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Divergence, exotic convergence and self-bumping in quasi-Fuchsian spaces ^(*)

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ABSTRACT. — In this paper, we study the topology of the boundaries of quasi-Fuchsian spaces. We first show for a given convergent sequence of quasi-Fuchsian groups, how we can know the end invariant of the limit group from the information on the behaviour of conformal structures at infinity of the groups. This result gives rise to a sufficient condition for divergence of quasi-Fuchsian groups, which generalises Ito's result in the once-punctured torus case to higher genera. We further show that quasi-Fuchsian groups can approach a b-group not along Bers slices only when the limit has isolated parabolic loci. This makes it possible to give a necessary condition for points on the boundaries of quasi-Fuchsian spaces to be self-bumping points. We use model manifolds invented by Minsky and their geometric limits studied by Ohshika–Soma to prove these results. This paper has been made as self-contained as possible so that the reader does not need to consult the paper of Ohshika–Soma directly.

RÉSUMÉ. — Dans cet article, on étudie la topologie des bords des espaces quasi-fuchsien. D'abord on montre comment on peut savoir les invariants des bouts du groupe limite pour une suite convergente de groupes quasi-fuchsien donnée, en utilisant les informations sur le comportement asymptotique des structures conformes à l'infini des groupes dans la suite. Ce résultat donne lieu à une condition suffisante pour la divergence des groupes quasi-fuchsien, laquelle est une généralisation du résultat d'Ito qui n'a traité que le cas des groupes du tore une fois perforé. On démontre de plus que des groupes quasi-fuchsien ne peuvent approcher un b-groupe hors de la tranche de Bers que si la limite admet un locus parabolique isolé. Ce résultat-ci permet également de donner une condition nécessaire pour qu'un point au bord de l'espace de déformations soit un point de « l'entrechoquement ». Pour démontrer ces résultats, on utilise des variétés modèles construites par Minsky et leurs limites géométriques étudiées par Ohshika–Soma. Pour que les lecteurs n'aient pas besoin de se reporter à l'article d'Ohshika–Soma, le présent article aussi contient les arguments simplifiés mais assez détaillés d'Ohshika–Soma qui sont nécessaires pour les démonstrations des théorèmes principaux.

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1. Introduction

In the theory of Kleinian groups, after the major problems like Marden's tameness conjecture and the ending lamination conjecture were solved, the attention is now focused on studying the topological structure of deformation spaces. Although we know, by the resolution of the Bers–Sullivan–Thurston density conjecture ([10, 16, 32, 33, 41, 42, 46]), that every finitely generated Kleinian group is an algebraic limit of quasi-conformal deformations of a (minimally parabolic) geometrically finite group, the structure of deformation spaces as topological spaces is far from completely understood. For instance, as was observed by work of Anderson–Canary [2] and McMullen [39], even in the case of Kleinian groups isomorphic to closed surface groups, the deformation spaces are fairly complicated, and in particular it is known that they are not manifolds since they have singularities caused by a phenomenon called “bumping”. Actually, this kind of phenomenon also makes the deformation spaces not locally connected as was shown by Bromberg [17] and was generalised by Magid [34].

The interior of a deformation space is known to be a disjoint union of quasi-conformal deformation spaces of minimally parabolic Kleinian groups, which is well understood by work of Ahlfors, Bers, Kra, Maskit, Marden, and Sullivan. In particular, their theory gives rise to a parametrisation of the quasi-conformal deformation space by the Teichmüller space of the boundary at infinity of the corresponding quotient hyperbolic 3-manifold. Therefore, to understand the global structure of a deformation space, what we need to know is how the boundary is attached to the quasi-conformal deformation space. More concretely, we need to determine, for a sequence of quasi-conformal deformations given as a sequence in the Teichmüller space using this parametrisation, first whether it converges or not, and if it does, what is the limit of the sequence. Also, we need to know in what cases sequences of quasi-conformal deformations can approach the same group from different directions as in the example of Anderson–Canary and McMullen. This paper tries to answer such problems for the case of Kleinian surface groups using the technique of model manifolds of geometric limits which we developed in Ohshika–Soma [50].

The interior of a deformation space of Kleinian surface group $AH(S)$ is the quasi-Fuchsian space $QF(S)$. The parametrisation in this case is the Ahlfors–Bers map qf from $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ to $QF(S)$. The celebrated work of Bers on compactification of Teichmüller space ([4]) shows that in $AH(S)$, the subspace of $QF(S)$ in the form $\mathcal{T}(S) \times \{\text{pt.}\}$ or $\{\text{pt.}\} \times \mathcal{T}(\bar{S})$ is relatively compact. Considering a direction in $QF(S)$ (with respect to the parametrisation of Ahlfors–Bers) quite different from that of Bers, Thurston proved that

a sequence $\{(m_i, n_i)\} \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ converges if $\{m_i\}$ converges to a projective lamination $[\lambda]$ and $\{n_i\}$ converges to another projective lamination $[\mu]$ both in the Thurston compactification of $\mathcal{T}(S)$ such that the supports of λ and μ are distinct and every component of $S \setminus (\lambda \cup \mu)$ is simply connected ([55]).

In contrast to these results on convergence, we showed in [45] that if $\{m_i\}$ and $\{n_i\}$ converge to arational laminations with the same support, then $\{qf(m_i, n_i)\}$ always diverges. Since the Teichmüller space of S is properly embedded in $AH(S)$ as a diagonal set of $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$, this may look very natural. However, there is an example by Anderson–Canary in [2] which shows that for a given hyperbolic structure m_0 , if we consider $(\tau^i(m_0), \tau^{2i}(m_0)) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, where τ denotes the deformation of the metric induced by the Dehn twist around a simple closed curve γ on S , then its image $qf(\tau^i(m_0), \tau^{2i}(m_0))$ in $AH(S)$ converges. These two results show that the situation is quite different depending on the types of the limit projective laminations.

In the case when $\mathcal{T}(S)$ has dimension 2, i.e., if S is either a once-punctured torus or a four-times punctured sphere, a measured lamination is either arational or a weighted simple closed curve, which means that there is nothing between these two situations above. In this case, Ito has given a complete criterion for convergence/divergence ([26]). In the general case when $\mathcal{T}(S)$ has dimension more than 2 however, there is a big room between these two extremes.

Therefore quite naturally, we should ask ourselves what would happen in the cases in between. One of our main theorems in this paper (Theorem 3.3) is an answer to this question. Consider sequences $\{m_i\}$ in $\mathcal{T}(S)$ and $\{n_i\}$ in $\mathcal{T}(\bar{S})$ converging to $[\mu^+]$ and $[\mu^-]$ such that their supports $|\mu^+|$ and $|\mu^-|$ share a component which is not a simple closed curve. We shall prove that $\{qf(m_i, n_i)\}$ diverges in $AH(S)$ in this setting. More generally, we shall show that if μ^+ and μ^- have components μ_0^+, μ_0^- whose minimal supporting surfaces share a boundary component up to isotopy, then $\{qf(m_i, n_i)\}$ diverges.

This theorem is derived from another of our main results, Theorem 3.2, which asserts that if $\{qf(m_i, n_i)\}$ converges, then we can determine the ending laminations of the limit group by considering the shortest pants decompositions of (S, m_i) and (S, n_i) and their Hausdorff limits. It also implies Theorem 3.1 stating that if we consider the limits of $\{m_i\}$ and $\{n_i\}$ in the Thurston compactification of the Teichmüller space, then every non-simple-closed-curve component of the limit of $\{m_i\}$ is an ending lamination of an upper end of the limit, and every non-simple-closed-curve component of the limit of $\{n_i\}$ is that of a lower end. These combined with Theorem 3.6 can be regarded as a partial answer to the problem of determining limit groups

of sequence of quasi-Fuchsian groups given in terms of the parametrisation by the Teichmüller spaces.

In the case when the laminations $|\mu^+|$ and $|\mu^-|$ share only simple closed curves, the convergence or divergence of the sequence depends on whether there is a simple closed curve component of either $|\mu^+|$ or $|\mu^-|$ which is contained in the boundary of the minimal supporting surface of a non-simple-closed-curve component of the other of $|\mu^-|$ or $|\mu^+|$. Theorem 3.5 asserts that if there is such a simple closed curve component, then the sequence diverges. Even in the case when such a component does not exist, $\{qf(m_i, n_i)\}$ can converge only in a special situation which is analogous to an example of Anderson–Canary [2]. Theorem 3.6 describes the situation where the sequence can converge. We should note that these are the best possible answers for divergence and convergence for sequence of quasi-Fuchsian groups when the asymptotic behaviour of the corresponding parameters are expressed in terms of the Thurston compactification of Teichmüller space.

The same example of Anderson–Canary also shows that there is a point in $AH(S)$ where $QF(S)$ bumps itself as explained above. In such a point, the self-bumping is caused by what is called the “exotic convergence”. A sequence of quasi-Fuchsian groups is said to converge to a b-group exotically when the groups in the sequence are not contained in Bers slices approaching the one containing the limit b-group. The construction of Anderson–Canary gives a sequence converging exotically to a regular b-group. We shall prove such a convergence can occur only for b-groups which have \mathbb{Z} -cusps not touching geometrically infinite ends.

As for self-bumping, we conjecture that such a phenomenon cannot occur for geometrically infinite b-groups all of whose \mathbb{Z} -cusps touch geometrically infinite ends. What we shall prove in Theorem 3.8 is a weaker form of this conjecture: if there are two sequences $\{qf(m_i, n_i)\}$ and $\{qf(m'_i, n'_i)\}$ both converging to the same geometrically infinite group all of whose \mathbb{Z} -cusps touch geometrically infinite ends, then for any small neighbourhood U of the quasi-conformal deformation space of the limit group, if we take large i , then $qf(m_i, n_i)$ and $qf(m'_i, n'_i)$ are connected by an arc in U . In particular this shows that under the same condition on \mathbb{Z} -cusps, if the limit group is either quasi-conformally rigid or a b-group whose upper conformal structure at infinity is rigid, it cannot be a self-bumping point. More generally, even when there is a cusp not touching a geometrically infinite end, the same argument shows that a Bers slice cannot bump itself at a b-group whose upper conformal structure at infinity is rigid. This latter result has been obtained independently by Brock–Bromberg–Canary–Minsky [12] by a different approach.

In Theorems 3.11 and 3.12, we shall generalise the results for quasi-Fuchsian groups in Theorems 3.2 and 3.3 to general Kleinian surface groups.

We also note that the results obtained in this paper has an application which has appeared in [49] (see also Papadopoulos [52]). There, we have considered a quotient space of a Bers boundary, which is called the reduced Bers boundary, and have proved its automorphism group coincides with the extended mapping class group.

Our tools for proving all these theorems are model manifolds invented by Minsky [40] and their geometric limits studied in Ohshika–Soma [50]. The technique which we develop in this paper shows that model manifolds are quite useful for studying the asymptotic behaviour of sequence in deformation spaces. We have tried to make this paper readable independently of [50]: each time we need arguments in [50], we provide their sketches or summaries so that the reader does not need to look into details of [50].

Recently, after the first version of the present paper had been put on the arXiv, there appeared two papers which made a further progress along the line of our results, using different techniques. The first is Brock–Bromberg–Canary–Minsky [12] and the other is Brock–Bromberg–Canary–Lecuire [11].

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2. Preliminaries

2.1. Generalities

Kleinian groups are discrete subgroups of $\mathrm{PSL}_2\mathbb{C}$. In this paper we always assume Kleinian groups to be torsion free. When we talk about deformation spaces, we only consider finitely generated Kleinian groups. However, we also need to consider infinitely generated Kleinian groups which will appear as geometric limits. We refer the reader to Marden [35, 36] for a general reference for the theory of Kleinian groups.

Let S be an oriented hyperbolic surface of finite area. In this paper, we focus on Kleinian groups which are isomorphic to $\pi_1(S)$ in such a way that punctures of S correspond to parabolic elements. We define the deformation

space $AH(S)$ to be the quotient space of

$$R(S) = \left\{ (G, \phi) \left| \begin{array}{l} \phi : \pi_1(S) \rightarrow \mathrm{PSL}_2\mathbb{C} \text{ is a faithful discrete} \\ \text{representation taking punctures to parabolic} \\ \text{elements with } \phi(\pi_1(S)) = G \end{array} \right. \right\}$$

by conjugacy in $\mathrm{PSL}_2\mathbb{C}$. The space $R(S)$ has a topology coming from the representation space and we endow $AH(S)$ with its quotient topology. We denote an element of $AH(S)$ also by (G, ϕ) for some representative of the equivalence class. We call ϕ a marking of the Kleinian group G .

The set of faithful discrete representations of $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$, which induce the same orientation as the one given on S , modulo conjugacy constitutes the Teichmüller space of S , which we denote by $\mathcal{T}(S)$. Therefore, $\mathcal{T}(S)$ is naturally contained in $AH(S)$. More generally, the space of quasi-Fuchsian groups $QF(S)$ lies in $AH(S)$. A quasi-Fuchsian group is a Kleinian group whose domain of discontinuity is a disjoint union of two simply connected components. By the theory of Ahlfors–Bers, $QF(S)$ is parametrised by a homeomorphism $qf : \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \rightarrow QF(S)$. Here both $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$ are the Teichmüller space of S , but the latter one is identified with $\mathcal{T}(S)$ by an orientation reversing automorphism of S . For $(m, n) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, its image $qf(m, n)$ is obtained by starting from a Fuchsian group and solving a Beltrami equation so that the conformal structure on the quotient of the lower Jordan domain is m and that on the quotient of the upper Jordan domain is n . We call m and n the lower and upper *conformal structures at infinity* of $qf(m, n)$ respectively. The set of Fuchsian groups corresponds to a slice of the form $\{qf(m, m)\}$. By the theory of Ahlfors–Bers combined with Sullivan’s stability theorem [53], we know that $QF(S)$ is the interior of the entire deformation space $AH(S)$. On the other hand, the Bers–Sullivan–Thurston density conjecture, which was solved by Bromberg [9], Brock–Bromberg [10] in this setting, or is obtained as a corollary of the ending lamination conjecture [13] combined with [43], $AH(S)$ is the closure of $QF(S)$.

By Margulis’ lemma, there is a positive constant ϵ_0 such that for any hyperbolic 3-manifold, its ϵ_0 -thin part is a disjoint union of cusp neighbourhoods and tubular neighbourhoods of short closed geodesics, which are called Margulis tubes. For a hyperbolic 3-manifold M , we denote by M_0 the complement of its cusp neighbourhoods.

Bonahon showed in [5] that for any $(G, \phi) \in AH(S)$, the hyperbolic 3-manifold \mathbb{H}^3/G is homeomorphic to $S \times (0, 1)$. We denote by Φ a homeomorphism from $S \times (0, 1)$ to \mathbb{H}^3/G inducing ϕ between the fundamental groups. In general, for an element in $AH(S)$, we denote a homeomorphism from $S \times (0, 1)$ to the quotient hyperbolic 3-manifold by the letter in the upper case corresponding to a Greek letter denoting the marking. Bonahon also

proved the every end of $(\mathbb{H}^3/G)_0$ is either geometrically finite or simply degenerate. Here an end is called *geometrically finite* if it has a neighbourhood which is disjoint from any closed geodesic, and *simply degenerate* if it has a neighbourhood of the form $\Sigma \times (0, \infty)$ for an incompressible subsurface Σ of S (i.e. a subsurface each of whose frontier components is non-contractible in S) and there are simple closed curves c_n on Σ which are homotopic to closed geodesics c_n^* going to the end. For a simply degenerate end, the *ending lamination* is defined to be the support of the projective lamination to which $[c_n]$ converges in the projective lamination space $\mathcal{PML}(\Sigma)$. We shall explain what are laminations and the projective lamination space below.

For $(G, \phi) \in AH(S)$, we choose an embedding $f : S \rightarrow \mathbb{H}^3/G$ inducing ϕ between the fundamental groups. Such an embedding is unique up to ambient isotopy in \mathbb{H}^3/G . An end of \mathbb{H}^3/G is called *upper* when it lies above $f(S)$ and *lower* when it lies below $f(S)$ with respect to the orientations of \mathbb{H}^3/G and $f(S)$. Let C be a compact core of $(\mathbb{H}^3/G)_0$ intersecting each component of $\text{Fr}(\mathbb{H}^3/G)_0$ by a core annulus. Let P denote $C \cap \text{Fr}(\mathbb{H}^3/G)_0$. Each component of P is called a *parabolic locus* of \mathbb{H}^3/G or C , and its core curve is called a *parabolic curve*. Parabolic loci corresponding to punctures of S are called *peripheral parabolic loci*. For non-peripheral parabolic loci, those contained in $S \times \{1\}$ are called upper and those contained in $S \times \{0\}$ lower. The ends of $(\mathbb{H}^3/G)_0$ correspond one-to-one to the components of $\text{Fr}C$. Since C is homeomorphic to $S \times [0, 1]$, each upper end faces a subsurface of $S \times \{1\}$, which is a component of $S \times \{1\} \setminus P$. A non-peripheral parabolic locus p in P is called *isolated* when the components of $S \times \{0, 1\} \setminus P$ adjacent to p (there are one or two such components) face geometrically finite ends, in other words, no component of $S \times \{0, 1\} \setminus P$ facing a simply degenerate end touches p at its frontier.

2.2. Laminations

A geodesic lamination on a hyperbolic surface S is a closed subset of S consisting of disjoint simple geodesics, which are called leaves. For a geodesic lamination λ , each component of $S \setminus \lambda$ is called a complementary region of λ . We say that a geodesic lamination is *arational* when every complementary region is either simply connected or in the case when S is not closed, a topologically open annulus whose core curve is homotopic to a puncture or a boundary component of S . A subset l of a geodesic lamination consisting of leaves is called a *minimal component* when for each leaf ℓ of l , its closure $\bar{\ell}$ coincides with l . Any geodesic lamination is decomposed into finitely many minimal components and finitely many non-compact isolated leaves.

A measured lamination is a (possibly empty) geodesic lamination endowed with a transverse measure which is invariant with respect to homotopies along leaves. The support of the transverse measure of a measured lamination μ , which is a geodesic lamination, is called the support of μ and is denoted by $|\mu|$. When we consider a measured lamination, we always assume that its support is the entire lamination. The measured lamination space $\mathcal{ML}(S)$ is the set of measured laminations on S endowed with the weak topology. Thurston proved that $\mathcal{ML}(S)$ is homeomorphic to $\mathbb{R}^{6g-6+2p}$, where g is the genus and p is the number of punctures. A weighted disjoint union of closed geodesics can be regarded as a measured lamination. It was shown by Thurston that the set of weighted disjoint unions of closed geodesics is dense in $\mathcal{ML}(S)$.

The projective lamination space $\mathcal{PML}(S)$ is the space obtained by taking a quotient of $\mathcal{ML}(S) \setminus \{\emptyset\}$ identifying scalar multiples. Thurston constructed a natural compactification of the Teichmüller space whose boundary is $\mathcal{PML}(S)$ in such a way that the mapping class group acts continuously on the compactification.

We need to consider one more space, the unmeasured lamination space. This space, denoted by $\mathcal{UML}(S)$, is defined to be the quotient space of $\mathcal{ML}(S)$, where two laminations with the same support are identified. An element in $\mathcal{UML}(S)$ is called an unmeasured lamination.

For a minimal geodesic lamination λ on S , its *minimal supporting surface* is defined to be an incompressible subsurface of S containing λ which is minimal up to isotopy among all such surfaces. When λ is a closed geodesic, we define its minimal supporting surface to be an annulus whose core curve is λ . It is obvious the minimal supporting surface of λ is uniquely determined up to isotopy.

2.3. Algebraic convergence and geometric convergence

When $\{(G_i, \phi_i)\}$ converges to $(\Gamma, \psi) \in AH(S)$, we say that the sequence converges *algebraically* to (Γ, ψ) . We can choose representatives (G_i, ϕ_i) so that ϕ_i converges to ψ as representations. As a convention, when we say that $\{(G_i, \phi_i)\}$ converges to (Γ, ψ) , we always take representatives so that $\{\phi_i\}$ converges to ψ .

We need to consider another kind of convergence: the geometric convergence. A sequence of Kleinian groups $\{G_i\}$ is said to converge to a Kleinian group G_∞ *geometrically* if (1) for any convergent sequence $\{\gamma_{i_j} \in G_{i_j}\}$, its limit lies in G_∞ , and (2) any element $\gamma \in G_\infty$ is a limit of some $\{g_i \in G_i\}$.

When (G_i, ϕ_i) converges to (Γ, ψ) algebraically, the geometric limit G_∞ contains Γ as a subgroup.

When $\{G_i\}$ converges to G_∞ geometrically, if we take a basepoint x in \mathbb{H}^3 and its projections $x_i \in \mathbb{H}^3/G_i$ and $x_\infty \in \mathbb{H}^3/G_\infty$, then $(\mathbb{H}^3/G_i, x_i)$ converges to $(\mathbb{H}^3/G_\infty, x_\infty)$ with respect to the pointed Gromov–Hausdorff topology: that is, there exists a (K_i, r_i) -approximate isometry $B_{r_i}(\mathbb{H}^3/G_i, x_i) \rightarrow B_{K_i r_i}(\mathbb{H}^3/G_\infty, x_\infty)$ with $K_i \rightarrow 1$ and $r_i \rightarrow \infty$, where $B_{r_i}(\mathbb{H}^3/G_i, x_i)$ denotes the r_i -metric ball centred at x_i . Here a (K_i, r_i) -approximate isometry ρ_i is a diffeomorphism from $B_{r_i}(\mathbb{H}^3/G_i, x_i)$ to $B_{K_i r_i}(\mathbb{H}^3/G_\infty, x_\infty)$ satisfying $K_i^{-1} d_{\mathbb{H}^3/G_i}(y, z) \leq d_{\mathbb{H}^3/G_\infty}(\rho_i(y), \rho_i(z)) \leq K_i d_{\mathbb{H}^3/G_\infty}(y, z)$ for every $y, z \in B_{r_i}(\mathbb{H}^3/G_i, x_i)$.

2.4. Bers slices and b-groups

We fix a point $m_0 \in \mathcal{T}(S)$ and consider a subspace $qf(\{m_0\} \times \mathcal{T}(\bar{S}))$ in $QF(S)$. This space is called the *Bers slice* over m_0 . Kleinian groups lying on its frontier are called *b-groups* with lower conformal structure m_0 .

Anderson–Canary [2] constructed a sequence of quasi-Fuchsian groups converging to a b-group whereas its coordinates in $QF(S)$ do not approach a Bers slice.

DEFINITION 2.1. — *We say a sequence of quasi-Fuchsian groups $\{(G_i, \phi_i) = qf(m_i, n_i)\}$ converges exotically to a b-group (Γ, ψ) if $\{(G_i, \phi_i)\}$ converges to (Γ, ψ) algebraically and both $\{m_i\}$ and $\{n_i\}$ go out from any compact set in the Teichmüller space.*

The existence of exotic convergence is related to singularities of $AH(S)$ at its boundary (as a subspace of the representation space modulo conjugacy). In fact, McMullen showed in [39] that $AH(S)$ has a singular point at the boundary where $QF(S)$ bumps itself. For a Kleinian surface group $(\Gamma, \psi) \in AH(S) \setminus QF(S)$, we say that $QF(S)$ *bumps itself* at (Γ, ψ) , when there is a neighbourhood V of (Γ, ψ) such that for any smaller neighbourhood $U \subset V$, its intersection with the quasi-Fuchsian space, $U \cap QF(S)$ is disconnected. McMullen showed the existence of points where $QF(S)$ bumps itself, making use of the construction of Anderson–Canary.

We say that a Bers slice $\mathcal{B}(m_0) = qf(\{m_0\} \times \mathcal{T}(\bar{S}))$ bumps itself at (Γ, ψ) when there is a neighbourhood V of (Γ, ψ) such that for any smaller neighbourhood $U \subset V$, its intersection with the Bers slice, $U \cap \mathcal{B}(m_0)$ is disconnected. Up to today, it is not known whether a Bers slice can bump itself or not.

2.5. Curve complexes and hierarchies

In this subsection, we shall give an explanation of the work of Masur–Minsky on what they called *hierarchies of tight geodesics*. Throughout this subsection, we fix an orientable hyperbolic surface S of finite type, and consider its subsurfaces. We say that a subsurface is *essential* when it is a proper subsurface and each of its frontier component is a non-contractible, non-peripheral curve on S . When we talk about curve complexes, we usually regard essential subsurfaces as open subsurfaces without boundary, and call them *domains* of S following Masur–Minsky. For a surface S of genus g with p punctures, we define $\xi(S)$ to be $3g + p$. We shall define curve complexes for orientable surfaces S with $\xi(S) \geq 4$ or $\xi(S) = 2$. The *curve complex* $\mathcal{CC}(S)$ of S with $\xi(S) \geq 5$ is defined as follows. (This notion was first introduced by Harvey [25].) The vertices of $\mathcal{CC}(S)$, whose set we denote by $\mathcal{CC}_0(S)$, are the homotopy classes of essential simple closed curves on S . A collection of $n + 1$ vertices $\{v_0, \dots, v_n\}$ spans an n -simplex if and only if they can be represented as pairwise disjoint simple closed curves on S .

In the case when $\xi(S) = 4$, the curve complex is a 1-dimensional simplicial complex. The vertices are the homotopy classes of essential simple closed curves as in the case when $\xi(S) \geq 5$. Two vertices v_0, v_1 are connected by an edge if their intersection number is 1 when S is a once-punctured torus, and 2 when S is a four-times-punctured sphere.

In the case when $\xi(S) = 2$, we consider a compactification of S to an annulus. The curve complex is a 1-dimensional and the vertices are the homotopy classes (relative to the endpoints) of essential arcs with both endpoints on the boundary. Two curves are connected by an edge if they can be made disjoint in their interiors.

A *tight sequence* in $\mathcal{CC}(S)$ with $\xi(S) \geq 5$ is a sequence of simplices $\{s_0, \dots, s_n\}$ with the first one and the last one being vertices such that for any vertices $v_i \in s_i$ and $v_j \in s_j$, we have $d_{\mathcal{CC}(S)}(v_i, v_j) = |j - i|$, and s_{j+1} is homotopic to the union of essential boundary components of a regular neighbourhood of $s_j \cup s_{j+2}$. We also consider an infinite tight sequence such as $\{s_0, \dots\}$ or $\{\dots, s_0\}$ or $\{\dots, s_0, \dots\}$. In the case of the surface with $\xi(S) = 2$ or 4, we consider a sequence of vertices and ignore the second condition.

For an essential simple closed curve c of S , we consider the covering A_c of S associated to the image of $\pi_1(c)$, which is an open annulus. If we fix a hyperbolic metric on S , we can compactify the hyperbolic annulus A_c to an annulus \bar{A}_c by regarding $\pi_1(c)$ as acting on \mathbb{H}^2 and considering the quotient of $\mathbb{H}^2 \cup \Omega_{\pi_1(c)}$ by $\pi_1(c)$, where $\Omega_{\pi_1(c)}$ denotes the region of discontinuity

of the action of $\pi_1(c)$ on the circle at infinity of \mathbb{H}^2 . We call an essential simple arc with endpoints on $\partial\bar{A}_c$ a *transversal* of c . A simple closed curve d intersecting c essentially induces $i(c, d)$ -many transversals of c . Any two transversals induced from d are within the distance 1 in $\mathcal{CC}(A_c)$.

For a non-annular domain Σ in S , we define $\pi_\Sigma : \mathcal{CC}_0(S) \rightarrow \mathfrak{P}(\mathcal{CC}(\Sigma)) \cup \{\emptyset\}$ to be a map sending $c \in \mathcal{CC}_0(S)$ to the set of essential simple closed curves obtained by connecting the endpoints of each component of $c \cap \Sigma$ by arcs on $\text{Fr}\Sigma$ in a consistent way if c intersects Σ essentially. We define $\pi_\Sigma(c)$ to be \emptyset when no essential simple closed curves are obtained from $c \cap \Sigma$. When Σ is an annulus we define $\pi_\Sigma(c)$ to be a transversal of the core curve of Σ induced from c if c intersects Σ essentially. (See Masur–Minsky [38, §2.3] for details.)

A *marking* μ on a surface S consists of a simplex in $\mathcal{CC}(S)$ and transversals on some of its vertices (at most one for each). The vertices of the simplex are called the *base curves* of μ , and their union is denoted by $\text{base}(\mu)$. A marking μ is said to be *clean* if every component c of $\text{base}(\mu)$ has a transversal and it is induced by a simple closed curve with intersection number 1 when c is non-separating and with intersection number 2 if c is separating, which is disjoint from the other components of $\text{base}(\mu)$. A clean marking μ' is said to be compatible with a marking μ when $\text{base}(\mu) = \text{base}(\mu')$, and every transversal of a component c in μ is within the distance 2 from the transversal of c in μ' as vertices in the curve complex of A_c defined above. A marking is called *complete* if its base curves constitute a pants decomposition of S and every base curve has a transversal.

For a marking μ and a non-annular domain Σ of S whose frontier does not intersect $\text{base}(\mu)$ transversely, we define $\mu|_\Sigma$ to be a marking on Σ whose base curves are those in $\text{base}(\mu) \cap \Sigma$ and whose transversals are those induced from the transversals of $\text{base}(\mu)$. If there is a component c of $\text{base}(\mu)$ intersecting $\text{Fr}\Sigma$ essentially, we let $\pi_\Sigma(c)$ be included in $\mu|_\Sigma$, forgetting the transversal of c . When Σ is an annulus, there are two cases where $\mu|_\Sigma$ is defined. One is when μ intersects the core curve of Σ transversely, in which case we define $\mu|_\Sigma$ to be $\pi_\Sigma(\mu)$. The other is when there is a component b of $\text{base}(\mu)$ which is a core curve of Σ , in which case we define $\mu|_\Sigma$ to be the transversal of b .

To deal with the case of geometrically infinite groups, we need a notion of generalised markings. A *generalised marking* consists of an unmeasured lamination on S and transversals on some of its components which are simple closed curves. Also for a generalised marking μ , we denote the unmeasured lamination by $\text{base}(\mu)$ and call it the *base lamination*. We say that a generalised marking μ is *complete*, if its base lamination is maximal, i.e. it is not a proper sublamination of another unmeasured lamination, and every simple

closed curve in $\text{base}(\mu)$ has a transversal. From now on, we always assume *markings and generalised markings to be complete*.

A finite *tight geodesic* on a surface Σ is a triple $(g, I(g), T(g))$, where g is a tight sequence, and $I(g)$ and $T(g)$ are generalised markings on Σ whose base laminations have at least one simple closed curve component, such that the first vertex is a simple closed curve component of the $\text{base}(I(g))$ and the last vertex is that of $\text{base}(T(g))$. The surface Σ is called the *support* of g and we write $\Sigma = D(g)$. An infinite tight geodesic is defined similarly just letting $T(g)$ be an arational unmeasured lamination to which s_i converges as $i \rightarrow \infty$ (in the quotient topology of the unmeasured lamination space induced from the measured lamination space) when g is in the form of $\{s_0, \dots\}$. Similarly, by letting $I(g)$ be an arational unmeasured lamination to which s_i converges as $i \rightarrow -\infty$ when g is in the form of $\{\dots, s_0\}$, and by letting both $I(g)$ and $T(g)$ be arational unmeasured laminations which are limits of $\{s_i\}$ as $i \rightarrow -\infty$ and as $i \rightarrow \infty$ respectively when g is in the form of $\{\dots, s_0, \dots\}$. Refer to Section 2.7 for more explanations on the boundary of $\mathcal{CC}(\Sigma)$.

Let Σ be a domain of S . For a simplex s in $\mathcal{CC}(\Sigma)$, a *component domain* of s is defined to be either a component of $\Sigma \setminus s$ or an annulus whose core curve is a component of s . We consider only one annulus for each component of s . Let g be a tight geodesic in $\mathcal{CC}(\Sigma)$, and suppose that s is a simplex on g . Let Σ' be a component domain of s . (Such a domain is also said to be a component domain of g .) Then, following Masur–Minsky [38], we define $T(\Sigma', g)$ to be $\text{succ}(s)|\Sigma'$ if s is not the last vertex of g , and to be $T(g)|\Sigma'$ if s is, where $\text{succ}(s)$ denotes the simplex of g following s . Similarly, we define $I(\Sigma', g)$ to be $\text{prec}(s)|\Sigma'$ if s is not the first vertex of g , and to be $I(g)|\Sigma'$ if s is, where $\text{prec}(s)$ denotes the simplex of g preceding s . We write $\Sigma' \searrow^d g$ or $\Sigma' \searrow^d (g, s)$, if $T(\Sigma', g)$ is non-empty, and $g \swarrow^d \Sigma'$ or $(g, s) \swarrow^d \Sigma'$ if $I(\Sigma', g)$ is non-empty. If a geodesic k is a tight geodesic supported on Σ' with $(g, s) \swarrow^d \Sigma'$ and $I(k) = I(\Sigma', g)$, then we write $g \swarrow^d k$ or $(g, s) \swarrow^d k$, and say that k is *directly backward subordinate* to g at s . Similarly, if $\Sigma' \searrow^d (g, s)$ and $T(k) = T(\Sigma', g)$, we write $k \searrow^d g$ or $k \searrow^d (g, s)$, and say that k is *directly forward subordinate* to g at s .

A *hierarchy* h on S , which was introduced by Masur–Minsky [38], is a family of tight geodesics supported on domains in S having the following properties.

- (1) There is a unique geodesic g_h supported on S .
- (2) For any $g \in h$ other than g_h , there are geodesics $b, f \in h$ with $b \swarrow^d g \searrow^d f$.
- (3) For any $b, f \in h$ and a component domain Σ of b, f with $b \swarrow^d \Sigma \searrow^d f$ (b and f may coincide), there is a unique geodesic k supported on Σ with $b \swarrow^d k \searrow^d f$.

A hierarchy h is said to be *complete* if every component domain of geodesics in h supports a geodesic in h , and *4-complete* if every non-annular component domain of geodesics in h supports a geodesic in h .

We write $g \searrow (f, v)$ if there is a sequence of geodesics in h such that $g = f_0 \searrow f_1 \searrow \dots \searrow (f_n, v) = (f, v)$, and say that g is forward subordinate to f . Similarly, we write $(b, u) \swarrow g$ if there is a sequence in h such that $(b, u) = (b_m, u) \swarrow \dots \swarrow b_1 \swarrow b_0 = g$, and say that g is backward subordinate to b . We use symbol $\searrow_{=}$ to mean either \searrow or $=$ and $\swarrow_{=}$ to mean either \swarrow or $=$.

In Section 9, we shall use the notions of slices and resolutions of hierarchies invented by Masur–Minsky [38]. We shall review them briefly here. Let h be a complete or 4-complete hierarchy. A *slice* σ of h is a set of pairs (g, v) , where g is a geodesic in h and v is a simplex on g satisfying the following conditions. (Masur and Minsky call σ satisfying the first three conditions a slice, and call it a complete slice if it also satisfies the fourth condition.)

- (1) A geodesic can appear at most in one pair of σ .
- (2) There is a pair whose first entry is the main geodesic of h .
- (3) For each pair (g, v) in σ such that g is not the main geodesic, $D(g)$ is a component domain of a simplex v' for some $(g', v') \in \sigma$.
- (4) For each component domain D of v for $(g, v) \in \sigma$ with $\xi(D) \neq 3$ if h is complete and $\xi(D) > 3$ if h is 4-complete, there is a pair $(g', v') \in \sigma$ with $D(g') = D$.

Masur and Minsky introduced two kinds of order, \prec_p between pairs of geodesics and simplices in h and \prec_s between slices. For two pairs (g, v) and (g', v') of a 4-complete hierarchy, we write $(g, v) \prec_p (g', v')$ if either $g = g'$ and v' comes after v , or there is a geodesic g'' with $(g, v) \searrow_{=} (g'', w)$ and $(g'', w) \swarrow_{=} (g', v')$ such that w' is a simplex coming after w . For two distinct slices σ and τ , we write $\sigma \prec_s \tau$ if for any $(g, v) \in \sigma$, either $(g, v) \in \tau$ or there is $(g', v') \in \tau$ with $(g, v) \prec_p (g', v')$.

