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## GIACOMO CANEVARI AND GIANDOMENICO ORLANDI Topological singularities for vector-valued Sobolev maps and applications

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## Topological singularities for vector-valued Sobolev maps and applications

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**ABSTRACT.** — We review the analysis of topological singularities of Sobolev maps into manifolds and their applications to variational problems of Ginzburg–Landau type and to the lifting problem for BV maps into manifolds. We describe in particular recent results obtained in the vector-valued case related to variational models of material science, more precisely the Landau–de Gennes model.

**RÉSUMÉ.** — Nous passons en revue certains résultats d'analyse des singularités topologiques des fonctions de Sobolev à valeurs dans des variétés, ainsi que leurs applications aux problèmes variationnels de type Ginzburg–Landau et au problème du relèvement dans l'espace BV. En particulier, nous présentons des résultats récents, portant sur les fonctions à valeurs vectorielles, qui trouvent leur application dans l'étude des modèles variationnels pour la science des matériaux, tels que le modèle de Landau–de Gennes.

### 1. Topological singularities of Sobolev maps into spheres

### 1.1. Motivating example: the Ginzburg–Landau energy

Consider the Ginzburg–Landau functional

$$u \in W^{1,2}(\Omega, \mathbb{C}) \mapsto E_{\varepsilon}^{\mathrm{GL}}(u) := \int_{\Omega} \left\{ \frac{1}{2} \left| \nabla u \right|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 \right\}, \quad (\mathrm{GL}_{\varepsilon})$$

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where  $\Omega$  is a smooth, bounded domain in  $\mathbb{R}^d$ ,  $d \ge 2$ , and  $\varepsilon > 0$  is a small parameter. Functionals of this form arise as variational models for the study of type-II superconductivity. In this context, u(x) represents the magnetisation vector at a point  $x \in \Omega$  and the energy favours configurations with |u(x)| = 1, which have a well-defined direction of magnetisation as opposed to the non-superconducting phase u = 0. Let  $\mathbb{S}^1$  denote the unit circle in the complex plane  $\mathbb{C}$ .

A rigorous mathematical analysis of the model, since the fundamental monograph by Bethuel, Brezis and Hélein [14], show that minimisers  $u_{\varepsilon}$  subject to a ( $\varepsilon$ -independent) boundary condition  $u_{\varepsilon|\partial\Omega} = u_{\rm bd} \in W^{1/2,2}(\partial\Omega, \mathbb{S}^1)$ satisfy the (sharp) energy bound  $E_{\varepsilon}^{\rm GL}(u_{\varepsilon}) \leq C|\log \varepsilon|$  for some  $\varepsilon$ -independent constant C. In particular,  $u_{\varepsilon}$  takes on average values closer and closer to  $\mathbb{S}^1$ as  $\varepsilon$  tends to 0, since  $\int_{\Omega} (1 - |u_{\varepsilon}|^2)^2 \leq C\varepsilon^2 |\log \varepsilon|$ . Despite the lack of uniform energy bounds, under suitable conditions on  $u_{\rm bd}$ , minimisers  $u_{\varepsilon}$  converge to a limit map  $u_0: \Omega \to \mathbb{S}^1$ , which is smooth except for a singular set of codimension two (see e.g. [2, 14, 15, 18, 50, 52, 65, 66]). Moreover, the singular set of  $u_0$  is itself a minimiser (in a suitable sense) of some "weighted area" functional. The emergence of singularities in the limit map  $u_0$  is related to topological obstructions, which may prevent the existence of a map in  $W^{1,2}(\Omega, \mathbb{S}^1)$  that satisfies the boundary conditions.

It should be remarked that the logarithmic energy bound

$$E_{\varepsilon}^{\mathrm{GL}}(v_{\varepsilon}) \leqslant C \left|\log \varepsilon\right|$$

does not guarantee compactness of the sequence  $(v_{\varepsilon})_{\varepsilon>0}$ , in any Sobolev norm. Indeed, the maps  $v_{\varepsilon}(x) := \exp(i\varphi(x)|\log \varepsilon|^{1/2})$ , where  $\varphi \in C^{\infty}(\Omega, \mathbb{R})$ is a fixed, non-constant function, satisfy  $|v_{\varepsilon}| = 1$  and

$$E_{\varepsilon}^{\mathrm{GL}}(v_{\varepsilon}) = \frac{\left|\log \varepsilon\right|}{2} \int_{\Omega} \left|\nabla \varphi\right|^{2} \leqslant C \left|\log \varepsilon\right|,$$

but  $|\nabla v_{\varepsilon}| = O(|\log \varepsilon|^{1/2})$  so the gradient diverges as  $\varepsilon \to 0$ . Actually, even for energy minimisers, no compactness can be expected even in  $L^1_{\text{loc}}$  (unless additional assumptions on the boundary datum are made). Indeed, Brezis and Mironescu [24] constructed a sequence of minimisers  $u_{\varepsilon_n}$ , on the unit ball  $B^d \subseteq \mathbb{R}^d$  with  $d \ge 2$ , that satisfies  $E^{\text{GL}}_{\varepsilon_n}(u_{\varepsilon_n}) = o(|\log \varepsilon_n|)$  as  $\varepsilon_n \to 0$ and yet has no subsequence that converges a.e. on a set of positive measure, as there holds  $\sup_{x \in B^d} |u_{\varepsilon_n}(x) - \exp(inx_1)| \to 0$ .

In the previous examples, the lack of compactness is due to *oscillations of* the phase and not to topological obstructions. In fact, it is possible to isolate the topological contribution to the energy and prove compactness results on that part alone. This is usually achieved by the use of distributional Jacobians.

### 1.2. Distributional Jacobian

Let  $d \ge k \ge 2$  be integers. Given a map  $u \colon \mathbb{R}^d \to \mathbb{R}^k$ ,  $u = (u^1, \ldots, u^k)$ , of class  $C^2$ , we compute that

$$\mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^k = \frac{1}{k} \mathrm{d}\left(\sum_{i=1}^k (-1)^{i+1} u^i \widehat{\mathrm{d}u^i}\right),\tag{1.1}$$

where  $\widehat{\mathrm{d}u^i} := \mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^{i-1} \wedge \mathrm{d}u^{i+1} \wedge \ldots \wedge \mathrm{d}u^k$ . In case d = k, we can rewrite the identity (1.1) using vector calculus instead of differential forms. More precisely, when d = k = 2 we have

$$\det(\nabla u) = \frac{1}{2}\partial_1\left(u^1\partial_2 u^2 - u^2\partial_2 u^1\right) + \frac{1}{2}\partial_2\left(u^2\partial_1 u^1 - u^1\partial_1 u^2\right)$$

and if d = k = 3 then

$$\det(\nabla u) = \frac{1}{3}\operatorname{div}(u \cdot \partial_2 u \times \partial_3 u, \, u \cdot \partial_3 u \times \partial_1 u, \, u \cdot \partial_1 u \times \partial_2 u)$$

Similar (but more involved) reformulations are possible if d = k > 3.

The left-hand side of (1.1) is well-defined for any  $u \in W^{1,k}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$ , while the right-hand side is well-defined (in the sense of distributions) if  $u \in (L^{\infty} \cap W^{1,k-1}_{\text{loc}})(\mathbb{R}^d, \mathbb{R}^k)$ . Therefore, we might use the right-hand side of (1.1) to define the *distributional Jacobian* of u:

$$\mathbf{J}u := \frac{1}{k} \mathbf{d} \left( \sum_{i=1}^{k} (-1)^{i+1} u^i \widehat{\mathbf{d}u^i} \right) \quad \text{for } u \in (L^{\infty} \cap W^{1,k-1})(\mathbb{R}^d, \mathbb{R}^k).$$
(1.2)

The rôle of the distributional Jacobian in connection with relaxation problems in the calculus of variations has been pointed out, for instance, by Ball [6] (distributional determinant in non-linear elasticity) and by Brezis, Coron and Lieb [23] (harmonic maps and minimal connections; see also Bethuel, Brezis and Coron [13]).

As a consequence of its definition (1.2), the Jacobian enjoys weak compactness properties. For instance, if  $(u_j)_{j \in \mathbb{N}}$  is a sequence of maps such that

$$\sup_{j\in\mathbb{N}}\|u_j\|_{L^{\infty}(\mathbb{R}^d,\mathbb{R}^k)}<+\infty$$

and  $u_j \to u$  strongly in  $W^{1,k-1}(\mathbb{R}^d,\mathbb{R}^k)$ , (1.3)

then  $Ju_j \rightharpoonup Ju$  in the distributional sense of  $\mathscr{D}'(\mathbb{R}^d)$ . The same conclusion holds if

$$\sup_{j \in \mathbb{N}} \|u_j\|_{L^{\infty}(\mathbb{R}^d, \mathbb{R}^k)} < +\infty$$
  
and  $u_j \to u$  weakly in  $W^{1,p}(\mathbb{R}^d, \mathbb{R}^k)$  for  $p > k - 1$ . (1.4)

A quantitative continuity estimate for the Jacobian was proved by Brezis and Nguyen.

