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Solvable groups of interval exchange transformations ^(*)

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ABSTRACT. — We prove that any finitely generated torsion free solvable subgroup of the group IET of all Interval Exchange Transformations is virtually abelian. In contrast, the lamplighter groups $A \wr \mathbb{Z}^k$ embed in IET for every finite abelian group A , and we construct uncountably many non pairwise isomorphic 3-step solvable subgroups of IET as semi-direct products of a lamplighter group with an abelian group.

We also prove that for every non-abelian finite group F , the group $F \wr \mathbb{Z}$ does not embed in IET.

RÉSUMÉ. — Nous démontrons que tout sous-groupe de type fini résoluble sans torsion du groupe IET des échanges d'intervalles est virtuellement abélien. À l'opposé, les groupes d'allumeurs de réverbères $A \wr \mathbb{Z}^k$ se plongent dans IET pour tout groupe abélien fini A , et nous construisons un nombre non dénombrable de sous-groupes résolubles de classe 3 dans IET non isomorphes entre eux comme produits semi-directs d'un groupe d'allumeurs de réverbères avec des groupes abéliens.

Nous démontrons aussi que pour tout groupe fini non-abélien F , le produit en couronne $F \wr \mathbb{Z}$ ne se plonge pas dans IET.

The group IET and its subgroups

The group IET of interval exchange transformations is the group of all bijections of the interval $[0, 1)$ that are piecewise translations with finitely many discontinuity points.

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Rather unexpectedly, the recent study of this group has given evidences that it is not as big as one could have thought, in several commonly accepted features. For instance, in [3] we established that IET does not have many free subgroups (if any at all), and that the connected Lie groups that can embed in it are only abelian. Another fact in this direction is that given any finitely generated subgroup of IET, and any point $x \in [0, 1)$ the orbit of x grows (in cardinality) at most polynomially in the word length of the elements of the subgroup [3, Lem. 6.2].

Yet another instance of these evidences is given by the main result of [7] of Juschenko and Monod which implies that certain natural subgroups of this group are amenable. More precisely, given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the subgroup IET_α of transformations whose translation lengths are all multiples of α modulo 1 is amenable. Indeed, given any finitely generated subgroup G of IET_α , G can be viewed as a group of homeomorphisms of the Cantor set K obtained by blowing up the R_α -orbit of the discontinuity points of the generators of G , where R_α is the rotation $x \mapsto x + \alpha \pmod{1}$ [1]. Denoting by \widehat{R}_α the homeomorphism of K induced by R_α , this embeds G in the full topological group of \widehat{R}_α , which is amenable by [7]. This has been extended in [6] to subgroups of rational rank ≤ 2 , i.e. such that the subgroup of \mathbb{Q}/\mathbb{Z} generated by the translation lengths of its elements does not contain \mathbb{Z}^3 .

Given these evidences, we chose to investigate the possible solvable subgroups of IET.

Results

In order to describe elementary examples of subgroups of IET, let us enlarge a bit the context, and instead of interval exchange transformations on the interval $[0, 1)$ we consider the group $\text{IET}(\mathcal{D})$ of interval exchange transformations on a domain \mathcal{D} consisting of a disjoint union of finitely many oriented circles, and oriented half-open intervals, closed on the left (see Section 1.1). This does not make change the isomorphism classes of subgroups encountered as $\text{IET} \simeq \text{IET}(\mathcal{D})$.

For all $n \in \mathbb{N}$, \mathbb{Z}^n embeds in $\text{IET}(\mathbb{R}/\mathbb{Z})$ as a group of rotations. The following general simple fact then implies that every finitely generated virtually abelian group embeds in IET.

PROPOSITION 1 (Proposition 1.2). — *Let G be a group, and assume that some finite index subgroup of G embeds in IET. Then so does G .*

It is then natural to ask which virtually polycyclic groups embed in IET. Our first result shows that only virtually abelian ones do. By a different method, Cornulier [2] also shows that a virtually polycyclic group in IET must be virtually abelian.

THEOREM 2 (See Corollary 3.2). — *Let H be a virtually polycyclic group. Then H embeds into IET if and only if it is virtually abelian.*

Since a polycyclic group is virtually torsion-free, this result is in fact a corollary of the following theorem which applies to all torsion-free solvable subgroups of IET.

THEOREM 3 (Theorem 3.1). — *Every finitely generated torsion-free solvable subgroup of IET is virtually abelian.*

If we allow torsion, a much greater variety of subgroups exists. The first interesting example is an embedding of the lamplighter group $L = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}^k$ in IET. Note that this group L is solvable (in fact metabelian), has exponential growth, and is not virtually torsion-free.

To describe this embedding, consider the domain $\mathcal{D} = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, a disjoint union of n circles. Choose $\Lambda \subset \mathbb{R}/\mathbb{Z}$ a subgroup isomorphic to \mathbb{Z}^k , and view Λ as a group of *synchronized rotations* in $\text{IET}(\mathcal{D})$, i.e. by making $\theta \in \Lambda$ act on $\mathcal{D} = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ by $(i, x) \mapsto (i, x + \theta)$. Consider the interval $I = [0, 1/2) \subset \mathbb{R}/\mathbb{Z}$, and let τ be the transformation that is the identity outside $(\mathbb{Z}/n\mathbb{Z}) \times I$, and sends (i, x) to $(i + 1, x)$ for $x \in I$. Then the subgroup of $\text{IET}(\mathcal{D})$ generated by τ and Λ is isomorphic to L . This is illustrated in Figure 4.1. More generally, taking $\mathcal{D} = A \times (\mathbb{R}/\mathbb{Z})$ for some finite abelian group A , this construction yields the following result:

PROPOSITION 4 (see Propositions 4.1 and 4.2). — *For any finite abelian group A and any $k \geq 1$, the wreath product $L = A \wr \mathbb{Z}^k$ embeds in IET.*

One could try a similar construction, replacing the abelian finite group A by a non-abelian one. But the group obtained would not be the wreath product. In fact, such a wreath product cannot embed in IET as the following result shows.

THEOREM 5 (Theorem 4.4). — *If F is a finite group and if $F \wr \mathbb{Z}$ embeds as a subgroup in IET, then F is abelian.*

It would be interesting to know which finitely generated wreath products embed in IET.

Starting from a subgroup $G < \text{IET}(\mathcal{D})$ and a finite abelian group A , the construction above allows to construct subgroups of $\text{IET}(\mathcal{D} \times A)$ isomorphic to $G \rtimes \mathcal{F}_A$ where \mathcal{F}_A is a subgroup of the abelian group $A^{\mathcal{D}}$. We then prove that, in contrast with the torsion-free case, this construction yields a huge variety of isomorphism classes of solvable subgroups in IET.

THEOREM 6 (Theorem 4.9). — *There exist uncountably many isomorphism classes of subgroups of IET that are generated by 3 elements, and that are solvable of derived length 3.*

The method we use consists of embedding many semidirect products in a way that is related to the twisted embeddings used in [6].

About proofs

The proof of Theorem 4.4 saying that $F \wr \mathbb{Z}$ does not embed into IET uses the fact that orbits in $[0, 1)$ by a finitely generated subgroup of IET have polynomial growth. On the other hand, if $F \wr \mathbb{Z} = (\bigoplus_{n \in \mathbb{Z}} F) \rtimes \mathbb{Z}$ embeds in IET, F and its conjugate have to commute with each other. This gives strong algebraic restrictions on the action on F . Using Birkhoff theorem, we show that if F itself is non-commutative, then the orbit growth of $F \wr \mathbb{Z}$ has to be exponential, a contradiction.

To prove that there are uncountably many groups as in Theorem 4.9, we start with a lamplighter group $G = (\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ constructed above on $\mathcal{D} = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, where \mathbb{Z} acts on the three circles by setting a generator to act as a synchronized rotation on them with irrational angle α . Then, we consider $\mathcal{D}' = (\mathbb{Z}/2\mathbb{Z}) \times \mathcal{D}$, on which we make G act diagonally. Then we choose an interval J in $\{0\} \times (\mathbb{R}/\mathbb{Z}) \subset \mathcal{D}$ and define τ_J on \mathcal{D}' by the identity outside $(\mathbb{Z}/2\mathbb{Z}) \times J$, and by $(i, x) \mapsto (i + 1, x)$ on $(\mathbb{Z}/3\mathbb{Z}) \times J \subset \mathcal{D}'$. This is illustrated in Figure 4.2. The group H generated by G and τ_J is a homomorphic image of the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr G$, but this is not an embedding. Still, the group H generated by G and τ_J has the structure of a semidirect product $G \rtimes \mathcal{F}$ where \mathcal{F} is an infinite abelian group of exponent 2. Using Birkhoff ergodic theorem, we prove that from the isomorphism class of H , one can read off the length of J modulo a countable additive group. This proves that by varying length of J , we get uncountably many distinct isomorphism classes of groups H .

