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# Moment problems related to Bernstein functions <sup>(\*)</sup>

THOMAS SIMON <sup>(1)</sup>

**ABSTRACT.** — We give a simple proof of the moment-indeterminacy on the half-line of the sequence  $(n!)^t$  for  $t > 2$ , using Lin’s condition. Under a logarithmic self-decomposability assumption, the method conveys to power moment sequences defined as the rising factorials of a given Bernstein function, and to more general infinitely divisible moment sequences. We also provide a very short proof of the infinite divisibility of all the integer moment sequences recently investigated in [16], including Fuss–Catalan’s.

**RÉSUMÉ.** — Nous donnons une preuve simple du caractère indéterminé sur la demi-droite de la suite de moments entiers  $(n!)^t$  pour  $t > 2$ , à l’aide de la condition de Lin. Sous une hypothèse d’auto-décomposabilité logarithmique, la méthode s’étend à des suites de puissances de puissances de moments entiers définis comme la factorielle croissante d’une fonction de Bernstein donnée, et plus généralement à d’autres suites infiniment divisibles de moments entiers. Nous donnons aussi une preuve très courte du caractère infiniment divisible de toutes les suites de moments entiers récemment étudiées dans [16] et en particulier de la suite de Fuss–Catalan.

## 1. A simple proof of a result by Berg

Let  $\mathbf{L}$  be the standard exponential random variable and  $\mathbf{G} = \log \mathbf{L}$  be the standard Gumbel random variable. For all  $s > -1$ , one has

$$\begin{aligned} \mathbb{E}[e^{s\mathbf{G}}] &= \mathbb{E}[\mathbf{L}^s] = \Gamma(1 + s) \\ &= \exp \left\{ -\gamma s + \int_{-\infty}^0 (e^{sx} - 1 - sx) \frac{dx}{|x|(e^{|x|} - 1)} \right\} \end{aligned} \quad (1.1)$$

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where  $\gamma$  is Euler's constant (see e.g. Formulæ 1.7.2(19) and 1.9(1) in [10] for the third equality). This shows that  $\mathbf{G}$  is infinitely divisible. Let  $\{\mathbf{G}_t, t \geq 0\}$  be the real Lévy process starting at zero such that  $\mathbf{G}_1 \stackrel{d}{=} \mathbf{G}$ . For every  $s > -1$  and  $t \geq 0$ , one has

$$\mathbb{E}[e^{s\mathbf{G}_t}] = \Gamma(1 + s)^t. \tag{1.2}$$

The family of positive random variables  $\mathbf{L}_t = e^{\mathbf{G}_t}, t \geq 0$ , induces a multiplicative convolution semi-group, that is

$$\mathbf{L}_u \mathbf{L}_t^{-1} \perp \mathbf{L}_t \quad \text{and} \quad \mathbf{L}_u \mathbf{L}_t^{-1} \stackrel{d}{=} \mathbf{L}_{u-t}$$

if  $t \leq u$ . For every  $t > 0$ , it is easy to see by (1.2) and Fourier inversion that the random variable  $\mathbf{L}_t$  has a smooth density  $f_t$  on  $(0, \infty)$ . It can also be shown, using (1.2) and Mellin inversion, that this density satisfies the integral equation

$$f_t(x) = \frac{1}{\Gamma(t)} \int_x^\infty f_t(y) (\log y - \log x)^{t-1} dy,$$

but we shall not need this in the sequel. The positive entire moments of  $\mathbf{L}_t$  are

$$m_n(t) = \mathbb{E}[\mathbf{L}_t^n] = (n!)^t$$

and the following observation was made in [2]:

$$\mathbf{L}_t \text{ is moment-determinate} \iff t \leq 2. \tag{1.3}$$

See the recent survey [15] for more details and references on the classical moment problem. Throughout, we will use the usual short notations M-det for moment-determinate and M-indet for moment-indeterminate.

As shown in [2], the if part of (1.3) is an immediate application of Carleman's criterion: if  $t \leq 2$  one has, by Stirling's formula,

$$\sum_{n \geq 1} m_n(t)^{-\frac{1}{2n}} \geq \sum_{n \geq 1} m_n(2)^{-\frac{1}{2n}} = \sum_{n \geq 1} (n!)^{-\frac{1}{n}} = \infty.$$

In [2], the proof of the only if part of (1.3) is however much more involved. It amounts to checking the classical Krein condition for a certain associate distribution (see Section 2 therein). Another proof was recently given in [4], using Krein's condition directly and an asymptotic analysis of  $f_t$  at infinity.

We begin this note with an alternative and simple argument for the only if part of (1.3), which relies on Lin's condition (see Condition  $L$  in Section 5 of [15]). We use the hyperbolically completely monotone (HCM) property of Bondesson and Thorin, which we will not introduce here in detail for the sake of concision. See Chapters 5 and 6 in [6] for all definitions and notations.

PROPOSITION 1.1. — *One has*

$$f_t \text{ is HCM} \iff t \geq 1.$$

*Proof.* — We begin with the only if part. Suppose  $t < 1$  and set  $\mathbf{M}_t = \mathbf{L}_t^{\frac{1}{t}}$ . One has

$$\frac{\mathbb{E}[\mathbf{M}_t^n]^{\frac{1}{n}}}{n} = \frac{\Gamma(1 + \frac{n}{t})^{\frac{t}{n}}}{n} \rightarrow \frac{1}{te}$$

by Stirling's formula. By Lemma 3.2 in [9], this implies

$$\log \mathbb{P}[\mathbf{L}_t > x] = \log \mathbb{P}[\mathbf{M}_t > x^{\frac{1}{t}}] \sim -tx^{\frac{1}{t}}.$$

Since  $t < 1$ , the upper tails of  $\mathbf{L}_t$  are superexponentially small and  $\mathbf{L}_t$  is not infinitely divisible, as is well-known. In particular, its density  $f_t$  cannot be HCM.

