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Cyclic covers of Stable curves and their moduli spaces $^{(*)}$

Binru Li⁽¹⁾

ABSTRACT. — We study the deformation of G-marked stable curves in the case where G is a cyclic group, and construct a parameterizing space for G-marked stable curves of a given numerical type.

This is then used in order to study the components of the locus of stable curves admitting the action of a cyclic group of non prime order d, extending the work of F. Catanese in the case where d is prime.

RÉSUMÉ. — Nous étudions la déformation des courbes stables marquées de G dans le cas où G est un groupe cyclique et construisons un espace de paramétrage pour les courbes stables marquées par G d'un type numérique donné.

Ceci est ensuite utilisé afin d'étudier les composantes de l'ensemble des courbes stables en admettant l'action d'un groupe cyclique de non-premier ordre d, extension du travail de F. Catanese dans le cas où d est premier.

Introduction

The purpose of this article is to study the structure of the locus $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(G)$ of (non-smooth) stable curves of genus g inside the compactified moduli space $\overline{\mathfrak{M}_g}$ admitting an effective action by a cyclic group G.

In [4] and [5] M. Cornalba determined the irreducible components of $\operatorname{Sing}(\mathfrak{M}_q)$, the singular locus of the moduli scheme of smooth projective

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curves of genus $g \ge 2$. The result was obtained by showing that $\mathfrak{M}_g(\mathbb{Z}/p)$, the locus inside $\operatorname{Sing}(\mathfrak{M}_g)$ of curves admitting an effective action by a cyclic group of prime order p, is irreducible and maximal (i.e. being not contained in another locus) except for finitely many cases. The main ingredient Cornalba used is that the locus corresponding to cyclic covers of prime order of smooth curves with a fixed combinatorial datum, called the numerical type (see Definition 2.1), is an irreducible Zariski closed subset of the moduli space \mathfrak{M}_g . Catanese in [1] extended this result to the case of cyclic groups of any order (cf. Theorem 2.3).

The studies of such loci can be continued in two directions:

In one direction more finite groups G are considered. For instance the case where $G = D_n$, the dihedral group of order 2n, was investigated in a series of papers by F. Catanese, M. Lönne and F. Perroni (cf. [2, 3]) and later by B. Li and S. Weigl (cf. [7]). The main difficulty there is that for general groups a numerical type might correspond to a reducible subset of the moduli space. In [3] the authors introduced a new homological invariant which enables them to distinguish the irreducible components asymptotically (i.e., when the genus of the quotient curve $\gg 0$).

The other direction is to consider the boundary of the compactified moduli space $\overline{\mathfrak{M}_g}$. In [1], Catanese determined the irreducible components of $\operatorname{Sing}(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)$ by studying the loci $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(\mathbb{Z}/p)$ and obtained analogous results as in the smooth case. In this case, the locus of stable curves with a given numerical type is not necessarily Zariski closed: if a stable curve C_1 is smoothable to another stable curve C_2 , then the corresponding locus of C_1 is contained in the closure of that of C_2 , hence one should look at the nonsmoothable stable curves. Hence in the boundary case the notion of maximal means that the Zariski closure of the locus is maximal (cf. Definition 2.13).

In this article we go into both directions, studying the loci $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(\mathbb{Z}/d)$ of non-prime order d and generalize several results in [1].

This article is organized as follows.

In Section 1 we give the definition of a G-marked stable curve (i.e., a stable curve C admitting an effective action by a finite group G, cf. Definition 1.1) and associated notions.

In Section 2 we study the G-equivariant deformation (cf. Definition 2.4) of G-marked stable curves, and determine when a G-marked stable curve is G-equivariantly smoothable. Then we define the associated numerical type for G-marked stable curves and prove the main result of this section that, for G-marked stable curves with a given numerical type, there is a parameterizing space (cf. Theorem 2.11):

THEOREM A. — Given a G-marked stable curve (C, G, ρ) , there exists a connected complex manifold $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$ parameterizing all G-marked stable curves with numerical type $[\mathcal{D}(C, G, \rho)].$

If moreover (C, G, ρ) is G-equivariantly non-smoothable, denoting by $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ the image of the natural morphism $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]} \to \overline{\mathfrak{M}_g} - \mathfrak{M}_g$, then each point inside $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ has finite inverse image in $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$, and the closure $\overline{\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}}$ consists of G-marked stable curves which can be G-equivariantly deformed into a curve with numerical type $[\mathcal{D}(C, G, \rho)]$.

In Section 3 we study the irreducible components of $(\overline{\mathfrak{M}_q} - \mathfrak{M}_q)(G)$ for the case $G = \mathbb{Z}/d$, the idea is to determine when a G-stratum (i.e., the image inside $(\overline{\mathfrak{M}_q} - \mathfrak{M}_q)(\mathbb{Z}/d)$ of the parameterizing space of a given numerical type, cf. Definition 2.13) is maximal. For this we need to compare all the order d cyclic subgroups of the stratum (cf. Definition 3.2).

Due to some phenomena arising from the smooth case (cf. Proposition 3.5), the automorphism group of a stratum might become very complicated, making it impossible to give a brief and explicit description for maximal strata. Hence we make some technical assumptions.

ASSUMPTION (cf. Assumption 3.6).

- (0) $(C = \sum_{i \in I} C_i, G, \rho)$ is G-equivariantly non-smoothable. (1) For a general stable curve (C, G, ρ) in the stratum we have $H_i =$ Aut (C_i) and $g(C_i) \ge 2$ for all *i*.
- (2) For any $i \in I$, the parameterizing space \mathcal{T}_{h_i,r_i} has dimension> 0.

With the above assumptions we prove the main result of this article (cf. Theorem 3.17):

THEOREM B. — Under the conditions of Assumption 3.6, we have the following:

- (1) For a G-equivariantly non-smoothable G-marked stable curve (C = $\sum_{i \in I} C_i, G, \rho$, the induced stratum $\mathfrak{M}_{C'}$, where C' = C/G, is maximal iff for a general stable curve (by abuse of notation we denote still by) (C, G, ρ) in the stratum:
 - (a) The cases in Lemma 3.10 do not occur.
 - (b) For any $\beta \in \operatorname{Aut}(C)$ (of order d) and any node p where Case (II-i) happens (cf. page 58), the following holds:

$$\zeta_{p,1}^{b(p,1)c(\beta,p,1)}\zeta_{p,2}^{b(p,2)c(\beta,p,2)} \neq 1.$$

(c) For any $\beta \in Aut(C)$ (of order d), there is no node p of type E with respect to β (see Definition 3.16).

(2) The Zariski closure of each maximal stratum in (1) is an irreducible component of $(\overline{\mathfrak{M}_q} - \mathfrak{M}_q)(G)$.

1. Notation

Let C be a (non-smooth) stable curve (i.e., C has at most nodes as singularities and Aut(C), the automorphism group of C, is finite),

$$C = \sum_{i \in I} C_i.$$

We define \mathcal{I} to be the graph whose set of vertices is the set I, and whose set of edges is the set \mathcal{N} of the nodes $P \in C$.

We let $\mathcal{N}_i := \mathcal{N} \cap C_i$, i.e., these are the edges of the graph containing the vertex *i*.

Note that if $P \in \mathcal{N}, P \in C_i \cap C_j, i \neq j$, then P yields an edge connecting two distinct vertices, else, if $P \in C_i$ and $P \notin C_j, \forall j \neq i, P$ yields a loop based at *i*. Hence we have

$$\mathcal{N}_i = \mathcal{N}_i^{(1)} \cup \mathcal{N}_i^{(2)},$$

where $\mathcal{N}_i^{(1)}$ corresponds to edges connecting two distinct vertices (one of the vertex is *i*) and $\mathcal{N}_i^{(2)}$ corresponds to loops based at *i*.

Set further $C - C_i = \overline{C \setminus C_i}$.

Definition 1.1.

- Let G be a finite group. A G-marked stable curve is a triple (C, G, ρ), where C is a stable curve, ρ : G → Aut(C) is an injective homomorphism, i.e., G acts effectively on the stable curve C. When ρ is clear, for instance if G is a subgroup of Aut(C), we write for short (C, G) instead of (C, G, ρ).
- (2) We call (C, G, ρ) a smooth (resp. irreducible) G-marked curve if C is smooth (resp. irreducible).

Remark 1.2. — In the case where ρ is clear from the context, we identify G with its image $\rho(G)$ and write $G \subset \operatorname{Aut}(C)$.

Given a G-marked curve (C, G, ρ) , then G acts naturally on the graph \mathcal{I} , and on the set I.

Definition 1.3.

 (i) Let K_v be the kernel of the action on I, and let instead G_i be the stabilizer of i ∈ I; in other words,

$$G_i := \{g \mid g(C_i) = C_i\}$$

and $K_v = \bigcap_{i \in I} G_i$.

- (ii) Let K be the kernel of the action on the graph I, and let, for P ∈ N, G_P be the stabilizer of P; hence K = K_v ∩ (∩_{P∈N}G_P). We let moreover G'_i be the subgroup of G_i which fixes the nodes in N_i, the nodes of C belonging to C_i, and we let G''_i be the subgroup which acts trivially on C_i. Hence K = ∩_{i∈I} G'_i. We denote by n_i the order of G/G_i.
- (iii) In the case G is an abelian group, let H_i be the quotient group G_i/G_i["], respectively H_i['] := G_i[']/G_i["]. Necessarily H_i['] is a cyclic subgroup if N_i ≠ Ø. We denote by d_i (resp. d_i[']) the order of H_i (resp. H_i[']).
- (iv) Setting where $I_0 = \{i \in I \mid G = G''_i\}$, $I_1 = \{i \in I \mid G = G_i \text{ and } G \neq G''_i\}$ and $I_2 = \{i \in I \mid G \neq G_i\}$, then the set I has a natural partial $I = I_0 \cup I_1 \cup I_2$.

In the rest of this article G shall denote a cyclic group \mathbb{Z}/d with generator γ and $\zeta_d := \exp\left(\frac{2\pi\sqrt{-1}}{d}\right)$. We work over the field of complex numbers \mathbb{C} .

2. Parametrizing space of cyclic coverings

In this section we will construct parameterizing spaces for G-marked stable curves, first we review the case of smooth G-marked curves.

Let (C, G) be a smooth irreducible *G*-marked curve. The action of *G* on *C* induces a (ramified) covering map $C \to C' := C/G$. For any $1 \leq i \leq d-1$, we define a divisor D_i as the sub-divisor of the branch locus $D \subset C'$ where the local monodromy is ζ_d^i (cf. [1, p. 4, l. 20]).

DEFINITION 2.1 ([1, Definition 2.2]). — Let C be a smooth irreducible projective curve of genus g on which $G = \mathbb{Z}/d$ acts faithfully, and set C' = C/G, h := genus(C').