The classification of torsion-free solvable subgroups of IET in Theorem 3.1 is based on the fact that centralizers of a *minimal* interval exchange transformation T is small. Indeed, if T is an irrational rotation on a circle, then its centralizer consists of the whole group of rotations on this circle; and if

T is not conjugate to such a rotation, a theorem by Novak [10] shows that its centralizer is virtually cyclic. If T is not minimal, its centralizer can be much larger: it will for instance contain a group isomorphic to IET if T fixes a non-empty subinterval.

Thus, we need to understand the orbit closures of a finitely generated group, and we also need to understand how it varies when we pass to a subgroup of finite index.

A result by Imanishi [5] about the holonomy of codimension 1 foliations shows that for each finitely generated group $G < \text{IET}(\mathcal{D})$, there is a partition of \mathcal{D} into finitely many G -invariant subdomains⁽¹⁾, such that in restriction to each subdomain, either every orbit is dense (such a subdomain is called an *irreducible component*), or every orbit is finite of the same cardinal (and one can say more, see Proposition 2.2). In particular, for each $x \in \mathcal{D}$, $\overline{G.x}$ is either finite, or is the closure of a subdomain (not a Cantor set).

When passing to a finite index subgroup G_0 of G , it could happen that an irreducible component for G splits into several irreducible components for G_0 .

For example, consider $\mathcal{D} = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, and consider the subgroup G of $\text{IET}(\mathcal{D})$ generated by the three following transformations τ, R_0, R_1 . Let τ be the involution $(i, x) \mapsto (i + 1, x)$; let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let R_0 be the rotation of angle α on the circle $\{0\} \times (\mathbb{R}/\mathbb{Z})$ and as the identity on the circle $\{1\} \times (\mathbb{R}/\mathbb{Z})$; and let $R_1 = \tau R_0 \tau^{-1}$. Then $\langle R_0, R_1 \rangle \simeq \mathbb{Z}^2$, and $G = \langle \tau, R_0 \rangle \simeq (\mathbb{Z} \oplus \mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and any orbit of G is dense.

However, $G_0 = \langle R_0, R_1 \rangle$ is a finite index subgroup which preserves each circle. The two circles are the irreducible components of G_0 . But this phenomenon cannot occur any more when passing to a further finite index G_1 of G_0 because G_1 has to contain an irrational rotation on each circle, thus ensuring that the two circles are still irreducible components of G_1 . This group G_0 is what we call *unfragmentable*. The following technical result of independent interest shows that this is a general fact.

THEOREM 7 (see Theorem 2.11). — *Given a finitely generated group $G < \text{IET}$, there exists a finite index subgroup G_0 which is unfragmentable in the following sense:*

- (1) G_0 acts by the identity outside its irreducible components
- (2) If G_1 is a finite index of G_0 , any irreducible component of G_0 is also an irreducible component of G_1 .

⁽¹⁾ A subdomain of \mathcal{D} is a subset that consists of finitely many semi-open intervals, closed on the left

1. Generalities

1.1. Definitions

DEFINITION 1.1. — *In all the following, a domain will be a non-empty disjoint union of finitely many oriented circles, and oriented half-open, bounded intervals (closed on the left).*

Given a domain \mathcal{D} , the group $\text{IET}(\mathcal{D})$ of interval exchange transformations of \mathcal{D} is the group of bijections of \mathcal{D} that are orientation preserving piecewise isometries, left continuous with finitely many discontinuity points.

By convention, we define $\text{IET} = \text{IET}([0, 1])$.

Given two disjoint domains $\mathcal{D}_1, \mathcal{D}_2$ having the same total length, there is an element of $\text{IET}(\mathcal{D}_1 \sqcup \mathcal{D}_2)$ that sends \mathcal{D}_1 on \mathcal{D}_2 . We call such an element an interval exchange bijection from \mathcal{D}_1 to \mathcal{D}_2 . This element then conjugates $\text{IET}(\mathcal{D}_2)$ to $\text{IET}(\mathcal{D}_1)$. Observe also that rescaling a domain doesn't change its group of interval exchange transformations. In particular, for any domains $\mathcal{D}_1, \mathcal{D}_2$, $\text{IET}(\mathcal{D}_1)$ and $\text{IET}(\mathcal{D}_2)$ are always isomorphic.

A subdomain $\mathcal{D}_0 \subset \mathcal{D}$ is a subset of \mathcal{D} which has finitely many connected components, and which is closed on the left. If \mathcal{D}_0 is a subdomain of \mathcal{D} that is invariant by the action of G , then G naturally maps to $\text{IET}(\mathcal{D}_0)$ by restriction. We denote by $G|_{\mathcal{D}_0}$ its image in $\text{IET}(\mathcal{D}_0)$. Moreover, if $\mathcal{D} = \mathcal{D}_0 \sqcup \mathcal{D}_1$ where both subdomains are invariant by G , then the induced morphism from G to $\text{IET}(\mathcal{D}_0) \times \text{IET}(\mathcal{D}_1)$ is injective.

1.2. Finite extensions

PROPOSITION 1.2. — *Let G be a group, and assume that some finite index subgroup of G embeds in IET . Then so does G .*

Proof. — Without loss of generality, consider $H < G$ a normal subgroup of finite index that embeds in $\text{IET}(\mathcal{D})$ for some domain \mathcal{D} . Let Q be the finite quotient $Q = G/H$. It is a classical algebraic fact (see [9]) that G embeds in the wreath product $H \wr Q = H^Q \rtimes Q$ (where Q acts on H^Q by permuting coordinates). Thus, it suffices to show that $H^Q \rtimes Q$ embeds in IET .

Consider the domain $\mathcal{D}' = Q \times \mathcal{D}$, and embed H^Q in $\text{IET}(\mathcal{D}')$ by making $(h_q)_{q \in Q}$ act on $\mathcal{D}' = Q \times \mathcal{D}$ by $(q, x) \mapsto (q, h_q.x)$. Then Q acts on \mathcal{D}' by left multiplication on the left coordinate. This naturally extends to a morphism $H^Q \rtimes Q \rightarrow \text{IET}(\mathcal{D}')$ which is clearly one-to-one. \square

2. Irreducibility and unfragmentability for finitely generated subgroups of IET

2.1. IET and irreducibility

Let \mathcal{D} be a domain. Let $G = \langle S \rangle$ be a finitely generated subgroup of the group $\text{IET}(\mathcal{D})$ of interval exchange transformations on \mathcal{D} , with S symmetric.

DEFINITION 2.1 (Irreducibility). — *We say that G is irreducible (on \mathcal{D}) if no subdomain of \mathcal{D} is invariant under G .*

We say that a subdomain J of \mathcal{D} is an irreducible component for G if it is G -invariant, and if G restricted to J is irreducible.

We will see that a finitely generated group G is irreducible if and only if every G -orbit is dense in \mathcal{D} (see Corollary 2.6).

If $s \in \text{IET}(\mathcal{D})$, we denote by $\text{Disc}(s) \subset \overset{\circ}{\mathcal{D}}$ the set of discontinuity points of s . If S is a set of elements of $\text{IET}(\mathcal{D})$, we denote by $\text{Disc}(S) = \bigcup_{s \in S} \text{Disc}(s)$ the set of discontinuity points of elements of S .

We say that $x, y \in \overset{\circ}{\mathcal{D}}$ are in the same *regular orbit* if there exist $g \in G$ continuous at x with $g(x) = y$. We say that $x, y \in \overset{\circ}{\mathcal{D}}$ are in the same *S -regular orbit* if there exists $g_1, \dots, g_n \in S$ such that $y = g_n \dots g_1(x)$, and for all $i \leq n$, $g_{i-1} \dots g_1(x) \in \overset{\circ}{\mathcal{D}} \setminus \text{Disc}(g_i)$. We denote by $\text{Reg}(x, G) \subset \overset{\circ}{\mathcal{D}}$ the regular orbit of x , and by $\text{Reg}(x, S) \subset \text{Reg}(x, G)$ its S -regular orbit. Although arguably less natural, the notion of S -regular orbit is the one that is needed to apply Imanishi Theorem below. On the other hand, Lemma 2.12 below will show that one can choose S so that $\text{Reg}(x, S) = \text{Reg}(x, G)$ for all $x \in \overset{\circ}{\mathcal{D}}$.

Let

$$\text{Sing}(G) = \{x \in \overset{\circ}{\mathcal{D}} \mid \exists g \in G, x \in \text{Disc}(g)\}.$$

By definition, $\text{Sing}(G)$ does not depend on any generating set, but one easily checks that for $x \in \overset{\circ}{\mathcal{D}}$, $x \in \text{Sing}(G)$ if and only if its G -orbit contains a point in $\text{Disc}(S)$, if and only if $\text{Reg}(x, S)$ contains a point in $\text{Disc}(S)$. Thus, $\text{Sing}(G)$ is a union of at most $\#\text{Disc}(S)$ S -regular orbits. We also note that for all $x \in \overset{\circ}{\mathcal{D}} \setminus \text{Sing}(G)$, $\text{Reg}(x, S) = \text{Reg}(x, G) = G.x \cap \overset{\circ}{\mathcal{D}}$.