We now prove the if part, which is clear for  $t = 1$  since  $f_1(x) = e^{-x}$ . If  $t > 1$ , introduce for every  $q > 0$  the associate random variables

$$\mathbf{L}_{q,t} = \mathbf{T} \left( \frac{q}{q+t-1}, \frac{1}{q+t-1}, \frac{t}{q+t-1} \right)$$

with the notation of [14]. It follows from (2.2) in [14] and an immediate asymptotic analysis that for every  $s \geq 0$ ,

$$\mathbb{E}[\mathbf{L}_{q,t}^s] \rightarrow c_t^s \exp \left\{ t \int_{-\infty}^0 (e^{sx} - 1 - sx) \frac{dx}{|x|(e^{|x|} - 1)} \right\}$$

as  $q \rightarrow \infty$ , for a constant  $c_t$  to be determined. The normalization

$$\mathbb{E}[\mathbf{L}_{q,t}] = \mathbb{E}[\mathbf{L}] = 1$$

for all  $q > 0$  and  $t > 1$  implies  $c_t = e^{-t\gamma}$  and we can deduce from (1.1) and (1.2) that

$$\mathbf{L}_{q,t} \xrightarrow{d} \mathbf{L}_t \tag{1.4}$$

as  $q \rightarrow \infty$ . Moreover, it follows from (2.4) and (2.7) in [14] that

$$\begin{aligned} \mathbf{L}_{q,t} &\stackrel{d}{=} \mathbf{T} \left( \frac{q}{q+t-1}, \frac{1}{q+t-1}, \frac{1}{q+t-1} \right) \times \mathbf{T} \left( \frac{q+1}{q+t-1}, \frac{1}{q+t-1}, \frac{t-1}{q+t-1} \right) \\ &\stackrel{d}{=} \left( \frac{q+t-1}{q} \right) \times \mathbf{\Gamma}_{\frac{q}{q+t-1}} \times \mathbf{T} \left( \frac{q+1}{q+t-1}, \frac{1}{q+t-1}, \frac{t-1}{q+t-1} \right) \end{aligned}$$

where  $\mathbf{\Gamma}_\lambda$  stands for the standard Gamma random variable with parameter  $\lambda > 0$ , and the products on the right-hand sides are independent. Applying Lemma 1 in [8] shows now that  $\mathbf{L}_{q,t}$  has a HCM density for every  $q > 0, t > 1$ . By (1.4) and Theorem 5.1.3 in [6], this is also the case for  $\mathbf{L}_t$ .  $\square$

**COROLLARY 1.2.** — *The random variable  $\mathbf{L}_t$  is M-indet for all  $t > 2$ .*

*Proof.* — If  $t > 2$ , combining Proposition 1.1 and Property (v) p. 68 in [6] shows that the function

$$x \mapsto -\frac{xf'_t(x)}{f_t(x)}$$

increases on  $(0, \infty)$ , to a limit which must be  $\infty$  since otherwise  $f_t$  would be regularly varying at infinity and some integer moments of  $\mathbf{L}_t$  would not be finite. Hence, Lin's condition is fulfilled. Moreover, we have

$$\frac{m_{n+1}(t)}{m_n(t)} = (n + 1)^t$$

and by Theorem 4 in [17], this implies that  $\mathbf{L}_t$  is M-indet.  $\square$

*Remarks 1.3.* — (a). In the terminology of [27], the equivalence (1.3) means that the moment-determinacy of  $\mathbf{L}_t$  is a time-dependent property for the Lévy process  $\{\mathbf{G}_t, t \geq 0\}$ . This temporal change comes from the multiplicative character of the semi-group associated to  $\mathbf{L}_t$ .

(b). For a positive random variable having all its integer moment finite and a density  $f$ , Lin's condition amounts to the non-decreasing character of

$$x \mapsto -\frac{x f'(x)}{f(x)}$$

on  $(0, \infty)$ , since the limit must then be  $\infty$ . An equivalent formulation is the log-concavity of the function  $x \mapsto f(e^x)$  on  $\mathbb{R}$ .

(c). The HCM property is sensitive to power transformations for a given random variable (see Chapter 5 in [6]). It is well-known that this is also the case for the moment problem (see the references in [15]). It is interesting to mention that Lin's condition is equivalent to the hyperbolic monotonicity (HM) property of Section 6.4 in [6], which is less stringent than the HCM property and does not depend on powers. Another pleasant feature of Lin's condition is that it only deals with the behaviour of the density at infinity, in accordance with Krein's condition. This will be used in the next section. See also [26] and the references therein for more detail on the relationships between Krein's and Lin's conditions.

## 2. A generalization to Bernstein functions

If  $\Phi$  is a Bernstein function, it was proved in [5] that the sequence

$$\Phi(1) \times \cdots \times \Phi(n)$$

is a determinate moment sequence on  $\mathbb{R}^+$ . The corresponding positive random variable  $\mathbf{R}$  is a multiplicative factor of  $\mathbf{L}$  which is called in [11] the Remainder. In Theorem 1.8 of [2], it is shown that  $\mathbf{S} = \log \mathbf{R}$  is infinitely divisible (see also Theorem 3.1 in [11] for a different proof). This leads to a real Lévy process  $\{\mathbf{S}_t, t \geq 0\}$  characterized by  $\mathbf{S}_1 = \mathbf{S}$ , and to the family of

positive random variables  $\mathbf{R}_t = e^{\mathbf{S}t}$ ,  $t \geq 0$ . The latter induces a multiplicative convolution semi-group exactly as above. The positive entire moments of  $\mathbf{R}_t$  are

$$\mu_n(t) = \mathbb{E}[\mathbf{R}_t^n] = (\Phi(1) \times \cdots \times \Phi(n))^t, \quad n \geq 0.$$

The unit drift case  $\Phi(x) = x$  yields  $\mathbf{R}_t = \mathbf{L}_t$ , which is for  $t \in (0, 1)$  the Remainder associated to a stable subordinator of parameter  $t$  (see Example 4.1 in [11]). In view of the previous section, it is natural to ask for the moment-determinacy of  $\mathbf{R}_t$ . This problem was recently addressed in [23]. Introduce the following parameter

$$\ell = \limsup_{x \rightarrow \infty} \left( \frac{\Psi(x)}{x \log x} \right) \quad \text{with} \quad \Psi(x) = \int_0^x \log \Phi(t) dt,$$

and observe from the concavity of  $\Phi$  that necessarily one has  $\ell \in [0, 1]$ .