THEOREM 1.1 ([25, Theorem 1]). — Let  $d = k \ge 2$ ,  $k - 1 \le p \le +\infty$ , and let  $1 \le q \le +\infty$  be such that (k - 1)/p + 1/q = 1. Then, for any u,  $v \in (L^q \cap W^{1,p}_{\text{loc}})(\mathbb{R}^k, \mathbb{R}^k)$  and any  $C^1$  function  $\varphi \colon \mathbb{R}^k \to \mathbb{R}$  supported in a ball  $B \subseteq \mathbb{R}^k$ , there holds

$$|\langle \mathbf{J}u - \mathbf{J}v, \varphi \rangle| \leqslant C \, \|u - v\|_{L^q(B)} \Big( \|\nabla u\|_{L^p(B)}^{k-1} + \|\nabla v\|_{L^p(B)}^{k-1} \Big) \|\nabla \varphi\|_{L^\infty(B)}$$

where C > 0 is a constant that only depends on k.

Another important feature of the Jacobian is its ability to capture topological information. To understand why this is the case, we introduce the (k-1)-form

$$\omega_{\mathbb{S}^{k-1}}(y) := \frac{1}{k} \sum_{i=1}^{k} (-1)^{i+1} y^i \widehat{\mathrm{d}y^i} \qquad \text{for } y \in \mathbb{R}^k,$$

which is (the 1-homogeneous extension of) a volume form on  $\mathbb{S}^{k-1}$ . The cohomology class of  $\omega_{\mathbb{S}^{k-1}}$ , restricted to  $\mathbb{S}^{k-1}$ , generates the de Rham cohomology  $H^{k-1}_{d\mathbb{R}}(\mathbb{S}^{k-1}) \simeq \mathbb{R}$ , as a real vector space. Then, we may rewrite (1.2) as

$$\mathbf{J}u = \mathbf{d}u^*(\omega_{\mathbb{S}^{k-1}}). \tag{1.5}$$

Consider now a sphere-valued map  $u: \mathbb{R}^k \to \mathbb{S}^{k-1}$ , possibly with point singularities (e.g. u(x) := x/|x|), and let  $B \subseteq \mathbb{R}^k$  be a ball whose boundary  $\partial B$  does not intersect any singularity of u. By formally integrating the identity (1.5) on B and applying Stokes' theorem, we obtain

$$\int_{B} \mathbf{J}u = \int_{B} \mathbf{d}u^{*}(\omega_{\mathbb{S}^{k-1}}) = \int_{\partial B} u^{*}(\omega_{\mathbb{S}^{k-1}}) = \alpha_{k} \operatorname{deg}(u, \partial B),$$

where  $\alpha_k = \operatorname{Vol}(\mathbb{S}^{k-1})/k$  is the volume of the unit ball of  $\mathbb{R}^k$  and deg $(u, \partial B)$  denotes the topological degree of  $u_{|\partial B} : \partial B \to \mathbb{S}^{k-1}$ . More precisely, we have the following property which was proven in [23]: suppose that  $u \in W_{\operatorname{loc}}^{1,k-1}(\mathbb{R}^k, \mathbb{S}^{k-1})$  is smooth except for a finite number of points  $x_1, \ldots, x_p$ . Then, there holds

$$\mathbf{J}u = \alpha_k \sum_{i=1}^p d_i \delta_{x_i} \qquad \text{in } \mathscr{D}'(\mathbb{R}^k), \tag{1.6}$$

where  $d_i \in \mathbb{Z}$  denotes the topological degree of u restricted to a small sphere around the point  $x_i$ . To prove this formula, one can approximate u with a sequence of smooth maps  $u_{\varepsilon} \colon \mathbb{R}^k \to \mathbb{R}^k$  such that  $u = u_{\varepsilon}$  out of small balls  $B_{\varepsilon}(x_i)$  around the singularities. By constructing suitable approximations, one can compute  $Ju_{\varepsilon}$  using Stokes' theorem as above, and make sure that  $u_{\varepsilon} \to u$  strongly in  $W^{1,k-1}$ , so to pass to the limit using the continuity of J, (1.3). We refer the reader to [21] and references therein for a comprehensive treatment of the relation between the jacobian and the topological degree.

In a similar spirit, if  $d \ge k \ge 2$  and  $u \in W^{1,k-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$  is smooth out of a smoothly embedded, closed, oriented (d-k)-manifold  $M \subseteq \mathbb{R}^d$ , then the distributional Jacobian Ju may be identified with a vector-valued measure supported on M. Indeed, we have (see [51])

$$\star \operatorname{J} u = \alpha_k \Delta \boldsymbol{\tau}_M \mathscr{H}^{d-k} \sqcup M, \tag{1.7}$$

where  $\Delta$  is an integer number and denotes the topological degree of u restricted to the boundary of a k-disk that intersects transversally M, while  $\tau_M$  is a unit, tangent (d-k)-vector field that orients M. Moreover,

$$\star \colon \Lambda^k \mathbb{R}^d \to \Lambda_{d-k} \mathbb{R}^d$$

is (a variant of) the Hodge star duality operator: for a k-covector  $\omega$ ,  $\star \omega$  is defined as the unique (d - k)-vector such that

$$\langle \tau, \star \omega \rangle = \langle \omega \wedge \tau, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_d \rangle$$
 for any  $(d-k)$ -covector  $\tau$ ,

where  $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$  is a positively oriented, orthonormal basis for  $\mathbb{R}^d$ . If u is smooth and x is a regular point for u (that is, the gradient  $\nabla u(x) \colon \mathbb{R}^d \to \mathbb{R}^k$  is surjective), then the level set  $u^{-1}(u(x))$  is locally a smooth (d-k)submanifold, and  $\star Ju(x)$  is a simple (d-k)-vector that spans the tangent space to  $u^{-1}(u(x))$  at x.

We will come back to the link between Jacobian and level sets, which is made precise by the coarea formula, in Section 1.3 below. For the time being, we consider an example. Let  $u: \mathbb{R}^k \to \mathbb{S}^{k-1}$  be defined by u(x) := x/|x|for  $x \in \mathbb{R}^k \setminus \{0\}$ . This map has an isolated singularity at the origin, which coincides with the support of the distributional Jacobian by (1.6), but is also the boundary of any level set  $u^{-1}(y)$ , for  $y \in \mathbb{S}^{k-1}$ . This is no coincidence, and in fact the distributional Jacobian of  $u \in W_{\text{loc}}^{1,k-1}(\mathbb{R}^d, \mathbb{S}^{k-1})$ , for  $d \ge k$ , may be characterised as the boundary of a generic level set  $u^{-1}(y)$ , for  $y \in \mathbb{S}^{k-1}$  [1, Theorem 3.8]. This fact, combined with the boundary rectifiability theorem by Federer and Fleming [37], implies the following rectifiability result: if  $d \ge k \ge 2$ ,  $u \in W_{\text{loc}}^{1,k-1}(\mathbb{R}^d, \mathbb{S}^{k-1})$ , and *if* Ju is a bounded measure, then Ju may be written in the form (1.7), where M is a (d - k)rectifiable set with orientation  $\tau_M$ , and  $\Delta$  is an integer-valued multiplicity function (see [1, Theorem 5.6] and [51, Theorem 1.1]).

### 1.3. The oriented coarea formula

In Section 1.2, we have described some properties of the distributional Jacobian of a sphere-valued map  $u \in W_{loc}^{1,k-1}(\mathbb{R}^d, \mathbb{S}^{k-1})$ . It turns out that

the study of Ju when  $u \in (L^{\infty} \cap W^{1,k-1}_{\text{loc}})(\mathbb{R}^d, \mathbb{R}^k)$  can be reduced to the previous case. Indeed, for  $y \in \mathbb{R}^k$ , we define the map  $u_y \colon \mathbb{R}^d \to \mathbb{S}^{k-1}$  by

$$u_y(x) := \frac{u(x) - y}{|u(x) - y|} \quad \text{for } x \in \mathbb{R}^d \setminus u^{-1}(y).$$

If u is smooth and y is a regular value of u, then by the discussion of Section 1.2 we might expect  $Ju_y$  to be a unit multiplicity rectifiable current supported on the smooth (d - k)-manifold  $u^{-1}(y)$ . The following property, sometimes referred to as the *oriented coarea formula*, relates  $Ju_y$  and Ju.