We denote by $E(G) \subset \text{Sing}(G)$ the set of points $x \in \text{Sing}(G)$ whose regular orbit is finite, and by $E(S)$ the set of points $x \in \text{Sing}(G)$ whose

S -regular orbit is finite. We note that $E(G) \subset E(S)$ and that these sets are finite since

$$E(S) = \bigcup_{x \in \text{Disc}(S), \# \text{Reg}(x, S) < \infty} \text{Reg}(x, S)$$

is a finite union of finite sets.

We will need to apply a decomposition theorem that applies for finite systems of isometries, so let us introduce the corresponding terminology (see [4]). Let $\overline{\mathcal{D}}$ be the obvious compactification of \mathcal{D} as a union of compact intervals and circles. A partial isometry on $\overline{\mathcal{D}}$ is an isometry $\varphi : I \rightarrow J$ between two closed subintervals $I, J \subset \overline{\mathcal{D}}$. Given a finite set S of interval exchanges on \mathcal{D} , one can construct a *system of isometries* on $\overline{\mathcal{D}}$ as follows: for each $s \in S$, let I_1, \dots, I_{n_s} be the maximal connected subdomains on which s is continuous. These intervals define a partition of \mathcal{D} . For each $i \leq n_s$, we let φ_i be the partial isometry defined on $\overline{I_i}$ that extends $s|_{I_i}$. Thus, for each element of s , we have a finite collection of partial isometries, and we denote by X the collection of all partial isometries of $\overline{\mathcal{D}}$ obtained from all the elements of S in this way. Orbits of X are defined in the natural way. Clearly, each G -orbit is contained in an X -orbit. One defines $\overset{\circ}{X}$ -orbits similarly using the restriction of all the partial isometries to the interior of their domains. Then for all $x \in \overset{\circ}{\mathcal{D}}$, its $\overset{\circ}{X}$ -orbit coincides exactly with its S -regular orbit $\text{Reg}(x, S)$.

Specifying [4, Thm. 3.1] to our setting, we get:

PROPOSITION 2.2 (Imanishi theorem [5], see [4, Thm. 3.1]). — *Let S be a finite symmetric set of interval exchanges on \mathcal{D} , and $G = \langle S \rangle < \text{IET}(\mathcal{D})$.*

Then $\overset{\circ}{\mathcal{D}} \setminus E(S) = \overset{\circ}{\mathcal{D}}_1 \sqcup \dots \sqcup \overset{\circ}{\mathcal{D}}_p$ where each $\overset{\circ}{\mathcal{D}}_i$ is an open subset of $\overset{\circ}{\mathcal{D}} \setminus E(S)$ invariant under S -regular orbits and such that for each $i \leq p$, one of the following holds:

- (1) *every S -regular orbit in $\overset{\circ}{\mathcal{D}}_i$ is dense in $\overset{\circ}{\mathcal{D}}_i$*
- (2) *every S -regular orbit in $\overset{\circ}{\mathcal{D}}_i$ is finite, of the same cardinality, and $\overset{\circ}{\mathcal{D}}_i \cap \text{Sing}(G) = \emptyset$.*

Remark 2.3. — Theorem 3.1 of [4] is stated for systems of isometries on a finite union of intervals, but generalizes immediately to the case where we allow circles in the domain (alternatively, we could restrict to this case by cutting \mathcal{D} along a point in each circle of \mathcal{D}). Our assertion 2 thus includes the possibility that $\overset{\circ}{\mathcal{D}}_i$ is a disjoint union of circles, and that all elements of G act continuously on $\overset{\circ}{\mathcal{D}}_i$. Since all elements of IET preserve the orientation, there cannot be any twisted family of finite orbits in the sense of [4, Thm. 3.1].

COROLLARY 2.4. — *Let $G < \text{IET}(\mathcal{D})$ be a finitely generated group.*

Then \mathcal{D} decomposes into G -invariant subdomains $\mathcal{D} = \mathcal{D}_\infty \sqcup \mathcal{D}_{\text{fin}}$ with $\mathcal{D}_\infty = I_1 \sqcup \cdots \sqcup I_r$ and $\mathcal{D}_{\text{fin}} = J_1 \sqcup \cdots \sqcup J_t$ where each I_i and J_j is a G -invariant subdomain whose boundary is contained in $E(S) \cup \partial\mathcal{D}$, and such that

- (1) *for all $i \leq r$, every G -orbit in I_i is dense in I_i*
- (2) *for all $j \leq r$, G acts on J_j with finite orbits, all of the same cardinality; the restriction to J_j of any $g \in G$ is continuous, and in particular, G permutes the connected components of J_j .*

Moreover, the collection of irreducible components of G , $\text{Irred}(G) = \{I_1, \dots, I_r\}$, is unique.

Remark 2.5. — Note that whereas $\text{Irred}(G) = \{I_1, \dots, I_r\}$ is uniquely defined, we don't claim that the decomposition of the complement $\mathcal{D} \setminus (\bigcup_i I_i)$ into G -invariant subdomains upon which G acts as a finite group is unique (although one could easily construct such a canonical decomposition).

Proof. — Take S a finite symmetric generating set of G . Write $\overset{\circ}{\mathcal{D}} \setminus E(S) = \overset{\circ}{\mathcal{D}}_1 \sqcup \cdots \sqcup \overset{\circ}{\mathcal{D}}_p$ as in Proposition 2.2. Let $\mathcal{D}_i \subset \mathcal{D}$ be the smallest subdomain of \mathcal{D} containing $\overset{\circ}{\mathcal{D}}_i$ (i.e. the set of points to which are limits of points in $\overset{\circ}{\mathcal{D}}_i$ from the right), so that $\mathcal{D} = \mathcal{D}_1 \sqcup \cdots \sqcup \mathcal{D}_p$. Then \mathcal{D}_i is G -invariant because if $g\mathcal{D}_i \cap \mathcal{D}_j \neq \emptyset$, then $\mathcal{D}_i \cap g^{-1}\mathcal{D}_j$ contains an interval and for any point $x \in \overset{\circ}{\mathcal{D}} \setminus \text{Sing}(G)$ in this interval, we have that $g(x) \in \text{Reg}(x, S) \cap \mathcal{D}_j$ contradicting that $\overset{\circ}{\mathcal{D}}_i$ is invariant under S -regular orbits. We denote by $I_1, \dots, I_r, J_1, \dots, J_t$ the subdomains of $\mathcal{D}_1, \dots, \mathcal{D}_p$ (with $r+t = p$) according to whether they satisfy the first or the second assertion of Proposition 2.2

Then for all $i \leq r$ and every $x \in I_i$, its G -orbit is dense in I_i because it contains $\text{Reg}(x, S)$. If $x \in I_i \setminus \overset{\circ}{I}_i$, then its G -orbit has to contain a point in $\overset{\circ}{I}_i$. Indeed, otherwise, the G -orbit of x would be finite, and so would be the G -orbits of points x' in a right neighbourhood of x , contradicting that regular orbits are dense in $\overset{\circ}{I}_i$.

For $j \leq t$, then since $\overset{\circ}{J}_j \cap \text{Sing}(G) = \emptyset$, G acts continuously on $\overset{\circ}{J}_j$, so $g|_{J_j}$ is continuous for every $g \in G$.

The uniqueness of $\{I_1, \dots, I_r\}$ is immediate from the fact that I_i is G -invariant and that every G -orbit is dense in I_i . \square

The following corollary is clear.

COROLLARY 2.6. — *A subgroup $G < \text{IET}(\mathcal{D})$ is irreducible if and only if every G -orbit is dense in \mathcal{D} .* \square

Remark 2.7. — In the case where $E(S) = \emptyset$, each subdomain I_i, J_j appearing in Corollary 2.4 is a union of connected components of \mathcal{D} . We can always cut the domain \mathcal{D} to reduce to this situation as follows.

We say that we cut a domain \mathcal{D}_1 along a finite set of points if we map it to some domain \mathcal{D}_2 by an interval exchange bijection $\tau : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ that is discontinuous exactly on this set of points, and such that τ^{-1} is continuous. For instance, if $\mathcal{D}_1 = [a, b)$, cutting along $c \in [a, b)$ yields a domain $\mathcal{D}_2 = [a, c) \sqcup [c, b)$, and if \mathcal{D}_1 is a circle, cutting along one point yields a domain that is a half-open interval. If $G_1 \subset \text{IET}(\mathcal{D}_1)$, we then get by conjugation a group $G_2 \subset \text{IET}(\mathcal{D}_2)$ (in particular, G_2 is isomorphic to G_1).

Now consider $G < \text{IET}(\mathcal{D})$, and S is a symmetric generating set of G . Then cutting \mathcal{D} along the (finitely many) points in $E(S)$ yields a domain \mathcal{D}' and a group $G' < \text{IET}(\mathcal{D}')$ conjugate to G by an interval exchange bijection from \mathcal{D} to \mathcal{D}' such that $E(S') = \emptyset$ (where S' is the conjugate of S).

2.2. Virtual unfragmentability for finitely generated groups of IET

Let G be a finitely generated subgroup of $\text{IET}(\mathcal{D})$.