We begin with an easy and general result.

PROPOSITION 2.1. — *The random variable  $\mathbf{R}_t$  is M-det for all  $t < 2/\ell$ .*

*Proof.* — Since  $\log \Phi$  increases, one has

$$\ell = \limsup_{n \rightarrow \infty} \left( \frac{\log(\Phi(1) \times \cdots \times \Phi(n))}{n \log n} \right).$$

Therefore,

$$\limsup_{n \rightarrow \infty} \left( \frac{\log \mu_n(t)}{n \log n} \right) < 2$$

whenever  $t < 2/\ell$ , whence the conclusion by Carleman’s criterion. □

In particular, we see that  $\mathbf{R}_t$  is M-det for every  $t > 0$  if  $\ell = 0$ , which is the case in Examples 4.2, 4.3 and 4.4 of [11]. In the following, we implicitly assume  $\ell > 0$ .

It is natural to ask for the M-indet character of  $\mathbf{R}_t$  when  $t > 2/\ell$ . A first difficulty is that in order to check either Krein’s or Lin’s condition, the absolute continuity of the law of  $\mathbf{R}_t$  is required. The latter is equivalent to that of  $\mathbf{S}_t$  but the problem of absolute continuity for marginals of a real Lévy process is hard in general, subject to temporal changes (see [27]). We will consider the following

ASSUMPTION 1. — *The random variable  $\mathbf{S}$  is self-decomposable.*

Under this assumption, it is well-known that  $\mathbf{S}_t$  is absolutely continuous for every  $t > 0$ . The self-decomposability of  $\mathbf{S}$  can be characterized by the Bernstein measure  $\kappa(dx)$  of the completely monotonic function  $\Phi'/\Phi$ . It is shown in Proposition 3.5 of [11] that  $\mathbf{S}$  is self-decomposable if and only if  $\kappa$  is absolutely continuous with a density  $\kappa$  such that  $x \mapsto (e^x - 1)^{-1}\kappa(x)$  is non-increasing on  $(0, \infty)$ . This is true for Examples 4.1, 4.5 and 4.6 of [11].

This is also true if  $\Phi$  is a complete Bernstein function, because it admits the representation

$$\Phi(x) = \Phi(1) \exp \left\{ \int_0^\infty \left( \frac{1}{1+t} - \frac{1}{x+t} \right) \eta(t) dt \right\}$$

for some measurable function  $\eta$  taking its values in  $[0, 1]$  (see e.g. Theorem 6.10 in [25]). A simple computation shows then that the measure  $\kappa$  has density

$$\kappa(x) = x \int_0^\infty e^{-xt} \eta(t) dt,$$

and the function  $(e^x - 1)^{-1} \kappa(x)$  is hence non-increasing on  $(0, \infty)$ . Let us also mention that by Corollary 1.11 of [1], the random variable  $\mathbf{S}$  is self-decomposable when the upper tail of the Lévy measure of  $\Phi$  is log-convex, which is less stringent than the complete Bernstein character of  $\Phi$ . See [1] and the references therein for further aspects of the measure  $\kappa$ . Introducing the further parameter

$$\bar{\ell} = \liminf_{x \rightarrow \infty} \left( \frac{\Psi(x)}{x \log x} \right),$$

we have the following counterpart to Proposition 2.1.

**PROPOSITION 2.2.** — *Under Assumption 1, the random variable  $\mathbf{R}_t$  is M-indet for every  $t > 2/\bar{\ell}$ .*

*Proof.* — If  $t > 2/\bar{\ell}$  we obtain, reasoning as in Proposition 2.1,

$$\liminf_{n \rightarrow \infty} \left( \frac{\log \mu_n(t)}{n \log n} \right) > 2.$$

Hence, by Theorem 7 in [15], we just need to check Lin's condition on  $\mathbf{R}_t$ . Recall now that the Lévy measure of the self-decomposable random variable  $-\mathbf{S}_t$  has, by Proposition 3.2 in [11], density

$$tx^{-1}(e^x - 1)^{-1} \kappa(x) \mathbf{1}_{(0, \infty)}(x).$$

Moreover, it follows from Lemma 1.3 in [1] that

$$\log \Phi(x) = \log \Phi(1) + \int_0^\infty (e^{-t} - e^{-xt}) \frac{\kappa(t)}{t} dt.$$

Therefore, one has necessarily

$$\int_0^1 (e^x - 1)^{-1} \kappa(x) dx = \infty$$

since otherwise  $\Phi$  would be bounded, which is clearly excluded by the positivity of  $\ell$ . All of this shows that the law of  $-\mathbf{S}_t$  is of the type  $\mathbf{I}_7$  in [24] and, by Theorem 1.3(xii) therein, that the density of  $\mathbf{S}_t$  is log-concave on some interval  $[a, \infty)$ . Changing the variable, the latter is equivalent to Lin's condition on  $\mathbf{R}_t$ .  $\square$

*Remarks 2.3.* — (a). The monotone density theorem applied to the increasing concave function  $\log \Phi$  shows that

$$\ell = \bar{\ell} \iff \log \Phi(x) \sim \ell \log x.$$

The latter holds true when  $\Phi$  is regularly varying at infinity, the index being then necessarily  $\ell$ . Observe also that this is a weaker condition than regular variation, given at the logarithmic level.

