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Special polynomials associated with the Painlevé equations I

HIROSHI UMEMURA ⁽¹⁾

ABSTRACT. — The Painlevé equations have rational or algebraic solutions on special parameters. We can find rational or algebraic solutions of the Painlevé equations as fixed points of the Bäcklund transformations. The τ function of the rational or algebraic solution can be written as the product of a special polynomial and an exponential factor. Since a series of τ functions satisfies the Toda equation, we obtain a recursive relation of the special polynomials. The coefficients of the special polynomials for the sixth Painlevé equation are described by the Young diagram.

(The original manuscript by the author was submitted to the proceeding of the Montreal conference in 1996, which were not published. The abstract was not part of the original manuscript and has not been written by the author.)

RÉSUMÉ. — Pour certaines valeurs spéciales des paramètres, les équations de Painlevé ont des solutions algébriques ou rationnelles. Elles sont associées aux points fixes des transformations de Bäcklund. La fonction τ de la solution rationnelle ou algébrique peut alors être écrite comme un produit de polynômes spéciaux et d'un facteur exponentiel. Puisque une série de fonctions τ satisfait l'équation de Toda, nous obtenons une relation de récurrence pour les polynômes spéciaux. Pour la sixième équation de Painlevé les coefficients des polynômes spéciaux sont décrits à l'aide de diagrammes de Young.

(Le manuscrit original a été soumis aux comptes-rendus de la conférence de Montréal en 1996, qui n'ont pas été publiés. Le résumé ne faisait pas partie du manuscrit original et il n'a pas été rédigé par l'auteur.)

1. Introduction

We have systems of classical orthogonal polynomials such as the Jacobi polynomials, the Laguerre polynomials, the Hermite polynomials, etc. They are solutions of linear differential equations of the second order: Namely the hypergeometric differential equations or their confluent.

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Example 1.1. — The Hermite polynomials. Let us set

$$H_n(t) := (-1)^n e^{t^2/2} \frac{d^n e^{-t^2/2}}{dt^n}$$

for every non-negative integer n . Then $H_n(t)$ is a polynomial of degree n , the Hermite polynomial and is a solution of the Hermite equation

$$y'' - ty' + ny = 0.$$

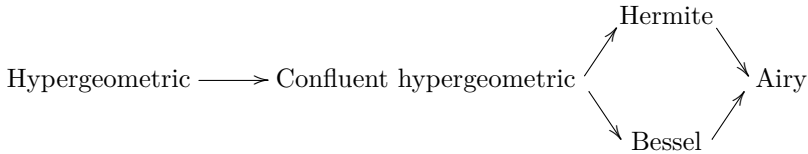
The Hermite polynomials are orthogonal each other, i.e.

$$\int_{-\infty}^{\infty} H_m(t)H_n(t)e^{-t^2/2} dt = \delta_{mn}n!\sqrt{2\pi}.$$

We have a recurrence formula

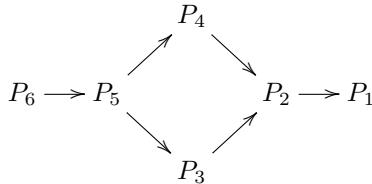
$$H_{n+1}(t) = H_n(t)t - nH_{n-1}(t). \tag{1.1}$$

As is well-known, we have the following diagram of confluence of the linear differential equations



(cf.[12]).

The diagram above is compatible with the degenerations of the Painlevé equations



Namely the hypergeometric equation is a part of the sixth Painlevé equation, the confluent hypergeometric the fifth and so on. Since the Jacobi polynomials and the Legendre polynomials satisfy hypergeometric equations, they are contained in the sixth Painlevé equation. Similarly the Laguerre polynomials that satisfy a confluent hypergeometric equation, and the Hermite polynomials are respectively included in the fifth and the fourth Painlevé equations.

The Painlevé equations, however, involve also non-classical polynomials. Yablonskii [13] and Vorob'ev [11] discovered such polynomials in P_2 and the Okamoto in P_4 . So the following natural question arises.

QUESTION 1.2. — *Are there non-classical special polynomials related with the other Painlevé equations?*

In fact, we can generate many special polynomials using the Painlevé equations. We present in this note some of them. The other polynomials and the details will appear in [8]. Things go miraculously well. Is it because the Painlevé functions (=the solutions of the Painlevé equations) deserve special functions? The most interesting series of polynomials arises with algebraic solutions of degree 2 of the sixth equation P_6 , which seem new (cf. Section 6). In Section 7 we observe that our polynomials are related with the Young diagrams.

2. Yablonskii–Vorob’ev polynomials (review)

The second Painlevé equation

$$P_2(\alpha) \quad y'' = 2y^3 + ty + \alpha$$

is equivalent to the system

$$S_2(b) \quad \begin{cases} q' = p - q^2 - \frac{t}{2}, \\ p' = 2pq + b, \end{cases}$$

where α and b are related by $b = \alpha + \frac{1}{2}$ (cf. [3]). The system $S_2(b)$ is a Hamiltonian system parameterized by $b \in \mathbb{C}$. Namely let $H = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}t)p - b$. Then $S_2(b)$ is written as

$$\begin{cases} dq/dt = \partial H/\partial p, \\ dp/dt = -\partial H/\partial q. \end{cases} \quad (2.1)$$

We denote by $Sol_2(b)$ the set of solutions of $S_2(b)$. We have a transformation

$$T(b, b-1) : Sol_2(b) \rightarrow Sol_2(b-1)$$

defined by

$$T(b, b-1)(q, p) = (-q + (b-1)/(p - 2q^2 - t), -p + 2q^2 + t)$$

and a transformation

$$I(b, -b) : Sol_2(b) \rightarrow Sol_2(-b)$$

given by $I(b, -b)(q, p) = (q + b/p, p)$ for $(q, p) \in Sol_2(b)$.

THEOREM 2.1.

- (1) $P_2(\alpha)$ has a 1-parameter family of classical solutions for $\alpha \in \frac{1}{2} + \mathbb{Z}$.
- (2) $P_2(\alpha)$ has a unique rational solution for $\alpha \in \mathbb{Z}$.
- (3) Except for those solutions in (1) and (2), other solutions are not classical.

This theorem is the irreducibility of the second Painlevé equation. For the definition of classical functions, see [8, 9] and for a proof of the theorem see [10]. Let us illustrate the theorem by examples.

Example 2.2.

- (1). — If $\alpha = -\frac{1}{2}$ or equivalently $b = 0$, then $S_2(0)$ reduces to

$$S_2(0) \quad \begin{cases} q' = p - q^2 - \frac{t}{2}, \\ p' = 2pq. \end{cases}$$

So if q is a solution of a Riccati equation $q' = -q^2 - t/2$, then $(q, 0)$ is a solution of $S_2(0)$. The solutions of this type form a 1-parameter family of classical solutions of $S_2(0)$. We notice that by setting $y'/y = q$, we linearize the Riccati equation to get the Airy equation

$$y'' + \frac{1}{2}ty = 0.$$

- (2). — If $\alpha = 0$ so that $b = \frac{1}{2}$, we have

$$S_2\left(\frac{1}{2}\right) \quad \begin{cases} q' = p - q^2 - \frac{t}{2}, \\ p' = 2pq + \frac{1}{2}. \end{cases}$$

Hence $(q, p) = (0, t/2)$ is the unique rational solution of $S(\frac{1}{2})$. It is natural to ask how we can calculate the unique rational solution of $P_2(n)$ for $n \in \mathbb{Z}$. The first idea that comes to our mind is to start from the rational solution $(0, t/2)$ of $S(\frac{1}{2})$ and use the transformations $T(b, b - 1)$ and $I(b, -b)$. There is a very elegant method due to Yablonskii and Vorob'ev.

Let us define rational functions $T_m(t)$ for $m \in \mathbb{N}$ recursively by

$$T_m(t) = \frac{tT_{m-1}^2(t) - 4(T_{m-1}''(t)T_{m-1}(t) - T_{m-1}'^2(t))}{T_{m-2}(t)}, \quad (2.2)$$

with $T_0(t) = T_1(t) = 1$. We can show that for every $m \in \mathbb{N}$, $T_m(t)$ is a monic polynomial with integral coefficients of degree $m(m-1)/2$, which we call the Yablonskii–Vorob'ev polynomials (see 8.1, [11] and [13]). We can show that $q_m = T_{m+1}'/T_{m+1} - T_m'/T_m$ is the unique rational solution of $P_2(-m-1)$ for

$m \in \mathbb{N}$. We want to generalize the Yablonskii–Vorob’ev polynomials. To this end, we must clarify the following points.

- (1) How to find rational or more generally algebraic solutions of the Painlevé equations?
- (2) Where does the recurrence formula (2.2) come from?
- (3) How to prove that the $T_m(t)$ ’s are in fact polynomials?

3. How to find rational or algebraic solutions

To this end let us review how the transformations $T(b, b - 1), I(b, -b)$ are introduced ([3]). As we explained above $S_2(b)$ is the Hamiltonian system (2.1) with the Hamiltonian $H(b, t, q, p) = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}t) - bq$. Let us set $h(b, q(t), p(t)) = H(b, t, q(t), p(t))$ for a solution $(q(t), p(t))$ of $S_2(b)$. Then $h(b, q, p)$, which is a function of t , satisfies

$$E_2(b) \quad \left(\frac{d^2h}{dt^2} \right)^2 + 4 \left(\frac{dh}{dt} \right)^3 + 2 \frac{dh}{dt} \left(t \frac{dh}{dt} - h \right) - \frac{1}{4}b^2 = 0.$$

The following lemmas are due to Okamoto [3].

LEMMA 3.1. — *Let (q, p) be a generic solution of $S_2(b)$. Namely we have*

$$\text{tr.d. } [\mathbb{Q}(b, t)\langle q, p \rangle : \mathbb{Q}(b, t)] = 2.$$

Then $h(b, q, p)$ is a generic solution of $E(b)$, i.e. we have

$$\text{tr.d. } [\mathbb{Q}(b, t)\langle h(b, q, p) \rangle : \mathbb{Q}(b, t)] = 2.$$

Here we denote by $\mathbb{Q}(b, t)\langle h(b, q, p) \rangle$ the differential field generated by $h(b, q, p)$ over $\mathbb{Q}(b, t)$.

LEMMA 3.2. — *If $(q(t), p(t))$ be a generic solution of $S_2(b)$, then $\mathbb{Q}(b, t)\langle q(t), p(t) \rangle$ coincides with $\mathbb{Q}(b, t)\langle h(t, q(t), p(t)) \rangle$. In fact we have*

$$\begin{cases} q(t) = \frac{2 \frac{d^2h(b, q(t), p(t))}{dt^2} + b}{4 \frac{dh(b, q(t), p(t))}{dt}}, \\ p(t) = -2 \frac{dh(b, q(t), p(t))}{dt} \end{cases} \quad (3.1)$$

(cf. [3, Formula (1.1), p. 227]).

