$ vanishes for each $\ell \geq 1$. Thus, by Proposition 4.2, it is sufficient to show that $\tilde{L}|_{\tilde{Y}} \cong \pi|_{\tilde{Y}}^* L$ holds. The line bundle $\mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1)$ corresponds to the divisor $\mathbb{P}(N_{Y/X}^*) \subset \mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)$, which does not intersect \tilde{Y} . Therefore we have $\tilde{L}|_{\tilde{Y}} = \mathcal{O}_{\mathbb{P}(\mathbb{I}_Y \oplus N_{Y/X}^*)}(1)|_{\tilde{Y}} \otimes \pi|_{\tilde{Y}}^* L = \pi|_{\tilde{Y}}^* L$. \square

4.2. Minimal singular metrics on L and \tilde{L}

Let X, Y , and L be those in Theorem 1.3, and let $\tilde{X}, \tilde{Y}, \tilde{L}, V$, and \tilde{V} be those in the previous subsection. Here we prove the following:

PROPOSITION 4.4. — *Let h be a metric of L with minimal singularities and \tilde{h} be a metric of \tilde{L} with minimal singularities. Then $h|_V \sim_{\text{sing}} (i^{-1})^* \tilde{h}|_{\tilde{V}}$ holds at each point in V .*

The proof of Proposition 4.4 is based on the ‘‘maximum construction technique’’ which is also used in the proof of [11, Theorem 1.2]. Fix a C^∞ metric h_∞ of L and denote by θ the curvature tensor Θ_{h_∞} . Set $\varphi_V := \sup\{\varphi \in PSH(V, \theta|_V) \mid \varphi \leq 0 \text{ on } V\}$ and $\varphi_X := \sup\{\varphi \in PSH(X, \theta) \mid \varphi \leq 0 \text{ on } X\}$. We first show the following:

LEMMA 4.5. — *It holds that $\varphi_V \sim_{\text{sing}} \varphi_X$. In particular, the restriction of a metric of L with minimal singularities to V has singularities equivalent to the metric $h_\infty|_V \cdot e^{-\varphi_V}$.*

Proof. — As the inequality $\varphi_X|_V \leq \varphi_V$ is easily obtained, all we have to do is to show the existence of a constant C with $\varphi_V \leq \varphi_X|_V + C$. As L is big and $\mathbf{B}(L) = Y$, we can take an integer $m \geq 1$ and sections $f_1, f_2, \dots, f_\ell \in H^0(X, L^m)$ such that the common zero of these sections is Y [14, 2.1.21]. Denote by h_a the Bergman type metric on L constructed from f_1, f_2, \dots, f_ℓ (see Example 2.4). Define the function φ_a by $h_a = h_\infty \cdot e^{-\varphi_a}$. We may assume that $\varphi_a \leq 0$ holds on V . Fix a relatively compact open neighborhood $V_0 \Subset V$ of Y and set $C_1 := -\inf_{V \setminus V_0} \varphi_a$. Consider a θ -plurisubharmonic function $\hat{\varphi} := \max\{\varphi_V - C_1, \varphi_a\}$. As $\varphi_V - C_1 \leq -C_1 \leq \varphi_a$ holds on $V \setminus V_0$, we have $\hat{\varphi} = \varphi_a$ on each point in $V \setminus V_0$. Thus we can extend $\hat{\varphi}$ to whole X by defining $\hat{\varphi}(x) := \varphi_a(x)$ for each $x \in X \setminus V_0$. It is clear from the construction that $\hat{\varphi} \in PSH(X, \theta)$. Set $C_2 := \max_X \hat{\varphi}$. Then, as $\hat{\varphi} - C_2 \leq 0$, we obtain that $\hat{\varphi} - C_2 \leq \varphi_X$. Therefore it holds that $\varphi_V - C_1 - C_2 \leq \hat{\varphi} - C_2 \leq \varphi_X$. \square