(b). If  $\ell = \bar{\ell}$ , one may ask for the moment-determinacy of  $\mathbf{R}_{2/\ell}$ . The behaviour at the threshold is usually a question where trouble begins in the literature on moment problems, in the absence of universal criteria. Under Assumption 1, a consequence of the above proof and Theorem 3 in [21] is

$$\mathbf{R}_t \text{ is M-det} \iff \sum_{n \geq 1} \mu_t(n)^{-\frac{1}{2n}} = \infty.$$

In particular, one has

- $\mathbf{R}_{2/\ell}$  is M-indet if

$$\liminf_{x \rightarrow \infty} \left( \frac{\Phi(x)}{x^\ell (\log x)^c} \right) = \infty \quad \text{for some } c > 1.$$

- $\mathbf{R}_{2/\ell}$  is M-det if

$$\limsup_{x \rightarrow \infty} \left( \frac{\Phi(x)}{x^\ell (\log x)^c} \right) < \infty \quad \text{for some } c < 1.$$

See Remark 3 below for another, Krein type, criterion.

(c). In the unit drift case  $\Phi(x) = x$ , Assumption 1 is fulfilled and Proposition 2.2 gives another quick proof of moment-indeterminacy of the sequence  $(n!)^t$  for  $t > 2$ . Our previous argument in Section 1 is more involved, but it is also more informative.

The above proof enhanced Lin's condition. We now show that it is possible to derive an analogous result with the help of the classical Krein's condition, under two different assumptions. The first one is weaker than Assumption 1.

**ASSUMPTION A.** — *There exists  $\varepsilon > 0$  such that for every  $t \in (2/\ell, 2/\ell + \varepsilon)$ , the random variable  $\mathbf{R}_t$  has a density which is ultimately monotone.*

By Yamazato's theorem on the unimodality of self-decomposable laws and a change of variable, Assumption A is implied by Assumption 1. The study of monotonicity properties of ID densities on  $\mathbb{R}$  can be a delicate problem, leading to pathological situations. For example, an ID density may have an infinite number of modes (see again [27] for more on this topic). We were not able to exhibit any absolutely continuous Remainder not fulfilling Assumption A, but we believe there should be some.

ASSUMPTION B. — *There exists  $c \in (0, \infty)$  such that*

$$\lim_{n \rightarrow \infty} \frac{(\Phi(1) \times \cdots \times \Phi(n))^{\frac{1}{n\ell}}}{n} = c.$$

By monotonicity, this second assumption implies  $\ell = \bar{\ell}$ . It is also a stronger condition, given at the natural level. Observe finally that Assumption B is fulfilled when  $\Phi$  is regularly varying at infinity, the index being then necessarily  $\ell$ .

PROPOSITION 2.4. — *Under Assumptions A and B, the random variable  $\mathbf{R}_t$  is M-indet for every  $t > 2/\ell$ .*

*Proof.* — As is well-known, it is enough to consider the case  $t \in (2/\ell, 2/\ell + \varepsilon)$ . Fix such a  $t$  and set

$$\nu_n(t) = \mathbb{E}[\mathbf{R}_t^{\frac{n}{\ell t}}]$$

for all  $n \geq 1$ . Since  $p \mapsto \mathbb{E}[X^p]^{\frac{1}{p}}$  is non-increasing on  $(0, \infty)$  for any positive random variable  $X$ , one has the bounds

$$\begin{aligned} \frac{[n(\ell t)^{-1}]}{n} \times \frac{\mathbb{E}[\mathbf{R}^{[n(\ell t)^{-1}]}]^{\frac{1}{\ell[n(\ell t)^{-1}]}}}{[n(\ell t)^{-1}]} &\leq \frac{\nu_n(t)^{\frac{1}{n}}}{n} \\ &\leq \frac{[n(\ell t)^{-1}] + 1}{n} \times \frac{\mathbb{E}[\mathbf{R}^{[n(\ell t)^{-1}] + 1}]^{\frac{1}{\ell[n(\ell t)^{-1}] + 1}}}{[n(\ell t)^{-1}] + 1} \end{aligned}$$

and Assumption B implies

$$\lim_{n \rightarrow \infty} \left( \frac{\nu_n(t)^{\frac{1}{n}}}{n} \right) = \frac{c}{\ell t}.$$

Reasoning as in Proposition 1.1 leads then to the estimate

$$\log \mathbb{P}[\mathbf{R}_t > x] \sim -c_t x^{\frac{1}{\ell t}}$$

with  $c_t = \ell t (ec)^{-1} \in (0, \infty)$ . By Assumption A, the random variable  $\mathbf{R}_t$  has a density  $f_t$  which is non-increasing at infinity, and the above estimate implies easily

$$x^2 f_t(x) \geq e^{-(c_t/2)x^{\frac{1}{\ell t}}}$$

for  $x$  large enough. Since  $t > 2/\ell$ , the relaxed Krein's condition given in Theorem 4 of [15] is in force, and  $\mathbf{R}_t$  is M-indet.  $\square$

We end this section with an example. Consider the Bernstein function

$$\Phi(\lambda) = \frac{\Gamma(\alpha\lambda + b)}{\Gamma(\alpha\lambda + c)}$$

with  $\alpha > 0$  and  $0 \leq c < b < c + 1$ . This function is studied in p. 102–103 of [5] for  $\alpha \in (0, 1)$ ,  $b = 1$  and  $c = 1 - \alpha$ , in Section 4.6 of [11] for  $c = 0$

and in Example 3.1 of [23] for  $\alpha \in (0, 1)$ . An affine change of variable in the expression of the Lévy measure given at the bottom of p. 1370 of [11] implies easily

$$\Phi(\lambda) = \frac{\Gamma(b)}{\Gamma(c)} + \int_0^\infty (1 - e^{-\lambda t}) \nu_{\alpha,b,c}(t) dt$$

where

$$\nu_{\alpha,b,c}(t) = \frac{(b - c)e^{-\frac{bt}{\alpha}}}{\alpha\Gamma(c - b + 1)(1 - e^{-\frac{t}{\alpha}})^{1+b-c}}$$