DEFINITION 2.8 (Unfragmentability). — *We say that G is unfragmentable if for any subdomain $J \subset \mathcal{D}$ which is invariant by a finite index subgroup of G , J is G -invariant.*

We say that an element $a \in \text{IET}(\mathcal{D})$ is *unfragmentable* if the cyclic group $\langle a \rangle$ is unfragmentable.

Here is an equivalent definition.

LEMMA 2.9. — *The subgroup G is unfragmentable if each of its finite orbits is trivial, and if, for every irreducible component J of G , and every finite index subgroup H of G , the restriction of H on J is irreducible.*

Proof. — Let $I_1, \dots, I_r \subset \mathcal{D}$ be the irreducible components of G , and $I' = \mathcal{D} \setminus (I_1 \cup \dots \cup I_r)$.

Assume that G is unfragmentable. If G has a non-trivial finite orbit (necessarily in I'), then there exists a subdomain $J \subset I'$ and $g \in G$ such that $gJ \neq J$. Then J is a subdomain that is invariant under the finite index subgroup of G acting trivially on I' , but not G -invariant, a contradiction. If some finite index subgroup $G_0 < G$ does not act irreducibly on some I_i , then

there is a G_0 -invariant subdomain $J \subsetneq I_i$ and J is not G -invariant because I_i is an irreducible component of G , a contradiction.

Conversely, assume that the statement in the lemma holds and let $J \subset \mathcal{D}$ be a subdomain invariant under a finite index subgroup $G_0 < G$. Since G acts trivially on I' , $J \cap I'$ is G -invariant. For each i , $J \cap I_i$ is G_0 -invariant, and since by assumption G_0 acts irreducibly on I_i , we either get that $J \cap I_i$ is empty or $J = I_i$, in particular, $J \cap I_i$ is G -invariant. Since this holds for each i , J is G -invariant. \square

Remark 2.10. — If \mathcal{D} and \mathcal{D}' are two domains with an interval exchange bijection τ from \mathcal{D} to \mathcal{D}' , then a group $G < \text{IET}(\mathcal{D})$ is unfragmentable (resp. irreducible) if and only if its conjugate by τ in $\text{IET}(\mathcal{D}')$ is unfragmentable (resp. irreducible).

The main theorem of this section is the following. We will use it only in the case of a cyclic group, but the general statement seems to be of interest.

THEOREM 2.11. — *If $G < \text{IET}(\mathcal{D})$ is a finitely generated subgroup of $\text{IET}(\mathcal{D})$, it admits a finite index subgroup that is unfragmentable.*

Before proving the theorem, we prove that one can find a finite symmetric generating set S of G such that for every $x \in \overset{\circ}{\mathcal{D}}$, $\text{Reg}(x, S) = \text{Reg}(x, G)$, and $E(S) = E(G)$.

LEMMA 2.12. — *Let G be a finitely generated subgroup of $\text{IET}(\mathcal{D})$.*

Then there exists a finite symmetric generating set S such that for all $x \in \overset{\circ}{\mathcal{D}}$, $\text{Reg}(x, S) = \text{Reg}(x, G)$.

Proof. — If $x \in \overset{\circ}{\mathcal{D}} \setminus \text{Sing}(G)$, then for all generating S of G , $\text{Reg}(x, S) = Gx = \text{Reg}(x, G)$. Let S be a finite symmetric generating set of G . We are going to increase S so that the Lemma holds. Fix a point $x \in \text{Sing}(G)$. Let R_1, \dots, R_n be the partition of $\text{Reg}(x, G)$ into S -regular orbits, and choose $x_i \in R_i$ for each $i \leq n$. By definition of $\text{Reg}(x, G)$, there exists $g_i \in G$ such that $g_i \cdot x_1 = x_i$ and g_i is continuous at x_1 . Adding $\{g_2^{\pm 1}, \dots, g_n^{\pm 1}\}$ to S yields a symmetric generating set S' such that $\text{Reg}(x, S') = \text{Reg}(x, G)$. Since $\text{Sing}(G)$ is a union of finitely many G -regular orbits, one can repeat this operation finitely many times and get a generating set satisfying the lemma. \square

Recall that $E(S)$ (resp. $E(G)$) is the set of points $x \in \text{Sing}(G)$ whose S -regular orbit $\text{Reg}(x, S)$ (resp. whose G -regular orbit $\text{Reg}(x, G)$) is finite. The previous Lemma gives the following.

COROLLARY 2.13. — *There exists a finite generating set S of G such that $E(G) = E(S)$.*

Proof of Theorem 2.11. — After cutting \mathcal{D} as in Remark 2.7, one can assume that $E_S(G) = \emptyset$. Apply Imanishi theorem, and write $\mathcal{D} = \mathcal{D}_{\text{fin}} \cup \mathcal{D}_{\infty}$, with $\mathcal{D}_{\infty} = I_1 \cup \dots \cup I_r$ as in Corollary 2.4.

Let G_1 be a finite index subgroup of G such that $(G_1)_{|\mathcal{D}_{\text{fin}}}$ is trivial, and therefore the map $G \rightarrow (G_1)_{|\mathcal{D}_{\infty}}$ is an isomorphism. Without loss of generality, and to keep readable notations, we assume that $\mathcal{D} = \mathcal{D}_{\infty}$, i.e. that every G_1 -orbit is infinite.

Let G_0 be an arbitrary finite index subgroup of G_1 . The fact that $E_S(G) = \emptyset$ means that for every $x \in \mathring{\mathcal{D}}_{\infty}$, $\text{Reg}(x, G)$ is infinite. We claim that $\text{Reg}(x, G_0)$ is also infinite. Indeed, let g_i be a sequence of elements of G that are continuous at x and such that the points $g_i.x$ are all distinct. Since $[G : G_0] < \infty$, up to extracting a subsequence, we may assume that there exists an element $a \in G$ such that $ag_i \in G_0$ for all i . Since $\text{Disc}(a)$ is finite, a is continuous at $g_i.x$ for i large enough, so $ag_i.x \in \text{Reg}(x, G_0)$. This proves our claim and shows that $E(G_0) = \emptyset$.

By Corollary 2.13, there exists a finite generating set S_0 of G_0 such that $E(S_0) = \emptyset$. Then Corollary 2.4 yields a decomposition of \mathcal{D} into finitely many G_0 -invariant subdomains I_1, \dots, I_r on which the action of G_0 is irreducible, and since $E(S_0) = \emptyset$, each I_i is a union of connected components of \mathcal{D} . The number r depends on G_0 but is bounded by the number of connected components of \mathcal{D} .

Among all possible choices of finite index subgroups $G_0 < G_1$, we choose G_0 so that r is maximal. Then for any $G'_0 < G_0$ of finite index, the decomposition of \mathcal{D} into G'_0 -irreducible components is its decomposition into G_0 -irreducible components. This shows that G_0 is unfragmentable. \square

3. Commutation and solvable subgroups

Let \mathcal{D} be a domain. The main result of this section is the following.

THEOREM 3.1. — *Let $G < \text{IET}(\mathcal{D})$ be a finitely generated torsion free solvable group. Then G is virtually abelian.*

Since virtually polycyclic groups are virtually torsion-free, we get

COROLLARY 3.2. — *Any virtually polycyclic subgroup of IET is virtually abelian.* \square

The theorem will be proved in several steps. We start with the following property of unfragmentable elements.

LEMMA 3.3. — *Let $a \in \text{IET}(\mathcal{D})$ be unfragmentable with irreducible components $\text{Irred}(a) = \{I_1, \dots, I_r\}$. If $g \in \text{IET}(\mathcal{D})$ is such that gag^{-1} commutes with a , then for all i , either $g(I_i)$ is disjoint from I_1, \dots, I_r , or $g(I_i)$ is equal to some I_j .*

Proof. — Observe that $\text{Irred}(gag^{-1}) = \{g(I_1), \dots, g(I_r)\}$. For readability we will write $a^g = gag^{-1}$. Since a commutes with a^g , it permutes the collection $\{g(I_1), \dots, g(I_r)\}$, and therefore some power $a^{r!}$ preserves each $g(I_j)$. By unfragmentability, any subdomain preserved by $a^{r!}$ is preserved by a , and therefore a itself preserves each $g(I_j)$. Similarly, a^g preserves each I_j .

The intersection $g(I_j) \cap I_i$ is a^g -invariant, and also a invariant. If $g(I_j) \cap I_i \neq \emptyset$, then $g(I_j) \cap I_i = I_i$ by irreducibility of a on I_i . Similarly, $g(I_j) \cap I_i = g(I_j)$ by irreducibility of a^g on $g(I_j)$. It follows that if $g(I_j) \cap I_i \neq \emptyset$ then $g(I_j) = I_i$. The lemma follows. \square

COROLLARY 3.4. — *Let $a \in G < \text{IET}(\mathcal{D})$ be a unfragmentable element with irreducible components I_1, \dots, I_n . Assume that for all $g \in G$, gag^{-1} commutes with a .*

Then for each $i \leq n$, $\{g(I_i), g \in G\}$ is a finite collection of disjoint subdomains.