A *resolution* $\tau = \{\sigma_i\}$ of a 4-complete hierarchy h is an ordered sequence of slices of h such that σ_{i+1} is obtained from σ_i by an elementary forward move. Here an elementary forward move is a change of pairs in σ_i as follows: We advance $(g, v) \in \sigma_i$ to $(g, \text{succ}(v))$ under the condition that for every pair (g', v') supported on a component domain of v into which $\text{succ}(v)$ is projected to an essential curve, the simplex v' is the last vertex, and after removing all such (g', v') we add pairs (g'', v'') such that g'' is supported on a component domain of $\text{succ}(v)$ into which v is projected to an essential curve and v'' is the first vertex of g'' .

2.6. Model manifolds

A model manifold for a Kleinian surface group was constructed in Minsky [40] as follows. Let G be a Kleinian surface group with $M = \mathbb{H}^3/G$. From an end invariant of G , we shall construct a hierarchy of tight geodesics, which we denote by h_G . In the case when G is quasi-Fuchsian, we construct a hierarchy by defining the initial and terminal markings to be the shortest clean markings with respect to the upper and lower conformal structures at infinity. When M has a totally degenerate end without accidental parabolics, we define the initial or terminal generalised marking to be its ending lamination. In general, we consider the union of ending laminations of M_0 , parabolic curves for upper or lower ends, and shortest clean markings on the remaining geometrically finite upper or lower ends, and let them be terminal or initial generalised markings. Here, we say that a clean marking μ on a hyperbolic surface S is shortest if the base curves of μ form a shortest pants decomposition of S , and transversals are chosen so that their total length is the smallest among the transversals obtained from them by performing Dehn twists around the base curves. Note that we are *not* assuming the transversals of μ are really shortest among all transversals.

Having defined the hierarchy h_G , we construct a resolution $\{\tau_i(G)\}$ of h_G . In the resolution, we look at each step $\tau_i(G) \rightarrow \tau_{i+1}(G)$ that advances a vertex on a 4-geodesic, from w_i to w_{i+1} . For such a step, we provide an *internal block*, which is topologically homeomorphic to $\Sigma \times [0, 1]$, where Σ is either a sphere with four holes or a torus with one hole. The block has two ditches, one on the top and the other on the bottom, corresponding to the two vertices w_i and w_{i+1} . To be more precise, we take annular neighbourhoods N_i, N_{i+1} of w_i, w_{i+1} on Σ , and set $B = \Sigma \times [0, 1] \setminus (N_i \times [0, 1/4] \cup N_{i+1} \times [3/4, 1])$. The top and bottom boundary of a block consists of pairs of sphere with three holes. We fix some constant ϵ_1 less than the Bers constant for S throughout the construction. We put a hyperbolic metric on Σ so that the lengths of the boundary components, w_i and w_{i+1} are all equal to ϵ_1 . We also assume that N_i and N_{i+1} are regular neighbourhoods whose boundaries have length ϵ_2 and deform their metrics to flat ones keeping their boundaries fixed. (Here ϵ_2 is chosen so that for pants decomposition of S by simple closed geodesics with length less than ϵ_1 , their annular neighbourhoods whose boundaries have length ϵ_2 are pairwise disjoint.) We define the metric on B to be the one induced by the product of the hyperbolic metric on Σ as above and the ordinary metric on $[0, 1]$. We note that the isometry type of B depends only on whether Σ is a sphere with four holes or a torus with one hole.

The model manifold for G is constructed by piling up such blocks by pasting a top component of one block to a bottom component of another, according to the information given by the resolution $\{\tau_i(G)\}$, attaching *boundary*

blocks to the top and the bottom of the piled up blocks if there are geometrically finite ends of M_0 , which have special forms and are constructed according to conformal structures at infinity of G , and then finally filling in “Margulis tubes”. (Here we abuse the term “Margulis tube”, which was defined before using the Margulis constant ϵ_0 . Our tubes here may have larger injectivity radii, but still are isometric to tubular neighbourhoods of closed geodesics.) In this paper, we define a boundary block to have topologically a form $\Sigma \times [s, t]$ or $\Sigma \times (t, s]$ for some incompressible subsurface Σ of S (i.e. a subsurface Σ such that every component of $\text{Fr}\Sigma$ is non-contractible in S), and do not put extra-annuli as in Minsky’s definition.

To be more precise, a boundary block has the following form. We describe it here only when the block corresponds to an upper end. We can deal with the case when the end is lower just by turning everything upside down. Let n_0 be a point in $\mathcal{T}(\Sigma)$, where each component of $\text{Fr}\Sigma$ is assumed to be a puncture, and regard it as a hyperbolic metric on $\text{Int}\Sigma$ with each component of $\text{Fr}\Sigma$ assumed to be the boundary of an ϵ_0 -cusp neighbourhood. Take a shortest pants decomposition of (Σ, n_0) and denote its components by c_1, \dots, c_p . A boundary block B is constructed by defining $B = \Sigma \times [-1, \infty) \setminus (\bigcup_{k=1}^p A(c_k) \times [-1, 0))$, where $A(c_k)$ is an open annular neighbourhood of c_k whose boundaries have length ϵ_1 . We now put a Riemannian metric on B as follows. Since each component of $\Sigma \times \{0\} \setminus (\bigcup_{k=1}^p A(c_k) \times \{0\})$ is a pair of pants, we put a standard hyperbolic metric n_0 so that each boundary component becomes a closed geodesic of length ϵ_0 . Now, as was shown in Section 3.4 of Minsky [40], there is a metric n'_0 on Σ conformal to n_0 in which the $A(c_k)$ are flat annuli and whose restriction to each component of $\Sigma \setminus (\bigcup_{k=1}^p A(c_i))$ coincides with the hyperbolic metric n_0 . We put this metric n'_0 on $\Sigma \times \{0\}$, and $n_0|_{(\Sigma \setminus (\bigcup_{k=1}^p A(c_i)))}$ on $\Sigma \times \{-1\} \setminus (\bigcup_{k=1}^p A(c_k) \times \{-1\})$. On $\Sigma \times [-1, 0) \setminus (\bigcup_{k=1}^p A(c_k) \times [-1, 0))$, we put the product of the metric of dn_0 and dt . Now, on $\Sigma \times [0, \infty)$, we put a metric defined by $e^{2t}d(n'_0)^2 + dt^2$.

We do not put extra-annuli which appeared in Minsky’s construction because we are constructing a model manifold of the *non-cuspidal part*, not of the entire manifold. By the same reason, in contrast to Minsky’s original construction, we do not fill in cusp neighbourhoods. Each slice in $\{\tau_i(G)\}$ corresponds to a *split level surface* in the model manifold which is a disjoint union of horizontal surfaces in blocks which are spheres with three holes. Taking split level surfaces into pleated surfaces and extending them over Margulis tubes, a homotopy equivalent map from the model manifold to M_0 is constructed. This can be modified to a uniform bi-Lipschitz map which is called a *model map* to M_0 . See Minsky [40] and Brock–Canary–Minsky [13] for more details. There is an alternative approach to constructing model manifolds by Bowditch [6, 7, 8].

2.7. The boundaries at infinity of curve complexes

It was proved by Masur–Minsky [37] that $\mathcal{CC}(S)$ is a Gromov hyperbolic space with respect to the path metric defined by setting every edge to have the unit length. For a Gromov hyperbolic space, its boundary at infinity can be defined as a topological space. (Refer for instance to Coornaert–Delzant–Papadopoulos [21].) Klarreich in [31] showed that the boundary at infinity of $\mathcal{CC}(S)$ is the space of ending laminations: that is, the space of arational unmeasured laminations with topology induced from $\mathcal{UML}(S)$. This space is denoted by $\mathcal{EL}(S)$.

We shall show the following lemma, which is an easy consequence of the definition of the topology of $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$.

LEMMA 2.2. — *Let $\{g_i\}$ be a sequence of geodesics in $\mathcal{CC}(\Sigma)$ converging to a geodesic ray g_∞ uniformly on every compact set. Then the last vertex of g_i converges to the endpoint at infinity of g_∞ with respect to the topology of $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$.*

Proof. — Let λ be a measured lamination whose support is the endpoint at infinity of g_∞ . We can assume that all the g_i have the same initial vertex, which we denote by v . Let w_i be the last vertex of g_i . Since the length of g_i goes to infinity, the distance between v and w_i goes to infinity. On the other hand, since $\{g_i\}$ converges to g_∞ on every compact set, there is a number $n_{i,j}$ going to ∞ such that the first $n_{i,j}$ simplices of g_i and g_j are the same. Since $(w_i|w_j)_v \geq n_{i,j}$, we see that $(w_i|w_j)_v$ goes to ∞ as $i, j \rightarrow \infty$. Therefore, $\{w_i\}$ converges to some ending lamination after passing to a subsequence. By the definition of the topology on $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$, there is a measured lamination μ and positive real numbers r_i such that $\{r_i w_i\}$ converges to μ .

We need to show that $|\mu| = |\lambda|$. Suppose not. Since $\{g_i\}$ converges to g_∞ uniformly on every compact set, we can take a simplex $v_i \in g_\infty$ which is also contained in g_i tending to $|\lambda|$ in $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$. Since $|\lambda|$ and $|\mu|$ are distinct points on the boundary at infinity, we have $\limsup_{i \rightarrow \infty} (v_i|w_i)_v < \infty$. This contradicts the facts that both v_i and w_i lie on the same geodesic g_i and that both $d(v, v_i)$ and $d(v, w_i)$ go to ∞ . \square

3. The main results

In this section, we shall state our main theorems.

3.1. End invariants of limit groups

We shall first state a theorem showing that for a limit of quasi-Fuchsian groups, the limit laminations of upper conformal structures at infinity appear as ending laminations of upper ends whereas the limit of lower ones appear as ending laminations of lower ends.

THEOREM 3.1. — *Let $\{(m_i, n_i)\}$ be a sequence in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ such that $\{qf(m_i, n_i)\}$ converges to (Γ, ψ) in $AH(S)$. Let $[\mu^+]$ and $[\mu^-]$ be projective laminations which are limits of $\{m_i\}$ and $\{n_i\}$ in the Thurston compactification of the Teichmüller space passing to subsequences. Then every component of $|\mu^+|$ that is not a simple closed curve is the ending lamination of an upper end of $(\mathbb{H}^3/\Gamma)_0$ whereas every component of $|\mu^-|$ that is not a simple closed curve is the ending lamination of a lower end.*

Theorem 3.1 will be obtained by combining the following theorem with a simple lemma regarding the Thurston compactification of Teichmüller space.

THEOREM 3.2. — *In the same setting as in Theorem 3.1, let c_{m_i} and c_{n_i} be shortest pants decompositions of (S, m_i) and (S, n_i) respectively. Let ν^-, ν^+ be the Hausdorff limits of $\{c_{m_i}\}$ and $\{c_{n_i}\}$ respectively after passing to subsequences. Then the minimal components of ν^+ that are not simple closed curves coincide with the ending laminations of upper ends of $(\mathbb{H}^3/\Gamma)_0$. Moreover, every upper parabolic curve is contained in ν^+ . Similarly the minimal components of ν^- that are not simple closed curves coincide with the ending laminations of lower ends of $(\mathbb{H}^3/\Gamma)_0$, and every lower parabolic curve is contained in ν^- .*

Conversely every simple closed curve contained in either ν^- or ν^+ that has isolated leaves spiralling around it is a parabolic curve. Every such simple closed curve in ν^+ that is not contained in ν^- is an upper parabolic curve whereas every such simple closed curve in ν^- that is not contained in ν^+ is a lower parabolic curve.

3.2. Divergence theorems

We shall next state our theorems on divergence of quasi-Fuchsian groups, where we shall give sufficient conditions for sequences of quasi-Fuchsian groups to diverge.

THEOREM 3.3. — *Let $\{(m_i, n_i)\}$ be a sequence in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ satisfying the following conditions.*

- (1) *$\{m_i\}$ converges to a projective lamination $[\mu^-] \in \mathcal{PML}(S)$ whereas $\{n_i\}$ converges to $[\mu^+] \in \mathcal{PML}(S)$ in the Thurston compactification of the Teichmüller space.*
- (2) *There are components μ_0^+ of μ^- and μ_0^- of μ^+ which are not weighted simple closed curves and have the minimal supporting surfaces sharing at least one boundary component up to isotopy.*

Then the sequence $\{qf(m_i, n_i)\} \subset QF(S)$ diverges in $AH(S)$.

Theorem 3.3 follows rather easily from Theorem 3.1. If we use Theorem 3.2 instead of Theorem 3.1, we get the following.

THEOREM 3.4. — *Let $\{m_i\}$ and $\{n_i\}$ be sequences in $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$ without convergent subsequences, and let c_{m_i} and c_{n_i} be shortest pants decompositions of the hyperbolic surfaces (S, m_i) and (S, n_i) respectively. Suppose that c_{m_i} and c_{n_i} converge to geodesic laminations μ^- and μ^+ in the Hausdorff topology respectively. Suppose that there are minimal components μ_0^- of μ^- and μ_0^+ of μ^+ which are not simple closed curves and have minimal supporting surfaces sharing at least one boundary component up to isotopy. Then $\{qf(m_i, n_i)\} \subset QF(S)$ diverges in $AH(S)$.*

In the setting of these two theorems above, the case when μ_0^- and μ_0^+ have the same support is most interesting. In fact, if they do not, it is much easier to prove the theorems just by using the continuity of length function in hyperbolic manifolds and the fact that on the ending lamination of an end e of the non-cuspidal part is maximal on a frontier component of a relative compact core facing e . Also the assumption that μ_0^- and μ_0^+ are not simple closed curves is essential. In the case when μ^- and μ^+ share simple closed curves, the construction of Anderson–Canary [2] gives an example of convergent sequence. Still, we can show the following theorem.

THEOREM 3.5. — *Let μ^- and μ^+ be two measured laminations on S such that the components shared by $|\mu^-|$ and $|\mu^+|$ are all simple closed curves or there are no components shared by $|\mu^-|$ and $|\mu^+|$. Suppose that either*

- (1) *there is a boundary component of the minimal supporting surface of a non-simple closed curve component of μ^+ which is contained in $|\mu^-|$ up to isotopy, or*
- (2) *there is a boundary component of the minimal supporting surface of a non-simple closed curve component of μ^- which is contained in $|\mu^+|$ up to isotopy.*

Then for every $\{m_i \in \mathcal{T}(S)\}$ converging to $[\mu^-]$ and $\{n_i \in \mathcal{T}(S)\}$ converging to $[\mu^+]$ in the Thurston compactification of the Teichmüller space, the sequence $\{qf(m_i, n_i)\} \subset QF(S)$ diverges in $AH(S)$.

In the case when a simple closed curve component of $|\mu^+|$ which does not lie on the boundary of minimal supporting surface of non-simple closed curve component, up to isotopy, is shared by $|\mu^-|$ as the same kind of component, we need to take into accounts the weights on c_1, \dots, c_r , as was done in Ito [26] in the case of once-punctured torus groups.

THEOREM 3.6. — *Consider, as in Theorem 3.5, sequences $\{m_i\}$ and $\{n_i\}$ converging to $[\mu^-]$ and $[\mu^+]$ respectively, and suppose that their supports share only simple closed curves c_1, \dots, c_r . Suppose that none of c_1, \dots, c_r is isotopic into the boundary of the minimal supporting surface of a component of μ^- or μ^+ . Then $\{qf(m_i, n_i)\}$ converges after taking a subsequence only if the following conditions are satisfied.*

- (1) For each c_j among c_1, \dots, c_r , neither $\text{length}_{m_i}(c_j)$ nor $\text{length}_{n_i}(c_j)$ goes to 0.
- (2) There are sequences of integers $\{p_i^1\}, \dots, \{p_i^r\}, \{q_i^1\}, \dots, \{q_i^r\}$ such that the following hold after passing to a subsequence:
 - (a) If $|\mu^-| \setminus \bigcup_{j=1}^r c_j$ is non-empty, then $(\tau_{c_1}^{p_i^1} \circ \dots \circ \tau_{c_r}^{p_i^r})^*(m_i)$ converges to $[\mu^- \setminus \bigcup_{j=1}^r w_j c_j]$ in the Thurston compactification, where w_j is the transverse measure on c_j which μ^- defines and $\tau_{c_j}^{p_i^j}$ denotes the Dehn twist around c_j . Otherwise, $(\tau_{c_1}^{p_i^1} \circ \dots \circ \tau_{c_r}^{p_i^r})^*(m_i)$ either stays in a compact set of the Teichmüller space or converges to a projective lamination $[\nu^-]$ which contains none of c_1, \dots, c_r as a leaf.
 - (b) In the same way, if $|\mu^+| \setminus \bigcup_{j=1}^r c_j$ is non-empty, $(\tau_{c_1}^{q_i^1} \circ \dots \circ \tau_{c_r}^{q_i^r})^*(n_i)$ converges to $[\mu^+ \setminus \bigcup_{j=1}^r v_j c_j]$ in the Thurston compactification, where v_j is the transverse measure on c_j which μ^+ defines. Otherwise, $(\tau_{c_1}^{q_i^1} \circ \dots \circ \tau_{c_r}^{q_i^r})^*(n_i)$ either stays in a compact set of the Teichmüller space or converges in the Thurston compactification to a projective lamination $[\nu^+]$ which contains none of c_1, \dots, c_r as a leaf.
 - (c) There exist $a_j \in \mathbb{Z}$ ($j = 1, \dots, r$) with $a_j \neq 0, -1$ and $k_i^j \in \mathbb{Z}$ with $|k_i^j| \rightarrow \infty$ such that $p_i^j = k_i^j a_j$ and $q_i^j = k_i^j (a_j + 1)$ for every $j = 1, \dots, r$ and large i .

If the sequence really converges, then c_j is an upper parabolic curve of the algebraic limit if $a_i > 0$, and a lower parabolic curve otherwise.

Conversely, let $a_j \in \mathbb{Z}$ ($j = 1, \dots, r$) be any number, and μ^-, μ^+ measured laminations whose supports share only (possibly empty) simple closed curves c_1, \dots, c_r and which satisfy the following conditions.

- (1*) The laminations μ^- and μ^+ do not have non-simple-closed-curve components whose minimal supporting surfaces share a boundary component up to isotopy.
- (2*) In the case when both μ^- and μ^+ are connected and the minimal supporting surfaces of μ^- and μ^+ are the entire surface S , the supports $|\mu^-|$ and $|\mu^+|$ do not coincide.
- (3*) No simple closed curve in $|\mu^-|$ is isotopic into the boundary of the minimal supporting surface of a non-simple-closed-curve component of μ^+ . In the same way, no simple closed curve in $|\mu^+|$ is isotopic into the boundary of the minimal supporting surface of a non-simple-closed-curve component of μ^- .

Then, there is a sequence of $\{(m_i, n_i)\}$ in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ with algebraically convergent $\{qf(m_i, n_i)\}$ such that $\{m_i\}$ converges to $[\mu^-]$ and $\{n_i\}$ converges to $[\mu^+]$ in the Thurston compactification, and the two conditions (1) and (2) above are satisfied.

The latter half of this theorem shows that *as sufficient conditions for divergence expressed in term of the limits of the conformal structures in the Thurston compactification, Theorems 3.3 and 3.5 together with the main theorem of [45] are best possible.*

3.3. Non-existence of exotic convergence

The assumptions in Theorem 3.5 is related to the fact that such a sequence as in the statement cannot converge exotically to a b-group. The condition that none of c_1, \dots, c_r lies on the boundary of the supporting surface of a component of μ^- or μ^+ is essential for the exotic convergence. We can prove the following related results. (Recall that a non-peripheral parabolic locus is said to be isolated if it does not touch a simply degenerate end.)

THEOREM 3.7. — *Let G be a b-group without isolated parabolic locus. Then there is no sequence of quasi-Fuchsian groups exotically converging to G .*

3.4. Self-bumping

As the results of McMullen [39], Bromberg [9] and Magid [34] suggest, the singularities of $AH(S)$ which are found thus far are all related to the construction of Kerckhoff–Thurston [30] and Anderson–Canary [2]. The following results show that convergence to geometrically infinite groups in $AH(S)$ without isolated parabolic loci is quite different from the situation for regular b-groups where $QF(S)$ bumps itself.

THEOREM 3.8. — *Let Γ be a geometrically infinite group with isomorphism $\psi : \pi_1(S) \rightarrow \Gamma$ in $AH(S)$. Suppose that Γ does not have an isolated parabolic locus. Let $\{(m_i, n_i)\}$ and $\{(m'_i, n'_i)\}$ be two sequences in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ such that both $\{qf(m_i, n_i)\}$ and $\{qf(m'_i, n'_i)\}$ converge to (Γ, ψ) . Then for any neighbourhood U of the quasi-conformal deformation space $QH(\Gamma, \psi)$ of (Γ, ψ) in $AH(S)$, we can take i_0 so that if $i > i_0$, then there is an arc α_i in $U \cap QF(S)$ connecting $qf(m_i, n_i)$ with $qf(m'_i, n'_i)$. In the case when Γ is a b-group whose lower conformal structure at infinity is m_0 , we can take α_i satisfying further the following condition. For any neighbourhood V of m_0 in $\mathcal{T}(S)$, we can take i_0 so that for any $i > i_0$, the arc α_i is also contained in $qf(V \times \mathcal{T}(\bar{S}))$.*

We shall then get a corollary as follows.

COROLLARY 3.9. — *In the setting of Theorem 3.8, suppose furthermore that each component of Ω_Γ/Γ that is not homeomorphic to S is a thrice-punctured sphere. Then $QF(S)$ does not bump itself at (Γ, ψ) , and in particular $AH(S)$ is locally connected at (Γ, ψ) .*

We can generalise this corollary by dropping the assumption that there are no isolated parabolic loci. The same result has been obtained by substantially different methods in Brock–Bromberg–Canary–Minsky [12]; see also Canary [19]. Also, a related result has been obtained by Anderson–Lecuire [3].

COROLLARY 3.10. — *Let Γ be a group on the boundary of $QF(S)$ with isomorphism $\psi : \pi_1(S) \rightarrow \Gamma$. This time we allow Γ to have isolated parabolic loci.*

- (1) *If every component of Ω_Γ/Γ is a thrice-punctured sphere, then $QF(S)$ does not bump itself at (Γ, ψ) .*
- (2) *If Γ is a b-group and every component of Ω_Γ/Γ corresponding to upper ends of $(\mathbb{H}^3/\Gamma)_0$ is a thrice-punctured sphere, then the Bers slice containing (Γ, ψ) on the boundary does not bump itself.*

3.5. General Kleinian surface groups

We can generalise Theorems 3.2 and 3.3 to sequences of Kleinian surface groups which may not be quasi-Fuchsian.

Let G be a Kleinian surface group with marking $\phi : \pi_1(S) \rightarrow G$, and set $M = \mathbb{H}^3/G$. The marking ϕ determines a homeomorphism $h : S \times \mathbb{R} \rightarrow M$. Let P_+ be the upper parabolic locus on S . We consider all the upper ends of the non-cuspidal part M_0 . For a geometrically finite end, we consider its minimal pants decomposition, and for a simply degenerate end, we consider its ending lamination. Take the union of all these curves and laminations together with core curves of P_+ , and denote it by e_+ . In the same way, we define e_- for the lower ends. We call e_+ and e_- the upper and the lower *generalised shortest pants decompositions* respectively.

We now state a generalisation of Theorem 3.2

THEOREM 3.11. — *Let $\{(G_i, \phi_i)\}$ be a sequence of Kleinian surface groups which have upper and lower generalised shortest pants decompositions $e(i)_+$ and $e(i)_-$. Suppose that $\{(G_i, \phi_i)\}$ converges algebraically to (Γ, ψ) . Consider the Hausdorff limit $e(\infty)_+$ of $\{e(i)_+\}$ and $e(\infty)_-$ of $\{e(i)_-\}$ after passing to subsequences. Then every minimal component of $e(\infty)_+$ (resp. $e(\infty)_-$) that is not a simple closed curve is the ending lamination of an upper end (resp. a lower end) of $(\mathbb{H}^3/\Gamma)_0$. Conversely any ending lamination of an upper end (resp. a lower end) of $(\mathbb{H}^3/\Gamma)_0$ is a minimal component of $e(\infty)_+$ (resp. $e(\infty)_-$). Moreover, every upper (resp. lower) parabolic curve is contained in $e(\infty)_+$ (resp. $e(\infty)_-$).*

Next we shall state a generalisation of Theorem 3.3.

THEOREM 3.12. — *Let $\{(G_i, \phi_i)\}$ be a sequence of Kleinian surface groups which have upper and lower generalised shortest pants decompositions $e(i)_+$ and $e(i)_-$. Let $e(\infty)_+$ and $e(\infty)_-$ be the Hausdorff limits of $\{e_+(i)\}$ and $\{e_-(i)\}$ respectively, after passing to subsequences. If there are minimal components λ of $e(\infty)_-$ and μ of $e(\infty)_+$ which are not simple closed curves and whose minimal supporting surfaces share a boundary component up to isotopy, then $\{(G_i, \phi_i)\}$ diverges in $AH(S)$.*

3.6. Application

We shall briefly explain here an application of the main results, in particular of Theorem 11 and Theorem 5.2, which appears in Ohshika [49]. For a point $m_0 \in \mathcal{T}(S)$, the subspace of $QF(S)$ in the form of $qf(\{m_0\} \times \mathcal{T}(\bar{S}))$ is

called the Bers slice based on m_0 , and is denoted by B_{m_0} . Bers proved in [4] that the closure of B_{m_0} in $AH(S)$ is compact for any $m_0 \in \mathcal{T}(S)$. The closure is called the Bers compactification of the Teichmüller space $\mathcal{T}(S)$ (identified with $\mathcal{T}(\bar{S})$) based on m_0 . We denote its boundary by $\partial_{m_0}^B \mathcal{T}(S)$. Kerckhoff and Thurston proved in [30] that there are two points $m_0, m_1 \in \mathcal{T}(S)$ such that the natural homeomorphism between B_{m_0} and B_{m_1} , which is obtained by identifying them with $\mathcal{T}(S)$, cannot extend continuously to a homeomorphism between their boundaries $\partial_{m_0}^B \mathcal{T}(S)$ and $\partial_{m_1}^B \mathcal{T}(S)$. This implies that the action of the mapping class group $MCG(S)$ on $\mathcal{T}(S)$ does not extend continuously to the Bers compactification (based on any point).

In [49], we considered a quotient space of the Bers boundary by collapsing each quasi-conformal deformation space lying there into a point, and showed that the mapping class group acts on this quotient space. (According to McMullen, Thurston already considered this space and conjectured this result.) We denote this quotient space obtained from $\partial_{m_0}^B \mathcal{T}(S)$ by $\partial_{m_0}^{RB} \mathcal{T}(S)$ and call it the reduced Bers boundary based on m_0 . Applying Theorem 11 in the present paper, we also showed that conversely every automorphism of $\partial_{m_0}^{RB} \mathcal{T}(S)$ is induced from an extended mapping class. The same kind of result was obtained for the unmeasured lamination space by Papadopoulos [51] and Ohshika [47], and for the geodesic lamination space with the Thurston topology by Charitos–Papadoperakis–Papadopoulos [20].

4. Models of geometric limits

4.1. Brick decompositions of geometric limits

In this section, we shall review the results in Ohshika–Soma [50] and show some facts derived from them, which are essential in our discussion. We shall give sketches of proofs for all necessary results so that the reader can understand them without consulting [50].

Throughout this section, we assume that we have a sequence $\{(G_i, \phi_i)\}$ in $AH(S)$ converging to (Γ, ψ) , and that $\{G_i\}$ converges geometrically to G_∞ , which contains Γ as a subgroup. We do not assume that G_i is quasi-Fuchsian, to make our argument work also for the proofs of Theorems 3.11 and 3.12. Recall that $M_\infty = \mathbb{H}^3/G_\infty$ is a Gromov–Hausdorff limit of $M_i = \mathbb{H}^3/G_i$ with basepoints y_i which are projections of some point fixed in \mathbb{H}^3 . We denote \mathbb{H}^3/Γ by M' . Let $p : M' \rightarrow M_\infty$ denote the covering associated to the inclusion of Γ to G_∞ . Let $\rho_i : B_{r_i}(M_i, y_i) \rightarrow B_{K_i r_i}(M_\infty, y_\infty)$ denote an approximate isometry corresponding to the pointed Gromov convergence of (M_i, y_i) to (M_∞, y_∞) .

In [50], we introduced the notion of brick manifolds. A brick manifold is a 3-manifold constructed from “bricks” which are defined as follows. We note that a brick is an entity different from a block introduced by Minsky which we explained in Preliminaries. Still, as we shall see, they are closely related, and actually, in our settings every brick is decomposed into blocks.

DEFINITION 4.1. — *A brick is a product interval bundle of the form $\Sigma \times J$, where Σ is an incompressible subsurface of S with $\chi(\Sigma) < 0$ and J is a closed or half-open interval in $[0, 1]$. We assume that Σ is a closed subset of S , i.e. $\text{Fr}\Sigma$ is contained in Σ . (Recall that we say that a (not necessarily proper) subsurface is incompressible if none of its boundary components is null-homotopic and at least one boundary component is non-peripheral, i.e. if it is either essential or S itself.) Note that it may be possible that two boundary components of Σ are parallel in S . For a brick $B = \Sigma \times J$, its lower front, denoted by ∂_-B , is defined to be $\Sigma \times \inf J$, and its upper front, denoted by ∂_+B , is defined to be $\Sigma \times \sup J$. When J is an half-open interval, one of them may not really exist, but corresponds to an end. In this case, it is called an ideal front. A brick naturally admits two foliations: one is a codimension-1 horizontal foliation whose leaves are horizontal surfaces $\Sigma \times \{t\}$, and the other is a codimension-2 vertical foliation whose leaves are vertical lines $\{p\} \times J$. We define $\xi(B)$ to be $\xi(\Sigma)$.*

A brick manifold is a manifold consisting of countably many bricks, whose boundary consists of tori and open annuli. Two bricks can intersect only at their (non-ideal) fronts in such a way that an essential subsurface, which is possibly disconnected but none of whose components is an annulus, in the upper front of one brick is pasted to an essential subsurface in the lower front of the other brick. In the case when the manifold is homeomorphic to $S \times (0, 1)$, we allow two bricks intersect at their entire non-ideal fronts. We also assume that an infinite sequence of bricks cannot accumulate inside the manifold, i.e. an infinite sequence of bricks must correspond to an end of the manifold after passing to a subsequence.

Since G_i is a Kleinian surface group, \mathbb{H}^3/G_i has a bi-Lipschitz model which was constructed by Minsky [40] and was proved to be bi-Lipschitz by Brock–Canary–Minsky [13]. We ignore cusp neighbourhoods in the model manifolds of Minsky to make them models for the non-cuspidal parts. Let \mathbf{M}_i be a model manifold for $(M_i)_0 = (\mathbb{H}^3/G_i)_0$ in the sense of Minsky with a bi-Lipschitz model map $f_i : \mathbf{M}_i \rightarrow (M_i)_0$. Minsky’s construction is based on complete hierarchies of tight geodesics which are determined by the end invariants of M_i , as we explained in Preliminaries. The model manifold has decomposition into blocks and Margulis tubes, which corresponds to a resolution of a complete hierarchy. When we talk about model manifolds \mathbf{M}_i , we always assume the existence of complete hierarchies h_i beforehand, and

that the manifolds are decomposed into blocks and Margulis tubes using resolutions. The metric of model manifolds are defined as the union of metrics on internal blocks and metrics determined by conformal structures at infinity on boundary blocks. We should note that as was shown in [40], the decomposition of \mathbf{M}_i into blocks and the metric on \mathbf{M}_i depend only on h_i and end invariants, and are independent of choices of resolutions.

We shall now see that geometric limits of algebraically convergent quasi-Fuchsian groups have also model manifolds. The following is one of the main theorems of [50] which is fitted into our present situation.

THEOREM 4.2 (Ohshika–Soma [50]). — *Let $\{(G_i, \phi_i)\}$ be a sequence in $AH(S)$ converging to (Γ, ψ) with $M' = \mathbb{H}^3/\Gamma$, and M_∞ a geometric limit of $M_i = \mathbb{H}^3/G_i$ with basepoint at y_i . Then, there are a model manifold \mathbf{M} of $(M_\infty)_0$, which has a structure of brick manifold and is a geometric limit of \mathbf{M}_i as a metric space, and a model map $f : \mathbf{M} \rightarrow (M_\infty)_0$ which is a K -bi-Lipschitz homeomorphism for a constant K depending only on $\chi(S)$. The model manifold \mathbf{M} has the following properties.*

- (0) *Each brick of \mathbf{M} is decomposed into blocks and Margulis tubes as in Minsky’s model manifolds, each of which is a limit of blocks and Margulis tubes in \mathbf{M}_i .*
- (1) *\mathbf{M} can be embedded in $S \times [0, 1]$ (with its image in $S \times (0, 1)$) in such a way that the vertical and horizontal foliations of the bricks are mapped into horizontal surfaces and vertical lines of $S \times [0, 1]$ respectively.*
- (2) *There is no essential properly embedded annulus in \mathbf{M} .*
- (3) *An end contained in a brick is defined to be either geometrically finite or simply degenerate. The model map takes geometrically finite ends to geometrically finite ends of $(M_\infty)_0$, and simply degenerate ends to simply degenerate ends of $(M_\infty)_0$.*
- (4) *Every geometrically finite end of \mathbf{M} corresponds to an incompressible subsurface of either $S \times \{0\}$ or $S \times \{1\}$.*
- (5) *An end not contained in any brick is neither geometrically finite nor simply degenerate. For such an end, there is no half-open incompressible annulus tending to the end which is not properly homotopic into a boundary component. We call such an end wild.*
- (6) *There are only countably many ends.*
- (7) *There is a π_1 -injective map $g : S \rightarrow \mathbf{M}$ which is π_1 -injective also as a map to $S \times [0, 1]$, such that $(f \circ g)_\#$ coincides with $\iota \circ \psi$, where ι is the monomorphism from $\Gamma = \pi_1(M')$ to $\pi_1(M_\infty)$ induced by the inclusion of the algebraic limit Γ into the geometric limit $\pi_1(M_\infty)$.*
- (8) *For any sequence of points $\{x_i \in \mathbf{M}\}$ tending to an end of \mathbf{M} , its image in $S \times [0, 1]$ converges, after passing to a subsequence, to a*

point in $S \times [0, 1] \setminus \mathbf{M}$. There are no two sequences in \mathbf{M} tending to distinct ends of \mathbf{M} whose images in $S \times [0, 1]$ converge to the same point.

Before starting the proof, we shall illustrate how a wild end as was described in (5) looks. Suppose that an end corresponding to $\Sigma \times \{t_0\}$ with an incompressible subsurface Σ of S is wild. Then there is a sequence of either torus cusps or simply degenerate ends (or both) whose images in $S \times [0, 1]$ accumulate to $\Sigma \times \{t_0\}$. In the case when torus cusps accumulate to $\Sigma \times \{t_0\}$, the vertical projections of its core curves to S converge to an arational geodesic lamination in the Hausdorff topology. In the case when simply degenerate ends e_i accumulate to $\Sigma \times \{t_0\}$, the end e_i corresponds to $\Sigma_i \times \{t_i\}$ with $t_i \rightarrow t_0$ such that Σ_i is a subsurface of Σ for large i , and each boundary component of Σ_i converges to the same arational lamination on Σ in the Hausdorff topology. The conditions that the Hausdorff limits are arational prevents the existence of an essential open annulus as described in the part (5) above. Figure 4.1 illustrates the case when simply degenerate ends accumulate to a wild end from below.

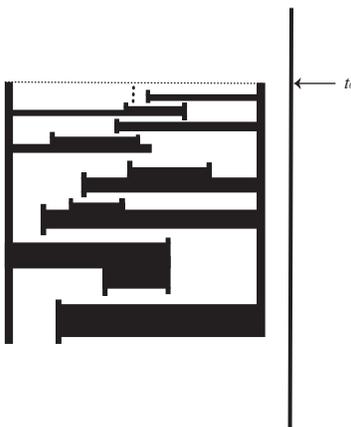


Figure 4.1. $\Sigma \times \{t_0\}$ is a wild end to which simply degenerate ends accumulate from below. The vertical line at the right end denotes the coordinate $[0, 1]$. Two vertically long rectangles at the left and right sides correspond to either frontier components or punctures of Σ . Small rectangles attached to black regions are sent into cusp neighbourhoods. Each long horizontal side of black regions corresponds to a simply degenerate end.

4.2. Proof of Theorem 4.2

Although this theorem was already proved in [50], we shall give an abridged version of its proof here, using sometimes an explanation a bit different from the one given in [50].