Let now h be a generic solution of $E(b)$. Then it follows from Lemma 3.1 that (q, p) being a generic solution of $S_2(b)$, we have a differential $\mathbb{Q}(b, t)$ -isomorphism $\mathbb{Q}(b, t)\langle h \rangle \rightarrow \mathbb{Q}(b, t)\langle h(b, q, p) \rangle, h \mapsto h(b, q, p)$. So if we set

$$\begin{cases} q(b, h) = (2d^2h/dt^2 + b)/(4dh/dt), \\ p(b, h) = -2dh/dt, \end{cases} \quad (3.2)$$

then $(q(h), p(h))$ is a generic solution of $S_2(b)$ and we have

$$h = h(b, q(b, h), p(b, h)) \tag{3.3}$$

so that

$$\mathbb{Q}(b, t)\langle h \rangle = \mathbb{Q}(b, t)(q(b, h), p(b, h)). \tag{3.4}$$

Let (q, p) be a generic solution of $S_2(b)$. Then by Lemma 3.1 $h(b, q, p)$ is a generic solution of $E(b)$ and hence is a generic solution of $E(-b)$. Therefore we have by Lemma 3.2 and (3.4).

$$\mathbb{Q}(b, t)(q, p) = \mathbb{Q}(b, t)\langle h(b, q, p) \rangle = \mathbb{Q}(b, t)(q(-b, h), p(-b, h)), \tag{3.5}$$

where h on the right should be read $h(b, q, p)$. A calculation using (3.1) shows

$$\begin{cases} q(-b, h(b, q, p)) = q + b/p, \\ p(-b, h(b, q, p)) = p, \end{cases} \tag{3.6}$$

This is the transformation $I(b, -b)$.

Remark 3.3. — The transformation $I(b, -b)(q, p)$ is defined for a generic solution (q, p) of $S_2(b)$ but we can show $I(b, -b)(q, p)$ is well-defined for every solution (q, p) of $S_2(b)$.

When $b = -b$ or $b = 0$, $I(0, 0)$ is an automorphism of $\mathbb{Q}(t)(q, p)$. It follows from (3.5) that $I(0, 0)$ is the identity automorphism.

LEMMA 3.4. — $I(0, 0)$ is identity automorphism of $\mathbb{Q}(t)(q, p)$.

Now we analyze the transformation $T(b, b - 1) : Sol_2(b) \rightarrow Sol_2(b - 1)$.

PROPOSITION 3.5 (Okamoto). — If (q, p) is a generic solution of S_2 , then $h_1(b, q, p) := h(b, q, p) + q$ is a generic solution of $E(b - 1)$. The differential subfield

$$\mathbb{Q}(b, t)\langle h_1(b, q, p) \rangle$$

coincides with $\mathbb{Q}(b, t)(q, p)$. Namely we have

$$\mathbb{Q}(b, t)\langle h_1(b, q, p) \rangle = \mathbb{Q}(b, t)(q, p). \tag{3.7}$$

In fact, we have

$$\begin{cases} q = \frac{2 \frac{d^2 h_1(b, q, p)}{dt^2} - b + 1}{4 \frac{dh_1(b, q, p)}{dt}}, \\ p = 2 \frac{dh_1(b, q, p)}{dt} + 2q^2 + t \end{cases} \tag{3.8}$$

(cf. [3, Formula (2.5), p. 235]). Now it follows from Proposition 3.5 and (3.4)

$$\mathbb{Q}(b, t)(q, p) = \mathbb{Q}(b, t)\langle h_1(b, q, p) \rangle = \mathbb{Q}(b, t)(q(b - 1, h_1), p(b - 1, h_1)), \tag{3.9}$$

where h_1 should be read $h_1(b, q, p)$. A calculation using (3.2), (3.8) shows

$$\begin{cases} q(b-1, h_1) = -q + \frac{b-1}{p-2q^2-t} \\ p(b-1, h_1) = -p + 2q^2 + t. \end{cases} \quad (3.10)$$

This is the transformation $T(b, b-1) : Sol_2(b) \rightarrow Sol_2(b-1)$. The value $T(b, b-1)(q, p)$ is defined for a generic solution (q, p) of $S_2(b)$ but we can show that it is well-defined for every solution (q, p) of $S_2(b)$. Let us now consider the composite of the translation $b \mapsto b-1$ and the reflection at 0 of $\mathbb{C} : b \mapsto b-1 \mapsto -(b-1) = 1-b$. So $\frac{1}{2}$ is its fixed point. This affine transformation $b \mapsto 1-b$ of \mathbb{C} lifts to the transformation

$$I(b-1, 1-b) \circ T(b, b-1) : \mathbb{Q}(b, t)(q(t), p(t)) \rightarrow \mathbb{Q}(t, b)(q(1-b), p(1-b)).$$

It follows from the definition of the transformations that $I(b-1, 1-b) \circ T(b, b-1)$ comes from

$$\mathbb{Q}(b, t)(q(b), p(b)) = \mathbb{Q}(b, t)\langle h_1(b, q, p) \rangle = \mathbb{Q}(b, t)(q(1-b, h_1), p(1-b, h_1)).$$

At the fixed point $b = \frac{1}{2}$, we get an automorphism $I(b-1, 1-b) \circ T(b, b-1) = I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2}) : Sol_2(\frac{1}{2}) \rightarrow Sol_2(\frac{1}{2})$. Let us look for a solution

$$\left(q\left(\frac{1}{2}, t\right), p\left(\frac{1}{2}, t\right) \right) = \left(q\left(\frac{1}{2}\right), p\left(\frac{1}{2}\right) \right)$$

fixed by this automorphism.

LEMMA 3.6. — *The following conditions for a solution $(q(\frac{1}{2}, t), p(\frac{1}{2}, t))$ of $S_2(\frac{1}{2})$ are equivalent.*

- (1) $I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2})(q(\frac{1}{2}, t), p(\frac{1}{2}, t)) = (q(\frac{1}{2}, t), p(\frac{1}{2}, t))$.
- (2) $h_1(\frac{1}{2}, q, p) = h(\frac{1}{2}, q, p)$.
- (3) $q(\frac{1}{2}, t) = 0$.
- (4) $(q(\frac{1}{2}, t), p(\frac{1}{2}, t)) = (0, t/2)$.

Proof. — It follows from (3.2) that we have

$$\mathbb{Q}(b, t)\langle h(b, q, p) \rangle = \mathbb{Q}(b, t)(q, p) \quad (3.11)$$

if $dh/dt \neq 0$. We notice $dh/dt = 0$ implies $b = 0$ by $E(b)$. So if $b \neq 0$, we have (3.10). Similarly, it follows from (3.8) that if $b \neq 1$, we have

$$\mathbb{Q}(b, t)\langle h_1(b, q, p) \rangle = \mathbb{Q}(b, t)(q, p). \quad (3.12)$$

Now since $b = \frac{1}{2}$, we have both (3.11) and (3.12). So $I(b-1, 1-b) \circ T(b, b-1)(q, p)$ is defined for every $(q, p) \in Sol_2(\frac{1}{2})$ by the equalities (3.11) and (3.12) as for a generic solution: We have $\mathbb{Q}(t)(q, p) = \mathbb{Q}(t)\langle h_1(\frac{1}{2}, q, p) \rangle = \mathbb{Q}(t)(q(\frac{1}{2}, h_1), p(\frac{1}{2}, h_1))$ and $I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2})(q, p) = (q(\frac{1}{2}, h_1), p(\frac{1}{2}, h_1))$. So since $h_1(\frac{1}{2}, q, p) = h(\frac{1}{2}, q(\frac{1}{2}, h_1), p(\frac{1}{2}, h_1))$, if $q(\frac{1}{2}, h_1) = q(\frac{1}{2}, t)$ and $p(\frac{1}{2}, h_1) = p(\frac{1}{2}, t)$, then we have $h_1(\frac{1}{2}, q(\frac{1}{2}), p(\frac{1}{2})) = h(\frac{1}{2}, q(\frac{1}{2}), p(\frac{1}{2}))$. This

shows the conditions (1) and (2) are equivalent. It follows from the definition that the second condition is equivalent to the third. By the first equation of $S_2(\frac{1}{2})$, $q(\frac{1}{2}) = 0$ if and only if $p(\frac{1}{2}) = t/2$. \square

Remarks 3.7.

(1). — We have seen that the rational solution $(q, p) = (0, t/2)$ of $S_2(\frac{1}{2})$ is the unique fixed solution of the automorphism

$$I\left(-\frac{1}{2}, \frac{1}{2}\right) \circ T\left(\frac{1}{2}, -\frac{1}{2}\right) : \mathbb{Q}(t)\left(q\left(\frac{1}{2}\right), p\left(\frac{1}{2}\right)\right) \rightarrow \mathbb{Q}(t)\left(q\left(\frac{1}{2}\right), p\left(\frac{1}{2}\right)\right).$$

As Theorem 2.1 shows, $(q, p) = (0, t/2)$ is the unique rational solution and hence $(0, t/2)$ is fixed by the birational automorphism $I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2})$.

(2). — Now we fix the parameter $b \in \mathbb{C}$. Let

$$\mathcal{X} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p},$$

which is a vector field on $X = \mathbb{C}^3$ with coordinate (t, q, p) . We denote by $p : X = \mathbb{C}^3 \rightarrow \mathbb{C}$ the first projection. A section $(t, q(t), p(t))$ of the projection p is tangent to the vector field \mathcal{X} if and only if $(q(t), p(t))$ satisfies $dq(t)/dt = \partial H/\partial p, dp(t)/dt = -\partial H/\partial q$, i.e. $(q(t), p(t))$ is a solution of $S_2(b)$. We consider the pair (X, \mathcal{X}) and a birational automorphism $f : (X, \mathcal{X}) \rightarrow (X, \mathcal{X})$ that commutes with the vector field \mathcal{X} and with the projection p . Let $S \subset X$ be the set of fixed points of the automorphism f . Then we can show that if a solution of $S_2(b)$ has initial conditions in S , then it remains in S or equivalently S is \mathcal{X} -invariant. So if the restriction $p|_S : S \rightarrow \mathbb{C}$ is finite algebraic over a Zariski open set of \mathbb{C} , then S gives an algebraic solution of $S_2(b)$. In our example, $f = I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2})$ and $S = \{(t, 0, t/2) \in X = \mathbb{C}^3 \mid t \in \mathbb{C}\}$ so that $p|_S : S \rightarrow \mathbb{C}$ is biregular and we find the rational solution $(q(t), p(t)) = (0, t/2)$.

