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# On the non existence of non negative solutions to a critical Growth-Fragmentation Equation (*)

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**Abstract.** — A growth fragmentation equation with constant dislocation density measure is considered, in which growth and division rates balance each other. This leads to a simple example of equation where the so called Malthusian hypothesis $(M_+)$ of J. Bertoin and A. Watson [8] is not necessarily satisfied. It is proved that, as it was first suggested by these authors, when that happens, no global non negative weak solution, satisfying some boundedness condition on several of its moments, exists. Non existence of local non negative solutions satisfying a similar condition, is proved to happen also. When a local non negative solution exists, the explicit expression is given.

**Résumé.** — Nous considérons une équation de croissance fragmentation dont les taux de croissance et de fragmentation s'équilibrent et dont le noyau de dislocation est constant. Suivant la valeur de cette constante l'équation vérifie ou non la condition $(M_+)$ introduite par J. Bertoin et A. Watson dans [8]. Nous démontrons que, comme ces auteurs l'avaient suggéré, lorsque la condition n'est pas vérifiée l'équation ne possède pas de solution globale non négative dont les moments satisfont certaines estimations naturelles. Nous montrons également que l'équation peut aussi ne pas avoir de solution locale vérifiant de telles estimations. Lorsqu'une telle solution existe, locale ou globale, une formule explicite est obtenue.

## 1. Introduction

Growth fragmentation equations have proved to be of interest due to their many applications in mathematical modeling and also for purely mathematical reasons (cf. [3, 4, 12, 20] and references therein). Motivated by the study of compensated growth-fragmentation stochastic processes (cf. [5]) and their

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occurrence in the construction of the Brownian map (cf. [6, 18, 21]), the Cauchy problem for the equation

$$\frac{\partial u(t,x)}{\partial t}+\frac{\partial}{\partial x}\big(x^{1+\gamma}u(t,x)\big)+x^{\gamma}u(t,x)=\int_x^{\infty}\frac{1}{y}k_0\left(\frac{x}{y}\right)y^{\gamma}u(t,y)\mathrm{d}y \quad (1.1)$$

in the domain $t>0$, $x>0$ is considered in [8] with initial data:

$$u(0)=\delta_1, \quad (1.2)$$

for $\gamma\in\mathbb{R}$ and $k_0$ a dislocation measure density, with support contained in $[0,1]$ and satisfying:

$$k_0(x)\mathrm{d}x=k_0(1-x)\mathrm{d}x,\quad \forall\, x\in[1/2,1);\quad \int_{[1/2,1)}(1-x)^2k_0(x)\mathrm{d}x<\infty. \quad (1.3)$$

The existence of solutions of growth fragmentation equations of a more general form:

$$\frac{\partial u(t,x)}{\partial t}+\frac{\partial}{\partial x}\big(\tau(x)u(t,x)\big)+B(x)u(t,x)=\int_x^{\infty}k(x,y)B(y)u(t,y)\mathrm{d}y \quad (1.4)$$

has been studied by several authors, with different motivations and by different methods (cf. for example [3, 4, 12, 20] and references therein). Several choices of the functions $B(x)$ and $\tau(x)$ have been used, based either on modeling considerations or for purely mathematical purpose. Functions $B$ and $\tau$ with power law behaviors at $x\to0$ and $x\to\infty$ seem to be acceptable approximations and are mathematically manageable. Suppose for the sake of simplicity that $B(x)=x^{\gamma}$ and $\tau(x)=x^{\nu}$. When $1+\gamma-\nu>0$, and under some other suitable conditions that may vary from an article to another, it has been proved that the equation has a kind of ground state, denoted as $(\lambda,N,\phi)$ whose components satisfy the stationary system:

$$\begin{gathered}
\frac{\partial}{\partial x}\big(\tau(x)N(x)\big)+B(x)N(x)=\int_x^{\infty}k(x,y)B(y)N(y)\mathrm{d}y-\lambda N(x)\\
\frac{\partial}{\partial x}\big(\tau(x)\phi(x)\big)-B(x)\phi(x)=-\int_x^{\infty}k(x,y)B(y)\phi(y)\mathrm{d}y+\lambda\phi(x)\\
N(0)=0, N(x)>0,\quad \forall\, x>0, \int_0^{\infty}N(x)\mathrm{d}x=1\\
\phi(x)>0,\quad \int_0^{\infty}\phi(x)N(x)\mathrm{d}x=1.
\end{gathered}$$

These eigenelements are important because, under some conditions, they have been proved to describe in some detail the dynamics of the evolution equation (1.4) and in particular its long time behavior as $t\to\infty$. They are particularly relevant in the mathematical modeling of population dynamics in biology (cf. [3, 25] and references therein).

The case $\nu = 0$ is treated in [25], and the cases $\nu \geqslant 0, \gamma \geqslant 0$ in [19]. In [20] the authors take $\nu = 0$ and the function $B$ is bounded from above and from below by positive constants. In [10] the function $\tau$ is compactly supported in $[0, x_M])$. The authors of [12] consider also more general kernels $k_0$ like $k_0(x) = (\delta_r + \delta_{1-r})/2$ for some $r \in [0, 1/2]$ or $k_0(x) = (\alpha+1)/2(x^\alpha + (1-x)^\alpha$. They also prove that if $\nu = 1$ and $\gamma = 0$ (and then $1 + \gamma - \nu = 0$) such eigenelements do not exist. That case also appears as critical in [15] when studying the long time behavior of fragmentation equations. It was considered in [11], where the existence of solutions was proved for a large set of initial data.

The growth fragmentation equation with $1 + \gamma - \nu = 0$ turns out to be of interest also in the study of compensated fragmentation processes in probability theory (cf. [5]). The equation is then rather specific, since the growth and dislocation rates balance each other in some sense. When $\gamma \neq 0$ the growth rate is not linear and the dislocation rate is unbounded. In that case, the existence of global, non negative, weak solutions of (1.1)–(1.3) has been proved in [8], under the condition (called Malthusian condition $(M_+)$ in [8]):

$$\inf_{s \geqslant 0} \Phi(s) < 0 \tag{1.5}$$

$$\Phi(s) = (K(s) + s - 2), \quad K(s) = \int_0^1 x^{s-1} k_0(s) \mathrm{d}s. \tag{1.6}$$

When property (1.5) is not satisfied it is shown in [8] and [7] that the particle system that corresponds to the stochastic version of (1.1)–(1.3) explode in finite time almost surely. The question has then been raised in [8] of the existence of non negative global solutions to (1.1)–(1.3) when the measure $k_0$ is such that:

$$\inf_{s \geqslant 0} \Phi(s) \geqslant 0 \tag{1.7}$$

and it was suggested that no such solutions exists when the inequality in (1.7) is strict. In order to obtain some insight into this question, we consider the simplest possible choice for $k_0$:

$$k_0(x) = \theta H(1 - x), \quad \theta > 0 \tag{1.8}$$

where $H$ is the Heaviside's function. This is of course a very particular example, but for which it is possible to obtain a rather explicit solutions, whose properties may be understood in detail. It is straightforward to check that for such a dislocation measure:

$$K(s) = \frac{\theta}{s} \tag{1.9}$$

$$\Phi(s) = \frac{\theta}{s} + s - 2 \equiv \frac{(s-\sigma_1)(s-\sigma_2)}{s}, \quad \forall\, s \in \mathbb{C}; \quad \Re e(s) > 0, \tag{1.10}$$

$$\sigma_1 = 1 - \sqrt{1-\theta}, \quad \sigma_2 = 1 + \sqrt{1-\theta}. \tag{1.11}$$

If $\theta \in (0,1)$ the two roots of $\Phi(s)$ are positive real numbers and condition (1.5) is satisfied. But, when $\theta \geqslant 1$, $\inf_{s>0} \Phi(s) = 2(\sqrt{\theta} - 1) \geqslant 0$ and (1.5) is not satisfied.

For $\theta \in (0,1)$ the existence of global non negative solutions follows from the results of [8]. We then focus on the case $\gamma \neq 0$, $\theta > 1$ and the question of the existence or not of non negative solutions.

For reasons due to the specific application of the growth fragmentation equation that one may be interested in, the dislocation measure $k_0$ is sometimes required to satisfy also:

$$\int_0^1 x k_0(x) \mathrm{d}x = 1,$$

(cf. for example in [3, 4, 11, 12, 20]). Were we willing to impose such condition in our case, it would force the value $\theta = 2$ and the condition (1.5) would not be satisfied.

### 1.1. Some notations

We denote $\mathbb{N}$ the set of non negative integers and $\Gamma(\,\cdot\,)$ the Gamma function. For a given interval $(a,b) \subset \mathbb{R}$ we define:

$$\mathscr{S}(a,b) = \{s \in \mathbb{C};\ \Re e(s) \in (a,b)\}\,. \tag{1.12}$$

We denote $\mathscr{D}'_1$ the set of distributions of order one and by $F(a,b,c,z)$ the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$. On the disc $|z| < 1$, the function $F(a,b,c,z)$ is defined by the series:

$$F(a,b,c,z) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} z^n \tag{1.13}$$

and it is defined by analytic continuation elsewhere (cf. [23]).

We say that the measure $u$ is a weak solution of (1.1), (1.8) on the time interval $(t_0, t_1)$ if

$$\forall\, \varphi \in C_c^1((t_0,t_1)\times(0,\infty)):$$
$$\int_0^\infty\int_0^\infty \left(\frac{\partial\varphi}{\partial t}+x^{\gamma+1}\frac{\partial\varphi}{\partial x}+x^\gamma\varphi(t,x)\right)u(t,x)\mathrm{d}x\mathrm{d}t$$
$$=-\theta\int_0^\infty\int_0^\infty u(t,y)y^{\gamma-1}\int_0^y\varphi(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t \quad (1.14)$$

We denote $\mathscr{M}_\rho$ the space of measures $u$ on $(0,\infty)$ such that

$$\int_0^\infty x^\rho u(x)\mathrm{d}x<\infty.$$

If $w$ is a measure, we denote $\mathcal{M}_w$ its Mellin transform, defined, when it makes sense, as

$$\mathcal{M}_w(t,s)=\int_0^\infty x^{s-1}w(t,x)\mathrm{d}x.$$

It follows from the definition of $K(s)$ in (1.6) that $K(s)=\mathcal{M}_{k_0}(s)$. The use of the Mellin transform makes the spaces $E'_{p,\,q}$ for $p<q$, presented for example in [22, Chapter 11], necessary. They are defined as the dual of the spaces $E_{p,\,q}$ of all the functions $\phi\in\mathscr{C}^\infty(0,\infty)$ such that:

$$N_{p,q,k}(\phi)=\sup_{x>0}\left(k_{p,q}(x)x^{k+1}\left|\phi^k(x)\right|\right)<\infty$$

where

$$k_{p,q}(x)=\begin{cases}x^{-p}, & \text{if } 0<x\leqslant 1\\ x^{-q}, & \text{if } x>1\end{cases}$$

with the topology defined by the numerable set of seminorms $\{N_{p,q,k}\}_{k\in\mathbb{N}}$. It follows that $E'_{p,\,q}$ is a subspace of $\mathscr{D}'(0,\infty)$. As indicated in [22], these are the spaces of Mellin transformable distributions.

### 1.2. Main results

In very short, when $\theta>1$ and $\gamma\neq 0$, global non negative solutions to (1.1), (1.2), (1.8), satisfying a boundedness condition on several of its moments, do not exist. More detailed statements depend on the sign of $\gamma$, as follows.

### 1.2.1. When $\gamma > 0$

Our first result is the following local existence of non negative solutions:

Theorem 1.1. — *For all $\theta > 0$ and $\gamma > 0$ there exists a unique, non negative weak solution $u \in \mathscr{D}'_1((0,\gamma^{-1}) \times (0,\infty))$ of* (1.1), (1.8) *on* $(0,\gamma^{-1})$ *such that, for some $\rho > 0$:*

$$u \in C\left(\left[0,\gamma^{-1}\right); E'_{\rho-\delta,\rho+\gamma+\delta}\right), \quad \text{for some } \delta > 0 \tag{1.15}$$

$$u(t) \rightharpoonup \delta_1 \quad \text{in } E'_{\rho,\rho+\gamma}, \quad \text{as } t \to 0. \tag{1.16}$$

*That solution is:*

$$u(t,x) = u^S(t,x) + u^R(t,x) H\left(1 - (1-\gamma t)^{\frac{1}{\gamma}}\, x\right) \tag{1.17}$$

$$u^S(t,x) = (1-\gamma t)^{\frac{1}{\gamma}}\, \delta\left(x - (1-\gamma t)^{-\frac{1}{\gamma}}\right) \tag{1.18}$$

$$u^R(t,x) = \theta\,(1-\gamma t)^{\frac{2}{\gamma}}\, tF\left(1 + \frac{\sigma_1}{\gamma}, 1 + \frac{\sigma_2}{\gamma}, 2, \gamma t\left(1 + (\gamma t - 1)\, x^\gamma\right)\right) \tag{1.19}$$

*and satisfies:*

$$u \in \mathscr{C}\left(\left[0,\gamma^{-1}\right); \mathscr{M}_{\rho-1}\right), \quad \forall\, \rho > 0. \tag{1.20}$$

The sense in which the initial data $\delta_1$ is taken in the hypothesis (1.16) ensures that the Mellin transform of $u(t)$ converges to 1 as $t$ goes to zero, for all $s \in \mathscr{S}(\rho,\rho+\gamma)$. Since $E'_{\rho,\rho+\gamma} \subset \mathscr{D}'(0,\infty)$ with continuous embedding, this condition is stronger than the convergence in the weak sense of measures.

Non uniqueness in some sense, of non negative solutions of (1.1) has been proved in [8] under some conditions on $k_0$. However the function $k_0$ given in (1.8) does not satisfy such conditions (cf. Remark 3.7).

As a consequence of Theorem 1.1 we deduce the following:

Corollary 1.2. — *The solution $u$ of* (1.1), (1.8) *defined in* (1.17)–(1.19) *satisfies:*

$$\forall\, x > 0: \quad \lim_{\gamma t \to 1^-} u(t,x) = \frac{\gamma \Gamma(\frac{2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})}(1+x^\gamma)^{-\frac{2}{\gamma}}, \tag{1.21}$$

$$\forall\, r > 1: \lim_{t \to \gamma^{-1}} (1-\gamma t)^{\frac{r-1}{\gamma}} \int_0^\infty x^r u(t,x)\mathrm{d}x = \frac{\Gamma(\frac{r+1}{\gamma})\Gamma(\frac{r-1}{\gamma})}{\Gamma(\frac{r+1-\sigma_1}{\gamma})\Gamma(\frac{r+1-\sigma_2}{\gamma})}, \tag{1.22}$$

$$\lim_{t \to \gamma^{-1}} \frac{-1}{\log(1-\gamma t)} \int_0^\infty x u(t,x)\mathrm{d}x = \frac{\Gamma(\frac{2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})}, \tag{1.23}$$

$$\forall\, r \in (0,1): \quad \lim_{t \to \gamma^{-1}} \int_0^\infty x^r u(t,x)\mathrm{d}x = \frac{\Gamma(\frac{r+1}{\gamma})\Gamma(\frac{1-r}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})}. \tag{1.24}$$

We prove in Theorem 5.3 that, when $\theta > 1$ and $\gamma > 0$, there is no possible extension of $u$ to a non negative global solution whose Mellin transform satisfies suitable conditions. When $\gamma \in (0, 2)$ and $\theta > 1$ the non existence result of non negative solutions for large times is the following:

Theorem 1.3. — *Suppose that $\theta > 1$, $\gamma \in (0, 2)$. Then there is no extension of the local solution $u$ to a non negative weak solution $w \in \mathscr{D}_1'((0, T) \times (0, \infty))$ of* (1.1), (1.8) *such that:*

$$w \in C\left([0, T) \,; \mathscr{M}_{\rho-1-\delta} \cap \mathscr{M}_{\rho+\gamma-1+\delta}\right), \quad \textit{for some} \quad \delta > 0 \tag{1.25}$$

*and satisfying the initial condition* (1.16)*, for any $T > \gamma^{-1}$ and $\rho \in (0, 2-\gamma)$.*

*Remark 1.4.* — It follows from Theorem 1.3 that, given $\gamma \in (0, 2)$, any $T > \gamma^{-1}$ and any $\varepsilon > 0$ arbitrarily small, the Cauchy problem for equation (1.1), (1.8) with initial condition (1.16) has no non negative weak solution $w \in \mathscr{D}_1'((0, T) \times (0, \infty))$ such that

$$w \in C\left([0, T) \,; \mathscr{M}_{1-\gamma-2\varepsilon} \cap \mathscr{M}_{1+\varepsilon}\right).$$

(Chose $\delta = 2\varepsilon$ and $\rho = 2 - \gamma - \varepsilon$.) Then, given any $r < 1 - \gamma$, any $\varepsilon > 0$ arbitrarily small and any $\rho > r$ it can't have a weak solution $w \in \mathscr{D}_1'((0, T) \times (0, \infty))$ such that

$$w \in C\left([0, T) \,; \mathscr{M}_{\rho} \cap \mathscr{M}_{\rho+\gamma+\varepsilon}\right)$$

### 1.2.2. When $\gamma < 0$.

When $\gamma < 0$ the existence of a local solution $v \in \mathscr{C}([0, -\gamma^{-1}), E'_{1+\gamma,\infty})$ of (1.1), (1.8), (1.16) on $(0, -\gamma^{-1})$ is proved in Theorem 6.5. But the following non existence of local nonnegative solutions holds:

Theorem 1.5. — *If $\gamma < 0$ and $\theta > 1$ there is no local, non negative weak solution $v$ of* (1.1), (1.8) *on $(0, T)$, for any $T > 0$, satisfying for some $\rho > 1 - \gamma$, the initial condition* (1.16) *and such that:*

$$v \in C\left([0, T) \,; \mathscr{M}_{\rho-1-\delta} \cap \mathscr{M}_{\rho-\gamma-1+\delta}\right), \quad \textit{for some} \quad \delta > 0. \tag{1.26}$$

In summary, for $\theta > 1$ and $\gamma \in (0, 2)$, the non negative solution $u$ of Theorem 1.1 blows up as $\gamma t \to 1^-$, in the sense given by Corollary (1.2), and can not be extended beyond $t = \gamma^{-1}$ to a non negative solution that still satisfies (1.25). If $\gamma < 0$, nonnegative solutions satisfying (1.26) do not exist, even locally in time. The non existence of global non negative solutions, for critical growth fragmentation equations where the condition (1.5) is not satisfied, was first suggested in [8]. Of course, Theorem 1.3 and Theorem 1.5 do not preclude the existence of non negative global solutions that do not satisfy (1.25) or (1.26). When $\theta \in (0, 1)$, the condition (1.5) is satisfied and then, as proved in [8], the problem (1.1), (1.2) has a global non negative solution $\mu$. It follows that $\mu$ coincides with the solution obtained in Section (5), when $\gamma < \sqrt{1-\theta}$ (cf. Proposition 5.4) or, when $\gamma < 0$, with the solution obtained in Section 6 (cf. Remark 6.6). If $\theta = 1$ the condition (1.5) is not satisfied, but our arguments do not prove the non existence of a non negative extension of $u$ beyond $t = \gamma^{-1}$ (cf. Remark 5.2).