This model manifold \mathbf{M} is obtained as the non-cuspidal part of a geometric limit of Minsky's model manifolds for \mathbb{H}^3/G_i with suitable basepoints chosen. Let \mathbf{M}_i be Minsky's model manifold for $(M_i)_0$ with a bi-Lipschitz model map f_i as explained above. We put a basepoint x_i in \mathbf{M}_i which is mapped to y_i by f_i . Recall that Minsky's model manifold is composed of blocks and tubes. As was explained in Section 2.5, there are two kinds of blocks: internal blocks and boundary blocks. An internal block is topologically homeomorphic to $\Sigma \times [0, 1]$, where Σ is either a sphere with four holes or a torus with one hole. Geometrically it has two ditches, and has form of $B = \Sigma \times [0, 1] \setminus (N_\alpha \times [0, 1/4] \cup N_\beta \times [3/4, 1])$, where N_α, N_β are annular neighbourhoods of simple closed curves α, β on Σ with $i(\alpha, \beta) = 2$ if Σ is a sphere with four holes and $i(\alpha, \beta) = 1$ if Σ is a torus with one hole. We should also recall that the isometry type of B is unique once we fix Σ to be a sphere with four holes or a torus with one hole. A boundary block corresponds to a geometrically finite end of $(\mathbb{H}^3/G_i)_0$ as was explained in Section 2.6. We also have an embedding of \mathbf{M}_i into $S \times [0, 1]$ such that the product structure of each block coincides with that of $S \times [0, 1]$, in particular each horizontal surface in a block is mapped into a horizontal surface of the form $S \times \{t\}$. Recall each Σ is an incompressible subsurface of S . For each Σ , we fix an embedding of Σ into S by fixing some hyperbolic metric on S and making each boundary component of Σ geodesic if no two boundary components of Σ are parallel on S , and a simple closed curve with small constant geodesic curvature otherwise. By this way, the vertical projection of B into S is determined. We note that this hyperbolic metric on S has nothing to do with the metric which we endow on model manifolds.

After gluing blocks in accordance with the information given by a hierarchy of tight geodesics associated to G_i , we get a 3-manifold whose boundary consists of two surfaces homeomorphic to S and countably many tori. We denote by $\mathbf{M}_i[0]$ this manifold before filling Margulis tubes, which is a subset of \mathbf{M}_i . For each torus boundary, we fill in a Margulis tube, which is a solid torus whose isometry type is determined by the flat structure and the marking on the boundary torus which is determined by the hierarchy.

Now, we shall see that a geometric limit of these model manifolds serves as a model manifold of $(\mathbb{H}^3/G_\infty)_0$ once cusp neighbourhoods are removed.

LEMMA 4.3. — Put a basepoint x_i in $\mathbf{M}_i[0]$. Then $\{(\mathbf{M}_i, x_i)\}$ converges geometrically to a 3-manifold $\widehat{\mathbf{M}}$ consisting of internal and boundary blocks, Margulis tubes, and cusp neighbourhoods, after passing to a subsequence. By removing the cusp neighbourhoods, we get a brick manifold \mathbf{M} . The bi-Lipschitz model map $f_i : \mathbf{M}_i \rightarrow (\mathbb{H}^3/G_i)_0$ also converges geometrically to a bi-Lipschitz map whose restriction to \mathbf{M} is a bi-Lipschitz model map to $(\mathbb{H}^3/G_\infty)_0$.

Proof. — For internal blocks, their isometry types depend only on whether Σ is a sphere with four holes or a torus with one hole. Therefore, their geometric limits are also internal blocks. We now turn to boundary blocks. Since topologically a boundary block is homeomorphic to $\Sigma \times [0, \infty)$ or $\Sigma \times (-\infty, 0]$ for an incompressible subsurface Σ of S , we can assume, passing to a subsequence, Σ does not depend on i . A boundary block as we explained in Section 2.6 has a metric uniquely determined by the conformal structure at infinity of the corresponding geometrically finite end. In the case when the conformal structures at infinity are bounded (in the moduli space) as i varies, by taking a subsequence, we can assume that the conformal structures converge in the moduli space, and it is evident that the geometric limit of boundary blocks is again a boundary block corresponding to the limit conformal structure at infinity.

Now we consider the situation where the hyperbolic metric m_0^i on the geometrically finite end corresponding to $\Sigma \times \{\infty\}$ varies with i and goes to an end in the moduli space. Let c_1, \dots, c_p be a shortest pants decomposition of (Σ, m_0^i) . Since $\{m_0^i\}$ is unbounded in the moduli space, the lengths of some of c_1, \dots, c_p with respect to m_0^i go to 0 as $i \rightarrow \infty$. By renumbering c_1, \dots, c_p , we can assume c_1, \dots, c_q to be the curves whose lengths go to 0. If the length of c_k with respect to m_0^i goes to 0, then the modulus of the annulus $A(c_k)$, whose boundaries we assumed to have length ϵ_2 in Section 2.6, goes to ∞ , and hence in a geometric limit, the surface Σ is cut along c_k . If we put a basepoint on $\Sigma \times \{-1\} \setminus (\bigcup_{k=1}^p A(c_k) \times \{-1\})$, then the boundary blocks have a geometric limit after passing to a subsequence, and the limit is identified with $(\Sigma \setminus \bigcup_{k=1}^q c_k) \times [-1, \infty) \setminus (\bigcup_{k=q+1}^p A(c_k) \times [-1, 0])$. By Lemma 3.4 in Minsky [40], there is a uniform bi-Lipschitz map (i.e. with bi-Lipschitz constants independent of i) from this boundary block to a component of the complement of the convex core of any hyperbolic structure on $\Sigma \times \mathbb{R}$ whose conformal structure at infinity is m_0^i , with tubular neighbourhoods of closed geodesics corresponding to the pants decomposition c_1, \dots, c_p removed. This implies that the geometric limit of boundary blocks is again a boundary block after its intersection with ϵ_0 -cusp neighbourhoods is removed.

Next we turn to Margulis tubes. Let V_i be a Margulis tube in the model manifold \mathbf{M}_i . We can parametrise \mathbf{M}_i by $S \times [0, 1]$ in such a way that V_i

corresponds to $A_i \times [s, t]$ for some essential annulus A_i on S . Each Margulis tube V_i has a coefficient $\omega_{\mathbf{M}_i}(V_i)$ which is defined as follows. The boundary of the tube ∂V_i has a flat metric induced from the metric of $\mathbf{M}_i[0]$ determined by the blocks. We can give a marking (longitude-meridian system) (α_i, β_i) to ∂V_i by defining α_i to be a horizontal curve in a block and β_i to be $\partial(a_i \times [s, t])$ for some essential arc a_i connecting the two boundary components of A_i . The flat metric and the marking determine a point in the Teichmüller space of a torus identified with $\{z \in \mathbb{C} \mid \Im z > 0\}$, which we define to be $\omega_{\mathbf{M}_i}(V_i)$. More concretely, we define the point $z_i = \omega_{\mathbf{M}_i}(V_i)$ in the upper half plane to be the point corresponding to ∂V_i if ∂V_i with marking (α_i, β_i) is conformal to $\mathbb{C}/(\mathbb{Z} + z_i\mathbb{Z})$ taking α_i to the curve coming from the first \mathbb{Z} and β_i to the curve coming from $z_i\mathbb{Z}$. We note that this parametrisation is continuous with respect to the geometric topology.

By our construction of metrics on blocks $\Im\omega_{\mathbf{M}_i}(V_i)$ is bounded away from 0. If $|\omega_{\mathbf{M}_i}(V_i)|$ is bounded, then passing to a subsequence, we can assume that $\{\omega_{\mathbf{M}_i}(V_i)\}$ converges to some number w , hence that $\{V_i\}$ converges to some Margulis tube whose coefficient is w . If $\Re\omega_{\mathbf{M}_i}(V_i) \rightarrow \infty$ whereas $\Im\omega_{\mathbf{M}_i}(V_i)$ is bounded, then the conformal structure on a torus corresponding to $\omega_{\mathbf{M}_i}$ is bounded in the moduli space as $i \rightarrow \infty$. Therefore, ∂V_i converges some flat torus passing to a subsequence. On the other hand, since $\Re\omega_{\mathbf{M}_i}(V_i)$ diverges, the length of the meridian for V_i diverges. This means that passing to a subsequence, $\{V_i\}$ converges to a torus cusp neighbourhood. If $\Im\omega_{\mathbf{M}_i}(V_i) \rightarrow \infty$, passing to a subsequence, the conformal structure on a torus corresponding to $\omega_{\mathbf{M}_i}(V_i)$ diverges in the moduli space. Since the length of longitude of ∂V_i is defined to be constant, this means that ∂V_i converges to an open annulus geometrically. This is possible only when $\{V_i\}$ converges geometrically to one or two rank-1 cusp neighbourhoods. Since we are considering the non-cuspidal part of the geometric limit, these cusp neighbourhoods are removed and what remain are torus or annulus boundaries. (What should be removed is an ϵ_0 -cusp neighbourhood, whereas the limit is an ϵ_2 -cusp neighbourhood. This does not matter since they are constant only depending on S and we can map the ϵ_2 -cusp neighbourhood to the ϵ_0 -cusp neighbourhood by a uniform bi-Lipschitz map.)

Thus we have seen that the non-cuspidal part of the geometric limit of \mathbf{M}_i with basepoint at x_i , which is defined to be \mathbf{M} , is a union of (internal and boundary) blocks and Margulis tubes. We denote by $\mathbf{M}[0]$ the part of \mathbf{M} which is the geometric limit of $\mathbf{M}_i[0]$ inside \mathbf{M} . Now, we shall see how a structure of brick decomposition of the geometric limit appears. Since the blocks of $\mathbf{M}_i[0]$ are embedded and they are pasted together along horizontal surfaces, we see that the notion of horizontal surfaces (and of horizontal foliation) makes sense in $\mathbf{M}_i[0]$. Taking their geometric limit, we see that

$\mathbf{M}[0]$ also has well-defined horizontal surfaces. Each Margulis tube of \mathbf{M}_i is foliated by horizontal annuli. Since Margulis tubes filled into $\mathbf{M}[0]$ are geometric limits of Margulis tubes in \mathbf{M}_i , they are also foliated by horizontal annuli. Therefore, horizontal surfaces intersecting a Margulis tube extend to annuli in the tube to form horizontal surfaces corresponding to subsurfaces of S . Taking geometric limits of these surfaces, we see that also in the geometric limit, horizontal surfaces intersecting Margulis tubes can be extended to annuli in the Margulis tubes to form subsurfaces of S , which we regard as horizontal surfaces. We consider a maximal union of vertically parallel horizontal surfaces, and define its closure to be a brick of \mathbf{M} . It is easy to see that two bricks can intersect each other only along (a part of) horizontal boundaries, and that the intersection consists of essential subsurfaces by the fact that it is decomposed into horizontal surfaces lying in blocks and horizontal annuli in Margulis tubes. We can also check that there is no annulus component among the intersection, since every boundary component corresponds to a cusp limit of Margulis tubes, and no two such components are homotopic. Thus we can see that the geometric limit admits a structure of a brick manifold. As the restriction of a geometric limit of the model maps $f_i : \mathbf{M}_i \rightarrow (M_i)_0$ to the non-cuspidal part, we get a model map $f_\infty : \mathbf{M} \rightarrow (M_\infty)_0$. \square

Thus we have obtained \mathbf{M} and f_∞ and by construction the condition (0) holds. We now check that the conditions (1)–(6) of the statement hold. Since f_i is a homeomorphism and takes Margulis tubes whose cores are sufficiently short to the same kind of Margulis tubes in \mathbb{H}^3/G_i , there is a one-to-one correspondence between the Margulis tubes with short cores of \mathbf{M}_i and those of $(\mathbb{H}^3/G_i)_0$. Since there are no two distinct Margulis tubes in $(\mathbb{H}^3/G_i)_0$ whose cores are homotopic, the same holds for Margulis tubes with short cores in \mathbf{M}_i . The condition (2) is derived from this property since two homotopic longitudes on $\partial\mathbf{M}$ give rise to two Margulis tubes with homotopic very short core curves for large i .

Next we turn to the condition (3). Let B be a brick in \mathbf{M} , and suppose that B has an end e . If the end e is contained in a boundary block of \mathbf{M} , we define e to be geometrically finite. Since f_i takes the end of every boundary block of \mathbf{M}_i to a geometrically finite end of M_i and a boundary block of \mathbf{M} is a geometric limit of boundary blocks of \mathbf{M}_i , as was seen in the proof of Lemma 4.3 above, which must also correspond to a geometrically finite end of $(M_\infty)_0$, the limit f also takes e to a geometrically finite end of $(M_\infty)_0$.

If e in B is not contained in a boundary block, we define e to be simply degenerate. Such an end appears only when there are infinitely many blocks constituting $B \cong \Sigma \times J$. This implies that there are infinitely many Margulis tubes tending to the end e . Since each core curve of Margulis tube is

homotopic to a simple closed curve on Σ , and f takes such a core curve to a closed geodesic in M_∞ , we see that the end of $(M_\infty)_0$ corresponding to e is simply degenerate.

Now we check the condition (5). Let e be an end of \mathbf{M} which is not contained in a brick. We need to show the following.

LEMMA 4.4. — *There is no essential half-open annulus whose end tends to the end e .*

Proof. — We prove this by contradiction. Suppose that A is an essential half-open annulus tending to e . Then A intersects infinitely many bricks B_n tending to e . Since the B_n are distinct, by using the condition (2), we can choose a vertical boundary of B_n with core curve c_n so that the $f(c_n)$ represent all distinct homotopy classes in M_∞ . Moreover the distance between the basepoint y_∞ and the closed geodesic homotopic to $f(c_n)$ goes to ∞ as $n \rightarrow \infty$. Let a be a core curve of A . Then a and c_n can be realised as disjoint curves on a horizontal surface on which c_n lies. Pulling back this situation to \mathbf{M}_i and M_i , we see that there is a pleated surface $k_n^i : S \rightarrow M_n$ homotopic to $f_i|S \times \{t\}$ which realises both a and c_n at the same time. This is impossible since the distance modulo the thin part between $k_n^i(a)$ and the one representing $k_n^i(c_n)$ goes to ∞ , which contradicts the compactness of pleated surfaces. (When $f(a)$ represents a parabolic class, we need some more care, but essentially the same kind of argument works, for $f(a)$ is homotoped into a cusp which does not touch the end e in this case.) \square

What now remain are (1), (4), (6) and (7).

Before starting to prove them, we shall describe general settings. First, we consider a monotone increasing exhaustion of \mathbf{M} by connected submanifolds consisting of finitely many bricks; that is, a sequence of submanifolds $\mathbf{N}_1 \subset \mathbf{N}_2 \subset \dots$ such that $\mathbf{M} = \bigcup_n \mathbf{N}_n$, each \mathbf{N}_n is a connected union of finitely many bricks in \mathbf{M} , and every brick of \mathbf{N}_n is attached to \mathbf{N}_{n-1} at either its upper front or lower front or both. (See Definition 4.1 for definitions of upper and lower fronts.) Each compact brick of \mathbf{N}_n is contained in the range of the approximate isometry from \mathbf{M}_i to \mathbf{M} for large i , and hence we can embed it preserving the vertical and horizontal foliations. In the same way, for each non-compact brick B of \mathbf{N}_n , its real front, which we denote by Σ , is contained in the range of the approximate isometry for large i . Since B is homeomorphic to either $\Sigma \times [0, 1)$ or $\Sigma \times (0, 1]$, by embedding its real front using the embedding of \mathbf{M}_i and the approximate isometry, we can embed B into $S \times [0, 1]$, again preserving the vertical and horizontal foliations. We can also adjust the image of its ideal front so that the closures of the images of two bricks do not intersect, i.e. so that even two ideal fronts do not intersect. Since \mathbf{N}_n has only finitely many bricks, by taking sufficiently large i , we can

embed \mathbf{N}_n into $S \times [0, 1]$ using these embeddings of its bricks induced by the embedding of \mathbf{M}_i into $S \times [0, 1]$, keeping the horizontal and vertical foliations. We denote this embedding of \mathbf{N}_n by η_n .

Passing to a subsequence and isotoping the η_n vertically, we can assume that for each brick B of \mathbf{M} , the horizontal level of $\eta_n(B)$ is independent of n if it is defined, without changing the condition that the upper ends of the upper boundary blocks lie on $S \times \{1\}$ and the lower ends of the lower boundary blocks lie on $S \times \{0\}$. Here we call a boundary block of the form $\Sigma \times [s, t]$ an upper and that of the form $\Sigma \times (s, t]$ lower, and its end is called an upper end and a lower end respectively. Now for each brick B of \mathbf{M} , let $\sup B$ and $\inf B$ be the levels of $\eta_n(\partial_+ B)$ and $\eta_n(\partial_- B)$ with respect to the second factor of $S \times [0, 1]$ (which we call *horizontal levels* from now on) under an embedding η_n for which $\eta_n(B)$ is defined. As noted above, these are independent of n . We consider the set $C = \{\sup B, \inf B \mid B \text{ are bricks in } \mathbf{M}\} \subset [0, 1]$ and its closure \bar{C} . We can further perturb η_n (for all n simultaneously) so that two distinct bricks share \inf or \sup only at 0 and 1.

For each $t \in [0, 1]$, we consider the surface $\text{Int}\eta_n(\mathbf{N}_n) \cap S \times \{t\}$, which we regard as a subsurface of S identifying $S \times \{t\}$ with S , and we denote it by $D_{\eta_n}(t)$. Since bricks are embedded with horizontal foliations preserved, $D_{\eta_n}(t)$ is an incompressible subsurface of S and there are no components of $D_{\eta_n}(t)$ which are discs or annuli. For convenience, we fix a hyperbolic structure on S , and we can assume that each boundary component of $D_{\eta_n}(t)$ either is geodesic or has small curvature, so that if two of such surfaces are homotopic, then they are vertically parallel to each other. Since there are neither annuli nor discs, for any $t \in [0, 1]$, the Euler characteristic of $D_{\eta_n}(t)$ is monotone non-increasing with respect to n . Therefore, the homeomorphism type of $D_{\eta_n}(t)$ does not change for large n . Furthermore, since there are only finitely many ways to embed a surface essentially into S up to automorphisms of S , we can assume that there is n_0 such that the homeomorphism type of $(S, D_{\eta_n}(t))$ as a pair does not change for $n \geq n_0$. We say that $D_{\eta_n}(t)$ is *stable* in this situation. For stable $D_{\eta_n}(t)$, we denote its Euler characteristic, which is independent of n , by $\chi_{\text{stab}}(D(t))$.

Now we start to prove (6). Since an end either lies in a brick or is a limit of bricks, it always corresponds to a number in \bar{C} . (Although we have not assumed that \mathbf{M} is embedded in $S \times [0, 1]$ at this stage, the horizontal level of an end makes sense because the horizontal levels of each brick are well defined.) Moreover, since there is a uniform bound for the number of components of $D_{\eta_n}(t)$, and one can approach $S \times \{t\}$ only from two sides, from above and from below, there is a uniform bound for the number of ends corresponding to $t \in \bar{C}$. Therefore, to show (6), what we need to show is the following.

LEMMA 4.5. — *The set \bar{C} is countable.*

Let t_∞ be an accumulation point of C . Then we can easily see the following. We define two subsurfaces $\Sigma_{\eta_n}^-(t_\infty)$ and $\Sigma_{\eta_n}^+(t_\infty)$ of S to be the sets of limit points of $S \setminus D_{\eta_n}(x - \epsilon)$ and $S \setminus D_{\eta_n}(x + \epsilon)$ as $\epsilon \searrow 0$ respectively. We denote the complements of $\Sigma_{\eta_n}^-(t_\infty)$ and $\Sigma_{\eta_n}^+(t_\infty)$ by $D_{\eta_n}^+(t_\infty)$ and $D_{\eta_n}^-(t_\infty)$ respectively.

CLAIM 4.6. — *There is $\delta > 0$ such that for any $t \in [t_\infty - \delta, t_\infty) \cup (t_\infty, t_\infty + \delta]$ we have $\chi_{\text{stab}}(D(t)) \leq \chi_{\text{stab}}(D(t_\infty))$ and $D_{\eta_n}(t_\infty)$ is vertically isotopic into $D_{\eta_n}(t)$ in $S \times [0, 1]$ for sufficiently large n . Moreover, for any $t \in [t_\infty - \delta, t_\infty)$, the surface $D_{\eta_n}^-(t_\infty)$ is vertically homotopic into either $D_{\eta_n}(t)$, and for any $t \in (t_\infty, t_\infty + \delta]$, the surface $D_{\eta_n}^+(t_\infty)$ is vertically homotopic into $D_{\eta_n}(t)$.*

Proof. — By the same reason as the argument just before the lemma, there are only finitely many bricks of \mathbf{M} the closures of whose images under η_n intersect the level surface $S \times \{t_\infty\}$, which we name $\{B_j^{t_\infty}\}$. (Again this property does not depend on n , and that both $\inf \eta_n(B_j^{t_\infty})$ and $\sup \eta_n(B_j^{t_\infty})$ are independent of n .) We set $t^+ = \min_j \{\sup B_j^{t_\infty} \mid \sup B_j^{t_\infty} \geq t_\infty\}$ and $t^- = \max_j \{\inf B_j^{t_\infty} \mid \inf B_j^{t_\infty} \leq t_\infty\}$. Then δ which is defined to be $\min\{t_+ - t_\infty, t_\infty - t_-\}$ has the desired property. \square

Proof of Lemma 4.5. — By Claim 4.6, for any $t \in [t_\infty - \delta, t_\infty)$, the surface $D_{\eta_n}^-(t_\infty)$ is vertically homotopic into $D_{\eta_n}(t)$. Moreover, if t_∞ is an accumulation point from below, since $\eta_n(\mathbf{N}_n)$ is connected and bricks intersect only along fronts, there is no $t \in [t_\infty - \delta, t_\infty)$ such that $\chi_{\text{stab}}(D_{\eta_n}(t)) = \chi(D_{\eta_n}^-(t_\infty))$, and hence for any $t \in [t_\infty - \delta, t_\infty)$, we have $\chi_{\text{stab}}(D_{\eta_n}(t)) < \chi(D_{\eta_n}^-(t_\infty))$. Therefore if there is an accumulation point t of C in $[t_\infty - \delta, t_\infty)$, then $\max\{\chi(D_{\eta_n}^-(t)), \chi(D_{\eta_n}^+(t))\} < \chi(D_{\eta_n}^-(t_\infty))$ for large n . The same holds when t_∞ is an accumulation point from above just changing $-$ to $+$. Repeating the same argument using Claim 4.6, we can take a neighbourhood $[t - \delta_t, t + \delta_t]$ such that for any $t' \in [t - \delta_t, t + \delta_t]$, we have $\max\{\chi(D_{\eta_n}^-(t')), \chi(D_{\eta_n}^+(t))\} < \max\{\chi(D_{\eta_n}^-(t)), \chi(D_{\eta_n}^+(t))\}$. Since $\max\{\chi(D_{\eta_n}^-(t)), \chi(D_{\eta_n}^+(t))\}$ is between $\chi(S)$ and 0, in finite steps, we reach a situation where there are no accumulation points in a neighbourhood. Thus we have shown that there are only countably many accumulation points of C , and hence \bar{C} is countable. \square

Thus we have proved (6).

For later use, using (6), we modify η_n further as follows. If there is a brick B whose front (real or ideal) is mapped by η_n into $S \times \{t\}$ for an accumulation point t of C , we can perturb $\eta_n|B$ off from $S \times \{t\}$ for all n

at the same time since there are only countably many accumulation points. In the same way, if there are two sequence of bricks $\{B_j\}$ and $\{B'_j\}$ such that $\lim_j \sup B_j = t = \lim_j \inf B'_j$, and if $\sup B_i$ converges to t from below whereas $\inf B'_i$ converges from above, then by using Claim 4.6, we can perturb $\eta_n|_{B'_n}$ so that $\lim_i \sup B'_j < t$ holds.

Thus we can assume the following.

ASSUMPTION 4.7. — *No point of C is an accumulation point of C . To each accumulation point of C points of C accumulate either only from below or only from above and cannot accumulate from both sides at the same time.*

Next we turn to the most difficult one, the condition (1), whose proof was given in [50, §3.1]. The subtle point is that we cannot construct an embedding of \mathbf{M} into $S \times [0, 1]$ simply as a limit of the embeddings η_n of \mathbf{N}_n . This is because for distinct m and n , the images of a brick by η_m and η_n may be different even homotopically. Therefore, instead of taking a limit of η_n , we construct embeddings h_n of \mathbf{N}_n inductively which stabilise if we restrict them to each brick. The embedding h_n is not an extension of the previous h_{n-1} , but an extension of an embedding obtained by “twisting” h_{n-1} . We shall use the original embeddings η_n in a step of induction to define “twisting maps”. Although h_n and η_n send each brick to the same horizontal levels, there is no simple direct relation between the two.

First we shall explain intuitively how to construct h_n provided h_{n-1} is already defined. We consider a decomposition of $[0, 1]$ into subintervals by setting the set of the sup and the inf of the bricks of \mathbf{N}_n to be the dividing points. Then we subdivide the bricks of \mathbf{N}_n so that each brick is contained in the product of S and one of the above subintervals. We give an order, defined using the horizontal levels, to the set \mathbf{B}_n of bricks (after the subdivision) which are contained in \mathbf{N}_n but not in \mathbf{N}_{n-1} . Let B be a brick in this set, and suppose that we have already defined h_n for bricks in \mathbf{B}_n which are smaller with respect to the given order, and denote this embedding by h_n^B and the union of bricks in \mathbf{B}_n smaller than B by $\mathbf{B}(B)$. We cannot always extend h_n^B to B since even if both $\partial_- B$ and $\partial_+ B$ are contained in $\mathbf{N}_{n-1} \cup \mathbf{B}(B)$, their images under h_n^B may not be vertically isotopic. (If they are vertically isotopic, then we can define $h_n|_B$ simply by extending h_n^B so that the horizontal level of $h_n(B)$ is the same as that of $\eta_n(B)$.) We shall avoid these difficulties by composing to h_n^B what we call a solid twisting map, which is a map obtained by cutting $S \times [0, 1]$ at two levels $\Sigma \times \{a\}$ and $\Sigma \times \{c\}$ with an essential subsurface Σ of S , and twisting the part $\Sigma \times [a, c]$ by using a homeomorphism from Σ to Σ fixing the boundary.

Now we start a formal discussion. We let C_n be a subset of C defined by $C_n = \{\sup B', \inf B' \mid \text{the } B' \text{ are bricks in } \mathbf{N}_n\}$, and number the elements

of C_n as a_1, \dots, a_s . We subdivide the brick decomposition of \mathbf{N}_n by cutting them along the level surfaces corresponding to $S \times \{a_j\}$ for $j = 1, \dots, s$ via η_n . By this operation, the image of each brick in \mathbf{N}_n under η_n is contained in $S \times [a_j, a_{j+1}]$ for some $j = 1, \dots, s$, where we set a_0 to be 0 and a_{s+1} to be 1. For each a_j among a_1, \dots, a_s , we shall define a twisting map $\varphi_{a_j} : S \rightarrow S$, which is a homeomorphism, and a solid twisting map $\widehat{\varphi}_{a_j} : S \times [0, 1] \rightarrow S \times [0, 1]$ for $j = 1, \dots, s$, which is discontinuous only along at most two horizontal subsurfaces embedded in $S \times [0, 1]$. Depending on the location of a_j , the map $\widehat{\varphi}_{a_j}$ twists either a compact submanifold of $S \times [0, 1]$ above $S \times \{a_j\}$ or below $S \times \{a_j\}$ and leaves the remaining part of $S \times [0, 1]$ unchanged. To determine which region we twist, we first need to define an accumulation point of C to which a_j “belongs”.

Recall that a subsurface $D_{\eta_n}(x)$ of S was defined for any $x \in [0, 1]$ and $n \in \mathbb{N}$. Suppose that d is an accumulation point of C . By Assumption 4.7, we see that d is not contained in C and that C accumulates to d either from above or from below, not from both at the same time. We set $\Sigma_{\eta_n}(d)$ to be the complement of $D_{\eta_n}(d)$ on S . By Claim 4.6, after passing to a subsequence of $\{\mathbf{N}_n\}$, we can find $\delta > 0$ such that for any n and $x \in [d - \delta, d + \delta]$, the subsurface $\Sigma_{\eta_n}(x)$ is vertically isotopic into $\Sigma_{\eta_n}(d)$. Furthermore, by Assumption 4.7, by taking smaller δ if necessary, we can assume the following.

ASSUMPTION 4.8. — *On the side from which C does not accumulate, $\Sigma_{\eta_n}(x)$ is vertically parallel to $\Sigma_{\eta_n}(d)$ for every $x \in [d - \delta, d + \delta]$ and large n .*

We note that δ above may depend on d but not on n if it is large enough. Now since there are only finitely many bricks whose images under the η_n intersect $S \times \{d\}$, by taking δ small enough, changing the order of appearance of finitely many bricks and passing to a subsequence, we can assume the following.

ASSUMPTION 4.9. — *For every accumulation point d of C and $\delta > 0$ as above, the first brick put in $S \times (d - \delta, d)$ or $S \times (d, d + \delta)$ appears after all bricks intersecting $S \times \{d\}$, that is, if a brick B with $\eta_n(B) \subset S \times (d - \delta, d) \cup S \times (d, d + \delta)$ is contained in \mathbf{N}_n , for any B' with $\eta_m(B') \cap S \times \{d\} \neq \emptyset$, there is $j < n$ with $B' \subset \mathbf{N}_j$.*

Let $\{d_i\}$ be the set of accumulation points of C , and for each d_i , we take δ_i so that $[d_i - \delta_i, d_i) \cup (d_i, d_i + \delta_i]$ has the properties of Claim 4.6 and Assumptions 4.8 and 4.9. We set I_i to be $(d_i - \delta_i, d_i) \cup (d_i, d_i + \delta_i)$. We note that it may be possible that two distinct I_{i_1} and I_{i_2} intersect, but we can choose smaller δ_i if necessary so that if $I_{i_1} \cap I_{i_2} \neq \emptyset$, then one of the two contains the other. By cutting bricks along $S \times \{d_i - \delta_i\}$ and $S \times \{d_i + \delta_i\}$, we can assume the following:

ASSUMPTION 4.10. — *For a brick B , if one of its sup and inf is contained in I_i , then the other is contained in its closure \bar{I}_i .*

We say that a point $a \in C$ belongs to an accumulation point d_i if a is contained in I_i and if I_i is the smallest with respect to the inclusion among the intervals I_i containing a . We say that a brick B belongs to d_i if \bar{I}_i is the smallest among the intervals $\{\bar{I}_i\}$ that contains both sup B and inf B .

Let a be a point in $C_n \setminus \{0, 1\}$. We say that a point $a \in C_n$ belonging to d_i is an *lower twisting point* when a lies in $(d_i - \delta_i, d_i)$, and a *upper twisting point* otherwise. Since there are only finitely many points of C which are contained in none of the I_i , by passing to a subsequence, we can assume the following.

ASSUMPTION 4.11. — *If a brick B has inf and sup neither of which contained in any I_i , then B is contained in \mathbf{N}_1 .*

Twisting maps (or solid twisting maps) at lower twisting points, which we shall define below, will be called lower (solid) twisting maps, and those at upper twisting points upper (solid) twisting maps. We shall define a new embedding h_n for \mathbf{N}_n for every $n \in \mathbb{N}$ inductively, starting from h_1 , which is set to be η_1 . The embedding h_n will be defined to each brick with the same sup and inf as the original embedding η_n , but our construction is based on the induction, and η_n will be used only to define twisting maps. Suppose that the embedding h_{n-1} is defined on \mathbf{N}_{n-1} in such a way that h_{n-1} sends each brick of \mathbf{N}_{n-1} to the same horizontal levels as η_{n-1} (and hence also η_n) does. We now start to define h_n on \mathbf{N}_n .

Let \mathbf{B}_n be the set of bricks contained in \mathbf{N}_n but not in \mathbf{N}_{n-1} . We let C'_n be a subset of C_n defined by $C'_n = \{\text{sup } B, \text{inf } B \mid B \in \mathbf{B}_n\} \cap (\bigcup_j I_j)$. We consider accumulation points of C to which some point of C'_n belongs. Since \mathbf{N}_n has only finitely many bricks, there are only finitely many such accumulation points. We renumber such accumulation points and the corresponding intervals as d_1, \dots, d_r and I_1, \dots, I_r in such a way that if I_{j_1} is (properly) contained in I_{j_2} for $d_{j_1}, d_{j_2} \in \{d_1, \dots, d_r\}$, then $j_1 > j_2$.

Starting from I_1 , and in the order according to the subscripts, we shall define twisting maps for each brick B in \mathbf{B}_n which belongs to d_j , and h_n on B after twisting the embedding defined up to that point. We let \mathbf{N}_n^{j-1} be the union of bricks of \mathbf{N}_n and bricks in \mathbf{B}_n belonging to d_1, \dots, d_{j-1} . Now, assuming that we have already defined the embedding h_n^{j-1} for \mathbf{N}_n^{j-1} , we shall define an embedding h_n^j for \mathbf{N}_n^j , which is not necessarily an extension of h_n^{j-1} . By Assumption 4.8, either all points of C'_n belonging to d_j are upper twisting points or all of them are lower twisting points. Since the argument is quite the same for both cases, we here assume that they are lower twisting

points. Let $\{a_1, \dots, a_p\}$ be the points in C'_n belonging to d_j , aligned in the increasing order. We shall proceed again inductively to define a lower twisting map φ_{a_k} for each a_k among a_1, \dots, a_p and the corresponding lower solid twisting map $\widehat{\varphi}_{a_k} : S \times [0, 1] \rightarrow S \times [0, 1]$

Let a_k be one among $\{a_1, \dots, a_p\}$, and suppose that we have already constructed solid twisting maps $\widehat{\varphi}_{a_1}, \dots, \widehat{\varphi}_{a_k}$ in such a way that $\widehat{\varphi}_{a_k} \circ \dots \circ \widehat{\varphi}_{a_1} \circ h_n^{j-1}$ extends to an embedding $h_n^{j-1}(k-1)$ of the union of \mathbf{N}_n^{j-1} and the bricks in \mathbf{B}_n whose suprema are in $\{a_1, \dots, a_{k-1}\}$, which we denote by $\mathbf{N}_n^{j-1}(k-1)$. We let B_{l_1}, \dots, B_{l_t} be the bricks with suprema at a_k which are contained in either \mathbf{B}_n or \mathbf{N}_{n-1} in such a way that their images under $h_n^{j-1}(k-1)$ are contained in $\Sigma_{h_n^{j-1}(d_j)} \times [0, a_k]$. We now define a lower twisting map $\varphi_{a_k} : S \rightarrow S$ at a_k , which is supported on $\Sigma_{h_n^{j-1}(k-1)}(d_j)$ and the corresponding lower solid twisting map $\widehat{\varphi}_{a_k}$ which is supported in $\Sigma_{h_n^{j-1}(k-1)}(d_j) \times [a_k, d_j]$.

Since both η_n and $h_n^{j-1}(k-1)$ embed $\mathbf{N}_{n-1}^j(k-1)$ into $S \times [0, 1]$ and they send each brick at the same horizontal level, there is a homeomorphism $\zeta_n^j(k) : h_n^{j-1}(k-1)(\mathbf{N}_{n-1}^j(k-1)) \rightarrow \eta_n(\mathbf{N}_{n-1}^j(k-1))$ preserving the horizontal levels. Since η_n also embeds $\mathbf{N}_{n-1}^j(k-1) \cup (B_{l_1}, \dots, B_{l_t})$, for each l among l_1, \dots, l_t , its bottom $\eta_n(\partial_- B_l)$ is vertically homotopic to its top $\eta_n(\partial_+ B_l)$ if $\partial_+ B_l$ is contained in $\mathbf{N}_{n-1}^j(k-1)$. Therefore there is a homeomorphism $\varphi_{a_k} : S \rightarrow S$ which is supported on $\Sigma_{h_n^{j-1}(k-1)}(d_j)$ such that $h_n^{j-1}(k-1)(\partial_- B_l)$ is vertically homotopic to $h_n^{j-1}(k-1)(\partial_+ B_l)$ if $\partial_+ B_l$ is contained in $\mathbf{N}_{n-1}^j(k-1)$. We let this φ_{a_k} be the lower twisting map at a_k , and define the corresponding lower solid twisting map $\widehat{\varphi}_{a_k} : S \times [0, 1] \rightarrow S \times [0, 1]$ to be $(\varphi_{a_k}(x), t)$ for $(x, t) \in \Sigma_{h_n^{j-1}(k-1)}(d_i) \times [a_k, d_j]$ and the identity elsewhere. Then $\widehat{\varphi}_{a_k} \circ h_n^{j-1}(k-1)$ extends to an embedding of $\mathbf{N}_{n-1}^j(k-1) \cup (B_{l_1}, \dots, B_{l_t})$. Repeating this construction inductively on a_1, \dots, a_p first then d_1, \dots, d_r next, we get an embedding h_n of \mathbf{N}_n .