is a completely monotonic function, so that  $\Phi$  is a complete Bernstein function. By Stirling's formula, one has

$$\Phi(\lambda) \sim (\alpha\lambda)^{b-c}, \quad \lambda \rightarrow \infty,$$

which implies that  $\Phi$  is regularly varying at infinity with index  $\ell = \bar{\ell} = b - c$ . If  $\mathbf{R}$  is the Remainder corresponding to  $\Phi$ , one can apply Proposition 2.1 and either Proposition 2.2 or Proposition 2.4, together with Remark 2.3(b), and obtain the following characterization:

$$\mathbf{R}_t \text{ is M-det} \iff t \leq \frac{2}{b - c}.$$

The limiting case  $b = c + 1$  gives  $\Phi(\lambda) = \alpha\lambda + c$  and the integer moment sequence of  $\mathbf{R}_t$  is then simply

$$\mu_n(t) = \left( \alpha^n \times \frac{\Gamma(1 + c\alpha^{-1} + n)}{\Gamma(1 + c\alpha^{-1})} \right)^t.$$

In this case, we will see in Section 3.1 below that one also has

$$\mathbf{R}_t \text{ is M-det} \iff t \leq 2.$$

### 3. Infinitely divisible moment sequences

If  $\{\mu_n, n \geq 1\}$  is the entire moment sequence of a positive random variable  $\mathbf{X}$ , we say that this sequence is infinitely divisible (ID for short) if  $\{\mu_n^t, n \geq 1\}$  is an entire moment sequence for every  $t > 0$ . It is clear from the considerations in Section 1 that this property is equivalent to the infinite divisibility of  $\log \mathbf{X}$  as a random variable. In particular, every moment sequence

$$\mu_n = \Phi(1) \times \dots \times \Phi(n)$$

with  $\Phi$  a Bernstein function, is ID (see the aforementioned Theorem 1.8 in [2] and Theorem 3.1 in [11]). For short, we will say that such an ID moment sequence is Bernstein. Observe that ID moment sequences need not be Bernstein, as shows the example  $\mu_n = (n!)^2$ .

An entire moment sequence  $\{\mu_n, n \geq 1\}$  which is both ID and M-det gives rise to two sets of positive random variables indexed by time, via the associate random variable  $\mathbf{X}$ . The first one is the multiplicative family  $\{\mathbf{X}_t, t > 0\}$  defined as in Section 1, and the second one is the family of positive power transformations  $\{\mathbf{X}^t, t > 0\}$ . A conjecture formulated in [16] is that  $\mathbf{X}_t$  is M-det if and only if  $\mathbf{X}^t$  is M-det, for every  $t > 0$ . The following proposition gives a partial answer. Recall that the Lévy measure of a real self-decomposable random variable has density  $k(x)/|x|$  on  $\mathbb{R}^*$ , where  $k$  is a function non-decreasing on  $(-\infty, 0)$  and non-increasing on  $(0, \infty)$ , which is called the spectral function. Henceforth, we implicitly exclude the case where  $\log \mathbf{X}$  has a non-trivial Gaussian component, since then both  $\mathbf{X}_t$  and  $\mathbf{X}^t$  have the log-normal distribution as multiplicative factor for every  $t > 0$  and are hence M-indet.

PROPOSITION 3.1. — *Assume that the random variable  $\log \mathbf{X}$  is self-decomposable and that its spectral function is not integrable at 0−. Then, for every  $t > 0$ , one has*

$$\mathbf{X}_t \text{ is M-det} \iff \mathbf{X}^t \text{ is M-det.}$$

*Proof.* — The assumption means that  $-\log \mathbf{X}$  is of the type  $I_7$  in [24], which implies as in the proof of Proposition 2.2 that Lin’s condition is satisfied by both  $\mathbf{X}_t$  and  $\mathbf{X}^t$ . Applying Theorem 3 in [21], we get

$$\mathbf{X}_t \text{ is M-det} \iff \sum_{n \geq 1} \mu_n^{-\frac{t}{2n}} = \infty \iff \sum_{n \geq 1} \mu_{[nt]}^{-\frac{t}{2[nt]}} = \infty \iff \mathbf{X}^t \text{ is M-det,}$$

the second equivalence being an easy consequence of the non-increasing character of  $n \mapsto \mu_n^{-\frac{t}{2n}}$ , whereas the third equivalence is obtained as for the bounds in the proof of Proposition 2.4.  $\square$

Remark 3.2. — For a Remainder  $\mathbf{R}$  satisfying Assumption 1, Proposition 3.1 combined with Theorems 4 and 10 in [15], and a change of variable, shows that at the threshold  $t = 2/\ell$ , one has the Krein type criterion

$$\mathbf{R}_{2/\ell} \text{ is M-det} \iff \mathbf{R}^{2/\ell} \text{ is M-det} \iff \int_0^\infty \frac{-\log f(x^\ell)}{1+x^2} dx = \infty$$

where  $f$  is the density of  $\mathbf{R}$ . This is useful since the asymptotic analysis at infinity of the density of  $\mathbf{R}_t$  for  $t \neq 1$  might be more involved than that of  $f$  (see [4] for the case  $\mathbf{R} = \mathbf{L}$ ).

In the recent paper [16], the infinite divisibility of several classical moment sequences was obtained for the first time, using essentially Theorem 1.8 in [2]. We can show this property for two larger families of moment sequences, and in a very simple way. We will also investigate some other interesting moment sequences.