Proof. — Let $g, h \in G$. Assume that $g(I_i) \cap h(I_i) \neq \emptyset$. Then, $h^{-1}g(I_i) \cap I_i \neq \emptyset$, and by Lemma 3.3, this implies that $h^{-1}g(I_i) = I_i$, hence $g(I_i) = h(I_i)$. Thus $\{g(I_i), g \in G\}$ is a collection of disjoint subdomains. It is finite because the measure of \mathcal{D} is finite. \square

PROPOSITION 3.5. — *Let G be a subgroup of $\text{IET}(\mathcal{D})$. If there exists a normal abelian subgroup of G , containing some irreducible and unfragmentable element, then G is either abelian or virtually cyclic. In particular, G is virtually abelian.*

Before starting the proof, let us recall the following, from [3]. Let $g \in \text{IET}(\mathcal{D})$ and $d(g)$ be the number of discontinuity points of g on \mathcal{D} . Let $\|g\| = \lim \frac{1}{n}d(g^n)$. In [3, Cor. 2.5], we proved that $\|g\| = 0$ if and only if g is conjugate to a continuous transformation of some domain \mathcal{D}' consisting only of circles, and one of its powers is a rotation on each circle.

Proof. — Let a be such an element in $A < G$ (with A abelian). If $\|a\| = 0$, then as we mentioned in the preceding discussion, a is conjugate to a continuous transformation on some \mathcal{D}' , and because it is irreducible and unfragmentable, \mathcal{D}' has to be a circle and a is an irrational rotation. Then, by [3, Lem. 1.1], its centralizer is conjugate to the rotation group on \mathcal{D}' , and since G normalises A , [3, Lem. 1.1] says that G must also be conjugate to a group of rotations on \mathcal{D}' , hence abelian.

Assume now that $\|a\| > 0$. Since a is irreducible, [10, Prop. 1.5] implies that its centraliser is virtually cyclic, hence so is A . Therefore A has a finite automorphism group. The group G acts by conjugation on A , and since the automorphism group of A is finite, the kernel of this action has finite index in G . This kernel is contained in the centralizer of a though, so G is virtually cyclic. \square

PROPOSITION 3.6. — *If $G < \text{IET}(\mathcal{D})$ is irreducible, and contains a normal abelian subgroup $A \triangleleft G$, with an element $a \in A$ of infinite order, then G is virtually abelian.*

In particular, if G is finitely generated, then so is A .

Proof. — Up to replacing a by a power, Theorem 2.11 allows us to assume that a is unfragmentable. Let I_1, \dots, I_k be its irreducible components.

By Corollary 3.4, $\{g(I_1), g \in G\}$ is a finite collection of disjoint subdomains. By irreducibility of G , it is a partition of \mathcal{D} , and we thus write $\mathcal{D} = M_1 \sqcup \dots \sqcup M_l$ with $M_1 = I_1$, and $M_i = g_i I_1$. There is a finite index subgroup G_0 of G that preserves each M_i . Let k be such that $a^k \in G_0$. Then, a^k is irreducible and unfragmentable on I_1 so Proposition 3.5 ensures that the image $G_{0|I_1} \subset \text{IET}(I_1)$ of G_0 under the restriction map is virtually abelian. Similarly, $(a^k)^{g_i}$ is irreducible on M_i , so G_0 has virtually abelian image in restriction to each M_i .

Since $\mathcal{D} = M_1 \sqcup \dots \sqcup M_l$, the product of restriction maps yields an injection of G_0 into $\prod_j \text{IET}(M_j)$. Thus, the image of G_0 is contained in a finite product of virtually abelian groups, so G_0 is virtually abelian and so is G . \square

We finish with this lemma before proving Theorem 3.1.

LEMMA 3.7. — *Let $G < \text{IET}(\mathcal{D})$ be a finitely generated group, and assume that it contains a torsion-free abelian normal subgroup A . Then A is finitely generated, and there is a finite index subgroup H in G , such that $A \cap [H, H] = \{1\}$.*

Proof. — Let $\text{Ired}(G) = \{I_1, \dots, I_k, I_{k+1}, \dots, I_r\}$ where one has ordered the components so that the image $A|_{I_i}$ of A in $\text{IET}(I_i)$ under the restriction map is a torsion group for all $i > k$ and contains an infinite order element for $i \leq k$. Let $\mathcal{D}_{\text{fin}} = \mathcal{D} \setminus (\bigcup_{i=1}^r I_i)$.

For every $i \leq k$, by Proposition 3.6 we get that the image $G|_{I_i}$ of G in $\text{IET}(I_i)$ under the restriction map is a virtually abelian group.

In other components, the image of A is a torsion group. Let us denote by G_1 the image of G in $\text{IET}(\bigcup_{i=1}^k I_i)$ and by G_2 the image of G in $\text{IET}((\bigcup_{j=k+1}^r I_j) \cup \mathcal{D}_{\text{fin}})$ under the restriction maps.

This gives an embedding $\iota : G \hookrightarrow G_1 \times G_2$, where G_1 is virtually abelian, and $p_2 \circ \iota(A)$ is torsion. In particular, because A is torsion-free, $p_1 \circ \iota$ is injective in restriction to A .

It already follows that A is finitely generated, since it embeds as a subgroup of a finitely generated virtually abelian group.

Consider H_1 an abelian finite index subgroup in G_1 , and H the preimage of H_1 in G , which is a finite index subgroup.

We saw that the map $p_1 \circ \iota$ is injective on A , but it vanishes on $[H, H]$ because $p_1 \circ \iota(H) = H_1$ is abelian. Therefore $A \cap [H, H] = \{1\}$, thus establishing the lemma. \square

Proof of Theorem 3.1. — Consider $G < \text{IET}(\mathcal{D})$ a finitely generated torsion free solvable group. Then its derived series has a largest index n for which $G^{(n)} \neq \{1\}$ (where $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ and $G^{(0)} = G$). If $n = 1$, G is abelian, we thus proceed by induction on n .

Since G is torsion free, $G^{(n)}$ is torsion free, infinite, and it is also abelian and normal in G . We may then apply Lemma 3.7 to $A = G^{(n)}$, to find that there is a finite index subgroup H of G such that $[H, H] \cap G^{(n)} = \{1\}$. However, $H^{(n)} \subset G^{(n)}$, and it follows that $H^{(n)}$ is trivial. The induction hypothesis implies that H is virtually abelian, hence so is G . \square

4. Lamps and lighters

4.1. A lamplighter group in IET

PROPOSITION 4.1. — *For all finite abelian group A , the group $A \wr \mathbb{Z}$ embeds in IET.*

Recall that in general, the group $A \wr G$ is the group $(\bigoplus_{i \in G} A) \rtimes G$, where G acts by shifting coordinates: if $g \in G$, and $(a_i)_{i \in G} \in (\bigoplus_{i \in G} A)$ is an almost null sequence, $g \cdot (a_i)_{i \in G} = (a_{g^{-1}i})_{i \in G}$.

We will actually describe an embedding in $\text{IET}(\mathcal{D})$ for a certain domain \mathcal{D} . However, IET and $\text{IET}(\mathcal{D})$ are isomorphic, and the choice of \mathcal{D} is only for convenience.

The construction is illustrated in Figure 4.1.

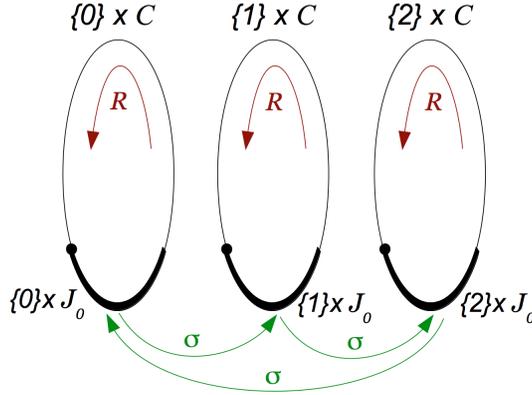


Figure 4.1. A Lamplighter group $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ in IET. The three circles $\{i\} \times C$ for $i \in \mathbb{Z}/3\mathbb{Z}$ are visible. The transformation R rotates each circle by the irrational angle θ . The support of the transformation $\sigma = \sigma_{1, J_0}$ is the union of the bold arcs.

Proof. — Consider the domain $\mathcal{D} = A \times C$ where $C = \mathbb{R}/\mathbb{Z}$. Given $a \in A$ and $J \subset C$ a subinterval, let $\sigma_{a, J}$ be the element of $\text{IET}(\mathcal{D})$ defined for all $(a', x) \in A \times C$ by

$$\sigma_{a, J} \cdot (a', x) = \begin{cases} (aa', x) & \text{if } x \in J \\ (a', x) & \text{if } x \notin J. \end{cases}$$

Note that the support of $\sigma_{a, J}$ is $A \times J$ when $a \neq 1$. We define \mathcal{A}_J as the subgroup of $\text{IET}(\mathcal{D})$ consisting of the elements $\sigma_{a, J}$ for $a \in A$ (note that \mathcal{A}_J is isomorphic to A as long as J is non-empty).