The equation (1.1) may be solved using the Mellin transform. The proof of the non existence of non negative solution is then done in two steps. The first is to prove the uniqueness of solutions that may take positive and negative values but some moments of which are suitably bounded. The second is to show that the solution that was previously obtained satisfies the regularity condition, but takes positive and negative values. That follows from its behavior as $x \to 0$ or $x \to \infty$, since they are given, up to some multiplicative factor depending on time, by $x^{-\sigma_2-\gamma}$ and $x^{-\sigma_1-\gamma}$. When $\sigma_2$ and $\sigma_1$ are complex numbers, this forces the solution to oscillate.

The choice of $k_0$ as in (1.8), is of course very particular and makes the solutions of equation (1.1) rather explicit. We emphasize however that the arguments used in Section 4 and Section 6, based on the Wiener Hopf method, permit to solve the growth fragmentation equation (1.1) for more general dislocation measures. More details will be presented elsewhere. We may recall at this point that explicit solutions to the Cauchy problem for the pure fragmentation equation, (i.e. without growth term and with $k_0$ such that $\int y k_0(y) \mathrm{d}y = 1$), with the initial data as in (1.2) where obtained in [26, 27] for several fragmentation rates and the same dislocation measure (1.8) with $\theta = 2$. Explicit solutions for the critical case $\gamma = 0, \nu = 1$ with $k_0(z) = \alpha \delta_{\frac{1}{\alpha}}(z)$ and $\alpha > 1$, which generalises the binary fission when $\alpha = 2$ have been obtained in [13].

The plan of this article is as follows. In Section 2 the Cauchy problem satisfied by $\mathcal{M}_u(t,s)$, the Mellin transform of suitable solutions $u$ of (1.1), (1.2), (1.8) is obtained. In Section 3 we prove Theorem (1.1 and Corollary 1.2. In Section 4 we study the extension of the local solution, and its uniqueness. The sign of the extension is studied in Section 5, where Theorem 1.3 is proved. Section 6 contains the case $\gamma < 0$ and the proof of Theorem (1.5. Several technical results are gathered in the Appendix. The content of Sections 2 and 3 where anounced and shortly presented in [14].

## 2. The problem in Mellin variables

We deduce in this Section the equation satisfied by the Mellin transform of a solution $u$ of (1.1) that would satisfy suitable conditions. To this end we suppose that $u(t,x)$ is a solution of equation (1.1) such that its Mellin transform $\mathcal{M}_u$ is well defined for $s$ and $s+\gamma$, where $s$ belongs to some domain $D$ of the complex plane $\mathbb{C}$. Applying the Mellin transform to both sides of equation (1.1) we arrive at:

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{M}_u(t,s)+\int_0^\infty \frac{\partial}{\partial x}\big(x^{1+\gamma}u(t,x)\big)x^{s-1}\mathrm{d}x+\mathcal{M}_u(t,s+\gamma)&\\ =\int_0^\infty x^{s-1}\int_x^\infty \frac{1}{y}k_0\left(\frac{x}{y}\right)y^\gamma u(t,y)\mathrm{d}y\mathrm{d}x&\\ =\int_0^\infty \mathrm{d}y y^{\gamma-1}u(t,y)\int_0^y \mathrm{d}x x^{s-1}k_0\left(\frac{x}{y}\right)&\\ =\mathcal{M}_u(t,s+\gamma)K(s).&\end{aligned}$$

If $\lim_{x\to 0} x^{\gamma+s}u(t,x)=\lim_{x\to\infty} x^{\gamma+s}u(t,x)=0$ we deduce that

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial x}\big(x^{1+\gamma}u(t,x)\big)x^{s-1}\mathrm{d}x&=-(s-1)\int_0^\infty \big(x^{1+\gamma}u(t,x)\big)x^{s-2}\mathrm{d}x\\ &=-(s-1)\mathcal{M}_u(t,s+\gamma)\end{aligned}$$

and finally,

$$\frac{\partial}{\partial t}\mathcal{M}_u(t,s)=(K(s)+s-2)\mathcal{M}_u(t,s+\gamma). \tag{2.1}$$

With our choice of the measure $k_0$ (cf. (1.8) and (1.10)), we are then led to consider the problem

$$\frac{\partial W}{\partial t}(t,s)=\Phi(s)W(t,s+\gamma),\quad \forall\, s\in\mathbb{C};\quad \Re e(s)=s_* \tag{2.2}$$

$$W(0,s)=1,\quad \forall\, s\in\mathscr{S}(\rho,\rho+\gamma). \tag{2.3}$$

for some $s_*\in\mathbb{R}$ and $\rho\in\mathbb{R}$, where

$$\Phi(s)=\left(\frac{\theta}{s}+s-2\right),\quad \forall\, s\in\mathbb{C}\setminus\{0\}. \tag{2.4}$$

Problem (2.2)–(2.3) was already considered in [8]. Equations like (2.2) have deserved some attention in the literature, for different functions $\Phi$ (cf. [2] and references therein, [16]). They may be Laplace transformed into a Carleman type problem and solved using the classical Wiener–Hopf method (cf. [9, 17, 24]). See also Section 4 for the same equation (2.2) but a different initial data.

## 3. $\gamma > 0$. Proof of Theorem 1.1

### 3.1. An explicit solution of (2.2)–(2.3)

It immediately follows from the identities 15.2.1 and 15.3.3 in [1] that the function

$$\Omega(t,s) = F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) \equiv (1-\gamma t)^{\frac{2-s}{\gamma}} F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) \quad (3.1)$$

satisfies the two identities in (2.2)–(2.3) for all $t \in (0, \gamma^{-1})$ and $s \in \mathbb{C}$ such that $s \neq 0, -\gamma, -2\gamma, \dots$. From the properties of hypergeometric functions it follows that, for all $\rho > 0$, $R > \rho$ and $T \in (0, \gamma^{-1})$:

$$\forall\, t \in (0,\gamma^{-1}) : \Omega(t,\cdot\,) \text{ is analytic in } \mathbb{C} \setminus \{-m\gamma, m \in \mathbb{N}\} \quad (3.2)$$

$$\Omega \in C\left([0,\gamma^{-1}) \times \mathscr{S}(0,\infty)\right) \quad (3.3)$$

$$\sup\left\{|\Omega(t,s)|, \;\; t \in [0,T), \;\; s \in \overline{\mathscr{S}(\rho,R)}\right\} < \infty \quad (3.4)$$

$$\lim_{t\to 0} \Omega(t,s) = 1, \quad \forall\, s \in \mathbb{C} \setminus \{-m\gamma,\; m \in \mathbb{N}\} \quad (3.5)$$

$$\forall\, s \in \mathscr{S}(-\infty, 2) \setminus \{-n\gamma,\; n \in \mathbb{N}\} :$$

$$\lim_{t\to\gamma^{-1}} \Omega(t,s) = \Omega(\gamma^{-1}, s) = \frac{\Gamma(\frac{s}{\gamma})\Gamma(\frac{2-s}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})}. \quad (3.6)$$

Our purpose is now to take the inverse Mellin transform of the function $\Omega(t,s)$. We first show:

Proposition 3.1. — *For any $\sigma_0 > 0$, $\sigma_1$, $\sigma_2$ and $x > 0$ and $T \in (0, \frac{1}{\gamma})$ there exists a positive constant $C = C(T, \sigma_0, \sigma_1, \sigma_2)$ such that, for all $t \in (0,T)$:*

$$\int_{\Re e s = \sigma_0} \left|\Omega(t,s) - (1-\gamma t)^{\frac{2-s}{\gamma}}\left(1 + \frac{2t}{s}\right) x^{-s}\right| \mathrm{d}s \leqslant C\left(x(1-\gamma t)^{\frac{1}{\gamma}}\right)^{-\sigma_0}.$$

*Proof.* — Using the series representation of the Hypergeometric function in (3.1) we deduce, for each $t \in (0, \gamma^{-1})$ fixed:

$$F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) - 1 - \frac{2t}{s} = \mathcal{O}_t\left(|s|^{-2}\right), \quad |s| \to \infty, \quad \Re e s = \sigma_0 > 0,$$

(cf. [23, (15.2.2)]) and there exists a constant $C = C(T, \sigma_0, \sigma_1, \sigma_2) > 0$ such that if $t \in (0,T)$ and $s = \sigma_0 + iv$, $v \in \mathbb{R}$ and $-\sigma_0 \notin \mathbb{N}$:

$$\left|F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) - 1 - \frac{2t}{s}\right| \leqslant C(1+|s|)^{-2}. \quad (3.7)$$

Then, by definition of $\Omega(t,s)$:

$$\left|\Omega(t,s) - (1-\gamma t)^{\frac{2-s}{\gamma}}\left(1+\frac{2t}{s}\right)\right| \leqslant C(1-\gamma t)^{\frac{2-s}{\gamma}}(1+|s|)^{-2}$$

and

$$\begin{aligned}\int_{\Re e s=\sigma_0}&\left|\Omega(t,s) - (1-\gamma t)^{\frac{2-s}{\gamma}}\left(1+\frac{2t}{s}\right)x^{-s}\right|\mathrm{d}s\\ &\leqslant C(1-\gamma t)^{\frac{2}{\gamma}}\int_{\Re e s=\sigma_0}(1+|s|)^{-3}\left|x(1-\gamma t)^{\frac{1}{\gamma}}\right|^{-s}\mathrm{d}s\\ &\leqslant C(1-\gamma t)^{\frac{2}{\gamma}}\left(x(1-\gamma t)^{\frac{1}{\gamma}}\right)^{-\sigma_0}\int_{\Re e s=\sigma_0}(1+|s|)^{-2}\mathrm{d}s. \quad \square\end{aligned}$$

It follows from Proposition 3.1 that $\Omega(t,s)$ has an inverse Mellin transform when $0<\gamma t<1$. Our next purpose is to obtain its explicit expression.

### 3.2. The inverse Mellin transform of $\Omega(t,s)$

We recall that, for suitable functions $V$, the classical inverse Mellin transform is defined as

$$\mathscr{M}^{-1}_{\sigma_0}(V) = \frac{1}{2i\pi}\int_{\Re e(s)=\sigma_0} x^{-s}V(s)\mathrm{d}s \tag{3.8}$$

for some $\sigma_0>0$ fixed. We first show the following:

Proposition 3.2. — *Suppose $\sigma_1\in\mathbb{C}$, $\sigma_2\in\mathbb{C}$, $\gamma>0$ and $t>0$ such that $0<\gamma t<1$ and define the function*

$$v(t,x) = F\left(1+\frac{\sigma_1}{\gamma}, 1+\frac{\sigma_2}{\gamma}, 2, \gamma t\left(1+(\gamma t-1)x^{\gamma}\right)\right) \times H\left(1-(1-\gamma t)^{\frac{1}{\gamma}}x\right) \tag{3.9}$$

*for $x>0$, where $H$ is the Heaviside function. Then, for all $s>0$, $v(s)\in E'_{0,q}$ for all $q>0$, the Mellin transform of $v$ is given by:*

$$\mathscr{M}_v(t,s) \equiv \int_0^\infty v(t,x)x^{s-1}\mathrm{d}x = (1-\gamma t)^{-\frac{s}{\gamma}}\frac{F(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t)-1}{\sigma_1\sigma_2 t}. \tag{3.10}$$

*Proof.* — For all $t<\gamma^{-1}$, $v(t)$ has compact support in $(0,(1-\gamma t)^{-1/\gamma}))$, is integrable on $(0,\infty)$ and bounded as $x\to 0$. Then $v\in\mathscr{C}([0,\gamma^{-1});E'_{0,q})$ for all $q>0$. The proof of (3.10) is a straightforward calculation using the expression of the hypergeometric function. Since $\gamma>0$ we have that $\gamma t>0$. Moreover, since $1-\gamma t>0$ and $x>0$, we have $(\gamma t-1)x^{\gamma}<0$ and then $(1+(\gamma t-1)x^{\gamma})<1$. Finally, due to the Heaviside function we only have to consider values of $(t,x)$ where $1+(\gamma t-1)x^\gamma>0$. It then follows from the expression of the function $F(1+\frac{\sigma_1}{\gamma},1+\frac{\sigma_2}{\gamma},2,\gamma t(1+(\gamma t-1)x^\gamma))$ as an absolutely convergent series (cf. [1, Definition 15.1.1]):

$$\mathcal{M}_v(t,s)=\sum_{n=0}^\infty \frac{\Gamma(1+\frac{\sigma_1}{\gamma}+n)\Gamma(1+\frac{\sigma_2}{\gamma}+n)\Gamma(2)(\gamma t)^n}{\Gamma(1+\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2}{\gamma})\Gamma(2+n)\Gamma(n+1)} \times \int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}}(1+(\gamma t-1)x^\gamma)^n x^{s-1}\mathrm{d}x.$$

We use now that, since $\gamma>0$, we have for all $s>0$:

$$\begin{aligned}\int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}}&(1+(\gamma t-1)x^\gamma)^n x^{s-1}\mathrm{d}x\\
&=\sum_{m=0}^n\binom{n}{m}(-1)^m(1-\gamma t)^m\times\int_0^{(1-\gamma t)^{-\frac{1}{\gamma}}} x^{m\gamma+s-1}\mathrm{d}x\\
&=(1-\gamma t)^{-\frac{s}{\gamma}}\sum_{m=0}^n\binom{n}{m}\frac{(-1)^m}{m\gamma+s}=(1-\gamma t)^{-\frac{s}{\gamma}}\frac{\Gamma(n+1)\Gamma(\frac{s}{\gamma})}{\gamma\Gamma(1+\frac{s}{\gamma}+n)}.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{M}_v(t,s)&=(1-\gamma t)^{-\frac{s}{\gamma}}\\
&\quad\times\sum_{n=0}^\infty\frac{\Gamma(1+\frac{\sigma_1}{\gamma}+n)\Gamma(1+\frac{\sigma_2}{\gamma}+n)\Gamma(2)(\gamma t)^n}{\Gamma(1+\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2}{\gamma})\Gamma(2+n)\Gamma(n+1)}\frac{\Gamma(n+1)\Gamma(\frac{s}{\gamma})}{\gamma\Gamma(1+\frac{s}{\gamma}+n)}\\
&=(1-\gamma t)^{-\frac{s}{\gamma}}\frac{F(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t)-1}{\sigma_1\sigma_2 t}.\end{aligned}$$

and this proves (3.10). □

The next Corollary follows from Proposition 3.2 and Theorem 11.10.1 in [22] on the uniqueness of the inverse Mellin transform:

Corollary 3.3. — *For all $\sigma_1\in\mathbb{C}$, $\sigma_2\in\mathbb{C}$, suppose that $\gamma>0$, $0<\gamma t<1$ and let $u$ be the measure:*

$$u(t,x)=(1-\gamma t)^{\frac{1}{\gamma}}\delta\big(x-(1-\gamma t)^{-\frac{1}{\gamma}}\big)+\sigma_1\sigma_2 t(1-\gamma t)^{\frac{2}{\gamma}}v(t,x).$$

*Then, for all $t\in\big(0,\gamma^{-1}\big)$:*

$$\mathcal{M}_u(t)=\Omega(t),\quad \textit{and}\quad u(t)=\mathcal{M}^{-1}(\Omega(t)).$$

We prove now the existence part in Theorem 1.1.

Proposition 3.4. — *The measure $u$ defined in* (1.17)–(1.19) *is a weak non negative solution of* (1.1), (1.8) *on* $(0,\gamma^{-1})$ *such that*

(i) $u\in\mathscr{C}\left([0,\gamma^{-1});\mathscr{M}_{\rho-1}\right),\quad\forall\,\rho>0.$
(ii) $\forall\,T\in\left(0,\gamma^{-1}\right),\quad\exists\,C_T>0;$

$$\forall\,t\in[0,T],\quad\forall\,s>0:\int_0^\infty u(t,x)x^{s-1}\mathrm{d}x\leqslant C_T$$

*and satisfies* (1.16).

*Proof of Proposition 3.4.* — The assertions (i) and (ii) follow from the explicit expression of $u$. It is easy to check that (1.16) holds true. Let us prove that $u$ is a weak solution of (1.1), (1.8) on $(0,\gamma^{-1})$.

We already know that $\Omega(t,s)$ solves (2.2) for all $t\in(0,\gamma^{-1})$ and all $s\in\mathscr{S}(0,\infty)$. Since $\Omega(t)$ and $\Phi\Omega(t)$ are analytic and bounded in $\mathscr{S}(0,q)$ for any $q>0$, we deduce from Theorem 11.10.1 in [22] that $u\in\mathscr{C}((0,\gamma^{-1}),E'_{0,q})$ for all $q>0$. Applying the inverse Mellin tranform (3.8) to both sides of the equation (2.2) we deduce the following identity:

$$\frac{\partial u}{\partial t}(t,x)=\mathcal{M}_{s_0}^{-1}\left(\left(\frac{\theta}{s}+(s-1)-1\right)\tau_\gamma\mathcal{M}_u\right)(t,x)\tag{3.11}$$

where all the terms are in $\mathscr{C}((0,\gamma^{-1}),E'_{0,q})$ and we have denoted

$$(\tau_\gamma\mathcal{M}_u)(t,s)=\mathcal{M}_u(t,s+\gamma).$$

We consider now each of the terms in the right and side separately. Since $\sigma_0>0$, $\gamma>0$, using that $\mathcal{M}_u(t,s)=\Omega(t,s)$ for all $\Re e(s)>0$ we have:

$$\mathcal{M}_{\sigma_0}^{-1}\left(\tau_\gamma\mathcal{M}_u\right)=x^\gamma u(t,x).\tag{3.12}$$

$$\mathcal{M}_{s_0}^{-1}\left((s-1)\tau_\gamma\mathcal{M}_u\right)(t,x)=-\frac{\partial}{\partial x}\left(x^{\gamma+1}u(t,x)\right)\tag{3.13}$$

In the last term in the right hand side of (3.11) we write as above:

$$\frac{1}{2i\pi}\int_{\Re es=\sigma_0}\frac{\theta}{s}\mathcal{M}_u(t,s+\gamma)x^{-s}\mathrm{d}s$$
$$=\int_0^\infty u(t,y)\left(\frac{1}{2i\pi}\int_{\Re es=\sigma_0}\frac{\theta}{s}y^{s+\gamma-1}x^{-s}\mathrm{d}s\right)\mathrm{d}y.$$

Using that for $\sigma_0>0$:

$$\frac{1}{2i\pi}\int_{\Re es=\sigma_0}\frac{1}{s}y^{s+\gamma-1}x^{-s}\mathrm{d}s=\begin{cases}0, & \text{if } y<x\\ y^{\gamma-1}, & \text{if } y>x\end{cases}\tag{3.14}$$

we deduce

$$\frac{1}{2i\pi}\int_{\Re es=\sigma_0}\frac{\theta}{s}\mathcal{M}_u(t,s+\gamma)x^{-s}\mathrm{d}s=\theta\int_x^\infty u(t,y)y^{\gamma-1}\mathrm{d}y.\tag{3.15}$$

Both sides of equation (1.1) are then equal in $\mathscr{C}((0,\gamma^{-1}),E'_{0,q})$. In particular, $u\in\mathscr{C}((0,\gamma^{-1}),\mathscr{D}'(0,\infty))$ and is a weak solution of (1.1), (1.8) on $(0,\gamma^{-1})$.