Geometrically, regarding now $b \in \mathbb{C}$ as a parameter, we consider the affine space $Z = \mathbb{C}^4$ of dimension 4 with coordinate $(b, t; q, p)$ so that \mathcal{X} is a vector field on Z . The transformation $T(b, b-1)$ induces a birational automorphism $T(-1) : (Z, \mathcal{X}) \rightarrow (Z, \mathcal{X}), (b, t; q, p) \mapsto (b-1, t; T(b, b-1)(q, p))$ lifting the translation $t(-1) : \mathbb{C} \rightarrow \mathbb{C}, b \mapsto b-1$ of the parameter space so that the diagram

$$\begin{array}{ccc} (Z, \mathcal{X}) & \xrightarrow{T(-1)} & (Z, \mathcal{X}) \\ p_1 \downarrow & & \downarrow p_1 \\ \mathbb{C} & \xrightarrow{t(-1)} & \mathbb{C} \end{array}$$

is commutative, where the vertical arrow p_1 is the first projection. Similarly the transformation $I(b, -b)$ defines a birational automorphism

$I : (Z, \mathcal{X}) \rightarrow (Z, \mathcal{X}), (b, t; q, p) \mapsto (-b, t; I(b, -b)(q, p))$ that lifts the reflection $i : \mathbb{C} \rightarrow \mathbb{C}, b \mapsto -b$ at the origine so that the diagram

$$\begin{array}{ccc} (Z, \mathcal{X}) & \xrightarrow{I} & (Z, \mathcal{X}) \\ p_1 \downarrow & & \downarrow p_1 \\ \mathbb{C} & \xrightarrow{i} & \mathbb{C} \end{array}$$

is commutative. Let G be the subgroup of affine transformations of \mathbb{C} generated by the translation $t(-1)$ and the reflection i at the origine of \mathbb{C} . So the subgroup $\langle t(-1) \rangle \simeq \mathbb{Z}$ generated by $t(-1)$ is the group of translations by integers and $G = \langle t(-1), i \rangle$ is the semi-direct product $\langle t(-1) \rangle \rtimes \langle i \rangle \simeq \mathbb{Z} \rtimes \mathfrak{S}_2$. Let \tilde{G} be the subgroup of \mathcal{X} -birational automorphism of Z generated by $T(-1)$ and I . Hence we have a canonical morphism $f : \tilde{G} \rightarrow G$ such that $f(T(-1)) = t(-1)$ and $f(I) = i$. Namely the projection $p_1 : Z \rightarrow \mathbb{C}$ is equivariant: For every $g \in \tilde{G}$, we have $p_1(g \cdot (b, t; q, p)) = f(g) \cdot (p_1(b, t; q, p))$ for all $(b, t; q, p) \in \mathbb{C}^4$. Here the dots mean the operations of \tilde{G} and G respectively on Z and \mathbb{C} . It follows from the definition $I^2 = \text{Id}$. A calculation shows $(I \circ T(-1))^2 = \text{Id}$ and hence $\tilde{G} = \langle T(-1) \rangle \rtimes \langle I \rangle$. Therefore the canonical morphism $f : \tilde{G} = \langle T(-1) \rangle \rtimes \langle I \rangle \rightarrow G = \langle t(-1) \rangle \rtimes \langle i \rangle$ is an isomorphism. In other words, G operates birationally on (Z, \mathcal{X}) .

Now we introduce a new parameter space $V := \{(v_1, v_2) \in \mathbb{C}^2 \mid v_1 + v_2 = 0\}$ so that $\{(1, -1), (-1, 1)\}$ is the root system of type A_1 . The root lattice is a subgroup $\mathbb{Z}(1, -1)$ of V generated by the roots $(1, -1), (-1, 1)$. The affine Weyl group is the group of affine transformations of the affine line V generated by the reflections with respect to the points $(n, -n) \in V, n \in \mathbb{Z}$. We know that the affine Weyl group of type A_1 is the subgroup of affine transformations of V generated by (i) the translations by the elements of the root lattice $\mathbb{Z}(1, -1)$ and (ii) the reflection at the origin $(0, 0)$. The weight lattice is the subgroup of V generated by $\frac{1}{2}(1, -1), \frac{1}{2}(-1, 1)$. The extended affine Weyl group is the subgroup of affine transformations of V generated by (i) the translation by the weight lattice and the reflection at the origin $(0, 0)$. As Okamoto did it, customarily we identify the parameter space \mathbb{C} of the Painlevé equation $P_2(\alpha)$ or of the system $S_2(b)$ with V by setting $v_1 = -v_2 = b = \alpha + \frac{1}{2}$ (cf. [3, 4, 5, 6, 7, 10]). So the subgroup G of the affine transformations of the parameter space \mathbb{C} of the system $S_2(b)$ generated by the translation $b \mapsto b + 1$ and the reflection at the origin is identified with the affine Weyl group of type A_1 . We had better, however, identify the subgroup G of affine transformations with the *extended affine Weyl group*. Namely in this paper we set $2v_1 = -2v_2 = b = \alpha + \frac{1}{2}$.

In real picture, $\{(v, -v) \in \mathbb{R}^2 \mid 0 \leq v \leq \frac{1}{2}\}$ is an alcove or a fundamental domain of the operation of the affine Weyl group on $V_{\mathbb{R}} = \{(v_1, v_2) \in \mathbb{R}^2 \mid$

$v_1 + v_2 = 0$ }. So with the above identification $\{b \in \mathbb{R} \mid 0 \leq b \leq 1\}$ is an alcove. Hence $\{b \mid b \in \mathbb{Z}\}$ is the set of walls of the affine Weyl group. So the affine Weyl group is generated by the reflections at $n \in \mathbb{Z}$ of \mathbb{R} . Similarly $\{b \in \mathbb{R} \mid 0 \leq b \leq \frac{1}{2}\}$ is a fundamental domain of the operation of the extended affine Weyl group on $\mathbb{R} = \{b \in \mathbb{R}\}$. Thus the extended affine Weyl group is generated by the reflections at the point of $\frac{1}{2} + \mathbb{Z}$. Now we can rephrase theorem as follows.

THEOREM 3.8.

- (1) $S_2(b)$ has a 1-parameter family of classical solutions if $b \in \mathbb{C}$ is a wall of the affine Weyl group of type A_1 .
- (2) $S_2(b)$ has the unique rational solution if $b \in \mathbb{C}$ is a wall of the extended affine Weyl group but b is not a wall of the affine Weyl group.
- (3) Except for those solutions in (1) and (2), no solutions of $S_2(b)$ are classical.

The above observation is summarized as follows. The point $b = \frac{1}{2}$ of the parameter space \mathbb{C} is the fixed point of the operation of $g = i \circ t(-1) \in G$. So g induces an automorphism of the fiber $p_1^{-1}(\frac{1}{2})$ of the first projection $p_1 : Z \rightarrow \mathbb{C}$ and consequently the automorphism $I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2}) : Sol_2(\frac{1}{2}) \rightarrow Sol_2(\frac{1}{2})$. The rational solution $(q, p) = (0, t/2)$ is the solution of $S_2(\frac{1}{2})$ invariant by the automorphism $I(-\frac{1}{2}, \frac{1}{2}) \circ T(\frac{1}{2}, -\frac{1}{2})$. So the principle is that first we look for an element g of the extended affine Weyl group that has a fixed point $b = \frac{1}{2}$ on the parameter space \mathbb{C} so that g induces an automorphism of $S_2(\frac{1}{2})$. Then we look for a solution of $S_2(\frac{1}{2})$ fixed by the operation of g .

4. Recurrence formula

We need theory of τ -function. Let us introduce a τ function of a solution $(q(t), p(t))$ of $S_2(b)$ by

$$\tau'(t)/\tau(t) = H(b, q(t), p(t)),$$

where $H = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}t)p - b$ is the Hamiltonian. Let us now fix the parameter b and the solution $(q(b, t), p(b, t))$ of $S_2(b)$. We set $\tau_0 = \tau_0(t)$ a τ -function of $q(b, t), p(b, t)$. By iteration of the operations $T(b, b-1), T(b-1, b-2), \dots$, we get from the given solution $(q(b), p(b)) = (q(b, t), p(b, t))$ of $S_2(b)$ a solution $(q(b-m), p(b-m))$ of $S_2(b-m)$. Let $\tau_m = \tau_m(t)$ be a τ -function of $(q(b-m), p(b-m))$ so that

$$\tau'_m/\tau_m = H(b-m, q(b-m), p(b-m)), \tag{4.1}$$

THEOREM 4.1 (Okamoto [3]). — *The τ_m 's satisfy the Toda equation*

$$d^2 \log \tau_m / dt^2 = c(m) \tau_{m-1} \tau_{m+1} / \tau_{m-1}^2, \quad (4.2)$$

where $c(m)$ is a non-zero constant.

Example 4.2. — We have a rational solution $(q(\frac{1}{2}), p(\frac{1}{2})) = (0, t/2)$ of $S_2(\frac{1}{2})$ so that $H(\frac{1}{2}, t, q(\frac{1}{2}), p(\frac{1}{2})) = -\frac{1}{8}t^2$. So since τ_0 is a solution of $\tau_0' = (-t^2/8) \tau_0$, $\tau_0 = \exp(-\int t^2/8dt) = \exp(-t^3/24)$. Now let us set

$$\tau_m = \exp(-t^3/24) T_m(t). \quad (4.3)$$

It follows from the Toda equation (4.3) with $c(m) = 1$ that

$$\begin{cases} T_{m+1}(t) = (T_m(t)^2 t - 4(T_m''(t)T_m(t) - T_m'(t)^2)) / T_{m-1}(t) \\ T_0(t) = T_1(t) = 1. \end{cases} \quad (4.4)$$

We find here the Yablonskii–Vorob'ev polynomials. It follows immediately from (4.3) that the $T_m(t)$'s are rational functions. In Section 5, we will discuss proofs that the $T_m(t)$'s are in fact polynomials. It follows from the definition of $q(b-1), p(b-1)$ we have

$$H(b-1, t, q(b-1), p(b-1)) = H(b, t, q(b), p(b)) + q(b). \quad (4.5)$$

Therefore by (4.1), (4.3) and (4.5) we get

$$\tau'_{m+1} / \tau_{m+1} - \tau'_m / \tau_m = q\left(-\frac{1}{2} - m\right). \quad (4.6)$$

So (4.6) shows that the Yablonskii–Vorob'ev polynomials give an effective method of calculating the rational solution $q(-\frac{1}{2} - m)$ of the second Painlevé equation $P_2(-m-1) : q'' = 2q^3 + tq - (m+1)$ for $m \in \mathbb{Z}$.

The conclusion is that the recurrence relation (2.2) comes from the Toda equation satisfied by the τ -functions.