We shall now show that for each brick B of \mathbf{M} , the restriction $h_n|_B$ stabilises for large n . (Here we are considering the original brick decomposition, not the subdivided one as the argument above.)

For a given brick B , take the smallest m such that B is contained in \mathbf{M}_m . We consider twisting maps at points $a_i \in C_{n_i}$ for some $n_i > m$ belonging to an accumulation point d_i , which appear in the construction of the embeddings above. We shall show that there are finitely many i and n_i for which the solid twisting map affects the embedding of B . We first consider the case where $h_m(B) \cap (S \times (a_i, d_i)) = \emptyset$ ($a_i < d_i$) or $h_m(B) \cap (S \times (d_i, a_i)) = \emptyset$ ($a_i > d_i$). For such a_i , the solid twisting map does not affect the embedding of B because the solid twisting map only moves $S \times (a_i, d_i)$ or $S \times (d_i, a_i)$ in this

case. Next suppose that either $h_m(B) \cap (S \times (a_i, d_i))$ or $h_m(B) \cap (S \times (d_i, a_i))$ is non-empty. Since the argument is the same for both cases, we only consider the case when a_i lies in $(d_i - \delta_i, d_i)$ and $h_m(B) \cap (S \times (a_i, d_i)) \neq \emptyset$. If d_i is contained in $(\inf B, \sup B]$, then $h_{n-1}(B) \cap S \times \{d_i\}$ is contained in $D_{h_{n-1}}(d_i)$, which implies that $h_{n-1}(B)$ is outside the support of the solid twisting map at a_i . Therefore, we can assume that $d_i > \sup B$. Since $a_i < \sup B < d_i$, if there are infinitely many such $a_i \in C_{n_i}$ with $n_i > m$, the sequence $\{a_i\}$ consisting of these points has an accumulation point $a_\infty \leq \sup B$. Since a_∞ is an accumulation point of C , there is j such that $a_\infty = d_j$. For sufficiently large i , the point a_i is contained in I_j . Since a_i belongs to d_i , this implies that I_i is contained in I_j . Since $d_i - \delta_i < a_i \leq d_j = a_\infty \leq \sup B < d_i$, we see that I_i , which contains $(d_i - \delta_i, d_i)$ cannot be contained in I_j which does not contain d_j . This is a contradiction. Thus, we have shown that there are only finitely many solid twisting maps which affect the embeddings of B . Therefore, $\{h_n\}$ stabilises on each brick for large n , and the limit of $\{h_n\}$ is a well-defined embedding of \mathbf{M} into $S \times [0, 1]$.

The condition (4) is much easier to see. As was shown in the proofs of (3) and (5), each geometrically finite end appears as a geometric limit of geometrically finite ends in boundary blocks of the \mathbf{M}_i . Since they can be assumed to lie on $S \times \{0, 1\}$ by the embedding h_i of \mathbf{M}_i constructed above, we can see the limit also lies on $S \times \{0, 1\}$.

We shall next check the condition (7). We fix a generator system of $\pi_1(S)$ so that none of the generators has image under ϕ_i which converges to a parabolic element in Γ as $i \rightarrow \infty$. Since $\{(G_i, \phi_i)\}$ converges algebraically, the geodesic loops representing the generators based at y_i have bounded lengths as $i \rightarrow \infty$. Therefore, we can construct a map $g_i : S \rightarrow M_i$ triangulated by ideal triangles whose image contains these geodesic loops so that $\{g_i\}$ converges geometrically to a map $\hat{g} : S \rightarrow M_\infty$ which can be lifted to a map from S to M' . Then $\{f_i^{-1} \circ g_i\}$ also converges to $f^{-1} \circ \hat{g}$, which we set to be g . By construction, g is π_1 -injective also as a map to $S \times [0, 1]$. Moreover, since $\hat{g} = f \circ g$ can be lifted to M' as a homotopy equivalence, we see that $(f \circ g)_\# \pi_1(S)$ corresponds to the image of $\pi_1(M')$.

Finally, we shall check the condition (8). Since we chose η_n so that no ideal front intersects another front in $S \times [0, 1]$, the embedding of \mathbf{M} which we constructed has the same property. Since we isotoped η_n so that there is no accumulation point of C contained in C itself and there is no accumulation point of C to which points of C accumulate from both below and above as in Assumption 4.7, there is no wild end which intersects a front of a brick or another wild end.

4.3. Block decomposition

Regard \mathbf{M} as a subset of $S \times [0, 1]$ which is embedded preserving the horizontal and vertical foliations as in Theorem 4.2. Then a brick of B has a form $\Sigma \times J$ with respect to the parametrisation of $S \times [0, 1]$. We denote $\sup J$ by $\sup B$ and $\inf J$ by $\inf B$ as in the previous subsection. Note that $\sup B$ is the level of the horizontal leaf on which the upper front of B lies and $\inf B$ that on which the lower front of B lies. Each end of \mathbf{M} , even if it is wild, corresponds to $\Sigma \times \{t\}$ for some incompressible subsurface Σ of S . By the condition (2), every geometrically finite end is contained in either $S \times \{0\}$ or $S \times \{1\}$. We call those contained in $S \times \{0\}$ *lower geometrically finite ends* and those in $S \times \{1\}$ *upper geometrically finite ends*. By moving the embedding vertically if necessary, we can assume that $S \times \{0\}$ and $S \times \{1\}$ consist of a union of ends, annuli homotopic in $S \times [0, 1] \setminus \text{Int}\mathbf{M}$ to the closure of open annulus boundary components of \mathbf{M} , and open annuli corresponding to punctures of S .

Occasionally, it is convenient to consider the complement of \mathbf{M} in $S \times (0, 1)$. Let C be a component of $S \times (0, 1) \setminus \mathbf{M}$. Then $\text{Fr}C \cap C$ consists of (countably many) horizontal surfaces, each corresponding to an end of \mathbf{M} which is either simply degenerate or wild. (Note that by the condition (8), each component of $\text{Fr}C \cap C$ corresponds to only one end of \mathbf{M} .) On the other hand, $\text{Fr}C \setminus C$ consists of either annuli or a single torus, which are boundary components of \mathbf{M} .

We can associate to each geometrically finite end of \mathbf{M} a marked conformal structure at infinity of the corresponding geometrically finite end of $(M_\infty)_0$. As was shown in the proof of Lemma 4.3, each geometrically finite end is contained in a geometric limit of boundary blocks of \mathbf{M}_i . Therefore, this conformal structure at infinity given on each geometrically finite end coincides with the conformal structure at infinity of the corresponding geometrically finite block. Similarly, we can associate to each simply degenerate end of \mathbf{M} the ending lamination of the corresponding simply degenerate end of $(M_\infty)_0$. We call these conformal structures and ending laminations *labels*.

In general a brick manifold each of whose non-wild ends has a label, an arational lamination for a simply degenerate end, and a conformal structure at infinity for a geometrically finite end, is called a *labelled brick manifold*. We showed in [50] that any labelled brick manifold is decomposed into blocks in the sense of Minsky [40] and tubes. This is just a generalisation of Minsky's construction of model manifolds in [40] based on hierarchies of tight geodesics defined in [38]. In particular in the case of Kleinian surface groups, our decomposition into blocks coincides with the construction of Minsky's model manifolds completely.

In our present situation, we do not need this general theory since $\mathbf{M}[0]$, which is a geometric limit of $\mathbf{M}_i[0]$, has a decomposition into blocks which is a limit block decomposition of $\mathbf{M}_i[0]$ as was stated in the part (0) of Theorem 4.2. We put a metric of a Margulis tube into each tube so that the flat metric induced on the boundary coincides with that induced from the metric on $\mathbf{M}[0]$ determined by blocks. As was explained in Proof of Theorem 4.2, each Margulis tube $V = A \times [s, t]$ has a coefficient $\omega_{\mathbf{M}}(V)$ lying in $\{z \in \mathbb{C} \mid \Im z > 0\}$. We define $\mathbf{M}[k]$ to be the complement of the tubes whose $\omega_{\mathbf{M}}$ have absolute value greater than or equal to k . By defining a brick to be the closure of a maximal union of parallel horizontal leaves, we can define a brick decomposition of $\mathbf{M}_i[k]$ and $\mathbf{M}[k]$. Such a brick decomposition is called the *standard brick decomposition*.

The following proposition was first shown in the proof of Theorem A in [50, §5.2]. The claims in this proposition were already proved in Theorem 4.2 above except for the existence of labels. The labels can be given by pulling back those on $(M_\infty)_0$ as we have just explained above.

PROPOSITION 4.12. — *In the situation of Theorem 4.2, let $x_i \in \mathbf{M}_i$ be a point in \mathbf{M}_i such that $f_i(x_i) = y_i$. Then $(\mathbf{M}_i[0], x_i)$ converges to $\mathbf{M}[0]$ after passing to a subsequence. The model manifolds \mathbf{M} and \mathbf{M}_i have structures of labelled brick manifolds admitting block decompositions with the following conditions.*

Let $\rho_i^{\mathbf{M}}$ be an approximate isometry between \mathbf{M}_i and the union of \mathbf{M} and cusp neighbourhoods, which corresponds to the geometric convergence. Then we can arrange $\rho_i^{\mathbf{M}}$ so that the following hold.

- (1) For any compact set K in \mathbf{M} , the restriction $\rho_i \circ f_i \circ (\rho_i^{\mathbf{M}})^{-1}|_K$ converges to $f|_K$ uniformly as $i \rightarrow \infty$.
- (2) For any block b of $\mathbf{M}[0]$, its pull-back $(\rho_i^{\mathbf{M}})^{-1}(b)$ is a block in $\mathbf{M}_i[0]$ for large i .
- (3) $\rho_i^{\mathbf{M}}$ preserves the horizontal foliations.

4.4. Algebraic limits in the models

By Theorem 4.2(7), there is an inclusion of $\pi_1(S)$ in $\pi_1(\mathbf{M})$ corresponding to the inclusion of Γ into G_∞ . We realise this inclusion by a π_1 -injective immersion $g : S \rightarrow \mathbf{M}$ so that $(f \circ g)_{\#}\pi_1(S)$ is equal to the image of $\pi_1(M')$ in $\pi_1(M_\infty)$ under the covering projection as in Theorem 4.2(7). We call such g an *algebraic locus*.

LEMMA 4.13. — *An algebraic locus g can be homotoped to a map g' as follows.*

- (1) *The surface S is decomposed into incompressible subsurfaces $\Sigma_1, \dots, \Sigma_m$, none of which is an annulus, and (possibly empty) annuli A_1, \dots, A_μ .*
- (2) *The restriction of g' to Σ_j is a horizontal embedding into $\Sigma_j \times \{t_j\} \subset \mathbf{M}$.*
- (3) *Each annulus $g'(A_j)$ is composed of $2n - 1$ horizontal annuli and $2n$ vertical annuli for some $n \in \mathbb{N}$ and goes around a torus boundary of \mathbf{M} n -times. See Figure 4.2.*

Proof. — Recall that g is π_1 -injective even as a map to $S \times [0, 1]$. Since every π_1 -injective map from S to $S \times [0, 1]$ is homotopic to a horizontal surface (see Waldhausen [57, Prop. 3.1]), g is homotopic to a horizontal surface $S \times \{t\}$ in $S \times [0, 1]$.

Since \mathbf{M} is a brick manifold, we can homotope g within \mathbf{M} so that $g(S)$ consists of horizontal leaves in bricks and vertical annuli. By the additivity of Euler characteristics, we see that the sum of the Euler characteristics of the horizontal leaves is equal to $\chi(S)$. We consider the projection of horizontal leaves to S . Since g is homotopic to $S \times \{t\}$ in $S \times I$, we see by the invariance of the algebraic intersection number that for each point $x \in S$ the surface $g(S)$ contains $x \times \{s\}$ for some $s \in I$. This implies that the horizontal leaves cannot overlap along a surface with negative Euler characteristic. It follows that only compact regions that $g(S)$ can bound in $S \times I$ are solid tori. If such a solid torus is contained in \mathbf{M} , we can eliminate it by a homotopy. The only remaining possibility is that such a solid torus contains components of $\partial\mathbf{M}$. By Theorem 4.2 (2), there is only one boundary component contained in each solid torus. Thus we have reached the situation as in our statement. \square

We call a map $g' : S \rightarrow \mathbf{M}$ as in Lemma 4.13 a *standard algebraic immersion*. Recall that there is a homotopy equivalence $\Phi_i : S \rightarrow M_i$ realising ϕ_i . By composing the inverse of the model map, we have a homotopy equivalence from S to \mathbf{M}_i , which we denote by $\Phi_i^{\mathbf{M}}$.

LEMMA 4.14. — *Let $\rho_i^{\mathbf{M}}$ be an approximate isometry between \mathbf{M}_i and \mathbf{M} with basepoints at the thick parts as in Theorem 4.2 and Proposition 4.12. For sufficiently large i , the immersion $(\rho_i^{\mathbf{M}})^{-1} \circ g'$ is homotopic to $\Phi_i^{\mathbf{M}}$ as a map to \mathbf{M}_i .*

Proof. — By the definition of g' , we see that $f \circ g'$ is homotopic to Ψ . Since $\rho_i^{-1} \circ \Psi$ is homotopic to Φ_i for large i , our lemma follows from the condition (1) of Proposition 4.12. \square

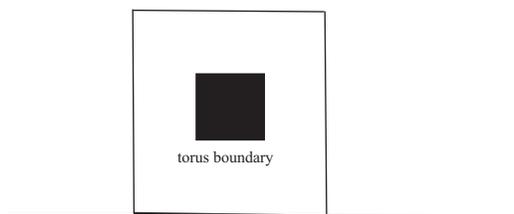


Figure 4.2. g' going around a torus boundary component (the case when $n = 1$).

DEFINITION 4.15. — Let e be a simply degenerate end of \mathbf{M} contained in a brick $B = \Sigma \times J$. We say that e is an upper algebraic end if $J = [s, s')$ and $\Sigma \times \{s' - \epsilon\}$ is freely homotopic in \mathbf{M} to $g'(\Sigma)$ which lies in a horizontal part of $g'(S)$ for sufficiently small $\epsilon > 0$. In the same way, we say that e is a lower algebraic end if $J = (s, s']$ and $\Sigma \times \{s + \epsilon\}$ is freely homotopic in \mathbf{M} to $g'(\Sigma)$ which lies in a horizontal part $g'(S)$ for sufficiently small $\epsilon > 0$. We also call the ending lamination of an algebraic simply degenerate end algebraic, and say that the ending lamination is upper or lower depending on whether the end is upper or lower.

Similarly, a core curve of an open annulus boundary component or a longitude (i.e. a horizontal curve) of a torus boundary component of \mathbf{M} is said to be an algebraic parabolic curve if it is homotopic in \mathbf{M} to a simple closed curve lying on a horizontal part of $g'(S)$. We also call its image of the vertical projection to S an algebraic parabolic curve. A parabolic curve is said to be upper or lower in the same way as simply degenerate ends, but we should note if it lies on a torus boundary component around which $g'(S)$ goes (by this we mean that $g'(S)$ goes around the torus boundary component k -times with $k \neq 0, -1$), then the curve is defined to be both upper and lower at the same time. An algebraic parabolic curve is said to be isolated if it is a core curve of an isolated parabolic locus of M'_0 .

LEMMA 4.16. — Any algebraic parabolic curve lying on a torus boundary component of \mathbf{M} is isolated.

Proof. — Let c be an algebraic parabolic curve lying on a torus boundary component of \mathbf{M} . Let P be a parabolic locus of a relative core C of $(M')_0$ into which the lift of $f(c)$ to M' is homotopic. Suppose, seeking a contradiction, that P is not isolated. Then, there is a simply degenerate end e touching the \mathbb{Z} -cusp corresponding to P . By Thurston's covering theorem (see [54] and Canary [18]) together with the argument in the proof of Lemma 2.3 in [44], there is a neighbourhood U of e which is projected in to M_∞ homeomorphically by the covering projection. This implies that \mathbf{M} has a simply degenerate brick which touches an open annulus boundary of \mathbf{M}

into which c is homotopic. Since no two distinct boundary components of \mathbf{M} have homotopic essential closed curves by Theorem 4.2(2), this contradicts the assumption that c lies on a torus boundary. \square

LEMMA 4.17. — *The algebraic simply degenerate ends of \mathbf{M} correspond one-to-one to the simply degenerate ends of M'_0 by mapping them by f and lifting them to M'_0 . The upper (resp. lower) ends correspond to upper (resp. lower) ends of M'_0 .*

Proof. — Consider a simply degenerate end corresponding to the upper ideal front of a brick $B = \Sigma \times [s, s']$ of \mathbf{M} . By the definition of the model map f , there is an infinite sequence of horizontal surfaces $\Sigma \times \{t_j\}$ in B tending to $\Sigma \times \{s'\}$ which are mapped to a sequence of pleated surfaces $f(\Sigma \times \{t_j\})$ tending to the corresponding simply degenerate end e of $(M_\infty)_0$. Since $\Sigma \times \{t\} \in B$ is freely homotopic into $g'(S)$, the pleated surfaces $f(\Sigma \times \{t_j\})$ lift to pleated surfaces \tilde{f}_i tending to an end of M'_0 .

Since the model map f has degree 1 with respect to the orientation of $(M_\infty)_0$, the end e is situated above $f \circ g'(S)$. Lifting this to M'_0 , we see that the end to which the \tilde{f}_i tend is an upper end. Similarly, we can show that if the simply degenerate brick B has the form $\Sigma \times (s, s']$, then the corresponding end of M'_0 is a lower end.

Conversely, suppose that e' is a simply degenerate end of M'_0 . By Thurston's covering theorem and some argument applying it ([18], [54] and the argument in the proof of Lemma 2.3 of [44]), there is a neighbourhood E of e' such that $p|_E$ is a proper embedding into $(M_\infty)_0$. Let \bar{e} denote the simply degenerate end of $(M_\infty)_0$ contained in $p(E)$. Then, there is a simply degenerate end \hat{e} of \mathbf{M} which is sent to \bar{e} by f . Since e' is simply degenerate, there is an essential subsurface Σ of S and a sequence of pleated surfaces $h_i : \Sigma \rightarrow M'_0$ taking $\partial\Sigma$ into $\partial M'_0$ which tend to e' . Their projections $p \circ h_i$ are pleated surfaces tending to \bar{e} . This implies that the end \hat{e} is contained in a simply degenerate brick $B_e \cong \Sigma \times J$, where J is a half-open interval. Since $f(\Sigma \times \{t\})$ is homotopic to $p \circ h_i$ and $p \circ h_i$ is homotopic into $f \circ g'(S)$, we see that $\Sigma \times \{t\}$ is freely homotopic to $g'(\Sigma)$. By Lemma 4.13, this is possible only when $\Sigma \times \{t\}$ is homotopic into a horizontal part of $g'(S)$, and we see that the end \hat{e} is algebraic. \square

4.5. Simply degenerate ends in limit models

As was shown in Theorem 4.2, except for the geometrically finite ends lying on $S \times \{0, 1\}$, all the tame ends of \mathbf{M} are simply degenerate. Next we shall see how simply degenerate ends in the model manifold \mathbf{M} are approximated in \mathbf{M}_i . Recall that the model manifold \mathbf{M}_i corresponds to a

hierarchy h_i of tight geodesics. Recall also that we have a homeomorphism $\widehat{\Phi}_i : S \times (0, 1) \rightarrow M_i$ inducing ϕ_i between the fundamental groups. This determines an embedding $\iota_i = f_i^{-1} \circ \widehat{\Phi}_i$ of the standard $S \times (0, 1)$ into $S \times [0, 1]$ in which \mathbf{M}_i is embedded. We identify the standard $S \times [0, 1]$ and $S \times [0, 1]$ in which \mathbf{M}_i is embedded so that this ι_i becomes an inclusion. In other words, by this identification, the model map f_i is homotopic to $\widehat{\Phi}_i$ if regarded as a map from $S \times (0, 1)$. We identify two $S \times [0, 1]$ in which \mathbf{M}_{i_1} and \mathbf{M}_{i_2} are embedded respectively for every pair i_1, i_2 using ι_{i_1} and ι_{i_2} .

We fix a complete marking μ on S once and for all. For a domain X in S , by considering a component of $\pi_X(\text{base}(\mu))$, we can define a basepoint in $\mathcal{CC}(X)$. We call this basepoint the *basepoint determined by μ* .

PROPOSITION 4.18. — *Let $B = \Sigma \times J$ be a simply degenerate brick in \mathbf{M} whose end e is algebraic. Then passing to a subsequence there is a tight geodesic γ_i contained in the hierarchy h_i (which was used to construct \mathbf{M}_i) as follows.*

- (1) *The support of γ_i is $\text{Int}\Sigma$.*
- (2) *Either all γ_i are geodesic rays, or they are finite geodesics whose lengths go to ∞ as $i \rightarrow \infty$.*
- (3) *In the case when the γ_i are geodesic rays, their endpoints at infinity converge to the ending lamination of e in $\mathcal{EL}(\text{Int}\Sigma)$ as $i \rightarrow \infty$, and γ_i contains a simplex whose distance from the basepoint determined by μ is bounded as $i \rightarrow \infty$.*
- (4) *Suppose that the γ_i are finite geodesics. In the case when the end e is upper, the last vertex of γ_i converges to the ending lamination of e as $i \rightarrow \infty$. In the case when the end is lower, the first vertex of γ_i converges to the ending lamination as $i \rightarrow \infty$. In both cases, γ_i contains a simplex whose distance from the basepoint determined by μ is bounded as $i \rightarrow \infty$.*

Before starting the proof of Proposition 4.18, we shall show the following lemma which is similar to Lemma 6.2 in Masur–Minsky [38].

LEMMA 4.19. — *There are constants M and P depending only on $\xi(S)$ with the following property. Let h be a hierarchy of tight geodesics on S , and D a domain (i.e. an open incompressible subsurface) of S . Suppose that there are two vertices v, w of $\mathcal{CC}(D)$ which are contained in simplices of tight geodesics constituting h such that $d_{\mathcal{CC}(D)}(v, w) > M$. Then there is a tight geodesic in h supported on D which contains simplices s_v and s_w such that $d_{\mathcal{CC}(D)}(v, s_v) \leq P$ and $d_{\mathcal{CC}(D)}(w, s_w) \leq P$.*

Proof. — Lemma 6.2 in Masur–Minsky [38] says that there is a constant M' depending only on $\xi(S)$ such that for if $d_{\mathcal{CC}(D)}(I(H), T(H)) > M'$ for some hierarchy H on S , then D supports a geodesic in H . Our lemma can be proved by repeating the argument of the proof Masur–Minsky’s lemma or modifying the hierarchy h so that we can apply Masur–Minsky’s lemma. We shall explain the latter here.

We set $M = M' + 2$ and let v, w be vertices in $\mathcal{CC}(D)$ as are given in the statement. Let σ_v and σ_w be simplices on geodesics of h containing v and w respectively. We construct a resolution of h , which we denote by $\tau = \{\tau_j\}$, where τ_j is a slice and j ranges in an interval in \mathbb{Z} . By the definition of resolutions, there are slices τ_{j_v} containing σ_v and τ_{j_w} containing σ_w . Since $d_{\mathcal{CC}(D)}(v, w) > 2$, they have non-zero intersection number, and hence cannot appear in the same slice, which implies that $j_v \neq j_w$. By interchanging v and w if necessary, we can assume that $j_v < j_w$.

Now, we consider a subsequence of the resolution defined to be $\tau' = \{\tau_j\}_{j_v \leq j \leq j_w}$. By the definition of elementary moves in a resolution (see [38, §5]), for any geodesic g in h , the simplices v on g such that (g, v) is contained in slices of τ' form a contiguous subset of the set of simplices on g . We denote this subset by $V(g)$, and a subgeodesic of g consisting of simplices contained in $V(g)$ by g' , which might be empty. Moreover, if $g_1 \searrow^d (g, v)$ or $(g, v) \swarrow^d g_1$ and $V(g_1) \neq \emptyset$, then (g, v) appears in some slice in τ' by the definition of slices. Therefore, the set of geodesics $\{g'\}_{g \in h}$ with the relation of subordination inherited from h forms a hierarchy on S by setting its initial marking to be τ_{j_v} and its terminal marking to be τ_{j_w} . (Strictly speaking, a geodesic of this hierarchy may have the first simplex or the last simplex which may not be vertices. This does not affect the argument for proving Lemma 6.2.) Let h' denote this hierarchy. Then we have $d_{\mathcal{CC}(D)}(I(h'), T(h')) = d_{\mathcal{CC}(D)}(\tau_{j_v}, \tau_{j_w}) \geq d_{\mathcal{CC}(D)}(v, w) - 2 > M'$. Now, by applying Masur–Minsky’s lemma for h' and D , we see that D supports a geodesic g'_D in h' . Since g'_D is a subgeodesic of a geodesic in h , we see that D supports a geodesic in h .

Now we shall show the existence of simplices s_v, s_w in g_D with the condition given in the statement. Lemma 6.1 of [38] implies that there is a constant M_1 depending only $\xi(S)$ which bounds both $d_{\mathcal{CC}(D)}(I(h'), I(g_D))$ and $d_{\mathcal{CC}(D)}(T(h'), T(g_D))$. We set P in the statement to be $M_1 + 1$. Let v_D be the first simplex of g'_D and w_D the last simplex of g_D . Since σ_v is contained in $I(h')$, we have $d_{\mathcal{CC}(D)}(\sigma_v, v_D) \leq M_1 + 1$. Since v_D is a simplex on g_D , this gives a bound as we wanted for v . In the same way, we have $d_{\mathcal{CC}(D)}(\sigma_w, w_D) \leq M_1 + 1$. This complete the proof. \square

Proof of Proposition 4.18. — Recall that $\{(\mathbf{M}_i, x_i)\}$ converges to the union of (\mathbf{M}, x_∞) and cusp neighbourhoods. We denote by $\rho_i^{\mathbf{M}}$ a (K_i, r_i) -approximate isometry associated to this convergence with domain $B_{r_i}(\mathbf{M}_i, x_i)$ as before. The intersection of the range of ρ_i and \mathbf{M} is the $K_i r_i$ -ball centred at x_∞ , which we denote by $B_{K_i r_i}(\mathbf{M}, x_\infty)$.

Since B is assumed to be simply degenerate brick, there is a sequence of Margulis tubes T_1, T_2, \dots appearing in the block decomposition of \mathbf{M} , which tend to the end of B and whose core curves projected into Σ , which we denote by c_1, c_2, \dots , converge to the ending lamination for the end. Passing to a subsequence, we can assume that $d_{\mathcal{CC}(\Sigma)}(c_{j_1}, c_{j_2}) \geq |j_1 - j_2|$ for any $j_1, j_2 \in \mathbb{N}$. For any $n \in \mathbb{N}$, we can take $i_0 \in \mathbb{N}$ such that for any $i \geq i_0$, the ball $B_{K_i r_i}(\mathbf{M}, x_\infty)$ contains all tubes T_1, \dots, T_n . Since the end of B is algebraic, the core curve of $(\rho_i^{\mathbf{M}})^{-1}(T_j)$ is homotopic in \mathbf{M}_i to c_j regarded as lying on $S \times \{1/2\}$. Since $(\rho_i^{\mathbf{M}})^{-1}(T_1), \dots, (\rho_i^{\mathbf{M}})^{-1}(T_n)$ are Margulis tubes of \mathbf{M}_i , the curves c_1, \dots, c_n are contained in simplices of geodesics constituting the hierarchy h_i . Since $d_{\mathcal{CC}(\Sigma)}(c_1, c_n) \geq n$, by letting n be greater than M given in Lemma 4.19, we see that Σ supports a geodesic in h_i , which we define to be γ_i . Then the part (1) holds automatically, and moreover by passing to a subsequence, we can assume either all the γ_i are finite geodesics or all of them are geodesic rays.

By Lemma 4.19, passing to a subsequence again, we can assume that γ_i contains a simplex s_i with $d_{\mathcal{CC}(\Sigma)}(c_i, s_i) \leq P$. Since $\{c_i\}$ tends to the ending lamination of e as $i \rightarrow \infty$, if the γ_i are finite geodesics then $\{\gamma_i\}$ converges to a geodesic ray which tends to the ending lamination of e , which implies that the last vertex of γ_i converges to the ending lamination of e . Suppose that passing to a subsequence all the γ_i are geodesic rays. Then again since γ_i contains s_i tending to the ending lamination, by the hyperbolicity of $\mathcal{CC}(\Sigma)$ the endpoint of γ_i converges to the ending lamination of e . In both cases, since γ_i contains s_1 for large i with $d_{\mathcal{CC}(\Sigma)}(c_1, s_1) \leq P$ by Lemma 4.19, we see that it contains a simplex with its distance from the basepoint determined by μ bounded as $i \rightarrow \infty$. \square

In the proof of Proposition 4.18, we used the assumption that the end in B is algebraic only to show that the support of γ_i is Σ . Even in the case when the end in B may not be algebraic, the argument above shows that we still have a geodesic as γ_i in h_i , and its support is the preimage of Σ , which may depend on i . Thus we get the following corollary.

COROLLARY 4.20. — *Let $B = \Sigma \times J$ be a simply degenerate brick of \mathbf{M} . Let \mathcal{V} be the union of all boundary components of \mathbf{M} touching the vertical boundary of B , and \mathcal{V}_i the union of Margulis tubes corresponding to $(\rho_i^{\mathbf{M}})^{-1}(\mathcal{V} \cap B_{K_i r_i}(\mathbf{M}, x_\infty))$. Then, there is a geodesic γ_i in h_i satisfying the following conditions.*

- (1) For sufficiently large i , the preimage $(\rho_i^{\mathbf{M}})^{-1}(B)$ is contained in a brick $B_i = \Sigma_i \times J_i$ in the standard brick decomposition of $\mathbf{M}_i \setminus \mathcal{V}_i$.
- (2) The geodesic γ_i is supported on Σ_i . Passing to a subsequence, we can assume that all γ_i either have finite lengths or are geodesic rays.
- (3) If the γ_i have finite lengths, their lengths go to ∞ as $i \rightarrow \infty$.
- (4) Let $\partial_{\text{real}} B$ be the real front of B . Let $k_i : \Sigma \rightarrow \Sigma_i$ be a homeomorphism induced from $(\rho_i^{\mathbf{M}})^{-1}|_{\partial_{\text{real}} B}$. If γ_i has finite length, for the last vertex v_i of γ_i , its image $k_i^{-1}(v_i)$ on Σ converges to the ending lamination of the simply degenerate end of B . If γ_i is a ray, then for the endpoint e_i of γ_i at infinity, $k_i^{-1}(e_i)$ converges to the ending lamination of the end in B in $\mathcal{EL}(\Sigma)$.

5. Limits of end invariants and ends of models

In this section, we consider the situation where $\{(G_i, \phi_i) = qf(m_i, n_i)\}$ converges to (Γ, ψ) in $AH(S)$ and $\{G_i\}$ converges geometrically to G_∞ . We assume that $\{m_i\}$ converges to $[\mu^-]$ and $\{n_i\}$ converges to $[\mu^+]$ in the Thurston compactification of the Teichmüller space. Let Σ^+ and Σ^- be the boundary components of the convex core of $M_i = \mathbb{H}^3/G_i$ facing the upper and lower ends respectively. We shall first recall the following fact, which follows from the continuity of length function.

LEMMA 5.1. — *Let ν be a component of either μ^- or μ^+ . If ν is a weighted simple closed curve, then $\Psi(|\nu|)$ represents a parabolic class of Γ . Otherwise, its image $\Psi(\nu)$ represents the ending lamination for an end of M'_0 .*

Proof. — This is just a combination of Thurston's theorem and the continuity of the length function proved by Brock [9] in general form. We can assume that ν is a component of μ^- since the argument for the case of μ^+ is exactly the same. Thurston's Theorem 2.2 in [55] (whose proof can be found in [23] and [56]) shows that there is a sequence of simple closed curves $r_i \gamma_i$ converging to μ^+ such that $\text{length}_{m_i}(r_i \gamma_i)$ goes to 0. By Bers' inequality [4], this implies that $\text{length}_{\Sigma_i^-}(r_i \gamma_i)$ also goes to 0. By the continuity of the length function with respect to the algebraic topology (see Brock [9]), we have $\text{length}_{M'} \Psi(\mu^-) = 0$, which means every component of $\Psi(\mu^-)$ represents either a parabolic class or an ending lamination. \square

We shall refine this lemma in Corollary 5.8 to show that in \mathbf{M} the components of the limit of $\{m_i\}$ appear as lower algebraic parabolic curves or lower algebraic ending laminations, whereas those of $\{n_i\}$ appear as upper algebraic parabolic curves or upper algebraic ending laminations. (See Definition 4.15 for the definitions of these terms.)

THEOREM 5.2. — *Let c_{m_i} and c_{n_i} constitute shortest (hyperbolic) pants decompositions of (S, m_i) and (S, n_i) respectively. Let ν^- and ν^+ be the Hausdorff limits of $\{c_{m_i}\}$ and $\{c_{n_i}\}$ respectively. Then the minimal components of ν^+ that are not simple closed curves coincide with the upper algebraic ending laminations of \mathbf{M} . Moreover, every upper algebraic parabolic curve of \mathbf{M} , regarded as a curve on S , is contained in ν^+ . Similarly the minimal components of ν^- that are not simple closed curves coincide with the lower algebraic ending laminations of \mathbf{M} , and every lower algebraic parabolic curve is contained in ν^- .*

Proof. — We shall only deal with ν^+ . The argument for ν^- is obtained only by turning \mathbf{M} upside down. Let h_i be a hierarchy corresponding to $qf(m_i, n_i)$, and consider the model manifold \mathbf{M}_i such that $\mathbf{M}_i[k]$ converges geometrically to $\mathbf{M}[k]$ as before. We regard \mathbf{M} as being embedded in $S \times [0, 1]$ as usual.

We shall first show that any upper algebraic ending lamination of \mathbf{M} is a minimal component of ν^+ . Let $B = \Sigma \times [s, t)$ be a algebraic simply degenerate brick of \mathbf{M} containing an end e . By Proposition 4.18, the hierarchy h_i contains a geodesic γ_i supported on Σ whose last vertex converges to the ending lamination λ_e of e in $\mathcal{UML}(\Sigma)$. Now, as was shown in Masur–Minsky [38, §6] using Theorem 3.1 in the same paper, the distance between the last vertex of γ_i and the projection of the terminal marking $T(h_i)$ of h_i to Σ is uniformly bounded. In particular, for the shortest pants decomposition c_{n_i} , which consists of the base curves of $T(h_i)$, its projection to Σ converges to λ_e in $\mathcal{UML}(\Sigma)$. Since the Hausdorff limit of $c_{n_i}|_\Sigma$ contains the limit of the projection of c_{n_i} in $\mathcal{UML}(\Sigma)$, this shows that any upper algebraic ending lamination is contained in ν^+ .

Secondly, we shall show that every upper algebraic parabolic curve is contained in ν^+ . Let c be an upper algebraic parabolic curve on S , and denote a standard algebraic immersion by $g' : S \rightarrow \mathbf{M}$. There are three cases which we have to consider. The first is the case (a) when $g'(c)$ is homotopic to a curve on a torus boundary of \mathbf{M} ; the second is the case (b) when $g'(c)$ is homotopic to a core curve of an open annulus boundary component of \mathbf{M} at least one of whose ends touches a geometrically finite end; and the third is the case (c) when $g'(c)$ is homotopic to a core curve of an open annulus boundary component whose ends touch only simply degenerate or wild ends.

(a). — We first consider the case when the curve $g'(c)$ is homotopic into a torus component T of $\partial\mathbf{M}$. For later use, we state here the result of the case (a) as a claim, taking into account also the case of ν^- .