THEOREM 1.2 ([51, Theorem 1.2], [1]). — Let  $d \ge k \ge 2$  and  $u \in (L^{\infty} \cap W^{1,k-1}_{\text{loc}})(\mathbb{R}^d, \mathbb{R}^k)$ . Then, for a.e.  $y \in \mathbb{R}^k$  we have  $u_y \in W^{1,k-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{S}^{k-1})$ ,  $Ju_y$  is supported on a (d-k)-rectifiable set, and there holds

$$\mathbf{J}u = \frac{1}{\alpha_k} \int_{\mathbb{R}^k} \mathbf{J}u_y \,\mathrm{d}y$$

in the sense of distributions. Here  $\alpha_k$  denotes the volume of the unit ball in  $\mathbb{R}^k$ .

To pave the way for the discussion in Section 2, it will be useful to recall here why we have  $u_y \in W_{\text{loc}}^{1,k-1}(\mathbb{R}^d, \mathbb{S}^{k-1})$  for a.e.  $y \in \mathbb{R}^k$ . This proof is based on a trick that was used by Hardt, Kinderlehrer and Lin [42, Lemma 2.3]. The chain rule implies that  $|\nabla u_y| \leq 2|u - y|^{-1}|\nabla u|$ . W.l.o.g., we might restrict our attention to the case  $|y| \leq M := ||u||_{L^{\infty}(\mathbb{R}^d)} + 1$ . By integrating over y in the ball  $B_M^k \subseteq \mathbb{R}^k$  of radius M, and letting  $B^d \subseteq \mathbb{R}^d$  be a ball, we obtain

$$\begin{split} \int_{B_M^k} \|\nabla u_y\|_{L^{k-1}(B^d)}^{k-1} \, \mathrm{d}y &\leq 2 \int_{B_M^k} \left( \int_{B^d} \frac{|\nabla u(x)|^{k-1}}{|u(x) - y|^{k-1}} \, \mathrm{d}x \right) \mathrm{d}y \\ &= 2 \int_{B^d} |\nabla u(x)|^{k-1} \left( \int_{B_M^k} \frac{\mathrm{d}y}{|u(x) - y|^{k-1}} \right) \mathrm{d}x \\ &\leq 2 \int_{B^d} |\nabla u(x)|^{k-1} \left( \int_{B_{2M}^k} \frac{\mathrm{d}z}{|z|^{k-1}} \right) \mathrm{d}x \\ &=: C_{k,M} \|\nabla u\|_{L^{k-1}(B^d)}^{k-1} \end{split}$$

We have made the change of variable z = u(x) - y in the inner integral, and used the fact that  $z \mapsto |z|^{-p}$  is locally integrable on  $\mathbb{R}^k$  for p < k. The constant  $C_{k,M}$  depends also on M, hence on  $||u||_{L^{\infty}(\mathbb{R}^d)}$ .

### 1.4. Applications to variational problems

The theory of distributional Jacobians can be applied to the asymptotic analysis, as  $\varepsilon \to 0$ , of variational problems of the form  $(\operatorname{GL}_{\varepsilon})$ . Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded, Lipschitz domain. For  $1 \leq p < +\infty$ , we define  $W^{-1,p}(\Omega, \Lambda_{d-2}\mathbb{R}^d)$ as the dual of  $W_0^{1,p'}(\Omega, \Lambda^{d-2}\mathbb{R}^d)$ , where p' := p/(p-1) is the Hölder conjugate of p. We have

THEOREM 1.3 ([2, 50]). — Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded, Lipschitz domain with  $d \ge 2$ , and let K > 0 be a fixed constant. Then, the following properties hold.

(1) Compactness and lower bound. For any sequence  $u_{\varepsilon} \in W^{1,2}(\Omega, \mathbb{C})$ such that  $E_{\varepsilon}^{\mathrm{GL}}(u_{\varepsilon}) \leq K |\log \varepsilon|$ , there exists a (non relabelled) subsequence and a (d-2)-current J such that  $\star Ju_{\varepsilon} \to \pi J$  in  $W^{-1,p}$  for every p < d/(d-1). The current J has the structure of a (d-2)rectifiable boundary in  $\Omega$  with finite mass  $|J|(\Omega) < +\infty$  and integer multiplicity. Moreover,

$$\liminf_{\varepsilon \to 0} \frac{E_{\varepsilon}^{\mathrm{GL}}(u_{\varepsilon})}{|\log \varepsilon|} \ge \pi |J|(\Omega).$$

(2) Upper bound. For any (d-2)-rectifiable boundary J in  $\Omega$  with finite mass and integer multiplicity, there exists a sequence  $u_{\varepsilon} \in W^{1,2}(\Omega, \mathbb{C})$  such that  $\star Ju_{\varepsilon} \to \pi J$  in  $W^{-1,p}$  for every p < d/(d-1)and

$$\lim_{\varepsilon \to 0} \frac{E_{\varepsilon}^{\mathrm{GL}}(u_{\varepsilon})}{|\log \varepsilon|} = \pi |J|(\Omega).$$

If the  $u_{\varepsilon}$ 's are critical points of  $E_{\varepsilon}^{\text{GL}}$  with  $E_{\varepsilon}^{\text{GL}}(u_{\varepsilon}) \leq K |\log \varepsilon|$ , and under suitable assumptions on the boundary data, the bounds on  $Ju_{\varepsilon}$  make it possible to obtain compactness for the  $u_{\varepsilon}$ 's themselves, by PDE arguments [15]. In this case, we have  $u_{\varepsilon} \to u_0$  in  $W^{1,p}$  for p < d/(d-1), and  $\pi J = \lim_{\varepsilon} \star Ju_{\varepsilon} = \star Ju_0$ .

Ginzburg–Landau type functionals of k-growth in the gradient (i.e., the term  $|\nabla u|^2$  in  $(\operatorname{GL}_{\varepsilon})$  is replaced by  $|\nabla u|^k$ , with  $k \ge 2$  an integer) and Dirichlet boundary conditions have also been studied [2]. In this case, the  $Ju_{\varepsilon}$ 's concentrate on a rectifiable set of codimension k, whose cobordism class is determined by the domain and the boundary condition. Other energy regimes arise naturally for Ginzburg–Landau type functionals and are interesting for applications. In particular the energy regime  $E_{\varepsilon}(u_{\varepsilon}) \approx |\log \varepsilon|^2$  corresponds to the onset of the mixed phase in type-II superconductors, and to the appearance of vortices in Bose–Einstein condensates. These situations have been

extensively studied in the two-dimensional case, especially by Sandier and Serfaty in the case of superconductivity (see [66] and references therein).

From a variational viewpoint, Theorem 1.3 shows that the Ginzburg– Landau functional itself can be considered an approximation of a (d-2)dimensional "weighted area" functional (see also [2, 3, 50, 66]). Therefore, the Ginzburg–Landau functional and its variants have been proposed as tools to construct "weak minimal surfaces" or, more precisely, stationary varifolds of codimension greater than one [5, 15, 53, 64, 67].

# 2. Manifold-valued Sobolev maps, topological singularities and applications

### 2.1. Motivation: variational problems for material science

There are other functionals, arising as variational models for material science, which share a common structure with the Ginzburg–Landau functional ( $GL_{\varepsilon}$ ), i.e. they can be written in the form

$$u \in W^{1,k}(\Omega, \mathbb{R}^m) \mapsto E_{\varepsilon}(u) := \int_{\Omega} \left\{ \frac{1}{k} \left| \nabla u \right|^k + \frac{1}{\varepsilon^2} f(u) \right\}.$$
 (2.1)

Here  $f: \mathbb{R}^m \to \mathbb{R}$  is a non-negative, smooth potential that satisfies suitable coercivity and non-degeneracy conditions, and  $\mathscr{N} := f^{-1}(0)$  is assumed to be a non-empty, smoothly embedded, compact, connected submanifold of  $\mathbb{R}^m$ without boundary. The elements of  $\mathscr{N}$  correspond to the ground states for the material, i.e. the local configurations that are most energetically convenient. An important example is the Landau–de Gennes model for nematic liquid crystals (in the so-called one-constant approximation of the uniaxial phase, see e.g. [39]). In this case, k = 2 and the distinguished manifold is a real projective plane  $\mathscr{N} = \mathbb{R}P^2$ , whose elements describe the locally preferred direction of alignment of the constituent molecules (which might be schematically described as un-oriented rods).

Minimisers of (2.1) subject to a boundary condition

$$u_{\mid \partial \Omega} = v \in W^{1-1/k,k}(\partial \Omega, \mathscr{N})$$

may not satisfy uniform energy bounds, due to topological obstructions carried by the boundary datum v. When this phenomenon occurs, the energy of minimisers is of order  $|\log \varepsilon|$  (see e.g. [14, 20, 65] in case k = 2,  $\mathcal{N} = \mathbb{S}^1$ ). A similar phenomenon arises for tangent vector fields on a closed manifold, due to the Poincaré-Hopf theorem (see e.g. [46]). Based on the analogy with the Ginzburg-Landau case (see e.g. [14, 15, 52, 66]), under suitable conditions on  $\mathscr{N}$  we expect the energy of minimisers to concentrate, to leading order, on a (d - k)-dimensional surface. Indeed, energy concentration results have been established for Landau–de Gennes minimisers [9, 26, 27, 33, 41, 48, 49, 54, 62]. To our best knowledge, minimisers of functionals associated with more general manifolds  $\mathscr{N}$ , in the logarithmic energy regime, have been studied only in case d = k = 2 so far [26, 58].