In order to prove the non negativity of the measure $u$ we use its definition (1.17)–(1.19) and the expression of the hypergeometric function in the right hand side of (1.19) as an absolutely convergent series:

$$F\left(1+\frac{\sigma_1}{\gamma},1+\frac{\sigma_2}{\gamma},2,z\right)=\sum_{n=0}^{\infty}\frac{\Gamma(1+\frac{\sigma_1}{\gamma}+n)\Gamma(1+\frac{\sigma_2}{\gamma}+n)z^n}{\Gamma(1+\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2}{\gamma})\Gamma(2+n)\Gamma(n+1)}$$

where we have denoted $\gamma t\left(1+\left(\gamma t-1\right)x^{\gamma}\right)=z$. When $\theta\in(0,1)$ all the terms of the series are obviously non negative since $\sigma_2>0$ and $\sigma_1>0$. When $\theta>1$, we use that, since $\sigma_1=\overline{\sigma_2}$, $\Gamma(1+\frac{\sigma_1}{\gamma}+n)=\overline{\Gamma(1+\frac{\sigma_2}{\gamma}+n)}$ for all $n\in\mathbb{N}$, and again all the term of the series are non negative. □

*Remark 3.5.* — The particular form of the measure $u(t)$ and a simple calculation with distributions in $(0,\infty)$ show that the measure $u^S$ solves:

$$\frac{\partial u^S(t)}{\partial t}+\frac{\partial}{\partial x}\left(x^{\gamma+1}u^S(t)\right)+x^{\gamma}u^S(t)=0,\quad\text{in }\mathscr{D}_1'\left(\left(0,\gamma^{-1}\right)\times(0,\infty)\right)$$

and the function $u^R$ satisfies, for all $t\in(0,\gamma^{-1})$ and $x\in\left(0,(1-\gamma t)^{-1/\gamma}\right)$:

$$\begin{aligned}\frac{\partial u^R(t,x)}{\partial t}&+\frac{\partial\left(x^{\gamma+1}u^R(t,x)\right)}{\partial x}+x^{\gamma}u^R(t,x)\\&=\theta\int_x^{(1-\gamma t)^{-\frac{1}{\gamma}}}u^R(t,y)y^{\gamma-1}\mathrm{d}y+\theta\left(1-\gamma t\right)^{-1}.\end{aligned}$$

*Remark 3.6.* — By the particular form of $u$ we deduce, for all $\varphi\in C_c^1([0,\gamma^{-1}]\times(0,\infty))$ :

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty}&\left(\frac{\partial\varphi}{\partial t}+x^{\gamma+1}\frac{\partial\varphi}{\partial x}+x^{\gamma}\varphi(t,x)\right)u(t,x)\mathrm{d}x\mathrm{d}t+\varphi(0,1)\\&=\int_0^{\infty}u\left(\gamma^{-1},x\right)\varphi\left(\gamma^{-1},x\right)\mathrm{d}x\\&\quad-\theta\int_0^{\infty}\int_0^{\infty}u(t,y)y^{\gamma-1}\int_0^y\varphi(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}\tag{3.16}$$

*Proof of Theorem 1.1.* — By the Proposition 3.4 only the uniqueness of non negative weak solutions satisfying (1.15)–(1.16) for some $\rho>0$ remains to be proved. Suppose on the contrary that $u$ and $v$ are two such solutions and $u(t)\neq v(t)$. Since $u_1$ and $u_2$ are nonnegative and satisfy (1.15) it follows that their Mellin transforms $\mathcal{M}_u(t)$ and $\mathcal{M}_v(t)$ are well defined and analytic on $\mathscr{S}(\rho-\delta,\rho+\gamma+\delta)$ for all $t\in(0,\gamma^{-1})$ and satisfy (7.1). By (1.15), $\mathcal{M}_u(t)$ and $\mathcal{M}_v(t)$ also satisfy (7.2). We check now that $u$ and $v$ also satisfy (7.3). Since the proof is of course the same for both, we only consider $u$. By (1.15) and the continuity of the Mellin transform on $E'_{\rho,\rho+\gamma}$, it follows that $\mathcal{M}_u(0,s)=1$ for all $s\in\mathscr{S}(\rho,\rho+\gamma)$ and $u$ satisfies (7.3). Therefore, $M_u$ and $M_v$ satisfy the hypotheses of Theorem 7.1 and are then equal. This contradicts our hypothesis that $u_1(t)\neq u_2(t)$, and proves the uniqueness. Assertion (1.20) has been shown in Proposition 3.4. □

*Remark 3.7.* — The existence of a non negative and non identically zero solution for the equation (1.1) with zero initial data has been proved in [8] for quite general dislocation measures $k_0$ under some conditions. One of these conditions, denoted $(M_-)$, requires to have $\sigma_1-1>0$. That is not possible in our case by our choice of $k_0$ and (1.11).

*Proof of Corollary 1.2.* — By (1.18), $u^s(t,x)\to 0$ as $\gamma t\to 1$. Then, by (1.17) the behavior of $u(t,x)$ as $t\to\gamma^{-1}$ is given by that of $F(1+\frac{\sigma_1}{\gamma}, 1+\frac{\sigma_2}{\gamma},2,z)$ as $z\to 1^-$ and depends on the values of $\frac{\sigma_1}{\gamma}$ and $\frac{\sigma_2}{\gamma}$.

$$\begin{aligned}
\lim_{\gamma t\to 1^-} u^R(t,x) &= \lim_{\gamma t\to 1^-}\sigma_1\sigma_2 t\,(1-\gamma t)^{\frac{2}{\gamma}}\\
&\qquad\times F\left(1+\frac{\sigma_1}{\gamma},1+\frac{\sigma_2}{\gamma},2,\gamma t\,(1+(\gamma t-1)\,x^\gamma)\right)\\
&= \lim_{\gamma t\to 1^-}\frac{\sigma_1\sigma_2 t\,(1-\gamma t)^{\frac{2}{\gamma}}}{(1-\gamma t\,(1+(\gamma t-1)\,x^\gamma))^{\frac{2}{\gamma}}}\\
&\qquad\times\lim_{\gamma t\to 1^-}\frac{F(1+\frac{\sigma_1}{\gamma},1+\frac{\sigma_2}{\gamma},2,\gamma t(1+(\gamma t-1)x^\gamma))}{(1-\gamma t\,(1+(\gamma t-1)\,x^\gamma))^{-\frac{2}{\gamma}}}
\end{aligned}$$

Since $\frac{\sigma_1+\sigma_2}{\gamma}=\frac{2}{\gamma}>0$, by 15.4.23 in [23] we have for all $x>0$:

$$\lim_{\gamma t\to 1^-}\frac{F(1+\frac{\sigma_1}{\gamma},1+\frac{\sigma_2}{\gamma},2,\gamma t(1+(\gamma t-1)x^\gamma))}{(1-\gamma t\,(1+(\gamma t-1)\,x^\gamma))^{-\frac{2}{\gamma}}}=\frac{\Gamma(\frac{2}{\gamma})}{\Gamma(1+\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2}{\gamma})}.$$

Since $1-\gamma t\,(1+(\gamma t-1)\,x^\gamma)=(1-\gamma t)(1+\gamma t x^\gamma)$ it follows:

$$\lim_{\gamma t\to 1^-}\frac{\sigma_1\sigma_2 t\,(1-\gamma t)^{\frac{2}{\gamma}}}{(1-\gamma t\,(1+(\gamma t-1)\,x^\gamma))^{\frac{2}{\gamma}}}=\lim_{\gamma t\to 1^-}\frac{\sigma_1\sigma_2 t}{(1+\gamma t x^\gamma)^{\frac{2}{\gamma}}}=\frac{\sigma_1\sigma_2}{\gamma\,(1+x^\gamma)^{\frac{2}{\gamma}}}$$

and finally,

$$\begin{aligned}
\lim_{\gamma t\to 1^-} u^R(t,x) &= \frac{\Gamma(\frac{2}{\gamma})\sigma_1\sigma_2}{\gamma\Gamma(1+\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2}{\gamma})}(1+x^\gamma)^{-\frac{2}{\gamma}}\\
&= \frac{\gamma\Gamma(\frac{2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})}(1+x^\gamma)^{-\frac{2}{\gamma}}.
\end{aligned}$$

The proofs of properties (1.22)–(1.24) follow from the explicit expression (3.1) of $\Omega(t,s)$ and formulas 15.4 (ii) in [23]. □

## 4. $\gamma > 0$. Extension of the local solution.

In the first of the two main results of this Section we extend the local solution $u$ obtained in Theorem 1.1. It uses the notation

$$\nu = \min(2, \Re e(\sigma_2) + \gamma). \tag{4.1}$$

THEOREM 4.1. — *For all $\theta > 0$ and $\gamma > 0$ there exists a global weak solution $w \in \mathscr{C}\left([0,\infty); E'_{0,\nu}\right)$ of* (1.1), (1.8) *on $t \in (0,\infty)$, such that $w(t) = u(t)$ for all $t \in (0, \gamma^{-1})$ and satisfying*

$$\begin{aligned} & w \in \mathscr{C}^\infty((\gamma^{-1},\infty)\times(0,\infty)) \cap \mathscr{C}([\gamma^{-1},\infty)\times(0,\infty)) \\ & w \text{ satisfies } (1.1),\ (1.8), \text{ pointwise, } \forall\, t > \gamma^{-1}, \forall\, x > 0 \\ & \forall\, t > \gamma^{-1}, \quad \mathcal{M}_w(t,\cdot\,) \text{ is analytic in } \mathscr{S}(0, \Re e(\sigma_2)+\gamma), \\ & \mathcal{M}_w(\gamma^{-1}, s) = \Omega(\gamma^{-1}, s) \quad \forall\, s \in \mathbb{C}, \quad \Re e(s) < 2 \end{aligned} \tag{4.2}$$

An expression of $w$ for $t > \gamma^{-1}$ is obtained in Remark 4.6.

Then, the following uniqueness result is proved, for the case $\gamma \in (0,2)$:

THEOREM 4.2. — *If $\gamma \in (0,2)$, the measure $w$ defined in Theorem 4.1 is the unique global weak solution of* (1.1), (1.8) *such that for all $T > 0$, $w \in \mathscr{C}([0,T); E'_{\rho-\delta,\rho+\gamma+\delta})$ for some $\rho \in (0, 2-\gamma)$, $\delta > 0$ and*

$$\begin{aligned} & \sup\{|\mathcal{M}_w(s,t)|; \quad s \in \mathscr{S}(\rho-\delta, \rho+\gamma+\delta), \quad t \in [0,T]\} < \infty \\ & w(0) = \delta_1. \end{aligned} \tag{4.3}$$

The uniqueness of the solution obtained in Theorem 4.1, valid for all $\gamma > 0$, is proved in Theorem 4.7.

In order to extend the solution $u$ of (1.1), (1.2) beyond $t = \gamma^{-1}$ we first obtain a solution of (2.2) for $\gamma t > 1$.

### 4.1. Another explicit solution of (2.2)

The new solution of the equation (2.2) is obtained as follows:

PROPOSITION 4.3. — *The function $U$ defined, for all $t > \gamma^{-1}$ and all $s \in \mathbb{C}$, as*

$$U(t,s) = \frac{(\gamma t)^{\frac{\sigma_1 - s}{\gamma}} \Gamma(\frac{s}{\gamma}) \Gamma(1 - \frac{s-\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma}) \Gamma(1 - \frac{\sigma_1 - \sigma_2}{\gamma})} \times F\left(1 - \frac{\sigma_1}{\gamma}, \frac{s-\sigma_1}{\gamma}; 1 + \frac{\sigma_2 - \sigma_1}{\gamma}; \frac{1}{\gamma t}\right) \tag{4.4}$$

*is such that:*

(i) $U$ *is meromorphic on* $(\gamma^{-1},\infty)\times S$ *and the set* $S$ *of its poles is:*

$$S=\{s=-m\gamma,\ m\in\mathbb{N}\}\cup\{s=\Re e(\sigma_2)+(m+1)\gamma,\ m\in\mathbb{N}\}.$$

(ii) $U$ *satisfies the equation* (2.2) *for all* $t>\gamma^{-1}$ *and all* $s\in\mathbb{C}\setminus S$*;*

(iii) *For all closed subinterval* $I\subset(0,\Re e(\sigma_2)+\gamma))$*, there exists a positive constant* $C=C(t,I)$ *such that, for all* $t\geqslant\gamma^{-1}$ *and all* $s\in\mathbb{C}$*,* $\Re e(s)\in I$

$$|U(t,s)|+|U_t(t,s)|\leqslant Ce^{-\frac{\pi|\Im m(s)|}{\gamma}}t^{\frac{\sigma_1-s_0}{\gamma}}\left((1+|s|)^{-1+\frac{\sigma_1}{\gamma}}+(1+|s|)^{-\frac{\sigma_2}{\gamma}}\right)$$

(iv) *If* $\Re e(s)<2$*:*

$$\lim_{t\to(\gamma^{-1})^+}U(t,s)=\lim_{t\to(\gamma^{-1})^-}\Omega(t,s)=\frac{\Gamma(\frac{s}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})}\frac{\Gamma(\frac{\sigma_2+\sigma_1-s}{\gamma})}{\Gamma(\frac{\sigma_2}{\gamma})}$$

*Proof.* — It is straightforward to check that the function

$$V(s)=\gamma^{\frac{s}{\gamma}}\frac{\Gamma(\frac{s-\sigma_1}{\gamma})}{\Gamma(\frac{s}{\gamma})\Gamma(1-\frac{s-\sigma_2}{\gamma})},\quad\forall\, s\in\mathbb{C}\setminus\{s\in\mathbb{C};s=\sigma_1-m\gamma,\,m\in\mathbb{N}\}$$

satisfies:

$$V(s+\gamma)=-\frac{(s-\sigma_1)(s-\sigma_2)}{s}V(s),\ \forall\, s\in\mathbb{C}\setminus\{s\in\mathbb{C};s=\sigma_1-m\gamma,\,m\in\mathbb{N}\}$$

We define now the function of $t$ and $s$:

$$\mathscr{U}(t,s)=\frac{1}{\gamma V(s)}\int_{\mathscr{C}}\frac{(-t)^{\frac{\sigma-s}{\gamma}}V(\sigma)\mathrm{d}\sigma}{\Gamma(1+\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)}\tag{4.5}$$

where the path of integration $\mathscr{C}$ is as follows. For $\sigma_0>\sigma_1$ fixed, we define:

$$\mathscr{C}=\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3\tag{4.6}$$

$$\mathscr{C}_1=\{s\in\mathbb{C};s=\sigma_0+iv,|v|\leqslant1\}\tag{4.7}$$

$$\mathscr{C}_2=\{s\in\mathbb{C};s=(u+iv),\ v=-u+(1+\sigma_0),\ v>1\}\tag{4.8}$$

$$\mathscr{C}_3=\{s\in\mathbb{C};s=(u+iv),\ v=u-(1+\sigma_0),\ v<-1\}.\tag{4.9}$$

Notice that when $\sigma\in\mathscr{C}$ is such that $|\sigma|\to\infty$ we have that $\Re e(\sigma)\to-\infty$. The function below the integral sign of (4.5) may be written as follows:

$$\begin{aligned}w(t,s,\sigma)&=\frac{(-t)^{\frac{\sigma-s}{\gamma}}V(\sigma)}{\Gamma(1+\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)}\\&=\frac{(-t)^{-\frac{s}{\gamma}}(-\gamma t)^{\frac{\sigma}{\gamma}}\Gamma(\frac{\sigma-\sigma_1}{\gamma})}{\Gamma(1+\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)\Gamma(\frac{\sigma}{\gamma})\Gamma(1-\frac{\sigma-\sigma_2}{\gamma})}\end{aligned}\tag{4.10}$$

For $s \in \mathbb{C}$ fixed, $\sigma \in \mathscr{C}$, $|\sigma| \to \infty$ we have, using Stirling's formula:

$$\left|\Gamma\left(\frac{\sigma-\sigma_1}{\gamma}\right)\right| \approx e^{\frac{\sigma-\sigma_1}{\gamma}\log\left(\frac{\sigma-\sigma_1}{\gamma}\right)} \approx e^{\frac{\Re e(\sigma)}{\gamma}\log|\sigma/\gamma|}$$
$$\left|\Gamma\left(\frac{\sigma}{\gamma}\right)\right| \approx e^{\frac{\sigma}{\gamma}\log\left(\frac{\sigma}{\gamma}\right)} \approx e^{\frac{\Re e(\sigma)}{\gamma}\log|\sigma/\gamma|}$$
$$\left|\Gamma\left(1-\frac{\sigma-\sigma_2}{\gamma}\right)\right| \approx e^{\left(1-\frac{\sigma-\sigma_2}{\gamma}\right)\log\left(1-\frac{\sigma-\sigma_2}{\gamma}\right)} \approx e^{\left(1-\frac{\Re e(\sigma)}{\gamma}\right)\log|\sigma/\gamma|}$$
$$\left|\Gamma\left(1+\frac{\sigma-s}{\gamma}\right)\right| \approx e^{\left(1+\frac{\sigma-s}{\gamma}\right)\log\left(1+\frac{\sigma-s}{\gamma}\right)} \approx e^{\left(1+\frac{\Re e(\sigma)}{\gamma}\right)\log|\sigma/\gamma|}$$
$$\left|e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1\right| = \left|e^{-\frac{2i\pi}{\gamma}(\Re e(\sigma)+i\Im m(\sigma))}-1\right| \geqslant \left|e^{\frac{2\pi\Im m(\sigma)}{\gamma}}-1\right|$$
$$(-\gamma t)^{\frac{\sigma-s}{\gamma}} = e^{\frac{\sigma-s}{\gamma}\log(-\gamma t)} \approx e^{\frac{\Re e(\sigma)}{\gamma}\log(\gamma t)}.$$

We deduce that, for each $s \in \mathbb{C}$ fixed there exists a constant $C = C(s) > 0$ such that for all $t > \gamma^{-1}$ and all $\sigma \in \mathscr{C}$:

$$|w(t,s,\sigma)| \leqslant C \frac{e^{\frac{\Re e(\sigma)}{\gamma}\log(\gamma t)}}{e^{\log|\sigma/\gamma|}\left|e^{\frac{2\pi v}{\gamma}}-1\right|}$$

where the constant $C$ depends on $s$. Since $\Re e(\sigma) \to -\infty$ as $|\sigma| \to \infty$ for $\sigma \in \mathscr{C}$ the function $w(t,s,\cdot\,)$ is exponentially decaying in $\sigma$ for $t > \gamma^{-1}$, and the integral in the right hand side of (4.5) is absolutely convergent. It follows that the function $\mathscr{U}(t,s)$ is well defined, continuous with respect to $t$ and analytic with respect to $s$ for all $t > \gamma^{-1}$ and $s$ in

$$D = \mathbb{C} \setminus \big(\{s \in \mathbb{C};\ s = -m\gamma,\ m \in \mathbb{N}\} \cup \{s \in \mathbb{C};\ s = \sigma_2 + (m+1)\gamma,\ m \in \mathbb{N}\}\big). \tag{4.11}$$

By the exponential decay of $w(t,s,\sigma)$ in $\sigma$ along $\mathscr{C}$ and its regularity in time, a simple calculation yields:

$$\frac{\partial \mathscr{U}}{\partial t}(t,s) = \frac{-1}{\gamma V(s)}\int_{\mathscr{C}} \frac{(-t)^{\frac{\sigma-s}{\gamma}-1}(\frac{\sigma-s}{\gamma})V(\sigma)\mathrm{d}\sigma}{\Gamma(1+\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)} \tag{4.12}$$
$$= \frac{-1}{\gamma V(s)}\int_{\mathscr{C}} \frac{(-t)^{\frac{\sigma-s}{\gamma}-1}V(\sigma)\mathrm{d}\sigma}{\Gamma(\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)} \tag{4.13}$$
$$= \frac{-1}{\gamma V(s)}\int_{\mathscr{C}} \frac{(-t)^{\frac{\sigma-(s+\gamma)}{\gamma}}V(\sigma)\mathrm{d}\sigma}{\Gamma(1+\frac{\sigma-(s+\gamma)}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-(s+\gamma))}-1)}. \tag{4.14}$$

By (6.2):

$$\frac{-1}{V(s)} = \frac{(s-\sigma_1)(s-\sigma_2)}{s}\frac{1}{V(s+\gamma)}.$$

We deduce

$$\frac{\partial \mathscr{U}}{\partial t}(t,s) = \frac{(s-\sigma_1)(s-\sigma_2)}{s}\mathscr{U}(t,s+\gamma).$$

and the function $\mathscr{U}(t,s)$ satisfies the equation (2.2) for $t > \gamma^{-1}$ and $s$ in $D$ given by (4.11).