CLAIM 5.3. — *Let c be a simple closed curve on S such that $g'(c)$ is homotopic into a torus boundary component of \mathbf{M} . Then c is a minimal*

component of ν^+ if c is an upper parabolic curve and is a minimal component of ν_- if c is a lower parabolic curve.

Proof. — Let V_i be the Margulis tube bounded by $(\rho_i^{\mathbf{M}})^{-1}(T)$ in \mathbf{M}_i for large i . Its boundary ∂V_i has a marked flat structure which is parametrised by $\omega_{\mathbf{M}_i}(V_i)$ as was explained in the proof of Lemma 4.3. The real part of $\omega_{\mathbf{M}_i}(V_i)$ corresponds to the difference of the marking on the top horizontal annulus and that of the bottom horizontal annulus, hence to the length of the tight geodesic of h_i supported on an annulus on S homotopic to the vertical projection of the horizontal annulus of V_i . Since ∂V_i converges to the boundary of a torus cusp neighbourhood, $\Re\omega_{\mathbf{M}_i}(V_i)$ goes to ∞ whereas the imaginary part is bounded as $i \rightarrow \infty$. Let c_i be a simple closed curve on S whose image by $\Phi_i^{\mathbf{M}}$ is homotopic to the longitude of V_i . Since the longitude of ∂V_i converges to that of T which is homotopic to the image of a simple closed curve under g' , the homotopy class of c_i is independent of i for large i . Therefore, by taking a subsequence, we can assume that c_i is constantly c . Let A be an annulus on S which is a regular neighbourhood of c . Since $\Re\omega_{\mathbf{M}_i}(V_i) \rightarrow \infty$, as was explained above, there is a geodesic γ_i in h_i supported on A whose length goes to ∞ as $i \rightarrow \infty$. Let $a(i)$ and $b(i)$ be the first and last vertices of γ_i , and let $n(a)_i$ and $n(b)_i$ be the (signed) numbers of times $a(i)$ and $b(i)$ respectively go around c compared to the transversal of the marking determined by $\Phi_i^{\mathbf{M}}$. Then $|n(b)_i - n(a)_i|$ goes to ∞ as $i \rightarrow \infty$. We set $n(i)$ to be $n(b)_i - n(a)_i$.

By the definition of hierarchy, there is a vertex of a geodesic g_i in h_i with $\xi(D(g_i)) = 4$ which represents c (and is denoted also by c), satisfying $\pi_A(\text{prec}(c)) = a(i)$ and $\pi_A(\text{succ}(c)) = b(i)$. As was shown above, the distance between $\pi_A(\text{prec}(c))$ and $\pi_A(\text{succ}(c))$ goes to ∞ . Since these $\text{prec}(c)$ and $\text{succ}(c)$ may depend on i , we denote $\text{prec}(c)$ in g_i by v_i and $\text{succ}(c)$ in g_i by w_i .

Since there is an elementary move changing v_i to c , there is a block b_i in \mathbf{M}_i realising this elementary move by the definition of model manifold by Minsky [40]. Let U_i be the Margulis tube in \mathbf{M}_i whose core curve represents v_i , and u_i a horizontal longitude on ∂U_i . Recall that the block decomposition of \mathbf{M}_i converges geometrically to that of \mathbf{M} as $i \rightarrow \infty$. Therefore, the block b_i can be pushed forward to a block b_∞ in \mathbf{M} for large i , and hence there is either a Margulis tube or a torus boundary in \mathbf{M} whose core curve or longitude, which we denote by u_∞ , is homotopic to $\rho_i^{\mathbf{M}}(u_i)$ for every large i . First suppose that g' does not go around T . Then, since c is upper parabolic, g' can be homotoped so that it passes b_∞ horizontally, and consequently, there is a simple closed curve u on S such that $g'(u)$ is homotopic to u_∞ . By pulling back this to \mathbf{M}_i , we see $\Phi_i^{\mathbf{M}}(u)$ is homotopic to $\text{prec}(c)$ for large i . This means that $n(a)_i$ is bounded as $i \rightarrow \infty$, and hence $|n(b)_i|$ goes to ∞ .

We next consider the case when g' goes k -times around T for an integer k with $k \neq 0, 1$. (We shall only consider the case when k is positive for simplicity. In the case when $k < -1$, we move $(\rho_i^{\mathbf{M}})^{-1} \circ g'$ in the negative direction to pass through V_i .) In this case, to homotope $(\rho_i^{\mathbf{M}})^{-1} \circ g'$ to an immersion g'_i which does not go around V_i , we have to make it pass k times through V_i in the upward direction. This g'_i converges geometrically to a surface which passes b_∞ horizontally, and therefore, in the same argument as the previous paragraph, there is a simple closed curve u on S as above such that $g'_i(u)$ is homotopic to u_i . Recall, as we explained in the proof of Lemma 4.3, the meridian β_i of V_i realises a homotopy between the markings below and above, and hence as the surface passes V_i in the positive direction, the transverse of c is twisted $n(i)$ -times. This implies that the $kn(i)$ -time Dehn twist of u_i around c represents the constant homotopy class u for large i . Therefore, $|n(a)_i|$ grows in the order of $|kn(i)|$, and $|n(b)_i|$ does in the order of $|(k+1)n(i)|$ in this case.

In either case, we see that $|n(b)_i|$ goes to ∞ . Therefore, by Section 6 of [38] again, the projection of the shortest pants decomposition c_{n_i} to $\mathcal{CC}(A)$ also goes around $n'(i)$ times around c with $n'(i) \rightarrow \infty$. This shows that the Hausdorff limit ν^+ of $\{c_{n_i}\}$ contains c as a minimal component. \square

(b). — Next we consider the second case when $g'(c)$ is homotopic to a core curve of an open annulus boundary component T which touches a geometrically finite end e of \mathbf{M} . Then we shall prove the following claim, which will also be used later.

CLAIM 5.4. — *Let Σ be an incompressible subsurface of S such that $\Sigma \times \{1\}$ corresponds to a geometrically finite end of \mathbf{M} with conformal structure n_Σ . Let a be a simple closed curve on S such that $g'(a)$ is homotopic to a curve a' on $\Sigma \times \{1\}$. Then the hyperbolic length $\text{length}_{n_i}(a)$ converges to that of a' with respect to n_Σ . In particular, if a' is peripheral, then $\text{length}_{n_i}(a)$ goes to 0.*

Proof. — By the definition of conformal structures on the geometrically finite bricks of \mathbf{M} in Section 4.3, there is a subsurface Σ_i in S such that (Σ_i, n_i) converges to (Σ, n_Σ) geometrically. Since g' is an algebraic locus, for a simple closed curve a' of Σ homotopic to $g'(a)$ is pulled back to a curve on Σ_i homotopic to $\Phi_i^{\mathbf{M}}(a)$. This implies the hyperbolic length of a with respect to n_i converges to that with respect to n_Σ . \square

This implies that the length of c with respect to n_i goes to 0, and the pants decomposition c_{n_i} must contain c . Therefore c is contained in the Hausdorff limit of $\{c_{n_i}\}$ also in this case.

(c). — Now, we consider the third case when $g'(c)$ is homotopic to a core curve of an open annulus boundary component T whose ends touch only simply degenerate or wild ends. Suppose, seeking a contradiction, that there is a minimal component d of ν^+ intersecting c transversely. By Claim 5.3, we see that d cannot intersect an upper algebraic parabolic curve on S whose image by g' is homotopic into a torus boundary component. We shall first show the following claim.

CLAIM 5.5. — *There is either a simply degenerate end or a horizontal annulus on a torus boundary component, corresponding to $\Sigma \times \{t\}$ lying above $g'(S)$ and an incompressible subsurface F of S containing d such that $g'(F \cap \Sigma)$ is vertically parallel into $\Sigma \times \{t - \epsilon\}$ for any small $\epsilon > 0$, and d intersects Σ essentially on F . (Here $F \cap \Sigma$ is assumed to have no inessential intersection.)*

Proof. — Suppose first that there is an upper algebraic simply degenerate end corresponding to $\Sigma \times \{t\}$ such that Σ intersects d essentially. (This is equivalent to saying that d intersects the minimal supporting surface of an upper ending lamination essentially.) Then, by letting F be the entire S , the condition above holds.

Suppose next that d can be homotoped so as to be disjoint from any minimal supporting surface of the upper ending lamination (i.e. from the surface Σ for any upper algebraic simply degenerate ends as described above). Let F be a component of the complement in S of the union of the minimal supporting surfaces of the upper algebraic ending laminations and annular neighbourhoods of parabolic curves of the case (a) above (i.e. those homotopic into a torus boundary component), which contains d . Since c , as well as d , cannot intersect the minimal supporting surface of an upper algebraic ending lamination and a parabolic curve of (a) above, c is also contained in F . Consider a simply degenerate end or a lower horizontal annulus of a torus boundary corresponding to $\Sigma' \times \{t\}$ above $g'(S)$ such that Σ' intersects d essentially. We can see that such an end or a torus exists since the boundary component into which $g'(c)$ is homotopic touches either a simply degenerate end or a wild end and there is no essential half-open annulus tending to a wild end. We take a lowest one among such $\Sigma' \times \{t\}$, and denote it by $\Sigma \times \{t\}$. Then $(\Sigma \cap F) \times \{t - \epsilon\}$ is homotopic to $g'(\Sigma \cap F)$ in \mathbf{M} for any small positive ϵ . (See Figure 5.1.) Thus we get subsurfaces F and Σ as we desired. \square

Suppose now that $\Sigma \times \{t\}$ corresponds to a simply degenerate end, and denote it by e . Let λ be the ending lamination of the end e , and B the brick of \mathbf{M} containing e . By Corollary 4.20, there are bricks $B_i \cong \Sigma_i \times J_i$ containing $(\rho_i^{\mathbf{M}})^{-1}(B)$ and geodesics γ_i supported on Σ_i whose lengths go to ∞ as $i \rightarrow \infty$. The approximate isometry induces a homeomorphism $f_i : \Sigma \rightarrow \Sigma_i$. Also, we know that for the last vertex t_i of γ_i , its image $(f_i)^{-1}(t_i)$ converge

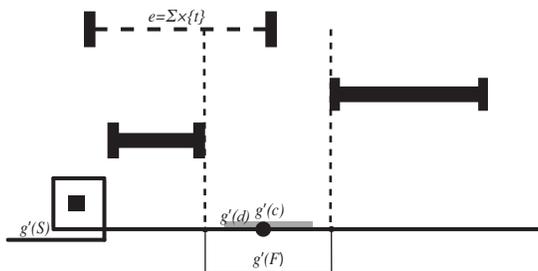


Figure 5.1. The definition of F and the lowest end

to the ending lamination λ . By the same argument as before using Section 6 of [38], $\{(f_i)^{-1}\pi_{\Sigma_i}(c_{m_i})\}$ converges to a geodesic lamination containing λ . We state this as a claim for later use.

CLAIM 5.6. — *Suppose that $\Sigma \times \{t\}$ corresponds to a simply degenerate end. Then there is a homeomorphism $f_i : \Sigma \rightarrow \Sigma_i$ induced from an approximate isometry between \mathbf{M} and \mathbf{M}_i , and the sequence of multicurves $\{(f_i)^{-1}\pi_{\Sigma_i}(c_{m_i})\}$ converges to a geodesic lamination containing λ .*

Since $g'(F \cap \Sigma)$ is vertically homotopic into $\Sigma \times \{t - \epsilon\}$ in \mathbf{M} for any small positive ϵ , we see that $f_i|(F \cap \Sigma)$ can be isotoped so that it is the identity map if restricted to $F \cap \Sigma$. Since λ is arational in Σ , and Σ intersects d essentially, we see that λ intersects d essentially. On the other hand, by Claim 5.6, $\{f_i^{-1}(c_{m_i})\}$ converges to a geodesic lamination containing λ whereas $\{c_{m_i}\}$ converges to ν containing d , both in the Hausdorff topology. Considering the fact that $f_i|F \cap \Sigma$ can be assumed to be the identity, we see that this is impossible.

Next suppose that $\Sigma \times \{t\}$ is a horizontal annulus lying on a torus boundary component T . Let Σ_i be the lower horizontal annulus on $\rho_i^{-1}(T)$. Then by the same argument as in the proof of Claim 5.3, we see that $\pi_{\Sigma_i}(c_{n_i})$ spirals around the core curve of Σ_i more and more as $i \rightarrow \infty$. Since $(\Sigma \cap F) \times \{t - \epsilon\}$ is homotopic to $g'(\Sigma \cap F)$ and $\Sigma \cap F$ intersects d essentially, this shows that the Hausdorff limit of $\{c_{n_i}\}$ intersects d transversely. This is a contradiction. Thus we have shown that an upper algebraic parabolic curve cannot intersect a minimal component of ν^+ transversely. Since ν^+ is a Hausdorff limit of pants decomposition, every closed geodesic either intersects ν^+ transversely or is contained in ν^+ . Therefore implies that any upper algebraic curve is contained in ν^+ .

Thus we have shown that both the upper algebraic ending laminations and the upper algebraic parabolic curves are contained in ν^+ . To complete

the proof, it remains to show that every minimal component of ν^+ that is not a simple closed curve is an upper algebraic ending lamination. For that we have only to show that ν^+ has no minimal component that is not a compact leaf and is disjoint from the minimal supporting surfaces of the upper algebraic ending laminations up to isotopy. We recall that no minimal component of ν^+ intersects an upper parabolic locus regarded as lying on S by the results of the cases (a)–(c) above. Let $\Sigma_1, \dots, \Sigma_{j_0}$ be the minimal supporting surfaces of the upper algebraic ending laminations. Let F be a component of the complement of the union of $\bigcup_{j=1}^{j_0} \Sigma_j$ and all upper parabolic loci. What we have to show is that every minimal component d of ν^+ contained in F is a simple closed curve.

The argument is quite similar to the proof of the case (c). Before dealing with the general situation, we begin with considering the special case when $g'(F)$ is homotopic into an end lying above $g'(S)$. The end cannot be wild since a wild end has no essential open annulus tending to the end. Hence the end is either geometrically finite or simply degenerate. Then there is an incompressible subsurface Σ of S containing F such that $g'(F)$ is homotopic into an end corresponding to $\Sigma \times \{s\}$ (i.e. into $\Sigma \times \{s - \epsilon\}$ for any small positive ϵ), where $s = 1$ if the end is geometrically finite. We first consider the case when $s = 1$ and $\Sigma \times \{1\}$ is geometrically finite. As shown in Claim 5.4, the surface $\Sigma \times \{1\}$ has a hyperbolic metric n_∞ which is a geometric limit of (S, n_i) with base point lying in the thick part of $(\Sigma_i, n_i | \Sigma_i)$ for some subsurface Σ_i homeomorphic to Σ . Since $F \times \{1 - \epsilon\}$ in \mathbf{M} is homotopic to $g'(F)$, we see that Σ_i contains F up to isotopy for large i and that n_i induces a hyperbolic structure $n_i | F$ on F with geodesic boundary, which converges to $n_\infty | F$ preserving the markings. Note that $\nu^+ | F$ is a Hausdorff limit of $c_{n_i} | F$ and d is contained in it. Since $n_i | F$ converges to $n_\infty | F$, we see that any minimal component of the Hausdorff limit ν^+ of $\{c_{n_i}\}$ contained in F must be a compact leaf.

Next suppose that there is a simply degenerate end e of \mathbf{M} of the form $\Sigma \times \{s\}$ with ending lamination λ , lying above $g'(S)$ into which $g'(F)$ is homotopic. Then this end cannot be algebraic since F lies in the complement of the minimal supporting surfaces of upper algebraic ending laminations up to isotopy. This implies that F is a proper subsurface of Σ up to isotopy. By Claim 5.6, there is a homeomorphism $f_i : \Sigma \rightarrow \Sigma_i$ for a subsurface Σ_i of S , and $\{f_i^{-1}(\pi_{S'_i}(c_{n_i}))\}$ converges to a geodesic lamination containing λ . Now, since $g'(F)$ is homotopic into $\Sigma \times \{s\}$ and λ is arational in Σ , this shows that the Hausdorff limit of $\{c_{n_i} | F\}$ consists only of arcs. Therefore, F cannot contain d , contradicting our assumption.

In general, as shown in Claim 5.5, there is a simply degenerate end or a horizontal annulus on a torus boundary component, corresponding to $\Sigma \times \{t\}$

situated above $g'(S)$, such that $g'(F \cap \Sigma)$ is homotopic into $\Sigma \times \{t - \epsilon\}$ in \mathbf{M} for any small positive ϵ , and such that $F \cap \Sigma$ intersects d essentially. (See Figure 5.1 again.) Suppose first that $\Sigma \times \{t\}$ corresponds to a simply degenerate end e . Then as in Claim 5.6, the Hausdorff limit of $c_{n_i}|(F \cap \Sigma)$ must contain the restriction of the ending lamination λ of e to F . Since $F \cap \Sigma$ intersects d essentially, λ is arational in Σ , and $\Sigma \times \{t\}$ is not algebraic, this shows that the Hausdorff limit of $c_{n_i}|(F \cap \Sigma)$ intersects d transversely. This is a contradiction.

Next suppose that $\Sigma \times \{t\}$ lies on a torus boundary component T . Then each component of $\Sigma \cap F$ is a strip. As in the proof of Claim 5.3, for each component Δ of $\Sigma \cap F$, the Hausdorff limit of $c_{n_i}|F$ has a leaf running to join the two components of $\text{Fr}_\Sigma \Delta$. Since $F \cap \Sigma$ intersects d essentially on F , each component of $d \cap \Sigma \cap F$ joins two boundary components of Σ . This implies that the Hausdorff limit of c_{n_i} intersects d transversely, which is a contradiction again. Thus we have shown that every minimal component of the Hausdorff limit of $\{c_{n_i}\}$ contained in F is a compact leaf. This completes the proof of Theorem 5.2. \square

Theorem 3.2 is obtained as a corollary of Theorem 5.2, as follows.

Proof of Theorem 3.2. — Each simply degenerate end of \mathbf{M} is mapped to that of $(M_\infty)_0$. Let $p : M' \rightarrow M_\infty$ be a covering associated to the inclusion of the algebraic limit Γ into the geometric limit G_∞ . By the covering theorem of Thurston [54] and Canary [18] with the argument in Lemma 2.3 of [44], each simply degenerate end of $(M')_0$ has a neighbourhood which is mapped homeomorphically to a neighbourhood of a simply degenerate end of $(M_\infty)_0$. Furthermore the ending lamination of an end of $(M')_0$ is identified with that of the corresponding end of $(M_\infty)_0$ by p , which follows immediately by the definition of ending laminations. Therefore the algebraic simply degenerate ends of \mathbf{M} correspond to simply degenerate ends of $(M')_0$ one-to-one preserving the ending laminations. It is also obvious that upper (resp. lower) ends of \mathbf{M} correspond to upper (resp. lower) ends of $(M')_0$. Similarly the algebraic parabolic curves correspond to the core curves of the parabolic loci of M' . Therefore, Theorem 5.2 implies the statement of Theorem 3.2 except for the last paragraph.

Suppose that c is a simple closed curve in ν^- or ν^+ that has isolated leaves around it. We assume that c is contained in ν_- for simplicity. Then there is a component d_i of c_{n_i} spiralling around c more and more as $i \rightarrow \infty$. Since the length of d_i is bounded, by the continuity of length function this shows that the geodesic length of $\psi(c)$ is 0, and hence c is a parabolic curve. Suppose further that c is not contained in ν_- . Then by Theorem 5.2, c cannot be a lower parabolic curve of \mathbf{M} , which means in particular that c is an upper parabolic curve of \mathbf{M} . Furthermore this implies that even if

$g'(c)$ is homotopic into a torus boundary component T of \mathbf{M} , the algebraic locus $g'(S)$ cannot go around T . Therefore, c is an upper parabolic curve of M' . \square

LEMMA 5.7. — *Let $\{g_i\}$ be a sequence in the Teichmüller space $\mathcal{T}(S)$ which converges to a projective lamination $[\mu]$ in the Thurston compactification. Let c_i be a pants decomposition on (S, g_i) whose total length is uniformly bounded independently of i . Then the Hausdorff limit of any subsequence of $\{c_i\}$ contains all the components of $|\mu|$ as minimal components.*

Proof. — Let λ be the Hausdorff limit of a convergent subsequence of $\{c_i\}$. Since c_i is a pants decomposition of S , every measured lamination on S except for the components of c_i intersects c_i essentially. Therefore every measured lamination on S other than those contained in λ intersects λ essentially. Let μ_0 be a component of μ , and suppose that its support is not a minimal component of λ . Then μ_0 must intersect λ transversely.

We shall first consider the case when μ_0 is not a simple closed curve. Let Σ be the minimal supporting surface of μ_0 . We consider a sequence of essential arcs and simple closed curves $c_i \cap \Sigma$ on Σ . Note that $c_i \cap \Sigma$ converges to $\lambda \cap \Sigma$ with respect to the Hausdorff topology, which is non-empty. If $\lambda \cap \Sigma$ has a minimal component contained in $\text{Int}\Sigma$, then there is a sequence of positive numbers r_i going to 0 such that $r_i c_i \cap \Sigma$ converges to a measured lamination γ in Σ . Otherwise we can find a bounded sequence of positive numbers r_i such that $\{r_i c_i\}$ converges to a non-empty union γ of essential arcs. (The limit is taken in the space of weighted essential curves in Σ with the weak topology of transverse measures.) In either case, let R be $\sup_i r_i$. Now, $\text{length}_{g_i}(c_i) \geq r_i \text{length}_{g_i}(c_i)/R$, where the right hand goes to ∞ since $i(\mu_0, \gamma) > 0$ and an arc with non-zero intersection with the Thurston limit has length going to ∞ if we consider hyperbolic structures on Σ with geodesic boundaries. This implies that $\text{length}_{g_i}(c_i)$ must also go to ∞ , which is a contradiction.

Next suppose that μ_0 is a simple closed curve. If the length of μ_0 with respect to g_i goes to 0, then we can take an annular neighbourhood $A_i(\mu_0)$ of μ_0 whose width (with respect to n_i) goes to ∞ as $i \rightarrow \infty$. Since λ intersects μ_0 essentially, c_i passes through $A_i(\mu_0)$ for large i . This implies the length of c_i in (S, g_i) goes to ∞ , which is a contradiction. Next suppose that the length of μ_0 is bounded from both above and below by positive constants. Then we can take an annular neighbourhood $A_i(\mu_0)$ whose width is bounded away from 0. Consider the shortest essential arc α_i in $A_i(\mu_0)$. Since μ_0 is contained in the limit lamination $[\mu]$ of $\{g_i\}$, the shortest arc α_i must spiral around μ_0 more and more as $i \rightarrow \infty$. Since λ does not contain μ_0 , the number of spiralling of c_i around μ_0 is bounded. This means twisting number between α_i and $c_i|_{A_i(\mu_0)}$ goes to ∞ . Therefore, the length of $c_i|_{A_i(\mu_0)}$ goes to ∞ as

$i \rightarrow \infty$, which is a contradiction. In the case when the length of μ_0 goes to ∞ , we take $A_i(\mu_0)$ whose width goes to 0 as $i \rightarrow \infty$. Also in this case, the shortest essential arc α_i spirals around μ_0 more and more as $i \rightarrow \infty$. Since the twisting number between α_i and $c_i|A_i(\mu_0)$ goes to ∞ also in this case, we see that the length of $c_i|A_i(\mu_0)$ goes to ∞ . This is a contradiction. \square

Combining this lemma with Theorem 5.2, we get the following corollary.

COROLLARY 5.8. — *In the setting of Theorem 3.2, let $[\mu^-]$ and $[\mu^+]$ be projective laminations to which $\{m_i\}$ and $\{n_i\}$ converge in the Thurston compactification of $\mathcal{T}(S)$ after taking subsequences. Then each minimal component of $[\mu^+]$ is either an upper algebraic ending lamination or an upper algebraic parabolic curve of \mathbf{M} . Similarly, each component of $[\mu^-]$ is either the ending lamination of a lower algebraic simply degenerate end or a lower algebraic parabolic curve of \mathbf{M} .*

Proof. — As usual, we shall only deal with $[\mu^+]$. Each component of $[\mu^+]$ that is not a simple closed curve is the ending lamination of an upper simply degenerate end by Theorem 5.2 and Lemma 5.7. Let c be a component of $[\mu^+]$ which is a simple closed curve. Then by Lemma 5.1, $\psi(c)$ is parabolic. Therefore c is an algebraic parabolic locus in \mathbf{M} . It remains to show that c is upper.

Suppose that c is not upper, seeking a contradiction. This assumption implies, in particular, that if c is isolated and is homotopic to a curve lying on a torus boundary T of \mathbf{M} , then the standard algebraic immersion g' does not go around T by our definition of the upperness. Since there is no essential half-open annulus tending to a wild end, there are only three possibilities for the curve c : (1) the first is when c lies in a domain F of S as a non-peripheral curve and $g'(F)$ is homotopic into some simply degenerate end above $g'(S)$, (2) the second is when there exists F containing c as above such that $g'(F)$ is homotopic into geometrically finite end, lying on $S \times \{1\}$, and (3) the third is when there are a domain F as above and either a simply degenerate end or a horizontal annulus on a torus boundary component above $g'(S)$, corresponding to $\Sigma \times \{t\}$, such that c intersects Σ essentially and $g'(F \cap \Sigma)$ can be homotoped into $\Sigma \times \{t - \epsilon\}$ for arbitrarily small $\epsilon > 0$.

(1). — Suppose that $g'(F)$ is homotopic into a simply degenerate end e corresponding to $\Sigma \times \{t\}$. Then its ending lamination λ intersects c essentially. As in Claim 5.6, there are bricks $B_i \cong \Sigma_i \times J_i$ and a homeomorphism $f_i : \Sigma \rightarrow \Sigma_i$ induced from the approximate isometry between \mathbf{M} and \mathbf{M}_i , such that $\{f_i^{-1}(c_{n_i}|\Sigma_i)\}$ converges to a geodesic lamination containing λ in the Hausdorff topology. Let $A(c)$ be an annulus with core curve c . Since $g'(F)$ is homotopic into e , we see that $f_i|F$ is isotopic to the identity. In particular, we can assume that $f_i|A(c)$ is the identity. Therefore $c_{n_i}|A(c)$ regarded as a

vertex of $\mathcal{CC}(A(c))$ converges a vertex represented by $\lambda|A(c)$. On the other hand, by Lemma 5.7, $\{c_{n_i}\}$ must converge to a lamination containing c in the Hausdorff topology; hence $c_{n_i}|A(c)$ diverges in $\mathcal{CC}(A(c))$. This is a contradiction.

(2). — In the second case, $g'(F)$ is homotopic into an upper geometrically finite end of \mathbf{M} . Take a simple closed curve δ on F intersecting c essentially. Since $g'(\delta)$ is homotopic to a curve in an upper geometrically finite end, by Claim 5.4, $\text{length}_{m_i}(\delta)$ is bounded as $i \rightarrow \infty$. This contradicts the assumption that c is contained in μ^+ .

(3). — Now we turn to the third case. Suppose first that $\Sigma \times \{t\}$ is a simply degenerate end e in a brick $B \cong \Sigma \times J$ with ending lamination λ . Then, as in the case (1), by Claim 5.6, there is a homeomorphism $f_i : \Sigma \rightarrow \Sigma_i$ induced from the approximate isometry between \mathbf{M} and \mathbf{M}_i such that the Hausdorff limit of $f_i^{-1}(c_{n_i}|\Sigma_i)$ contains λ . Since $F \cap \Sigma$ is homotopic into $\Sigma \times \{t - \epsilon\}$, and f_i fixes $F \cap \Sigma$, we see that the Hausdorff limit of $f_i^{-1}(c_{n_i}|\Sigma_i)$ contains $c \cap \Sigma$. This is a contradiction since c intersects λ transversely. The same kind of argument works also for the case when $\Sigma \times \{t\}$ is a horizontal annulus on a torus boundary using Claim 5.3 instead of Claim 5.6. \square

6. Proofs of Theorem 3.1, Theorem 3.3 and Theorem 3.5

We can now prove Theorem 3.1, Theorem 3.3 and Theorem 3.5 making use of our results obtained in the previous section.

Proof of Theorem 3.1. — We consider the geometric limit M_∞ of M_i and the model \mathbf{M} of its non-cuspidal part as before. By the definition of algebraic simply degenerate ends of \mathbf{M} , they are mapped by the model map f to simply degenerate ends of $(M_\infty)_0$ which lift to those of the algebraic limit $(M')_0$. Upper ends among them are mapped to those lifted to upper ends of $(M')_0$, and lower ones to those lifted to lower ends of $(M')_0$. Now, by Corollary 5.8, every component of $|\mu^+|$ that is not a simple closed curve is an upper algebraic ending lamination. Therefore, it is the ending lamination of a upper simply degenerate end of $(M')_0$. The same argument works for $|\mu^-|$.

The second paragraph of the statement also follows immediately from Corollary 5.8. \square

Proof of Theorem 3.3. — Suppose, seeking a contradiction, that $\{qf(m_i, n_i)\}$ as in the statement of Theorem 3.3 converges after taking a subsequence. Then by Theorem 3.1, μ_0^+ is an ending lamination of an upper end of $(M')_0$ and μ_0^- is that of a lower end of $(M')_0$. Let Σ^+ and Σ^- be the

minimal supporting surfaces of μ_0^+ and μ_0^- respectively, which were assumed to share at least one boundary component c . Since c lies on the boundary of both Σ^- and Σ^+ , it represents a \mathbb{Z} -cusp both above and below $\Psi(S)$. This is impossible since no two distinct cusps have homotopic core curves. \square

Proof of Theorem 3.5. — As in the statement, let μ^- and μ^+ be two measured laminations on S and suppose that there is a boundary component c of a non-simple closed curve component of the minimal supporting surface of a non-simple closed curve component μ_0 of μ^+ which is contained up to isotopy in $|\mu^-|$ as in the part (1) of the statement. We can argue in the same way also when the part (2) of the statement holds just by exchanging $+$ and $-$.

Suppose, seeking a contradiction, that $\{qf(m_i, n_i)\}$ converges passing to a subsequence. By Corollary 5.8, there is an upper algebraic simply degenerate end of \mathbf{M} in the form $\Sigma(\mu_0) \times \{t\}$ with $\Sigma(\mu_0)$ the minimal supporting surface of μ_0 , which has $|\mu_0|$ as the ending lamination. This implies that there is a boundary component of \mathbf{M} which is an open annulus with core curve homotopic to c , and one of whose ends tends to this simply degenerate end. By Lemmas 4.16 and 4.17, this shows that c is an upper algebraic parabolic curve of \mathbf{M} and moreover that it cannot be a lower algebraic parabolic curve at the same time since c does not lie on a torus boundary component. In the same way, by Corollary 5.8, if c is contained in $|\mu^-|$, it must be a lower algebraic parabolic curve of \mathbf{M} . This is a contradiction. Thus we have completed proof of Theorem 3.5. \square

7. Proof of Theorem 3.6

7.1. Necessity

We shall first show that the condition (1) is necessary. Suppose, on the contrary, that there is c_j among c_1, \dots, c_r such that $\text{length}_{n_i}(c_j)$ goes to 0 whereas $\{(G_i, \phi_i) = qf(m_i, n_i)\}$ converges. (The argument for the case when $\text{length}_{m_i}(c_j)$ goes to 0 is quite the same.) Let G_∞ be the geometric limit of a subsequence of $\{G_i\}$ as before, and set $M_i = \mathbb{H}^3/G_i$ and $M_\infty = \mathbb{H}^3/G_\infty$. Consider the model manifold \mathbf{M} of $(M_\infty)_0$. By Corollary 5.8, c_j is an upper algebraic parabolic curve of \mathbf{M} . Let $g' : S \rightarrow \mathbf{M}$ be a standard algebraic immersion. Since $\text{length}_{n_i}(c_j) \rightarrow 0$, the boundary blocks of \mathbf{M}_i corresponding to the upper boundary are pinched along an annulus with core curve c_j , and the one in the geometric limit is split along the annulus. By pushing forward this core curve and $\Phi_i(c_j)$ to M_∞ and pull it back to \mathbf{M} , we get a homotopy between a core curve of an open annulus component of $\partial\mathbf{M}$

and $g'(c_j)$. By Lemma 4.13, this shows that $g'(S)$ cannot go around a torus boundary component whose longitude corresponds to c_j . In particular, c_j cannot be a lower algebraic parabolic curve of \mathbf{M} . This contradicts, by way of Corollary 5.8, the fact that $|\mu_-|$ also contains c_j . This completes the proof of the necessity of the condition (1).

Next we turn to showing the necessity of the condition (2). By Corollary 5.8 again, we see that if $\{(G_i, \phi_i)\}$ converges algebraically, each of c_1, \dots, c_r must be both upper and lower algebraic parabolic curves of \mathbf{M} . By Lemma 4.13, this is possible only when $g'(S)$ goes around a torus boundary component T_j of \mathbf{M} whose longitude is homotopic to $g'(c_j)$ for each $j = 1, \dots, r$. Suppose that $g'(S)$ goes a_j times around T_j for an integer a_j , where we define the counter-clockwise rotation in Figure 4.2 (when viewed from the right to the left) to be the positive direction. As before, we define the condition $a_j = 0$ means that $g'(S)$ passes below T_j , and if $g'(S)$ passes above T_j not going around it, we define a_j to be -1 . Since $g'(S)$ goes around T_j , we have $a_j \neq 0, -1$.

Let \mathbf{M}_i be a model manifold of $(M_i)_0$. Since $\mathbf{M}_i[0]$ converges to $\mathbf{M}[0]$ geometrically, there is a torus boundary $T_j(i)$ of $\mathbf{M}_i[0]$ which is mapped to T_j by the approximate isometry $\rho_i^{\mathbf{M}}$. The torus $T_j(i)$ consists of two horizontal annuli and two vertical annuli. We choose a meridian-longitude system of T_j in such a way that the longitude l_j lies on a horizontal annulus and the meridian m_j is shortest among all the simple closed curves on T_j intersecting the longitude at one point. We choose orientations on l_j and m_j so that the three-dimensional orientation determined by the frame formed by l_j, m_j and the normal vector of T_j pointing inward coincides with that of \mathbf{M} . By pulling back this system using the approximate isometry $\rho_i^{\mathbf{M}}$ between $\mathbf{M}_i[0]$ and $\mathbf{M}[0]$, we get an oriented longitude $l_j(i)$ and an oriented meridian $m_j(i)$ on $T_j(i)$. There is a Margulis tube $V_j(i)$ attached to $T_j(i)$ in \mathbf{M}_i . The compressing curve of $V_j(i)$ intersects the longitude $l_j(i)$ only at one point. Therefore we can express the homology class of the compressing curve as $k_i^j[l_j(i)] + [m_j(i)]$ if we choose an orientation on the compressing curve. Since $T_j(i)$ converges geometrically to a torus boundary component of \mathbf{M} , we have $|k_i^j| \rightarrow \infty$.

Fix some j , and consider c_j . In \mathbf{M} , there is a block B^+ intersecting T_j by an annulus A^+ containing the upper horizontal annulus of T_j in the middle. Similarly, there is a block B^- intersecting T_j by an annulus A^- containing the lower horizontal annulus of T_j in the middle. One or both of these may be boundary blocks. We shall only consider the case when $a_j > 0$, i.e. the case when $g'(S)$ goes around T_j counter-clockwise in Figure 4.2 if it proceeds from left to right. Since c_j is an algebraic parabolic curve, the standard immersion can be homotoped to pass through both B^- and B^+ .

Now, consider a simple closed curve γ^- on the *lower* horizontal annulus of B^- which is homotopic to a core curve if B^- is an internal block. When B^- is a boundary block, we consider horizontal upper boundary components of B^- adjacent to A^- . If there are two such surfaces, we denote their union by Δ^- , and if there is only one such surface, we denote it by Δ^- . We take a simple closed curve γ^- which lies in $A^- \cup \Delta^-$ and intersects the core curve of A^- at two points when Δ^- consists of two components and at one point when Δ^- is connected. We define γ^+ in the same way. By pulling back γ^+ and γ^- by $(\rho_i^{\mathbf{M}})^{-1}$, we get simple closed curves $\gamma^+(i)$ and $\gamma^-(i)$, which are horizontal except for vertical parts passing through vertical annuli contained in A^- or A^+ . Using the vertical projection to S in \mathbf{M}_i , we regard $\gamma^+(i)$ and $\gamma^-(i)$ also as curves on S .