Denote by $k_i = \deg(D_i)$ for $i = 1, \ldots, d-1$, and by (k_1, \ldots, k_{d-1}) the branching sequence of γ . A change of generator of \mathbb{Z}/d corresponds to a $(\mathbb{Z}/d)^*$ -action on the set of sequences, we denote the resulting equivalence class by $[(k_1, \ldots, k_{d-1})]$, and call it the numerical type of the cyclic cover $C \to C'$.

DEFINITION 2.2 ([1, Definition 2.3]). — Given a branching datum corresponding to a sequence $[(k_1, \ldots, k_{d-1})]$, set

$$h := 1 + \frac{2(g-1)}{2d} - \frac{1}{2} \sum_{i=1}^{d-1} k_i \left(1 - \frac{\gcd(i,d)}{d}\right).$$

The branching datum is said to be admissible for d and q if the following two conditions are satisfied:

- (1) $\sum_{i=1}^{d-1} k_i i \equiv 0 \pmod{d}$, (2) *h* is an positive integer; h = 0, $\gcd\{d, \gcd\{i \mid k_i \neq 0\}\} = 1$.

Note that the branching datum of a cyclic cover $C \to C'$ is admissible.

The main result for the parameterizing space of smooth G-marked curves is the following:

THEOREM 2.3 ([1, Theorem 2.4]). — The pairs (C,G), where C is a complex projective curve of genus $g \ge 2$, and G is a finite cyclic group of order d acting faithfully on C with a given branching datum $[(k_1, \ldots, k_{d-1})]$ are parametrized by a connected complex manifold $\mathcal{T}_{g,d;[(k_1,\ldots,k_{d-1})]}$ of dimension 3(h-1) + k, where $k := \sum_i k_i$.

The image $\mathfrak{M}_{g,d;[(k_1,\ldots,k_{d-1})]}$ of $\mathcal{T}_{g,d;[(k_1,\ldots,k_{d-1})]}$ inside the moduli space \mathfrak{M}_{q} is a closed subset of the same dimension 3(h-1)+k.

We will give an analogous result for G-marked stable curves.

DEFINITION 2.4. — Let (C, G, ρ) be a G-marked stable curve: a G-equivariant deformation of (C, G, ρ) is a triple $(p : \mathfrak{C} \to B, G, \eta)$ such that

- (1) $p: \mathfrak{C} \to B$ is a deformation of C over an irreducible base B with all fibres stable curves and the central fibre $\mathfrak{C}_O \simeq C$ $(O \in B)$.
- (2) $\eta: G \to \operatorname{Aut}(\mathfrak{C})$ is an injective homomorphism inducing an effective action on \mathfrak{C} such that p is G-invariant (where the action of G on B is trivial) and $\eta|_{\mathfrak{C}_{\mathcal{O}}} \simeq \rho$.

DEFINITION 2.5. — We say that a G-marked stable curve (C, G, ρ) is G-equivariantly non-smoothable (or has no G-equivariant smoothing) if (C, G, ρ) can not be G-equivariantly deformed to (C', G, ρ') such that C' has less nodes than C.

We have the following criterion which tells when a G-marked stable curves is *G*-equivariantly non-smoothable, and generalizes the prime case in [1, Lemma 4.3]:

PROPOSITION 2.6. — Let $P \in C = \sum_i C_i$ be a node, set $G_P := \operatorname{Stab}(P)$ the stabilizer group of P in G, then the following are equivalent:

- (1) All points in G(P)(:= the orbit of P) can by simultaneously G-equivariantly smoothed.
- (2) The induced group homomorphism $G_P \to GL(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_P) \simeq \mathbb{C}^*$ is trivial.

Proof. — Recall the local to global spectral sequence:

$$0 \longrightarrow \bigoplus_{i} H^{1} \left(\Theta_{C_{i}} \left(-\sum_{j \neq i} (C_{i} \cap C_{j}) \right) \right) \longrightarrow \operatorname{Ext}^{1}(\Omega_{C}^{1}, \mathcal{O}_{C}) \longrightarrow \bigoplus_{P \in \mathcal{N}} \mathcal{E}\operatorname{xt}^{1}(\Omega_{C}^{1}, \mathcal{O}_{C})_{P} \longrightarrow 0, \quad (*)$$

where Ω_C^1 is the sheaf of differentials of C and Θ_{C_i} denotes the tangent sheaf of C_i .

Here I shall explain why the first term of (*) has this form, since in the original sequence the first term should be $H^1(\mathcal{H}om(\Omega_C, \mathcal{O}_C))$. Denote by $\iota_i : C_i \to C$ the natural embedding. As C has only nodes as singularities, a local computation yields the following exact sequence:

$$0 \longrightarrow \Omega_C \longrightarrow \bigoplus_{i \in I} \iota_{i*}(\iota_i^* \Omega_C) \longrightarrow Q \longrightarrow 0$$

and

$$\iota_i^* \Omega_C \simeq \Omega_{C_i} \oplus Q_i,$$

where $Q(\text{resp. } Q_i)$ is supported on the nodes of C (resp. $\sum_{j \neq i} C_j \cap C_i$). Now Applying the dual functor $\mathcal{H}om(-, \mathcal{O}_C)$, since Q and Q_i are supported on 0-dimensional sets, we have

$$0 \longrightarrow \mathcal{H}om\left(\bigoplus_{i \in I} \iota_{i*}\Omega_{C_i}, \mathcal{O}_C\right) \stackrel{\iota}{\longrightarrow} \mathcal{H}om(\Omega_C, \mathcal{O}_C).$$

After some local computation (again note that C has only nodes as singularities), we get the following:

- (1) ι is in fact an isomorphism.
- (2) $\mathcal{H}om(\iota_{i*}\Omega_{C_i}, \mathcal{O}_C) \simeq \iota_{i*}(\Theta_i(-\sum_{j\neq i} C_j \cap C_i)).$

As $H^1(C_i, \Theta_i(-\sum_{j\neq i} C_j \cap C_i)) \simeq H^1(C, \iota_i (\Theta_i(-\sum_{j\neq i} C_j \cap C_i)))$, we complete the explanation of the first term of (*).

The exact sequence (*) remains exact after taking the subspaces of *G*-invariant vectors. Hence we have a surjection:

$$\operatorname{Ext}^{1}(\Omega_{C}, \mathcal{O}_{C})^{G} \longrightarrow \left(\bigoplus_{P \in \mathcal{N}} \mathcal{E}\operatorname{xt}^{1}(\Omega_{C}^{1}, \mathcal{O}_{C})_{P} \right)^{G}.$$
 (**)

Since G_P is a subgroup of $G = \mathbb{Z}/d$, we have $G_P \simeq \mathbb{Z}/m$ for some m|d, define r := m/d. Denoting by $\bar{\gamma}$ the image of γ in G/G_P , clearly $\bar{\gamma}$ is a generator of G/G_P . Up to a change of index, we can assume that G(P) = $\{P_1 = P, \ldots, P_r\}$, such that $\bar{\gamma}(P_i) = P_{i+1}$. The surjection (**) means that all combinations of possible local G-equivariant deformation can be realized by global ones, hence we only need to consider the local deformation at each node P. On the other hand if a G-equivariant deformation smooths a node P, it must smooth all the points in G(P) simultaneously. Therefore we see that all points in G(P) can be simultaneously smoothed $\Leftrightarrow \exists v = (v_1, \ldots, v_r) \in$ $(\bigoplus_{P_i \in G(P)} \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i})^G$, such that $v \neq 0$ (since v is G-invariant, $v \neq 0$ is equivalent to $v_i \neq 0$ for all i).

Assume $\gamma(v_1, \ldots v_r) = (\lambda_r v_r, \lambda_1 v_1, \ldots \lambda_{r-1} v_{r-1})$ for some $\lambda_i \in \mathbb{C}^*$. It is easy to see that $\gamma v = v \Leftrightarrow \prod_{i=1}^r \lambda_i = 1$.

If there exists $0 \neq v \in (\bigoplus_{P_i \in G(P)} \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i})^G$, then we have $\gamma v = v$. Noting that γ^r is a generator of G_P , the induced homomorphism $G_P \to \mathbb{C}^*$ is then given by $\gamma^r \mapsto \prod \lambda_i = 1$, hence trivial. Conversely, if $G_P \to \mathbb{C}^*$ is trivial, let $v = (v_1, \lambda_1 v_1, \dots, (\prod_{i=1}^{r-1} \lambda_i)v_1)$ with $v_1 \neq 0$, it is clear that $\gamma v = v$, since $\prod \lambda_i = 1$.

Remark 2.7. — From the proof of Proposition 2.6 we see that a *G*-marked curve *C* is *G*-equivariantly non-smoothable iff $(\bigoplus_{P \in \mathcal{N}} \mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C)_P)^G = 0.$

DEFINITION 2.8. — A G-marked stable curve (C, G, ρ) has the following associated combinatorial datum $\mathcal{D}(C, G, \rho)$:

- (1) A G-marked graph $(\mathcal{I}, G, \tilde{\rho})$, i.e., the graph \mathcal{I} with induced G-action $\tilde{\rho}$ from the action $\rho: G \to \operatorname{Aut}(C)$.
- (2) For any $i \in I$, recall that $H_i = G_i/G''_i$ and $d_i = \operatorname{Ord}(H_i)$. The image of an element $\beta \in G_i$ in H_i is denoted as $\overline{\beta}$. We get a H_i -marked curve (C_i, H_i, ρ_i) , denote by $\widetilde{C_i}$ the normalization of C_i and set $g_i = \operatorname{genus}(\widetilde{C_i})$, $h_i = \operatorname{genus}$ of $\widetilde{C'_i} := \widetilde{C_i}/H_i$. The element $\overline{\gamma^{n_i}}$ generates H_i (recall that $n_i = \operatorname{Ord}(G/G_i)$), it induces an action $\widetilde{\rho_i}$ of H_i on $\widetilde{C_i}$, and hence a branching sequence $(k_1(i), \ldots, k_{d_i-1}(i))$ on $\widetilde{C_i}$.
- (3) Note that the ramification points of C_i consists of three parts:
 (a) points which are not from nodes of C, (b) points which are inverse images of intersection points of two distinct C_i, C_j, and

(c) points which are inverse images of nodes of some C_i . Hence we record the following data:

- (a) For each $i \in I_1 \cup I_2$, we record the branching sequence $(k'_1(i), \ldots, k'_{d_i-1}(i))$ corresponding to the ramification points of $\widetilde{\rho}_i$ on \widetilde{C}_i which are not coming from nodes of C.
- (b) For each node p which is the intersection of two different components C_i and C_j, we record the monodromy m(p,i) (resp. m(p,j)) induced by γn_i (resp. γn_j) at p.
- (c) For each node p which is a node of some C_i, we record the monodromy n₁(p,i) and n₂(p,i) (an unordered pair) induced by γ^{n_i} at p.