Our next step is to prove that $\mathscr{U} = U$, using the residue's method. To this end, we notice that for $s$ fixed as in (4.11), the poles of the function $w(t,s,\sigma)$ to integrate are located at the following points:

$$\{\sigma = \sigma_1 - \gamma m,\ m \in \mathbb{N}\} \cup \{\sigma = s + \gamma m,\ m \in \mathbb{N}\}$$

For values of $t$ such that $\gamma t > 1$ we must use the residues at the points $\sigma = \sigma_1 - \gamma m$. We deform the integration contour, always in the region where $\Re s \to -\infty$. Since $\sigma_0 > \Re e(\sigma_1)$:

$$\begin{aligned}\mathscr{U}(t,s) &= \frac{2i\pi}{V(s)(e^{-\frac{2i\pi}{\gamma}(\sigma_1-s)}-1)}\\ &\quad\times\sum_{m=0}^{\infty}\frac{(-t)^{\frac{\sigma_1-s}{\gamma}-m}\gamma^{\frac{\sigma_1}{\gamma}-m}(-1)^m}{\Gamma(\frac{\sigma_1}{\gamma}-m)\Gamma(1+\frac{\sigma_1-s}{\gamma}-m)\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma}+m)\Gamma(m+1)}\\ &= \frac{2i\pi(-t)^{\frac{\sigma_1-s}{\gamma}}\gamma^{\frac{\sigma_1}{\gamma}}}{V(s)(e^{-\frac{2i\pi}{\gamma}(\sigma_1-s)}-1)}\\ &\quad\times\sum_{m=0}^{\infty}\frac{(\gamma t)^{-m}}{\Gamma(\frac{\sigma_1}{\gamma}-m)\Gamma(1+\frac{\sigma_1-s}{\gamma}-m)\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma}+m)\Gamma(m+1)}\\ &= \frac{2i\pi(-t)^{\frac{\sigma_1-s}{\gamma}}\gamma^{\frac{\sigma_1}{\gamma}}F(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t})}{V(s)(e^{-\frac{2i\pi}{\gamma}(\sigma_1-s)}-1)\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1-\frac{s-\sigma_1}{\gamma})\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma})}.\end{aligned}$$

This may we writen:

$$\mathscr{U}(t,s) = \frac{2i\pi(-\gamma t)^{\frac{\sigma_1-s}{\gamma}}\Gamma(\frac{s}{\gamma})\Gamma(1-\frac{s-\sigma_2}{\gamma})F(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma})\Gamma(\frac{s-\sigma_1}{\gamma})\Gamma(1-\frac{s-\sigma_1}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma_1-s)}-1)}$$

and using the identity $\Gamma(x)\Gamma(1-x)(e^{2i\pi x}-1) = 2i\pi e^{i\pi x}$:

$$\begin{aligned}\mathscr{U}(t,s) &= \frac{(-\gamma t)^{\frac{\sigma_1-s}{\gamma}}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma})}\Gamma\left(\frac{s}{\gamma}\right)\Gamma\left(1-\frac{s-\sigma_2}{\gamma}\right)\\ &\quad\times F\left(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t}\right)e^{-\frac{i\pi}{\gamma}(s-\sigma_1)} \qquad (4.15)\end{aligned}$$

from where, using that $e^{-i\pi} = -1$ it follows that $\mathscr{U}(t,s) = U(t,s)$ for $\gamma t > 1$ and $s \in \mathbb{C}$. Property (ii) immediately follows. In order to prove the property (iii) we use again Stirling's formula. For $s\in\mathbb{C}$ such that $\Re e(s)$ remains bounded and $|\Im m(s)|\to\infty$:

$$\left|\Gamma\left(\frac{s}{\gamma}\right)\right|\approx e^{\frac{iv}{\gamma}i\,\mathrm{Arg}(s/\gamma)}=e^{-\frac{v}{\gamma}\,\mathrm{Arg}(s/\gamma)}$$

$$\left|\Gamma\left(1-\frac{s-\sigma_2}{\gamma}\right)\right|\approx e^{\frac{-iv}{\gamma}i\,\mathrm{Arg}(-s/\gamma)}=e^{\frac{v}{\gamma}\,\mathrm{Arg}(-s/\gamma)}$$

Then,

$$\left|\Gamma\left(\frac{s}{\gamma}\right)\Gamma\left(1-\frac{s-\sigma_2}{\gamma}\right)\right|\approx e^{-\frac{v}{\gamma}\,\mathrm{Arg}(s/\gamma)}e^{\frac{v}{\gamma}\,\mathrm{Arg}(-s/\gamma)}.$$

If $v\to\infty$, $\mathrm{Arg}(s/\gamma)\to\pi/2$, $\mathrm{Arg}(-s/\gamma)\to-\pi/2$ and then

$$\left|\Gamma\left(\frac{s}{\gamma}\right)\Gamma\left(1-\frac{s-\sigma_2}{\gamma}\right)\right|\approx e^{-\frac{v}{\gamma}\frac{\pi}{2}}e^{\frac{v}{\gamma}\frac{(-\pi)}{2}}\approx e^{-\pi v}.$$

If $v\to-\infty$, $\mathrm{Arg}(s/\gamma)\to-\pi/2$ and $\mathrm{Arg}(-s/\gamma)\to\pi/2$ from where

$$\left|\Gamma\left(\frac{s}{\gamma}\right)\Gamma\left(1-\frac{s-\sigma_2}{\gamma}\right)\right|\approx e^{\frac{-v}{\gamma}(\frac{-\pi}{2})}e^{\frac{v}{\gamma}\frac{\pi}{2}}\approx e^{\pi v}.$$

On the other hand, by 15.7.2 in [1], if $|\Im m(s)|\to\infty$:

$$\left|F\left(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1+\frac{\sigma_2-\sigma_1}{\gamma};\frac{1}{\gamma t}\right)\right|=\mathcal{O}\left(|s|^{-1+\frac{\sigma_1}{\gamma}}+|s|^{-\frac{\sigma_2}{\gamma}}\right).$$

Using $|(\gamma t)^{\frac{\sigma_1-s}{\gamma}}|=(\gamma t)^{\frac{\sigma_1-s_0}{\gamma}}$ we deduce:

$$\begin{aligned}|U(t,s)|&\sim Ce^{-\frac{v}{\gamma}\frac{\pi}{2}}e^{\frac{v}{\gamma}(-\frac{\pi}{2})}t^{\frac{\sigma_1-s_0}{\gamma}}\left|F\left(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1+\frac{\sigma_2-\sigma_1}{\gamma};\frac{1}{\gamma t}\right)\right|\\&=e^{-\frac{\pi v}{\gamma}}t^{\frac{\sigma_1-s_0}{\gamma}}\mathcal{O}\left(|s|^{-1+\frac{\sigma_1}{\gamma}}+|s|^{-\frac{\sigma_2}{\gamma}}\right),\quad v\to\infty\\|U(t,s)|&\sim Ce^{-\frac{v}{\gamma}(-\frac{\pi}{2})}e^{\frac{v}{\gamma}\frac{\pi}{2}}t^{\frac{\sigma_1-s_0}{\gamma}}\left|F\left(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1+\frac{\sigma_2-\sigma_1}{\gamma};\frac{1}{\gamma t}\right)\right|\\&=e^{\frac{\pi v}{\gamma}}t^{\frac{\sigma_1-s_0}{\gamma}}\mathcal{O}\left(|s|^{-1+\frac{\sigma_1}{\gamma}}+|s|^{-\frac{\sigma_2}{\gamma}}\right),\quad v\to-\infty\end{aligned}$$

and (iii) follows.

The property (iv) is directly deduced from the definitions of $\Omega$ and $U$, (3.1) and (4.4) respectively, and the application of identity 15.4.20 in [23] when $s<\Re e(\sigma_2+\sigma_1)\equiv 2$:

$$\lim_{\gamma t\to 1^+}F\left(1-\frac{\sigma_1}{\gamma},\frac{s-\sigma_1}{\gamma};1+\frac{\sigma_2-\sigma_1}{\gamma};\frac{1}{\gamma t}\right)=\frac{\Gamma(1+\frac{\sigma_2-\sigma_1}{\gamma})\Gamma(\frac{\sigma_2+\sigma_1-s}{\gamma})}{\Gamma(\frac{\sigma_2}{\gamma})\Gamma(1+\frac{\sigma_2-s}{\gamma})}$$

$$\lim_{\gamma t\to 1^-}F\left(\frac{s-\sigma_1}{\gamma},\frac{s-\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t\right)=\frac{\Gamma(\frac{s}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})}\frac{\Gamma(\frac{\sigma_2+\sigma_1-s}{\gamma})}{\Gamma(\frac{\sigma_2}{\gamma})}.\qquad\square$$

*Remark 4.4.* — The solution of (2.2)–(2.3) defined in (3.1) may be obtained in the same way as the solution $U$ has been obtained in Proposition 4.3. It is enough to this end to start with the following function $\widetilde{V}(s)$:

$$\widetilde{V}(s) = (-\gamma)^{\frac{s}{\gamma}} \frac{\Gamma(\frac{s-\sigma_1}{\gamma})\Gamma(\frac{s-\sigma_2}{\gamma})}{\Gamma(\frac{s}{\gamma})} \tag{4.16}$$

that also satisfies the equation (6.2). It may then be checked that:

$$\Omega(t,s) = \frac{-1}{\gamma\widetilde{V}(s)} \int_{\Re e\sigma=\sigma_0} \frac{(-t)^{\frac{\sigma-s}{\gamma}}\widetilde{V}(\sigma)d\sigma}{\Gamma(1+\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)} \tag{4.17}$$

for any $\sigma_0 > 0$.

## 4.2. Inverse Mellin transform of the function $U$ in (4.4)

By the Proposition 4.3 it is possible to apply to the function $U(t)$ the inverse Mellin transform defined as in (3.8) with $s_0 \in (0, \Re e(\sigma_2))$. We then define:

$$\omega(t,x) = \mathcal{M}_{s_0}^{-1}(U(t))(x). \tag{4.18}$$

Proposition 4.5. — *For any $s_0 \in (0, \Re e(\sigma_2))$ the function $\omega$ defined in* (4.18) *is such that:*

(i) $\omega \in C^\infty\left((\gamma^{-1},\infty)\times(0,\infty)\right)$
(ii) $\omega \in \mathscr{C}((\gamma^{-1},\infty); E'_{0,\Re e(\sigma_2)+\gamma})$ *and* $\mathcal{M}_\omega(t,s) = U(t,s)$*, for all* $t > \gamma^{-1}, s \in \mathscr{S}(0, \Re e(\sigma_2)+\gamma)$*,*
(iii) $\omega$ *satisfies the equation* (1.1)*,* (1.8) *pointwise for all* $t > \gamma^{-1}$ *and* $x > 0$*.*
(iv) *For all* $x > 0$*:*

$$\lim_{t\to(\gamma^{-1})^+} \omega(t,x) = \frac{\gamma\Gamma(\frac{2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})}(1+x^\gamma)^{-\frac{2}{\gamma}}$$

(v) *For all* $t > \gamma^{-1}$ *:*

$$\omega(t,x) = \frac{\gamma\Gamma(1+\frac{\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2-\sigma_1}{\gamma})} \frac{(\gamma t-1)^{\frac{\sigma_1}{\gamma}-1}}{(\gamma t)^{\frac{\sigma_2}{\gamma}}} x^{-\sigma_2-\gamma} + o\left(x^{-\sigma_2-\gamma}\right), \; x\to\infty$$

*Proof.* — The assertion (i) It follows from the regularity of the function $U$ with respect to $t$ and Proposition 4.3 (i).

For the proof of assertion (ii) we first notice that, by Proposition 4.3 (i) and (iii), the function $U(t,s)$ is analytic and exponentially decaying the strip $s \in \mathscr{S}(0, \Re e(\sigma_2)+\gamma)$ as $|\Im m(s)| \to \infty$. Then, by classical properties of the Mellin transform (cf. [22, Theorem 11.10.1]) assertion (ii) follows.

In order to prove assertion (iii) we first notice that, from the analyticity and boundedness properties of $\Phi U(t)$ on the strip $s \in \mathscr{S}(0, \Re e(\sigma_2)+\gamma)$ and by the same general properties of the Mellin transform, we have $\Phi U \in \mathscr{C}((\gamma^{-1},\infty); E'_{0,\Re e(\sigma_2)+\gamma})$. We may then apply the inverse Mellin transform as defined in (3.8) with $s_0 \in (0, \Re e(\sigma_2))$ to both sides of the equation (2.2)

$$\frac{\partial}{\partial t}\mathcal{M}^{-1}_{s_0}(U(t))(x) = \mathcal{M}^{-1}_{s_0}\left(\Phi U(t,\cdot+\gamma)\right)(x). \tag{4.19}$$

By the regularity of $U(t,s)$ with respect to $t$ and the decay properties of $U(t,s)$ and $U_t(t,s)$ along the integration curve $\Re e(s) = s_0$ we have

$$\frac{\partial}{\partial t}\mathcal{M}^{-1}_{s_0}(U(t))(x) = \frac{\partial \omega}{\partial t}(t,x). \tag{4.20}$$

Arguing now as in the proof of Proposition 3.4 we deduce that $\omega$ satisfies the equation (1.1), (1.8) where all the terms belong to $\mathscr{C}((\gamma^{-1},\infty); E'_{0,\Re e(\sigma_2)+\gamma})$. Therefore, the equation is satisfied in the weak sense. By the assertion (i) it follows that it is satisfied pointwise in $(\gamma^{-1}) \times (0,\infty)$.