5. Proof that the $T_m(t)$'s are polynomials

The quickest way of proving that the $T_m(t)$'s are polynomials is to use the following fundamental

THEOREM 5.1. — *For the second Painlevé equation, a τ -function $\tau(t)$ is holomorphic over \mathbb{C} .*

If we admit the Theorem, since $\tau_m = T_m(t) \exp(-t^3/24)$, the rational function $T_m(t)$ is holomorphic over \mathbb{C} and hence $T_m(t)$ is a polynomial. Theorem 5.1 is a deep and transcendental result, which we would like to avoid for the following reason. When we want to generalize the Yablonskii–Vorob'ev

polynomials to the other Painlevé equations, the analogous theorems for the other Painlevé equations do not allow us to prove that the rational functions in question are polynomials but they show only that the rational functions are regular over $\mathbb{C} \setminus \{0, 1\}$ for example if we deal with the P_6 . The best way of showing that the $T_m(t)$'s are polynomials is as follows. As we noticed above, the $T_m(t)$'s are rational functions by the recurrence relation (4.4). So $\tau_m = T_m \exp(-t^3/24)$ is meromorphic over \mathbb{C} . Consequently $\tau'_m/\tau_m = H(-\frac{1}{2} - m, t, q(-\frac{1}{2} - m), p(-\frac{1}{2} - m)) = h(-\frac{1}{2} - m, q(-\frac{1}{2} - m), p(-\frac{1}{2} - m))$ is meromorphic over \mathbb{C} and satisfies the differential equation $E(-\frac{1}{2} - m)$. We can easily show

LEMMA 5.2. — *If a solution h of $E(-\frac{1}{2} - m)$ is meromorphic at a point $a \in \mathbb{C}$, then $t = a$ is at most a pole of order 1. Moreover if the point a is a pole of order 1 of h , then the residue is equal to 1.*

Now Lemma 5.2 shows that τ_m is regular over \mathbb{C} so that $T_m(T)$ is a polynomial.

6. Special polynomials

We apply the method of the previous sections to the Painlevé equations P_j ($3 \leq j \leq 6$). As we illustrated in Section 3 taking the second equation as an example, the Painlevé equation P_j has a Hamiltonian representation

$$S_j(\mathbf{v}) \quad \begin{cases} dq/dt = \partial H_j / \partial p, \\ dp/dt = -\partial H_j / \partial q, \end{cases}$$

where $H_j = H_j(\mathbf{v}, t, q, p)$ is a polynomial of q, p and \mathbf{v} with coefficients in $\mathbb{Q}(t)$. Namely $S_j(\mathbf{v})$ is a Hamiltonian system parametrized by \mathbf{v} which belongs to a complex vector space V_j . An appropriate extended Weyl group G_j operates not only on the parameter space V_j but also on the set of solutions of the system $S_j(\mathbf{v})$.

Remark 6.1. — Precisely speaking, we must enlarge G_j to \tilde{G}_j or we must consider an extension \tilde{G}_j of the group G_j so that \tilde{G}_j operates on the set of solutions of $S_j(\mathbf{v})$. Namely we can find a group \tilde{G}_j such that G_j is a quotient group of \tilde{G}_j and such that G_j operates on the set of solutions of the system $S_j(\mathbf{v})$ as well as on the parameter space V_j . We know, however, $\tilde{G}_j = G_j$ for $j = 2, 4$.

6.1. The fourth Painlevé equation. The fourth equation

$$P_4(\alpha, \beta) \quad y'' = \frac{y'^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

is equivalent to the Hamiltonian system

$$S_4(v_1, v_2, v_3) \quad \begin{cases} dq/dt = 4pq - q^2 - 2tq + 2(v_1 - v_2), \\ dp/dt = -2p^2 + 2q + 2t + (v_1 - v_3), \end{cases}$$

with Hamiltonian $H_4 = 2qp^2 - (q^2 + 2tq - 2(v_1 - v_2))p - (v_1 - v_3)q$, where $(v_1, v_2, v_3) \in \mathbb{C}^3$ belongs to the parameter space $V_4 = \{(v_1, v_2, v_3) \in \mathbb{C}^3 \mid v_1 + v_2 + v_3 = 0\}$ (cf. [3, 7]). Here the parameters v_1, v_2, v_3 are related with the usual parameters α, β of the Painlevé equation $P_4(\alpha, \beta)$ by $\alpha = 3v_3 + 1, \beta = -2(v_1 - v_2)^2$. The extended affine Weyl group of type A_2 operates on both the parameter space V_4 and the set of solutions of the system $S_4(v_1, v_2, v_3)$. We consider a translation $v \mapsto v + \frac{1}{3}(-1, -1, 2)$ and a permutation $(v_1, v_2, v_3) \mapsto (v_3, v_1, v_2)$, which are affine transformations of the parameter space and belong to the extended affine Weyl group. Their composite is the affine transformation $(v_1, v_2, v_3) \rightarrow (v_3 + 2/3, v_1 - 1/3, v_2 - 1/3)$ so that $\frac{1}{3}(1, 0, -1)$ is the fixed point. Let $(q(t), p(t))$ be a solution of $S_4(1/3, 0, -1/3)$. Let $h = H_4((1/3, 0, -1/3), t, q(t), p(t))$. Then the argument of Lemma 3.16 shows $(q(t), p(t))$ is fixed by the transformation $\mathbf{v} \mapsto \mathbf{v} + \frac{n}{3}(-1, -1, 2)$ if and only if $h + q + 2t/3 = h$ or equivalently $q = -2t/3$. If $q = -2t/3$, then $p = t/3 + 1/(2t)$ by the first equation of $S_4(1/3, 0, -1/3)$. We discover the fixed solution $(q(t), p(t)) = (-2t/3, t/3 + 1/(2t))$ at $\mathbf{v} = \frac{1}{3}(1, 0, -1)$. Substituting the solution into the Hamiltonian, we get $H_4(1/3(1, 0, -1), t, q(t), p(t)) = 4t^3/27 + 2t/3$ and the τ -function $\tau_0(t)$ satisfies $\tau_0'(t)/\tau_0(t) = 4t^3/27 + 2t/3$ so that $\tau_0 = \text{const. exp}(\int (4t^3/27 + 2t/3)dt) = \text{const. exp}(t^4/27 + t^2/3)$. We consider the solutions translated by the above translations $v \mapsto v + \frac{n}{3}(-1, -1, 2)$ with $n \in \mathbb{Z}$ as well as their τ -functions $\tau_n(t)$. It follows from the Toda equation ([3, Formula (3.18), p. 242])

$$d^2 \log \tau_n / dt^2 + 2(n - 2/3) = c(n)\tau_{n-1}\tau_{n+1}/\tau_n^2$$

with $c(n) = 1$ that we have

$$T_{n+1}(t) = (T_n''(t)T_n(t) - T_n'(t)^2 + (4t^2/9 + 2(2n - 1)/3) T_n(t)^2) / T_{n-1}(t)$$

with $T_0(t) = T_1(t) = 1$. If we set $x = \sqrt{3/2t}$ and $T_n(t) = S_n(x)$ with $c(m) = 2/3$, we get

$$S_{n+1}(x) = (S_n''(x)S_n(x) - S_n'(x)^2 + (x^2 + (2n - 1)) S_n(x)^2) / S_{n-1}(x),$$

with $S_0(x) = S_1(x) = 1$. $S_n(x)$ is a polynomial of degree $n(n - 1)$, which is called the Okamoto polynomial (cf. 8.2). The Okamoto polynomials appear at the barycenters $\frac{1}{3}(1 - k, -k, 2k - 1), k \in \mathbb{N}$ of the alcoves, whereas the Hermite functions are on the wall $v_1 - v_3 = 0$. In particular the Hermite

polynomial appear at the points $\frac{k}{3}(-1, 2, -1), k = 0, 1, 2, \dots$. We notice here that the reflection with respect to a wall induces the identity transformation of the solutions on the wall as for P_2 .

6.2. The third Painlevé equation. The third equation

$$P_3(\alpha, \beta, \gamma, \delta) \quad y'' = \frac{y'^2}{y} - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}$$

is equivalent to the Hamiltonian system

$$S_3(v_1, v_2) \quad \begin{cases} tdq/dt = 2q^2p - q^2 - v_1q + t \\ tdp/dt = -2qp^2 + 2qp + v_1p - (v_1 + v_2)/2 \end{cases}$$

with Hamiltonian $H_3(\mathbf{v}, t, q, p) = (q^2p^2 - (q^2 + v_1q - t)p + \frac{1}{2}(v_1 + v_2)q)/t$, where (v_1, v_2) belongs to the parameter space \mathbb{C}^2 (cf. [6, 7]). In this case the equivalence is exceptional. Namely we replace t in the third equation $P_3(\alpha, \beta, \gamma, \delta)$ by t_1 and introduce t by $t_1^2 = t$. Then $S_3(\mathbf{v})$ is equivalent to $P_3(\alpha, \beta, \gamma, \delta)$. Namely after the field extension $\mathbb{Q}(t) \rightarrow \mathbb{Q}(t_1)$, $S_3(\mathbf{v})$ is equivalent to $P_3(\alpha, \beta, \gamma, \delta)$. Here we normalized $\gamma = 4, \delta = -4$ and v_1, v_2 and α, β are related by $\alpha = -4v_2, \beta = 4(v_1 + 1)$. The corresponding Lie algebra is B_2 . We consider a translation $(v_1, v_2) \mapsto (v_1 + 1, v_2 + 1)$ and the reflection with respect to a line $v_1 + v_2 = 0$. So their composite is the reflection with respect to a line $v_1 + v_2 + 1 = 0$. Let $(q(t), p(t))$ be a solution of $S_4(v_1, v_2)$ with $v_1 + v_2 + 1 = 0$. We set $h = tH_3(v_1, v_2, t, q(t), p(t)) + v_1^2/4 - t/2$. Then the argument of Lemma 3.6 shows that the solution $(q(t), p(t))$ is fixed by the reflection with respect to the line $v_1 + v_2 + 1 = 0$ if and only if $h + (2v_1 + 1)/4 - q(t)(p(t) - 1) = h$ or equivalently

$$q(t)(p(t) - 1) = \frac{2v_1 + 1}{4}. \tag{6.2.1}$$

Differentiating (6.3.1) and eliminating $q', p'(t)$ by using $S_3(v_1, v_2)$, we get

$$q + 2q^2 - 2pq^2 - 2t + 2pt + 2qv_1 = 0. \tag{6.2.2}$$

Eliminating p from (6.2.1) and (6.2.2), we get

$$(1 + 2v_1)(q^2 + t) = 0$$

We have either $v_1 = -\frac{1}{2}$ or $q^2 = -1$. In the first case, it follows from (6.2.2) that we have $q(t) = 0$ or $p(t) = 1$. Since $q = 0$ contradicts the first equation of system $S(v_1, v_2)$, we conclude $p(t) = 1$. Then it follows from the first equation of system $S(v_1, v_2) \quad tdq/dt = 2q^2 - q^2 - v_1q + t$ so that we find a classical solution $(q, p) = (q, 1)$, where q satisfies $tdq/dt = 2q^2 - q^2 - v_1q + t$. Since we are looking for a fixed algebraic solution, let us study the second case $q^2 = -t$. Then it follows from (6.2.1) $p = 1 + (2v_1 + 1)/(4q)$. In other words, we find the fixed solution $(q(t), p(t)) =$