Unfortunately, the theory of Jacobians does not carry over directly to this setting. Consider the following example: let S be a (d-k)-plane that intersects  $\Omega$ , and let  $u: \Omega \setminus S \to \mathscr{N}$  be a map that is smooth everywhere, except at S. Then, each point of S can be encircled by a (k-1)-dimensional sphere  $\Sigma \subseteq \Omega \setminus S$ , contained in the k-plane orthogonal to S. The (based) homotopy class of  $u_{|\Sigma}: \Sigma \to \mathscr{N}$  defines an element of  $\pi_{k-1}(\mathscr{N})$  which, roughly speaking, characterises the behaviour of the material configuration around the defect. (This is the basic idea of the topological classification of defects in ordered materials; see e.g. [55] for more details.) If  $\pi_{k-1}(\mathscr{N})$  contains elements of finite order, these cannot be realised by integration of a differential form. Indeed, suppose that the homotopy class of  $u_{|\Sigma}$  has finite order q > 1in  $\pi_{k-1}(\mathscr{N})$ . Let  $[\Sigma]$  be a generator for  $H_{k-1}(\Sigma) \simeq \mathbb{Z}$ , corresponding to a choice of the orientation. By the Hurewicz homomorphism, the homology class  $u_*[\Sigma] \in H_{k-1}(\mathscr{N})$  satisfies  $qu_*[\Sigma] = 0$ . As a consequence, for any closed differential (k-1)-form  $\omega$  in  $\mathscr{N}$ , we have

$$\int_{\Sigma} u^*(\omega) = \int_{u_*[\Sigma]} \omega = q^{-1} \int_{qu_*[\Sigma]} \omega = 0$$

and hence, no notion of Jacobian that can be expressed as a differential form (such as (1.2)) is able to capture such homotopy classes of defects.

In the following sections, our aim is to construct an object that (i) brings topological information and (ii) enjoys compactness properties even when the distributional Jacobian is not defined, in particular when  $\pi_{k-1}(\mathcal{N})$  contains elements of finite order. A notion of "set of topological singularities" for a manifold-valued Sobolev map was already introduced by Pakzad and Rivière [63], using the language of flat chains. In the context of manifoldconstrained problems, the use of flat chains with coefficients in an abelian group traces its roots back in the earlier literature on the subject: the very notion of "minimal connection", introduced by Brezis, Coron and Lieb [23], can be interpreted as the flat norm of the distributional Jacobian. Very roughly, flat chains can be thought of as sets equipped with multiplicities, which belong to a given coefficient group. Pakzad and Rivière first defined the topological singular set  $\mathbf{S}^{\mathrm{PR}}(u)$  of a map  $u: \Omega \to \mathcal{N}$  that is discontinuous on a polyhedral set of dimension d-k at most. In this case,  $\mathbf{S}^{\mathrm{PR}}(u)$  was defined as a (d-k)-dimensional flat chain with multiplicities in  $\pi_{k-1}(\mathcal{N})$ , supported by the polyhedral set where u is discontinuous. Maps with polyhedral singularities as above are dense in  $W^{1,k-1}(\Omega, \mathcal{N})$  [12, Theorem 2]. Then, Pakzad and Rivière were able to define  $\mathbf{S}^{PR}(u)$  for any  $u \in W^{1,k-1}(\Omega, \mathcal{N})$  by a density argument, using deep results from Geometric Measure Theory and a refined topological construction (a variant of the "dipole insertion" proposed in [13]).

In [28], we carry out a different construction. In a first step, we consider a smooth (not necessarily  $\mathscr{N}$ -valued) map  $u: \Omega \to \mathbb{R}^m$  and define a family of flat chains  $\mathbf{S}_y(u)$ , depending on a parameter  $y \in \mathbb{R}^m$ . We prove a continuity estimate, which allows us to define  $\mathbf{S}_y(u)$  for any  $u \in (L^{\infty} \cap W^{1,k-1})(\Omega, \mathbb{R}^m)$ , by density; the arguments rely essentially on projection "à la Hardt, Kinderlehrer and Lin" ([42], see Section 1.3) and the coarea formula. Although the construction is different, we recover Pakzad and Rivière's operator  $\mathbf{S}^{\mathrm{PR}}$  in case u is  $\mathscr{N}$ -valued; more precisely, if  $u \in W^{1,k-1}(\Omega, \mathscr{N})$ , then  $\mathbf{S}^{\mathrm{PR}}(u) = \mathbf{S}_u(u)$  for a.e. y of norm small enough.

Before giving a few more details on the construction of [28], let us recall some basic definitions and facts about flat chains, following the approach in [38, 69, 70].

### 2.2. Flat chains with coefficients in an abelian group

Let  $(\mathbf{G}, |\cdot|)$  be a normed abelian group, that is, an abelian group together with a non-negative function  $|\cdot|: \mathbf{G} \to [0, +\infty)$  that satisfies

- (1) |g| = 0 if and only if g = 0
- (2) |-g| = |g| for any  $g \in \mathbf{G}$
- (3)  $|g+h| \leq |g| + |h|$  for any  $g, h \in \mathbf{G}$ .

In addition, we assume that there exists a constant c > 0 such that

$$|g| \ge c \qquad \text{for any } \mathbf{G} \setminus \{0\}. \tag{2.2}$$

For  $n \in \mathbb{Z}$ ,  $1 \leq n \leq d$ , a polyhedral *n*-chain with coefficients in **G** is a linear combination, with coefficients in **G**, of compact, convex, oriented *n*-dimensional polyhedra in  $\mathbb{R}^d$ , modulo a suitable equivalence relation  $\sim$ . We define  $\sim$  by requiring  $-\sigma \sim \sigma'$  if the polyhedra  $\sigma'$  and  $\sigma$  only differ for the orientation, and  $\sigma \sim \sigma_1 + \sigma_2$  if  $\sigma$  is obtained by gluing  $\sigma_1$ ,  $\sigma_2$  along a common face (with the correct orientation). The set of polyhedral *n*-chains with coefficients in **G** is a group, with a naturally defined addition operation, and is denoted  $\mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$ . Every element  $S \in \mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$  can be represented as a finite sum

$$S = \sum_{i=1}^{p} g_i \llbracket \sigma_i \rrbracket, \qquad (2.3)$$

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where  $g_i \in \mathbf{G}$ , the  $\sigma_i$ 's are compact, convex, non-overlapping *n*-dimensional polyhedra, and  $\llbracket \cdot \rrbracket$  denotes the equivalence class modulo the relation  $\sim$  defined above. Thus, *S* may be identified with a finite collection of polyhedra as above, endowed with multiplicities in  $\mathbf{G}$ .

Polyhedral chains enjoy a notion of boundary: the boundary is a linear operator  $\partial \colon \mathbb{P}_n(\mathbb{R}^d; \mathbf{G}) \to \mathbb{P}_{n-1}(\mathbb{R}^d; \mathbf{G})$ , identified by its actions on single polyhedra, which satisfies  $\partial(\partial S) = 0$  for any chain S. The mass of a polyhedral chain  $S \in \mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$ , presented in the form (2.3), is defined by  $\mathbb{M}(S) := \sum_i |g_i| \mathscr{H}^n(\sigma_i)$ . The flat norm of a polyhedral *n*-dimensional chain S is defined by

$$\mathbb{F}(S) := \inf \left\{ \mathbb{M}(P) + \mathbb{M}(Q) \colon \begin{array}{c} P \in \mathbb{P}_{n+1}(\mathbb{R}^d; \mathbf{G}), \ Q \in \mathbb{P}_n(\mathbb{R}^d; \mathbf{G}), \\ S = \partial P + Q \end{array} \right\}$$

Thus, two chains  $S_1$ ,  $S_2$  are close with respect to the flat norm if  $S_2 - S_1$  is, up to small errors, the boundary of a chain of small mass. It can be showed (see e.g. [38, Section 2]) that  $\mathbb{F}$  indeed defines a norm on  $\mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$ , in such a way that the group operation on  $\mathbb{P}_n(\mathbb{R}^d; \mathbf{G})$  is  $\mathbb{F}$ -Lipschitz continuous. The completion of  $(\mathbb{P}_n(\mathbb{R}^d; \mathbf{G}), \mathbb{F})$  as a metric space will be denoted  $\mathbb{F}_n(\mathbb{R}^d; \mathbf{G})$ . It can be given the structure of a  $\mathbf{G}$ -module, and it is called the group of flat *n*-chains with coefficients in  $\mathbf{G}$ . Moreover, the mass  $\mathbb{M}$  extends to a  $\mathbb{F}$ -lower semi-continuous functional  $\mathbb{F}_n(\mathbb{R}^d; \mathbf{G}) \to [0, +\infty]$ , still denoted  $\mathbb{M}$ , and it remains true that

$$\mathbb{F}(S) = \inf \left\{ \mathbb{M}(P) + \mathbb{M}(Q) \colon \frac{P \in \mathbb{F}_{n+1}(\mathbb{R}^d; \mathbf{G}), \ Q \in \mathbb{F}_n(\mathbb{R}^d; \mathbf{G}), \\ S = \partial P + Q \right\}$$
(2.4)

for any  $S \in \mathbb{F}_n(\mathbb{R}^d; \mathbf{G})$  [38, Theorem 3.1].