The property (iv) is a direct consequence of Proposition 4.3 (iv). More precisely, since $s_0 \in (0, \Re e(\sigma_2))$, it follows that $s < \Re e(\sigma_2+\sigma_1) \equiv 2$ and then, using the Lebesgue's convergence Theorem and Proposition 4.3 (iv):

$$\lim_{\gamma t \to 1^+} \omega(t,x) = \mathcal{M}^{-1}_{s_0}(U(\gamma^{-1}))(x) = u(\gamma^{-1},x).$$

In order to prove (v) we use the definition of $\omega$ and deformation of the contour integration. For $t > \gamma^{-1}$ fixed and $x \to \infty$ we have:

$$\omega(t,x) = \frac{\gamma\Gamma(1+\frac{\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2-\sigma_1}{\gamma})}\frac{(\gamma t-1)^{\frac{\sigma_1}{\gamma}-1}}{(\gamma t)^{\frac{\sigma_2}{\gamma}}}x^{-\sigma_2-\gamma} + \frac{1}{2i\pi}\int_{\Re e s = s_*} U(t,s)x^{-s}\mathrm{d}s$$

for $s_* > \sigma_2+\gamma$. Then,

$$\omega(t,x) = \frac{\gamma\Gamma(1+\frac{\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2-\sigma_1}{\gamma})}\frac{(\gamma t-1)^{\frac{\sigma_1}{\gamma}-1}}{(\gamma t)^{\frac{\sigma_2}{\gamma}}}x^{-\sigma_2-\gamma} + \mathcal{O}\left(x^{-s_*}\right), \quad x\to\infty$$

and (v) follows. □

*Remark 4.6.* — It is possible to obtain an explicit expression of $\omega(t,x)$ for all $\gamma t > 1$ and $x > 0$ by deforming the integration contour and the residue's

Theorem. For $\gamma x^\gamma t<1$ we must use the residues at $s=-m\gamma$, $m\in\mathbb{N}$:

$$\begin{aligned}\omega(t,x)&\equiv\frac{1}{2i\pi}\int_{\Re es=s_0}U(t,s)x^{-s}\mathrm{d}s\\&=-\frac{\gamma\,(\gamma t)^{\frac{\sigma_1}{\gamma}}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma})}\times\sum_{m=0}^{\infty}\frac{(\gamma x^\gamma t)^m\Gamma(1+m+\frac{\sigma_2}{\gamma})(-1)^{m+1}}{\Gamma(m+1)}\\&\qquad\times F\left(1-\frac{\sigma_1}{\gamma},-m-\frac{\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t}\right).\end{aligned}$$

If $\gamma x^\gamma t>1$ we use the poles at $s=\sigma_2+\gamma(m+1)$, $m\in\mathbb{N}$ and obtain

$$\begin{aligned}\omega(t,x)&\equiv\frac{1}{2i\pi}\int_{\Re es=s_0}U(t,s)x^{-s}\mathrm{d}s\\&=-\frac{\gamma\,(\gamma t)^{\frac{\sigma_1-\sigma_2}{\gamma}-1}x^{-\sigma_2-\gamma}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1-\frac{\sigma_1-\sigma_2}{\gamma})}\times\sum_{m=0}^{\infty}\frac{(\gamma x^\gamma t)^{-m}\Gamma(1+m+\frac{\sigma_2}{\gamma})(-1)^{m+1}}{\Gamma(m+1)}\\&\qquad\times F\left(1-\frac{\sigma_1}{\gamma},1+m+\frac{\sigma_2-\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t}\right).\end{aligned}\tag{4.21}$$

The function $\omega$ may then be written as follows:

$$\omega(t,x)=\frac{\gamma(\gamma t)^{\frac{\sigma_1}{\gamma}}H(t,\gamma t x^\gamma)}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1+\frac{\sigma_2-\sigma_1}{\gamma})},\quad\forall\,t>\gamma^{-1},\quad\forall\,x>0,$$

where

$$H(t,z)=\begin{cases}\sum_{m=0}^\infty\frac{(-z)^m\Gamma\left(1+m+\frac{\sigma_2}{\gamma}\right)}{\Gamma(m+1)}\\\qquad\times F\left(1-\frac{\sigma_1}{\gamma},-m-\frac{\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t}\right), & z\leqslant 1\\ z^{-\frac{\sigma_2}{\gamma}-1}\sum_{m=0}^\infty\frac{\Gamma\left(1+m+\frac{\sigma_2}{\gamma}\right)}{(-z)^m\Gamma(m+1)}\\\qquad\times F\left(1-\frac{\sigma_1}{\gamma},1+m+\frac{\sigma_2-\sigma_1}{\gamma};1-\frac{\sigma_1-\sigma_2}{\gamma};\frac{1}{\gamma t}\right), & z\geqslant 1.\end{cases}$$

*Proof of Theorem 4.1.* — We claim that the measure defined as

$$w(t)=\begin{cases}u(t), & \text{if },t\in(0,\gamma^{-1})\\ \omega(t), & \text{if } t\geqslant\gamma^{-1}.\end{cases}\tag{4.22}$$

satisfies all the requirements. In order to prove that $w$ is a global weak solution of (1.1), (1.8) we must prove that for all $\varphi\in C_c^1((0,\infty)\times(0,\infty))$:

$$\begin{aligned}&\int_0^\infty\int_0^\infty\left(\frac{\partial\varphi}{\partial t}+x^{\gamma+1}\frac{\partial\varphi}{\partial x}+x^\gamma\varphi(t,x)\right)w(t,x)\mathrm{d}x\mathrm{d}t\\&\qquad=-\theta\int_0^\infty\int_0^\infty\varphi(t,x)\int_0^\infty w(t,y)y^{\gamma-1}\mathrm{d}y\varphi(t,x)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.23}$$

This is easily shown by splitting the time integrals in (4.23) in the two domains $(0,\gamma^{-1})$ and $(\gamma^{-1},\infty)$, use (3.16) in Remark 3.6, assertion (iii) of Proposition 4.5 and the continuity of $w$ at $t=\gamma^{-1}$.

Let us prove now $w\in\mathscr{C}((0,\infty);E'_{0,\Re e(\sigma_2)+\gamma})$. By Theorem 1.1, for all $t\in[0,\gamma^{-1})$ the function $\mathcal{M}_w(t)=\mathcal{M}_u(t)$ is analytic and bounded in $\mathscr{S}(0,\infty)$. By Theorem 4.7, for $t>\gamma^{-1}$, $\mathcal{M}_w(t)=\mathcal{M}_\omega(t)$ is analytic and bounded in $\mathscr{S}(0,\Re e(\sigma_2)+\gamma)$. But, for $t=\gamma^{-1}$, $\lim_{t\to\gamma^{-1}}\mathcal{M}_w(\gamma^{-1},s)$ is analytic and bounded only on $\mathscr{S}(0,\nu)$. Then, $\mathcal{M}_w(t)$ is analytic and bounded in $\mathscr{S}(0,\nu)$. Again by Theorem 1.1 and Theorem 4.7, $\mathcal{M}_w\in\mathscr{C}((0,\infty)\times\mathscr{S}(0,\nu)$. Therefore $w\in\mathscr{C}((0,\infty);E'_{0,\nu})$ using Theorem 11.10.1 in [22].

The properties (4.2) for $t>\gamma^{-1}$ follow directly from Theorem 4.7 since $w=\omega$ for $t>\gamma^{-1}$. On the other since $u=w$ in $t\in(0,\gamma^{-1})$ it follows from (3.2)–(3.4) that $w$ also satisfies (4.2) in that interval of time. □

### 4.3. Uniqueness of the extension to $t\geqslant\gamma^{-1}$

We are now concerned with the question of uniqueness of global solutions to (1.1), (1.8), (1.2). After the existence and uniqueness of a local solution $u$ on $(0,\gamma^{-1})$ proved in Theorem 1.1, and since, by Corollary 1.2, this local solution has a limit as $\gamma t\to 1^-$, this question is reduced in some sense to the uniqueness of the solutions of (1.1), (1.8) with initial data $u\left(\gamma^{-1}\right)$. How is this on the side of the Mellin variables?

By general properties of hypergeometric functions, the limit when $\gamma t\to 1^-$ of $\mathcal{M}_u$ only exists for $\Re e(s)<2$. Therefore, the data at $t=\gamma^{-1}$ of $\mathcal{M}_\omega$ is only defined for $\Re e(s)<2$. On the other hand, $\mathcal{M}_\omega$ is meromorphic, with a countable set of poles located at $s=-m\gamma$ and $s=\sigma_2+(m+1)\gamma$ for $m\in\mathbb{N}$. Since, in order to uniquely determine $\mathcal{M}_\omega$, we need the data to be given in a strip of width strictly larger than $\gamma$, when $\gamma>2$ this forces to use an argument of uniqueness in a strip where $\mathcal{M}_\omega$ has a pole. That is why the hypotheses in Theorem 4.7 are given in terms of $s\mathscr{P}_\omega(t,s)$ on the strip $\mathscr{S}(-\gamma,\varepsilon)$, where $\mathscr{P}_\omega$ is the analytic extension of $\mathcal{M}_\omega$ to $\mathscr{S}(-\gamma,\varepsilon)$.

Theorem 4.7. — *For all $\gamma>0$ and $\theta>0$, the measure $\omega$ defined in* (4.18) *is the unique real valued weak solution $\omega\in\mathscr{D}'_1((\gamma^{-1},\infty)\times(0,\infty))$ of* (1.1), (1.8) *for all $t>\gamma^{-1}$ such that its Mellin transform $\mathcal{M}_\omega$ satisfies the following properties:*

$$\forall\, T>0,\exists\,\varepsilon>0:\mathcal{M}_\omega(t)\text{ is analytic on }\mathscr{S}(0,\varepsilon),\ \ \forall\, t\in(\gamma^{-1},T)$$

$$\forall\, t\in(0,T),\quad \mathcal{M}_\omega(t,s)\text{ has an extension }\mathscr{P}_\omega,\text{ to }(-\gamma,\varepsilon)\tag{4.24}$$

$$\mathscr{P}_\omega\text{ satisfies (2.2), on }(\gamma^{-1},T)\text{ for some }s_*\in(-\gamma,\varepsilon)\tag{4.25}$$

$$\forall\, t \in (0,T), s\mathscr{P}_\omega(t,s) \textit{ is analytic in } (-\gamma,\varepsilon) \tag{4.26}$$

$$s\mathscr{P}_\omega \in \mathscr{C}([\gamma^{-1},\infty) \times \mathscr{S}(-\gamma,\varepsilon)) \tag{4.27}$$

$$\sup\left\{|s\mathscr{P}_\omega(t,s)|;\quad s \in \mathscr{S}(-\gamma,\varepsilon),\quad t \in \left[\gamma^{-1},T\right]\right\} < \infty \tag{4.28}$$

$$\mathscr{P}_\omega(\gamma^{-1},s) = \Omega(\gamma^{-1},s),\quad \forall\, s \in \mathscr{S}(-\gamma,\varepsilon). \tag{4.29}$$

*Moreover this solution is such that:*

$$\omega \in C^\infty((\gamma^{-1},\infty)\times(0,\infty)) \cap C([\gamma^{-1},\infty)\times(0,\infty)). \tag{4.30}$$

*For all* $t > \gamma^{-1}$, $\omega$ *satisfies* (1.1), (1.8) *pointwise and its Mellin transform* $\mathcal{M}_\omega$ *is such that:*

$$\mathcal{M}_\omega \textit{ is analytic on } (\gamma^{-1},\infty) \times \mathscr{S}(0, \Re e(\sigma_2)+\gamma), \tag{4.31}$$

$$\mathcal{M}_\omega(\gamma^{-1},s) = \Omega(\gamma^{-1},s) \quad \forall\, s \in \mathbb{C},\quad \Re e(s) < 2 \tag{4.32}$$

$$\mathcal{M}_\omega \in \mathscr{C}([\gamma^{-1},\infty) \times \mathscr{S}(0,\nu)). \tag{4.33}$$

*Remark 4.8.* — The condition (4.28) is not satisfied in general by $\mathcal{M}_u(t)$ for $\gamma t < 1$, but is satisfied by $\mathcal{M}_u(\gamma^{-1})$, the initial data of $\omega$ at $t = \gamma^{-1}$, as it follows using (3.6) and Stirling's formula.

*Proof of Theorem 4.7.* — By Proposition 4.3 (ii), $\mathcal{M}_\omega(t,s) = U(t,s)$ for all $s \in \mathscr{S}(0, \Re e(\sigma_2)+\gamma)$. It follows from Proposition 4.3) and Proposition 4.5 that $\omega$ satisfies (4.30)–(4.33). On the other hand, the function $\mathcal{M}_\omega(t,s)$ has a meromorphic extension to the complex plane, given by $U(t,s)$ that, by Proposition 4.3 (ii) satisfies (4.25). By Proposition 4.3 (i), $U$ as a simple pole at $s = 0$ and $sU(t,s)$ is analytic on $(-\gamma, \Re e(\sigma_2)+\gamma)$. By Proposition 4.3 (i) and (iv), $sU(t,s)$ satisfies (4.27) and it satisfies (4.28) and (4.29) by points (iii) and (iv).

We prove now the uniqueness of weak solutions satisfying (4.24)–(4.29). Suppose that two such solutions $\omega_1$ and $\omega_2$ exist and let $\mathscr{P}_{\omega_1}$, $\mathscr{P}_{\omega_2}$ be the extensions of their Mellin transforms. Then, the two functions $W_1(t,s) = \mathscr{P}_{\omega_1}(t-\gamma^{-1},s)$ and $W_2(t,s) = \mathscr{P}_{\omega_2}(t-\gamma^{-1},s)$ satisfy the hypotheses of Theorem 7.2 and are then equal. We deduce in particular that $\mathcal{M}_{\omega_1}(t,s) = \mathcal{M}_{\omega_2}(t,s)$ for $t \in (\gamma^{-1},T)$ and $s \in \mathscr{S}(0,\varepsilon)$ which is a contradiction.

Suppose now that $\omega$ is complex valued. Since the coefficients of the equation (1.1), (1.8) are real, the conjugate $\overline{\omega}$ is also a solution, with the same initial data. Moreover, just by definition, its Mellin transform $\mathcal{M}_{\overline{\omega}}$ is such that $\mathcal{M}_{\overline{\omega}}(t,s) = \overline{\mathcal{M}_\omega(t,\overline{s})}$ and $s\mathcal{M}_{\overline{\omega}}(t,s) = \overline{\overline{s}\mathcal{M}_\omega(t,\overline{s})}$ for $s \in \mathscr{S}(0,\varepsilon)$. By hypothesis, for all $t > \gamma^{-1}$, $\mathcal{M}_\omega(t,s)$ has an extension $\mathscr{P}_\omega(t,s)$ such that the function $h(t,s) = s\mathscr{P}_\omega(t,s)$ is analytic in $s \in \mathscr{S}(-\gamma,\varepsilon)$. Therefore, $\mathscr{Q}_{\overline{\omega}}(t,s) = \overline{\mathscr{P}_\omega(t,\overline{s})} =$ is an extension of $\mathcal{M}_{\overline{\omega}}(t,s)$ such that $\overline{h(t,\overline{s})} = \overline{s}\mathscr{Q}_{\overline{\omega}}(t,s)$ is analytic in the domain:

$$\overline{\mathscr{S}} = \{s \in \mathbb{C};\ \overline{s} \in \mathscr{S}(-\gamma,\varepsilon)\} \equiv \mathscr{S}(-\gamma,\varepsilon).$$

Moreover, $\mathscr{Q}_{\overline{\omega}}$ satisfies the conditions (4.27)–(4.29) since, by hypothesis, so does $\mathscr{P}_\omega$. We deduce that $u = \overline{u}$ by the uniqueness property that has been proved just above and $u$ is then real valued. This ends the proof of Theorem 4.7. $\square$

The following Proposition is used in the proof of Theorem 4.2,

Proposition 4.9. — *Suppose $\gamma \in (0,2)$. There exists a unique real valued weak solution $\omega \in \mathscr{D}_1'((\gamma^{-1},\infty)\times(0,\infty))$ of* (1.1), (1.8) *for all $t > \gamma^{-1}$ such that for some $\rho \in (0, 2-\gamma)$ and all $T > \gamma^{-1}$ its Mellin transform $M_\omega$ solves* (2.2) *and satisfies, for some $\delta > 0$:*

$$\forall\, t \in (\gamma^{-1}, T), \mathcal{M}_\omega(t,s) \text{ is analytic on } \mathscr{S}(\rho-\delta, \rho+\gamma+\delta) \tag{4.34}$$

$$\mathcal{M}_\omega(t,s) \text{ is continuous on } [\gamma^{-1},\infty)\times\mathscr{S}(\rho-\delta,\rho+\gamma+\delta) \tag{4.35}$$

$$\sup\left\{|\mathcal{M}_\omega(s,t)|;\ \ s\in\mathscr{S}(\rho-\delta,\rho+\gamma+\delta),\ \ t\in\left[\gamma^{-1},T\right]\right\}<\infty \tag{4.36}$$

$$\mathcal{M}_\omega(\gamma^{-1},s) = \Omega(\gamma^{-1},s),\ \ \forall\, s\in\mathscr{S}(\rho,\rho+\gamma). \tag{4.37}$$

*This solution is the function $\omega$ obtained in Theorem 4.7.*

*Proof of Proposition 4.9.* — By Theorem 4.7 and the hypothesis $\gamma \in (0,2)$, the function $\omega$ obtained in Theorem 4.7 satisfies (4.34)–(4.37). Suppose that $\omega_1$ and $\omega_2$ are two different weak solution in $\mathscr{D}_1'((\gamma^{-1},\infty)\times(0,\infty))$ of (1.1), (1.8) satisfying (4.34)–(4.37). Then the functions $W_1(t,s) = \mathcal{M}_{\omega_1}(t+\gamma^{-1},s)$ and $W_2(t,s) = \mathcal{M}_{\omega_2}(t+\gamma^{-1},s)$ satisfy the hypotheses of Theorem 7.1 for all $T > 0$ for $W_0(s) = U(\gamma^{-1},s)$ and are then equal. This contradiction concludes the proof. $\square$

We now prove Theorem 4.2 whose hypotheses are simpler than (4.24)–(4.29) in Theorem 4.7, but that requires the condition $\gamma \in (0,2)$.

*Proof of Theorem 4.2.* — Suppose first that $T < \gamma^{-1}$. By Theorem 1.1 the measure $u$ is a weak non negative solution of (1.1), (1.8) such that $u \in \mathscr{C}([0,\gamma^{-1}); E_{0,q}')$ for all $q > 0$ and satisfies (4.3). If we suppose now that $u_1$ and $u_2$ are two different solutions satisfying these conditions, $\mathcal{M}_{u_1}$ and $\mathcal{M}_{u_2}$ would both satisfy the hypotheses of Theorem 7.1 for all $T \in (0,\gamma^{-1})$ and therefore would be equal. This contradiction proves the uniqueness for $T \in (0,\gamma^{-1})$.

Suppose now that $T > \gamma^{-1}$ and $\widetilde{w}$ satisfies the hypotheses of Theorem 4.2. We already know by the previous step that $\widetilde{w} = u$ for $t \in (0,\gamma^{-1})$. Let us prove that $\widetilde{w} = \omega$ for $t \in (\gamma^{-1},T)$. By hypothesis, $\widetilde{w} \in \mathscr{C}((0,T); E_{\rho-\delta,\rho+\gamma+\delta}')$ for some $\rho \in (0, 2-\gamma)$, $\delta > 0$ and is a weak solution of (1.1), (1.8). Since $x_+^{s-1} \in E_{p,q}$ for all $s \in \mathbb{C}$ such that $\Re e(s) \in (p,q)$ and $\mathscr{D}(0,\infty)$ is dense in $E_{p,q}$ for all $p < q$, we deduce that $\mathcal{M}_{\widetilde{w}}$ solves the equation (2.2). By (4.3), it also satisfies (4.34)–(4.36). Since, by the continuity of $\mathcal{M}_{\widetilde{w}}$ at $t = \gamma^{-1}$ we also have $\mathcal{M}_w(\gamma^{-1},s) = \mathcal{M}_u(\gamma^{-1},s) = \Omega((\gamma^{-1},s))$ for all $s \in \mathscr{S}(\rho,\rho+\gamma)$.

It follows that $\widetilde{w}$ satisfies all the hypotheses of Proposition 4.9. We deduce from that Proposition that $\widetilde{w} = \omega$ for $t \in (\gamma^{-1}, T)$. We deduce by continuity that $\widetilde{w} = w$ for $t \in (0, T)$. □

## 5. $\gamma > 0$. Non existence of non negative solutions for large time.

The measure $w$ defined in Theorem 4.7 is a global weak solution of (1.1), (1.8) for all values of the parameter $\theta$ and all possible values of the roots $\sigma_1$ and $\sigma_2$. However, as we prove in this Section, it is not always a non negative solution. By uniqueness of the possible extensions of the local solution $u$, as stated in Theorem 4.7, it follows that it can not be extended to a suitable weak solution beyond $t = \gamma^{-1}$.

Our next result is concerned with the sign of the solution $\omega$ obtained in Theorem 4.7, and the possible extension of the local solution $u$ to a global non negative solution.