Fix $J_0 = [0, 1/2[\subset C$, and let $\mathcal{A} = \mathcal{A}_{J_0}$. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $R \in \text{IET}(\mathcal{D})$ be the rotation by θ on each circle: $R(a, x) = (a, x + \theta)$. We claim that the group generated by R and \mathcal{A} is isomorphic to $A \wr \mathbb{Z}$.

First, one easily checks that for all $J, J' \subset C$, any element of \mathcal{A}_J commutes with any element of $\mathcal{A}_{J'}$ (this is because A is abelian). Denote by t a generator of the factor \mathbb{Z} in $A \wr \mathbb{Z}$. Since $R^k \mathcal{A} R^{-k} = \mathcal{A}_{R^k J_0}$, there is a homomorphism $\varphi : A \wr \mathbb{Z} \rightarrow \langle R, \mathcal{A} \rangle$ sending t to R and sending the almost null sequence $(a_i)_{i \in \mathbb{Z}}$ to $\prod_{i \in \mathbb{Z}} \sigma_{a_i, R^i J_0}$.

To prove that φ is injective, consider an element $g = ((a_i)_{i \in \mathbb{Z}}, t^k)$ of its kernel. Since $\varphi(g)$ sends (a, x) to some $(a', x + k\theta)$, we get that $k = 0$. This means that $\varphi(g)$ is a commuting product $\varphi(g) = \sigma_{a_1, J_1} \dots \sigma_{a_n, J_n}$ where J_1, \dots, J_n are distinct translates of J_0 by a multiple of θ . We can assume

that $n > 0$ and that no a_i is trivial. For all $x \in \mathcal{C}$, consider $\mathcal{J}_x \subset \{1, \dots, n\}$ the set of indices $j \in \{1, \dots, n\}$ such that $x \in J_j$. Note that when x varies, \mathcal{J}_x changes only when x crosses an endpoint of some J_i , and that \mathcal{J}_x then changes by exactly one element (this is because the $2n$ endpoints of the J_i 's are distinct since θ is irrational). Thus, there exist $x, x' \in \mathcal{C}$ and $i_0 \in \{1, \dots, n\}$ such that $\mathcal{J}_{x'} = \mathcal{J}_x \cup \{J_{i_0}\}$. Now, for all $a \in A$, $\varphi(g)(a, x) = (a \prod_{j \in \mathcal{J}_x} a_j, x)$, and since $\varphi(g)$ is the identity, $\prod_{j \in \mathcal{J}_x} a_j = 1$. Similarly, $\prod_{j \in \mathcal{J}_{x'}} a_j = 1$, so $a_{i_0} = 1$, a contradiction. \square

The argument above immediately generalizes to $A \wr \mathbb{Z}^d$, by replacing the rotation of angle θ by d rotations of rationally independent angles. We thus get:

PROPOSITION 4.2. — *For all $d \geq 1$, and all finite abelian group A , $A \wr \mathbb{Z}^d$ embeds in IET.*

Remark 4.3. — Given $G < \text{IET}$, and A a finite abelian group, we don't know when the group $A \wr G$ embeds in IET.

4.2. Lamps must commute

We now put restrictions on which wreath products may embed in IET. Note that if a group A contains an infinite order element, then $A \wr \mathbb{Z}$ contains the torsion-free solvable group $\mathbb{Z} \wr \mathbb{Z}$ which does not embed in IET by Theorem 3.1.

THEOREM 4.4. — *Let $L = F \wr \mathbb{Z}$ with F finite non-abelian. Then L does not embed in IET.*

We will use several times the observation (see [3, Lem. 6.2]) that for any finitely generated subgroup G of $\text{IET}(\mathcal{D})$, the orbit of any point of the domain \mathcal{D} has polynomial growth in the following sense: given a finite generating set and the corresponding word metric on G , denoting by B_R the ball of radius R in G , there exists a polynomial P such that for all $x \in \mathcal{D}$ and all $R \geq 0$, $\#(B_R.x) \leq P(R)$.

The theorem will be proved by showing that if L did embed in IET, there would exist an orbit with exponential growth.

If $E \subset [0, 1)$ has positive measure and if T is any element of IET, then by Poincaré recurrence Theorem, the orbit of almost every $x \in E$ comes back to E . We will need a more precise estimate, that comes from Birkhoff theorem.

LEMMA 4.5. — *Let $E \subset [0, 1)$ have positive measure, and let T be any element in IET.*

Then there exists $x_0 \in [0, 1)$, and constants $\alpha > 0, \beta > 0$, such that for all n ,

$$\#\{i \mid 0 \leq i < n, T^i(x) \in E\} \geq \alpha n - \beta.$$

Proof. — Assume first that the Lebesgue measure μ is ergodic, and apply Birkhoff theorem to the characteristic function f of E . Then for almost every x , $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \rightarrow \mu(E)$, so for n large enough, $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \geq \frac{1}{2} \mu(E)$, so $\#\{i \mid 0 \leq i < n, T^i(x) \in E\} \geq \frac{1}{2} \mu(E)n$ for n large enough, and the result holds.

If the Lebesgue measure is not ergodic, Birkhoff theorem still says that the limit exists almost everywhere, and the limit is a T -invariant function $l(x)$ having the same average as f , that is $\int l(x)dx = \mu(E)$ (see [8] for instance). It follows that there exist points where $l(x) \geq \mu(E)/2$, and the proof works the same. \square

We prove an abstract algebraic lemma that relates the stabilizer of a point in a product of groups to coordinate-wise stabilizers.

LEMMA 4.6. — *Consider some groups F_i , and an action of $F = F_1 \times \cdots \times F_n$ on a set X . Let $x \in X$, $\text{Stab}(x)$ its stabilizer in F , and $S_i = F_i \cap \text{Stab}(x)$. Let N_i the normalizer of S_i in F_i .*

Then $\text{Stab}(x) \subset N_1 \times \cdots \times N_n$.

In particular, if no S_i is normal in F_i , then $\#F.x \geq 2^n$.

Proof. — Note that $N_1 \times \cdots \times N_n = \bigcap_i N_F(S_i)$, so we have to prove that for each i , any $g \in \text{Stab}(x)$ normalizes S_i . If $g \in \text{Stab}(x)$, then g normalizes both $\text{Stab}(x)$ and F_i (because $F_i \triangleleft F$), hence normalizes their intersection, namely S_i .

Let us prove the last comment. The given assumption says that for all i , $[F_i : N_i] \geq 2$ so $\#F.x = [F : \text{Stab}(x)] \geq 2^n$. \square

To prove the theorem, consider $L = F_0 \wr \mathbb{Z}$ for some finite non-abelian group F_0 . We assume that L embeds in IET and argue towards a contradiction. Let t be a generator of \mathbb{Z} viewed as a subgroup of L , and write $F_i = F_0^{t^i}$. We equip the group G with the word metric corresponding to the generating set $\{t\} \cup F_0$. The subgroup $F_0 \times F_1 \times \cdots \times F_{n-1}$ is contained in the ball of radius $3n$. We identify L with the corresponding subgroup of IET. Since orbits grow polynomially [3, Lem. 6.2], for any $x \in [0, 1)$, the orbit of x under $F_0 \times \cdots \times F_{n-1}$ has to be bounded by a polynomial in n .

PROPOSITION 4.7. — *For all $x \in [0, 1)$, $\text{Stab}_{F_0}(x) \triangleleft F_0$.*

Proof. — Otherwise, let $E \subset [0, 1)$ be the set of points where $\text{Stab}_{F_0}(x) \not\triangleleft F_0$. Since F_0 is a finite group, this is a subdomain of $[0, 1)$, and it has positive measure. We apply Birkhoff theorem (in the form of Lemma 4.5), and get that there exists $x \in [0, 1)$, $\alpha, \beta > 0$, such that, for all n , there exists $k_n \geq \alpha n - \beta$, and some indices $0 \leq i_1 < \dots < i_{k_n} \leq n$ such that $t^{i_j}(x) \in E$. Applying the algebraic Lemma 4.6 to $F = F_{i_1} \times \dots \times F_{i_{k_n}}$ we get that $\#(F.x) \geq 2^{k_n} \geq 2^{\alpha n - \beta}$. Since F is contained in a ball of linear radius in L , this contradicts polynomial growth of orbits. \square

We now prove another algebraic lemma.

LEMMA 4.8. — *Consider an action of $F = F_1 \times \dots \times F_n$ on a set X . Let $x \in X$, $\text{Stab}(x)$ its stabilizer in x , and $S_i = \text{Stab}(x) \cap F_i$ its stabilizer in F_i .*

Assume that $\text{Stab}(x) \triangleleft F$ (in particular $S_i \triangleleft F_i$). Consider $Z(F_i/S_i)$ the center of the quotient group F_i/S_i , and $Q_i = (F_i/S_i)/Z(F_i/S_i)$.

Then the natural epimorphism $F_1 \times \dots \times F_n \rightarrow Q_1 \times \dots \times Q_n$ factors through an epimorphism $F/\text{Stab}(x) \twoheadrightarrow Q_1 \times \dots \times Q_n$.