The automorphism group $\operatorname{Aut}(G) = (\mathbb{Z}/d)^*$ acts naturally on the set of data $\{\mathcal{D}(C,G,\rho)\}$, we call the resulting equivalence class, $[\mathcal{D}(C,G,\rho)]$ the numerical type of (C,G,ρ) .

Remark 2.9.

- (1) As in the smooth case, we can determine an "admissible condition" for the above combinatorial data (for the case G has a prime order, see [1, Definition 4.8]), which we will not use in later discussion.
- (2) For the H_i -marked curve (C_i, H_i, ρ_i) , it is important to consider the branching sequence $(k_1(i), \ldots, k_{d_i-1}(i))$ (induced by $\overline{\gamma^{n_i}}$) instead of the equivalent H_i -class $[(k_1(i), \ldots, k_{d_i-1}(i))]$. Later we will see the differences.
- (3) For a non-smoothable *G*-marked curve, using Proposition 2.6, we see that $\forall i \in I_0$, the component C_i is smooth (i.e. $\mathcal{N}_i^{(2)} = \emptyset$).

Now we come to the main result of this section, which is a partial generalization of [1, Theorem 4.10].

We denote by Orb the set of G-orbits in I, for any $o \in Orb$, we define a subcurve of C consisting of all components in the orbit o,

$$C(o) := \bigcup_{i \in o} C_i.$$

We have an induced $G_o := G/G''_i$ -marked (nodal)-curve $(C(o), G_o, \rho_o)$ (note that C(o) might be disconnected). The following lemma shows that we have a "canonical form" for $(C(o), G_o, \rho_o)$.

LEMMA 2.10. — The G_o -marked curve $(C(o), G_o, \rho_o)$ is G_o -equivariantly isomorphic to the canonical form $(\bigcup_{j=1}^{n_o} C_o^{(j)}, G_o, \tilde{\rho}_o)$, where $n_o = \#|o|(=n_i)$, $C_o^{(j)}$ are n_o copies of an irreducible component C_i in C(o), and $\bigcup_{i=1}^{n_o} C_o^{(j)}$ is a

quotient of $\bigsqcup_{j=1}^{n_o} C_o^{(j)}$ by identifying a finite set of (unordered) pair of points \mathcal{P}_o and $\tilde{\rho}_o$ is determined by the following morphisms:

 $\mathrm{id}: C_o^{(j)} \longrightarrow C_o^{(j+1)} \ \forall \ 1 \leqslant j \leqslant n_o - 1; \ \overline{\gamma^{n_o}}: C_o^{(n_o)} \longrightarrow C_o^{(1)}.$

Proof. — It is clear that the morphisms given in the lemma define an action of G_o on $\bigsqcup_{j=1}^{n_o} C_o^{(j)}$. It is easy to check that the morphisms $\overline{\gamma^{j-1}}|_{C_i}$: $C_o^{(j)}(=C_i) \rightarrow \overline{\gamma^{j-1}}(C_i)$ induce a surjective G_o -equivariant morphism $\bigsqcup_{j=1}^{n_o} C_o^{(j)} \rightarrow C(o)$. Denoting by \mathcal{P}_o the set of inverse images of nodes in C(o) which do not have two branches on the same irreducible curve, we obtain a quotient curve $\cup_{j=1}^{n_o} C_o^{(j)}$ by identifying the pairs of points lying in the same inverse image in \mathcal{P}_o , then we have a G_o -equivariant isomorphism $\phi_o: \cup_{j=1}^{n_o} C_o^{(j)} \rightarrow C(o)$, since C(o) is a subset of a stable curve and no three components can meet in a point. □

THEOREM 2.11. — Given a G-marked stable curve (C, G, ρ) , there exists a connected complex manifold $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$ parameterizing all G-marked stable curves with numerical type $[\mathcal{D}(C,G,\rho)]$, i.e., there is a family of G-marked curves over $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$, such that

- (i) each fiber of this family is a G-marked stable curves with numerical type [D(C, G, ρ)],
- (ii) every G-marked stable curves with numerical type $[\mathcal{D}(C, G, \rho)]$ is G-equivariantly isomorphic to a fiber of this family.

Denoting by $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ the image set of the natural morphism $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]} \rightarrow \overline{\mathfrak{M}_g} - \mathfrak{M}_g$, then each point inside $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ has finite inverse image in $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$, and the closure $\overline{\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}}$ consists of curves with a faithful action of G which can be G-equivariantly deformed into a curve with numerical type $[\mathcal{D}(C,G,\rho)]$.

If moreover (C, G, ρ) is G-equivariantly non-smoothable, then $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$ has the same dimension as the Kuranishi space $\operatorname{Ext}^{1}(\Omega_{C}^{1}, \mathcal{O}_{C})^{G}$.

Proof. — $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$ is a product of three products of Teichmüller spaces, corresponding to the partition $I = I_0 \cup I_1 \cup I_2$:

$$\mathcal{T}_0 := \prod_{i \in I_0} \mathcal{T}_{h_i, r_i},$$

where $r_i = \#|\mathcal{N}_i^{(1)}| + 2\#|\mathcal{N}_i^{(2)}|$, and \mathcal{T}_{h_i,r_i} is a parametrizing space of smooth irreducible curves of genus h_i with r_i marked points. Over each \mathcal{T}_{h_i,r_i} we have a family of curves of genus $h_i = g_i$ with r_i marked points.

$$\mathcal{T}_1 := \prod_{i \in I_1} \mathcal{T}_{h_i, r_i},$$

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where $r_i = \sum_{l=1}^{d_i-1} k_l(i)$ is the number of ramification points of the covering $\widetilde{C}_i \to \widetilde{C}_i/H_i$. Here \mathcal{T}_{h_i,r_i} coincides with the parametrizing space for smooth H_i -marked curves of type $\mathcal{D}(\widetilde{C}_i, H_i, \widetilde{\rho}_i)$ constructed in [1, Theorem 2.4], and in the proof of [1, Theorem 2.4] Catanese has also shown the existence of the family of H_i -covers with numerical type $\mathcal{D}(\widetilde{C}_i, H_i, \widetilde{\rho}_i)$ over \mathcal{T}_{h_i,r_i} . Hence over each \mathcal{T}_{h_i,r_i} we have a family of H_i -marked curves of genus g_i with the branching sequence $(k_1(i), \ldots, k_{d_i-1}(i))$ with respect to a fixed generator $\gamma_i := \overline{\gamma}$ of H_i (See [1, Theorem 2.4]).

$$\mathcal{T}_2 := \prod_{[i]\in \bar{I}_2} \mathcal{T}_{h_i, r_i},$$

where \bar{I}_2 is the set of orbits in I_2 , $r_i = \sum_{l=1}^{d_i-1} k_l(i)$. In each orbit [i] we pick one \mathcal{T}_{h_i,r_i} , over which we construct a family of n_i disjoint copies of H_i -marked curves of genus g_i with the branching datum $(k_1(i), \ldots, k_{d_i-1}(i))$ with respect to a fixed generator $\gamma_i := \overline{\gamma^{n_i}}$ of H_i .

Define

$$\mathcal{T}_{[\mathcal{D}(C,G,\rho)]} := \mathcal{T}_0 \times \mathcal{T}_1 \times \mathcal{T}_2$$

Now we can glue the pull back of the families over each factor, by identifying the sections according to the numerical type $[\mathcal{D}(C,G,\rho)]$, to get a family $\mathcal{C}_{[\mathcal{D}(C,G,\rho)]}$ over $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$.

Each fibre of $\mathcal{C}_{[\mathcal{D}(C,G,\rho)]}$ is a stable curve, on which we will define an action of G, making it a G-marked stable curve with numerical type $[\mathcal{D}(C,G,\rho)]$.

We pick a fibre $C = \sum_{i \in I} C_i$, first we use part (3) of the numerical type $[\mathcal{D}(C, G, \rho)]$ to define the action on each orbit of the curves:

If $i \in I_0$, γ acts trivially.

If $i \in I_1$, we have a natural action of H_i on C_i which is induced by the branching datum $(k_1(i), \ldots, k_{d_i-1}(i))$ with respect to γ_i , the chosen generator of H_i (by abuse of notation, the corresponding automorphism is also denoted as γ_i). Then the action of G is defined by the homomorphism $G \to H_i$ which sends γ to γ_i .

If $i \in I_2$, we have to define the action of G on C([i]). First we have the action of H_i on C_i which is determined by the branching datum $(k_1(i), \ldots, k_{d_i-1}(i))$ with respect to γ_i . The action of G, equivalently the automorphism corresponding to γ , is defined as follows:

$$\begin{split} \gamma : C_{\gamma^{l-1}(i)} &\longrightarrow C_{\gamma^{l}(i)}, \ x \longmapsto x \quad \text{if } 1 \leqslant l \leqslant n_{[i]} - 1, \\ \gamma : C_{\gamma^{n_{[i]} - 1}(i)} &\longrightarrow C_{i}, \ x \longmapsto \gamma_{i} x. \end{split}$$

By Lemma 2.10, this should be the expected action.

Then we use conditions (1) and (3) of the numerical type $[\mathcal{D}(C, G, \rho)]$ to glue the action on each orbits and obtain an action of G on C.

For two *G*-marked stable curves (C, G, ρ) and (D, G, ρ') with the same numerical type, *C* and *D* are isomorphic iff for each $i \in I$, $(\tilde{C}_i, H_i, \tilde{\rho}_i)$ and $(\tilde{D}_i, H_i, \tilde{\rho}'_i)$ are isomorphic. In the smooth case, by [1, Theorem 2.4], \mathcal{T}_{h_i, r_i} parametrizes all the H_i -marked curves with numerical type $[\mathcal{D}(\tilde{C}_i, H_i, \tilde{\rho}_i)]$, hence the same holds for $\mathcal{T}_{[\mathcal{D}(C, G, \rho)]}$.

The finiteness of the morphism $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]} \to \mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ follows from the stability of curves and the fact that the automorphism group of a stable curve is finite.

If (C, G, ρ) is G-equivariantly non-smoothable, our parameterizing space has the expected maximal dimension. By Remark 2.7 we have that

$$\left(\bigoplus_{p\in\mathcal{N}}\mathcal{E}\mathrm{xt}^1(\Omega_C^1,\mathcal{O}_C)_p\right)^G=0.$$

Taking the G-invariant subspaces of (*), we get

$$\left(\bigoplus_{i} H^{1}\left(\Theta_{C_{i}}\left(-\sum_{j\neq i} C_{i}\cap C_{j}\right)\right)\right)^{G}\simeq \operatorname{Ext}^{1}(\Omega_{C}^{1},\mathcal{O}_{C})^{G}.$$

It is easy to see that

$$\left(\bigoplus_{i} H^{1}\left(\Theta_{C_{i}}\left(-\sum_{j\neq i} C_{i} \cap C_{j}\right)\right)\right)^{G}$$
$$= \bigoplus_{o\in \operatorname{Orb}}\left(\bigoplus_{i\in o} H^{1}\left(\Theta_{C_{i}}\left(-\sum_{j\neq i} C_{i} \cap C_{j}\right)\right)\right)^{G}.$$

For each $i \in I_0$, it is clear that

$$H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i} C_i \cap C_j\right)\right)^G = H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i} C_i \cap C_j\right)\right),$$

hence has dimension equal to $\dim_{\mathbb{C}} \mathcal{T}_{g_i,r_i}$.