A flat chain S is said to be supported in a closed set  $K \subseteq \mathbb{R}^d$  if, for any open set  $U \supseteq K$ , S is the  $\mathbb{F}$ -limit of a sequence of polyhedral chains supported in U. If M is a smooth n-dimensional manifold, respectively a nrectifiable set, then we can define a chain [M] supported on M with constant multiplicity  $1 \in \mathbf{G}$  by approximating M with polyhedral sets, considering the associated polyhedral chains (with unit multiplicity), and passing to the limit in the flat norm. The chain [M] is an example of a *smooth*, respectively, *rectifiable* chain. More generally, Equation (2.4) shows that the boundary of a n-rectifiable chain of finite mass is a (n - 1)-flat chain; for instance, the "Koch's snowflake", which is a planar set of Hausdorff dimension greater than 1 that bounds a finite area, can be seen as the support of a 1-dimensional flat chain. In fact, under the assumption (2.2), any (n-1)-flat chain has the form (boundary of a rectifiable n-chain) + (rectifiable (n-1)-chain) [38, 69].

In case  $\mathbf{G} = \mathbb{Z}$ , rectifiable chains may be identified with rectifiable currents with integer multiplicity, by integration. The class of *n*-chains of finite

mass with coefficients in  $\mathbb{Z}$  may be interpreted as bounded measures with values in the space of *n*-vectors, and in general flat *n*-chains with coefficients in  $\mathbb{Z}$  may be regarded as elements of  $W_0^{1,\infty}(\mathbb{R}^d, \Lambda^n \mathbb{R}^d)'$ .

Finally, we define the group of flat *n*-chains relative to an open set  $\Omega \subseteq \mathbb{R}^d$ as the quotient group

 $\mathbb{F}_n(\Omega; \mathbf{G}) := \mathbb{F}_n(\mathbb{R}^d; \mathbf{G}) / \{ S \in \mathbb{F}_n(\mathbb{R}^d; \mathbf{G}) \colon S \text{ is supported in } \mathbb{R}^d \setminus \Omega \}.$ 

The quotient norm may equivalently be rewritten as

$$\mathbb{F}_{\Omega}(S) := \inf \left\{ \mathbb{M}(P \sqcup \Omega) + \mathbb{M}(Q \sqcup \Omega) \colon \begin{array}{l} P \in \mathbb{F}_{n+1}(\mathbb{R}^d; \mathbf{G}), \ Q \in \mathbb{F}_n(\mathbb{R}^d; \mathbf{G}), \\ S - \partial P - Q \text{ is supported in } \mathbb{R}^d \setminus \Omega \right\}$$
(2.5)

where  $P \sqcup \Omega$  denotes the restriction of P to  $\Omega$  (see [28, Section 2] for more details).

### 2.3. Sketch of the construction

Let  $\mathscr{N} \subseteq \mathbb{R}^m$  be a smoothly embedded manifold without boundary; let  $k \ge 2$  be an integer. We make the following assumption on  $\mathscr{N}$  and k:

(H)  $\mathscr{N}$  is compact and (k-2)-connected, that is  $\pi_0(\mathscr{N}) = \pi_1(\mathscr{N}) = \dots = \pi_{k-2}(\mathscr{N}) = 0$ . In case k = 2, we also assume that  $\pi_1(\mathscr{N})$  is abelian.

The integer k is thus related to the topology of  $\mathscr{N}$ . The condition (H) guarantees that  $k \leq \dim \mathscr{N} + 1$  and in case  $\mathscr{N}$  is a sphere, we can indeed choose  $k = \dim \mathscr{N} + 1$ ; however, the inequality may be strict in general. For instance, if  $\mathscr{N}$  is a real projective plane,  $\mathscr{N} \simeq \mathbb{R}P^2$ , then (H) is satisfied if and only if k = 2. Under the assumption (H), there is no topological obstruction associated with defects of codimension  $\langle k; \mathscr{N}$ -valued maps may have singularities of codimension  $\langle k,$  but these can be removed by local surgery. On the other hand, singularities of codimension k (or higher) may be associated with topological obstructions, and are classified by elements of  $\pi_{k-1}(\mathscr{N})$ . As a consequence of (H), the group  $\pi_{k-1}(\mathscr{N})$  is abelian and may be endowed with a norm that satisfies (2.2) (see e.g. (2.11) below). It will be the coefficient group for our flat chains.

While it is impossible to construct a smooth projection of  $\mathbb{R}^m$  onto a closed manifold  $\mathscr{N}$ , under the assumption (H) it is possible to construct a smooth projection  $\varrho \colon \mathbb{R}^m \setminus \mathscr{X} \to \mathscr{N}$ , where  $\mathscr{X}$  is a finite union of polyhedra of dimension m - k at most. Moreover, we can make sure that

$$|\nabla \varrho(y)| \leqslant \frac{C}{\operatorname{dist}(y, \mathscr{X})} \quad \text{for any } y \in \mathbb{R}^m \setminus \mathscr{X}.$$
 (2.6)

(In Section 1.3, we used the radial projection  $\rho: \mathbb{R}^k \setminus \{0\} \to \mathbb{S}^{k-1}$ ,  $\rho(y) := y/|y|$ .) The existence of such  $\rho$  was obtained by Hardt and Lin as a consequence of more general results of topology [43, Lemma 6.1]; self-contained approaches are presented in [22, 44].

Let  $d \ge k \ge 2$ , let  $\Omega \subseteq \mathbb{R}^d$  be a bounded, smooth domain, and let  $u \in$  $C^{\infty}(\mathbb{R}^d, \mathbb{R}^m)$ . One could be tempted to identify the set of topological singularities of u with  $u^{-1}(\mathscr{X})$ , which is exactly the set where the reprojection  $\rho(u)$  fails to be well-defined, but  $u^{-1}(\mathscr{X})$  may be very irregular even if u is smooth. However, Thom transversality theorem implies that, for a.e.  $y \in \mathbb{R}^m$ , the set  $(u-y)^{-1}(\mathscr{X})$  is indeed a finite union of (possibly disconnected) manifolds of dimension  $\leq d-k$ . For each (m-k)-manifold  $K \subseteq \mathscr{X}$ , we consider the inverse image  $(u-y)^{-1}(K) \cap \Omega$ , we assign an orientation to  $(u-y)^{-1}(K)$  and we equip it with a multiplicity, in the following way. Let  $x \in (u-y)^{-1}(K)$  be a point such that u(x) - y lies in the interior of the polyhedron K. We consider the normal k-plane  $\Pi$  to  $(u-y)^{-1}(K)$  at the point x, and let  $\Sigma := \partial B^d_{\rho}(x) \cap \Pi$  be a small (k-1)-sphere around x. The given orientation of  $(u - y)^{-1}(K)$  induces an orientation of  $\Sigma$  and hence, an orientation-preserving diffeomorphism  $\Sigma \simeq \mathbb{S}^{k-1}$ . By means of this diffeomorphism, the homotopy class of  $(u-y)_{|\Sigma}$  can be identified with an element  $\alpha_y(u, K) \in \pi_{k-1}(\mathcal{N})$ . The homotopy class  $\alpha_y(u, K)$  is actually independent of  $\rho$  and x, but it does depend on the orientation of  $(u-y)^{-1}(K)$ . Now, we define the smooth chain

$$\mathbf{S}_{y}(u) := \sum_{K} \alpha_{y}(u, K) \llbracket (u - y)^{-1}(K) \rrbracket \in \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathscr{N}))$$

The chain  $\mathbf{S}_y(u)$  is independent on the choice of the orientation on each  $(u-y)^{-1}(K)$ : if we change the orientation, then both  $\alpha_y(u, K)$  and  $\llbracket (u-y)^{-1}(K) \rrbracket$  will change their sign, so their product remains unaffected. Moreover, even if  $\pi_{k-1}(\mathcal{N})$  is an infinite group, it is possible to show [28, Section 3.2] that the multiplicities  $\alpha_y(u, K)$  belong to a finite subset of  $\pi_{k-1}(\mathcal{N})$ , which depends on  $\varrho, \mathscr{X}$  but not on u and y. (This is a consequence of transversality, just as the local degree of a vector field  $u \colon \mathbb{R}^k \to \mathbb{R}^k$  at a point  $x \in \mathbb{R}^k$  such that u(x) = 0, det  $\nabla u(x) \neq 0$  can only be 1 or -1.) We have disregarded the contributions coming from manifolds  $K \subseteq \mathscr{X}$  of dimension < m - k: this is because no  $S \in \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathcal{N}))$  can be supported on a set of dimension < d - k, unless S = 0 [68, Theorem 3.1].