THEOREM 5.1. — *If $\theta > 1$, the solution $w$ obtained in Theorem 4.7 is not always non negative for $t > \gamma^{-1}$.*

*Proof of Theorem 5.1.* — If we define the two following functions of $t > 0$:

$$A(t) = \gamma \frac{(\gamma t)^{\frac{\sigma_1 - \sigma_2}{\gamma} - 1}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(1 + \frac{\sigma_2 - \sigma_1}{\gamma})}$$

$$H(t) = \Gamma\left(1 + \frac{\sigma_2}{\gamma}\right) F\left(1 - \frac{\sigma_1}{\gamma}, 1 + \frac{\sigma_2 - \sigma_1}{\gamma}; 1 - \frac{\sigma_1 - \sigma_2}{\gamma}; \frac{1}{\gamma t}\right)$$

$$= \Gamma\left(1 + \frac{\sigma_2}{\gamma}\right)\left(1 - \frac{1}{\gamma t}\right)^{-1 + \frac{\sigma_1}{\gamma}}$$

the right hand side of (4.21) may be written as follows:

$$\omega(t, x) = A(t) x^{-\sigma_2 - \gamma}\left(H(t) + B(t, x)\right)$$

$$B(t, x) = -(\gamma x^{\gamma} t)^{-1}\left(\sum_{m=0}^{\infty} \frac{(\gamma x^{\gamma} t)^{-m}\Gamma(2 + m + \frac{\sigma_2}{\gamma})(-1)^m}{\Gamma(2 + m)} \times F\left(1 - \frac{\sigma_1}{\gamma}, 2 + m + \frac{\sigma_2 - \sigma_1}{\gamma}, \frac{1}{\gamma t}\right)\right),$$

where, by (1.11), $A(t) \geqslant 0$ and $H(t) \geqslant 0$ for all $t > \gamma^{-1}$. By 15.7.2 in [1], there exists a constant $C = C(\sigma_1, \sigma_2, \gamma)$ such that for all $t > \frac{1}{\gamma}$ and $m \geqslant 0$:

$$\left| F\left(1 - \frac{\sigma_1}{\gamma}, 2 + m + \frac{\sigma_2 - \sigma_1}{\gamma}, \frac{1}{\gamma t}\right)\right| \leqslant C e^{\left(2 + m + \frac{\sigma_2 - \sigma_1}{\gamma}\right)\frac{1}{\gamma t}}$$

Then, for all $t > \gamma^{-1}$, all $x > 1$ and $m \geqslant 0$

$$\left| \frac{(\gamma x^\gamma t)^{-m}\Gamma(2+m+\frac{\sigma_2}{\gamma})(-1)^m}{\Gamma(2+m)} F\left(1-\frac{\sigma_1}{\gamma}, 2+m+\frac{\sigma_2-\sigma_1}{\gamma}, \frac{1}{\gamma t}\right) \right| \leqslant C e^{-m\left(\log(\gamma x^\gamma t)-\frac{1}{\gamma t}\right)} \frac{\Gamma(2+m+\frac{\sigma_2}{\gamma})}{\Gamma(2+m)}.$$

But, from Stirling's formula:

$$\frac{\Gamma(2+m+\frac{\sigma_2}{\gamma})}{\Gamma(2+m)} = \mathcal{O}\left(\left(2+m+\frac{\sigma_2}{\gamma}\right)^{\frac{\sigma_2}{\gamma}}\right), \quad m \to \infty.$$

and therefore:

$$\left| \sum_{m=0}^{\infty} \frac{(\gamma x^\gamma t)^{-m}\Gamma(2+m+\frac{\sigma_2}{\gamma})(-1)^m}{\Gamma(2+m)} \times F\left(1-\frac{\sigma_1}{\gamma}, 2+m+\frac{\sigma_2-\sigma_1}{\gamma}, \frac{1}{\gamma t}\right) \right| \varepsilon C \sum_{m=0}^{\infty} e^{-m\left(\log(\gamma x^\gamma t)-\frac{1}{\gamma t}\right)} m^{\frac{\sigma_2}{\gamma}}.$$

The series in the right hand side defines a bounded function on the domain $t \geqslant \gamma^{-1}$, $x \geqslant R_\gamma$ for $R_\gamma > 0$ fixed large enough to have

$$\log(\gamma R_\gamma^\gamma t) - \frac{1}{\gamma t} \geqslant \delta > 0, \quad \forall\, t \geqslant \frac{1}{\gamma}.$$

We then deduce that, for every $t > \gamma^{-1}$:

$$\omega(t,x) = A(t) x^{-\sigma_2-\gamma} \left(H(t) + \mathcal{O}\left(\frac{1}{x^\gamma t}\right)\right), \quad x \to \infty. \tag{5.1}$$

Suppose now that $\theta > 1$, from where $\sigma_2 \in \mathbb{C} \setminus \mathbb{R}$, and fix any $t_0 > \gamma^{-1}$. There exists $R = R(t_0)$ large enough such that:

$$\begin{aligned} A(t_0)(H(t_0) + B(t,x)) &\geqslant A(t_0)\left(H(t_0 - |B(t,x)|\right) \\ &\geqslant \frac{1}{2} A(t_0) H(t_0) > 0, \quad \forall\, x > R. \end{aligned}$$

Since $\sigma_2 \in \mathbb{C} \setminus \mathbb{R}$, the function $x^{-\sigma_2-\gamma}$ is oscillatory. Therefore by (5.1), $u(t_0,x)$ can not remain non negative for all $x > R$. □

*Remark 5.2.* — If $\theta = 1$, then $\sigma_1 = \sigma_2 = 1$ and our final argument in the proof of Theorem 5.1 fails.

*Proof of Theorem 1.3.* — Suppose that there exists a non negative weak solution $\widetilde{w}$ satisfying (1.16), (1.25) for some $T > \gamma^{-1}$. Since $u \geqslant 0$, it follows from (1.25) that $u \in \mathscr{C}([0,T); E'_{\rho-\delta,\rho+\gamma+\delta})$. The function, $\mathcal{M}_u(t)$ is then well defined and analytic on $\mathscr{S}(\rho-\delta, \rho+\gamma+\delta)$ and satisfies (2.2) for any $s_* \in (\rho, \rho+\gamma)$. By (1.25) again, we deduce that $\mathcal{M}_u$ satisfies (4.3). Then we have, by Theorem 4.2, $\widetilde{w}(t) = w(t)$ for $t \in (T, \gamma^{-1})$. But, by Theorem 5.1, $w$ is not non negative on $(0, \infty)$ when $t > \gamma^{-1}$, and this contradiction concludes the proof. □

The following Theorem on non existence of global solutions follows from the uniqueness result of Theorem 4.7 and Theorem 5.1, in the same way as Theorem 1.3 follows from Theorem 4.2 and Theorem 5.1.

THEOREM 5.3. — *Suppose that* $\theta > 1$ *and* $T > \gamma^{-1}$. *Then there is no possible extension of the local solution* $u$ *to a non negative weak solution* $\widetilde{w} \in \mathscr{D}_1'((0,T) \times (0,\infty))$ *of* (1.1), (1.8) *satisfying the initial condition* (1.16) *and such* $\mathcal{M}_{\widetilde{w}}$ *satisfies the conditions* (4.24)–(4.29).

When $\theta \in (0,1)$ the condition (1.5) is satisfied and by Corollary 4.2 in [8] a global nonnegative solution $\mu$ exists. Although we do not know if $\mu = w$, the solution obtained in Theorem 4.1, in general, we have:

PROPOSITION 5.4. — *Suppose that* $\theta \in (0,1)$ *and* $0 < \gamma < \sqrt{1-\theta}$. *Let* $\mu$ *be the global non negative solution of* (1.1), (1.8) *obtained in* [8] *and* $w$ *the solution obtained in Theorem 4.1. Then* $\mu = w$.

*Proof.* — When $\theta \in (0,1)$ condition (1.5) is satisfied and by Corollary 4.2 in [8] a global nonnegative solution $\mu$ exists. Moreover, it follows from Lemma 4.3 (i) and Corollary 4.2 in [8] that $\mu \in \mathscr{C}([0,T); E'_{1,\sigma_2})$ and $\mathcal{M}_\mu \in$ satisfies (4.3) for any $\rho \in (1, \sigma_2 - \gamma)$. By the uniqueness of such solutions proved in Theorem 4.2 it follows that $\mu = w$. □

*Remark 5.5.* — The results in [8] are proved for general dislocation measures, for which the corresponding function $\Phi(s)$ could be defined only for $\Re e s \geqslant 2$. When $k_0$ as in (1.8), $\Phi(s)$ is defined for all $\Re e s > -1$ and the solutions in [8] may then be expected to have moments of order $s$ in a larger interval than $\Re e(s) \in (1, \sigma_2)$. That could make Proposition 5.4 to be true under a weaker condition than $\gamma < \sqrt{1-\theta}$.

## 6. The case $\gamma < 0$.

When $\gamma < 0$ the function obtained in Section 3.1

$$\Omega(t,s) = F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right) \equiv (1-\gamma t)^{\frac{2-s}{\gamma}} F\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right).$$

is a solution of (2.2) for all $t > 0$. If, to obtain a solution to (1.1), (1.8), our purpose was still to take its inverse Mellin transform, the inverse Mellin transform should be defined along a vertical integration curve contained in the half plane $\Re e(s) > 0$, as in the case $\gamma > 0$. But now the poles of $\Omega(t)$, namely $s=-m\gamma, m\in\mathbb{N}$, are non negative real numbers. It follows that the moments $\mathcal{M}_u(t,r)$ of any solution $u$ of (1.1), (1.8) whose Mellin transform is $\Omega(t,s)$ will be bounded only for $r$ in an interval $(-m\gamma,-(m+1)\gamma)$ for some $m\in\mathbb{N}$. For that reason we look for another solution of (2.2)–(2.3)

### 6.1. Still another solution of (2.2)–(2.3)

The function

$$V_2(s)=\frac{\gamma^{\frac{s}{\gamma}}\Gamma(1-\frac{s}{\gamma})}{\Gamma(1-\frac{s-\sigma_1}{\sigma})\Gamma(1-\frac{s-\sigma_2}{\sigma})} \tag{6.1}$$

satisfies:

$$V_2(s+\gamma)=-\frac{(s-\sigma_1)(s-\sigma_2)}{s}V_2(s) \tag{6.2}$$

for all $s\in\mathbb{C}\setminus\{s\in\mathbb{C}; s=-m\gamma,\ m\in\mathbb{N}\}$. For $\sigma_0>0$ fixed let $\widetilde{\mathscr{C}_\theta}$ be the following curve in the complex plane:

$$\begin{aligned}\widetilde{\mathscr{C}_\theta}&=\widetilde{\mathscr{C}_{1,\theta}}\cup\widetilde{\mathscr{C}_2}\\ \widetilde{\mathscr{C}_{1,\theta}}&=\{s=\xi+i\zeta\in\mathbb{C};\xi=\sigma_0+\theta\zeta,\ \ \zeta\geqslant 0\}\\ \widetilde{\mathscr{C}_2}&=\{s=\xi+i\zeta\in\mathbb{C};\xi=\sigma_0,\ \ \zeta<0\}\end{aligned}$$

and define

$$U_2(t,s)=\frac{1}{\gamma V_2(s)}\int_{\widetilde{\mathscr{C}_\theta}}\frac{(-t)^{\frac{\sigma-s}{\gamma}}V_2(\sigma)\mathrm{d}\sigma}{\Gamma(1+\frac{\sigma-s}{\gamma})(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1)}. \tag{6.3}$$

Proposition 6.1. — *For all $t\in(0,(-\gamma)^{-1})$, the integral in the right hand side of (6.3) is absolutely convergent and defines the following meromorphic function $U_2(t)$ in the complex plane:*

$$U_2(t,s)=\Omega_1(t,s)-\Omega_2(t,s) \tag{6.4}$$

$$\Omega_1(t,s)=\frac{(-\gamma t)^{-\frac{s}{\gamma}}}{(e^{\frac{2i\pi}{\gamma}s}-1)}\frac{\gamma t\,\Gamma(1-\frac{s-\sigma_1}{\gamma})\Gamma(1-\frac{s-\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(1-\frac{s}{\gamma})}\times\frac{F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2-\frac{s}{\gamma},\gamma t)}{\Gamma(2-\frac{s}{\gamma})} \tag{6.5}$$

$$\Omega_2(t,s)=\frac{i}{2\pi}(1-\gamma t)^{\frac{2-s}{\gamma}}F\left(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t\right) \tag{6.6}$$

*and*

$$U_2\in\mathscr{C}([0,-\gamma^{-1})\times\mathscr{S}(1-\gamma,\infty)). \tag{6.7}$$

*Remark 6.2.* — By 15.3.3 in [1]:

$$\Omega_2(t,s) \equiv \frac{i}{2\pi} F\left(\frac{s-\sigma_1}{\gamma}, \frac{s-\sigma_2}{\gamma}, \frac{s}{\gamma}, \gamma t\right).$$

*Proof.* — Using Stirling's formulas in the same way as in Section 4.1, we obtain by straightforward calculation that for all $t \in (0, (-\gamma)^{-1})$ and $s \in \mathbb{C}$, there exists a positive constant $C = C(s,\gamma)$:

$$\left|\frac{(-t)^{\frac{\sigma}{\gamma}} V_2(\sigma) \mathrm{d}\sigma}{\Gamma\left(1+\frac{\sigma-s}{\gamma}\right)\left(e^{-\frac{2i\pi}{\gamma}(\sigma-s)}-1\right)}\right| \leqslant \begin{cases} Ce^{\frac{\xi}{\gamma}\log(-\gamma t)} e^{-\frac{\zeta\pi}{\gamma}}, & v > 0 \\ Ce^{\frac{\xi}{\gamma}\log(-\gamma t)} e^{\frac{2\zeta\pi}{\gamma}}, & v < 0. \end{cases} \tag{6.8}$$

The right hand side is exponentially decaying as $\zeta \to -\infty$ for any $\xi$ fixed. On the other hand, suppose that $-\gamma t \in (0,\tau)$ with $\tau < 1$. Then $\log(-\gamma t) < \log \tau < 0$ and

$$0 < \frac{\log \tau}{\gamma} < \frac{\log(-\gamma t)}{\gamma}.$$

Therefore, if $\sigma = \xi + i\zeta \in \widetilde{\mathscr{C}_{1,\theta}}$, $\xi = \sigma_0 + \theta\zeta$ and:

$$\begin{aligned} \frac{\xi}{\gamma}\log(-\gamma t) - \frac{\zeta\pi}{\gamma} &= \frac{(\sigma_0+\theta\zeta)}{\gamma}\log(-\gamma t) - \frac{\zeta\pi}{\gamma} \\ &= \frac{\sigma_0}{\gamma}\log(-\gamma t) + \theta\zeta\frac{\log(-\gamma t)}{\gamma} - \frac{\zeta\pi}{\gamma} \\ &= \frac{\sigma_0}{\gamma}\log(-\gamma t) + \zeta\left(\theta\frac{\log(-\gamma t)}{\gamma} - \frac{\pi}{\gamma}\right). \end{aligned}$$

If we choose now $\theta < 0$ we deduce:

$$\frac{\xi}{\gamma}\log(-\gamma t) - \frac{\zeta\pi}{\gamma} < \frac{\sigma_0}{\gamma}\log(-\gamma t) + \zeta\left(\theta\frac{\log\tau}{\gamma} - \frac{\pi}{\gamma}\right)$$

and, if

$$\theta < \frac{\pi}{\log\tau}$$

the right hand side of (6.8) is also exponentially decreasing as $\zeta \to \infty$ and $\sigma \in \mathscr{C}_\theta$. The function under the integral in (6.3) is then absolutely integrable.

The integral may now be obtained using the method of residues. The poles of the function to integrate are:

$$\sigma = \gamma(m+1), \quad \sigma = s + \gamma m, \quad m \in \mathbb{N}.$$

and then, for $-\gamma t \in (0,1)$:

$$U(t,s) = \frac{(-t)^{-\frac{s}{\gamma}}}{\gamma V(s)} \sum_{m=0}^{\infty} \frac{(-t)^{m+1}\gamma^{m+1} Res(\Gamma(1-\frac{\sigma}{\gamma});\sigma=\gamma(m+1))}{\Gamma(\frac{\sigma_1}{\gamma}-m)\Gamma(\frac{\sigma_2}{\sigma}-m)\Gamma(2+m-\frac{s}{\gamma})(e^{\frac{2i\pi}{\gamma}s}-1)}$$

$$+\frac{1}{\gamma V(s)} \sum_{m=0}^{\infty} \frac{(-t)^m \gamma^{\frac{s}{\gamma}+m}\Gamma(1-\frac{s}{\gamma}-m) Res((e^{\frac{2i\pi}{\gamma}(\sigma-s)}-1)^{-1};\sigma=s+\gamma m)}{\Gamma(1-\frac{s-\sigma_1}{\gamma}-m)\Gamma(1-\frac{s-\sigma_2}{\sigma}-m)\Gamma(1+m)}.$$

We use that, for all $m \in \mathbb{N}$:

$$Res\left(\Gamma\left(1-\frac{\sigma}{\gamma}\right);\sigma=\gamma(m+1)\right) = \frac{(-1)^{m+1}\gamma}{\Gamma(m+1)}$$

$$Res\left(\left(e^{\frac{2i\pi}{\gamma}(\sigma-s)}-1\right)^{-1};\sigma=s+\gamma m\right) = -\frac{i\gamma}{2\pi}$$

and obtain, for $-\gamma t \in (0,1)$:

$$U(t,s) = \frac{(-t)^{-\frac{s}{\gamma}}}{V(s)(e^{\frac{2i\pi}{\gamma}s}-1)} \sum_{m=0}^{\infty} \frac{(t)^{m+1}\gamma^{m+1}}{\Gamma(\frac{\sigma_1}{\gamma}-m)\Gamma(\frac{\sigma_2}{\sigma}-m)\Gamma(2+m-\frac{s}{\gamma})\Gamma(m+1)}$$

$$-\frac{i\gamma^{\frac{s}{\gamma}}}{2\pi V(s)} \sum_{m=0}^{\infty} \frac{(-t)^m\gamma^m\Gamma(1-\frac{s}{\gamma}-m)}{\Gamma(1-\frac{s-\sigma_1}{\gamma}-m)\Gamma(1-\frac{s-\sigma_2}{\sigma}-m)\Gamma(1+m)}.$$

The two series may be summed when $-\gamma t \in (0,1)$:

$$\sum_{m=0}^{\infty} \frac{(t)^{m+1}\gamma^{m+1}}{\Gamma(\frac{\sigma_1}{\gamma}-m)\Gamma(\frac{\sigma_2}{\sigma}-m)\Gamma(2+m-\frac{s}{\gamma})\Gamma(m+1)} = \frac{\gamma t\, F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2-\frac{s}{\gamma},\gamma t)}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(2-\frac{s}{\gamma})}$$

$$\sum_{m=0}^{\infty} \frac{(-t)^m\gamma^m\Gamma(1-\frac{s}{\gamma}-m)}{\Gamma(1-\frac{s-\sigma_1}{\gamma}-m)\Gamma(1-\frac{s-\sigma_2}{\sigma}-m)\Gamma(1+m)} = \frac{\Gamma(1-\frac{s}{\gamma})F(\frac{s-\sigma_1}{\gamma},\frac{s-\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t)}{\Gamma(1-\frac{s-\sigma_1}{\gamma})\Gamma(1-\frac{s-\sigma_2}{\gamma})}.$$

We deduce:

$$U_2(t,s) = \frac{(-t)^{-\frac{s}{\gamma}}}{V_2(s)(e^{\frac{2i\pi}{\gamma}s}-1)} \frac{\gamma t\, F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2-\frac{s}{\gamma},\gamma t)}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(2-\frac{s}{\gamma})}$$

$$-\frac{i\gamma^{\frac{s}{\gamma}}}{2\pi V_2(s)} \frac{\Gamma(1-\frac{s}{\gamma})F(\frac{s-\sigma_1}{\gamma},\frac{s-\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t)}{\Gamma(1-\frac{s-\sigma_1}{\gamma})\Gamma(1-\frac{s-\sigma_2}{\gamma})}.$$

and the explicit expression of $U_2(t)$ in (6.4)–(6.6) follows using the expression (6.1) of $V_2(s)$. The property (6.7) follows now from (6.4)–(6.6) and the well known properties of the Gamma and hypergeometric functions □

### 6.2. The solution $v$

We wish now to define a solution $u$ of (1.1), (1.8) by means of a suitable inverse Mellin transform of $U_2(t)$. The inverse Mellin transform of $\Omega_2$ has been obtained in Section (3.2). Some useful properties of $\Omega_1$ are now given in the next Proposition.