### 3.1. Gamma moment sequences of order 2

We consider the moment sequence

$$\mu_n = \frac{\Gamma(a + sn)\Gamma(a + b)}{\Gamma(a)\Gamma(a + b + sn)}$$

for  $a, b, s > 0$ . The associate random variable is the power transformation  $\mathbf{B}_{a,b}^s$  of the standard Beta random variable  $\mathbf{B}_{a,b}$  with density

$$\frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1} \mathbf{1}_{(0,1)}(x).$$

This sequence is ID by the well-known fact that  $\log \mathbf{B}_{a,b}^s = s \log \mathbf{B}_{a,b}$  is an ID random variable. The latter amounts to the standard Malmstén type formula

$$\begin{aligned} \mathbb{E}[e^{\lambda \log \mathbf{B}_{a,b}^s}] &= \frac{\Gamma(a + s\lambda)\Gamma(a + b)}{\Gamma(a)\Gamma(a + b + s\lambda)} \\ &= \exp \left\{ - \int_0^\infty (1 - e^{-\lambda x}) \left( \frac{e^{-as^{-1}x}(1 - e^{-bs^{-1}x})}{x(1 - e^{-s^{-1}x})} \right) dx \right\}. \end{aligned}$$

Clearly, one has  $\mu_n^t \rightarrow 0$  as  $n \rightarrow \infty$  so that  $\{\mu_n^t\}$  is M-det for every  $t > 0$ , by Carleman's criterion. The same is true for  $\{\mu_{nt}\}$  since the associate random variable  $\mathbf{B}_{a,b}^{st}$  has bounded support.

Taking  $a = 1/2, b = 3/2$  and  $s = 1$ , we have

$$\mu_n = \frac{1}{4^n(n+1)} \binom{2n}{n} = 4^{-n} C_n$$

which is the Catalan number sequence up to a multiplicative constant. Hence, the previous discussion encompasses Theorem 1 in [16]. Moreover, letting  $b \rightarrow \infty$ , we get

$$b^{sn} \mu_n \rightarrow \frac{\Gamma(a + sn)}{\Gamma(a)}$$

for every  $a, s > 0$ , which is the Gamma sequence of order 1 recently studied in [3]. Since the ID property of a moment sequence is preserved under point-wise limit, the latter sequence is also ID. The associate random variable is the power transformation  $\mathbf{\Gamma}_a^s$  of the standard Gamma random variable  $\mathbf{\Gamma}_a$  with density

$$\frac{1}{\Gamma(a)} x^{a-1} e^{-x} \mathbf{1}_{(0,\infty)}(x).$$

Carleman's criterion and Krein's condition show at once that the sequence

$$\frac{\Gamma(a + stn)}{\Gamma(a)}$$

is M-det if and only if  $st \leq 2$ . By Proposition 3.1, the same is true for

$$\left(\frac{\Gamma(a + sn)}{\Gamma(a)}\right)^t$$

because, similarly as for (1.1), one has

$$\begin{aligned} \mathbb{E}[e^{\lambda \log \Gamma_a^s}] &= \mathbb{E}[\mathbf{\Gamma}_a^{s\lambda}] \\ &= \exp \left\{ -\psi(a)s\lambda + \int_{-\infty}^0 (e^{\lambda x} - 1 - \lambda x) \frac{e^{-as^{-1}|x|}}{|x|(1 - e^{-s^{-1}|x|})} dx \right\} \end{aligned}$$

where  $\psi$  is the digamma dunction: this implies that  $\log \mathbf{\Gamma}_a^s$  is self-decomposable with a spectral function non-integrable at  $0-$ . Putting everything together, we have got a very simple proof of Conjecture 2 in [16], which is also Theorem 1.1 in [3].

*Remarks 3.3.* — (a). A natural and more involved question, which is connected to the approach of [16], is whether the sequence  $\{\mu_n\}$  is Bernstein, with our above notation. This question was actually already adressed in [7], for other purposes. It follows easily from the hypergeometric transformations carried out in Section 2.3 of [7] that

$$\{\mu_n\} \text{ is Bernstein} \iff \inf\{b, s\} \leq 1 \quad \text{and} \quad a \geq s.$$

The corresponding Bernstein function is

$$\Phi(\lambda) = \frac{\Gamma(a)\Gamma(a + b - s)}{\Gamma(a + b)\Gamma(a - s)} + \int_0^\infty (1 - e^{-\lambda x})\rho(x) dx$$

with

$$\begin{aligned} \rho(x) &= be^{-as^{-1}x} {}_2F_1 \left[ \begin{matrix} 1 + s, 1 - b \\ 2 \end{matrix}; 1 - e^{-s^{-1}x} \right] \\ &= be^{(1-(a+b)s^{-1})x} {}_2F_1 \left[ \begin{matrix} 1 + b, 1 - s \\ 2 \end{matrix}; 1 - e^{-s^{-1}x} \right], \end{aligned}$$

a non-negative integrable function which simplifies into  $\rho(x) = be^{(1-(a+b)s^{-1})x}$  for  $s = 1$  and into  $\rho(x) = be^{-as^{-1}x}$  for  $b = 1$ . We refer to Section 2.3 in [7] for some other interesting aspects of the function  $\rho$ , connected to the zeroes of the classical hypergeometric series. In particular, it can be shown that  $\Phi$  is a complete Bernstein function if and only if  $b = 1$  or  $s = 1$ , and that it belongs to the Jurek class if and only if  $2a + b + s + bs \geq 1$ .

(b). For the Catalan moment sequence, one has  $C_n = \Phi(1) \times \dots \times \Phi(n)$ , where

$$\Phi(\lambda) = 2 \left( 2 - \frac{3}{1 + \lambda} \right)$$

is not Bernstein in the strict sense, since it takes negative values on  $(0, 1/2)$ . On the other hand, the function  $\tilde{\Phi}(\lambda) = \Phi(\lambda + 1/2)$  is Bernstein and the factorization  $C_n = \tilde{\Phi}(1/2) \times \cdots \times \tilde{\Phi}(n - 1/2)$  was used in [16] together with a previous Lemma of [4] to show the ID character of  $\{C_n\}$ .