 $\forall i \in I_1$, we have the following:

$$\dim\left(H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i}C_i\cap C_j\right)\right)^G\right) = \dim H^1\left(\Theta_{\widetilde{C}'_i}(-B_i)\right) = \dim \mathcal{T}_{h_i,r_i},$$

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where B_i is the branching locus of the covering $\widetilde{C_i} \to \widetilde{C'_i}$. The second equality follows easily from the Riemann–Rock formula, while the first one needs some explanation.

First note that

$$H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i} C_i \cap C_j\right)\right)^G = H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i} C_i \cap C_j\right)\right)^{H_i}$$

Let $\sigma_i: \widetilde{C}_i \to C_i$ be the normalization map, since C_i as only nodes, we have

$$(\sigma_i)_* \Theta_{\widetilde{C}_i} \left(-\sum_{p \in \mathcal{N}_i^{(2)}} \sigma_i^{-1}(p) \right) = \Theta_{C_i},$$

here recall that $\mathcal{N}_i^{(2)}$ is the set of nodes which are nodes of C_i . Using the projection formula we get

$$(\sigma_i)_* \left(\Theta_{\widetilde{C}_i} \left(-\sum_{p \in \mathcal{N}_i^{(2)}} \sigma_i^{-1}(p) - \sum_{j \neq i} C_i \cap C_j \right) \right) = \Theta_{C_i} \left(-\sum_{j \neq i} C_i \cap C_j \right).$$

Since the action of H_i on \widetilde{C}_i is induced by the one on C_i , we have

$$\dim H^1 \left(\Theta_{\widetilde{C}_i} \left(-\sum_{p \in \mathcal{N}_i^{(2)}} \sigma_i^{-1}(p) - \sum_{j \neq i} C_i \cap C_j \right) \right)^{H_i} \\ = \dim H^1 \left(\Theta_{C_i} \left(-\sum_{j \neq i} C_i \cap C_j \right) \right)^{H_i}.$$

Therefore the first equality is equivalent to showing

$$\dim H^1 \left(\Theta_{\widetilde{C}_i} \left(-\sum_{p \in \mathcal{N}_i^{(2)}} \sigma_i^{-1}(p) - \sum_{j \neq i} C_i \cap C_j \right) \right)^{H_i} = \dim H^1 \Big(\Theta_{\widetilde{C}'_i}(-B_i) \Big),$$

which follows from the following lemma:

LEMMA 2.12. — Let $\pi : C \to C/G =: C'$ be a cyclic cover of order d between smooth projective curves. Denote by \mathcal{R} (resp. \mathcal{B}) the ramification divisor (resp. the branching divisor) and $R = \text{Supp}(\mathcal{R})$ (resp. $B = \text{Supp}(\mathcal{B})$) the ramification locus (resp. the branching locus). Let $D \subset R$ be a G-invariant effective divisor, then we have

$$H^1(C,\Theta_C(-D))^G \simeq H^1(C',\Theta_{C'}(-B)).$$

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Proof of the lemma. — For any *G*-invariant line bundle L on C, we have a decomposition

$$\pi_*(L) = \bigoplus_{i=0}^{d-1} L^{(i)},$$

where we fix a generator γ of G, $L^{(i)}$ is the γ^i -eigen-subsheaf of $\pi_*(L)$ (specially, we have $\pi_*(L)^G = L^{(0)}$). Moreover, in this case all $L^{(i)}$ are line bundles on C'. Using this decomposition, we have

$$H^{1}(\pi_{*}(L))^{G} = H^{1}\left(\bigoplus_{i=0}^{d-1} L^{(i)}\right)^{G} = \left(\bigoplus_{i=0}^{d-1} H^{1}(L^{(i)})\right)^{G} = H^{1}(L^{(0)}). \quad (\star)$$

Recall (dual version of) the Riemann–Hurwitz formula:

$$\Theta_C = \pi^*(\Theta_{C'}) - \mathcal{R}$$

where, setting e_P to be the ramification index at a point $P \in B$ (to be precise, the ramification index should refer to a point $Q \in \pi^{-1}(P)$, but in the case of cyclic covers, for all $Q \in \pi^{-1}(P)$, e_Q remain the same, here we write e_P instead.),

$$\mathcal{R} = \sum_{P \in B} \sum_{Q \in \pi^{-1}(P)} (e_P - 1)Q.$$

Since by assumption D is a reduced divisor, we have

$$\sum_{P \in B} \sum_{Q \in \pi^{-1}(P)} (e_P - 1)Q = \mathcal{R} \leqslant \mathcal{R} + D \leqslant \sum_{P \in B} \sum_{Q \in \pi^{-1}(P)} e_P Q = \pi^*(B),$$

therefore

$$\pi^*(\Theta_{C'}) - \pi^*(B) \leqslant \Theta_C - D \leqslant \Theta_C = \pi^*(\Theta_{C'}) - \mathcal{R}.$$

Using the fact

$$(\pi_*(\pi^*(\Theta_{C'}) - \pi^*(B)))^G = \Theta_{C'}(-B) \otimes (\pi_*\mathcal{O}_C)^G$$
$$= \Theta_{C'}(-B) \otimes \mathcal{O}_{C'} = \Theta_{C'}(-B),$$

the statement is then reduced to the case of D = 0.

We do local computation to show that $(\pi_*(\Theta_C))^G = \Theta_{C'}(-B)$. Let $P \in B$ and $\pi^{-1}\{P\} = \{Q_1, Q_2, \dots, Q_s\}$ for some s|d and set e = d/s. Let z (resp. z_1, \dots, z_s) be a local coordinate at the point P (resp. Q_1, \dots, Q_s), in local coordinates π is expressed as $z = z_i^e$ for $1 \leq i \leq s$. Up to a change of indices and coordinates (see the proof of Lemma 2.10), we may assume the action of γ sends (z_1, \dots, z_s) to $(\zeta_d^s z_s, z_1, \dots, z_{s-1})$. We compute when a vector field is G-invariant, consider $\eta = \sum_i f_i(z_i) \frac{\partial}{\partial z_i}$, then

$$\gamma(\eta) = \sum_{i=1}^{s-1} f_i(z_{i+1}) \frac{\partial}{\partial z_{i+1}} + \frac{f_s(\zeta_d^s z_1)}{\zeta_d^s} \frac{\partial}{\partial z_1}.$$

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If η is *G*-invariant, we have $f_1(u) = f_2(u) = \cdots = f_s(u)$ and $f_1(u) = f_1(\zeta_d^s u)/\zeta_d^s$. This implies $f_i(u) = ug(u^e)$ for all *i*, noting that $z_i \frac{\partial}{\partial z_i} = ez \frac{\partial}{\partial z}$, we can rewrite η as $\eta = (e \sum_{i=1}^s g(z_i^e)) z \frac{\partial}{\partial z}$ and this finishes the proof of the lemma.

Now we continue the proof of Theorem 2.11.

For each $i \in I_2$, consider the map

$$H^{1}\left(\Theta_{C_{i}}\left(-\sum_{j\neq i}C_{i}\cap C_{j}\right)\right)\longrightarrow \bigoplus_{j\in[i]}H^{1}\left(\Theta_{C_{j}}\left(-\sum_{l\neq j}C_{j}\cap C_{l}\right)\right)$$
$$v\longmapsto (v,\gamma(v),\ldots,\gamma^{n_{[i]}-1}(v)).$$

It is easy to see that this induces an isomorphism between the subspaces

$$H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i} C_i \cap C_j\right)\right)^{H_i} \simeq \left(\bigoplus_{j\in[i]} H^1\left(\Theta_{C_j}\left(-\sum_{l\neq j} C_j \cap C_l\right)\right)\right)^G.$$

Therefore we obtain that

$$\dim\left(\bigoplus_{j\in[i]} H^1\left(\Theta_{C_j}\left(-\sum_{l\neq j} C_j \cap C_l\right)\right)\right)^G$$
$$=\dim H^1\left(\Theta_{C_i}\left(-\sum_{j\neq i} C_i \cap C_j\right)\right)^{H_i} = \dim \mathcal{T}_{h_i,r_i}.$$

We see that the family $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]}$ has the same dimension as the Kuranishi space $\operatorname{Ext}^1(\Omega^1_C, \mathcal{O}_C)^G$.

DEFINITION 2.13.

- (1) We call the image of the natural map $\mathcal{T}_{[\mathcal{D}(C,G,\rho)]} \to \overline{\mathfrak{M}_g} \mathfrak{M}_g$ a stratum with numerical type $[\mathcal{D}(C,G,\rho)]$, which we denote by $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$.
- (2) A stratum $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ is called maximal, if it is not contained in the Zariski closure of another stratum $\overline{\mathfrak{M}}_{[\mathcal{D}(C',G,\rho')]}$, such that $\dim \mathfrak{M}_{[\mathcal{D}(C,G,\rho)]} < \dim \mathfrak{M}_{[\mathcal{D}(C',G,\rho')]}$.

It is clear that $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(G)$ is a union of all the strata (with group G). By Theorem 2.11 we see that the closure of any stratum is an irreducible Zariski closed subset of $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(G)$. Therefore to understand the components of $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(G)$, is equivalent to understanding the maximal strata.

Remark 2.14. — Note that when we say a "stratum" we remember the numerical type, the possibility of two strata of different numerical types having the same support is not excluded.

3. The maximal strata

In the previous section we have interpreted the problem of determining irreducible components of $(\overline{\mathfrak{M}_g} - \mathfrak{M}_g)(G)$ into determining the maximal *G*-strata.

In this section we first discuss in general when a stratum is maximal. Then with certain additional conditions we give an explicit description via the associated combinatoric data.

Remark 3.1. — If a stable curve (C, G, ρ) can be smoothed to another stable curve (C', G, ρ') , then inside the moduli space, $[C] \notin \mathfrak{M}_{[\mathcal{D}(C', G, \rho')]}$ and $[C] \in \overline{\mathfrak{M}_{[\mathcal{D}(C', G, \rho')]}}$, in fact we have a stronger result that

$$\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]} \subset \mathfrak{M}_{[\mathcal{D}(C',G,\rho')]} - \mathfrak{M}_{[\mathcal{D}(C',G,\rho')]}.$$

Hence we have $\dim \overline{\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}} < \dim \overline{\mathfrak{M}_{[\mathcal{D}(C',G,\rho')]}}.$

The fact that $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]} \cap \mathfrak{M}_{[\mathcal{D}(C',G,\rho')]} = \emptyset$ is clear since the number of nodes of C' is strictly less than that of C, we only need to show $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]} \subset \overline{\mathfrak{M}_{[\mathcal{D}(C',G,\rho')]}}$. Note that this is enough to show that general (in fact, every) $[C_1] \in \mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$, with $[\mathcal{D}(C_1,G,\rho_1)] = [\mathcal{D}(C,G,\rho)]$, can be deformed Gequivalently to a curve (C_2, G, ρ_2) such that $[\mathcal{D}(C_2, G, \rho_2)] = [\mathcal{D}(C', G, \rho')]$. Recall that part (3) of the numerical type remembers the local monodromy at each node P, hence by Proposition 2.6, whether a node is smoothable only depends on the numerical type $[\mathcal{D}(C,G,\rho)]$.