Let $g'_i : S \rightarrow \mathbf{M}_i$ be a pull-back of the standard immersion g' obtained by composing $(\rho_i^{\mathbf{M}})^{-1}$. We consider to homotope g'_i to unwrap it around $T_j(i)$ and make the surface lie under $T_j(i)$, by making it pass a_j times through $V_j(i)$. Let g''_i be a surface obtained by modifying the part of g'_i going around $T_j(i)$ to a horizontal annulus and giving a natural marking coming from the structure of $S \times I$, which is equal to a marking determined by a pull-back of a horizontal surface in \mathbf{M} obtained by removing the parts of g' going around torus boundaries. Note that g''_i and g'_i are not homotopic as maps because of the difference of markings. (This means that g''_i is not homotopic to $\Phi_i^{\mathbf{M}}$.)

Recall that the homology class of compressing curves for $V_j(i)$ is expressed as $k_i^j[l_j(i)] + [m_j(i)]$. We fix an orientation on the compressing disc whose normal vector points towards the orientation given on $l_j(i)$. Recall that the torus $T_j(i)$ consists of four annuli; two horizontal annuli, the lower annulus and the upper annulus, which are expressed as $A_j(i) \times \{u\}$ and $A_j(i) \times \{v\}$ with respect to the inclusion of \mathbf{M}_i into $S \times [0, 1]$, and two vertical annuli $\partial A_j(i) \times [u, v]$. Consider an essential simple arc $\alpha : [0, 1] \rightarrow A_j(i)$ such that its image in the lower annulus $\alpha_u(t) = (\alpha(t), u)$ is a part of the meridian: $m_j(i) \cap (A_j(i) \times \{u\})$. Then by considering the homotopy class of a compressing curve as above, we see that α_u is homotopic (relative to the endpoints) to an arc obtained by joining a vertical arc expressed as $\alpha(0) \times [u, v]$ oriented upward, a horizontal arc $\alpha_v(t) = (\alpha(t), v)$, another horizontal arc representing $-k_i^j[l_j(i)] \times \{v\}$, and another vertical arc expressed as $\alpha(1) \times [u, v]$ oriented downward. Therefore, each time $g'_i(S)$ passes through $V_j(i)$ in the positive direction, a curve δ on S intersecting c_j is twisted $-k_i^j$ -times around c_j . Here the positive direction is the direction to which a horizontal surface below $V_j(i)$ passes to one above $V_j(i)$. We fix such a transverse orientation for immersions of S . If $g'_i(S)$ passes through $V_j(i)$ in the negative direction δ is twisted k_i^j -times around c_j . (It goes $|k_i^j|$ -times around c_i in the direction of the right hand Dehn twist if $k_i^j > 0$ and in the opposite direction if

$k_i^j < 0$.) Since $\gamma^-(i)$ is homotopic to $g_i''(\gamma^-)$, we see that $g_i'(\tau_{c_j}^{k_i^j a_j}(\gamma^-))$ is homotopic to $\gamma^-(i)$. Similarly, since $\gamma^+(i)$ is homotopic to $g_i''(\tau_{c_j}^{k_i^j}(\gamma^+))$, we see that $g_i'(\tau_{c_j}^{k_i^j(a_j+1)}(\gamma^+))$ is homotopic to $\gamma^+(i)$.

Let h_i be a hierarchy of tight geodesics for \mathbf{M}_i as before. Since $V_j(i)$ appears as a Margulis tube in \mathbf{M}_i , we see that an annular neighbourhood $A(c_j)$ of c_j supports a geodesic $h_i(c_j)$ in h_i . By the definition of gluing blocks in Minsky [40], we see that $\pi_{A(c_j)}(\gamma^-(i))$ is the initial marking and $\pi_{A(c_j)}(\gamma^+(i))$ is the terminal marking of $h_i(c_j)$ if both B^- and B^+ are internal blocks. By Section 6 of Masur–Minsky [38], we see that $\pi_{A(c_j)}(I(h_i))$ is in a uniformly bounded distance from $\pi_{A(c_j)}(\gamma^-(i))$ and $\pi_{A(c_j)}(T(h_i))$ is in a uniformly bounded distance from $\pi_{A(c_j)}(\gamma^+(i))$. Even when B^- or B^+ is a boundary block, we have the same properties: for, since the length of $\gamma^-(i)$ with respect to m_i or that of $\gamma^+(i)$ with respect to n_i is bounded, its projection to $A(c_j)$ is within uniformly bounded distance from those of $I(h_i)$ or $T(h_i)$. Since g_i' is homotopic to Φ_i for large i and \mathbf{M}_i is identified with a subset of $S \times (0, 1)$ using Φ_i as a marking, we see that $\pi_{A(c_j)}(I(h_i))$ is within a uniformly bounded distance from $\pi_{A(c_j)}(\tau_{c_j}^{k_i^j a_j}(\gamma^-))$, whereas $\pi_{A(c_j)}(T(h_i))$ is within a bounded distance from $\pi_{A(c_j)}(\tau_{c_j}^{k_i^j(a_j+1)}(\gamma^+))$. If we consider the pulled-back metrics $(\tau_{c_j}^{k_i^j a_j})^* m_i$ and $(\tau_{c_j}^{k_i^j(a_j+1)})^* n_i$ instead of m_i and n_i , the initial and the terminal markings are twisted around c_i by $-k_i a_j$ times and $-k_i^j(a_j + 1)$ times respectively. Therefore for the quasi-Fuchsian representation $qf((\tau_{c_j}^{k_i^j a_j})^* m_i, (\tau_{c_j}^{k_i^j(a_j+1)})^* n_i)$, the tight geodesic supported on c_j has length bounded as $i \rightarrow \infty$. Therefore, the shortest pants decompositions of $(S, (\tau_{c_j}^{k_i^j a_j})^* m_i)$ and $(S, (\tau_{c_j}^{k_i^j(a_j+1)})^* n_i)$ have Hausdorff limits which do not spiral around c_j . Since the lengths of c_j with respect to $(\tau_{c_j}^{k_i^j a_j})^* m_i$ and $(\tau_{c_j}^{k_i^j(a_j+1)})^* n_i$ do not go to 0, by the proof of Lemma 5.7, this implies that the limits of $(\tau_{c_j}^{k_i^j a_j})^* m_i$ and $(\tau_{c_j}^{k_i^j(a_j+1)})^* n_i$ in the Thurston compactification of $\mathcal{T}(S)$ do not contain c_j as a leaf. We repeat the same argument for every c_j , and let p_i^j and q_i^j be $k_i^j a_j$ and $k_i^j(a_j + 1)$ respectively. This completes the proof of the necessity.

If $a_j > 0$, i.e. g' goes around the torus containing c_j in the counter-clockwise in Figure 4.2 as it proceeds from left to right, then the torus lies in the positive direction viewed from $g'(S)$. Therefore $f(c_j)$ is lifted to a curve lying above the core surface obtained the lift of $f \circ g'$ in M' . This shows that c_j is a core curve of an upper parabolic locus if $a_j > 0$. We can argue in the same way also when $a_j < 0$.

7.2. Existence

We shall next show that the existence of limits of quasi-Fuchsian groups satisfying the conditions (1) and (2). Our construction just follows the argument of Anderson–Canary [2].

We first construct a geometrically finite Kleinian group Γ_0 such that $N = \mathbb{H}^3/\Gamma_0$ is homeomorphic to the complement of $c_j \times \{1/2\}$ ($j = 1, \dots, r$) in $S \times (0, 1)$, and the conformal structures corresponding to the ends $S \times \{0\}$ and $S \times \{1\}$ are the same point $m_0 \in \mathcal{T}(S)$. (Here we identify S with $S \times \{0\}$ and $S \times \{1\}$ by the natural inclusions.) We consider an immersion $g_0 : S \rightarrow N_0$ which is in the standard form in the sense of Lemma 4.13, and wraps a_j times around each $c_j \times \{1/2\}$ counted counter-clockwise as it proceeds from left to right in Figure 4.2 when we identify N_0 with its embedding in $S \times [0, 1]$.

Next, we consider a quasi-conformal deformation of Γ_0 . Let μ_1^-, \dots, μ_s^- and μ_1^+, \dots, μ_t^+ be the components of μ^- and μ^+ that are not shared simple closed curves. We consider the minimal supporting surfaces $\Sigma(\mu_1^-), \dots, \Sigma(\mu_s^-)$ of μ_1^-, \dots, μ_s^- and isotope them so that if two boundary components are isotopic, they coincide. We move the supporting surfaces of μ_1^+, \dots, μ_t^+ in the same way. Let d_1^-, \dots, d_σ^- be the simple closed curve components of $|\mu^-|$ which are not shared by $|\mu^+|$, and d_1^+, \dots, d_τ^+ those of $|\mu^+|$ not shared by $|\mu^-|$. We let e_1^-, \dots, e_q^- be the frontier components of $\bigcup_{j=1}^s \Sigma(\mu_j^-)$ which do not appear in d_1^-, \dots, d_σ^- , and in the same way, we let e_1^+, \dots, e_r^+ be the frontier components of $\bigcup_{j=1}^t \Sigma(\mu_j^+)$ which do not appear in d_1^+, \dots, d_τ^+ . We define a Kleinian group Γ_k for $k \in \mathbb{N}$ to be the one obtained by quasi-conformally deforming the conformal structures m_0 on $S \times \{0\}$ by the earthquake with respect to $k(\bigcup_{j=1}^s \mu_j^-)$ and pinching along $d_1^-, \dots, d_\sigma^-; e_1^-, \dots, e_q^-$ so that the ϵ_0 -thin part around each of d_1^-, \dots, d_σ^- has height k whereas that around each of e_1^-, \dots, e_q^- has height \sqrt{k} ; and m_0 on $S \times \{1\}$ by the earthquake with respect to $k(\bigcup_{j=1}^t \mu_j^+)$ and pinching along $d_1^+, \dots, d_\tau^+; e_1^+, \dots, e_r^+$ in the same way. The pinching is performed so that the conformal structures on $S \setminus ((\bigcup_{j=1}^s \Sigma(\mu_j^-)) \cup d_1^- \cup \dots \cup d_\sigma^-)$ and $S \setminus ((\bigcup_{j=1}^t \Sigma(\mu_j^+)) \cup d_1^+ \cup \dots \cup d_\tau^+)$ do not change. Let N_k be \mathbb{H}^3/Γ_k and $h_k : N \rightarrow N_k$ a natural homeomorphism derived from the quasi-conformal deformation. We regard N_k also as embedded in $S \times [0, 1]$ in such a way that the images of drilled out curves lie on $S \times \{1/2\}$, and the natural identification of this $S \times [0, 1]$ with the one in which N is embedded is compatible with h_k . Then we get an immersion $g_k : S \rightarrow N_k$ which is defined to be the composition $h_k \circ g_0$.

In $S \times [0, 1]$ where N is embedded, we regard μ^+ as lying on $S \times \{1\}$ and μ^- as lying on $S \times \{0\}$. Then every essential annulus in $S \times [0, 1]$

either intersects a torus cusp or $\mu^+ \cup \mu^-$, or has a boundary component contained as a non-peripheral curve in a component of either $S \times \{0\} \setminus \mu^-$ or $S \times \{1\} \setminus \mu^+$, where the conformal structure is not deformed, by the conditions (1*), (2*), and (3*). Therefore, by the main theorem of [43], we see that N_k with marking determined by h_k converges algebraically to a hyperbolic 3-manifold $N_\infty = \mathbb{H}^3/\Gamma_\infty$ with a homeomorphism $h_\infty : N \rightarrow N_\infty$. The laminations $\mu_1^-, \dots, \mu_s^-; \mu_1^+, \dots, \mu_t^+$ represent ending laminations of simply degenerate ends of N_∞ . Since we pinched all frontier curves of $S \setminus (\bigcup_{j=1}^s \Sigma(\mu_j^-) \cup d_1^- \cup \dots \cup d_\sigma^-)$ and of $S \setminus (\bigcup_{j=1}^t \Sigma(\mu_j^+) \cup d_1^+ \cup \dots \cup d_\tau^+)$, by Lemma 3 of [1] each component Σ^f of $S \setminus (\bigcup_{j=1}^s \Sigma(\mu_j^-) \cup d_1^- \cup \dots \cup d_\sigma^-)$ or $S \setminus (\bigcup_{j=1}^t \Sigma(\mu_j^+) \cup d_1^+ \cup \dots \cup d_\tau^+)$ appears as a surface with punctures on the boundary at infinity of N_∞ . On the other hand, the same surface at infinity also appears in the geometric limit since a subregion of a component of Ω_{G_i} invariant under $\pi_1(\Sigma^f)$ converges to the corresponding component of Ω_Γ in the sense of Carathéodory. (See the proof of Proposition 4.2 of Jørgensen-Marden [27].) This shows that each geometrically finite end of $(\mathbb{H}^3/\Gamma_\infty)_0$ has a neighbourhood which is projected homeomorphically into the geometric limit. The covering theorem of Thurston and Canary ([54], [18]) and the argument of Lemma 2.3 of [44] implies that every geometrically infinite end also has a neighbourhood descending homeomorphically to the geometric limit. Therefore the convergence is strong. Let $g_\infty : S \rightarrow N_\infty$ be an immersion which is defined to be $h_\infty \circ g_0$.

Let l_j and m_j be respectively a longitude which lies in a tubular neighbourhood of $c_j \times \{1/2\}$ lying on a level surface along $c_j \times \{1/2\}$, and any meridian intersecting l_j at one point. Let $m_j(k)$ and $l_j(k)$ be a meridian and a longitude in N_k obtained by pulling back m_j and l_j using approximate isometries. We orient them so that the coordinate system $l_j - m_j$ determines an orientation of a torus around $c_j \times \{1/2\}$ whose normal vectors point to the inside of N_∞ . Now, we consider a Dehn filling of N_k such that the compressing disc is attached along a curve represented by $k[l_j(k)] + [m_j(k)]$. Since N_k converges geometrically to N_∞ , we see, by passing to a subsequence, that the filling corresponding to $k[l_j(k)] + [m_j(k)]$ gives rise to a convex cocompact hyperbolic structure. (A general version of hyperbolic Dehn surgery theorem, which applies to this case, was proved by Bromberg [14].) We define M_k to be the thus obtained geometrically finite hyperbolic 3-manifold, which is homeomorphic to $S \times (0, 1)$. We let G_k be the corresponding quasi-Fuchsian group, and $\phi_k : \pi_1(S) \rightarrow \pi_1(M_k)$ an isomorphism derived from the pull-back of g_k by an approximate isometry between N_k and M_k . By the same argument as in [2], we see that the conformal structure at infinity of M_k on the end corresponding to $S \times \{0\}$, denoted by m_k , is obtained by performing the $-ka_j$ -time (right-hand) Dehn twist around the c_j , the earthquake along

$k(\mu_1^- \cup \cdots \cup \mu_s^-)$ and pinching along $d_1^-, \dots, d_\sigma^-; e_1^-, \dots, e_q^-$ from m_0 , and that on the end corresponding to $S \times \{1\}$, denoted by n_k , by performing the $-k(a_j + 1)$ -Dehn twist around the c_j , the earthquake along $k(\mu_1^+ \cup \cdots \cup \mu_t^+)$ and pinching along $d_1^+, \dots, d_\tau^+; e_1^+, \dots, e_r^+$. (Note that the deformation by $-ka_j$ -time Dehn twist is the same as the pull-back by ka_j -time Dehn twist.) By Kerckhoff's cosine formula (see [29, Cor. 3.4] and also [48, §2.2]) we see the contribution of the part of μ_j^- or μ_j^+ to the growth of the length for a curve γ traversing it is asymptotically in the same order as $ki(\gamma, \mu_j^-)$ or $ki(\gamma, \mu_j^+)$ as $k \rightarrow \infty$. Therefore the divergence transverse to μ_j^- (resp. μ_j^+) has the same order as that of d_1^-, \dots, d_σ^- (resp. d_1^+, \dots, d_τ^+), whereas that transverse to e_1^-, \dots, e_q^- (resp. e_1^-, \dots, e_r^+) has lower order. This shows that the limits in the Thurston compactification of the conformal structures m_k and n_k are $[\mu^-]$ and $[\mu^+]$ respectively, and that those of $(\tau_{c_1}^{-ka_1} \circ \cdots \circ \tau_{c_r}^{-ka_r})^*(m_k)$ and $(\tau_{c_1}^{-k(a_1+1)} \circ \cdots \circ \tau_{c_r}^{-k(a_r+1)})^*(n_k)$ are $[\mu_1^- \cup \cdots \cup \mu_s^- \cup d_1^- \cup \cdots \cup d_\sigma^-]$ and $[\mu_1^+ \cup \cdots \cup \mu_t^+ \cup d_1^+ \cup \cdots \cup d_\tau^+]$ respectively. In the case when one of these latter two projective laminations is empty, the corresponding conformal structures stay in a compact set of the Teichmüller space.

By the diagonal argument, we see that $\{(G_k, \phi_k)\}$ converges algebraically to a subgroup of Γ_∞ corresponding to the covering of N_∞ associated to $(g_\infty)_*\pi_1(S)$. Thus we have obtained a sequence of quasi-Fuchsian groups $\{(G_k, \phi_k)\}$ as we wanted.

For the example which we have constructed above, if either μ^- or μ^+ consists only of c_1, \dots, c_r , then $\{(\tau_{c_1}^{-ka_1} \circ \cdots \circ \tau_{c_r}^{-ka_r})^*(m_k)\}$ or $\{(\tau_{c_1}^{-k(a_1+1)} \circ \cdots \circ \tau_{c_r}^{-k(a_r+1)})^*(n_k)\}$ stays in a compact set of the Teichmüller space. We can also make it converge to a projective lamination ν^- or ν^+ not containing c_1, \dots, c_r as leaves, by composing the earthquake along $\sqrt{k}\nu^-$ or $\sqrt{k}\nu^+$.

8. Non-existence of exotic convergence

We shall prove Theorem 3.7 in this section.

Let Γ be a b-group as in the statement and $\psi : \pi_1(S) \rightarrow \Gamma$ an isomorphism giving the marking. Let $\{(G_i, \phi_i)\}$ be a sequence of quasi-Fuchsian groups converging to (Γ, ψ) . What we need to show is that the conformal structures at infinity of the bottom ideal boundaries of the $M_i = \mathbb{H}^3/G_i$ are bounded in the Teichmüller space then.

Let M_∞ be a geometric limit (of a subsequence) of M_i with basepoint coming from a fixed basepoint in \mathbb{H}^3 as usual. Let \mathbf{M} be a model manifold of $(M_\infty)_0$ with a model map $f_\infty : \mathbf{M} \rightarrow (M_\infty)_0$. Let $g' : S \rightarrow \mathbf{M}$ be

a standard algebraic immersion. By our assumption, there is no isolated algebraic parabolic loci in \mathbf{M} . This implies by Lemma 4.16 that there is no torus boundary around which g' can go. If there is an algebraic simply degenerate end below $g'(S)$, it is mapped to a simply degenerate end of $(M_\infty)_0$ which is lifted to a lower simply degenerate end of $(\mathbb{H}^3/\Gamma)_0$. This is a contradiction since Γ is a b-group. Similarly, there is no lower algebraic parabolic locus. Since g' does not go around a torus boundary component, an end closest to $g'(S)$ among those below $g'(S)$ must be algebraic. These imply that the only possible end below $g'(S)$ is a geometrically finite end corresponding to the entire $S \times \{0\}$.

The diameter of the manifold cobounded by $f_\infty \circ g'(S)$ and the lower boundary of the convex core in $(M_\infty)_0$ is finite since the manifold cobounded by $g'(S)$ and $S \times \{0\}$ in \mathbf{M} has finite diameter. This cobordism in $(M_\infty)_0$ can be pulled back to $(M_i)_0$ for large i . Note that there is no closed geodesic in $(M_\infty)_0$ below the lower boundary of the convex core. This implies that the distance from the lower boundary of the convex core of M_i and the pull-back of the cobordism must go to 0. It follows that the lower boundary of the convex core of M_i converges geometrically to that of M_∞ . Since the cobordism above gives a marking on the lower boundary of the convex core homotopic to g' , the marked hyperbolic structure on the lower boundary component of the convex core of M_i converges to that of M_∞ . This shows that the lower conformal structure at infinity of M_i is bounded in the Teichmüller space as $i \rightarrow \infty$ by Sullivan's theorem (see Epstein–Marden [22]).

9. Self-bumping

In this section, we shall prove Theorem 3.8 and Corollary 3.9. For that, we shall show that for $\{qf(m_i, n_i)\}$ as in the statement of Theorem 3.8, there is a continuous deformation to a strong convergent sequence whose algebraic limit is a quasi-conformal deformation of Γ . This will be done by using a model manifold of $qf(m_i, n_i)$ with a special property, which we shall construct in Lemma 9.2. This model does not come from a hierarchy of tight geodesics h_i as before, but its geometric limit can be better understood. Let us state the existence of such a deformation as a proposition.

PROPOSITION 9.1. — *We consider quasi-Fuchsian groups $qf(m_i, n_i)$ and their algebraic limit (Γ, ψ) as in Theorem 3.8. Let c_1, \dots, c_s be upper parabolic curves and c'_1, \dots, c'_t lower parabolic curves on S of (Γ, ψ) . Then, passing to a subsequence of $\{(m_i, n_i)\}$, there is an arc $\alpha_i : [0, 1] \rightarrow QF(S)$ with the following properties. Let (\bar{m}_i, \bar{n}_i) denote a point in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ such that $qf(\bar{m}_i, \bar{n}_i) = \alpha_i(1)$.*

- (1) $\alpha_i(0) = qf(m_i, n_i)$.
- (2) $\{\alpha_i(1) = qf(\bar{m}_i, \bar{n}_i)\}$ converges strongly to a quasi-conformal deformation (Γ', ψ') of (Γ, ψ) as $i \rightarrow \infty$.
- (3) The length of each of c_1, \dots, c_s with respect to \bar{n}_i goes to 0, and the length of each of c'_1, \dots, c'_t with respect to \bar{m}_i also goes to 0 as $i \rightarrow \infty$.
- (4) In the case when Γ is a b-group, the lower conformal structure at infinity of $\alpha_i(t)$ is constant with respect to t , for every i .
- (5) For any neighbourhood U in $AH(S)$ of the quasi-conformal deformation space $QH(\Gamma, \psi)$, there exists i_0 such that for $i > i_0$, the arc α_i is contained in U .

9.1. Proof of Proposition 9.1

Our basic strategy for the proof of this proposition is as follows. We first consider to deform continuously the model manifold of the geometric limit of $\{qf(m_i, n_i)\}$ to that of the strong limit of $\{qf(\bar{m}_i, \bar{n}_i)\}$. Corresponding to this, we get a continuous deformation of the model manifold of $qf(m_i, n_i)$ to that of $qf(\bar{m}_i, \bar{n}_i)$. This will give rise to an arc α_i as desired.

Set $(G_i, \phi_i) = qf(m_i, n_i)$ and $M_i = \mathbb{H}^3/G_i$. We consider the geometric limit M_∞ of M_i with basepoints coming from some fixed basepoint in \mathbb{H}^3 . Let \mathbf{M}_i be a bi-Lipschitz model of $(M_i)_0$ with a model map f_i and \mathbf{M} that of $(M_\infty)_0$ as before. Let $g' : S \rightarrow \mathbf{M}$ be a standard algebraic immersion as in Lemma 4.13. Let E^1, \dots, E^p be the algebraic simply degenerate ends of \mathbf{M} . Recall that \mathbf{M}_i converges geometrically to the union of \mathbf{M} and cusp neighbourhoods, and \mathbf{M}_i corresponds to a hierarchy of tight geodesics h_i determined by G_i .

We renumber E^1, \dots, E^p so that E^1, \dots, E^q are upper ends whereas E^{q+1}, \dots, E^p are lower. (It is possible that $q = 0$ or $q = p$.) We let Σ^j be a subsurface of S such that E^j is contained in a brick B^j of the form $\Sigma^j \times J^j$. Since we assumed that Γ has no isolated parabolic loci, each of c_1, \dots, c_s is homotopic to a component of $\text{Fr}\Sigma^j$ for $j = 1, \dots, q$ and each of c'_1, \dots, c'_t is homotopic to a component of $\text{Fr}\Sigma^j$ for $j = q + 1, \dots, p$.

We shall show that we can modify model manifolds \mathbf{M}_i of M_i to \mathbf{M}'_i so that in its geometric limit \mathbf{M}' the ends E^1, \dots, E^q lie on the same horizontal levels, and the same holds for E^{q+1}, \dots, E^p . This manifold \mathbf{M}'_i will be constructed by removing product regions from $S \times (0, 1)$ which should converge to neighbourhoods of these ends, and considering the standard brick decomposition. We shall construct a uniform Lipschitz map from \mathbf{M}'_i to M_i in Claim 9.3 just by taking split level surfaces to pleated surfaces as in

the construction of model manifolds by Minsky [40]. We need some more argument to show this map can be homotoped to a uniform bi-Lipschitz homeomorphism in Claim 9.4.

In the following argument, we shall only describe the case when both $q > 1$ and $p > q$; that is both upper algebraic simply degenerate ends and lower algebraic simply degenerate ends exist. When one of these does not exist, we can modify the argument below easily just regarding it as an empty set.

LEMMA 9.2. — *There are uniform bi-Lipschitz model manifolds \mathbf{M}'_i for $(M_i)_0$ and \mathbf{M}' for $(M_\infty)_0$, both of which are embedded in $S \times [0, 1]$ preserving horizontal and vertical foliations, and have the following properties.*

- (1) *We can choose basepoints in the thick parts so that $\mathbf{M}'_i[0]$ converges geometrically to $\mathbf{M}'[0]$ with these basepoints.*
- (2) *Under the same choice of the basepoints, \mathbf{M}'_i converges geometrically to the union of \mathbf{M}' and cusp neighbourhoods, which are geometric limits of Margulis tubes.*
- (3) *There is a homeomorphism from \mathbf{M} to \mathbf{M}' taking an algebraic locus of \mathbf{M} to that of \mathbf{M}' . (See Section 4.4 for the definition of algebraic locus.)*
- (4) *We use the same symbol E^j ($j = 1, \dots, p$) to denote the end of \mathbf{M}' corresponding to E^j of \mathbf{M} . Then, E^j is contained in a brick $\bar{B}^j = \Sigma^j \times J^j$ of \mathbf{M}' such that $\inf \bar{B}^1 = \dots = \inf \bar{B}^q, \sup \bar{B}^1 = \dots = \sup \bar{B}^q, \sup \bar{B}^{q+1} = \dots = \sup \bar{B}^p$, and $\inf \bar{B}^{q+1} = \dots = \inf \bar{B}^p$, with $\inf \bar{B}^1 > \sup \bar{B}^p$ unless $q = 0$ or $p = q$.*
- (5) *An algebraic locus in \mathbf{M}' can be homotoped to a horizontal surface lying between the horizontal levels of $\sup \bar{B}^p$ and $\inf \bar{B}^1$.*

Proof. — For each component c of $\text{Fr}\Sigma^j$ for Σ^j among $\Sigma^1, \dots, \Sigma^q$, which were defined above, we consider a solid torus $V(c)$ in $S \times (0, 1)$ which has the form of $A(c) \times J_c$, where $A(c)$ is an annulus with core curve c and J_c is a closed interval in $[0, 1]$ such that $\min J_c = \inf B^j$ and $\max J_c = \sup B^j$, where B^j is a brick in \mathbf{M} as described above. Let $\bar{\mathcal{T}}^+$ be the union of the $V(c)$ for all frontier components c of $\Sigma^1, \dots, \Sigma^q$. Even if a curve c is homotopic to frontier components of two distinct Σ^j and $\Sigma^{j'}$, we take only one solid torus. In such a case, we take J_c so that $\sup J_c = \max\{\sup B^j, \sup B^{j'}\}$, and $\inf J_c = \min\{\inf B^j, \inf B^{j'}\}$. In the same way, we take $\bar{\mathcal{T}}^-$ for $\Sigma^{q+1}, \dots, \Sigma^p$.

Recall that there is an approximate isometry $\rho_i^{\mathbf{M}}$ between $\mathbf{M}_i[0]$ and $\mathbf{M}[0]$. For each component T of $\bar{\mathcal{T}}^+$ or $\bar{\mathcal{T}}^-$, the preimage of its boundary $(\rho_i^{\mathbf{M}})^{-1}(\partial T \cap \mathbf{M}[0])$ lies on a boundary component of $\mathbf{M}_i[0]$, which bounds a solid torus in \mathbf{M}_i . We denote the union of such solid tori for $\bar{\mathcal{T}}^+$ by $\bar{\mathcal{T}}_i^+$, and that for $\bar{\mathcal{T}}^-$ by $\bar{\mathcal{T}}_i^-$. Although $(\rho_i^{\mathbf{M}})^{-1}$ is not defined on the entire

$S \times (0, 1)$, since $\rho_i^{\mathbf{M}}$ takes the algebraic locus g' to an immersion homotopic to Φ_i , it induces a homeomorphism ϱ_i between $S \times (0, 1) \setminus (\overline{\mathcal{T}}^+ \cup \overline{\mathcal{T}}^-)$ and $\mathbf{M}_i \setminus (\overline{\mathcal{T}}_i^+ \cup \overline{\mathcal{T}}_i^-)$. The model map f_i takes $\overline{\mathcal{T}}_i^+$ to Margulis tubes in M_i , which we denote by V_1^i, \dots, V_s^i and $\overline{\mathcal{T}}_i^-$ to other Margulis tubes which we denote by $V_1^{i'}, \dots, V_t^{i'}$, whose core geodesics $c_1^i, \dots, c_s^i; c_1^{i'}, \dots, c_t^{i'}$ correspond to $\Phi_i(c_1), \dots, \Phi_i(c_s); \Phi_i(c_1') \dots, \Phi_i(c_t')$. Note that the length of each of $c_1^i, \dots, c_s^i; c_1^{i'}, \dots, c_t^{i'}$ goes to 0 as $i \rightarrow \infty$. Since f_i is a homeomorphism, $f_i \circ \varrho_i$ is also a homeomorphism between $S \times (0, 1) \setminus (\overline{\mathcal{T}}^+ \cup \overline{\mathcal{T}}^-)$ and $M_i \setminus (V_1^i \cup \dots \cup V_s^i \cup V_1^{i'} \cup \dots \cup V_t^{i'})$.

Now for each $V(c) = A(c) \times J_c$ in $\overline{\mathcal{T}}^+$, we let $V'(c)$ be $A(c) \times [5/8, 3/4]$ in $S \times [0, 1]$, and denote the union of such solid tori by \mathcal{T}^+ . Similarly, for each $V(c)$ in $\overline{\mathcal{T}}^-$, we let $V'(c)$ be $A(c) \times [1/4, 3/8]$, and denote the union of such solid tori by \mathcal{T}^- . Then, we see that there is a homeomorphism from $S \times (0, 1) \setminus (\overline{\mathcal{T}}^- \cup \overline{\mathcal{T}}^+)$, in which \mathbf{M} is embedded, to $S \times (0, 1) \setminus (\mathcal{T}^- \cup \mathcal{T}^+)$ taking a standard algebraic immersion g' to $S \times \{1/2\}$ since \mathbf{M} has no torus boundary around which g' can go (Lemma 4.16) and $\overline{\mathcal{T}}^-$ lies in the lower component of $S \times [0, 1] \setminus g'(S)$ whereas $\overline{\mathcal{T}}^+$ lies in the upper component.

We let a new brick manifold $\widetilde{\mathbf{M}}$ be the one obtained by the standard brick decomposition of $S \times (0, 1) \setminus (\mathcal{T}^- \cup \mathcal{T}^+)$. Then $\widetilde{\mathbf{M}}$ consists of bricks lying on five levels: the top one $S \times [3/4, 1)$, those touching \mathcal{T}^+ along vertical boundaries, the middle one $S \times [3/8, 5/8]$, those touching \mathcal{T}^- along vertical boundaries, and the bottom one $S \times (0, 1/4]$. For each i , we define a labelled brick manifold $\widetilde{\mathbf{M}}_i$ to be the one obtained by giving the conformal structures m_i to the bottom and n_i to the top.

Corresponding to $\widetilde{\mathbf{M}}_i$, we shall construct a geometrically finite hyperbolic manifold from M_i using the drilling theorem of Bromberg [15] and Brock–Bromberg [10]. First, recall that we have closed geodesics $c_1^i, \dots, c_s^i; c_1^{i'}, \dots, c_t^{i'}$ in M_i , which are core curves of Margulis tubes. The complement of the core curves $\widetilde{M}_i = M_i \setminus (\bigcup_{j=1}^s c_j^i \cup \bigcup_{j=1}^t c_j^{i'})$, regarded as a 3-manifold topologically, admits a geometrically finite hyperbolic structure with conformal structures m_i at the bottom and n_i at the top, by Thurston's uniformisation theorem. Since the lengths of $c_1^i, \dots, c_s^i; c_1^{i'}, \dots, c_t^{i'}$ go to 0 as $i \rightarrow \infty$, we can apply the drilling theorem to see that there are a constant K independent of i and a K -bi-Lipschitz homeomorphism \check{f}_i between $M_i \setminus (\bigcup_{j=1}^s V_j^i \cup \bigcup_{j=1}^t V_j^{i'})$ and $(\widetilde{M}_i)_0$. Composing this with the homeomorphism $f_i \circ \varrho_i$ from $S \times (0, 1) \setminus (\overline{\mathcal{T}}^- \cup \overline{\mathcal{T}}^+)$ to $M_i \setminus (\bigcup_{j=1}^s V_j^i \cup \bigcup_{j=1}^t V_j^{i'})$ and the one from $\widetilde{\mathbf{M}}_i$ to $S \times (0, 1) \setminus (\overline{\mathcal{T}}^- \cup \overline{\mathcal{T}}^+)$ described above, we get a homeomorphism from $\widetilde{\mathbf{M}}_i$ to $(\widetilde{M}_i)_0$.

Now, we shall show that for sufficiently large i , this homeomorphism is taken to be a K' -bi-Lipschitz homeomorphism for a constant K' independent of i . Any labelled brick manifold has a decomposition into blocks and geometrically finite bricks, obtained by putting unions of solid tori corresponding to tight geodesics, which we call *tube unions*, as we mentioned in Section 4.3 and can be found in [50, §3]. We now show how this decomposition is obtained in our specific situation. We first take shortest pants decompositions of (S, m_i) and (S, n_i) , and denote their decomposing curves by e_1^-, \dots, e_{3g-3}^- and e_1^+, \dots, e_{3g-3}^+ respectively, where g denotes the genus of S . We put tubes V_1^-, \dots, V_{3g-3}^- , each of which is bounded by two horizontal annuli and two vertical annuli, into the bottom brick corresponding to $S \times (0, 1/4]$ so that $\inf V_j^- = 1/16, \sup V_j^- = 3/16$ for every $j = 1, \dots, 3g-3$ and the vertical projection of a core curve of V_j^- is e_j^- . In the same way we put V_1^+, \dots, V_{3g-3}^+ into the top brick corresponding to $S \times [3/4, 1)$ so that $\inf V_j^+ = 13/16, \sup V_j^+ = 15/16$ and the vertical projection of a core curve of V_j^+ is e_j^+ .