In particular, if all groups F_i/S_i are non-abelian, each Q_i is non-trivial, and $\#F.x = \#F/\text{Stab}(x) \geq 2^n$.

Proof. — Consider $g = (g_1, \dots, g_n) \in \text{Stab}(x)$, and denote by \bar{g}_i the image of $g_i \in F_i/S_i$. We have to prove that \bar{g}_i is central in F_i/S_i , in other words that for all $a \in F_i$, $[g_i, a] \in S_i$. Since $S_i = \text{Stab}(x) \cap F_i$ is normal in F , $[g_i, a] = g_i(ag_i^{-1}a^{-1}) \in S_i$. The lemma follows. \square

Proof of Theorem 4.4. — Fix $k \geq 1$ and consider the subgroup $L' \subset L$ generated by t^k , and $F_0 \times \dots \times F_{k-1}$. Clearly, $L' \simeq F'_0 \wr \mathbb{Z}$ where $F'_0 = F_0 \times \dots \times F_{k-1}$. Applying Proposition 4.7 to L' , we get that for all x , and all k , the stabilizer of x in $F_0 \times \dots \times F_{k-1}$ is normal in $F_0 \times \dots \times F_{k-1}$.

Let $E \subset [0, 1)$ be the set of points x where $F_0/\text{Stab}(x)$ is non-abelian. If E is empty, any commutator of F_0 acts trivially in $[0, 1)$, so F_0 does not act faithfully, a contradiction. So E is a non-empty subdomain and has positive measure. By Lemma 4.5, there exists $x \in [0, 1)$, $\alpha, \beta > 0$, such that, for all n , there exists $k_n \geq \alpha n - \beta$, and some indices $0 \leq i_1 < \dots < i_{k_n} \leq n$ such that $t^{i_j}(x) \in E$.

Applying the second algebraic Lemma 4.8 to $F = F_{i_1} \times \dots \times F_{i_{k_n}}$ we get that $\#F.x \geq 2^{\alpha n - \beta}$. This contradicts polynomial growth of orbits as above. \square

4.3. Uncountably many solvable groups, via lamplighter-lighters

We finish by an abundance result among solvable subgroups of IET (necessarily with a lot of torsion in view of Theorem 3.1).

THEOREM 4.9. — *There exist uncountably many isomorphism classes of 3-generated solvable subgroups of derived length 3 in IET.*

These will be obtained as iterated lamplighter-like constructions. Again we will be free to choose the domain \mathcal{D} for convenience.

For the time being, take \mathcal{D} an arbitrary domain. Let $G < IET(\mathcal{D})$ and $(A, +)$ a finite abelian group. The following construction generalizes the construction showing that lamplighter groups embed. In this case, G would be the cyclic group generated by an irrational rotation on a circle \mathcal{D} .

Let $\mathcal{F} = A^{\mathcal{D}}$ be the additive group of all functions on \mathcal{D} with values in A . Given $a \in A$ and $J \subset \mathcal{D}$, we denote by $a\mathbb{1}_J \in \mathcal{F}$ the function defined by $a\mathbb{1}_J(x) = a$ if $x \in J$, and $a\mathbb{1}_J(x) = 0$ otherwise. The group G acts on \mathcal{F} by precomposition. Given a subdomain $J \subset \mathcal{D}$, let $\mathcal{F}_J \subset \mathcal{F}$ be the smallest G -invariant additive subgroup containing the functions $a\mathbb{1}_J$ for $a \in A$.

PROPOSITION 4.10 (Lamplighter-like construction). — *Let G be a subgroup of $IET(\mathcal{D})$, A a finite abelian group, and $J \subset \mathcal{D}$. Then $\mathcal{F}_J \rtimes G$ embeds in IET.*

Remark 4.11. — In the particular case of Proposition 4.1, we proved that \mathcal{F}_J is isomorphic to $\bigoplus_{g \in G} A$. In general, there is a natural morphism from $\bigoplus_{g \in G} A$ to \mathcal{F}_J , but it might be non-injective. One easily constructs examples using a group G that does not act freely on \mathcal{D} .

Proof. — Take $\mathcal{D}' = A \times \mathcal{D}$. Embed G in $IET(\mathcal{D}')$ by setting for all $g \in G$, $g(a, x) = (a, g(x))$.

For all $b \in A$, consider $\sigma_b \in IET(\mathcal{D}')$ defined by $\sigma_b(a, x) = (b + a, x)$ if $x \in J$, and by $\sigma_b(a, x) = (a, x)$ if $x \notin J$.

We claim that the subgroup G' generated by G and $\{\sigma_b | b \in A\}$ is isomorphic to $\mathcal{F}_J \rtimes G$.

Consider the subset $H \subset IET(\mathcal{D}')$ of all $T \in IET(\mathcal{D}')$ such that there exists some transformation $g \in G$ and some $f \in \mathcal{F}_J$ such that $T(a, x) = (a + f(x), g(x))$. It is clear that H is a subgroup of $IET(\mathcal{D}')$, that H contains G' , and that $G' = H$. Finally, the fact that $H \simeq \mathcal{F}_J \rtimes G$ is also clear. \square

Let $\mathcal{D}_0 = \mathbb{R}/\mathbb{Z}$, and $I = [0, 1/2[\subset \mathcal{D}_0$. Fix once and for all $\alpha \notin \mathbb{Q}$, ρ_α the rotation of \mathcal{D}_0 of angle α . Let G be the lamplighter group $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ realised

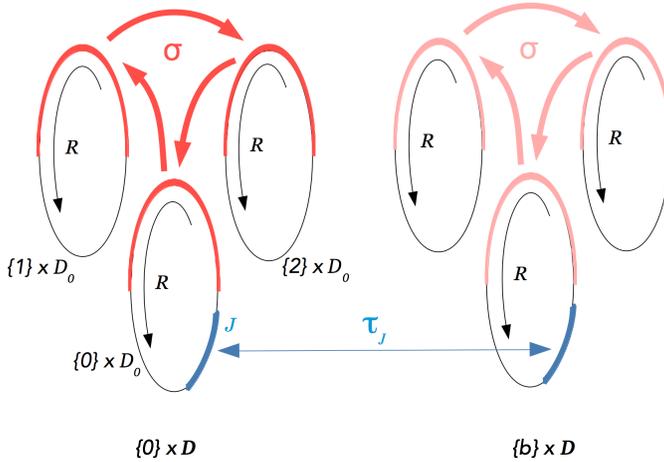


Figure 4.2. A Lamplighter-like construction in IET producing the examples of Theorem 4.9: on the left, the domain $\mathcal{D} = \mathbb{Z}/3\mathbb{Z} \times \mathcal{D}_0$, and the Lamplighter group $G = \mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}$ generated by a cyclic swap on the three intervals $\mathbb{Z}/3\mathbb{Z} \times I$ (in bold red) and by the simultaneous rotation R . On the right, $\{b\} \times \mathcal{D}$ is the duplicated copy of \mathcal{D} , with the swap τ_J on J and its copy. The group G acts diagonally on $\mathcal{D}' = \mathbb{Z}/2\mathbb{Z} \times \mathcal{D}$. The group $(\bigoplus_G \mathbb{Z}/2\mathbb{Z}) \rtimes G$ acts on \mathcal{D}' but its image H_J in $\text{IET}(\mathcal{D}')$ depends on J .

as a subgroup of $\text{IET}(\mathcal{D})$ for $\mathcal{D} = (\mathbb{Z}/3\mathbb{Z}) \times \mathcal{D}_0$ as in Proposition 4.1. We denote by R the rotation $(a, x) \mapsto (a, \rho_\alpha(x))$ on \mathcal{D} .

We now perform this lamplighter-like construction a second time, starting with G acting on \mathcal{D} , with $A = \mathbb{Z}/2\mathbb{Z}$ and with some $J \subset \{0\} \times \mathcal{D}_0 \subset \mathcal{D}$. We recall that \mathcal{F}_J is the subgroup of the group of functions from \mathcal{D} to $A = \mathbb{Z}/2\mathbb{Z}$ as above. Define $H_J = \mathcal{F}_J \rtimes G$ the lamplighter-like group thus obtained, on $\mathcal{D}' = (\mathbb{Z}/2\mathbb{Z}) \times \mathcal{D}$. See Figure 4.2.

PROPOSITION 4.12. — *Assume that $J_1, J_2 \subset \{0\} \times \mathcal{D}_0 \subset \mathcal{D}$, and that $|J_1|, |J_2| < \frac{1}{2}$.*

If $H_{J_1} \simeq H_{J_2}$ then $|J_1| \in \text{Vect}_{\mathbb{Q}}(1, \alpha, a_2, b_2)$, where $a_2, b_2 \in \mathbb{R}/\mathbb{Z}$ are the endpoints of J_2 .

Denote by \mathcal{E}_I and $\mathcal{E}_{J_i} \subset \mathbb{R}/\mathbb{Z}$ the orbit under the rotation ρ_α of the endpoints of I and J_i respectively (i.e. $\mathcal{E}_I = \{0, \frac{1}{2}\} + \alpha\mathbb{Z}$, $\mathcal{E}_{J_i} = \{a_i, b_i\} + \alpha\mathbb{Z}$).