PROPOSITION 6.3. — *For all $t \in (0, -\gamma^{-1})$, the function $\Omega_1(t)$ is analytic for $s \in \mathbb{C}$ such that $\Re e(s) > 1+\gamma$. For all $t \in (0, -\gamma^{-1})$ there exists a positive constant $C$ such that*

$$|\Omega_1(t,s)| \leqslant \begin{cases} C(t)e^{\frac{2\pi \Im m(s)}{\gamma}}, & \Im m(s) > 0 \\ C(t)(1+|\Im m(s)|)^{-1+\frac{2}{\gamma}}, & \Im m(s) < 0. \end{cases} \tag{6.9}$$

*Proof.* — The estimate follows from the expression of $\Omega_1$ and Stirling's formula. □

We deduce from the Proposition 6.3 that we may set

$$v(t,x) = \frac{1}{2i\pi} \int_{\Re e(s)=s_0} x^{-s} U_2(t,s) \mathrm{d}s, \quad s_0 > 1+\gamma \tag{6.10}$$

and, by classical results on the Mellin and inverse Mellin transform this expression defines a measure $v \in \mathscr{C}\left([0,-\gamma^{-1}); E'_{1+\gamma,\infty}\right)$.

PROPOSITION 6.4. — *For all $t \in (0, -\gamma^{-1})$, $U_2(t)$ is meromorphic on $\mathbb{C}$ with poles located at:*

$$s = \sigma_\ell + (m+1)\gamma, \quad \ell = 1,2, \quad m \in \mathbb{N}.$$

*If we call:*

$$A_m(t) = Res\left(U(t,s); s = \sigma_1 + \gamma(m+1)\right)$$
$$B_m(t) = Res\left(U(t,s); s = \sigma_2 + \gamma(m+1)\right),$$

*we have:*

$$A_m(t) = \frac{(-\gamma t)^{-\frac{\sigma_1}{\gamma}-(m+1)}}{(e^{\frac{2i\pi}{\gamma}\sigma_1}-1)} \frac{\gamma t\, \Gamma(-\frac{\sigma_1-\sigma_2}{\gamma}-m)(-1)^{m+1}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(-\frac{\sigma_1}{\gamma}-m)\Gamma(m+1)} \times \frac{F(1-\frac{\sigma_1}{\gamma}, 1-\frac{\sigma_2}{\gamma}, 1-\frac{\sigma_1}{\gamma}-m, \gamma t)}{\Gamma(1-\frac{\sigma_1}{\gamma}-m)}$$

$$B_m(t) = \frac{(-\gamma t)^{-\frac{\sigma_2}{\gamma}-(m+1)}}{(e^{\frac{2i\pi}{\gamma}\sigma_2}-1)} \frac{\gamma t\, \Gamma(-\frac{\sigma_2-\sigma_1}{\gamma}-m)(-1)^{m+1}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(-\frac{\sigma_2}{\gamma}-m)\Gamma(m+1)} \times \frac{F(1-\frac{\sigma_1}{\gamma}, 1-\frac{\sigma_2}{\gamma}, 1-\frac{\sigma_2}{\gamma}-m, \gamma t)}{\Gamma(1-\frac{\sigma_2}{\gamma}-m)}.$$

*Moreover, $U_2$ satisfies (2.2) for $t > \gamma^{-1}$ and $s$ such that $\Re e(s) > 1+\gamma$.*

*Proof.* — For each $t\in(0,-\gamma^{-1})$ the functions $\Omega_1(t)$ and $\Omega_2(t)$ are meromorphic on $\mathbb{C}$ with poles located respectively at $s=-\gamma m,\quad s=\sigma_\ell+\gamma(m+1)$ and $s=-\gamma m$, $m\in\mathbb{N}$. But, at poles $s=-m\gamma$ we have:

$$\begin{aligned}Res(\Omega_1(t),s=-\gamma m)&=Res\left(\left(e^{\frac{2i\pi}{\gamma}s}-1\right)^{-1};s=-\gamma m\right)(-\gamma t)^m\\&\quad\times\frac{\gamma t\,\Gamma(1+\frac{\sigma_1}{\gamma}+m)\Gamma(1+\frac{\sigma_2}{\gamma}+m)}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(1+m)}\frac{F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2+m,\gamma t)}{\Gamma(2+m)}\\&=-\frac{i\gamma}{2\pi}(-\gamma t)^m\frac{\gamma t\,\Gamma(1+\frac{\sigma_1}{\gamma}+m)\Gamma(1+\frac{\sigma_2}{\gamma}+m)}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(1+m)}\\&\quad\times\frac{F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2+m,\gamma t)}{\Gamma(2+m)}.\end{aligned}$$

On the other hand, since

$$F\left(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t\right)=\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\sigma_1}{\gamma}+n)\Gamma(\frac{\sigma_2}{\gamma}+n)\Gamma(\frac{s}{\gamma})(\gamma t)^n}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(\frac{s}{\gamma}+n)\Gamma(n+1)}.$$

we deduce:

$$\begin{aligned}&Res\left(F\left(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t\right),s=-\gamma m\right)\\&\qquad=\gamma\sum_{n=m+1}^{\infty}\frac{\Gamma(\frac{\sigma_1}{\gamma}+n)\Gamma(\frac{\sigma_2}{\gamma}+n)(\gamma t)^n(-1)^m}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(-m+n)\Gamma(n+1)\Gamma(m+1)}.\end{aligned}$$

This series may still be summed,

$$\begin{aligned}&Res\left(F\left(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t\right),s=-\gamma m\right)\\&\qquad=\gamma\frac{(\gamma t)^{m+1}(-1)^m\Gamma(1+m+\frac{\sigma_1}{\gamma})\Gamma(1+m+\frac{\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(m+2)\Gamma(m+1)}\\&\qquad\quad\times F\left(1+m+\frac{\sigma_1}{\gamma},1+m+\frac{\sigma_2}{\gamma},2+m,\gamma t\right).\end{aligned}$$

We use now 15.3.3 in [1] to write:

$$\begin{aligned}&F\left(1+m+\frac{\sigma_1}{\gamma},1+m+\frac{\sigma_2}{\gamma},2+m,\gamma t\right)\\&\qquad=(1-\gamma t)^{-m-\frac{2}{\gamma}}F\left(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2+m,\gamma t\right)\end{aligned}$$

from where:

$$Res\left(F\left(\frac{\sigma_1}{\gamma},\frac{\sigma_2}{\gamma},\frac{s}{\gamma},\gamma t\right),s=-\gamma m\right)$$
$$=\gamma\frac{(\gamma t)^{m+1}(-1)^m\Gamma(1+m+\frac{\sigma_1}{\gamma})\Gamma(1+m+\frac{\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(m+2)\Gamma(m+1)}$$
$$\times(1-\gamma t)^{-m-\frac{2}{\gamma}}F\left(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2+m,\gamma t\right)$$

and then,

$$Res\left(\Omega_2(t),s=-\gamma m\right)$$
$$=\frac{i}{2\pi}\gamma\frac{(\gamma t)^{m+1}(-1)^m\Gamma(1+m+\frac{\sigma_1}{\gamma})\Gamma(1+m+\frac{\sigma_2}{\gamma})}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(m+2)\Gamma(m+1)}$$
$$\times F\left(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},2+m,\gamma t\right).$$

Therefore, at $s=-m\gamma$, the residues of $\Omega_1(t)$ and $\Omega_2(t)$ are equal and therefore, they cancel when combined to obtain the residue of $U(t)$. On the other hand, the residues of $\Omega_1(t)$ at $s=\sigma_2+\gamma(m+1)$:

$$Res\left(\Omega_1(t,s);s=\sigma_2+\gamma(m+1)\right)$$
$$=\frac{(-\gamma t)^{-\frac{\sigma_2}{\gamma}-(m+1)}}{(e^{\frac{2i\pi}{\gamma}\sigma_2}-1)}\frac{\gamma t\,\Gamma(-\frac{\sigma_2-\sigma_1}{\gamma}-m)(-1)^{m+1}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(-\frac{\sigma_2}{\gamma}-m)\Gamma(m+1)}$$
$$\times\frac{F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},1-\frac{\sigma_2}{\gamma}-m,\gamma t)}{\Gamma(1-\frac{\sigma_2}{\gamma}-m)}$$

and similarly:

$$Res\left(\Omega_1(t,s);s=\sigma_1+\gamma(m+1)\right)$$
$$=\frac{(-\gamma t)^{-\frac{\sigma_1}{\gamma}-(m+1)}}{(e^{\frac{2i\pi}{\gamma}\sigma_1}-1)}\frac{\gamma t\,\Gamma(-\frac{\sigma_1-\sigma_2}{\gamma}-m)(-1)^{m+1}}{\Gamma(\frac{\sigma_1}{\gamma})\Gamma(\frac{\sigma_2}{\gamma})\Gamma(-\frac{\sigma_1}{\gamma}-m)\Gamma(m+1)}$$
$$\times\frac{F(1-\frac{\sigma_1}{\gamma},1-\frac{\sigma_2}{\gamma},1-\frac{\sigma_1}{\gamma}-m,\gamma t)}{\Gamma(1-\frac{\sigma_1}{\gamma}-m)}.$$

Arguing as in the proof of Proposition 4.3, we deduce that $U_2$ satisfies (2.2) for $t>\gamma^{-1}$ and $s$ such that $\Re e(s)>1+\gamma$. □

Theorem 6.5. — *The measure* $u\in\mathscr{C}\left([0,-\gamma^{-1});E'_{1+\gamma,\infty}\right)$ *defined in* (6.10) *is a weak solution of* (1.1), (1.8) *on* $t\in(0,-\gamma^{-1})$. *It satisfies*

$$\int_0^\infty v(t,x)x^{s-1}\mathrm{d}x=U_2(t,s),\quad\forall\, s\in\mathbb{C};\quad\Re e(s)>1+\gamma.\tag{6.11}$$

*If $\theta > 1$, for all $t \in (0, -\gamma^{-1})$ the measure $v(t)$ takes positive and negative values on $(0, \infty)$.*

*Proof.* — The identity (6.11) follows from Theorem 11.10.1 in [22] and by classical properties of the Mellin transform, $u$ is a weak solution of (1.1), (1.8).

It follows from (6.10) and the properties of $U_2(t)$ that as $x \to 0$:

$$v(t,x) = \Re e\left(A_0(t)x^{-\sigma_2-\gamma}\right) + \Re e\left(B_0(t)x^{-\sigma_1-\gamma}\right) + o\left(x^{-1-\gamma}\right), \quad x \to 0$$

If $\theta > 1$, $\sigma_2 = 1 + i\zeta$, $\sigma_1 = 1 - i\zeta$ with $\zeta = \sqrt{\theta - 1}$,

$$x^{-\sigma_2-\gamma} = x^{-1-i\zeta-\gamma} = x^{-1-\gamma}\left(\cos(\zeta \log x) - i\sin(\zeta \log x)\right)$$
$$x^{-\sigma_1-\gamma} = x^{-1-i\zeta-\gamma} = x^{-1-\gamma}\left(\cos(\zeta \log x) + i\sin(\zeta \log x)\right)$$

and we deduce, for all $t$ fixed, as $x \to 0$:

$$v(t,x) = x^{-1-\gamma}\left(h_1(t)\cos(\zeta \log x) + h_2(t)\sin(\zeta \log x)\right) + o\left(x^{-1-\gamma}\right) \quad (6.12)$$
$$h_1(t) = \Re e A_0(t) + \Re e B_0(t)$$
$$h_2(t) = \Im m A_0(t) - \Im m B_0(t).$$

Consider now two values of $x$:

$$x_1 = e^{-\frac{2\ell\pi}{\zeta}}, \quad x_2 = e^{-\frac{(2\ell+1)\pi}{\zeta}}$$

where $\ell \in \mathbb{N}$ has to be fixed. From (6.12):

$$v(t,x_1) = x_1^{-1-\gamma}h_1(t) + o(x_1^{-1-\gamma}), \qquad x_1 \to 0$$
$$v(t,x_2) = -x_1^{-1-\gamma}h_1(t) + o(x_1^{-1-\gamma}), \quad x_2 \to 0.$$

We chose now $\ell$ large enough to have:

$$v(t,x_1) \geqslant \frac{x_1^{-1-\gamma}h_1(t)}{2} > 0, \quad u(t,x_2) \leqslant -\frac{x_1^{-1-\gamma}h_1(t)}{2} < 0 \quad \text{if } h_1(t) > 0$$
$$v(t,x_1) \leqslant \frac{x_1^{-1-\gamma}h_1(t)}{2} < 0, \quad u(t,x_2) \geqslant -\frac{x_1^{-1-\gamma}h_1(t)}{2} > 0 \quad \text{if } h_1(t) < 0. \quad \square$$

*Proof of Theorem 1.5.* — We argue by contradiction and suppose that such a local solution, that we denote $\widetilde{v}$, exists on some time interval $(0, T)$. We may suppose without loss of generality that $T < -\gamma^{-1}$. By hypothesis, $\mathcal{M}_{\tilde{v}}$ satisfies all the assumptions in Theorem 7.1. By (6.7), $U_2$ satisfies (7.1) for any $\rho > 1-\gamma$. By (6.9) and the property (3.5) of $\Omega_2$, $U_2$ also satisfies (7.2) for any $\rho > 1 - \gamma$. It follows by (6.11) that $\mathcal{M}_v$ also satisfies the hypotheses of Theorem 7.1 for any $\rho > 1 - \gamma$. Then $\mathcal{M}_{\tilde{v}}(t,s)=\mathcal{M}_v(t,s)$ for $t \in [0,T)$, $s \in \mathscr{S}(\rho, \rho - \gamma)$ and therefore $v(t) = \widetilde{v}(t)$ for $t \in [0, T)$ but this is not possible since $v$ takes positive and negative values in $(0, \infty)$. This contradiction concludes the proof. $\square$

*Remark 6.6.* — If $\theta \in (0,1)$ the existence of a unique global non negative solution $\mu$ is proved in Theorem 4.1 of [8]. It immediately follows from this and related results in Section 4 of [8] that the Mellin transform of $\mu$ is such that $\mathcal{M}_\mu \in \mathscr{C}([0,\infty); E'_{1+\gamma,\infty})$ and satisfies (7.2)–(7.3) for any $\rho > 1+\gamma$ and $W_0(s) = 1$. We deduce by Theorem 7.1 that $\mu = v$, the solution obtained in Theorem 6.5, on $t \in (0, -\gamma^{-1})$.

## 7. Appendix

### 7.1. Uniqueness of bounded analytic solutions of (2.2)

THEOREM 7.1. — *Given any $T > 0$ and $W_0(s)$ a bounded and analytic function on a strip $\mathscr{S}(\rho, \rho+|\gamma|)$ for some $\rho > 0$, there exists at most one solution $W$ to the equation* (2.2) *for $t \in (0,T)$, such that, for all $t \in (0,T)$, $W(t,s)$ is analytic on the strip $\mathscr{S}(\rho, \rho+|\gamma|)$, satisfying*

$$W \in C\left([0,T) \times \overline{\mathscr{S}(\rho, \rho+|\gamma|)}\right) \tag{7.1}$$

$$\sup\left\{|W(t,s)|;\ \ 0 \leqslant t \leqslant T,\ s \in \overline{\mathscr{S}(\rho, \rho+|\gamma|)}\right\} < \infty. \tag{7.2}$$

$$W(0,s) = W_0(s),\ \ \forall\, s \in \mathscr{S}(\rho, \rho+|\gamma|) \tag{7.3}$$

*Proof.* — Suppose that $W_\ell(t,s)$, $\ell = 1,2$ are two solutions, analytic on the strip $\mathscr{S}(\rho, \rho+|\gamma|)$, satisfying (7.1)–(7.3) and denote $W = W_1 - W_2$. The function $W$ satisfies the same conditions and $W(0) = 0$. Given any $T' < T$, let $\alpha(t)$ be a $C^\infty$ cutt-off function satisfying $\alpha(t) = 1$ for $0 \leqslant t \leqslant T'$ and $\alpha(t) = 0$ if $t \geqslant T$. If we define:

$$\widehat{W}(t,s) = W(t,s)\alpha(t)$$

we have

$$\frac{\partial \widehat{W}}{\partial t}(t,s) = \Phi(s)\widehat{W}(t,s+\gamma) + r(t,s) \tag{7.4}$$

where the function $r$ is bounded in $(0,T) \times \mathscr{S}(\rho, \rho+|\gamma|)$ and $r(t) \equiv 0$ for $0 \leqslant t \leqslant T'$. We apply now the Laplace transform in $t$ at both sides of (7.4) and obtain, for $\Re e(z) > 0$ and $s \in \mathscr{S}(\rho, \rho+|\gamma|)$:

$$z\widetilde{W}(z,s) = \Phi(s)\widetilde{W}(z,s+\gamma) + \widetilde{r}(z,s), \tag{7.5}$$

where, for some constant $C > 0$,

$$|\widetilde{r}(z,s)| \leqslant Ce^{-T'\Re e(z)},\quad \forall\, s \in \mathscr{S}(\rho, \rho+\gamma),\quad \Re e(z) > 0. \tag{7.6}$$