We remove the interiors of these tubes from $\widetilde{\mathbf{M}}_i$ and regard the remaining manifold $\widetilde{\mathbf{M}}_i^1$ as a labelled brick manifold by considering the standard brick decomposition and keeping the conformal structures at infinity m_i and n_i . The bricks corresponding to $S \times (0, 1/16]$ and $S \times [15/16, 1)$ turn into geometrically finite bricks. For each brick $B = \Sigma \times J$ of $\widetilde{\mathbf{M}}_i^1$, where Σ is either S or one of $\Sigma^1, \dots, \Sigma^p$, such that J is one of $[3/16, 1/4]$, $[1/4, 3/8]$, $[3/8, 5/8]$, $[5/8, 3/4]$, and $[3/4, 13/16]$, we consider $\mathcal{A}^- = \partial_- B \cap \partial \widetilde{\mathbf{M}}_i^1$ and $\mathcal{A}^+ = \partial_+ B \cap \partial \widetilde{\mathbf{M}}_i^1$, both of which consist of disjoint horizontal annuli. We consider a tight geodesic g_B in $\mathcal{CC}(\Sigma)$ by setting $I(g_B)$ to be the vertical projections of core curves of \mathcal{A}^- , and $T(g_B)$ to be the vertical projections of core curves of \mathcal{A}^+ . We put a tube union into B corresponding to g_B , i.e. for each simplex s of g_B , we put a union V_s of solid tori bounded by two horizontal annuli and two vertical annuli, whose core curves are vertically homotopic to s , in such a way that $\inf V_{v_I} = \inf B$ for the initial vertex v_I , $\sup V_{v_T} = \sup B$ for the terminal vertex v_T , and $\inf V_s = \sup V_{\text{precs}}$ when $\xi(B) > 4$, whereas $\inf V_s > \sup V_{\text{precs}}$ if $\xi(S) = 4$. If two tubes contained in different bricks have core curves homotopic to each other without touching other tubes, we fuse them into one by putting a tube between the two. We remove the interior of this tube union from $\widetilde{\mathbf{M}}_i^1$ for every B as above and get a brick manifold $\widetilde{\mathbf{M}}_i^2$ by considering the standard brick decomposition. We repeat the same construction for every brick of $\widetilde{\mathbf{M}}_i^1$ that is neither a geometrically finite brick nor with $\xi(B) = 3$, and remove the interiors of the tube unions to get a new brick manifold $\widetilde{\mathbf{M}}_i^2$. We stop this process when we reach the situation where every brick B either is geometrically finite or $\xi(B) = 3$

or $\xi(B) = 4$ but g_B has length 1. We denote the brick manifold at the final stage by $\widetilde{\mathbf{M}}_i[0]$. By cutting every brick of $\widetilde{\mathbf{M}}_i[0]$ with $\xi = 3$ into halves and paste them to bricks above it and below it, we get a decomposition of $\widetilde{\mathbf{M}}_i[0]$ into blocks in the sense of Minsky.

We decompose $\widetilde{\mathbf{M}}_i[0]$ into blocks in this way, and put a metric on $\widetilde{\mathbf{M}}_i[0]$ defined by them, i.e. from the standard metric on each block. Next we attach a Margulis tube to each boundary component of $\widetilde{\mathbf{M}}_i[0]$ except for those corresponding to \mathcal{T}^- and \mathcal{T}^+ in the same way as was explained in the proof of Lemma 4.3. For this model manifold, we have the following:

CLAIM 9.3. — *There are a constant K'_0 depending only on $\xi(S)$ and a K'_0 -Lipschitz map $\widetilde{f}'_i : \widetilde{\mathbf{M}}_i \rightarrow (\widetilde{M}_i)_0$ which takes every split level surface to a pleated surface realising the closed geodesics corresponding to core curves of Margulis tubes which the split level surface touches.*

Proof. — Recall that by construction, $\widetilde{\mathbf{M}}_i$ consists of the following parts: two geometrically finite bricks, which are also boundary blocks, at the top and the bottom; the second bottom part, the second top part, and the middle part each of which is homeomorphic to $S \times I$; and the parts homeomorphic to $\Sigma^j \times I$ for $j = 1, \dots, p$. The decomposition into blocks constructed above induces a hierarchy of tight geodesics in each of these parts, except for the two geometrically finite bricks. In particular, by Lemma 7.9 of Minsky [40], the closed geodesics in M_i corresponding to core curves of Margulis tubes in $\widetilde{\mathbf{M}}_i$ have lengths bounded by a constant D depending only on $\xi(S)$, for Σ^j is also a subsurface of S . We then construct a map taking every split level surface to the corresponding pleated surface as in [40], and this map extends to a Lipschitz map $\widetilde{f}'_i : \widetilde{\mathbf{M}}_i \rightarrow (\widetilde{M}_i)_0$ by the technique of “interpolating pleated surfaces”, whose Lipschitz constant depends only on $\xi(S)$ by the same argument as Minsky [40]. \square

Next we shall show that this map can be homotoped to a uniform bi-Lipschitz homeomorphism.

CLAIM 9.4. — *The Lipschitz map \widetilde{f}'_i can be homotoped to a K_0 -bi-Lipschitz homeomorphism $f'_i : \widetilde{\mathbf{M}}_i \rightarrow (\widetilde{M}_i)_0$, where K_0 depends only on $\xi(S)$.*

Proof. — For Kleinian surface groups, it was shown in Brock–Canary–Minsky [13] that the map is homotoped to a homeomorphism keeping the Lipschitz property, by rearranging the order of pleated surfaces to make it accord with the order of split level surfaces, and then homotoping each pleated surface to an embedding. Some argument involving geometric limits was then used to show the resulting map is in fact uniformly bi-Lipschitz,

which works without changes in our situation. In each of the first two processes, to guarantee the Lipschitz property of the map, it was necessary to show that we can choose a homotopy which does not pass through a Margulis tube around a very short closed geodesic. To homotope a pleated surface to an embedding, the technique of Freedman–Hass–Scott [24] was used, and we can use the same argument in our situation. What we only need to check in our settings is that we can rearrange the order of pleated surfaces without passing through Margulis tubes around very short closed geodesics.

In each of the parts of $\widetilde{\mathbf{M}}_i$ that we have described above except for the two geometrically finite blocks, the block decomposition of $\widetilde{\mathbf{M}}_i[0]$ gives a sequence of split level surfaces $\{F_k\}$ in which F_{k+1} is obtained from F_k by changing two adjacent thrice-punctured spheres to another pair of thrice-punctured spheres in such a way the step corresponds to an elementary move of slices of a hierarchy. In the construction of the Lipschitz map \bar{f}'_i above, the homotopy between the pleated surface corresponding to F_k and F_{k+1} was chosen to move only such a four-times punctured sphere or an once-punctured torus so that intermediate surfaces have curvature less than -1 in the moving part. The standard argument using the area of meridional discs in Margulis tubes implies that for sufficiently small $\epsilon > 0$ depending only on $\xi(S)$, the image of \bar{f}'_i can intersect any ϵ -Margulis tube V in $\widetilde{\mathbf{M}}_i$ only as the image of the corresponding Margulis tube in $\widetilde{\mathbf{M}}_i$.

Now, we see how we can rearrange the order of pleated surfaces without passing through ϵ -Margulis tubes. Since the argument is the same for every part, we only consider the case of the second bottommost part corresponding to $S \times [3/16, 1/4]$. The uppermost split level surface is a union of three-holed spheres corresponding to pants decomposition of S obtained by deleting from $S \times \{1/4\}$ the tube unions which we put into $\widetilde{\mathbf{M}}_i$, and each of the pairs of pants is mapped to a pleated surface which is a thrice-punctured sphere with each of its punctures lying on the axes of Margulis tubes or extends to a torus cusp. We note that each of c'_1, \dots, c'_t is homotopic to a boundary component of one of these pairs of pants. We denote by $U(\Sigma)$ the pleated surface which is the union of (the closures of) these thrice-punctured spheres. Let \bar{V} be a Margulis tube appearing in the part of $\widetilde{\mathbf{M}}_i$ corresponding to $S \times [3/16, 1/4]$ below those which $S \times \{1/4\}$ passes through, and suppose that it is mapped by \bar{f}'_i to a Margulis tube V with axis having length less than ϵ in $\widetilde{\mathbf{M}}_i$ which lies above $U(\Sigma)$, i.e. such that there is no proper half-open arc starting from V and tending to $S \times \{0\}$ without algebraic intersection with $U(\Sigma)$. We can take a sub-solid torus \bar{V}' of \bar{V} which is mapped by \bar{f}'_i onto the ϵ -Margulis tube V' in V . Since each vertex can appear only once in the hierarchy and by our choice of ϵ , we see that $\bar{f}'_i((\widetilde{\mathbf{M}}_i \cap S \times (0, 1/4]) \setminus \bar{V}')$ is disjoint from V' .

Therefore, $\bar{f}'_i|((\widetilde{\mathbf{M}}_i \setminus (f'_i)^{-1}(\text{Int}V'))$ is a proper map to $(\widetilde{M}_i)_0 \setminus \text{Int}V'$. Since \bar{V}' lies below $S \times \{1/4\}$, its longitude can be homotoped towards the lower end corresponding to $S \times \{0\}$ without algebraic intersection with the split level surface lying on $S \times \{1/4\}$ nor $(f'_i)^{-1}(\text{Int}V')$. This property must be preserved by the map \bar{f}'_i , whereas the longitude of V' cannot be homotoped towards the lower end without touching $U(\Sigma)$. This is a contradiction.

Next suppose that there is a split level surface F of $\widetilde{\mathbf{M}}_i \cap S \times [3/16, 1/4]$ which is mapped into a pleated surface $P(F)$ above $U(\Sigma)$. Since F is homotopic to the lower end of $\widetilde{\mathbf{M}}_i$, the pleated surface $P(F)$ can be homotoped below $U(\Sigma)$. If the image of the homotopy can intersect an ϵ -Margulis tube, it must be one coming from a Margulis tube lying below F in $\widetilde{\mathbf{M}}_i$. Since we have already shown that such an ϵ -Margulis tube cannot lie above $U(\epsilon)$, we see that $P(F)$ can be homotoped below $U(\Sigma)$ without passing through ϵ -Margulis tubes.

By replacing the uppermost split level surface with any split level surface in the part corresponding to $S \times [3/16, 1/4]$ and repeating the argument above, we see that we can homotope \bar{f}'_i without passing ϵ -Margulis tubes to make it preserve the order of split level surfaces. This proves the most important step of the proof of our claim.

As we explained at the beginning of the proof, we can then homotope \bar{f}'_i to make the image of each split level surface an embedding using the result of Freedman–Hass–Scott [24], and the entire map a homeomorphism, both without passing through ϵ -Margulis tubes. The fact that the homotopy which we constructed does not pass through ϵ -Margulis tubes implies the resulting homeomorphism is again K'' -Lipschitz for some K'' depending only on $\xi(S)$. Finally, by applying the argument of [13] involving geometric limits, we can show that the resulting map is a K_0 -bi-Lipschitz homeomorphism f'_i for some constant K_0 depending only on $\xi(S)$. \square

Now, by composing $(\bar{f}'_i)^{-1}$ with f'_i , we get a K' -bi-Lipschitz embedding of $\widetilde{\mathbf{M}}'_i$ into M_i , with K' depending only on K and K_0 . By filling appropriate Margulis tubes into $\widetilde{\mathbf{M}}'_i$, we get a K'' -bi-Lipschitz model manifold \mathbf{M}'_i for $(M_i)_0$, with K'' independent of i , for sufficiently large i .

It remains to verify that this \mathbf{M}'_i has a geometric limit with the desired properties. Since \mathbf{M}'_i has decomposition into blocks and geometrically finite bricks, and is a uniform bi-Lipschitz model for $(M_i)_0$, by the same argument as the proof of Theorem 4.2 (see also [50, §5] for the original argument), there is a labelled brick manifold \mathbf{M}' which is a bi-Lipschitz model manifold of $(M_\infty)_0$ such that \mathbf{M}'_i converges to the union of \mathbf{M}' and cusp neighbourhoods geometrically. This shows the conditions (1) and (2). Since both \mathbf{M}

and \mathbf{M}' are model manifolds of the same geometric limit $(M_\infty)_0$, there is a homeomorphism taking an algebraic locus to an algebraic locus between them, which shows the condition (3).

Since the geometric convergence of \mathbf{M}'_i to \mathbf{M}' preserves horizontal foliations, and the embedding of \mathbf{M}' also preserves the horizontal levels, we see that the two horizontal annuli of each component of $\overline{\mathcal{T}}^-$ lie on $S \times \{1/4\}$ and $S \times \{3/8\}$, whereas those of $\overline{\mathcal{T}}^+$ lies on $S \times \{5/8\}$ and $S \times \{3/4\}$. Therefore, we have $\inf \overline{B}^1 = \dots = \inf \overline{B}^q$ and $\sup \overline{B}^{q+1} = \dots = \sup \overline{B}^p$. We can modify the embedding of \mathbf{M}' into $S \times [0, 1]$ only at these bricks to make them have the same height and to make the condition $\sup \overline{B}^1 = \dots = \sup \overline{B}^q, \inf \overline{B}^{q+1} = \dots = \inf \overline{B}^p$ hold. Finally, we verify the condition (5). Since E^1, \dots, E^q are upper ends and E^{q+1}, \dots, E^p are lower ends, an algebraic locus must pass under $\overline{B}^1, \dots, \overline{B}^q$ and above $\overline{B}^{q+1}, \dots, \overline{B}^p$. By our assumption, there are no other algebraic ends, neither torus boundary components containing algebraic parabolic curves. Therefore, there is no obstruction to homotope an algebraic locus to a horizontal surface in this region above $\overline{B}^{q+1}, \dots, \overline{B}^p$ and below $\overline{B}^1, \dots, \overline{B}^q$. \square

We shall use the symbol g' to denote the horizontal algebraic locus in \mathbf{M}' as in (5) of the above lemma. (This is the same symbol as the standard immersion in \mathbf{M} , but there is no fear of confusion since we can distinguish them by model manifolds in which they are lying.)

Take t and t' such that $\overline{B}^j = \Sigma^j \times [5/8, t)$ for $j = 1, \dots, q$ and $\overline{B}^j = \Sigma^j \times (s', 3/8]$ for $j = q + 1, \dots, p$. Let $\rho_i^{\mathbf{M}'}$ denote an approximate isometry between \mathbf{M}'_i and \mathbf{M}' which is associated to the geometric convergence of \mathbf{M}'_i to the union of \mathbf{M}' and cusp neighbourhoods. We denote by x'_i and x'_∞ basepoints in the thick parts of \mathbf{M}'_i and \mathbf{M}' , which we used for the geometric convergence. By our construction of \mathbf{M}'_i , there are bricks $B_i^j \cong \Sigma^j \times [5/8, 3/4]$ ($j = 1, \dots, q$) and $B_i^j \cong \Sigma^j \times [1/4, 3/8]$ ($j = q + 1, \dots, p$) of $\widetilde{\mathbf{M}'_i}$ which contains $(\rho_i^{\mathbf{M}'})^{-1}(\overline{B}^j \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$.

Now, we shall construct two sequences of markings on S starting from c_{m_i} and c_{n_i} respectively, and a sequence of markings on Σ^j for $j = 1, \dots, p$, in both of which a marking advances by an elementary move. Recall that \mathbf{M}'_i , which we defined in the proof of Lemma 9.2 above, consists of interior bricks lying on five levels, a disjoint union of the product I -bundles over three-holed spheres lying at two levels, and two geometrically finite bricks. As was described there, there are tube unions in \mathbf{M}'_i which decompose it into blocks. Since the hierarchies which we used for the decomposition are all complete, by perturbing tubes vertically if necessary, each real front of every brick can be assumed to intersect the tubes constituting the decomposition in such

a way that the complement of the tubes in the front is a disjoint union of thrice-punctured spheres.

We first consider the bottom interior brick corresponding to $S \times [3/16, 1/4]$. We denote this brick by \bar{B}_i . By our construction of tube unions in the proof of Lemma 9.2, we have a 4-complete hierarchy $\check{h}(\bar{B}_i)$ on S corresponding to tube unions in \bar{B}_i , whose initial marking is c_{m_i} . In each annular component domain in $\check{h}(\bar{B}_i)$ except for those corresponding to tubes intersecting the upper front of \bar{B}_i , we can put a tight geodesic, which is uniquely determined. The length of such a geodesic determines the $\omega_{\widetilde{M}_i}$ of the Margulis tube which was attached to the corresponding torus boundary. For each annular component domain corresponding to a tube intersecting the upper front, we put a geodesic of length 0 at this stage, for the terminal markings for such geodesics are not determined if we look only at \bar{B}_i . We denote by $h(\bar{B}_i)$ the hierarchy obtained by adding such annular geodesics to $\check{h}(\bar{B}_i)$. A resolution $\tau(\bar{B}_i) = \{\tau(\bar{B}_i)_k\}$ of $h(\bar{B}_i)$ gives rise to a sequence of split level surfaces starting from the one lying on the lower front and ending at the one lying on the upper front. On the other hand, forward steps in the resolution $\tau(\bar{B}_i)$ correspond to elementary moves of markings which can be assumed to be clean. (See Minsky [40].)

The situation is quite similar for the top interior brick, which corresponds to $S \times [3/4, 13/16]$, and we denote by \hat{B}_i . As in the case of \bar{B}_i , we have a hierarchy $h(\hat{B}_i)$. Reversing the order of slices of this hierarchy, we get a resolution, which we denote by $\tau(\hat{B}_i) = \{\tau(\hat{B}_i)_k\}$, giving rise to split level surfaces starting from the one on the upper front and ending at the one on the lower front, and a sequence of clean markings on S advancing by elementary moves. Since \mathbf{M}'_i is obtained from $\widetilde{\mathbf{M}}'_i$ by filling Margulis tubes, the split level surfaces as above for \bar{B}_i and \hat{B}_i can be also regarded as lying in \mathbf{M}'_i .

Next we consider a brick B_i^j for $j = q + 1, \dots, p$, which corresponds to \bar{B}^j by $\rho_i^{\mathbf{M}'}$. Again, we have a complete hierarchy $h(B_i^j)$ supported on Σ^j corresponding to the decomposition of B_i^j into blocks and filled-in Margulis tubes. We can take a resolution $\tau(B_i^j) = \{\tau(B_i^j)_k\}$ starting from the restriction of the last slice of $\tau(\bar{B}_i)$ to Σ^j . Similarly, for each brick B_i^j for $j = 1, \dots, p$, we have a complete hierarchy $h(B_i^j)$. We reverse the order of slices in this case, and consider a resolution $\tau(B_i^j) = \{\tau(B_i^j)_k\}$ starting from the restriction of the last slice of $\tau(\hat{B}_i)$.

For each $n \in \mathbb{N}$, we consider the n -th slice $\tau(B_i^j)_{-n}$ counting from the last one of each $\tau(B_i^j)$ for $j = q + 1, \dots, p$. Corresponding to the slice $\tau(B_i^j)_{-n}$,

we have a split level surface $S(B_i^j)(n)$ embedded in B_i^j . Taking the union of all $S(B_i^j)(n)$ for $j = q + 1, \dots, p$ together with the thrice-punctured spheres lying on $\partial_+ \bar{B}_i \setminus (\bigcup_{j=p+1}^q B_i^j)$ which are parts of the last slice of $\tau(\bar{B}_i)$, we get a split level surface $S_i^-(n)$. From this, we construct an *extended level surface* $\widehat{S}_i^-(n)$ as follows.

Let \mathbf{T}_- be the union of Margulis tubes intersecting $S_i^-(n)$. Each torus T in $\partial \mathbf{T}_-$ is split into two annuli by cutting it along $T \cap S_i^-(n)$, which we call the upper annulus and the lower annulus depending on their locations. We paste the upper one to $S_i^-(n)$ for each T and get a surface homeomorphic to S , which we define to be $\widehat{S}_i^-(n)$. We also regard a union of slices, taken one from each $\tau(B_i^j)$ ($j = q + 1, \dots, p$) as above, as a marking on the entire S by defining its restriction to $S \setminus \bigcup_{j=q+1}^p \Sigma^j$ to be the marking defined by the last slice of $\tau(\bar{B}_i)$, and setting a transversal of a component c of $\text{Fr}\Sigma^j$ to be the one determined by the first vertex of a geodesic in $h(\bar{B}_i)$ supported on an annular neighbourhood $A(c)$ of c , which is uniquely determined by the Margulis tube of \mathbf{M}_i' corresponding to c . We note that every curve in the $\text{Fr}B_i^j$ is a base curve of the last slice of $\tau(\bar{B}_i)$ since it is a core curve of the intersection of the upper front of \bar{B}_i and a torus boundary component of $\widetilde{\mathbf{M}}_i$. We repeat the same construction for B_i^j with $j = 1, \dots, q$, considering the n -th slice $\tau(B_i^j)_n$ $j = 1, \dots, q$ counting from the first one, and get a split level surface $S_i^+(n)$ and an extended split level surface $\widehat{S}_i^+(n)$, this time using lower annuli. Also, we can regard a union of slices taken one from each $\tau(B_i^j)$ ($j = 1, \dots, q$) as a marking on the entire S in the same way, by setting a transversal of each component of $\text{Fr}\Sigma^j$ to be the last vertex this time.

Recall that in the construction of model manifolds, Minsky defined the initial and terminal markings to be the shortest markings for the conformal structures at infinity. We consider a correspondence in the opposite direction. We can choose some positive constant δ_0 such that for any clean marking, there is a marked conformal structure on S for which the marking is shortest and in which all the curves (both base curves and transversals) of the marking have hyperbolic lengths greater than δ_0 . (The existence of such a constant is easy to see since there are only finitely many configurations of clean markings up to homeomorphisms of S . Recall that a clean marking is said to be shortest when its base curves constitute a shortest pants decomposition and the transversals are shortest among those obtained by Dehn twists around the base curves.)

DEFINITION 9.5. — *For each marking μ , we define the marked conformal structure $m(\mu)$ to be one for which a clean marking μ_0 compatible with μ is shortest and such that every curve of μ_0 has hyperbolic length greater than δ_0 .*

Although there are more than one such structures, we just choose any one of them. By using Kerckhoff's formula in [28] for instance, we can easily see that there is a universal constant K depending only on δ_0 (and S) such that if μ' is obtained from μ by one step in the resolutions $\tau(\overline{B}_i)$ or $\tau(\widehat{B}_i)$ or $\tau(B_i^j)$, which corresponds to an elementary move of markings on S or Σ_j , then the Teichmüller distance between $m(\mu)$ and $m(\mu')$ is bounded by K , whatever choices of $m(\mu)$ and $m(\mu')$ we make.

Now, we shall consider two sequences of markings: the first one connects the marking corresponding to first marking of $\tau(\overline{B}_i)$ with that corresponding to the union of $\tau(B_i^j)_{-n}$ ($j = q + 1, \dots, p$); and the other, which proceeds in the negative direction, connects the marking corresponding to the last slice of $\tau(\widehat{B}_i)$ with that corresponding to the union of $\tau(B_i^j)_n$ $j = 1, \dots, q$. The first one is obtained by combining a sequence of markings corresponding to $\tau(\overline{B}_i)$ with those corresponding to the $\tau(B_i^j)$ ($j = q + 1, \dots, p$). By our definition of markings corresponding to slices in $\tau(B_i^j)$, if we choose the first slice from every $\tau(B_i^j)$, the corresponding marking coincides with the one corresponding to the last slice of $\tau(\overline{B}_i)$, where vertices on annular geodesics are set to be the first ones. Therefore after the sequence of markings corresponding to $\tau(\overline{B}_i)$, we can append the one obtained by advancing slices in the $\tau(B_i^j)$ ($j = q + 1, \dots, p$), so that we proceed at each step by advancing a slice in $\tau(B_i^j)$ which is farthest from the goal, i.e. the n -th slice counted from the last one, up to the point where all the slices are the n -th counting from the last one. We denote the thus obtained sequence of markings by $\mu^-(i, n) = \{\mu_k^-(i, n)\}_k$. In the same way, we define a sequence of markings obtained by combining one corresponding to $\tau(\widehat{B}_i)$ with those corresponding to $\tau(B_i^j)$ ($j = 1, \dots, q$). We denote this sequence by $\mu^+(i, n) = \{\mu_l^+(i, n)\}_l$. To simplify the notation, we denote the last markings in $\mu^-(i, n)$ and $\mu^+(i, n)$ by $\mu_\infty^-(i, n)$ and $\mu_\infty^+(i, n)$ respectively.

For any n , if we take a sufficiently large i , the component of $(\rho_i^{\mathbf{M}'})^{-1}(\mathbf{M}' \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$ intersecting $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ contains all $\widehat{S}_i^-(k)$ and $\widehat{S}_i^+(k)$ with $k \leq n$, since the distance between B_i^j and $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ is uniformly bounded, and so are the diameters of extended level surfaces. Moreover making i larger if necessary, we can make $\widehat{S}_i^-(k)$ and $\widehat{S}_i^+(k)$ with $k \leq n$ all homotopic to $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ in $(\rho_i^{\mathbf{M}'})^{-1}(\mathbf{M}' \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$ since there is no torus boundary inside the \overline{B}^j , which implies that if we fix k , then the diameters of Margulis tubes intersecting essentially a homotopy between $\widehat{S}_i^-(k)$ (or $\widehat{S}_i^+(k)$) and $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ in \mathbf{M}'_i are bounded independently of i . Now, for each i , let \mathbf{n}_i be the largest number such that both $S_i^-(k)$ and $S_i^+(k)$ with all $k \leq \mathbf{n}_i$ are contained in the component of $(\rho_i^{\mathbf{M}'})^{-1}(\mathbf{M}' \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$ intersecting $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ and are homotopic to $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ there. By

the above observation, we have $\mathbf{n}_i \rightarrow \infty$ as $i \rightarrow \infty$. We then take another \mathbf{n}'_i which also goes to ∞ , such that $\mathbf{n}_i - \mathbf{n}'_i$ is positive and goes to ∞ as $i \rightarrow \infty$.

With this preparation in hand, we can now construct arcs in the Teichmüller space as follows. Since $\mu_k^-(i, \mathbf{n}'_i)$ proceeds by advancing the slice among those in $\tau(B_i^j)$ which is farthest from the \mathbf{n}'_i -th one counting from the last one, we see that for some $k = k_0$ the marking $\mu_k^-(i, \mathbf{n}'_i)$ reaches the split level surface $S^-(\mathbf{n}_i)$. Recall that for each $\mu_k^-(i, \mathbf{n}'_i)$ in $\mu^-(i, \mathbf{n}'_i)$, the point $m(\mu_k^-(i, \mathbf{n}'_i))$ in the Teichmüller space was defined.

DEFINITION 9.6. — *We define a new function m' by setting $m'(\mu_k^-(i, \mathbf{n}'_i))$ to be $m((\mu_k^-(i, \mathbf{n}'_i))$ for $k \leq k_0$, and for $k > k_0$ to be the hyperbolic structure obtained from $m((\mu_k^-(i, \mathbf{n}'_i))$ by pinching the curves c'_1, \dots, c'_t so that their extremal lengths become $e^{k_0 - k}$ and the lengths of other curves in $\text{base}(\mu_k^-(i, \mathbf{n}'_i))$ remain bounded below by δ_0 .*

Starting from $\mu_1^-(i, \mathbf{n}'_i)$, which is a shortest marking for m_i , we consider for each step $\mu_k^-(i, \mathbf{n}'_i) \rightarrow \mu_{k+1}^-(i, \mathbf{n}'_i)$ in $\mu^-(i, \mathbf{n}'_i)$, a Teichmüller geodesic arc connecting $m'(\mu_k^-(i, \mathbf{n}'_i))$ with $m'(\mu_{k+1}^-(i, \mathbf{n}'_i))$, and then construct a broken geodesic arc $\alpha^-(i, \mathbf{n}'_i)$ connecting $m'(\mu_1^-(i, \mathbf{n}'_i))$ to $m'(\mu_\infty^-(i, \mathbf{n}'_i))$ by joining them. We note that by the definition of m' , each constituting geodesic arc has uniformly bounded length. Both $m'(\mu_1^-(i, \mathbf{n}'_i))$ and m_i have $\mu_1^-(i, \mathbf{n}'_i)$ as a shortest marking, but the hyperbolic lengths of base curves of $\mu_1^-(i, \mathbf{n}'_i)$ in m_i may be different from those in $m'(\mu_1^-(i, \mathbf{n}'_i))$. We can connect m_i with $m'(\mu_1^-(i, \mathbf{n}'_i))$ by a Teichmüller quasi-geodesic so that $\text{base}(\mu_1^-(i, \mathbf{n}'_i))$ remains a shortest pants decomposition throughout the points on the geodesic. (The quasi-geodesic constant depends only on S .) We define $\bar{\alpha}^-(i, \mathbf{n}'_i)$ to be a broken quasi-geodesic arc obtained by joining this quasi-geodesic with $\alpha^-(i, \mathbf{n}'_i)$. In the same way, we construct a broken quasi-geodesic arc $\bar{\alpha}^+(i, \mathbf{n}'_i)$, which connects n_i with $m(\mu_\infty^+(i, \mathbf{n}'_i))$ by pinching the curves c_1, \dots, c_s . In the case when either lower or upper algebraic simply degenerate ends do not exist, we define the corresponding $\bar{\alpha}^-(i, \mathbf{n}'_i)$ or $\bar{\alpha}^+(i, \mathbf{n}'_i)$ to be a constant map.

Let $\alpha_i^\pm : [0, k_i^\pm] \rightarrow \mathcal{T}(S)$ be broken quasi-geodesic arcs $\bar{\alpha}^\pm(i, \mathbf{n}_i)$ connecting $m'(\mu_\infty^\pm(i, \mathbf{n}'_i))$ with m_i, n_i which are constructed as above by joining Teichmüller geodesics and one quasi-geodesic. We parametrise α_i^\pm so that $\alpha_i^\pm|_{[0, 1]}$ is the appended quasi-geodesics connecting m_i or n_i with $m'(\mu_1^\pm(i, \mathbf{n}'_i))$, and $\alpha_i^\pm|_{[s, s+1]}$ for $s \in \mathbb{N}$ corresponds to the Teichmüller geodesic constituting α_i^\pm which connects $m'(\mu_s^\pm(i, \mathbf{n}'_i))$ with $m'(\mu_{s+1}^\pm(i, \mathbf{n}'_i))$. We define $\beta_i : [0, \bar{k}_i] \rightarrow QF(S)$ by setting $\beta_i(t)$ to be $qf(\alpha_i^-(t), \alpha_i^+(t))$, where $\bar{k}_i = \max\{k_i^+, k_i^-\}$ and assuming that the arcs α_i^\pm stay at the endpoint after t gets out of the domain.

CLAIM 9.7. — *A sequence of quasi-Fuchsian group $\{qf(m'(\mu_\infty^-(i, \mathbf{n}'_i)), m'(\mu_\infty^+(i, \mathbf{n}'_i)))\}$ converges strongly to a quasi-conformal deformation of (Γ, ψ) as $i \rightarrow \infty$ after passing to a subsequence.*

Proof. — By our definition of the function m' and the argument in the proof of Lemma 9.2, a uniformly bi-Lipschitz model manifold for the quasi-Fuchsian group $qf(m'(\mu_\infty^-(i, \mathbf{n}'_i)), m'(\mu_\infty^+(i, \mathbf{n}'_i)))$, which we denote by $\mathbf{M}'(\mathbf{n}'_i)$, can be obtained from the submanifold of $\mathbf{M}'_i[0]$ cobounded by $S_i^-(\mathbf{n}'_i)$ and $S_i^+(\mathbf{n}'_i)$ by pasting boundary blocks corresponding to $m'(\mu_\infty^-(i, \mathbf{n}'_i))$ and $m'(\mu_\infty^+(i, \mathbf{n}'_i))$ respectively and filling Margulis tubes. For large i , this model manifold contains $(\rho_i^{\mathbf{M}'})^{-1} \circ g'(S)$ and $(\rho_i^{\mathbf{M}'})^{-1} \circ g'$ is homotopic to Φ_i . It follows that for a fixed generator system of $\pi_1(S)$ with base point on $(\rho_i^{\mathbf{M}'})^{-1} \circ g'(S)$, the length of the shortest closed loop in $\mathbf{M}'(\mathbf{n}'_i)$ representing each generator is bounded as $i \rightarrow \infty$, and hence that $qf(m'(\mu_\infty^-(i, \mathbf{n}'_i)), m'(\mu_\infty^+(i, \mathbf{n}'_i)))$ converges algebraically after passing to a subsequence. Let (Γ', ψ') be the algebraic limit, and denote \mathbb{H}^3/Γ' by $M_{\Gamma'}$.

Let $\mathbf{M}'(\mathbf{n}'_\infty)$ be the geometric limit of the complement of the boundary blocks in $\mathbf{M}'(\mathbf{n}'_i)$ as $i \rightarrow \infty$. Then $\mathbf{M}'(\mathbf{n}'_\infty)$ is embedded in \mathbf{M}' as a submanifold, by our definition of $\mathbf{M}'(\mathbf{n}'_i)$. For each $j = 1, \dots, q$, the intersection of $\rho_i^{\mathbf{M}'}(S_i^+(\mathbf{n}'_i))$ with \bar{B}^j goes deeper and deeper into \bar{B}^j to the direction of the end E_j as $i \rightarrow \infty$ since $\mathbf{n}'_i \rightarrow \infty$. The same holds for $\rho_i^{\mathbf{M}'}(S_i^-(\mathbf{n}'_i)) \cap \bar{B}^j$ for $j = q+1, \dots, p$. Therefore the entire \bar{B}^j is contained in $\mathbf{M}'(\mathbf{n}'_\infty)$ for every $j = 1, \dots, p$.

Now we look at the two boundary blocks of $\mathbf{M}'(\mathbf{n}'_i)$, which we denote by b_i^- and b_i^+ , where b_i^- has conformal structure at infinity $m'(\mu_\infty^-(i, \mathbf{n}'_i))$ and b_i^+ has conformal structure at infinity $m'(\mu_\infty^+(i, \mathbf{n}'_i))$, and consider their geometric limits. By our definition of m' and from the fact that $\mathbf{n}_i - \mathbf{n}'_i \rightarrow \infty$, we see that the hyperbolic lengths of $c_1^{i'}, \dots, c_t^{i'}$ with respect to $m'(\mu_\infty^-(i, \mathbf{n}'_i))$ and those of c_1^i, \dots, c_s^i with respect to $m'(\mu_\infty^+(i, \mathbf{n}'_i))$ go to 0. Let F^1, \dots, F^r be the components of $S \setminus \bigcup_{k=1}^q \Sigma^k$. Then $m'(\mu_\infty^+(i, \mathbf{n}'_i))$ restricted to each one of F^1, \dots, F^r has pants decomposition whose lengths are bounded below by δ_0 , and the boundary components of F^1, \dots, F^r correspond to c_1^i, \dots, c_s^i . Therefore, the geometric limit of b_i^+ with basepoint in the part of F^k is homeomorphic to $\text{Int}F^k \times I$ with conformal structure at infinity corresponding to a complete hyperbolic structure on $\text{Int}F^k$ making each frontier component a cusp. The same holds for b_i^- considering the components of $S \setminus \bigcup_{k=q+1}^p \Sigma^k$.

Therefore, the geometric limit of $\mathbf{M}'(\mathbf{n}'_i)$ has upper simply degenerate ends corresponding to $\Sigma^1, \dots, \Sigma^q$, lower simply degenerate ends corresponding to $\Sigma^{q+1}, \dots, \Sigma^p$, upper geometrically finite ends corresponding to the components of $S \setminus \bigcup_{k=1}^q \Sigma^k$, and lower geometrically finite ends corresponding to the components of $S \setminus \bigcup_{k=q+1}^p \Sigma^k$, all of which are algebraic. This

shows that the geometric limit has fundamental group isomorphic to $\pi_1(S)$, and hence that the convergence is strong. We denote a representative of this strong limit by (Γ', ψ') . The model manifold shows that the ends of $M_{\Gamma'}$ consists of simply degenerate ends corresponding to E^1, \dots, E^p and geometrically finite ends and that every parabolic locus touches one of E^1, \dots, E^p . Since the ending laminations of E^1, \dots, E^p are the same as those of the corresponding ends of $M' = \mathbb{H}^3/\Gamma$, by the ending lamination theorem due to Brock–Canary–Minsky [13], we see that (Γ', ψ') is a quasi-conformal deformation of (Γ, ψ) . \square

CLAIM 9.8. — *For any sequence $\{t_i \in [0, \bar{k}_i]\}$, the sequence $\{\beta_i(t_i) \in QF(S)\}$ converges algebraically to a quasi-conformal deformation of (Γ, ψ) .*

Proof. — Recall that $\beta_i(t_i) = qf(\alpha_i^-(t_i), \alpha_i^+(t_i))$, and that α^- and α^+ are broken quasi-geodesic arcs consisting of Teichmüller geodesics with bounded lengths except for the first quasi-geodesics; $\alpha^\pm \ll [0, 1]$, which may be long.