(c). Similarly as above, one can show that

$$\left\{ \frac{\Gamma(a + sn)}{\Gamma(a)} \right\} \text{ is Bernstein} \iff \inf\{1, a\} \geq s.$$

The corresponding Bernstein functions are

$$\Phi(\lambda) = \frac{\Gamma(a)}{\Gamma(a-s)} + \frac{1}{\Gamma(1-s)} \int_0^\infty (1 - e^{-\lambda x}) \left( \frac{e^{-as^{-1}x}}{(1 - e^{-s^{-1}x})^{1+s}} \right) dx$$

for  $s < 1$  and  $\Phi(\lambda) = a - 1 + \lambda$  for  $s = 1$ .

### 3.2. Binomial and Raney moment sequences

We consider the sequence

$$\mu_n = \binom{pn + r}{n}$$

which is known (see [18]) to be a moment sequence on  $\mathbb{R}^+$  if and only if  $p \geq 1$  and  $r \in [-1, p - 1]$ . The associate random variable is here more complicated than above. It follows from Theorem 3.1 in [18] that for  $p \geq 1$  rational and  $r \in (-1, p - 1]$ , it is a renormalized finite product of independent Beta random variables, so that its logarithm is infinitely divisible. A density argument shows then immediately that the sequence  $\{\mu_n\}$  is ID for all  $p \geq 1$  and  $r \in [-1, p - 1]$ . Carleman's criterion implies, as above, that the power sequences  $\{\mu_n^t\}$  and  $\{\mu_{nt}\}$  are M-det for every  $t > 0$ .

Taking now  $r = 0$  and  $p \geq 2$  an integer, we obtain a quick proof of Theorems 2 and 6 in [16], and also of Theorem 2', 3 and 5 therein by the factorization argument given in Lemma 2 of [16]. For instance, the Fuss-Catalan sequence of order  $k$

$$C_{k,n} = \left( \frac{1}{1 + kn} \right) \times \binom{(k+1)n}{n}$$

is the product of two ID moment sequences taking  $p = k + 1, r = 0$  and, in the previous paragraph,  $a = b = 1, s = k$ .

In the same vein, it is interesting to mention that the above discussion also implies the ID character of the Raney sequence

$$\mu_n = \frac{r}{np + r} \binom{pn + r}{n},$$

which is a moment sequence on  $\mathbb{R}^+$  if and only if  $p \geq 1$  and  $r \in [0, p]$  (see [18, 19]). Indeed, in the non-trivial case  $r \neq 0$ , we have the factorization

$$\mu_n = \left( \frac{1}{1 + r^{-1}(p-1)n} \right) \times \binom{pn + r - 1}{n}$$

and we can again apply Lemma 2 in [16], the first factor being ID by the case  $a = b = 1$  and  $s = r^{-1}(p-1)$  of the previous paragraph. Taking  $p = 2$  and  $r = 1$ , we recover the ID character of the Catalan sequence.

*Remark 3.4.* — Characterizing the Bernstein property of the binomial and the Raney moment sequences is an open problem, which is apparently not easy. Indeed, in both situations the ratio

$$\frac{\mu_n}{\mu_{n-1}}$$

involves six Gamma functions in general. Trying an hypergeometric summation argument as in Section 2.3 in [7] should lead to generalized hypergeometric series, and it is well-known that exact summation formulæ are rather rare in this broader context.

### 3.3. Gamma moment sequences of higher order

Consider the general sequence

$$\mu_n = \prod_{i=1}^p \frac{\Gamma(a_i + A_i n)}{\Gamma(a_i)} \times \prod_{j=1}^q \frac{\Gamma(b_j)}{\Gamma(b_j + B_j n)}$$

with all parameters positive. Characterizing the positive definiteness in  $\mathbb{R}^+$  of this sequence, that is whether it is the moment sequence of a positive random variable, as was done in [18, 19] for binomial and Raney sequences, seems to be a difficult task which has not been undertaken as yet. On the other hand, the ID moment sequence character of  $\{\mu_n\}$  in the case of a compact support can be characterized from the recent results in [13]. To be more precise, it follows from Lemma 1 and Theorem 4 in [13] that  $\{\mu_n\}$  is the moment sequence of a positive random variable  $\mathbf{X}$  with compact support and that this sequence is ID, if and only if

$$\sum_{i=1}^p A_i = \sum_{j=1}^q B_j \quad \text{and} \quad \sum_{i=1}^p \frac{e^{-a_i A_i^{-1} x}}{1 - e^{-A_i^{-1} x}} - \sum_{j=1}^q \frac{e^{-a_j B_j^{-1} x}}{1 - e^{-B_j^{-1} x}} \geq 0 \quad \forall x > 0.$$

The support of  $\mathbf{X}$  is then the interval  $[0, \rho]$  with

$$\rho = \prod_{i=1}^p A_i^{A_i} \times \prod_{j=1}^q B_j^{-B_j},$$

see (16) of [13]. However, as mentioned in the introduction to [13], the non-negativity condition is not easy to check directly on the parameters, even in the special cases of Raney and binomial sequences. In these two cases, the above density argument via the Beta distribution is much quicker.

*Remark 3.5.* — Other ID moment sequences with compact support can be built on two sequences of positive numbers via the multiple Gamma function. The associate random variables have the so-called Barnes Beta distribution (see [20], especially Theorem 2.4 therein, for more detail).