For our purpose, it suffices to consider the case where C' is obtained by smoothing a single orbit G(P) of some node P. We will see that $[\mathcal{D}(C', G, \rho')]$ is then uniquely determined by $[\mathcal{D}(C, G, \rho)]$.

- Let I' be the dual graph of C', then I' is obtained by contracting the edges in I which correspond to nodes in G(P), the action of G on I' is then naturally induced from the one on I.
- We denote by Con : $\mathcal{I} \to \mathcal{I}'$ the contraction map. For each vertex $i' \in I'$, the arithmetic genus associated to i' (here for i' we remember the loops based at i', i.e., the set of nodes $\mathcal{N}_{i'}^{(2)}$) is just the arithmetic genus of Con⁻¹(i'). $G_{i'}$ is the subgroup which leaves the sub-graph Con⁻¹(i') invariant and $G''_{i'}$ is the subgroup which acts trivially on Con⁻¹(i'), thus we obtain $H_{i'}$.

Let $G(P) = \{P = P_1, \ldots, P_s\}$, we choose a small G_P -invariant analytic open neighborhood U of P, such that \overline{U} (here we mean the analytic closure) do not contain any other nodes of C nor fixed points of some H_i on C_i . Possibly after shrinking U, we require moreover G(U) is a disjoint union of connected U_1, \ldots, U_s , each U_j is a connected G_{P_j} -invariant neighborhood of P_j . Then $(C - \overline{G(U)}, G, \rho)$ is G-equivalently isomorphic to a G-invariant open subset V of (C', G, ρ') , therefore, the fixed points in V with their monodromies, and the monodromy at each node in V are inherited from those on $C - \overline{G(U)}$, which is determined by $\mathcal{D}(C, G, \rho)$. Note that C' - V is smooth (since it is the local smoothing of G(P) in G(U)), we only need to determine the fixed points in C' - V and their monodromies. Locally the deformation is given by xy = t, where C corresponds to $\{t = 0\}, P = (0, 0)$ and C' is given by $xy = t_0$ for some $t_0 \neq 0$, fix a generator γ_P of G_P . There are two cases:

 $-G_P$ dose not exchange the two branches, the action on the family is given by

$$\gamma_P: (x, y; t) \longmapsto (\alpha x, \alpha^{-1}y; t), \text{ for some } \alpha \neq 1.$$

Then it is clear on C', there is no new fixed points, we are done in this case.

 $-G_P$ exchanges the two branches (this also implies $2||G_P|$), the action on the family is given by

$$\gamma_P: (x, y; t) \longmapsto (\alpha y, \alpha^{-1} x, t).$$

We find two new fixed points on C': $Q_1 = (\alpha y_0, y_0)$ and $Q_2 = (-\alpha y_0, -y_0)$, where y_0 satisfies $\alpha y_0^2 = t_0$. Then $u = y - y_0$ is a local parameter near Q_1 , the action of γ sends

$$u \longmapsto \frac{t_0}{\alpha(y_0 + u)} - y_0 = -u \left(1 - \frac{u}{y_0} + \frac{u^2}{y_0^2} - \cdots\right).$$

We see the monodromy of γ_P at Q_1 is -1, a similar computation shows this holds also for Q_2 .

Combining the above arguments, we see that $[\mathcal{D}(C', G, \rho')]$ is completely determined by $[\mathcal{D}(C, G, \rho)]$.

Hence we only need to consider the strata of *G*-equivariantly non-smoothable curves. In the rest of the article, we do only consider strata whose general curve is *G*-equivariantly non-smoothable.

We first recall some results on Teichmüllar space of smooth curves. Given a smooth *H*-marked curve (C, H, ρ) , there is an induced map $\tilde{\rho} : H \to \text{Map}_q(\text{cf.} [3, \text{Section 2}])$, which induces an *H*-action on \mathcal{T}_g . Fix(H) is an

irreducible analytic subset of \mathcal{T}_g , whose image inside \mathfrak{M}_g equals $\mathfrak{M}_{g,H,\rho}$. (cf. [3, Appendix B]).

Set $G(H) = \bigcap_{C \in \operatorname{Fix}(H)} \operatorname{Stab}_{C}$, $(\operatorname{Aut}(C) \simeq \operatorname{Stab}_{C} \subset \operatorname{Map}_{g})$, a general curve in $\operatorname{Fix}(H)$ has automorphism group G(H) (cf. [3], Appendix B). Since both $\mathcal{T}_{g,H,\rho}$ and $\operatorname{Fix}(H)$ maps finitely onto $\mathfrak{M}_{g,H,\rho}$, a general curve in $\mathcal{T}_{g,H,\rho}$ has the same property as the one in $\operatorname{Fix}(H)$. Therefore, given a subgroup H'of a general curve C in $\mathcal{T}_{g,H,\rho}$, which induces an H'-marked curve (C, H', ρ'') , H' can be considered as a subgroup of general curves in $\mathcal{T}_{g,H,\rho}$, which induce the same numerical type as (C, H', ρ'') , this implies that $\mathfrak{M}_{g,H,\rho} \subset \mathfrak{M}_{g,H',\rho''}$.

Given a G-marked stable curve (C, G, ρ) , write

$$C = \sum_{i} C_{i} = \sum_{\lambda \in \Lambda} \sum_{t=1}^{s_{\lambda}} C_{\lambda,t},$$

where Λ is the index set of isomorphism classes of the irreducible components with marked points $(C_i, \mathcal{N}_i^{(1)}, \mathcal{N}_i^{(2)})$ and s_{λ} is the number of curves $(C_i, \mathcal{N}_i^{(1)}, \mathcal{N}_i^{(2)})$ belonging to the isomorphism class λ .

Clearly Aut(C) is a subgroup of $\prod_{\lambda \in \Lambda} ((\prod_{t=1}^{s_{\lambda}} \operatorname{Aut}(C_{\lambda,t})) \rtimes \mathfrak{S}_{s_{\lambda}})$ consisting of elements preserving the nodes of C, where for each class λ we fix an identification of Aut($C_{\lambda,t}$) for all curves $C_{\lambda,t}$ and the semi-direct product is determined by the following group homomorphism:

$$\mathfrak{S}_{s_{\lambda}} \longrightarrow \operatorname{Aut}\left(\prod_{t=1}^{s_{\lambda}} \operatorname{Aut}(C_{\lambda,t})\right), \sigma \longmapsto \phi_{\sigma} : (g_1, \dots, g_{s_{\lambda}}) \longmapsto (g_{\sigma(1)}, \dots, g_{\sigma(s_{\lambda})}).$$

Now assume C is a general curve of a stratum, and D is another curve inside the same stratum. Applying the previous mentioned results on smooth curves to the components of C and D, we may identify $\prod_{\lambda \in \Lambda} ((\prod_{t=1}^{s_{\lambda}} \operatorname{Aut}(C_{\lambda,t})) \rtimes \mathfrak{S}_{s_{\lambda}})$ with $\prod_{\lambda \in \Lambda} ((\prod_{t=1}^{s_{\lambda}} \operatorname{Aut}(D_{\lambda,t})) \rtimes \mathfrak{S}_{s_{\lambda}})$ and consider $\operatorname{Aut}(C)$, $\operatorname{Aut}(D)$ as subgroups. For an element $(g_{\lambda,t}, \sigma_{\lambda}) \in \operatorname{Aut}(C)$, each $g_{\lambda,t}$ corresponds to a $g'_{\lambda,t} \in \operatorname{Aut}(D_i)$, such that the numerical type induced by $g'_{\lambda,t}$ is the same as that of $g_{\lambda,t}$, hence $\{g'_{\lambda,t}\}$ satisfy the compatibility condition and yields an element $(g'_{\lambda,t}, \sigma_{\lambda}) \in \operatorname{Aut}(D)$, moreover, the numerical type of $(g_{\lambda,t}, \sigma_{\lambda})$ and $(g'_{\lambda,t}, \sigma_{\lambda})$ are the same.

In fact, we have deduced the following proposition:

Proposition 3.2.

(1) Given a stratum, the automorphism groups of general curves in the stratum remain the same, hence we define the automorphism group of a stratum to be the automorphism group of a general curve inside the stratum.

(2) Let G' be the automorphism group of a stratum $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$, for any cyclic subgroup H of G', the induced H-numerical types on general curves are identical, say $[\mathcal{D}(C',H,\rho')]$, this means there is an open subset U of $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ with $U \subset \mathfrak{M}_{[\mathcal{D}(C',H,\rho')]}$.

DEFINITION 3.3. — Given a stratum, we say that (the action of) G is maximal if for any general curve (C, G, ρ) inside the stratum, there is no subgroup $G' \subset \operatorname{Aut}(C)$ isomorphic to G (including G itself) such that the induced G'-marked stable curve (C, G') is G'-equivariantly smoothable or the dimension of the stratum corresponding to (C, G') is larger than the dimension of the given one.

Remark 3.4. — Let $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ and $\mathfrak{M}_{[\mathcal{D}(C',G,\rho')]}$ be two strata whose general curves are *G*-equivariantly non-smoothable. If $\overline{\mathfrak{M}}_{[\mathcal{D}(C,G,\rho)]} \subset \overline{\mathfrak{M}}_{[\mathcal{D}(C',G,\rho')]}$, we have two cases:

- (1) If there is a general curve [C] ∈ 𝔅_[𝔅(C,G,ρ)] such that [C] ∈ 𝔅_{𝔅(C',G,ρ')}, then this means there is a subgroup G' of Aut(C) isomorphic to G, such that the induced G'-numerical type is exactly [𝔅(C,G,ρ)], by Proposition 3.2(2), there is an open subset U of 𝔅_[𝔅(C,G,ρ)] such that U ⊂ 𝔅_[𝔅(C',G,ρ')].
 (2) If no general curve in 𝔅_[𝔅(C,G,ρ)] is contained in 𝔅_[𝔅(C',G,ρ')], since
- (2) If no general curve in $\mathfrak{M}_{[\mathcal{D}(C,G,\rho)]}$ is contained in $\mathfrak{M}_{[\mathcal{D}(C',G,\rho')]}$, since (C,G,ρ) is *G*-equivariantly non-smoothable, the only possibility is that there is a subgroup G' of $\operatorname{Aut}(C)$ isomorphic to G, such that the induced G'-marked curve $(C,G'\simeq G,\rho'')$ can be *G*-equivariantly deformed to a *G*-marked stable curve with numerical type $[\mathcal{D}(C',G,\rho')]$.