The chain  $\mathbf{S}_y(u)$  satisfies the following topological property. Let  $D \subseteq \Omega$ be a smoothly embedded, oriented k-disk, such that  $\partial D$  does not intersect  $(u-y)^{-1}(\mathscr{X})$  (hence,  $\varrho(u-y)$  is well defined on  $\partial D$ ). Generically, Dintersects the support of  $\mathbf{S}_y(u)$  at a finite number of points. By summing up the multiplicities of  $\mathbf{S}_y(u)$  at the intersection points, with a sign accounting for the relative orientations of D and  $\mathbf{S}_{y}(u)$ , we define the so-called intersection index, denoted  $\mathbb{I}(\mathbf{S}_{y}(u), \llbracket D \rrbracket) \in \pi_{k-1}(\mathscr{N})$  (see e.g. [28, Section 2.1] for more details). Then, a simple topological argument shows that

$$\mathbb{I}(\mathbf{S}_{y}(u), \llbracket D \rrbracket) = \text{homotopy class of } \varrho(u-y) \text{ on } \partial D.$$
(2.7)

In this sense, the chain  $\mathbf{S}_{y}(u)$  carries topological information on u.

Thanks to the estimate (2.6) on  $\nabla \varrho$ , we can now integrate over  $y \in \mathbb{R}^m$ and apply a strategy similar to that devised by Hardt, Kinderlehrer and Lin (sketched in Section 1.3). In particular, by applying the coarea formula, we obtain a continuity estimate on  $\mathbf{S}_y(u)$  depending on the Sobolev norms of u. Then, by density, one can define  $\mathbf{S}_y(u)$  in case u is a Sobolev map.

We let  $X := (L^{\infty} \cap W^{1,k-1})(\Omega, \mathbb{R}^m)$  and endow this set with a topology, in such a way that a sequence  $(u_j)_{j \in \mathbb{N}}$  converges to u in X if and only if  $u_j \to u$  strongly in  $W^{1,k-1}$  and  $\sup_{j \in \mathbb{N}} ||u_j||_{L^{\infty}} < +\infty$ . Let us take a number  $\delta_* \in (0, \operatorname{dist}(\mathscr{N}, \mathscr{X}))$  and define  $B^* := B^m(0, \delta^*)$ . We consider the set  $Y := L^1(B^*, \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathscr{N})))$ , whose elements are measurable maps  $y \in B^* \mapsto S_y \in \mathbb{F}_{d-k}(\Omega; \pi_{k-1}(\mathscr{N}))$  such that

$$||S||_Y := \int_{B^*} \mathbb{F}_{\Omega}(S_y) \, \mathrm{d}y < +\infty.$$

The set Y is a complete normed modulus, with respect to the norm  $\|\cdot\|_{Y}$ .

THEOREM 2.1 ([28, 30]). — Suppose that (H) is satisfied. Then, there exists a unique continuous map  $\mathbf{S}: X \to Y$  such that, for any  $u \in X \cap C^{\infty}(\Omega, \mathbb{R}^m)$ , a.e.  $y \in B^*$ , and any smoothly embedded, oriented k-disk  $D \subseteq \Omega$  such that  $\partial D \cap (u-y)^{-1}(\mathscr{X}) = \emptyset$ , the property (2.7) holds. In addition, for any  $u_0, u_1 \in X$  and a.e.  $y \in B^*$ , we can write  $\mathbf{S}_y(u_1) - \mathbf{S}_y(u_0) = \partial R_y$  in  $\Omega$ , where  $R_y$  is a (d-k+1)-chain that satisfies

$$\int_{B^*} \mathbb{M}(R_y) \, \mathrm{d}y \leq C \int_{\Omega} |u_1 - u_0| \left( |\nabla u_1|^{k-1} + |\nabla u_0|^{k-1} \right)$$
(2.8)

and C is a constant that only depends on  $\mathcal{N}$ , k,  $\varrho$ ,  $\mathscr{X}$ ,  $\delta^*$  and  $\Omega$ . Finally, if  $u \in W^{1,k-1}(\Omega, \mathcal{N})$  then for a.e.  $y, y' \in B^*$  there holds

$$\mathbf{S}_{y}(u) = \mathbf{S}_{y'}(u). \tag{2.9}$$

Actually, Property (2.7) holds for any  $u \in X$ , provided that both sides of the identity are suitably defined (we refer to [28, Section 2 and Theorem 3.1]). The inequality (2.8), together with (2.5), implies the continuity estimate

$$\|\mathbf{S}(u_1) - \mathbf{S}(u_0)\|_Y \leq C \int_{\Omega} |u_1 - u_0| \left( |\nabla u_1|^{k-1} + |\nabla u_0|^{k-1} \right),$$

which is analougous to Theorem 1.1. In particular, we have stability of **S** with respect to strong and weak convergence, as in (1.3)–(1.4). Therefore,

some of the compensation compactness properties that are typical of the Jacobian are retained by **S**. By choosing  $u_0$  equal to a constant (so that  $\mathbf{S}_y(u_0) = 0$  for a.e. y), we also see that  $\mathbf{S}_y(u_1)$  may be written as a relative boundary:  $\mathbf{S}_y(u_1) = \partial R_y$  inside  $\Omega$ , where  $R_y$  is a (d - k + 1)-flat chain that satisfies

$$\int_{B^*} \mathbb{M}(R_y) \, \mathrm{d}y \leqslant C \, \|\nabla u_1\|_{L^{k-1}(\Omega)}^{k-1} \,. \tag{2.10}$$

In case u is  $\mathscr{N}$ -valued, (2.9) states that the map  $y \mapsto \mathbf{S}_y(u)$  is locally constant; we denote its constant value by  $\mathbf{S}^{\mathrm{PR}}(u)$ . The chain  $\mathbf{S}^{\mathrm{PR}}(u)$  coincides with the topological singular set as introduced by Pakzad and Rivière in [63].

The inequality (2.8) is the main item in the statement of Theorem 2.1. To prove (2.8), first we construct a chain  $R_y$  such that  $\partial R_y = \mathbf{S}_y(u_1) - \mathbf{S}_y(u_1)$ , then we estimate the mass of  $R_y$  using the coarea formula. The construction of  $R_y$ , which is inspired by [2, Section 6], is fairly straightforward because, contrarily to [63], we are now working with  $\mathbb{R}^m$ -valued maps: we reduce to the case  $u_0, u_1$  are smooth, consider the affine interpolant U(x, t) := $(1-t)u_0(x) + tu_1(x)$  for  $(x, t) \in \Omega \times [0, 1]$  and define  $R_y$  as the projection of  $\mathbf{S}_y(U)$  onto  $\Omega$ . Topological arguments show that  $\partial R_y = \mathbf{S}_y(u_1) - \mathbf{S}_y(u_0)$ .

In the special case  $\mathscr{N} = \mathbb{S}^{k-1} \subseteq \mathbb{R}^k$ , we have  $\pi_{k-1}(\mathbb{S}^{k-1}) \simeq \mathbb{Z}$  and so  $\mathbf{S}_y(u)$  has an alternative description in terms of currents. If we make the choice  $\mathscr{X} = \{0\} \subseteq \mathbb{R}^k$  and  $\varrho(y) = y/|y|$ , then Theorem 1.2 implies

$$Ju = \frac{1}{\alpha_k} \int_{\mathbb{R}^m} \mathbf{S}_y(u) \, dy \qquad \text{for any } u \in (L^{\infty} \cap W^{1,k-1})(\Omega, \mathbb{R}^k),$$

where  $\alpha_k$  is the volume of the unit k-disk and the integral in the right-hand side is intended in the sense of distributions. However, if  $\pi_{k-1}(\mathscr{N})$  is a finite group (or, more generally, if it only contains elements of finite order), then there is no meaningful way to define the integral of  $\mathbf{S}_y(u)$  with respect to the Lebesgue measure dy. In order to define such an integral, we would need to define the "product" between an element of  $\pi_{k-1}(\mathscr{N})$  (the multiplicity of a flat chain) and a real number (the volume element). It is natural to require that this "product" be linear (i.e., a  $\mathbb{Z}$ -module homomorphism) in both arguments, so as to be compatible with the group structures on  $\pi_{k-1}(\mathscr{N})$ and  $\mathbb{R}$ . However, if  $\pi_{k-1}(\mathscr{N})$  is a finite group and  $\mathbf{G}$  is any group, then any bilinear form  $\pi_{k-1}(\mathscr{N}) \times \mathbb{R} \to \mathbf{G}$  is identically equal to zero.