LEMMA 4.13. — *If $H_{J_1} \simeq H_{J_2}$, then there exist two subdomains $K, K' \subset \mathbb{R}/\mathbb{Z}$ with endpoints in \mathcal{E}_{J_2} and \mathcal{E}_I respectively and $\varepsilon \in \{\pm 1\}$ such that*

$$\forall n \in \mathbb{Z}, \quad \rho_\alpha^n(I) \cap J_1 \neq \emptyset \iff \rho_\alpha^{\varepsilon n}(K) \cap K' \neq \emptyset.$$

Proof of Proposition 4.12 from Lemma 4.13. — We have that $\rho_\alpha^n(I) \cap J_1 \neq \emptyset$ if and only if $n\alpha \in J_1 - I$, and that $\rho_\alpha^{\varepsilon n}(K) \cap K' \neq \emptyset$ if and only if $n\alpha \in \varepsilon(K' - K)$.

By Birkhoff theorem, we get that $|J_1 - I| = |K' - K|$. Since $|J_1| < \frac{1}{2} = |I|$, $|J_1 - I| = \frac{1}{2} + |J_1|$.

On the other hand, since $K' - K$ is a union of intervals whose endpoints are in $\mathcal{E}_I - \mathcal{E}_{J_2} \subset \text{Vect}_{\mathbb{Q}}(1, \alpha, a_2, b_2)$, the proposition follows. \square

We view H_J as the semidirect product $\mathcal{F}_J \rtimes G$. Thus each element $h \in H_J$ can be written (uniquely) under the form $h = g\tau$ with $\tau \in \mathcal{F}_J$, $g \in G$. Now we view G as a group of interval exchange transformations on $\mathcal{D} = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$. It is generated by the rotation $R : (a, x) \mapsto (a, x + \alpha)$, and by the lamp element $\sigma \in \mathcal{S}$ (of order 3) that sends $(a, x) \in (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ to $(a + 1, x)$ for $x \in I$, and is the identity otherwise (see left part of figure 4.2). Since G is a lamplighter group, any $g \in G$ can be written uniquely as $R^n S_f$ where S_f is an IET of the form $(a, x) \rightarrow (f(x) + a, x)$ for some function $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$. We denote by $\mathcal{S} = \{S_f\}$ the 3-torsion abelian group of lamps of G . It is freely generated by the $\langle R \rangle$ -conjugates of σ . Thus any $h \in H_J$ is written in a unique way as $h = R^n S_f \tau$ as above.

The kernel of the natural map $H_J \rightarrow \mathbb{Z}$ is the torsion group $N = \mathcal{F}_J \mathcal{S}$. It is exactly the set of elements of finite order.

Denote by b be the generator of $\mathbb{Z}/2\mathbb{Z}$. Given $K \subset \mathcal{D}$ denote by $\tau_K = b\mathbf{1}_K \in \mathcal{F}$. For certain K (for instance $K = J$), $\tau_K \in \mathcal{F}_J$ and thus is in H_J .

FACT 4.14. — *If $K \subset \{0\} \times \mathcal{D}_0$, then $[\sigma, \tau_K] = 1$ if and only if $K \cap I = \emptyset$.*

Proof. — From the definition of the semidirect product, $\sigma\tau_K\sigma^{-1} = b\mathbf{1}_{\sigma(K)}$. Hence $[\sigma, \tau_K] = 1$ if and only if $\sigma(K) = K$. If $K \cap I \neq \emptyset$, then $\sigma(K)$ contains a point outside $\{0\} \times \mathcal{D}_0$, so $\sigma(K) \neq K$. \square

FACT 4.15. — *For any $\tau, \tau' \in \mathcal{F}_J$ and any $h \in H_J$, $[h\tau, \tau'] = [h, \tau']$.*

Proof. — This is an immediate consequence of the commutation of τ with τ' . \square

Proof of Lemma 4.13. — Let $\varphi : H_{J_1} \rightarrow H_{J_2}$ be an isomorphism. We first note that $\mathcal{F}_J \triangleleft H_J$ is precisely the set of elements $g \in H_J$ such that $g^2 = 1$. Moreover, the set of elements of finite order in H_J is the subgroup $N = \mathcal{F}_J \mathcal{S}$ (but there are exotic elements of order 3).

Thus $\varphi(\tau_{J_1}) \in \mathcal{F}_{J_2}$ can be viewed as a function $\tau : \mathcal{D} \rightarrow \mathbb{Z}/2\mathbb{Z}$, and $\varphi(\sigma)$ as an element $S'\tau'$ for some $S' \in \mathcal{S}$, and $\tau' \in \mathcal{F}_J$. Now $\varphi(R)$ generates G_{J_2} modulo the torsion subgroup, so \mathcal{F}_{J_2} , so $\varphi(R) = R^\varepsilon S''\tau''$ for some function $\tau'' \in \mathcal{F}_{J_2}$, $S'' \in \mathcal{S}$, and $\varepsilon = \pm 1$.

In what follows, we use the notation g^h for hgh^{-1} . Fix $n \in \mathbb{Z}$ and consider the commutator $C = [\sigma, \tau_{J_1}^{R^n}] = [\sigma, \tau_{R^n(J_1)}]$. By Fact 4.14, C is trivial if and only if $R^n(J_1) \cap I = \emptyset$. On the other hand, by Fact 4.15,

$$\varphi(C) = [S'\tau', \tau^{(R^\varepsilon S''\tau'')^n}] = [S', \tau^{(R^\varepsilon S''\tau'')^n}].$$

There exists $\tau''' \in \mathcal{F}_{J_2}$ such that $(R^\varepsilon S''\tau'')^n = (R^\varepsilon S''\tau''')^n$, and since \mathcal{F}_{J_2} is abelian, $\tau^{(R^\varepsilon S''\tau'')^n} = \tau^{(R^\varepsilon S''\tau''')^n}$. Hence, $\varphi(C) = 1$ if and only if $[S'^{(R^\varepsilon S''\tau''')^{-n}}, \tau] = 1$. A similar calculation in G shows that $S'^{(R^\varepsilon S''\tau''')^{-n}} = S'^{R^{-\varepsilon n}} S''' = S'^{R^{-\varepsilon n}}$, so $\varphi(C) = 1$ if and only if $[S', \tau^{R^{\varepsilon n}}] = 1$.

Given $x \in \mathbb{R}/\mathbb{Z}$, we call the *fiber* of x the 3 point set $F_x = \{(a, x) \mid a \in \mathbb{Z}/3\mathbb{Z}\} \subset \mathcal{D}$. Let $K \subset \mathbb{R}/\mathbb{Z}$ be the set of points x such that τ is non constant on the fiber of x . Since $\tau \in \mathcal{F}_{J_2}$, K is a union of intervals with endpoints in \mathcal{E}_{J_2} . Write the transformation S' as $(a, x) \mapsto (a + f(x), x)$ for some function $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, and let K' be the support of f . It is a union of intervals with endpoints in \mathcal{E}_I .

We claim that $[S', \tau^{R^{\varepsilon n}}] = 1$ if and only if $R^{\varepsilon n}(K) \cap K' = \emptyset$. Note that S' preserves each fiber. We view τ as an element in the additive group \mathcal{F} of functions on \mathcal{D} with values in $\mathbb{Z}/2\mathbb{Z}$, and S', R as transformations of \mathcal{D} , so that $[S', \tau^{R^{\varepsilon n}}] = 1$ if and only if $\tau \circ R^{\varepsilon n} \circ S' = \tau \circ R^{\varepsilon n}$. So fix $x \in \mathbb{R}/\mathbb{Z}$, and F_x its fiber. If $x \notin K'$, then S' acts as the identity on F_x , so $\tau \circ R^{\varepsilon n} \circ S' = \tau \circ R^{\varepsilon n}$ in restriction to F_x . If $x \notin R^{\varepsilon n}(K)$, then $\tau^{R^{\varepsilon n}}$ is a constant function on F_x , and the same conclusion holds. If on the contrary $x \in R^{\varepsilon n}(K) \cap K'$, then $\tau^{R^{\varepsilon n}}$ is not constant on F_x and since S' acts transitively on F_x , $\tau^{R^{\varepsilon n}} \circ S'$ does not coincide with $\tau^{R^{\varepsilon n}}$ on F_x . This proves the claim and concludes the proof. \square

We can finally prove Theorem 4.9.

Proof of Theorem 4.9. — Let $J_x = [0, x[$ for $x < \frac{1}{2}$. Proposition 4.12 shows that given x , there are at most countably many $y < \frac{1}{2}$ such that $H_{J_x} \simeq H_{J_y}$. This gives uncountably many isomorphism classes of groups.

Since G is metabelian, H_{J_x} is solvable of derived length at most 3. Since there are only countably many isomorphism classes of finitely generated metabelian groups, H_{J_x} has derived length exactly 3 for uncountably many values of x . \square

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