Hence we see that the notion of maximal G-action is enough to determine the maximal strata (although a maximal stratum may correspond to more than one numerical type).

Once we know the automorphism group of the stratum, we can find the subgroups which are isomorphic to G and hence determine whether the stratum is maximal.

Now for fixed genus g and group G, we can determine the irreducible components (equivalently, the maximal strata) of $(\overline{\mathfrak{M}}_g - \mathfrak{M}_g)(G)$ since the possible configurations are finite. However if we do not fix the genus g, due to the following phenomena, it is not so easy to obtain a brief description of the irreducible components even for cyclic groups.

Recall that in the smooth case a stratum with group G is called *full* if the automorphism group of the stratum equals G. Now for a G-marked stable

curve $(C = \sum_{i} C_i, G, \rho)$, if for some *i*, the group H_i is not full for the induced action of H_i on \widetilde{C}_i , the complicity of Aut(C) increases.

We give an example for the automorphism group of a non-full stratum of smooth curves. By [8, Lemma 4.1] we know that for a general smooth curve C inside a non-full stratum, G is a normal subgroup of $\operatorname{Aut}(C)$ and $\operatorname{Aut}(C)/G$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, etc. In the case of G being cyclic and $\operatorname{Aut}(C)/G \simeq (\mathbb{Z}/2\mathbb{Z})^2$, by [8, Lemma 4.1] there are three elements $b_1, b_2, b_3 \in \operatorname{Aut}(C) - G$, such that b_i has order 2 and the product $b_1b_2b_3$ is contained in G. The following proposition tells us in this case all the possibilities for $\operatorname{Aut}(G): ([6, Lemma 5.7])$

PROPOSITION 3.5. — Let G(H) be a group containing a normal cyclic subgroup H of order d such that $G(H)/H \simeq (\mathbb{Z}/2)^2$. Assume in addition that there exist three elements $b_1, b_2, b_3 \in G(H) - H$ such that b_i has order 2 and the product $b_1b_2b_3$ is contained in H. Then G(H) has the presentation:

$$\{\alpha, \beta_1, \beta_2 \,|\, \alpha^d = 1, \beta_1^2 = \beta_2^2 = 1, \beta_1 \alpha = \alpha^{l_1} \beta_1, \beta_2 \alpha = \alpha^{l_2} \beta_2, \beta_1 \beta_2 = \beta_2 \beta_1 \alpha^{e_{1,2}} \}$$

such that $0 \leq l_1, l_2, e_{1,2} < d$, $gcd(l_i, d) = 1$, $l_i^2 \equiv 1 \mod d$, $d|(l_i + 1)e_{1,2}$, for i = 1, 2 and $gcd(d, l_1l_2 + 1)|e_{1,2}$.

Moreover, $\gamma := \bar{\alpha}$ is a generator of H; $b_i = \bar{\beta}_i$, $b_i \gamma b_i = \gamma^{l_i}$ for i = 1, 2and $b_2 b_1 b_2 = b_1 \gamma^{e_{1,2}}$; $b_3 = b_1 b_2 \gamma^f$, where f is an integer such that $0 \leq f < d$ and $d|((l_1 l_2 + 1)f + e_{1,2})$.

In the smooth case, in order to determine if a stratum is maximal, we only need to compute the subgroups of G(H) which are isomorphic to H. However, for stable curves, we have to compute all the cyclic subgroups of $\operatorname{Aut}(\widetilde{C}_i)$ and solve a combinatoric problem concerning the dual graph and all the cyclic subgroups of $\operatorname{Aut}(\widetilde{C}_i)$ for each C_i .

In order to have a more detailed discussion, for the rest of the article we make the following assumptions:

Assumption 3.6.

- (0) $(C = \sum_{i \in I} C_i, G, \rho)$ is G-equivariantly non-smoothable.⁽¹⁾
- (1) For a general stable curve (C, G, ρ) in the stratum we have $H_i = \operatorname{Aut}(\widetilde{C_i})$ and $q(\widetilde{C_i}) \ge 2$ for all *i*.
- (2) For any $i \in I$, the parameterizing space \mathcal{T}_{h_i,r_i} has dimension> 0.

 $^{^{(1)}}$ Of course, this is a necessary condition for having a maximal stratum by Remark 3.1.

Remark 3.7.

- (1) By assumption (2), for a general curve (C, G, ρ) in the stratum, two irreducible components coming from different *G*-orbits must be non-isomorphic, hence we have $\text{Orb} = \Lambda$. Therefore any $\beta \in \text{Aut}(C)$ fixes the *G*-orbits and induces $\beta_o := \beta|_{C(o)} \in \text{Aut}(C(o))$, conversely $(\beta_o)_{o \in \text{Orb}}$ determines β .
- (2) Given $\beta = (\beta_o) \in \operatorname{Aut}(C)$, the order of β is $\operatorname{lcm}\{\operatorname{Ord}(\beta_o)\}$. Using the isomorphism in Lemma 2.10 and regarding β_o as an element in $(\prod_{j=1}^{n_o} \operatorname{Aut}(C_o^{(j)})) \rtimes \mathfrak{S}_{n_o}$, we can write $\beta_o = ((\beta_{o,1}, \ldots, \beta_{o,n_o}), \overline{\beta_o})$. What is the order of β_o ? Assume that $\overline{\beta_o}$ has $\mu_o(\beta)$ orbits in o with lengths $l_1, \ldots, l_{\mu_o(\beta)}$, then we have

$$\overline{\beta_o} = (i_1, \dots, i_{l_1})(i_{l_1+1}, \dots, i_{l_1+l_2}) \dots (i_{n_o-l_{\mu_o(\beta)}+1}, \dots, i_{n_o})$$

and

$$\operatorname{Ord}(\beta_o) = \operatorname{lcm}\left(\operatorname{Ord}(\beta_{o,i_1}\dots\beta_{o,i_{l_1}})l_1,\dots,\right.$$
$$\operatorname{Ord}\left(\beta_{o,(i_{n_o}-l_{\mu_o(\beta)}+1)}\dots\beta_{o,i_{n_o}}\right)l_{\mu_o(\beta)}\right).$$

We want to understand when the stratum is maximal. For this purpose we study first quotient curves of type $C/\langle\beta\rangle$, where $\beta \in \operatorname{Aut}(C) - G$ is an element of order d.

LEMMA 3.8. — For any $\beta \in \operatorname{Aut}(C)$, the quotient map $\pi : C \to C' := C/G$ factors through the quotient map $C \to C/\langle \beta \rangle$.

Proof. — Note that we have the following decomposition of C' into irreducible components: $C' = \sum_{o \in \text{Orb}} C'_o$.⁽²⁾

For the lemma, it suffices to show that for any $P \in C$, $\pi(P) = \pi(\beta(P))$. By assumption (2) we have that $\beta(C(o)) = C(o)$, therefore it suffices to consider the map $\pi|_{C(o)} : C(o) \to C'_o$ and $\beta_o := \beta|_{C(o)}$. Using Lemma 2.10 we see this is equivalent to considering the map $\pi_o : \bigsqcup_{j=1}^{n_o} C_o^{(j)} \to C'_o$, where π_o is the composition of $\pi|_{C(o)}$ with the natural map $\bigsqcup_{j=1}^{n_o} C_o^{(j)} \to C(o)$.

We determine first the fibre of π_o : noting that $\overline{\gamma}$ acts transitively on the vertices inside o, hence any fibre of π_o must contain at least a point in $C_o^{(1)}$, say $x^{(1)} \in C_o^{(1)}$. Here we only discuss in detail the case where $x^{(1)}$ does not lie in the inverse image of a node of C(o), the other case is similar. Then using the isomorphism of Lemma 2.10 we see that $\pi_o^{-1}(\pi_o(x^{(1)})) = \{x^{(1)}, \gamma^{n_o}(x^{(1)}), \dots, \gamma^{n'_o - n_o}(x^{(1)}); x^{(2)}, \dots, \gamma^{n'_o - n_o}(x^{(2)});$ $\dots; x^{(n_o)}, \dots, \gamma^{n'_o - n_o}(x^{(n_o)})\}$, where $n'_o := d/|G''_i|$ for any $i \in o$ and $x^{(j)}$

 $^{^{(2)}}$ For any $i\in o,$ $\widetilde{C}'_i=$ normalization of C'_o

denotes the point on $C_o^{(j)}$ which equals to $x^{(1)}$ via the identification $C_o^{(j)} = C_o^{(1)}$. Now since $\beta_o = ((\beta_{o,1}, \dots, \beta_{o,n_o}), \overline{\beta_o})$ and $\forall x^{(j)} \in C_o^{(j)}, \ \beta_o(x^{(j)}) = \beta_{o,\overline{\beta_o}(j)} x^{(\overline{\beta_o}(j))}$, by assumption (1) we have that $\beta_{o,\overline{\beta_o}(j)} \in \langle \gamma^{n_o} \rangle$ and hence $\beta_o(x^{(j)}) \in \pi_o^{-1}(\pi_o(x^{(1)}))$.

Remark 3.9. — For simplicity we denote by $\mathfrak{M}_{C'}$ the stratum corresponding to $C \to C'$ and by \mathfrak{M}_{β} the stratum corresponding to $C \to C/\langle \beta \rangle$ (similarly for $\mathcal{T}_{C'}$ and \mathcal{T}_{β}), Lemma 3.8 says that an open subset U of $\mathfrak{M}_{C'}$ is contained in \mathfrak{M}_{β} (not just in $\overline{\mathfrak{M}_{\beta}}$!). If $\mathfrak{M}_{C'}$ is not maximal, by Remark 3.4 there are two cases:

- (1) there exists a $\beta \in \operatorname{Aut}(C)$ of order d, such that $\dim \mathfrak{M}_{\beta} > \dim \mathfrak{M}_{C'}$;
- (2) there exists a $\beta \in \operatorname{Aut}(C)$ of order d, such that $\dim \mathfrak{M}_{\beta} = \dim \mathfrak{M}_{C'}$ and (C, β) is *G*-equivariantly smoothable.