If the gradient of  $u \in X$  has better integrability, i.e.  $u \in W^{1,k}(\Omega, \mathbb{R}^m)$ , then  $\mathbf{S}_y(u)$  can be identified with a finite-mass chain, i.e.

$$\mathbf{S}_y(u) \in \mathbb{F}_{d-k}(\mathbb{R}^d; \pi_{k-1}(\mathscr{N})) \text{ and } \mathbb{M}(\mathbf{S}_y(u)) < +\infty \qquad \text{for a.e. } y \in B^*$$

(see [28, Theorem 3.1] and [30, Proposition 2.3]). In particular,  $\mathbf{S}_y(u)$  is well-defined not only as a relative flat chain in  $\Omega$ , but also "up to the boundary".

(Something analogous happens with the Jacobian determinant, too: the Jacobian of a map  $u: \Omega \subseteq \mathbb{R}^k \to \mathbb{R}^k$  is well defined in a distributional sense if  $u \in L^{\infty} \cap W^{1,k-1}$ , and in a pointwise sense if  $u \in W^{1,k}$ ; in the latter case, Ju is a finite measure.) Then, we can extend  $\mathbf{S}_{|L^{\infty} \cap W^{1,k}}$  to a continuous operator

$$W^{1,k}(\Omega, \mathbb{R}^m) \to L^1(B^*, \mathbb{F}_{d-k}(\mathbb{R}^d; \pi_{k-1}(\mathscr{N}))),$$

still denoted  $\mathbf{S}$ , for simplicity. This operator satisfies an important topological property:

PROPOSITION 2.2. — Let 
$$u_0$$
,  $u_1 \in W^{1,k}(\Omega, \mathbb{R}^m)$  be such that  
 $u_0(x) = u_1(x) \in \mathscr{N}$  for  $\mathscr{H}^{d-1}$ -a.e.  $x \in \partial \Omega$ 

(in the sense of traces). Then, for a.e.  $y_0, y_1 \in B^*$  there exists a finite-mass (d-k+1)-chain R, supported in  $\overline{\Omega}$ , such that  $\mathbf{S}_{y_1}(u_1) - \mathbf{S}_{y_0}(u_0) = \partial R$ .

It is worth noticing that the proofs of Theorem 2.1 and Proposition 2.2 do not strictly rely upon the manifold structure of  $\mathscr{N}$ . What is needed, is the existence and regularity of the exceptional set  $\mathscr{X}$  and the retraction  $\varrho$ , in order to be able to apply Thom transversality theorem. This suggests a possible extension to more general targets  $\mathscr{N} \subseteq \mathbb{R}^m$  such as, for instance, finite simplicial complexes.

The results we presented in this section are valid for any group norm on  $\pi_{k-1}(\mathcal{N})$  that satisfies (2.2). However, for the variational applications we will give in Section 2.4 below, we need to choose a specific norm. A natural attempt, motivated by the analogy with the functional (2.1), is to define

$$E_{\min}(\sigma) := \inf_{v \in W^{1,k}(\mathbb{S}^k, \mathscr{N}) \cap \sigma} \left( \frac{1}{k} \int_{\mathbb{S}^{k-1}} |\nabla_{\top} v|^k \right)$$

for any  $\sigma \in \pi_{k-1}(\mathscr{N})$ . Here  $\nabla_{\top}$  denotes the tangential gradient on  $\mathbb{S}^{k-1}$ , that is, the restriction of the Euclidean gradient  $\nabla$  to the tangent plane to the sphere. However,  $E_{\min}$  does not satisfy the triangle inequality, in general. Instead, following [32], for any  $\sigma \in \pi_{k-1}(\mathscr{N})$  we define

$$|\sigma|_* := \inf_{(\sigma_1,\dots,\sigma_q)} \sum_{i=1}^q E_{\min}(\sigma_i), \qquad (2.11)$$

where the infimum is taken over all the q-uples  $(\sigma_1, \ldots, \sigma_q) \in (\pi_{k-1}(\mathscr{N}))^q$ , with arbitrary q, such that  $\sum_{i=1}^q \sigma_i = \sigma$ . Under the assumption (H), the function  $|\cdot|_*$  is a norm on  $\pi_{k-1}(\mathscr{N})$  that satifies (2.2) (and the infimum at the right-hand side of (2.11) is achieved; see [30, Proposition 2.1]). In case  $\mathscr{N} = \mathbb{S}^{k-1}$ , the group  $\pi_{k-1}(\mathbb{S}^{k-1})$  is isomorphic to  $\mathbb{Z}$  and  $|d|_* = (k-1)^{k/2}\alpha_k |d|$  for any  $d \in \mathbb{Z}$  (where  $\alpha_k$  is the Lebesgue measure of the unit ball in  $\mathbb{R}^k$ ).

### 2.4. An application: asymptotics for $E_{\varepsilon}$

Let us consider again the functional  $E_{\varepsilon}$  defined by (2.1). In [30] we show that the re-scaled functional  $|\log \varepsilon|^{-1} E_{\varepsilon}$  does converge to a (d - k)dimensional weighted area functional as  $\varepsilon \to 0$ , thus extending Theorem 1.3 to more general potentials f. The key tool is the topological singular set of vector-valued maps, that is, the operator  $\mathbf{S}$  we introduced above, which identifies the appropriate topology of the  $\Gamma$ -convergence. The operator  $\mathbf{S}$ effectively serves as a replacement, or rather a generalisation, of the distributional Jacobian.

We can now state our main  $\Gamma$ -convergence result ([30, Theorem C]) and its application to the asymptotic analysis of minimisers of (2.1) in the limit as  $\varepsilon \to 0$ . We make the following assumptions on the potential f and the exponent k:

- (H<sub>1</sub>)  $f \in C^1(\mathbb{R}^m)$  and  $f \ge 0$ .
- (H<sub>2</sub>) The set  $\mathscr{N} := f^{-1}(0) \neq \emptyset$  is a smooth, compact manifold without boundary. Moreover,  $\mathscr{N}$  is (k-2)-connected, that is  $\pi_0(\mathscr{N}) = \pi_1(\mathscr{N}) = \ldots = \pi_{k-2}(\mathscr{N}) = 0$ , and  $\pi_{k-1}(\mathscr{N}) \neq 0$ . In case k = 2, we also assume that  $\pi_1(\mathscr{N})$  is abelian.
- (H<sub>3</sub>) There exists a positive constant  $\lambda_0$  such that  $f(y) \ge \lambda_0 \operatorname{dist}^2(y, \mathscr{N})$  for any  $y \in \mathbb{R}^m$ .

The assumption (H<sub>2</sub>) is consistent with (H) and is satisfied, for instance, when k = 2 and  $\mathscr{N} = \mathbb{S}^1$  (the Ginzburg–Landau case) or k = 2 and  $\mathscr{N} = \mathbb{R}P^2$  (the Landau–de Gennes case). The assumption (H<sub>3</sub>) is both a nondegeneracy condition around the minimising set  $\mathscr{N}$  and a growth condition, because it implies that f grows at least quadratically at infinity. We define flat chains with coefficients in  $\pi_{k-1}(\mathscr{N})$  using the group norm defined by (2.11). We impose a Dirichlet boundary condition u = v on  $\partial\Omega$ , where

(H<sub>4</sub>) 
$$v \in W^{1-1/k,k}(\partial\Omega, \mathscr{N})$$
, that is,  $v \in W^{1-1/k,k}(\partial\Omega, \mathbb{R}^m)$  and  $v(x) \in \mathscr{N}$  for  $\mathscr{H}^{d-1}$ -a.e.  $x \in \partial\Omega$ .

Let  $W_v^{1,k}(\Omega, \mathbb{R}^m)$  be the set of maps  $u \in W^{1,k}(\Omega, \mathbb{R}^m)$  with trace v at the boundary. Given an arbitrary  $u_0 \in W_v^{1,k}(\Omega, \mathbb{R}^m)$  and a generic  $y_0 \in B^*$ , we define

$$\mathscr{C}(\Omega, v) := \left\{ \mathbf{S}_{y_0}(u_0) + \partial R \colon \frac{R \text{ is a } (d-k+1)\text{-chain}}{\text{ of finite mass, supported in } \overline{\Omega}} \right\}.$$

By Proposition 2.2, we have  $\mathbf{S}_{y}(u) \in \mathscr{C}(\Omega, v)$  for any  $u \in W_{v}^{1,k}(\Omega, \mathbb{R}^{m})$  and a.e.  $y \in B^{*}$ .

THEOREM 2.3 ([30, Theorem C]). — Suppose that the assumptions  $(H_1)$ – $(H_4)$  are satisfied. Then, the following properties hold.