By the linearity of the equation in (7.5) we may write $\widetilde{W} = \widetilde{W}_{part} + \widetilde{W}_{hom}$ where $\widetilde{W}_{hom}$ solves

$$z\widetilde{W}_{hom}(z,s) = \Phi(s)\widetilde{W}_{hom}(z,s+\gamma),\quad \forall\, s \in \mathscr{S}(\rho, \rho+\gamma),\quad \Re e(z) > 0 \tag{7.7}$$

and $\widetilde{W}_{part}$ is a particular solution of (7.5). Arguing as with the function $\mathscr{U}$ defined by (4.5) in the Proof of Proposition 4.3 it follows that, if $\widetilde{V}(s)$ is the function defined by (4.16), then

$$\widetilde{W}_{part}(z,s)=\frac{i}{\gamma z\,\widetilde{V}(s)}\int_{\Re e(\sigma)=\sigma_0}\frac{(-z)^{\frac{\sigma-s}{\gamma}}\widetilde{V}(\sigma)\widetilde{r}(z,\sigma)\mathrm{d}\sigma}{(1-e^{-\frac{2i\pi}{\gamma}(s-\sigma)})} \tag{7.8}$$

satisfies (7.5) and, by (7.6),

$$|\widetilde{W}_{part}(z,s)|\leqslant Ce^{-T'\Re e(z)},\quad \forall\, s\in\mathscr{S}(\rho,\rho+|\gamma|),\quad \Re e(z)>0. \tag{7.9}$$

It is simpler to write our next argument if we distinguish now the cases $\gamma>0$ and $\gamma<0$, although the proof is completely similar in both cases. Let us then assume from now on that $\gamma>0$. We first perform the change of variables:

$$\zeta=e^{-\frac{2i\pi}{\gamma}(s-\rho)},\quad G(z,\zeta)=\widetilde{W}(z,s). \tag{7.10}$$

For all $z\in\mathbb{C}$ such that $\Re(z)>z_0$, the function $G(z,\zeta)$ is now analytic with respect to $\zeta$ for $\zeta\in\mathbb{C}\setminus\overline{\mathbb{R}^+}$ and bounded on $\mathbb{C}\setminus\overline{\mathbb{R}^+}$. We also have, using that $\widetilde{W}\in C((0,\infty)\times\overline{\mathscr{S}(\rho+|\gamma|)})$:

$$\begin{cases}\forall\, s=\rho+iv:\widetilde{W}(z,s)=G(z,x-i0), & x:=e^{\frac{2\pi v}{\gamma}}\\ \forall\, s=\rho+\gamma+iv:\widetilde{W}(z,s)=G(z,x+i0), & x:=e^{\frac{2\pi v}{\gamma}}.\end{cases} \tag{7.11}$$

We also define:

$$\begin{cases}\widetilde{\varphi}(\zeta)=\Phi(s)=\dfrac{(\rho-\sigma_1-\frac{\gamma}{2i\pi}\log\zeta)(\rho-\sigma_2-\frac{\gamma}{2i\pi}\log\zeta)}{(\rho-\frac{\gamma}{2i\pi}\log\zeta)}, & \forall\,\zeta\in\mathbb{C}\\ \varphi(x)=\lim\limits_{\varepsilon\to0}\widetilde{\varphi}\left(xe^{-i\varepsilon}\right), & \forall\, x>0.\end{cases} \tag{7.12}$$

where $\log(\zeta)=\log|\zeta|+i\arg(\zeta)$, and $\arg(\zeta)\in[0,2\pi)$. The equation reads:

$$G(z,x-i0)=\frac{\varphi(x)}{z}G(z,x+i0),\quad \forall\, x>0.$$

If we denote:

$$m(z,\zeta)=\frac{1}{2i\pi}\int_0^\infty \mathrm{Log}\left(\frac{\varphi(\lambda)}{z}\right)\left(\frac{1}{\lambda-\zeta}-\frac{1}{\lambda-\lambda_0}\right)\mathrm{d}\lambda$$

where $\mathrm{Log}(\zeta)=\log|\zeta|+i\,\mathrm{Arg}(\zeta)$, and $\mathrm{Arg}(\zeta)\in(-\pi/2,\pi/2]$, this is an analytic function on $\mathbb{C}\setminus\overline{\mathbb{R}^+}$ and by Plemej–Sojoltski formulas:

$$\frac{\varphi(x)}{z}=\frac{e^{m(z,x+i0)}}{e^{m(z,x-i0)}},\quad \forall\, x>0.$$

We deduce from the equation:

$$e^{m(z,x-i0)}G(z,x-i0)=e^{m(z,x+i0)}G(z,x+i0),\quad \forall\, x>0$$

and therefore, since $e^{m(z,\zeta)}G(z,\zeta)$ is analytic with respect to $\zeta$ for $\zeta \in \mathbb{C}\backslash\mathbb{R}^+$, the function

$$C(z,\zeta) = e^{m(z,\zeta)}G(z,\zeta)$$

is analytic on $\mathbb{C} \setminus \{0\}$. It remains to check the behavior of $C(z,\zeta)$ as $\zeta \to 0$ and $|\zeta| \to \infty$.

By definition:

$$\begin{aligned}\varphi(\zeta) &= \frac{(\rho - \sigma_1 - \frac{\gamma}{2i\pi}\log\zeta)(\rho - \sigma_2 - \frac{\gamma}{2i\pi}\log\zeta)}{(\rho - \frac{\gamma}{2i\pi}\log\zeta)} \\ &= -\frac{\gamma}{2i\pi}(\log|\zeta|)\frac{(1+\frac{\theta_1}{\log|\zeta|})(1+\frac{\theta_2}{\log|\zeta|})}{(1+\frac{\theta_0}{\log|\zeta|})}\end{aligned}$$

where

$$\theta_\ell = -\frac{2i\pi(\rho-\sigma_\ell)}{\gamma} + i\arg(\zeta), \quad \ell = 1,2; \quad \theta_0 = -\frac{2i\pi\rho}{\gamma} + i\arg(\zeta).$$

Then, as $|\zeta| \to 0$ or $|\zeta| \to \infty$

$$\varphi(\zeta) = -\frac{\gamma}{2i\pi}(\log|\zeta|)\left(1 + \mathcal{O}\left(\frac{1}{|\log|\zeta||}\right)\right). \tag{7.13}$$

It follows that

$$\frac{1}{2i\pi}\operatorname{Log}\left(\frac{\varphi(\lambda)}{z}\right) = \frac{1}{2i\pi}\operatorname{Log}\left(\frac{\gamma|\log\lambda|}{2\pi|z|}\right) + \frac{1}{2\pi}\arg\left(-\frac{\gamma\log\lambda}{2i\pi z}\right).$$

Since $\operatorname{Arg} z \in (-\pi/2, \pi/2)$ we have

$$\operatorname{Arg}\left(-\frac{\gamma\log\lambda}{2i\pi z}\right) = \operatorname{Arg}\left(i\frac{\gamma\log\lambda}{2\pi}\right) - \operatorname{Arg}(z)$$

where

$$\operatorname{Arg}\left(i\frac{\gamma\log\lambda}{2\pi}\right) = \begin{cases} -\frac{\pi}{2}, & \text{if } 0 < \lambda < 1 \\ \frac{\pi}{2}, & \text{if } \lambda > 1. \end{cases} \tag{7.14}$$

We may then write

$$\begin{aligned} m(z,\zeta) &= I_1(z,\zeta) + I_2(z,\zeta) + I_3(z,\zeta) \\ I_1(z,\zeta) &= \frac{1}{2i\pi}\int_0^\infty \log\left(\frac{\gamma|\log\lambda|}{2\pi|z|}\right)\left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right)\mathrm{d}\lambda \\ I_2(z,\zeta) &= \frac{1}{2\pi}\left(-\frac{\pi}{2} - \operatorname{Arg}(z)\right)\int_0^1 \left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right)\mathrm{d}\lambda \\ I_3(z,\zeta) &= \frac{1}{2\pi}\left(\frac{\pi}{2} - \operatorname{Arg}(z)\right)\int_1^\infty \left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right)\mathrm{d}\lambda \end{aligned} \tag{7.15}$$

If we take $\lambda_0 = i$:

$$\int_0^1 \left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right) \mathrm{d}\lambda = -\frac{i\pi}{4} - \frac{\operatorname{Log} 2}{2} + \operatorname{Log}\left(\frac{\zeta-1}{\zeta}\right)$$
$$\int_1^\infty \left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right) \mathrm{d}\lambda = -\frac{i\pi}{4} + \frac{\operatorname{Log} 4}{4} - \operatorname{Log}(1-\zeta)$$

and then,

$$(I_2+I_3)(z,\zeta) = \frac{1}{2\pi}\Big\{ \left(\frac{\pi}{2} + \operatorname{Arg}(z)\right) \operatorname{Log}\left(\frac{\zeta}{\zeta-1}\right) - \left(\frac{\pi}{2} - \operatorname{Arg}(z)\right) \operatorname{Log}(1-\zeta) \Big\} + R(z)$$
$$R(z) = \frac{1}{2\pi}\Big\{ -\left(\frac{\pi}{2} + \operatorname{Arg}(z)\right)\left(\frac{i\pi}{4} + \frac{\operatorname{Log} 2}{2}\right) + \left(\frac{\pi}{2} - \operatorname{Arg}(z)\right)\left(\frac{\operatorname{Log} 2}{2} - \frac{i\pi}{4}\right) \Big\}.$$

As $|\zeta| \to 0$,

$$\begin{aligned} \operatorname{Log}\left(\frac{\zeta}{\zeta-1}\right) &= \operatorname{Log}(\zeta) - \log(\zeta-1) \\ &= \log|\zeta| + i \operatorname{Arg}(\zeta) - \operatorname{Log}(\zeta-1) \\ &= \log|\zeta| \left(1 + \frac{i \operatorname{Arg}(\zeta) - \log(\zeta-1)}{\log|\zeta|}\right) \\ &= \log|\zeta| \left(1 + \mathcal{O}\left(\frac{1}{\log|\zeta|}\right)\right), |\zeta| \to 0 \end{aligned}$$

and then, as $|\zeta| \to 0$:

$$(I_2+I_3)(z,\zeta) = \frac{1}{2\pi}\left(\frac{\pi}{2} + \operatorname{Arg}(z)\right)\left(\log|\zeta| + \mathcal{O}\left(\frac{1}{\log|\zeta|}\right)\right).$$

Similarly, as $|\zeta| \to \infty$

$$\begin{aligned} \operatorname{Log}(1-\zeta) &= \log|\zeta-1| + i \operatorname{Arg}(1-\zeta) \\ &= \log|\zeta| + \log\left|1 - \frac{1}{\zeta}\right| + i \operatorname{Arg}(1-\zeta) \\ &= \log|\zeta| \left(1 + \frac{\log\left|1-\frac{1}{\zeta}\right| + i \operatorname{Arg}(1-\zeta)}{\log|\zeta|}\right) \\ &= \log|\zeta| \left(1 + \mathcal{O}\left(\frac{1}{\log|\zeta|}\right)\right) \end{aligned}$$

and, as $|\zeta| \to \infty$:

$$(I_2 + I_3)(z,\zeta) = -\frac{1}{2\pi}\left(\frac{\pi}{2} - \text{Arg}(z)\right)\left(\log|\zeta| + \mathcal{O}\left(\frac{1}{\log|\zeta|}\right)\right).$$

By (7.15), $e^{m(z,\zeta)} = e^{I_1(z,\zeta)}e^{(I_2+I_3)(z,\zeta)}$. We notice that $\left|e^{I_1(z,\zeta)}\right| = 1$ and, as $|\zeta| \to 0$:

$$\begin{aligned}\left|e^{(I_2+I_3)(z,\zeta)}\right| &= e^{\frac{1}{2\pi}\left(\frac{\pi}{2}+\text{Arg}(z)\right)\left(\log|\zeta|+\mathcal{O}\left(\frac{1}{\log|\zeta|}\right)\right)} \leqslant e^{\frac{1}{2\pi}\left(\frac{\pi}{2}+\text{Arg}(z)\right)(\log|\zeta|+1)}\\ &= e^{\frac{1}{2\pi}\left(\frac{\pi}{2}+\text{Arg}(z)\right)}e^{\frac{1}{2\pi}\left(\frac{\pi}{2}+\text{Arg}(z)\right)\log|\zeta|}\\ &= e^{\frac{1}{2\pi}\left(\frac{\pi}{2}+\text{Arg}(z)\right)}|\zeta|^{\frac{1}{2\pi}\left(\frac{\pi}{2}+\text{Arg}(z)\right)}.\end{aligned} \tag{7.16}$$

A similar argument gives, as $|\zeta| \to \infty$:

$$\left|e^{(I_2+I_3)(z,\zeta)}\right| \leqslant e^{-\frac{1}{2\pi}\left(\frac{\pi}{2}-\text{Arg}(z)\right)}|\zeta|^{-\frac{1}{2\pi}\left(\frac{\pi}{2}-\text{Arg}(z)\right)}. \tag{7.17}$$

Since $\text{Arg}(z) \in (-\pi/2, \pi/2)$:

$$0 < \frac{1}{2\pi}\left(\frac{\pi}{2} + \text{Arg}(z)\right) < \frac{1}{2}. \tag{7.18}$$

From the boundedness of the function $G(z,\cdot)$ on $\mathbb{C}$ and (7.16)–(7.18), we deduce that for all $z \in \mathbb{C}$, $\Re e(z) > z_0$, the function $C(z,\zeta) = G(z,\zeta)e^{m(z,\zeta)}$ is bounded as $|\zeta| \to 0$ and $|\zeta| \to \infty$. It follows that $C(z,\cdot)$ is independent of $\zeta$. Using (7.16) again

$$\lim_{\zeta\to 0} C(z,\zeta) = 0$$

and we deduce that $C(z,\zeta) = 0$ for all $\zeta$, then $G(z) \equiv 0$ for all $z \in \mathbb{C}$, $\Re e(z) > z_0$. Therefore $\widetilde{W}_{hom} = 0$ and then $\widetilde{W} = \widetilde{W}_{part}$. Laplace's inversion then yields:

$$\widehat{W}(t,s) = \frac{1}{2i\pi}\int_{b-i\infty}^{b+i\infty} \widehat{W}_{part}(t,s)e^{zt}\mathrm{d}z \tag{7.19}$$

for any $b > 0$. Then, (7.9) implies $\widehat{W}(t,s) = W(t,s) = 0$ for all $0 \leqslant t \leqslant T'$ and $s \in \mathscr{S}(\rho, \rho + |\gamma|)$. □

It is not always possible to apply Theorem 7.1 to the solutions of a Cauchy problem associated to (2.2). That is the case when $\gamma > 0$ and consider the solution $U$, obtained in Section 4, Proposition 4.3. Our next result is then useful:

Theorem 7.2. — *Suppose $\gamma > 0$. Given any $T > 0$ and $W_0(s)$ such that $sW_0(s)$ is a bounded and analytic function on the strip $\mathscr{S}(-\gamma, \varepsilon)$ for some $\varepsilon > 0$, there exists at most one solution $W$ to the equation* (2.2) *for*

*$t \in (0,T)$, such that, for all $t \in (0,T)$, $sW(t,s)$ is analytic on the strip $\mathscr{S}(-\gamma,\varepsilon)$, satisfying*

$$\begin{aligned} & sW \in C\left([0,T) \times \mathscr{S}(-\gamma+\delta, \varepsilon-\delta)\right), \quad \textit{for some } \delta \in (0,\varepsilon) \\ & \sup\left\{|sW(t,s)|;\quad 0 \leqslant t \leqslant T,\, s \in \mathscr{S}(-\gamma+\delta,\varepsilon-\delta)\right\} < \infty. \\ & W(0,s) = W_0(s), \quad \forall\, s \in \mathscr{S}(-\gamma,\varepsilon) \end{aligned}$$

*Proof of Theorem 7.2.* — Assume the existence of two such solutions to (2.2) and call $W$ their difference. Then, we define the function:

$$H(t,s) = sW(t,s)$$

If $\rho > 0$ is such that $(\rho, \rho+\gamma) \subset (-\gamma, \varepsilon)$, by our hypotheses on $W$:

$$\begin{aligned} \frac{\partial H(t,s)}{\partial t} &= \frac{(s-\sigma_1)(s-\sigma_2)}{s+\gamma} H(t,s+\gamma) \\ H(0,s) &= sW_0(s), \quad \forall\, s \in \mathscr{S}(\rho,\rho+\gamma). \end{aligned}$$

The proof follows now the same arguments used in the proof of Theorem 7.1, applied to $H$ instead of $W$. This amounts just to consider the new function

$$\Psi(s) = \frac{(s-\sigma_1)(s-\sigma_2)}{s+\gamma}$$

instead of $\Phi$.

We first consider the case where $T < \gamma^{-1}$. Then, all the beginning of the proof of Theorem 7.1 may be exactly reproduced until the formula (7.7), with the interval $(\rho, \rho+\gamma)$. In order to obtain a particular solution $\widetilde{H}_{part}$ of

$$z\widetilde{W}(z,s) = \Psi(s)\widetilde{W}(z,s+\gamma) + \widetilde{r}(z,s), \tag{7.20}$$

we consider the function:

$$\mathscr{V}(s) = \frac{(-\gamma)^{\frac{s}{\gamma}}\Gamma(\frac{s-\sigma_1}{\gamma})\Gamma(\frac{s-\sigma_2}{\gamma})}{\Gamma(1+\frac{s}{\gamma})}.$$

It is straightforward to check that $\mathscr{V}(s)$ satisfies:

$$\mathscr{V}(s+\gamma) = -\Psi(s)\mathscr{V}(s), \quad s \in \mathbb{C} \setminus \{s \in \mathbb{C}; s = \sigma_1 - m\gamma,\, m = 0,1,2,\ldots\}$$

and, arguing as for the function $V$ in (4.1), its behavior as $|\Im m(s)| \to \infty$ with $\Re e(s)$ bounded is such that the function:

$$\widetilde{H}_{part}(t,s) = \frac{i}{\gamma z\,\mathscr{V}(s)} \int_{\Re e(\sigma)=\sigma_0} \frac{(-z)^{\frac{\sigma-s}{\gamma}}\mathscr{V}(\sigma)\widetilde{r}(z,\sigma)\mathrm{d}\sigma}{(1-e^{-\frac{2i\pi}{\gamma}(s-\sigma)})}$$

satisfies the equation (7.20) and the estimate (7.9) for $t \in (0,T)$.

The argument for $\widetilde{H}_{hom}$ is now very similar using the new functions $\mathscr{G}(t,\zeta) = H(t,s)$ instead of $G$ in (7.10), and $\widetilde{\psi}(\zeta)$, $\psi(x)$ instead of $\widetilde{\varphi}(\zeta)$, $\varphi(x)$ in (7.11), (7.12). Since $\psi(\zeta)$ may still be estimated as in (7.13) the end of the argument follows straightforwardly in the same way to prove that $H(t) \equiv 0$ for $t \in [0, T)$. This proves Theorem 7.2 if $T < \gamma^{-1}$.

The result for $T > \gamma^{-1}$ follows by iteration of the uniqueness of solutions on intervals $[m\gamma/2, (m+1)\gamma/2)$ for $m = 0, 1, 2, \ldots, M$ where $M = [2T/\gamma]$, the integer part of $2T/\gamma$, and finally on $[M\gamma/2, T)$. □

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