Recall that $\mathcal{T}_{C'}$ is the product of the parameterizing spaces of all the coverings $C(o) \to C'_o$, which is isomorphic to the parameterizing space $\mathcal{T}_{C'_o}$ of the covering $C_o^{(1)} \to C'_o$ (strictly speaking, of the covering $\widetilde{C_o^{(1)}} \to \widetilde{C'_o}$). Denoting by $(C/\langle\beta\rangle)(o)$ the inverse image of C'_o in $C/\langle\beta\rangle$, the number of irreducible components of $(C/\langle\beta\rangle)(o)$ is $\mu_o(\beta)$. Hence the parameterizing space of $C(o) \to (C/\beta)(o)$ is a product of $\mu_o(\beta)$ parameterizing spaces of the irreducible components of $(C/\beta)(o)$, all of which have dimension greater than or equal to dim $\mathcal{T}_{C'_o}$. Now we can characterize case (1) of Remark 3.9:

LEMMA 3.10. — Case (1) of Remark 3.9 happens if and only if $\exists o \in Orb$, such that one of the following cases occurs:

- (a) $\mu_o(\beta) > 1;$
- (b) $\mu_o(\beta) = 1$ and $3 \operatorname{Ord}(\beta_o^{n_o}) \leq \operatorname{Ord}(H_i)$ for any $i \in o$;
- (c) $\mu_o(\beta) = 1$, $2 \operatorname{Ord}(\beta_o^{n_o}) = \operatorname{Ord}(H_i)$ for any $i \in o$ and we are not in the exceptional cases of Proposition 3.13.

Proof. — First note that $\operatorname{Ord}(H_i)$ does not depend on the choice of $i \in o$. By assumption (2) dim $\mathcal{T}_{C'_o} \geq 1$, hence (a) follows from the preceding discussion. If $\mu_o(\beta) = 1$ for all o, we have that $C/\langle\beta\rangle(o)$ is an irreducible curve, in case (b) by Proposition 3.13 we always have dim $\mathcal{T}_{C/\langle\beta\rangle(o)} > \dim \mathcal{T}_{C'_o}$; if $2\operatorname{Ord}(\beta_o^{n_o}) = \operatorname{Ord}(H_i)$ then dim $\mathcal{T}_{(C/\langle\beta\rangle)(o)} > \dim \mathcal{T}_{C'_o}$ except for the cases in Proposition 3.13.

Now we discuss briefly the exceptional cases mentioned in 3.10, since the technique we use here is independent of the other part of this paper, we only sketch the proof and for reader who are interested in details we refer to [2, 8, 7]. First we introduce a few settings (cf. [8, Section 2]).

DEFINITION 3.11. — Let $C \to C/G =: C'$ be a cyclic cover and $B = \{P_1, \ldots, P_r\} \subset C'$ the branching locus, then topologically this is determined by a surjective group homomorphism

$$f: \pi_1(C'-B) \longrightarrow G.$$

Fix a set of small circles $\{\gamma_j\}_{1 \leq j \leq r}$ with each γ_j around the point P_j , denote by c_j the order of $f(\gamma_j)$ in G. We call the (unordered) sequence (c_1, \ldots, c_r) the signature of the G-marked curve C (or equivalently, of the G-cover $C \rightarrow C'$).

Remark 3.12. — Under the condition of Definition 3.11, assume that d = |G| and $P_j \in D_{i_j}$. Since $f(\gamma_j) = [i_j]$ (possibly after replacing some $[\gamma_j]$ by $[\gamma_j]^{-1}$), we have $c_j \gcd(d, i_j) = d$.

PROPOSITION 3.13. — Given an admissible branching sequence $[(k_1, \ldots, k_{d-1})]$ for $g \ge 2$, assume that $\dim \mathcal{T}_{g;d,[(k_1,\ldots,k_{d-1})]} \ge 1$ and for any general curve $C \in \mathcal{T}_{g;d,[(k_1,\ldots,k_{d-1})]}$, $G = \operatorname{Aut}(C)$. For any proper subgroup G' of G, we have an induced cyclic cover of degree $d' := \operatorname{order}(G'): C \to C/G'$ and hence an admissible sequence $[(k'_1, k'_2, \ldots, k'_{d'-1})]$ for d' and g. Then $\dim_{\mathbb{C}} \mathcal{T}_{g;d,[(k_1,\ldots,k_{d-1})]} > \dim_{\mathbb{C}} \mathcal{T}_{g;d',[(k'_1,\ldots,k'_{d'-1})]}$ except for three cases:

(1) $d = 2d' \ge 4, 2 \mid d', C/G \simeq \mathbb{P}^1, [(k_1, \dots, k_{d-1})] = [(1, 0, \dots, 0, k_{d'} = 2, 0, \dots, 0, 1)] and [(k'_1, \dots, k'_{d'-1})] = [(1, 0, \dots, 0, k_{d'/2} = 2, 0, \dots, 0, 1)];$ (2) $d = 2d' \ge 6, 2 \nmid d', C/G \simeq \mathbb{P}^1, [(k_1, \dots, k_{d-1})] = [(0, 1, 0, \dots, 0, k_{d'} = 2, 0, \dots, 0, 1, 0)] and [(k'_1, \dots, k'_{d'-1})] = [(1, 0, \dots, 0, 1)].$

(3)
$$d = 2, g(C) = 2, C/G \simeq \mathbb{P}^1, \lfloor (k_1) \rfloor = \lfloor (6) \rfloor$$

Proof. — If dim_C $\mathcal{T}_{g;d,[(k_1,...,k_{d-1})]} = \dim_{\mathbb{C}} \mathcal{T}_{g;d',[(k'_1,...,k'_{d'-1})]}$, then we have that $\mathcal{T}_{g;d,[(k_1,...,k_{d-1})]} = \mathcal{T}_{g;d',[(k'_1,...,k'_{d'-1})]}$. Now since the condition of [8, Lemma 4.1], is satisfied, we see that the pair (C, G') must be one of the cases there.

By assumption dim_C $\mathcal{T}_{g;d',[(k'_1,\ldots,k'_{d'-1})]} \ge 1$, hence only the cases I, II, IIIin [8, Lemma 4.1] may happen. Note that in [8, Lemma 4.1], indices of signatures satisfy $c_j \le c_{j+1}$.

Case III-c is excluded since here $\operatorname{Aut}(C)$ is a cyclic group, which can not have a quotient group isomorphic to $(\mathbb{Z}/2)^2$.

For the remaining cases, we have d = 2d' and $C/\operatorname{Aut}(C) \simeq \mathbb{P}^1$. We now treat the cases separately, for a branching point $P_j \in \mathbb{P}^1$, denote by i_j the index satisfying $P_j \in D_{i_j}$.

Case I. — The covering map has 6 branching points P_1, \ldots, P_6 with signature (2, 2, 2, 2, 2, 2, 2). By Remark 3.12 we get $i_j = d'$ for all $1 \leq j \leq 6$, Definition 2.2(1) is automatically satisfied, and 2.2(2) implies d' = 1, i.e.,

 $G \simeq \mathbb{Z}/2\mathbb{Z}$. Then by Riemann–Hurwitz formula we see that g = 2 and the branching sequence is (6).

Case II. — The covering map has 5 branching points P_1, \ldots, P_5 with signature $(2, 2, 2, 2, c_5)$ $(c_5 \ge 2)$. This implies $i_j = d'$ for $1 \le j \le 4$, by 2.2(1) we must have $d|i_5$, which is impossible, since by definition we have $1 \le i_5 \le d-1$.

Cases III-a. $C \to C/G$ has 4 branching points (P_1, P_2, P_3, P_4) with signature $(2, 2, 2, c_4)$ $(c_4 \ge 3$ since we assume $g \ge 2$), this implies $i_1 = i_2 = i_3 = d'$ and $(i_4, d) = d/c_4$. Now if Definition 2.2(1) is satisfied, we must have $i_4 = d'$, then Definition 2.2(2) implies d' = 1 and d = 2, which contradicts the fact $d \ge c_4 \ge 3$.

Case III-b. — $C \to C/\operatorname{Aut}(C)$ has four branching points P_1, P_2, P_3, P_4 with signature $(2, 2, c_3, c_4)$ such that $2 < c_3 \leq c_4$ (since we assume $g \geq 2$). Using Remark 3.12, we get $i_1 = i_2 = d'$, $c_3 = d/\operatorname{gcd}(d, i_3)$ and $c_4 = d/\operatorname{gcd}(d, i_4)$, since $2 < c_3 \leq c_4$, we have $i_3, i_4 \neq d'$ and $d \geq c_3 > 2$. Moreover Definition 2.2(1) says that $d|(d'+d'+i_3+i_4)$, which implies that $d|(i_3+i_4)$; 2.2(2) says that $\operatorname{gcd}(d, i_3) = 1$ or 2, where the = 2 case happens only if $2 \nmid d'$. Since we are interested in the equivalent class of branching sequences, after possibly a change of generator of G, the above numerical restriction on $\{i_3, i_4\}$ yields two possibilities:

$$\{i_3, i_4\} = \{1, d-1\}$$
 and $[(k_1, \dots, k_{d-1})] = [(1, 0, \dots, 0, k_{d'} = 2, 0, \dots, 0, 1)]$

or

$$2 \nmid d', \{i_3, i_4\} = \{2, d-2\}, \\ [(k_1, \dots, k_{d-1})] = [(0, 1, 0, \dots, 0, k_{d'} = 2, 0, \dots, 0, 1, 0)].$$

Noting that there is one more restriction in case III-b that $C/G' \to \mathbb{P}^1$ is a double cover branched in two points on \mathbb{P}^1 , we see that if

$$[(k_1,\ldots,k_{d-1})] = [(1,0,\ldots,0,k_{d'}=2,0,\ldots,0,1)],$$

then we must have 2|d', otherwise $C/G' \to \mathbb{P}^1$ is branched on four points on \mathbb{P}^1 ; in this case

$$[(k'_1,\ldots,k'_{d'-1})] = [(1,0,\ldots,0,k_{d'/2}=2,0,\ldots,0,1)].$$

For the other case where $2 \nmid d'$ and

$$[(k_1,\ldots,k_{d-1})] = [(0,1,0,\ldots,0,k_{d'}=2,0,\ldots,0,1,0)],$$

we get $[(k'_1, \ldots, k'_{d'-1})] = [(1, 0, \ldots, 0, 1)].$

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Remark 3.14. — Let us look at the case when dim $\mathcal{T}_{\beta} = \dim \mathcal{T}_{C'}$: first $\mu_o(\beta) = 1$ for all $o \in \text{Orb}$, which means that $\overline{\beta_o}$ acts transitively on the vertices in o.

For the subcurve $C/(\beta)(o)$, if we are not in the exceptional cases, then we have $(C/\langle\beta\rangle)(o) \simeq C'_o$ and $\beta_o^{n_o}|_{C_o^{(1)}} = \prod_{k=1}^{n_o} \beta_{o,k}$ is a generator of $\operatorname{Aut}(C_o^{(1)})$. Otherwise C'_o is rational and $C/\langle\beta\rangle(o) \to C'_o$ is a double cover and $\langle \prod_{k=1}^{n_o} \beta_{o,k} \rangle$ is the (unique) index 2 subgroup of $\operatorname{Aut}(C_o^{(1)})$ which arises from the exceptional cases in Proposition 3.13.