$(s, 1 + (1 + 2v_1)/(4s))$, which is algebraic, on the line $v_1 + v_2 + 1 = 0$, where $s^2 = -t$. So we assume from now on $v_1 + v_2 + 1 = 0$. Substituting the solution into the Hamiltonian, we get $H_3(v_1, v_2, t, q(t), p(t)) = 1 - (v_1 + \frac{1}{2})st^{-1} + ((1 + 2v_1)(1 - 2v_1)/16)t^{-1}$. We consider the translations by $n(1, 1)$ ($n \in \mathbb{N}$) of the fixed solution on the line $v_1 + v_2 + 1 = 0$ and their τ -functions $\tau_n(t)$. We set $\tau(t) = T_n(t) \exp(\int (H_3(v_1, v_2, t, q(t), p(t)) - \frac{n}{4}(2v_1 + 1)t^{-1}) dt)$. Then it follows from the Toda equation ([6, Formula (3.12), p. 328])

$$\frac{d}{dt}t \frac{d}{dt} \log \tau_n = c(n) \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}$$

with $c(n) = 1$ that we get

$$T_{n+1}(s) = \frac{-\left(\frac{dT_n(s)}{ds} + s\frac{d^2T_n(s)}{ds^2}\right)T_n(s) + s\frac{dT_n(s)^2}{ds} + (4s + 2v_1 + 1)T_n(s)^2}{4sT_{n-1}(s)}$$

with $T_0(s) = T_1(t) = 1$. The polynomials $T_n(s)$'s contain the parameter v_1 so that we may denote $T_n(s)$ by $T_n(v_1, s)$. If we set $u = 1/(4s)$ and $S_n(a, u) = T_n(v_1, s)$ and $a = 2v_1 + 1$, we get

$$S_{n+1}(a, u) = \frac{-\left(4u^3\frac{\partial S_n(a, u)}{\partial u} + 4u^4\frac{\partial^2 S_n(a, u)}{\partial u^2}\right)S_n(a, u) + 4u^4\frac{\partial S_n(a, u)^2}{\partial u} + (1 + au)S_n(a, u)^2}{S_{n-1}(a, u)}.$$

The general theory of the τ -function tells us that $\tau_n(t)$ is regular over the universal covering space of $\mathbb{C} \setminus \{0\}$ so that $S_n(u)$ is a Laurent polynomial of u . We can show, however, that $S_n(a, u)$ is a polynomial of a and u such that $\deg_u S_n(a, u) = n(n - 1)/2$ (cf. (8.3)).

6.3. The fifth Painlevé equation. The fifth equation

$$P_5(\alpha, \beta, \gamma, \delta) \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \delta \frac{y(y+1)}{y-1}$$

is equivalent to the Hamiltonian system

$$S_5(\mathbf{v}) \quad \begin{cases} tdq/dt = 2q(q-1)^2p + (v_1 - v_2)(q-1)^2 + 2(v_1 + v_2)q(q-1) - tq, \\ tdp/dt = -(3q^2 - 4q + 1)p^2 - ((6v_1 + 2v_2)q - 4v_1 - t)p \\ \qquad \qquad \qquad - (v_1 - v_3)(v_1 - v_4), \end{cases}$$

the Hamiltonian H_5 being

$$\frac{q(q-1)^2p^2 + ((v_1 - v_2)(q-1)^2 + 2(v_1 + v_2)q(q-1) - tq)p}{t} + \frac{(v_1 - v_3)(v_1 - v_4)(q-1)}{t},$$

where (v_1, v_2, v_3, v_4) belongs to the parameter space $\{\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{C}^4 \mid v_1 + v_2 + v_3 + v_4 = 0\}$. Here we normalize $\delta = -\frac{1}{2}$, and the parameters v_i 's and α, β, γ are related by $\alpha = (v_3 - v_4)^2/2, \beta = -(v_1 - v_2)^2/2, \gamma = 2(v_1 + v_2 + 1)$. The corresponding Lie algebra is A_3 . So the extended affine Weyl group of type A_3 operates on the parameter space. We consider the translation $\mathbf{v} \mapsto \mathbf{v} + 1/4(1, -3, 1, 1)$ and a permutation (1432) that naturally operates on the parameter space. These are operations of elements of the extended affine Weyl group. Their composite f sends $\mathbf{v} = (v_1, v_2, v_3, v_4)$ to $(v_2 - 3/4, v_3 + 1/4, v_4 + 1/4, v_1 + 1/4)$. The transformation f is of order 4 and $1/8(-3, 3, 1, -1)$ is the fixed point. The fixed solution at the fixed point and its translations by $n/4(1, -3, 1, 1)$, $n \in \mathbb{N}$ give special polynomials but f^2 yields more general and interesting polynomials involving a parameter. Namely the set of fixed points of f^2 is a line $\{(v, \frac{1}{2} - v, v - \frac{1}{2}, -v) \in \mathbb{C}^3 \mid v \in \mathbb{C}\}$ of the parameter space. The fixed solution of the Hamiltonian system on this line is $(q(t), p(t)) = (-1, t/8 + v)$ which contains a parameter v . Substituting $(p(t), q(t))$ in the Hamiltonian, we get $H_5(v, \frac{1}{2} - v, v - \frac{1}{2}, -v; t, q(t), p(t)) = t/16 + v + 2v(2v - 1)/t$ so that $\tau_0 = \exp(\int (t/16 + v + 2v(2v - 1)/t) dt) = 2v(2v - 1) \exp(t^2/32 + vt)$. Let τ_n be a τ -function of the solution obtained from the solution $(q(t), p(t))$ of $S_5(v, \frac{1}{2} - v, v - \frac{1}{2}, -v)$ by the translation $\mathbf{v} \mapsto \mathbf{v} + n/4(1, -3, 1, 1)$.

We set $\tau_n(t) = T_n(t) \exp(\int (t/16 + v + 2v(2v - 1)t - n(t/4 - 2v - 1)/t) dt)$. Then it follows from the Toda equation

$$\frac{d}{dt} t \frac{d}{dt} \log \tau_n + v_4 - v_1 + n = c(n) \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}$$

([5, p. 68, Formula (2.9)])

$$T_{n+1} = ((t/8 - v + 3n/4)T_n^2 + T_n' T_n + tT_n'' T_n - tT_n'^2) / T_{n-1}$$

with $T_0 = T_1 = 1$. The general theory of the τ -functions shows that $T_n(v, t)$ is a Laurent polynomial of t . We can, however, prove that $T_n(v, t)$ is a polynomial in t and v of degree $n(n - 1)/2$ (cf. 8.4).

6.4. The sixth Painlevé equation. The sixth equation

$$q'' = \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) q'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) q' + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{q^2} + \gamma \frac{t-1}{(q-1)^2} + \delta \frac{t(t-1)}{(q-t)^2} \right)$$

is equivalent to the Hamiltonian system

$$S_6(\mathbf{b}) \begin{cases} \frac{tdq}{dt} = \frac{\partial H_6}{\partial p} \\ \frac{tdp}{dt} = -\frac{\partial H_6}{\partial q} \end{cases}$$

with Hamiltonian $H_6(\mathbf{b}, t, q, p) = [q(q-1)(q-t)p^2 - \{(b_1+b_2)(q-1)(q-t) + (b_1-b_2)q(q-t) + (b_3+b_4)q(q-1)\}p + (b_1+b_3)(b_1+b_4)]/(t(t-1))$, where (b_1, b_2, b_3, b_4) belongs to the parameter space \mathbb{C}^4 . The parameters are related by $\alpha = (b_3 - b_4)^2/2$, $\beta = -(v_1 + b_2)^2/2$, $\gamma = (b_1 - b_2)^2/2$, $\delta = -(b_3 + b_4)^2/2$. The corresponding Lie algebra is D_4 . We consider a translation $\mathbf{v} \mapsto \mathbf{v} + (0, 0, 1, 0)$ and a linear map $(b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, -b_3, -b_4)$, which are operations of elements of the extended Weyl group of the Lie algebra of type D_4 . Their composite is $(b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, -b_3 - 1, -b_4)$ and hence $(b_1, b_2, -\frac{1}{2}, 0)$ is the fixed point. We look for fixed solutions for $(b_1, b_2, -\frac{1}{2}, 0)$. To this end, let us set according to [4] $h(\mathbf{b}, t) = H_6(\mathbf{b}, t, q(t), p(t)) + \sigma'_2[\mathbf{b}]t - \sigma_2[\mathbf{b}]$ for a solution $(q(t), p(t))$ of $S_6(\mathbf{v})$, where σ'_2 is the second symmetric polynomial of b_1, b_3, b_4 and $\sigma_2[\mathbf{b}]$ is the second symmetric polynomial of b_1, b_2, b_3, b_4 . We denote $h(\mathbf{b}, t) - q(t)(q(t)-1)p(t) + (b_1+b_4)q(t) - (b_1+b_2+b_4)/2$ by $h^+(\mathbf{b}, t)$. Then it follows from [4, Formula (1.13), p. 352] a solution $(q(t), p(t))$ of $S_6(b_1, b_2, -\frac{1}{2}, 0)$ is fixed by the above transformation if and only if $h^+(b_1, b_2, -\frac{1}{2}, 0) = h(b_1, b_2, -\frac{1}{2}, 0)$ or equivalently

$$-q(t)(q(t)-1)p(t) + b_1q(t) - \frac{1}{2}(b_1+b_2) = 0 \quad (6.4.1)$$

by definition. Differentiating (6.4.1) and eliminating $q'(t), p'(t)$ by using the system $S_6(b_1, b_2, -\frac{1}{2}, 0)$, we get

$$\begin{aligned} & -p^2q^3 + p^2q^4 - p^2qt + 3p^2q^2t - 2p^2q^3t + pq^2b_1 - 2pq^3b_1 + ptb_1 \\ & - 4pqt b_1 + 4pq^2t b_1 + q^2b_1^2 + tb_1^2 - 2qtb_1^2 + pq^2b_2 + ptb_2 \\ & - 2pqt b_2 - qb_1b_2 + tb_1b_2 = 0. \end{aligned} \quad (6.4.2)$$

It follows from (6.4.1)

$$p(t) = \frac{1}{q(t)(q(t)-1)} \left(b_1q(t) - \frac{1}{2}(b_1+b_2) \right). \quad (6.4.3)$$