Proof of Proposition 4.4. — Take a C^∞ Hermitian metric h_∞ on L and \tilde{h}_∞ on \tilde{L} with $h_\infty|_V = i^* \tilde{h}_\infty$ (here we used Proposition 4.1). By Lemma 4.5, it follows that $\varphi_V \sim_{\text{sing}} \varphi_X$, where φ_V and φ_X are as above. Set $\varphi_{\tilde{V}} := \sup\{\varphi \in PSH(\tilde{V}, \Theta_{\tilde{h}_\infty}|_{\tilde{V}}) \mid \varphi \leq 0 \text{ on } \tilde{V}\}$ and $\varphi_{\tilde{X}} := \sup\{\varphi \in PSH(\tilde{X}, \Theta_{\tilde{h}_\infty}) \mid \varphi \leq 0 \text{ on } \tilde{X}\}$. By the arguments in Section 3.2, we can apply Lemma 4.5 also to the projective space bundle model (\tilde{X}, \tilde{L}) to obtain that $\varphi_{\tilde{V}} \sim_{\text{sing}} \varphi_{\tilde{X}}$. As $h_\infty|_V = i^* \tilde{h}_\infty$, we have that $\varphi_V = i^* \varphi_{\tilde{V}}$. Therefore it follows that $\varphi_X \sim_{\text{sing}} i^* \varphi_{\tilde{X}}$. \square

4.3. Proof of Theorem 1.3

Theorem 1.3 follows from Theorem 3.1 and Proposition 4.4. □

5. A sufficient condition for the existence of a holomorphic tubular neighborhood and Proof of Theorem 1.4

Let X be a complex manifold and let $Y \subset X$ be a compact complex submanifold of codimension r . In this section, we investigate when Y admits a holomorphic tubular neighborhood V in X . In particular, we here study a higher codimensional analogue of Grauert's theorem ([8], the case of $r = 1$, see also Theorem 5.4 below).

5.1. A higher codimensional analogue of Grauert's theorem

In this subsection, we show the following:

PROPOSITION 5.1. — *Assume that $N_{Y/X}$ admits a direct decomposition $N_{Y/X} = N_1 \oplus N_2 \oplus \cdots \oplus N_r$ into r negative line bundles. Assume also that $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ and $H^1(E, T_E \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ hold for every $\nu \geq 1$, where E is the total space of the projective space bundle $\mathbf{P}(N_{Y/X})$. Then Y admits a holomorphic tubular neighborhood.*

Note that Proposition 5.1 is the Grauert's theorem when $r = 1$. We will consider a blow-up $p: W \rightarrow X$ of X along Y and apply the Grauert's theorem to $E \subset W$ to show this proposition, where we are regarding E as the exceptional divisor (see [5, Proposition 12.4]). For this purpose, we first show the following:

LEMMA 5.2. — *Assume that E admits a holomorphic tubular neighborhood in W . Then Y also admits a holomorphic tubular neighborhood in X .*

Proof. — Denote by \tilde{Y} the zero section of $\pi: N_{Y/X} \rightarrow X$. Take a neighborhood \tilde{V} of \tilde{Y} . We denote by \tilde{W} the blow-up $\tilde{p}: \tilde{W} \rightarrow \tilde{V}$ of \tilde{V} along \tilde{Y} and by $\tilde{E} \subset \tilde{W}$ the exceptional set. By the assumption (and by shrinking X if necessary), we may assume that there exists a biholomorphic map $F: \tilde{W} \rightarrow W$ with $F|_{\tilde{E}} = q|_{\tilde{E}}$, where $q: \mathcal{O}_{\mathbf{P}(N_{Y/X})}(-1) \rightarrow E (= \mathbf{P}(N_{Y/X}))$ is the natural projection (here we are regarding \tilde{W} as a neighborhood of the zero section E of the line bundle $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(-1)$, see also [5, Proposition 12.4]).

First, let us construct a holomorphic map $g: V \rightarrow Y$ with $g|_Y = \text{id}_Y$ such that the diagram

$$\begin{array}{ccccc}
 \tilde{W} & \xrightarrow{F} & W & \xrightarrow{p} & V \\
 \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow g \\
 \tilde{E} & \xrightarrow{q|_{\tilde{E}}} & E & \longrightarrow & Y
 \end{array}$$

is commutative. It is clear that such function g is uniquely determined in the set-theoretic sense. It also follows from a standard argument that this g is a continuous map. It is also clear that $g|_{V \setminus Y}$ is biholomorphic and that $g|_Y = \text{id}_Y$. Thus the existence of the holomorphic map g follows from Riemann's extension theorem (see Lemma 5.3 below).