### 3.4. Other moment sequences of Gamma type

In this last paragraph we come back to the moment sequence  $(n!)^t$  of Section 1. The third equality in (1.1) yields the exponential representation

$$\frac{\Gamma(1+s)^t}{\Gamma(1+st)} = \exp \left\{ \int_{-\infty}^0 (e^{sx} - 1 - sx) \left( \frac{t}{e^{|x|} - 1} - \frac{1}{e^{|x|t^{-1}} - 1} \right) \frac{dx}{|x|} \right\}$$

for every  $s, t > 0$ . Besides, some analysis shows that the function  $z \mapsto t(z-1) - (z^t-1)$  is positive on  $(1, \infty)$  for  $t \in (0, 1)$  and negative on  $(1, \infty)$  for  $t > 1$ . By Mellin inversion, this shows the identities in law

$$\mathbf{L}_t \stackrel{d}{=} \mathbf{L}^t \times \mathbf{M}_t \quad \text{for } t \in (0, 1) \quad \text{and} \quad \mathbf{L}^t \stackrel{d}{=} \mathbf{L}_t \times \mathbf{M}_t \quad \text{for } t > 1,$$

where  $\mathbf{M}_t$  is a positive random variable with fractional moments

$$\mathbb{E}[\mathbf{M}_t^s] = \exp \left\{ \int_{-\infty}^0 (e^{sx} - 1 - sx) \left| \frac{t}{e^{|x|} - 1} - \frac{1}{e^{|x|t^{-1}} - 1} \right| \frac{dx}{|x|} \right\},$$

$$s > -\inf\{1, t^{-1}\}.$$

Stirling's formula implies that  $\mathbf{M}_t$  has a compact support which is  $[0, t^{-t}]$  for  $t \in (0, 1)$  and  $[0, t^t]$  for  $t > 1$ . The positive entire moments of  $\mathbf{M}_t$  are

$$\mu_n = \frac{(n!)^t}{\Gamma(1+nt)} \quad \text{for } t < 1 \quad \text{and} \quad \mu_n = \frac{\Gamma(1+nt)}{(n!)^t} \quad \text{for } t > 1.$$

Since  $\log \mathbf{M}_t$  is infinitely divisible, these moment sequences are ID. Moreover, by compactness of the support, the sequences  $\{\mu_{ns}\}$  and  $\{\mu_n^s\}$  are M-det for every  $s > 0$ . For  $t > 1$ , the sequence  $\{\mu_n\}$  is not Bernstein since the corresponding Remainder would then have Laplace exponent

$$\frac{\Gamma(1+\lambda t)}{\lambda^t \Gamma(1-t+\lambda t)}$$

and this function takes negative values on  $(0, \infty)$ . For  $t \in (0, 1)$ , the Bernstein character of  $\{\mu_n\}$  amounts to that of the function

$$\Phi(\lambda) = \frac{\Gamma(1-t+\lambda)}{\lambda^{1-t} \Gamma(\lambda)}$$

but I was not able to give an answer to this puzzling question. Combining Formulæ 1.7.2(22) and 1.9(1) in [10] yields the exponential representation

$$\Phi(\lambda) = \exp \left\{ \int_0^\infty e^{-\lambda x} \left( \frac{(1-t)(1-e^{-x}) + e^{-(1-t)x} - 1}{x(1-e^{-x})} \right) dx \right\}$$

and it can be shown, using Theorem 6.10 in [25], that the function on the right-hand side is not a complete Bernstein function. This representation also implies that  $1/\Phi(\lambda)$  is logarithmically completely monotone, which is necessary but not sufficient for  $\Phi(\lambda)$  to be Bernstein (see Proposition 5.17 in [25] and the remark thereafter).

#### 4. A further example with generalized stable laws

We conclude this paper with another example of strict dichotomy between moment-determinacy and moment-indeterminacy, in the spirit of Section 1. The framework is that of the  $r$ -gstable( $a, m$ ) laws recently studied in [12, 22] (see also the references therein). These laws are well-defined if and only if  $0 < a < m$  and they form a generalization of the inverse positive stable laws which correspond to the case  $m = 1$ . We refer to [12, 22] for more details. The entire moments of the corresponding random variable  $\mathbf{Y}_{a,m}$  are given by

$$\mu_n = \mathbb{E}[\mathbf{Y}_{a,m}^n] = a^{\frac{(m-a)n}{a}} \times \frac{G(m+n, a)G(a, a)}{G(a+n, a)G(m, a)}$$

where  $G$  is the double Gamma function (see (11) in [12]). It follows from the main Theorem in [12] that  $\log \mathbf{Y}_{a,m}$  is infinitely divisible and the moment sequence  $\{\mu_n\}$  is hence always ID. It is also easy to see from the concatenation formula  $G(z+1, \tau) = \Gamma(z\tau^{-1})G(z, \tau)$  that

$$\{\mu_n\} \text{ is Bernstein} \iff 1 \leq a < m \leq 2a.$$

The corresponding Bernstein functions are

$$\Phi(\lambda) = a^{\frac{m-a}{a}} \times \frac{\Gamma(a^{-1}(\lambda + m - 1))}{\Gamma(a^{-1}(\lambda + a - 1))}$$

for  $m < 2a$  and  $\Phi(\lambda) = \lambda + a - 1$  for  $m = 2a$ . Observe that those are special instances of the example discussed at the end of Section 2.

PROPOSITION 4.1. — *The random variable  $\mathbf{Y}_{a,m}$  is M-det if and only if  $m \leq 3a$ .*

*Proof.* — An easy consequence of (11) and (13) in [12] is

$$\mathbb{E}[\mathbf{Y}_{a,m}^n]^{\frac{1}{n}} \sim \left( \frac{n}{ae} \right)^{\frac{m-a}{a}},$$

so that

$$\frac{\log \mathbb{E}[\mathbf{Y}_{a,m}^n]}{2n \log n} \rightarrow \frac{m-a}{2a}.$$

Moreover, we know by Corollary (b) in [12] that  $\mathbf{Y}_{a,m}$  has a HCM density for  $m \geq 2a$ . A combination of Carleman's criterion and Theorem 7 in [15] shows that  $\mathbf{Y}_{a,m}$  is M-det if and only if

$$\frac{m-a}{2a} \leq 1 \iff m \leq 3a. \quad \square$$

This result was obtained for  $m$  integer in Theorem 8.2 of [22], whose proof of the only if part is a consequence of Krein's condition and of the subexponential tail behaviour at infinity of the density of  $\sqrt{Y_{a,m}}$ , which is obtained therein by means of a certain class of special functions. It is shown in the Proposition of [12] that the latter subexponentiality property holds true for every  $m > 3a$  non necessarily an integer, so that one can conclude as in [22]. Overall, this proof is however more involved than the above HCM argument.

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