Substituting (6.4.3) into (6.4.2), we get

$$(-b_1^2 + 2b_1b_2 - b_2^2 - 4tb_1b_2)q^2 + 2t(b_1+b_2)^2q - t(b_1+b_2)^2 = 0.$$

So we have

$$q(t) = \frac{t(b_1+b_2)^2 + (b_1+b_2)(b_1-b_2)\sqrt{t(t-1)}}{(b_1-b_2)^2 + 4tb_1b_2}$$

and consequently

$$p(t) = \frac{b_1 q - \frac{1}{2}(b_1 + b_2)}{q(q-1)}$$

by (6.4.3). Substituting $(q(t), p(t))$ into the Hamiltonian H_6 , we get

$$\begin{aligned} H_6 \left(b_1, b_2, -\frac{1}{2}, 0; t, q(t), p(t) \right) \\ = -b_1 + 2tb_1 + b_1^2 - 2tb_1^2 + 2\sqrt{t(t-1)}b_1^2 \\ - b_2 + 2b_1b_2 + b_2^2 - 2tb_2^2 - \frac{2\sqrt{t(t-1)}b_2^2}{4t(t-1)}, \end{aligned}$$

which we denote by \mathcal{H}_0 . So we have $\tau_0 = \exp(\int \mathcal{H}_0 dt)$. Let τ_n be a τ -function of the solution of $S_6(b_1, b_2, -\frac{1}{2} + n, 0)$ obtained from the solution $(q(t), p(t))$ by the translation $\mathbf{v} \mapsto \mathbf{v} + n(0, 0, 1, 0)$. We set

$$\tau_n(t) = T_n(t) \exp \left(\int \left(\mathcal{H}_0 - \frac{n(b_1 t - \frac{1}{2}(b_1 + b_2))}{t(t-1)} \right) dt \right).$$

Then it follows from the Toda equation

$$\frac{d}{dt} t(t-1) \frac{d}{dt} \log \tau_n + (b_1 + b_3 + n)(b_3 + b_4 + n) = c(n) \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}$$

with $c(n) = 1$ (cf. [4, Formula (4.4), p. 368]),

$$\begin{aligned} T_{n+1} = \frac{\left\{ \frac{1}{4} \left(-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2) \frac{2t-1}{\sqrt{t(t-1)}} \right) + \left(n - \frac{1}{2} \right)^2 \right\} T_n^2}{T_{n-1}} \\ + \frac{(2t-1)T'_n T_n + t(t-1)(T''_n T_n - T_n'^2)}{T_{n-1}}. \quad (6.5) \end{aligned}$$

Now we introduce a new variable v by $v = \sqrt{t/(t-1)} + \sqrt{(t-1)/t}$. Then the recurrence formula (6.5) is translated into

$$\begin{aligned} T_{n+1} = \frac{\left\{ \frac{1}{4} (-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2)v) + \left(n - \frac{1}{2} \right)^2 \right\} T_n^2}{T_{n-1}} \\ + \frac{\frac{1}{4}(v^2 - 4) \left\{ (v^2 - 4) \frac{d^2 T_n}{dv^2} + v \frac{dT_n}{dv} \right\} T_n - \left(\frac{1}{2}(v^2 - 4) \frac{dT_n}{dv} \right)^2}{T_{n-1}}. \end{aligned}$$

Since $T_n(v)$ contains parameters b_1, b_2 , we denote it by $T(b_1, b_2, v)$. We can show that $T_n(b_1, b_2, v)$ is a polynomial in b_1, b_2, v and $\deg_v T_n = n(n-1)/2$ (see 8.5).

7. Special polynomials and Young diagrams

We do not know any significance of our polynomials. Attending at our talk at Montreal, a weekend is enough for Noumi and Okamoto to reveal a hidden structure of our polynomials for P_6 .

7.1. Polynomials in (6.4) for P_6 . We already need several pages to print the polynomial $T_5(b_1, b_2, v)$ of Section 6 for P_6 . Let $z = (2 - v)/4$, $w = (2 + v)/4$, $d = -4b_1^2$, $c = -4b_2^2$ and

$$d_n := (d + 1^2) z \cdot (d + 3^2) z \cdots (d + (2n - 1)^2) z,$$

$$c_n := (c + 1^2) w \cdot (c + 3^2) w \cdots (c + (2n - 1)^2) w$$

for $n \in \mathbb{N}$. Denoting $2^{n(n-1)}T_n(b_1, b_2, u)$ by $U_n(w, z)$, they observed $U_2 = d_1 + c_1$, $U_3 = d_1d_2 + 3c_1d_2 + 3c_2d_1 + c_1c_2$, $U_4 = d_1d_2d_3 + 6c_1d_2d_3 + 15c_2d_1d_3 + 10c_1c_2d_3 + 10c_3d_1d_2 + 15c_1c_3d_2 + 6c_2c_3d_1 + c_1c_2c_3$. Here it is convenient to attach a pair of Young diagrams

$$(\emptyset, \square)$$

for d_1 ,

$$(\square, \emptyset)$$

for c_1 ,

$$(\emptyset, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array})$$

for d_1d_2, \dots , etc. so that for example the corresponding pair for $c_1d_2d_3$ is

$$(\square, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}).$$

So

$$U_4 = (\emptyset, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}) + 6(\square, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}) + 15(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}) + 10(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array})$$

$$+ 10(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}) + 15(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}) + 6(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \square) + (\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \emptyset).$$

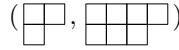
Here two monomials with coefficients 10 give the coefficients of w^3z^3 . The question is whether the U_n 's have such expressions for every $n \in \mathbb{N}$ and if so how the coefficients are determined. For example, in U_4 the binomial coefficients (1, 6, 15, 20, 15, 6,1) appear. Why 20 is divided into 10 for

$$(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array})$$

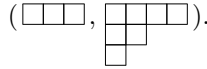
and 10 for

$$(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array})?$$

In this case, $20 = 10 + 10$ is reasonable because two diagrams are dual and because the polynomial is symmetric for the permutation of (d, w) and (c, z) but we have a non-trivial decomposition say in U_5 : 120 is divided into 50 for



and 70 for



A thorough investigation on this decomposition will be done in [2].

Noumi noticed further the constant terms of the Okamoto polynomials have simple factorizations. Let $S_n(x)$ be the Okamoto polynomials in 6.1. Then we have $S_0(0) = S_1(0) = S_2(0) = 1$, $S_3(0) = 5$, $S_4(0) = 5^2 \cdot 7$, $S_5(0) = 5^3 \cdot 7^2 \cdot 11$, $S_6(0) = 5^4 \cdot 7^3 \cdot 11^2 \cdot 13$, $S_7(0) = 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17$, $S_8(0) = 5^6 \cdot 7^5 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19, \dots$. So looking at the exponents, we have a feeling that they are related to the Young diagram too. The constant terms of the Yablonskii–Vorob’ev polynomials have similar factorizations. Namely let $T_n(t)$ be the Yablonskii–Vorob’ev polynomial. We have $T_2(0) = 2^2$, $T_3(0) = -2^4 \cdot 5$, $T_4(0) = 0$, $T_5(0) = -2^{10} \cdot 5^3 \cdot 7^2$, $T_6(0) = -2^{14} \cdot 5^4 \cdot 7^3 \cdot 11$, $T_7(0) = 0$, $T_8(0) = -2^{24} \cdot 5^6 \cdot 7^5 \cdot 11^3 \cdot 13^2$, $T_9(0) = 2^{30} \cdot 5^7 \cdot 7^6 \cdot 11^4 \cdot 13^3 \cdot 17, \dots$.

7.2. Our polynomials for P_3 and P_5 resemble and they look also related with the Young diagrams. To illustrate this, let us take our polynomials $T_n(v, t)$ of (6.3) for P_5 . It is convenient to introduce $l(i) := (i - 4v)/4$. Their constant terms $T_n(v, 0)$ are 1, 1, $l(3)$, $l(3)l(4)l(5)$, $l(3)l(4)l(5)^2l(6)l(7)$, $l(3)l(4)l(5)^2l(6)^2l(7)^2l(8)l(9), \dots$. Moreover if we set $z = t/8$, we can interpret a first few of them as $T_0(v, t) = T_1(v, t) = 1$, $T_2(v, t) = z + l(3)$, $T_3(v, t) = z^3 + 3l(4)z^2 + 3l(3)l(5)z + l(3)l(4)l(5)$, $T_4(v, t) = z^6 + 6l(5)z^5 + 15l(4)l(6)z^4 + 20(\frac{1}{2}l(4)l(5)l(6) + \frac{1}{2}l(3)l(5)l(7))z^3 + 15l(3)l(5)^2l(7)z^2 + 6l(3)l(4)l(5)l(6)l(7)z + l(3)l(4)l(5)^2l(6)l(7)$. It is easy to attach a Young diagram to a monomial $l(i)l(j) \dots z^k$ as we did for P_6 but we do not understand how their coefficients are determined.

8. Table

We give a first few terms of the polynomials in Section 6.

8.1. The second Painlevé equation.

The recurrence formula for the Yablonskii–Vorob’ev polynomials:

$$T_n(t) = \frac{tT_{n-1}^2(t) - 4(T_{n-1}''(t)T_{n-1}(t) - T_{n-1}'^2(t))}{T_{n-2}}$$

$$T_0(t) = T_1(t) = 1,$$

$$T_2(t) = 4 + t^3,$$

$$T_3(t) = -80 + 20t^3 + t^6,$$

$$T_4(t) = t(11200 + 60t^6 + t^9),$$

$$T_5(t) = -6272000 - 3136000t^3 + 78400t^6 + 2800t^9 + 140t^{12} + t^{15},$$

$$T_6(t) = -38635520000 + 19317760000t^3 + 1448832000t^6 - 17248000t^9 \\ + 627200t^{12} + 18480t^{15} + 280t^{18} + t^{21},$$

$$T_7(t) = -3093932441600000t - 49723914240000t^7 - 828731904000t^{10} \\ + 13039488000t^{13} + 62092800t^{16} + 5174400t^{19} \\ + 75600t^{22} + 504t^{25} + t^{28},$$

$$T_8(t) = -991048439693312000000 - 74328632976998400000t^3 \\ + 37164316488499200000t^6 + 1769729356595200000t^9 \\ + 126696533483520000t^{12} + 407736096768000t^{15} \\ - 6629855232000t^{18} + 124309785600t^{21} + 2018016000t^{24} \\ + 32771200t^{27} + 240240t^{30} + 840t^{33} + t^{36},$$

$$T_9(t) = 5396694815443548897280000000 - 404752111158261672960000000t^3 \\ - 404752111158266167296000000t^6 + 9636955027577765888000000t^9 \\ - 55850538189165977600000t^{12} - 18430315465453731840000t^{15} \\ - 318108851847987200000t^{18} + 483515342069760000t^{21} \\ + 8403341506560000t^{24} + 46132742656000t^{27} + 1876485811200t^{30} \\ + 21525504000t^{33} + 160160000t^{36} + 646800t^{39} + 1320t^{42} + t^{45},$$

$$T_{10}(t) = -558360178500369459236883660800000000t \\ - 17947291451797589761185546240000000t^7 \\ - 299121524196626496019759104000000t^{10} \\ + 11766143871370797483294720000000t^{13} \\ + 56029256530337130872832000000t^{16} \\ + 9157516514955772035072000000t^{19} \\ + 136012478911949832192000000t^{22} \\ + 716670308138534830080000t^{25} - 2587344319342182400000t^{28} \\ + 4351638078627840000t^{31} + 182934280488960000t^{34} \\ + 2227039406592000t^{37} + 24080581324800t^{40} + 155387232000t^{43} \\ + 639038400t^{46} + 1544400t^{49} + 1980t^{52} + t^{55}.$$