Next we show that the biholomorphic map $f: \widetilde{V} \setminus \widetilde{Y} \cong V \setminus Y$ induced by $F|_{\widetilde{W} \setminus \widetilde{E}}$ extends to the biholomorphism $\widehat{f}: \widetilde{V} \cong V$ with $\widehat{f}|_{\widetilde{Y}} = \pi|_{\widetilde{Y}}$. Note that, by construction, the fibration structures $\pi|_{\widetilde{Y}}: \widetilde{V} \rightarrow \widetilde{Y}$ and $g: V \rightarrow Y$ are preserved by f . Therefore, by a simple topological argument, it follows that there uniquely exists a continuous map $\widehat{f}: \widetilde{V} \rightarrow V$ with $\widehat{f}|_{\widetilde{V} \setminus \widetilde{Y}} = f$, and that this \widehat{f} satisfies $\widehat{f}|_{\widetilde{Y}} = \pi|_{\widetilde{Y}}$. The regularity of \widehat{f} and \widehat{f}^{-1} is shown again by Lemma 5.3 below. \square

LEMMA 5.3. — *Let M and N be complex manifolds, let $Z \subset M$ be a submanifold with codimension greater than or equal to 1, and let $h: M \rightarrow N$ be a continuous map. Assume that $h|_{M \setminus Z}$ is holomorphic. Then h is holomorphic on M .*

Proof. — Take a point $z \in Z$ and an open ball $U' \subset N$ with coordinate system $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ around $h(z)$ ($n := \dim N$). We may assume that $U' = \{|\eta| < \varepsilon\}$ for some $\varepsilon > 0$. Take a sufficiently small neighborhood $U \subset M$ of z so that $U \subset h^{-1}(U')$ (and thus $h(U) \subset U'$). In what follows, we show the lemma by replacing M with U , N with U' , and h with $h|_U$ (in particular, we are regarding N as an open ball of \mathbb{C}^n). It is sufficient to show each function h_λ is holomorphic on z , where $h = (h_1, h_2, \dots, h_n)$ is the decomposition by the coordinate system $\eta = (\eta_1, \eta_2, \dots, \eta_n)$. As h_λ is continuous (and thus it is locally bounded), we may assume that the L^2 -norm of $h_\lambda|_{U \setminus Z}$ is bounded by shrinking U if necessary. Therefore it follows from Riemann's extension theorem that we can extend $h_\lambda|_{U \setminus Z}$ to a holomorphic function $\widehat{h}_\lambda: U \rightarrow \mathbb{C}$. Since $U \setminus Z \subset U$ is a dense subset, we conclude that $h_\lambda = \widehat{h}_\lambda$, which proves the lemma. \square

By Lemma 5.2, all we have to do is to investigate when E admits a holomorphic tubular neighborhood in W . We apply the following Grauert's theorem to this problem:

THEOREM 5.4 ([8, Satz 7, p. 363], see also [3, Theorem 4.4]). — *Let M be a complex manifold and let $Z \subset M$ be a strongly exceptional subvariety of pure codimension 1. Assume that $H^1(Z, N_{Z/M}^{-\nu}) = 0$ and $H^1(Z, T_Z \otimes N_{Z/M}^{-\nu}) = 0$ hold for every $\nu \geq 1$. Then Z has a holomorphic tubular neighborhood in M .*

Proof of Proposition 5.1. — By the assumption, Lemma 5.2 and Theorem 5.4, all we have to do is to show that $E \subset W$ is an exceptional subset (in the sense of Grauert). By [13, Theorem 4.9, 6.12], [8, Satz 8, p. 353], and [9, Lemma 11] (see also [3, Theorem 3.6]), it is sufficient to see the following two conditions:

- (a) $N_{E/W}$ is negative, and
- (b) $\mathcal{O}_W/I_E^2 \cong \mathcal{O}_{\widetilde{W}}/I_{\widetilde{E}^2}$, where $I_E \subset \mathcal{O}_W$ and $I_{\widetilde{E}} \subset \mathcal{O}_{\widetilde{W}}$ are the defining ideal sheaves of E and \widetilde{E} , respectively.