We assume to be in one of the above cases, that is under the condition dim $\mathcal{T}_{\beta} = \dim \mathcal{T}_{C'}$, and determine when (C, β) is *G*-equivariantly nonsmoothable. Here we apply Proposition 2.6 to a node $p \in \mathcal{N}_{i_1} \cap \mathcal{N}_{i_2}$. We have two cases: $i_1 = i_2 =: i, p \in \mathcal{N}_i^{(2)}$ and $i_1 \neq i_2, p \in \mathcal{N}_{i_1}^{(1)} \cap \mathcal{N}_{i_2}^{(1)}$, which we treat separately.

Case (I). — If $p \in \mathcal{N}_i^{(2)}$, we must have $G_p \subset G_i$. Denoting by $\{p_1, p_2\}$ the inverse image of p of the normalization map $\widetilde{C}_i \to C_i$, we have the following easy lemma:

LEMMA 3.15. — For any $g \in H_i$, (regarding g also as an automorphism of $\widetilde{C_i}$,) there are three possibilities:

- (a) $g(p_i) = p_i$ for i = 1, 2.
- (b) $g(p_1) = p_2$ and $g(p_2) = p_1$.
- (c) $g(p_1) = p'_1$ and $g(p_2) = p'_2$, where $\{p'_1, p'_2\}$ is the inverse image of $g(p) (\neq p)$.

Proof. — Obvious.

We apply Proposition 2.6 to $(C_i, H_i = \langle \overline{\gamma^{n_i}|_{C_i}} \rangle)$ and $(C_i, \langle \overline{\beta^{n_i}|_{C_i}} \rangle)$. If $\langle \overline{\beta^{n_i}|_{C_i}} \rangle = H_i$, then by assumption (0) it is clear that p is non-smoothable for $(C_i, \langle \overline{\beta^{n_i}|_{C_i}} \rangle)$ and hence non-smoothable for $(C, \langle \beta \rangle)$.

Assume we are in the exceptional cases of Proposition 3.13. First we consider exceptional case (1), recall that $\pi_i : \widetilde{C}_i \to \widetilde{C}_i/H_i$ is branched on four points P_1, \ldots, P_4 with signature $(2, 2, d_i, d_i)$ with $d_i \ge 4$. Note that p_1 or p_2 does not belong to either $\pi_i^{-1}(P_1)$ or $\pi_i^{-1}(P_2)$:

Case (a) in Lemma 3.15 does not occur since $\#\pi_i^{-1}(P_1) = \#\pi_i^{-1}(P_2) = d_i/2 \ge 2$.

For the same reason, if $d_i \ge 6$, then Case (b) does not occur; if $d_i = 4$, Case (b) does not occur, either, otherwise p is H_i -equivariantly smoothable.

Case (c) does not occur, otherwise p is H_i -equivariantly smoothable.

Hence we may assume $p_1 = \pi_i^{-1}(P_3)$ and $p_2 = \pi_i^{-1}(P_4)$. Let z_j be a local coordinate near p_j , j=1,2, the action of H_i near p is

$$\overline{\gamma_i^{n_i}}: z_1 \longmapsto \zeta_{d_i} z_1, \ z_2 \longmapsto \zeta_{d_i}^{-1} z_2.$$

This implies that p is H_i -equivariantly smoothable, a contradiction. Therefore we see that exceptional case (1) does not occur.

Using a similar argument we see that exceptional cases (2) and (3) do not occur, either.

Case (II): $i_1 \neq i_2, p \in \mathcal{N}_{i_1}^{(1)} \cap \mathcal{N}_{i_2}^{(1)}$. — We have two subcases:

- (i) G_p fixes i_1 and i_2 respectively.
- (ii) G_p exchanges i_1 with i_2 .

Subcase (i). — Let x (resp. y) be a local parameter on C_{i_1} (resp. on C_{i_2}) near p. Denoting by a_l the smallest positive integer such that $\gamma^{n_{i_l}a_l}(p) = p$ for l = 1, 2 (note that we necessarily have $n_{i_1}a_1 = n_{i_2}a_2$), then locally the action of $\gamma^{n_{i_l}a_l}$ around p is given by $(x, y) \mapsto (\zeta_{p,1}^{b_1}x, \zeta_{p,2}^{b_2}y)$ for some natural numbers b_1, b_2 , where $\zeta_{p,l}$ is a primitive $n''_{i_l}/(n_{i_l}a_l)$ -th root of unity and $n''_{i_l} := d/|G''_{i_l}|$ (we require that $b_1, b_2 \leq n''_{i_l}/(n_{i_l}a_l)$). The condition that (C, G, ρ) is non-smoothable implies that $\zeta_{p,1}^{b_1}\zeta_{p,2}^{b_2} \neq 1$.

By our assumption (1) we have $\overline{\beta^{n_{i_l}}|_{C_{i_l}}} \in \operatorname{Aut}(C_{i_l}) = \langle \overline{\gamma^{n_{i_l}}|_{C_{i_l}}} \rangle$, hence we get that $\overline{\beta^{n_{i_l}}|_{C_{i_l}}} = (\overline{\gamma^{n_{i_l}}})^{c_l}$ for some $0 \leq c_l < \operatorname{Ord}(H_{i_l})$. By Proposition 3.13 we have two possibilities:

- $\langle \overline{\beta^{n_{i_l}}}|_{C_{i_l}} \rangle = \operatorname{Aut}(C_{i_l})$, which is equivalent to $\operatorname{gcd}(c_l, \operatorname{Ord}(H_{i_l})) = 1$.
- We are in the exceptional cases where $\langle \overline{\beta^{n_{i_l}}|_{C_{i_l}}} \rangle$ is the index 2 subgroup of Aut (C_{i_l}) , and $c_l = 2c'_l$ for some $0 \leq c'_l < \operatorname{Ord}(H_{i_l})/2$ with $\operatorname{gcd}(c'_l, \operatorname{Ord}(H_{i_l})) = 1$.

The action of $\overline{\beta^{n_i}a_i}$ is given by $(x, y) \to (\zeta_{p,1}^{b_1c_1}x, \zeta_{p,2}^{b_2c_2}y)$. We see easily that p is non-smoothable for $(C_i, \langle \overline{\beta^{n_i}}|_{C_i}\rangle)$ iff $\zeta_{p,1}^{b_1c_1}\zeta_{p,2}^{b_2c_2} \neq 1$.

Subcase (ii). — Observe that i_1 and i_2 lie in the same orbit, hence we have $d_{i_1} = d_{i_2}$ and we may identify H_{i_1} with H_{i_2} . For any element in H_{i_1} fixing the node p, the induced monodromies on two branches of p are the same.

First we deduce a restriction on l_{p,i_1} , the length of the G_{i_1} -orbit of p. Assume $G_p = \langle \gamma^a \rangle$ for some a | d and consider the group $K_{p,i_1} := G_p \cap G_{i_1} = \langle \gamma^b \rangle$ for some b | d, which is the subgroup of G simultaneously fixing p and leaving C_{i_1} invariant. Regarding K_{p,i_1} as a subgroup of G_p , we should have b = 2a; on the other hand regarding K_{p,i_1} as a subgroup of G_{i_1} , we have $b = n_{i_1}l_{p,i_1}$. Hence we have $2a = n_{i_1}l_{p,i_1}$. if $2|l_{p,i_1}$, we see that $n_{i_1}|a$, which is absurd since in subcase (ii) G_p is not contained in G_{i_1} , therefore l_{p,i_1} must be an odd number.

If $\langle \overline{\beta^{n_{i_1}}}|_{C_{i_1}} \rangle = H_{i_1}$, then the same holds for C_{i_2} and for G_p . For the same reason as in case (I), p is non-smoothable for $(C, \langle \beta \rangle)$.

Now we consider the exceptional cases, where $\langle \overline{\beta^{n_{i_1}}|_{C_{i_1}}} \rangle \subset H_{i_1}$ is the unique index 2 subgroup of H_{i_1} (the same for C_{i_2}), let x (resp. y) be a local parameter on C_{i_1} (resp. on C_{i_2}) near p.

We discuss according to the monodromy of H_{i_1} at the node p, there are two cases:

- (a) The monodromy order is greater than or equal to 3, including the ramification points over P_3 , P_4 in the exceptional cases (1) (the case of ramification points over P_3 , P_4 in the exceptional cases (2) dose not occur because there P_3 or P_4 has inverse images of even length), locally the action of $G_p = \langle \gamma_a \rangle$ is $\gamma_a : (x, y) \mapsto (\zeta_{d_{i_1}} y, x)$, the action corresponding to $\langle \overline{\beta^{n_i}} \rangle$ is then $(x, y) \mapsto (\zeta_{d_{i_1}}^2 y, x)$. Since in the exceptional case (1) $d_{i_1} \ge 4$, we see that p is non-smoothable.
- (b) The monodromy order equals 2, and the length of the H_{i_1} -orbit of p is an odd number. This includes the following:
 - exceptional case (2), p is a ramification point over P_1 or P_2 . (note that in exceptional case (1), d' is even, hence the case that p is a ramification point over P_1, P_2 of exceptional case (1) does not occur.)
 - exceptional case (3), p is a ramification point.

With a similar argument as in (a), we see that p is smoothable.

DEFINITION 3.16. — We say a node p is of type E with respect to $\beta \in Aut(C)$, if it is in the case of (b) above.

Now we fix the local parameters for each nodes where subcase (II-i) happens, then we obtain an unordered pair $(\zeta_{p,1}^{b(p,1)}, \zeta_{p,2}^{b(p,2)})$ at each node p which is determined by γ . For any $\beta \in \operatorname{Aut}(C)$ with degree d such that $\dim \mathcal{T}_{\beta} = \dim \mathcal{T}_{C'}$, we get a pair of integers $(c(\beta, p, 1), c(\beta, p, 2))$ at each node.

Combining with the previous argument, we obtain our main theorem:

THEOREM 3.17. — Under the conditions of Assumption 3.6, we have the following:

- (1) For a G-equivariantly non-smoothable G-marked stable curve $(C = \sum_{i \in I} C_i, G, \rho)$, the induced stratum $\mathfrak{M}_{C'}$, where C' = C/G, is maximal iff for a general stable curve (by abuse of notation we denote still by) (C, G, ρ) in the stratum:
 - (a) The cases in Lemma 3.10 do not occur.
 - (b) For any β ∈ Aut(C) (of order d) and any node p where Case (II-i) happens, the following holds:

$$\zeta_{p,1}^{b(p,1)c(\beta,p,1)}\zeta_{p,2}^{b(p,2)c(\beta,p,2)} \neq 1.$$

- (c) For any $\beta \in Aut(C)$ (of order d), there is no node p of type E with respect to β .
- (2) The Zariski closure of each maximal stratum in (1) is an irreducible component of $(\overline{\mathfrak{M}_q} \mathfrak{M}_q)(G)$.

Remark 3.18. — The above argument also shows that, unlike in the smooth case (cf. [4, Theorem 1] and [1, Theorem 3.4]), for stable curves, usually a component corresponds to more than one numerical types.

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