8.2. The fourth equation. The recurrence formula of the Okamoto polynomials:

$$S_n(x) = (S''_{n-1}(x)S_{n-1}(x) - S'_{n-1}(x)^2 + (x^2 + 2n - 3)S_n(x)^2) / S_{n-2}(x),$$

$$S_0(x) = S_1(x) = 1,$$

$$S_2(x) = x,$$

$$S_3(x) = 1 + x^2,$$

$$S_4(x) = 5 + 5x^2 + 5x^4 + x^6,$$

$$S_5(x) = 175 + 350x^2 + 175x^4 + 140x^6 + 65x^8 + 14x^{10} + x^{12},$$

$$S_6(x) = 67375 + 134750x^2 + 202125x^4 + 107800x^6 + 42350x^8 + 20020x^{10} \\ + 8050x^{12} + 2200x^{14} + 355x^{16} + 30x^{18} + x^{20},$$

$$S_7(x) = 337211875 + 1011635625x^2 + 1011635625x^4 + 1146520375x^6 \\ + 838212375x^8 + 371896525x^{10} + 118243125x^{12} + 36295875x^{14} \\ + 11836825x^{16} + 3570875x^{18} + 854315x^{20} + 147525x^{22} \\ + 17325x^{24} + 1295x^{26} + 55x^{28} + x^{30},$$

$$S_8(x) = 28691672384375 + 86075017153125x^2 + 172150034306250x^4 \\ + 149196696398750x^6 + 100830734379375x^8 + 69023966136125x^{10} \\ + 40183246103000x^{12} + 17590898325000x^{14} + 5725722752750x^{16} \\ + 1496006762250x^{18} + 356797140700x^{20} + 86586199700x^{22} \\ + 21345073750x^{24} + 4874150050x^{26} + 939138200x^{28} + 143826760x^{30} \\ + 16851835x^{32} + 1461425x^{34} + 90090x^{36} + 3710x^{38} + 91x^{40} + x^{42},$$

$$S_9(x) = 46383387951666390625 + 185533551806665562500x^2 \\ + 278300327709998343750x^4 + 408173813974664237500x^6 \\ + 404198095007378546875x^8 + 265047931152379375000x^{10} \\ + 134451586893661537500x^{12} + 63494998891449125000x^{14} \\ + 29435148214202355625x^{16} + 12470663251834287500x^{18} \\ + 4461474719087636250x^{20} + 1306669436585012500x^{22} \\ + 317280487275626875x^{24} + 66794754468442000x^{26} \\ + 13050553160645000x^{28} + 2532929501102000x^{30} \\ + 500062887198875x^{32} + 96562055089500x^{34} + 17076096487450x^{36} \\ + 2618834718500x^{38} + 336055329225x^{40} + 35218183000x^{42} \\ + 2954107100x^{44} + 194128200x^{46} + 9725275x^{48} \\ + 357140x^{50} + 9030x^{52} + 140x^{54} + x^{56}.$$

8.3. The third equation. The recurrence formula:

$$S_n(a, u) = \frac{-4u^3 \left(\frac{\partial S_{n-1}}{\partial u} + u \frac{\partial^2 S_{n-1}}{\partial u^2} \right) S_{n-1} + 4u^4 \left(\frac{\partial S_{n-1}}{\partial u} \right)^2 + (1 + au)S_{n-1}^2}{S_{n-2}},$$

$$S_0(a, u) = S_1(a, u) = 1,$$

$$S_2(a, u) = 1 + au,$$

$$S_3(a, u) = 1 + 3au + 3a^2u^2 + (-2 + a)a(2 + a)u^3,$$

$$S_4(a, u) = 1 + 6au + 15a^2u^2 + 20(-1 + a)a(1 + a)u^3 \\ + 15(-2 + a)a^2(2 + a)u^4 + 6(-2 + a)a(2 + a)(-6 + a^2)u^5 \\ + (-4 + a)(-2 + a)a^2(2 + a)(4 + a)u^6,$$

$$S_5(a, u) = 1 + 10au + 45a^2u^2 + 60a(-1 + 2a^2)u^3 + 210a^2(-2 + a^2)u^4 \\ + 252(-2 + a)(-1 + a)a(1 + a)(2 + a)u^5 \\ + 210(-2 + a)a^2(2 + a)(-6 + a^2)u^6 \\ + 60(-2 + a)a(2 + a)(60 - 27a^2 + 2a^4)u^7 \\ + 45(-4 + a)(-2 + a)a^2(2 + a)(4 + a)(-8 + a^2)u^8 \\ + 10(-4 + a)(-2 + a)a^3(2 + a)(4 + a)(-22 + a^2)u^9 \\ + (-6 + a)(-4 + a)(-2 + a)^2a^2(2 + a)^2(4 + a)(6 + a)u^{10},$$

$$S_6(a, u) = 1 + 15au + 105a^2u^2 + 35a(-4 + 13a^2)u^3 + 105a^2(-16 + 13a^2)u^4 \\ + 21a(192 - 440a^2 + 143a^4)u^5 + 385(-2 + a)a^2(2 + a)(-28 + 13a^2)u^6 \\ + 45(-2 + a)a(2 + a)(720 - 968a^2 + 143a^4)u^7 \\ + 45(-2 + a)a^2(2 + a)(5424 - 1892a^2 + 143a^4)u^8 \\ + 35(-2 + a)a(2 + a)(-20160 + 18416a^2 - 3124a^4 + 143a^6)u^9 \\ + 21(-4 + a)(-2 + a)a^2(2 + a)(4 + a)(7632 - 2420a^2 + 143a^4)u^{10} \\ + 105(-4 + a)(-2 + a)a^3(2 + a)(4 + a)(2592 - 400a^2 + 13a^4)u^{11} \\ + 35(-4 + a)(-2 + a)a^2(2 + a)(4 + a)(-5760 + 7312a^2 - 620a^4 + 13a^6)u^{12} \\ + 105(-6 + a)(-4 + a)(-2 + a)^2a^3(2 + a)^2(4 + a)(6 + a)(-28 + a^2)u^{13} \\ + 15(-6 + a)(-4 + a)(-2 + a)^2a^2(2 + a)^2(4 + a)(6 + a)(240 - 52a^2 + a^4)u^{14} \\ + (-8 + a)(-6 + a)(-4 + a)^2(-2 + a)^2a^3(2 + a)^2(4 + a)^2(6 + a)(8 + a)u^{15}.$$

8.4. The fifth equation. The recurrence formula:

$$T_n(v, t) = \frac{\left(\frac{t}{8} - v + \frac{3(n-1)}{4} \right) T_{n-1}^2 + \frac{\partial T_{n-1}}{\partial t} T_{n-1} + t \frac{\partial^2 T_{n-1}}{\partial t^2} T_{n-1} - t \left(\frac{\partial T_{n-1}}{\partial t} \right)^2}{T_{n-2}},$$

$$T_0 = T_1 = 1,$$

$$2^3T_2(v, t) = 6 - 8v + t,$$

$$2^9T_3(v, t) = 32(1 - v)(-5 + 4v)(-3 + 4v) \\ + 12(-5 + 4v)(-3 + 4v)t + 24(1 - v)t^2 + t^3,$$

$$2^{18}T_4(v, t) = 512(-1 + v)(-3 + 2v)(-7 + 4v)(-5 + 4v)^2(-3 + 4v) \\ + 1536(1 - v)(-3 + 2v)(-7 + 4v)(-5 + 4v)(-3 + 4v)t \\ + 240(-7 + 4v)(-5 + 4v)^2(-3 + 4v)t^2 \\ + 80(-5 + 4v)(-45 + 80v - 32v^2)t^3 \\ + 480(-1 + v)(-3 + 2v)t^4 + 12(5 - 4v)t^5 + t^6,$$

$$2^{30}T_5(v, t) \\ = 65536(-2 + v)(-1 + v)(-3 + 2v)^2(-9 + 4v)(-7 + 4v)^2(-5 + 4v)^2(-3 + 4v) \\ + 163840(1 - v)(-2 + v)(-3 + 2v)(-9 + 4v)(-7 + 4v)^2(-5 + 4v)^2(-3 + 4v)t \\ + 737280(-2 + v)(-1 + v)(-3 + 2v)^2(-9 + 4v)(-7 + 4v)(-5 + 4v)(-3 + 4v)t^2 \\ + 7680(15 - 8v)(-3 + 2v)(-9 + 4v)(-7 + 4v)(-5 + 4v)(-3 + 4v)(-9 + 8v)t^3 \\ + 6720(-7 + 4v)(-5 + 4v)(1959 - 6048v + 6624v^2 - 3072v^3 + 512v^4)t^4 \\ + 8064(-3 + 2v)(-7 + 4v)(-5 + 4v)(-59 + 96v - 32v^2)t^5 \\ + 1680(2235 - 6432v + 6752v^2 - 3072v^3 + 512v^4)t^6 \\ + 480(-3 + 2v)(-133 + 192v - 64v^2)t^7 \\ + 180(-7 + 4v)(-5 + 4v)t^8 + 40(3 - 2v)t^9 + t^{10},$$