(a) follows from $N_{E/W}^{-1} = \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) = \mathcal{O}_{\mathbb{P}(N_1^{-1} \oplus N_2^{-1} \oplus \dots \oplus N_r^{-1})}(1)$ and the assumption that N_λ is negative. (b) follows from the condition $H^1(E, T_E \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1)) = 0$ (see [3, Proposition 1.10, 1.11]). \square

5.2. A sufficient condition for the existence of a holomorphic tubular neighborhood

In this subsection, we show the following lemma as an application of Proposition 5.1:

LEMMA 5.5. — *Let X be a complex manifold and let Y be a compact complex submanifold. Assume that $N_{Y/X}$ admits a direct decomposition $N_{Y/X} = N_1 \oplus N_2 \oplus \dots \oplus N_r$ into r negative line bundles. Assume also the following three conditions:*

- (i) $N_\lambda \cong N_\mu$ for each λ and μ ,
- (ii) $N_\lambda^{-1} \otimes K_Y^{-1} \otimes T_Y$ is Nakano positive, and
- (iii) $N_\lambda^{-1} \otimes K_Y^{-1}$ is ample for each λ .

Then Y admits a holomorphic tubular neighborhood in X .

Note that, when T_Y is holomorphically trivial, conditions (ii) and (iii) are automatically satisfied.

Proof. — By Proposition 5.1, it is sufficient to show $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ and $H^1(E, T_E \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$ for every $\nu \geq 1$, where $E := \mathbf{P}(N_{Y/X})$. Note that it follows from condition (i) that $E \cong Y \times \mathbf{P}^r$. By the relative Euler sequence

$$0 \rightarrow \mathbb{I}E \rightarrow p|_E^* N_{Y/X} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) \rightarrow T_{E/Y} \rightarrow 0,$$

it turns out that it is sufficient to show the following four vanishing assertions: $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$, $H^2(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu)) = 0$, $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu + 1) \otimes p|_E^* N_\lambda) = 0$, and $H^1(E, \mathcal{O}_{\mathbf{P}(N_{Y/X})}(\nu) \otimes p|_E^* T_Y) = 0$ for each $\nu \geq 1$. By Nakano's vanishing theorem, the problem is reduced to show Nakano positivity for the following three vector bundles: $K_E^{-1} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1)$, $K_E^{-1} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(2) \otimes p|_E^* N_\lambda$, and $K_E^{-1} \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) \otimes p|_E^* T_Y$. As

$$K_E^{-1} \cong p|_E^*(N^r \otimes K_Y^{-1}) \otimes \mathcal{O}_{\mathbf{P}(N_{Y/X})}(r)$$

holds ($N := N_1 \cong N_2 \cong \dots \cong N_r$), these three bundles are written as $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(N^r \otimes K_Y^{-1})$, $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+2) \otimes p|_E^*(N^{r+1} \otimes K_Y^{-1})$, and $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(T_Y \otimes N^r \otimes K_Y^{-1})$, respectively. In what follows, we show the Nakano positivity for these three bundles.

First let us note that $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(1) \otimes p|_E^* N = \text{Pr}_1^* \mathcal{O}_{\mathbf{P}^r}(1)$, where Pr_1 is the first projection $E \cong \mathbf{P}^r \times Y \rightarrow \mathbf{P}^r$. Let us denote by h the metric on this line bundle which is the pull-back of the Fubini–Study metric by Pr_1 . By tensoring h and a metric on $N^{-1} \otimes K_Y^{-1}$ with positive curvature, we have the positivity for the

bundles $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(N^r \otimes K_Y^{-1}) = \mathrm{Pr}_1^* \mathcal{O}_{\mathbf{P}^r}(r+1) \otimes p|_E^*(N^{-1} \otimes K_Y^{-1})$ and $\mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+2) \otimes p|_E^*(N^{r+1} \otimes K_Y^{-1}) = \mathrm{Pr}_1^* \mathcal{O}_{\mathbf{P}^r}(r+2) \otimes p|_E^*(N^{-1} \otimes K_Y^{-1})$.