We first assume that neither $\alpha_i^-(t_i)$ nor $\alpha^+(t_i)$ lies on the first quasi-geodesics, which connect $m'(\mu_1^-(i, \mathbf{n}_i))$ with and $m'(\mu_1^+(i, \mathbf{n}_i))$ with respectively. In this case, since $\alpha_i^-(t_i)$ and $\alpha^+(t_i)$ lie on Teichmüller geodesics with bounded length, we have only to consider the case when both $\alpha^+(t_i)$ and $\alpha_i^-(t_i)$ are endpoints of Teichmüller geodesic arcs constituting α^- and α^+ , i.e. the case when t_i is an integer. The marking $\alpha^-(t_i)$ corresponds to either a slice in $\tau(\bar{B}_i)$ or a union of slices, taken one from each of $\tau(B_i^j)$ ($j = q + 1, \dots, p$). This corresponds in turn to an extended level surface $\hat{S}_i^-(t_i)$. Similarly, we have an extended level surface $\hat{S}_i^+(t_i)$ corresponding to $\alpha^+(t_i)$. Then, we can see that a uniform bi-Lipschitz model manifold $\mathbf{M}'(\beta_i(t_i))$ for $\beta_i(t_i)$ is obtained from the submanifold of \mathbf{M}'_i cobounded by $S_i^-(t_i)$ and $S_i^+(t_i)$ by pasting boundary blocks and filling Margulis tubes by the argument of Lemma 9.2. Now, by the same argument as in the proof of the previous claim, we see that $\{\beta_i(t_i)\}$ converges algebraically after passing to a subsequence, and the geometric limit $\mathbf{M}'_\infty(\beta)$ of $\mathbf{M}'(\beta_i(t_i))$ contains bricks $\bar{B}^1, \dots, \bar{B}^p$. Since the internal blocks of $\mathbf{M}'_\infty(\beta)$ also lie in \mathbf{M}' , the ends of the algebraic limit other than those corresponding to E^1, \dots, E^p are geometrically finite and there are no extra parabolic loci. This implies that the algebraic limit is a quasi-conformal deformation of (Γ, ψ) by the ending lamination theorem, as in the previous claim.

Next suppose that $\alpha^+(t_i)$ lies in the first quasi-geodesic, i.e. $t_i \in [0, 1]$. Then all the internal blocks of \mathbf{M}'_i above $(\rho_i^{\mathbf{M}'})^{-1} \circ g'(S)$ lie also in a model manifold $\mathbf{M}'(\beta_i(t_i))$ for $\beta_i(t_i) = qf(\alpha_i^-(t_i), \alpha_i^+(t_i))$ defined in the same way as above. Therefore, the algebraic ends of \mathbf{M}'_i lie also in the model manifold $\mathbf{M}'(\beta_i(t_i))$. The above argument for showing that there is no extra parabolic loci works also in this case.

We can argue in the same way also in the case when $\alpha^-(t_i)$ lies (or both $\alpha^-(t_i)$ and $\alpha^+(t_i)$ lie) in the first quasi-geodesic. \square

Now, we shall complete the proof of Proposition 9.1, by setting $\alpha_i(t) = \beta_i(\bar{k}_i t)$ for β_i defined above, and showing thus constructed arc α_i satisfies the conditions in the statement. The conditions (1) and (2) have already been shown. Let us show (3). By our definition of m' , the lengths of c_1, \dots, c_s with respect to $m'(\mu_\infty^+(i, \mathbf{n}'_i))$ go to 0, and also those of c'_1, \dots, c'_t with respect to $m'(\mu_\infty^-(i, \mathbf{n}'_i))$ go to 0. This means that the length of each of c_1, \dots, c_s with respect to \bar{n}_i and that of each of c'_1, \dots, c'_t with respect to \bar{m}_i goes to 0.

Next we turn to the condition (4). In the case when Γ is a b-group, there is no lower algebraic simply degenerate ends for \mathbf{M}' . By our definition of β_i , in this case the lower conformal structure $\alpha_i^-(t)$ is the same as m_0 for every t .

Finally, we verify the condition (5). Suppose, seeking a contradiction, that the condition (5) does not hold. Then, there exist a neighbourhood U of $QH(\Gamma, \psi)$ and $t_i \in [0, k_i]$ such that $\beta_i(t_i)$ stays outside U for every i after passing to a subsequence. Now, we apply Claim 9.8 to see that $\beta(t_i)$ converges to a point in $QH(\Gamma, \psi)$ after passing to a subsequence. This is a contradiction. Thus we have completed the proof of Proposition 9.1.

9.2. Proofs of Theorem 3.8 and Corollaries 3.9, 3.10

Proof of Theorem 3.8. — Having proved Proposition 9.1, to prove Theorem 3.8, it remains to show that two sequences strongly converging to groups in the quasi-conformal deformation space of (Γ, ψ) as constructed in the proof of Proposition 9.1 can be joined by arcs in small neighbourhoods of the deformation space, fixing the conformal structure at bottom when Γ is a b-group.

Let $\{(G_i, \phi_i)\}$ and $\{(H_i, \varphi_i)\}$ be two sequences of quasi-Fuchsian groups both of which converge algebraically to quasi-conformal deformations of (Γ, ψ) . Let \mathbf{M}'_i and \mathbf{N}'_i be model manifolds for \mathbb{H}^3/G_i and \mathbb{H}^3/H_i constructed as in Lemma 9.2, converging geometrically to model manifolds \mathbf{M}' and \mathbf{N}' for the geometric limits of $\{G_i\}$ and $\{H_i\}$ respectively. Since the algebraic limits of $\{(G_i, \phi_i)\}$ and $\{(H_i, \varphi_i)\}$ are quasi-conformally equivalent, \mathbf{M}' and \mathbf{N}' have the same number of algebraic simply degenerate ends with the same ending laminations $\lambda_1, \dots, \lambda_p$. As in Proposition 9.1, we renumber them so that those having $\lambda_1, \dots, \lambda_q$ as ending laminations are upper whereas the rest are lower. In particular, we can assume that the boundary components touching these ends are the same open annuli in $S \times [0, 1]$ for \mathbf{M}' and \mathbf{N}' . We denote the union of these annuli by \mathcal{T} .

Let $\{(G'_i, \phi'_i) = qf(\bar{m}_i, \bar{n}_i)\}$ and $\{(H'_i, \varphi'_i) = qf(\bar{\mu}_i, \bar{\nu}_i)\}$ be two strongly convergent sequences as constructed in Proposition 9.1 for the two sequences $\{(G_i, \phi_i)\}$ and $\{(H_i, \varphi_i)\}$. Recall that in the construction of such a sequence in Proposition 9.1, we took a number \mathbf{n}'_i . Note that the construction works if we take a number smaller than \mathbf{n}'_i for each i provided that it goes to ∞ . Therefore, we can let the number \mathbf{n}'_i be common to both G'_i and H'_i . Let $\mathbf{M}'(\mathbf{n}'_i)$ and $\mathbf{N}'(\mathbf{n}'_i)$ be model manifolds for them as in the proof of Proposition 9.1. Then both $\mathbf{M}'(\mathbf{n}'_i)$ and $\mathbf{N}'(\mathbf{n}'_i)$, regarded as embedded in $S \times [0, 1]$, have the following properties. There are unions of Margulis tubes \mathbf{V}_i and \mathbf{V}'_i coming from \mathcal{T} in $\mathbf{M}'(\mathbf{n}'_i)$ and $\mathbf{N}'(\mathbf{n}'_i)$ respectively, which can be assumed to correspond to the same union of solid tori in $S \times [0, 1]$. As in the proof of Proposition 9.1, the complements $\mathbf{M}'(\mathbf{n}'_i) \setminus \mathbf{V}_i$ and $\mathbf{N}'(\mathbf{n}'_i) \setminus \mathbf{V}'_i$ have decompositions into bricks among which there are $B_i^{j'} = \Sigma^j \times J_i^{j'}$ in $\mathbf{M}'(\mathbf{n}'_i)$ and $B_i^{j''} = \Sigma^j \times J_i^{j''}$ in $\mathbf{N}'(\mathbf{n}'_i)$ on the same side of the preimages of the standard algebraic immersions. The bricks $B_i^{j'}$ and $B_i^{j''}$ contain tube unions corresponding to tight geodesics $\gamma_i^{j'}$ and $\gamma_i^{j''}$ supported on Σ^j whose lengths go to ∞ as $i \rightarrow \infty$. Furthermore, for each of $\gamma_i^{j'}$ and $\gamma_i^{j''}$, one of the endpoints stays in a compact set and the other endpoint (which we denote by $b_i^{j'}$ for $\gamma_i^{j'}$ and $b_i^{j''}$ for $\gamma_i^{j''}$) goes to the ending lamination λ_j as $i \rightarrow \infty$. Also, the tube unions put in $B_i^{j'}$ and $B_i^{j''}$ induce hierarchies $h_i^{j'}$ and $h_i^{j''}$ on Σ^j with main geodesics $\gamma_i^{j'}$ and $\gamma_i^{j''}$ respectively.

In the following, we only consider the case when the end of \mathbf{M}' (hence also that of \mathbf{N}') having λ_j as the ending lamination is an upper end, i.e. $j = 1, \dots, q$. The case when it is a lower end can be dealt with just by turning everything upside down as usual. Recall that we constructed a sequence of markings $\mu^+(i, \mathbf{n}'_i)$ in the proof of Proposition 9.1. We construct its counterpart $\mu^{+'}(i, \mathbf{n}'_i)$ for \mathbf{N}' . As can be seen in the construction of $\mathbf{M}'(\mathbf{n}'_i)$ in the proof of Proposition 9.1, the last slices of $h_i^{j'}$ and $h_i^{j''}$ correspond to the restrictions of the last terms $\mu_\infty^+(i, \mathbf{n}'_i)$ and $\mu_\infty^{+'}(i, \mathbf{n}'_i)$ of the sequences of markings $\mu^+(i, \mathbf{n}'_i)$ and $\mu^{+'}(i, \mathbf{n}'_i)$.

Since the length of the frontier of Σ^j goes to 0 with respect to both \bar{n}_i and $\bar{\nu}_i$, we can assume that the length of each component of $\text{Fr}\Sigma^j$ with respect to \bar{n}_i is equal to that with respect to $\bar{\nu}_i$ without changing the algebraic limit and the structure of the model manifolds except for the boundary blocks, by deforming \bar{m}_i and $\bar{\nu}_i$ locally in thin annular neighbourhoods of $\text{Fr}\Sigma^j$.

Now, we connect the endpoint $b_i^{j'}$ of $\gamma_i^{j'}$ with the endpoint $b_i^{j''}$ of $\gamma_i^{j''}$ by a tight sequence $\gamma_i = \{s_1^i, \dots, s_{u_i}^i\}$ in $\mathcal{CC}(\Sigma^j)$. By the hyperbolicity of $\mathcal{CC}(\Sigma^j)$, for any choice of an integer v_i between 1 and u_i , the simplex $s_{v_i}^i$ also converges to the lamination λ_j as $i \rightarrow \infty$. By letting $\mu_\infty^+(i, \mathbf{n}'_i)$ and

$\mu_\infty^+(i, \mathbf{n}'_i)$ be the initial and the terminal markings respectively, we can regard γ_i as a tight geodesic, and there is a hierarchy $h(\gamma_i)$ on Σ^j which has γ_i as its main geodesic. Considering a resolution of this hierarchy $h(\gamma_i)$, we can connect $\mu_\infty^+(i, \mathbf{n}'_i)$ with $\mu_\infty^{+'}(i, \mathbf{n}'_i)$ by elementary moves of markings $\bar{\mu}_\infty^i(r), r = 0, \dots, w_i$. Recall that $\mu_\infty^+(i, \mathbf{n}'_i)$ is a shortest marking for (S, \bar{n}_i) and $\mu_\infty^{+'}(i, \mathbf{n}'_i)$ is a shortest marking for $(S, \bar{\mu}_i)$ and that both of them contain every component of $\text{Fr}\Sigma^j$. Therefore we can connect \bar{n}_i with $\bar{\mu}_i|(S \setminus \Sigma^j) \cup \bar{\nu}_i|\Sigma^j$ by a piecewise Teichmüller geodesic arc $\delta_i : [0, w_i] \rightarrow \mathcal{T}(S)$ so that for any integer r , the restriction $\bar{\mu}_\infty^i(r)|\Sigma^j$ is a shortest marking for $\delta_i(r)|\Sigma^j$ whereas $\delta_i(r)|(S \setminus \Sigma^j) = \bar{n}_i|(S \setminus \Sigma^j)$. Since the base curve of $\bar{\mu}_\infty^i(r)|\Sigma^j$ lies among $s_1^i, \dots, s_{u_i}^i$, every one of which converges to λ_j , and the structure of $\mathbf{M}'_i(\mathbf{n}'_i)$ outside \bar{B}^j is unchanged, we see that $qf(\bar{m}_i, \delta_i(t_i))$ converges to (Γ, ψ) strongly for any sequence $\{t_i \in [0, w_i]\}$. By the same argument as in the proof of Proposition 9.1, we see that for any neighbourhood U of $QH(\Gamma, \psi)$, the arc $qf(\bar{m}_i, \delta_i[0, w_i])$ is contained in U for large i .

We repeat the same procedure for each $j = 1, \dots, p$. Then, we get an arc α'_i connecting $qf(\bar{m}_i, \bar{n}_i)$ with $qf(\bar{m}_i|(S \setminus \bigcup_{j=q+1}^p \Sigma^j) \cup \bar{\mu}_i|\bigcup_{j=q+1}^p \Sigma^j, \bar{n}_i|(S \setminus \bigcup_{j=1}^q \Sigma^j) \cup \bar{\nu}_i|\bigcup_{j=1}^p \Sigma^j)$ such that for any neighbourhood U of $QH(\Gamma, \psi)$, the arc α'_i is contained in U for sufficiently large i . Since all of the $\bar{m}_i|(S \setminus \bigcup_{j=q+1}^p \Sigma^j)$, the $\bar{\mu}_i|(S \setminus \bigcup_{j=q+1}^p \Sigma^j)$, the $\bar{n}_i|(S \setminus \bigcup_{j=1}^p \Sigma^j)$, and the $\bar{\nu}_i|(S \setminus \bigcup_{j=1}^p \Sigma^j)$ are bounded in the Teichmüller spaces with free boundary, and the lengths of the boundary components are the same for \bar{m}_i and $\bar{\mu}_i$ and for \bar{n}_i and $\bar{\nu}_i$ and go to 0 as $i \rightarrow \infty$, we can deform $qf(\bar{m}_i|(S \setminus \bigcup_{j=q+1}^p \Sigma^j) \cup \bar{\mu}_i|\bigcup_{j=q+1}^p \Sigma^j, \bar{n}_i|(S \setminus \bigcup_{j=1}^q \Sigma^j) \cup \bar{\nu}_i|\bigcup_{j=1}^p \Sigma^j)$ to $qf(\hat{\mu}_i, \hat{\nu}_i)$ by a uniformly bounded quasi-conformal deformation, where the differences between $\hat{\mu}_i$ and $\bar{\mu}_i$ and between $\hat{\nu}_i$ and $\bar{\nu}_i$ are just compositions of Fenchel–Nielsen twists around c_1, \dots, c_s and c'_1, \dots, c'_t respectively. This quasi-conformal deformation gives an arc connecting $qf(\bar{m}_i|(S \setminus \bigcup_{j=q+1}^p \Sigma^j) \cup \bar{\mu}_i|\bigcup_{j=q+1}^p \Sigma^j, \bar{n}_i|(S \setminus \bigcup_{j=1}^q \Sigma^j) \cup \bar{\nu}_i|\bigcup_{j=1}^p \Sigma^j)$ to $qf(\hat{\mu}_i, \hat{\nu}_i)$ which is contained in the neighbourhood U for large i .

Now, we connect $\hat{\mu}_i$ with $\bar{\mu}_i$ and $\hat{\nu}_i$ with $\bar{\nu}_i$ by arcs $\boldsymbol{\mu}_i(x), \boldsymbol{\nu}_i(x)$ in the Teichmüller space realising these compositions of Fenchel–Nielsen twists corresponding to their differences. Then the lengths of c_1, \dots, c_s with respect to $\boldsymbol{\mu}_i(x)$ also go to 0 as $i \rightarrow \infty$ for any x , and so do the lengths of c'_1, \dots, c'_t with respect to $\boldsymbol{\nu}_i(x)$. Therefore, by the same argument as before, we can connect $qf(\hat{\mu}_i, \hat{\nu}_i)$ with $qf(\bar{\mu}_i, \bar{\nu}_i)$ by an arc which is contained in U for large i .

Thus, joining the arcs obtained this way, we have shown that we can connect $(G'_i, \phi'_i) = qf(\bar{m}_i, \bar{n}_i)$ with $(H'_i, \psi'_i) = qf(\bar{\mu}_i, \bar{\nu}_i)$ by an arc α_i in $QF(S)$ such that for any neighbourhood U of $QH(\Gamma, \psi)$, the arc α_i is contained in U large i . On the other hand, by our definition of (\bar{m}_i, \bar{n}_i) and $(\bar{\mu}_i, \bar{\nu}_i)$ which

uses Proposition 9.1, there are arcs with such a property connecting (G_i, ϕ_i) with (G'_i, ϕ'_i) and (H_i, ψ_i) with (H'_i, ψ'_i) . Therefore, connecting these three arcs, we get an arc as we wanted.

In the case when Γ is a b-group, by Theorem 3.7, the lower conformal structures at infinity of both (G_i, ϕ_i) and (H_i, ψ_i) converge to m_0 . Therefore, we can construct an arc as above keeping the lower conformal structures in an arbitrarily small neighbourhood of m_0 for large i . This shows the second paragraph of our theorem. \square

Corollary 3.9 follows easily from this as follows.

Proof of Corollary 3.9. — Suppose that every small neighbourhood of (Γ, ψ) intersects more than one component of $QF(S)$. Then for every small neighbourhood U of (Γ, ψ) in $AH(S)$, there are sequences $\{qf(m_i, n_i)\}$ and $\{qf(\hat{m}_i, \hat{n}_i)\}$ both converging to (Γ, ψ) such that $qf(m_i, n_i)$ and $qf(\hat{m}_i, \hat{n}_i)$ belong to different component of $U \cap QF(S)$. Applying Theorem 3.8, we see that for any small neighbourhood V of $QF(\Gamma, \psi)$ in $AH(S)$, for sufficiently large i , the two points $qf(m_i, n_i)$ and $qf(\hat{m}_i, \hat{n}_i)$ must be connected in $V \cap QF(S)$. In the case when every component of Ω_Γ/Γ is a thrice-punctured sphere, this is a contradiction since $QH(\Gamma, \psi)$ consists of only (Γ, ψ) then and we can let V be U .

In the case when there is a component of Ω_Γ/Γ which is homeomorphic to S , the Kleinian group Γ is a b-group. Then the second paragraph of Theorem 3.8 shows that we can connect $qf(m_i, n_i)$ to $qf(\hat{m}_i, \hat{n}_i)$ keeping the lower conformal structure within a small neighbourhood. This means that they can be connected in a small neighbourhood of (Γ, ψ) in $QF(S)$. This again is a contradiction. \square

To get Corollary 3.10, we need to review the argument of the proof of Theorem 3.8.

Proof of Corollary 3.10. — In Corollary 3.10, we have allowed isolated parabolic loci to exist. We consider a sequence of quasi-Fuchsian groups $\{qf(m_i, n_i)\}$ converging to (Γ, ψ) in this generalised setting, and shall show that we can deform $\{qf(m_i, n_i)\}$ continuously to a sequence which converges strongly to (Γ, ψ) . In the proof of Theorem 3.8, we used the assumption that Γ has no isolated parabolic loci in two places; first to show that the standard immersion g' can be isotoped to a horizontal embedding, and secondly in the proofs of Claims 9.7 and 9.8 to show that the limits are quasi-conformal deformations of (Γ, ψ) . We shall show how to modify the argument in the latter part first.

(1). *How to modify the proofs of Claims 9.7 and 9.8.* — In the proofs of these claims, we used the fact that all the parabolic loci of the algebraic

limit Γ' of the new sequence touch simply degenerate ends to imply that the parabolic loci of Γ' and Γ are the same. In our setting now, if we do the same construction, this is not the case any more. To preserve the parabolic loci of algebraic limits throughout the modification of quasi-Fuchsian groups, we consider to modify the model manifolds \mathbf{M}' and \mathbf{M}'_i constructed in Lemma 9.2 and the function m as follows.

We divide the isolated parabolic loci of Γ into two categories. Let c be a core curve of an isolated parabolic locus P of Γ . We can regard c as lying on the boundary of the model \mathbf{M}' of the non-cuspidal part of the geometric limit $(M_\infty)_0$. We say that the parabolic locus P and its core curve c are of torus type if c can be regarded as lying on a torus boundary component of \mathbf{M}' and are of annulus type if c is regarded as lying on an open annulus component of \mathbf{M}' .

Let d_1, \dots, d_u be core curves of the isolated parabolic loci, not caring whether they are of torus type or annulus type. We renumber them so that d_1, \dots, d_v are upper and d_{v+1}, \dots, d_u are lower. These curves correspond to algebraic parabolic curves lying on boundary components in the original model manifold \mathbf{M} . We take annular neighbourhoods $A(d_1), \dots, A(d_v)$ and $A(d_{v+1}), \dots, A(d_u)$ of these curves on S so that both $A(d_1), \dots, A(d_v)$ and $A(d_{v+1}), \dots, A(d_u)$ are pairwise disjoint. We set $V(d_j) = A(d_j) \times [5/8, 3/4]$ for $j = 1, \dots, v$ and $V(d_j) = A(d_j) \times [1/4, 3/8]$ for $j = v + 1, \dots, u$. We set \mathcal{U}^+ to be $\bigcup_{j=1}^v V(d_j)$ and \mathcal{U}^- to be $\bigcup_{j=v+1}^u V(d_j)$.

We then define $\widetilde{\mathbf{M}}$ to be $S \times [1/8, 7/8] \setminus (\mathcal{T}^- \cup \mathcal{T}^+ \cup \mathcal{U}^- \cup \mathcal{U}^+)$, where \mathcal{T}^- and \mathcal{T}^+ are unions of solid tori corresponding to non-isolated parabolic curves which were defined in the proof of Lemma 9.2. This new $\widetilde{\mathbf{M}}$ also admits standard brick decomposition having five stages although the bottom and top levels are changed to $1/8$ and $7/8$. The bottom brick and the top brick are denoted \widetilde{B}_i and \widehat{B}_i as before. By the same construction as in the proof of Lemma 9.2, we can construct labelled brick manifolds $\widetilde{\mathbf{M}}_i$ by attaching geometrically finite blocks corresponding to m_i and n_i , and then model manifolds \mathbf{M}'_i by filling in Margulis tubes. As in the proof of Lemma 9.2, we put tube unions in \widetilde{B}_i and \widehat{B}_i , which are parts of block decomposition of $\widetilde{\mathbf{M}}_i$, and decompose them into blocks. These give rise to resolutions $\tau(\widetilde{B}_i)$ and $\tau(\widehat{B}_i)$ as before.

Next we turn to the levels $[1/4, 3/8]$ and $[5/8, 3/4]$. On these levels, there lie bricks B_i^j , among which B_i^1, \dots, B_i^q are contained in $S \times [5/8, 3/4]$ and B_i^{q+1}, \dots, B_i^p are contained in $S \times [1/4, 3/8]$. For these, hierarchies $h(B_i^j)$ resolutions $\tau(B_i^j)$ are defined in the same way as before. On the other hand, the decomposition of $\widetilde{\mathbf{M}}_i$ into blocks determines a geodesic $h_i(d_j)$ supported

on $A(d_j)$ for $j = 1, \dots, u$. If the length of $h_i(d_j)$ goes to ∞ as $i \rightarrow \infty$, then d_j is of torus type, otherwise it is of annulus type. Now, recall that for $n \in \mathbb{N}$, we defined a slice on S extending the n -th slices on the $\tau(B_i^j)$, and then defined a sequence of markings $\mu^-(i, n)$. In the present situation where there are isolated algebraic parabolic loci, we also have to take into account geodesics supported on the $A(d_j)$. For each d_j ($j = v + 1, \dots, u$) of torus type, we take the vertex of $h_i(d_j)$ which is the n_0 -th counting from the last one, and denote it by $\tau(d_j)_{-n_0}$. Starting from the last slice of $\tau(\overline{B}_i)$, we advance, by elementary moves, vertices on geodesics supported on Σ^j ($j = q + 1, \dots, p$) and $A(d_j)$ ($j = v + 1, \dots, u$) for d_j of torus type until it gets to $\tau(B_i^j)_{-n}$ on Σ^j and to $\tau(d_j)_{-n_0}$ on $A(d_j)$, and obtain a sequence of markings $\{\mu_-(i, n, n_0)\}$ as before. Note that we do not move other vertices from those lying on the last slice of $\tau(\overline{B}_i)$. In particular if d_j ($j = v + 1, \dots, u$) is of annulus type, the vertex on $A(d_j)$ remains to be the first one. In the same way, we define $\tau(d_j)_n$ for d_j ($j = 1, \dots, v$) of torus type and a sequence of markings $\{\mu_+(i, n, n_0)\}$.

We also need to modify the definition of the function m in Definition 9.5. For each μ in $\mu^-(i, n, n_0)$, we define $m(\mu)$ to be a point in $\mathcal{T}(S)$ such that a clean marking compatible with μ is a shortest marking in $(S, m(\mu))$, the length of each curve d of torus type among d_{v+1}, \dots, d_u is equal to $\delta_0/d_{A(d)}(I(h_i(d)), \mu|A(d))$, where $\mu|A(d)$ is the marking on $A(d)$ determined by μ , and the lengths of curves of annulus type among d_{v+1}, \dots, d_u are equal to length_{m_i} of the curves. We define m for $\mu \in \mu^+(i, n)$ similarly. Before Claim 9.7, we introduced the number \mathbf{n}'_i and considered the last marking in the sequence $\mu^\pm(i, \mathbf{n}'_i)$, which we denoted by $\mu_\infty^\pm(i, \mathbf{n}'_i)$. In the present case, we define \mathbf{n}'_i in the same way as before, and fix n_0 independently of i . We define the function m' in the same way as in Definition 9.6 by decreasing gradually lengths of curves corresponding to non-isolated parabolic loci. Then the rest of the construction in the proof of Proposition 9.1 works without any change, and we can define broken quasi-geodesics α_i^- and α_i^+ .

We need to show that the algebraic limits appearing Claims 9.7 and 9.8 has d_1, \dots, d_u as parabolic elements whatever point you choose on α^- or α^+ by our definitions of m and m' as above. Suppose first that c is a curve in d_1, \dots, d_u of annulus type. Let A be a boundary component of \mathbf{M}' having c as a core curve. Since c is an isolated curve this means that the geometrically finite block is split along c , which implies that either $\text{length}_{m_i}(c) \rightarrow 0$ or $\text{length}_{n_i}(c) \rightarrow 0$ depending on c is lower or upper. Therefore, by our modified definition of m above, we see that the length of c with respect to the sequences appearing Claims 9.7 and 9.8 goes to 0, and hence c is parabolic in their algebraic limits.

Suppose next that c is of torus type. We first consider the algebraic limit of $qf(m'(\mu_\infty^-(i, \mathbf{n}'_i, n_0)), m'(\mu^+(i, \mathbf{n}'_i, n_0)))$, corresponding to Claim 9.7.

By our definition of the function m (and m') above, we see that if c is among d_1, \dots, d_v , then $\text{length}_{m'(\mu^+(i, \mathbf{n}'_i, n_0))}(c)$ goes to 0 as $i \rightarrow \infty$, and if it is among d_{v+1}, \dots, d_u , then $\text{length}_{m'(\mu^-(i, \mathbf{n}'_i, n_0))}(c)$ goes to 0. This shows that c is parabolic in the algebraic limit of this sequence.

We now turn to considering the algebraic limit of the sequence $\{\beta_i(t_i)\}$, where $\beta_i(t) = qf(\alpha_i^-(t), \alpha_i^+(t))$. Suppose that c is a curve of torus type among d_{v+1}, \dots, d_u , and we consider the point $\alpha_i^-(t_i)$ in the Teichmüller space. There are essentially three cases to consider. In the first and the second cases, we assume that $\alpha_i^-(t_i)$ lies on a Teichmüller geodesic with endpoints $m'(\mu_1(i)), m'(\mu_2(i))$ for some adjacent slices $\mu_1(i), \mu_2(i)$ in $\mu^-(i, \mathbf{n}_i, n_0)$. In the first case, we further assume that $\mu_1(i)$ contains the $k(i)$ -th vertex on $h_i(c)$, where $\text{length}(\gamma(i)) - k(i) \rightarrow \infty$. Then the same holds for $\mu_2(i)$ since they are adjacent. Then for the hierarchy of tight geodesics connecting for $\alpha_i^-(t_i)$ with $\alpha_i^+(t_i)$ has a geodesic supported on $A(c)$ with length at least $\text{length}(\gamma(i)) - k(i) - 1$, which goes to ∞ . Therefore in the geometric limit $\mathbf{M}_\infty(\beta)$, the curve c lies on a torus boundary. This implies that c is parabolic in the algebraic limit. In the second case, we assume on the contrary that $\text{length}(\gamma(i)) - k(i)$ is bounded, which implies $k(i) \rightarrow \infty$. Then by our modified definition of the function m (and m'), we have $\text{length}_{m'(\mu_1(i))}(c)$ and $\text{length}_{m'(\mu_2(i))}(c)$ go to 0, which implies that $\text{length}_{\alpha_i^-(t_i)}(c)$ also goes to 0. Therefore c is a parabolic curve in the algebraic limit also in this case. The third case is when $\alpha_i^-(t_i)$ lies a quasi-geodesic connecting m_i with the first marking $\mu_1^-(i, \mathbf{n}'_i, n_0)$. Then the situation is the same as the first case and c lies on a torus boundary of the geometric limit $\mathbf{M}_\infty(\beta)$ since the initial and the terminal markings are the same as the case of connecting $\mu_1^-(i, \mathbf{n}'_i, n_0)$ with $\mu_1^+(i, \mathbf{n}'_i, n_0)$ and the hierarchy of tight geodesics connecting them has a geodesic of length $h_i(c) \rightarrow \infty$ supported on $A(c)$. Thus we have shown that c is a parabolic curve in all the cases. The same argument works also when c is of torus type among d_1, \dots, d_v , and we can prove them to be upper parabolic of the algebraic limit of $\{\beta_i(t_i)\}$. We note that $\mathbf{M}_\infty(\beta)$ has the geometrically infinite ends corresponding to E^1, \dots, E^p by our definition of α_i^\pm . This implies that $\mathbf{M}_\infty(\beta)$ has no isolated algebraic parabolic curves other than d_1, \dots, d_u . Therefore we also see that there are no other isolated parabolic curves other than d_1, \dots, d_u in the algebraic limit of $\{\beta_i(t_i)\}$.

Thus we have shown that Claims 9.7 and 9.8 holds under the assumption that g' does not go around a torus boundary component. In particular, this shows that the algebraic limit of $\beta_i(t_i)$ is a quasi-conformal deformation of (Γ, ψ) .

(2). *To show that g' does not go around a torus boundary component.* — It remains to show that the standard immersion g' does not go around a torus boundary component in \mathbf{M}' in our setting. We consider first the case when

Γ is a b-group. Let $\{(G_i, \phi_i) = qf(m_0, n_i)\}$ and $\{(\bar{G}_i, \bar{\phi}_i) = qf(m_0, \bar{n}_i)\}$ be sequences in the Bers slice converging to (Γ, ψ) . Let \mathbf{M}' and $\bar{\mathbf{M}}'$ be model manifolds for non-cuspidal parts of geometric limits G_∞ and G'_∞ of $\{G_i\}$ and $\{G'_i\}$, which are non-cuspidal parts of geometric limits of the model manifolds \mathbf{M}'_i and $\bar{\mathbf{M}}'_i$ constructed as in Lemma 9.2 respectively. Let $g' : S \rightarrow \mathbf{M}'$ and $\bar{g}' : S \rightarrow \bar{\mathbf{M}}'$ be standard algebraic immersions. Since the lower geometrically finite ends must be lifted to the algebraic limits, neither g' nor \bar{g}' can go around torus boundary components, hence homotopic to horizontal surfaces in \mathbf{M}' and $\bar{\mathbf{M}}'$ respectively. Therefore the argument of the proof of Theorem 3.8 works, and we see that for any neighbourhood U of (Γ, ψ) , there is an arc α_i connecting (G_i, ϕ_i) and $(\bar{G}_i, \bar{\phi}_i)$ in the Bers slice $U \cap B_{m_0}$.

Next suppose that Γ is not a b-group. Let $\{(G_i, \phi_i) \in QF(S)\}$ be a sequence converging to (Γ, ψ) . Again, we let \mathbf{M}' be a model manifold of the non-cuspidal part of the geometric limit G_∞ , which is the non-cuspidal part of the geometric limit of \mathbf{M}'_i . Suppose that g' goes around a torus component T of \mathbf{M}' counter-clockwise. Let c be a longitude of T . Then, c , regarded as lying on S , is an upper parabolic curve of $M_\Gamma = \mathbb{H}^3/\Gamma$.

Since we assumed that Ω_Γ/Γ consists of thrice-punctured spheres, there is either a lower parabolic curve d or an ending lamination λ of a lower end, intersecting c essentially. If there is a lower parabolic curve d , then there is a boundary component T' of \mathbf{M}' whose core curve or a longitude is homotopic to $g'(d)$ and which is situated below $g'(S)$. This is impossible since g' goes around T whose longitude intersects d essentially on S . If there is an ending lamination λ , then there is a algebraic simply degenerate end contained in $\Sigma \times (s, t]$ in \mathbf{M} having λ as the ending lamination. Then $\Sigma \times \{s + \epsilon\}$ must be homotopic to $g'(\Sigma)$. Again this is impossible since g' goes around T whose longitude intersects λ essentially on S . Thus, we are lead to a contradiction in both cases, and see that g' cannot go around a torus boundary component counter-clockwise. The case when g' goes around a torus boundary component clockwise can be deal with in the same way just by turning everything upside down. \square

10. Proof of Theorems 3.11 and 3.12

10.1. Proof of Theorem 3.11

After passing to a subsequence, $\{G_i\}$ can be assumed to converge geometrically to a Kleinian group G_∞ containing Γ as before. This induces a pointed Gromov convergence of (M_i, y_i) to (M_∞, y_∞) with $M_\infty = \mathbb{H}^3/G_\infty$.

Let \mathbf{M} be a model manifold of $(\mathbb{H}^3/G_\infty)_0$, and \mathbf{M}_i that of $(M_i)_0 = (\mathbb{H}^3/G_i)_0$ as before. Then $\mathbf{M}_i[0]$ converges geometrically to $\mathbf{M}[0]$ by taking a basepoint x_i in \mathbf{M}_i which is mapped by the model map to a point within uniformly bounded distance from the basepoint y_i of M_i . As in Lemma 4.13, we take a standard algebraic immersion $g' : S \rightarrow \mathbf{M}$.

Let E be an algebraic simply degenerate end of \mathbf{M} with ending lamination λ_E , and $B = \Sigma \times J$ a brick of \mathbf{M} containing E . We assume that E is an upper end. The case when E is a lower end can be argued in the same way by just turning everything upside down as usual. Then by Proposition 4.18, there is a tight geodesic γ_i in $\mathcal{CC}(\Sigma)$ one of whose endpoints (possibly at infinity) converges to λ_E as $i \rightarrow \infty$. In the case when γ_i is a geodesic ray, its endpoint at infinity is an ending lamination of an upper end of \mathbf{M}_i , which is contained in $e_+(i)$. Therefore, we see that λ_E is contained in the Hausdorff limit of $e_+(i)$.

Next suppose that γ_i is a finite geodesic. Then, by Theorem 3.1 and Section 6 of Masur–Minsky [38], the last vertex of γ_i and $\pi_\Sigma(e_+(i))$ are within uniformly bounded distance, and hence the endpoint of γ_i and $\pi_\Sigma(e_+(i))$ converge to the same lamination λ_E with respect to the topology of $\mathcal{UML}(\Sigma)$ as $i \rightarrow \infty$. Since λ_E is arational, we see that the Hausdorff limit of $e_+(i)|\Sigma$ contains λ_E .

The converse can be shown by the same argument as the proof of Theorem 5.2.

10.2. Proof of Theorem 3.12

Suppose, seeking a contradiction, that a sequence $\{(G_i, \phi_i)\}$ as in the statement converges. Then by Theorem 3.11, λ is an ending lamination of an upper end and μ is an ending lamination of a lower end of the algebraic limit. This implies the shared boundary component of the minimal supporting surface of μ is a parabolic locus which is both upper and lower. This is impossible, and we have completed the proof of Theorem 3.12.

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