$$2^{45}T_6(v, t) \\ = 16777216(1 - v)(-2 + v)^2(-5 + 2v)(-3 + 2v)^2(-11 + 4v) \\ \times (-9 + 4v)^2(-7 + 4v)^3(-5 + 4v)^2(-3 + 4v) \\ + 125829120(-2 + v)^2(-1 + v)(-5 + 2v)(-3 + 2v)^2(-11 + 4v) \\ \times (-9 + 4v)^2(-7 + 4v)^2(-5 + 4v)^2(-3 + 4v)t \\ + 55050240(1 - v)(-2 + v)(-5 + 2v)(-3 + 2v)(-11 + 4v) \\ \times (-9 + 4v)^2(-7 + 4v)^3(-5 + 4v)^2(-3 + 4v)t^2 \\ + 9175040(-2 + v)(-1 + v)(-5 + 2v)(-3 + 2v)(-11 + 4v) \\ \times (-9 + 4v)(-7 + 4v)^2(-5 + 4v)(-3 + 4v)(609 - 728v + 208v^2)t^3 \\ + 1720320(-2 + v)(-3 + 2v)(-11 + 4v)(-9 + 4v)(-7 + 4v)(-5 + 4v) \\ \times (-3 + 4v)(-25815 + 65072v - 59360v^2 + 23296v^3 - 3328v^4)t^4 \\ + 172032(-2 + v)(-3 + 2v)(-9 + 4v)(-7 + 4v)^2(-5 + 4v) \\ (205725 - 612304v + 623392v^2 - 256256v^3 + 36608v^4)t^5 \\ + 17920(-9 + 4v)(-7 + 4v)(-5 + 4v)(-10739205 + 43244712v - 69722480v^2 \\ + 57894144v^3 - 26208512v^4 + 6150144v^5 - 585728v^6)t^6 \\ + 11520(-9 + 4v)(-7 + 4v)^2(-5 + 4v) \\ \times (239805 - 661584v + 637472v^2 - 256256v^3 + 36608v^4)t^7$$

$$\begin{aligned}
 &+92160(-2+v)(-3+2v)(-7+4v) \\
 &\quad \times (-123135+333872v-319616v^2+128128v^3-18304v^4)t^8 \\
 &+17920(1613115-6065682v+9304924v^2 \\
 &\quad -7465920v^3+3308800v^4-768768v^5+73216v^6)t^9 \\
 &+1344(-7+4v)(-142695+358512v-326656v^2+128128v^3-18304v^4)t^{10} \\
 &+3360(13905-33712v+30016v^2-11648v^3+1664v^4)t^{11} \\
 &+2240(-7+4v)(-75+91v-26v^2)t^{12} \\
 &+3360(-2+v)(-3+2v)t^{13}+30(7-4v)t^{14}+t^{15}.
 \end{aligned}$$

8.5. The sixth equation

The recurrence formula:

$$\begin{aligned}
 T_n(b_1, b_2, v) = & \frac{\left\{ \frac{1}{4}(-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2))v + \left(n - \frac{1}{2}\right)^2 \right\} T_{n-1}^2}{T_{n-2}} \\
 & + \frac{\frac{1}{4}(v^2 - 4) \left\{ (v^2 - 4) \frac{\partial^2 T_{n-1}}{\partial v^2} + v \frac{\partial T_{n-1}}{\partial v} \right\} T_{n-1} - \left(\frac{1}{2}(v^2 - 4) \frac{\partial T_{n-1}}{\partial v}\right)^2}{T_{n-2}},
 \end{aligned}$$

$$T_0(b_1, b_2, v) = T_1(b_1, b_2, v) = 1,$$

$$4T_2(b_1, b_2, v) = 1 - 2b_1^2 - 2b_2^2 + b_1^2v - b_2^2v,$$

$$\begin{aligned}
 64T_3(b_1, b_2, v) = & 9 - 38b_1^2 + 28b_1^4 - 8b_1^6 - 38b_2^2 + 120b_1^2b_2^2 - 24b_1^4b_2^2 + 28b_2^4 \\
 & - 24b_1^2b_2^4 - 8b_2^6 + 15b_1^2v - 36b_1^4v + 12b_1^6v - 15b_2^2v + 12b_1^4b_2^2v \\
 & + 36b_2^4v - 12b_1^2b_2^4v - 12b_2^6v + 15b_1^4v^2 - 6b_1^6v^2 - 30b_1^2b_2^2v^2 \\
 & + 6b_1^4b_2^2v^2 + 15b_2^4v^2 + 6b_1^2b_2^4v^2 - 6b_2^6v^2 + b_1^2v^3 - 2b_1^4v^3 \\
 & + b_1^6v^3 - b_2^2v^3 - 3b_1^4b_2^2v^3 + 2b_2^4v^3 + 3b_1^2b_2^4v^3 - b_2^6v^3,
 \end{aligned}$$

$$\begin{aligned}
 4096T_4(b_1, b_2, v) = & 2025 - 13212b_1^2 + 24364b_1^4 - 14656b_1^6 + 4272b_1^8 - 832b_1^{10} \\
 & + 64b_1^{12} - 13212b_2^2 + 74648b_1^2b_2^2 - 105280b_1^4b_2^2 + 47552b_1^6b_2^2 - 6720b_1^8b_2^2 \\
 & + 384b_1^{10}b_2^2 + 24364b_2^4 - 105280b_1^2b_2^4 + 66080b_1^4b_2^4 - 16000b_1^6b_2^4 + 960b_1^8b_2^4 \\
 & - 14656b_1^6b_2^6 + 47552b_1^2b_2^6 - 16000b_1^4b_2^6 + 1280b_1^6b_2^6 + 4272b_2^8 - 6720b_1^2b_2^8 \\
 & + 960b_1^4b_2^8 - 832b_2^{10} + 384b_1^2b_2^{10} + 64b_2^{12} + 5130b_1^2v - 24252b_1^4v + 23520b_1^6v \\
 & - 10656b_1^8v + 2400b_1^{10}v - 192b_1^{12}v - 5130b_2^2v + 74400b_1^4b_2^2v - 51264b_1^6b_2^2v \\
 & + 11040b_1^8b_2^2v - 768b_1^{10}b_2^2v + 24252b_2^4v - 74400vb_1^2b_2^4v \\
 & + 8640b_1^6b_2^4v - 960b_1^8b_2^4v - 23520b_2^6v + 51264b_1^2b_2^6v - 8640b_1^4b_2^6v \\
 & + 10656b_2^8v - 11040b_1^2b_2^8v + 960b_1^4b_2^8v - 2400b_2^{10}v + 768b_2^{12}v + 192b_2^{12}v \\
 & + 5175b_1^4v^2 - 15720b_1^6v^2 + 11160b_1^8v^2 - 2880b_1^{10}v^2 + 240b_1^{12}v^2 \\
 & - 10350b_1^2b_2^2v^2 + 15720b_1^4b_2^2v^2 + 4800b_1^6b_2^2v^2 - 4800b_1^8b_2^2v^2 + 480b_1^{10}b_2^2v^2
 \end{aligned}$$

$$\begin{aligned}
 &+ 5175b_1^4v^2 + 15720b_1^2b_2^4v^2 - 31920b_1^4b_2^4v^2 + 7680b_1^6b_2^4v^2 - 240b_1^8b_2^4v^2 \\
 &- 15720b_2^6v^2 + 4800b_1^2b_2^6v^2 + 7680b_1^4b_2^6v^2 - 960b_1^6b_2^6v^2 + 11160b_2^8v^2 \\
 &- 4800b_1^2b_2^8v^2 - 240b_1^4b_2^8v^2 - 2880b_2^{10}v^2 + 240b_2^{12}v^2 \\
 &+ 45b_1^2v^3 - 280b_1^4v^3 + 6940b_1^6v^3 - 6360b_1^8v^3 + 1840b_1^{10}v^3 - 160b_1^{12}v^3 \\
 &- 45b_2^2v^3 - 18420b_1^4b_2^2v^3 + 10800b_1^6b_2^2v^3 - 1200b_1^8b_2^2v^3 + 280b_2^4v^3 \\
 &+ 18420b_1^2b_2^4v^3 - 4320b_1^6b_2^4v^3 + 480b_1^8b_2^4v^3 - 6940b_2^6v^3 - 10800b_1^2b_2^6v^3 \\
 &+ 4320b_1^4b_2^6v^3 + 6360b_2^8v^3 + 1200b_1^2b_2^8v^3 - 480b_1^4b_2^8v^3 - 1840b_2^{10}v^3 \\
 &+ 160b_2^{12}v^3 + 975b_1^4v^4 - 2490b_1^6v^4 + 2115b_1^8v^4 - 660b_1^{10}v^4 + 60b_1^{12}v^4 \\
 &- 1950b_1^2b_2^2v^4 + 2490b_1^4b_2^2v^4 - 3900b_1^6b_2^2v^4 + 1500b_1^8b_2^2v^4 - 120b_1^{10}b_2^2v^4 \\
 &+ 975b_2^4v^4 + 2490b_1^2b_2^4v^4 + 3570b_1^4b_2^4v^4 - 840b_1^6b_2^4v^4 - 60b_1^8b_2^4v^4 \\
 &- 2490b_2^6v^4 - 3900b_1^2b_2^6v^4 - 840b_1^4b_2^6v^4 + 240b_1^6b_2^6v^4 + 2115b_2^8v^4 \\
 &+ 1500b_1^2b_2^8v^4 - 60b_1^4b_2^8v^4 - 660b_2^{10}v^4 - 120b_1^2b_2^{10}v^4 + 60b_2^{12}v^4 + 81b_1^2v^5 \\
 &- 342b_1^4v^5 + 543b_1^6v^5 - 396b_1^8v^5 + 126b_1^{10}v^5 - 12b_1^{12}v^5 - 81b_2^2v^5 \\
 &- 45b_1^4b_2^2v^5 + 504b_1^6b_2^2v^5 - 390b_1^8b_2^2v^5 + 48b_1^{10}b_2^2v^5 + 342b_2^4v^5 + 45b_2^2b_2^4v^5 \\
 &+ 540b_1^6b_2^4v^5 - 60b_1^8b_2^4v^5 - 543b_2^6v^5 - 504b_1^2b_2^6v^5 - 540b_1^4b_2^6v^5 + 396b_2^8v^5 \\
 &+ 390b_1^2b_2^8v^5 + 60b_1^4b_2^8v^5 - 126b_2^{10}v^5 - 48b_1^2b_2^{10}v^5 + 12b_2^{12}v^5 + 16b_1^4v^6 \\
 &- 40b_1^6v^6 + 33b_1^8v^6 - 10b_1^{10}v^6 + b_1^{12}v^6 - 32b_1^2b_2^2v^6 + 40b_1^4b_2^2v^6 - 68b_1^6b_2^2v^6 \\
 &+ 30b_1^8b_2^2v^6 - 6b_1^{10}b_2^2v^6 + 16b_2^4v^6 + 40b_1^2b_2^4v^6 + 70b_1^4b_2^4v^6 - 20b_1^6b_2^4v^6 \\
 &+ 15b_1^8b_2^4v^6 - 40b_2^6v^6 - 68b_1^2b_2^6v^6 - 20b_1^4b_2^6v^6 - 20b_1^6b_2^6v^6 + 33b_2^8v^6 \\
 &+ 30b_1^2b_2^8v^6 + 15b_1^4b_2^8v^6 - 10b_2^{10}v^6 - 6b_1^2b_2^{10}v^6 + b_2^{12}v^6,
 \end{aligned}$$

Remark. — [2, 8] was published as [1]

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