Finally we show the Nakano positivity for $F := \mathcal{O}_{\mathbf{P}(N_{Y/X})}(r+1) \otimes p|_E^*(T_Y \otimes N^r \otimes K_Y^{-1}) = \mathcal{O}_{\mathbf{P}(\mathbb{P}_Y)}(r+1) \otimes p|_E^*(T_Y \otimes N^{-1} \otimes K_Y^{-1})$. Take a metric h' on $T_Y \otimes N^{-1} \otimes K_Y^{-1}$ with Nakano positive curvature. Then $h^{r+1} \otimes p|_E^* h'$ is a metric on F , with curvature $(r+1)\Theta_h \otimes \mathrm{Id}_F + p|_E^* \Theta_{h'}$, which is easily seen to be Nakano positive. \square

5.3. Proof of Theorem 1.4

By Lemma 5.5, Y admits a holomorphic tubular neighborhood in X .

By Theorem 1.3, there exists a metric $h_{\min,L}$ with minimal singularities whose local weight $\varphi_{\min,L}$ is written in the form

$$\varphi_{\min,L}(z, y) = \log \max_{\alpha \in \square_L} |z^\alpha|^2 e^{(\varphi_\alpha)_e(y)} + O(1).$$

By choosing metrics $e^{\varphi_\alpha(y)}$ as in [10, §2.2], we obtain that $(\varphi_\alpha)_e(y) = \varphi_\alpha(y)$ holds and $\varphi_\alpha(y)$ depends continuously on (y, α) , which proves the assertion. \square

6. Examples

6.1. Nakayama's example

Nakayama's example (X, L, Y) is the example which admits no Zariski decomposition even after modifications [15, IV, §2.6] (see also [10, §1]). In this example, the manifold X is a total space of the projective space bundle $\pi: X := \mathbb{P}(M_1 \oplus M_2 \oplus M_3) \rightarrow Y$ over an abelian surface Y , where M_1 and M_2 are ample line bundles on Y and M_3 is a line bundle on Y . The line bundle L is the inverse of the tautological line bundle: i.e. $L := \mathcal{O}_{\mathbb{P}(M_1 \oplus M_2 \oplus M_3)}(1)$. Then the stable base locus of L is the subset $\mathbb{P}(M_3) \subset X$, which we are regarding as Y here. As it is clearly observed, this example (X, L, Y) is a special case of those in §3. Therefore we can apply Theorem 3.1 to this example. In particular, by using the metrics as in the proof of Theorem 1.4, we can reprove the main result in [10].

6.2. Zariski's example and its higher (co-)dimensional analogues and proof of Theorem 1.1

In [11, §4.2], the second-named author applied its main result (=Theorem 1.4, Theorem 1.3 of this paper for $r = 1$) to Zariski's and Mumford's example (X, L, Y) , in which L is nef and big however not semi-ample, and showed the semi-positivity of

L (i.e. the existence of a C^∞ Hermitian metric on L with semi-positive curvature). Here we construct an example which can be regarded as a higher-codimensional analogue of Zariski's example and apply Theorem 1.4 to it. In what follows, we only consider the case of $r = 2$ for simplicity.

Take two general quadric surfaces Q_1 and Q_2 in \mathbb{P}^3 . Then we may assume that the intersection $C := Q_1 \cap Q_2$ is a smooth elliptic curve and Q_1 and Q_2 intersects transversally along C . Fix N points p_1, p_2, \dots, p_N in C . Denote by $\pi: X := \text{Bl}_{\{p_1, p_2, \dots, p_N\}} \mathbb{P}^3 \rightarrow \mathbb{P}^3$ the blow-up of \mathbb{P}^3 at these N points, by Y the strict transform of C , by D_1 and D_2 the strict transform of Q_1 and Q_2 , respectively, by E_λ the exceptional divisor $\pi^{-1}(p_\lambda)$ for each λ , by E the divisor $\sum_{\lambda=1}^N E_\lambda$, and by H the pull-back $\pi^* \mathcal{O}_{\mathbb{P}^3}(1)$. Note that $D_\lambda \in |2H - E|$.

Let us consider the line bundle $L := \mathcal{O}_X(H + D_1) = \mathcal{O}_X(3H - E)$ on X . As H is big and D_1 is effective, L is also big. It is also observed that $\text{Bs}|L| \subset Y$ holds, since H is base point free and $\text{Bs}|H| \subset Y$ by construction. As a simple computation shows that the intersection number $(L.Y)$ is equal to $12 - N$, we conclude that L is nef if and only if $12 \geq N$.

First let us consider the case of $N = 12$. In this case, we may assume that $L|_Y$ is a general (and thus non-torsion) element of $\text{Pic}^0(Y)$ by choosing p_1, p_2, \dots, p_{12} generically. Then it is easily observed that $\mathbf{B}(L) = Y$ holds, and therefore that L is not semi-ample, however L is nef and big. In this sense, we can regard this example (X, Y, L) as an analogue of Zariski's example with $r = 2$. As D_1 and D_2 intersects transversally along Y , we obtain the decomposition

$$\begin{aligned} N_{Y/X} &= N_{D_1/X}|_Y \oplus N_{D_2/X}|_Y = \mathcal{O}_X(D_1)|_{D_1}|_Y \oplus \mathcal{O}_X(D_2)|_{D_2}|_Y \\ &= \mathcal{O}_X(D_1)|_Y \oplus \mathcal{O}_X(D_2)|_Y \end{aligned}$$

By denoting $N_\lambda := \mathcal{O}_X(D_\lambda)|_Y$ for each $\lambda = 1, 2$, it holds that $N_1 \cong N_2$, $(D_\lambda.Y) = 2(H.Y) - (E.Y) = 8 - 12 < 0$, and $\deg_Y L|_Y \otimes N_\lambda^{-1} = 0 - (8 - 12) > 0$. Therefore we can apply Theorem 1.4, Corollary 1.5, and also Lemma 5.5 to our (X, L, Y) . By Corollary 1.5, we have that $h_{\min, L}|_Y$ is bounded, where $h_{\min, L}$ is a metric of L minimal singularities (here we use that fact that $L|_Y$ admits a C^∞ Hermitian metric with zero curvature, since $L|_Y$ is a flat line bundle). Therefore we can conclude that $h_{\min, L}$ is bounded. Note that we can moreover show that the semi-positivity of L (i.e. that we can choose $h_{\min, L}$ as a C^∞ Hermitian metric) by applying Lemma 5.5 and use the ‘‘regularized maximum construction’’ technique as in [11, Corollary 3.4].

Next let us consider the case of $N > 12$. In this case, L is not nef. By the argument as above, we also have $\mathbf{B}(L) = Y$, $\deg N_\lambda = 8 - N < 0$, and $\deg(L|_Y \otimes N_\lambda^{-1}) = (12 - N) - (8 - N) = 4 > 0$. Thus we can apply Theorem 1.4 to (X, Y, L) also in this case. As the computation shows that

$$\square_L = \left\{ \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_{\geq 0}^2 \mid \left| \frac{N-12}{N-8} \leq |\alpha| \leq 1 \right. \right\},$$

it follows from Theorem 1.4 that the local weight function $\varphi_{\min,L}$ of a metric $h_{\min,L}$ with minimal singularities can be written as

$$\varphi_{\min,L}(z, y) = \log \max_{\alpha \in \square_L} \prod_{\lambda=1}^r |z_\lambda|^{2\alpha_\lambda} + O(1) = \frac{N-12}{N-8} \cdot \log(|z_1|^2 + |z_2|^2) + O(1)$$

on a neighborhood of any point of Y , where y is a coordinate of Y and $z = (z_1, z_2, \dots, z_r)$ is a system of local defining functions of Y . In particular, in this case, $\varphi_{\min,L}$ has analytic singularities along Y .

Note that similar example can be constructed in general dimension by considering some points blow-up of a del Pezzo manifold of degree 1 (see [7] for example. For the choice of the counterpart of the divisors D_ν 's above, see [12, §6.3]).

6.3. An example in [2]

The above two examples satisfies Condition 1.2(ii) and the condition that Y admits a holomorphic tubular neighborhood. On the other hand, the example (X, Y, L) in [2, Example 5.4] does not satisfy these conditions. In [2]'s example, a metric of L with minimal singularities is unbounded and actually has singularities along Y (i.e. local weight function equals to $-\infty$ on Y), however the Lelong number of the local weight is 0 for every point in X (see also [11, Example 4.2]). In particular, the conclusion of Theorem 1.3 does not hold in this example.

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