(1) Let  $(u_{\varepsilon})_{\varepsilon>0}$  be a sequence in  $W^{1,k}_{v}(\Omega, \mathbb{R}^{m})$  that satisfies

$$\sup_{\varepsilon>0} \frac{E_{\varepsilon}(u_{\varepsilon})}{|\log \varepsilon|} < +\infty.$$

Then, there exists a (non relabelled) countable subsequence and a finite-mass chain  $S \in \mathscr{C}(\Omega, v)$  such that  $\mathbf{S}(u_{\varepsilon}) \to S$  in Y and, for any open subset  $A \subseteq \mathbb{R}^d$ ,

$$\mathbb{M}(S \, {\mathrel{\sqsubseteq}}\, A) \leqslant \liminf_{\varepsilon \to 0} \frac{E_{\varepsilon}(u_{\varepsilon}, \, A \cap \Omega)}{|\log \varepsilon|}$$

(2) For any finite-mass chain  $S \in \mathscr{C}(\Omega, v)$ , there exists a sequence of maps  $u_{\varepsilon} \in W_{v}^{1,k}(\Omega, \mathbb{R}^{m})$  such that  $\mathbf{S}(u_{\varepsilon}) \to S$  in Y and

$$\limsup_{\varepsilon \to 0} \frac{E_{\varepsilon}(u_{\varepsilon})}{|\log \varepsilon|} \leq \mathbb{M}(S).$$

As an application of Theorem 2.3, we can characterise the energy concentration set for minimisers of (2.1), in the limit as  $\varepsilon \to 0$ . Let  $u_{\varepsilon,\min}$  be a minimiser of (2.1) in  $W_v^{1,k}(\Omega, \mathbb{R}^m)$ . Under the assumptions (H<sub>1</sub>)–(H<sub>4</sub>), the rescaled energy densities

$$\mu_{\varepsilon,\min} := \left(\frac{1}{k} |\nabla u_{\varepsilon,\min}|^k + \frac{1}{\varepsilon^k} f(u_{\varepsilon,\min})\right) \frac{\mathrm{d}x \, {\rm L}\, \Omega}{|\log \varepsilon|}$$

have uniformly bounded mass, by Theorem 2.3 (here,  $dx \perp \Omega$  denotes the Lebesgue measure restricted to  $\Omega$ ). Up to extraction of a subsequence, we may assume that  $\mu_{\varepsilon,\min}$  converges weakly<sup>\*</sup> (as measures in  $\mathbb{R}^d$ ) to a non-negative measure  $\mu_{\min}$ , as  $\varepsilon \to 0$ . We provide a variational characterisation of  $\mu_{\min}$  in terms of flat chains with coefficients in  $(\pi_{k-1}(\mathcal{N}), |\cdot|_*)$ .

THEOREM 2.4 ([30, Theorem A]). — Under the assumptions  $(H_1)$ – $(H_4)$ , there exists a finite-mass chain  $S_{\min} \in \mathscr{C}(\Omega, v)$ , such that

$$\mu_{\min}(E) = \mathbb{M}(S_{\min} \sqcup E)$$

for any Borel set  $E \subseteq \mathbb{R}^d$ . Moreover,  $S_{\min}$  minimises the mass in  $\mathscr{C}(\Omega, v)$ , that is, for any  $S \in \mathscr{C}(\Omega, v)$ , we have

$$\mathbb{M}(S_{\min}) \leqslant \mathbb{M}(S).$$

In other words, in the limit as  $\varepsilon \to 0$  the energy of minimisers concentrates, to leading order, on the support of a flat chain  $S_{\min}$  that solves a least area problem in a given homology class. In particular, we expect  $S_{\min}$  to be (at least partially) regular. In fact, since regularity is a local property, most of the regularity theory for solutions to the Plateau problem extend

to mass-minimisers in a given homology class. If the coefficient group is  $\mathbb{Z}$  or  $\mathbb{Z}/2\mathbb{Z}$ , mass-minimising (d-k)-chains (with  $k \ge 2$ ) are smooth, embedded manifolds on their interior, except for a singular set of Hausdorff dimension d-k-2 at most; this bound is sharp. This follows from results by Federer [35, 36], in case the coefficient group is  $\mathbb{Z}/2\mathbb{Z}$ , and by Almgren [4], in case the coefficient group is  $\mathbb{Z}$ .

The proof of the  $\Gamma$ -lower bound in Theorem 2.3 is based on the same strategy as in [2]. However, the construction of a recovery sequence is rather different from [2]. The main building block is inspired by the "dipole construction" [11, 13, 23]. In our situation, dipoles are suitably inserted into a non-constant and, in fact, singular background.

### 2.5. Another application: lifting of BV maps into manifolds

The methods described in Section 2.3 also find application to the socalled lifting problem. Let  $\mathscr{N}$  be a smooth, compact, connected Riemannian manifold without boundary. Let

$$\pi\colon \mathscr{E}\to \mathscr{N}$$

be the (smooth) universal covering of  $\mathscr{N}$ . We endow  $\mathscr{E}$  with the pull-back metric, so that  $\pi$  is a local isometry. Given a bounded, smooth domain  $\Omega \subseteq \mathbb{R}^d$  and measurable maps  $u: \Omega \to \mathscr{N}, v: \Omega \to \mathscr{E}$ , we say that v is a *lifting* for u if  $\pi \circ v = u$  a.e. on  $\Omega$ . We are interested in the

LIFTING PROBLEM. — Given a regular map  $u: \Omega \to \mathcal{N}$ , is there a lifting  $v: \Omega \to \mathscr{E}$  of u that is as regular as u?

Of course, the answer depends on what we mean precisely by "regular". If u is of class  $C^k$  (with  $k = 0, 1, ..., \infty$ ) and  $\Omega$  is simply connected, then the lifting problem has a positive answer. If other notions of regularity (for instance, Sobolev regularity) are considered, the problem may be more delicate. The lifting problem for non-continuous maps has been studied first when  $\mathscr{N}$  is the unit circle,  $\mathscr{N} = \mathbb{S}^1$ , in connection with the Ginzburg–Landau theory of superconductivity. In this case,  $\mathscr{E} = \mathbb{R}$  and the covering map  $\pi \colon \mathbb{R} \to \mathbb{S}^1$  is given by  $\pi(\theta) = \exp(i\theta)$ . The study of this case was initiated in [17, 19] and culminated with the work by Bourgain, Brezis and Mironescu [21], who gave a complete answer to the lifting problem when  $u \in W^{s,p}(\Omega, \mathbb{S}^1), s > 0, 1 . Their results have been extended to the Besov setting by Mironescu, Russ and Sire [56]. Another particular instance of the lifting problem is the case when <math>\mathscr{N}$  is the real projective plane,  $\mathscr{N} = \mathbb{R}P^2$ , which is obtained from the 2-dimensional sphere  $\mathbb{S}^2$  by identifying pairs of antipodal points. The covering space  $\mathscr{E}$  is then the sphere  $\mathbb{S}^2$ 

and  $\pi: \mathbb{S}^2 \to \mathbb{RP}^2$  is the natural projection.  $\mathbb{RP}^2$ -valued maps and their lifting have a physical interpretation e.g. in materials science, as they serve as models for a class of materials known as (uniaxial) nematic liquid crystals (see e.g. [7, 8] for more details). The lifting problem for  $\mathbb{RP}^2$ -valued maps, in the context of Sobolev  $W^{1,p}$ -spaces, has been studied e.g. by Ball and Zarnescu [8] and Mucci [59].

For more general target manifolds  $\mathscr{N}$ , the lifting problem in  $W^{s,p}(\Omega, \mathscr{N})$ with  $s \neq 1$  was studied by Bethuel and Chiron [16], and only very recently it has been completely settled by Mironescu and Van Schaftingen [57]. Among other results, Bethuel and Chiron proved that, if  $\Omega$  is simply connected and  $p \geq 2$ , then every map  $u \in W^{1,p}(\Omega, \mathscr{N})$  has a lifting  $v \in W^{1,p}(\Omega, \mathscr{E})$ . However, there exist maps that belong to  $W^{1,p}(\Omega, \mathscr{N})$  for any p < 2, and yet have no lifting in  $W^{1,p}(\Omega, \mathscr{E})$  (for instance, we can take  $\mathscr{N} = \mathbb{S}^1$ ,  $\Omega$ the unit disk in  $\mathbb{R}^2$ , and u(x) := x/|x|). Bethuel and Chiron raised the conjecture [16, Remark 1] that any map  $u \in W^{1,p}(\Omega, \mathscr{N})$ , with  $p \geq 1$ , has a lifting of bounded variation (BV).

In [30] we consider the lifting problem when u is a BV-map. Previous works showed that the lifting problem for  $u \in BV(\Omega, \mathcal{N})$  has a positive answer in case  $\mathcal{N} = \mathbb{S}^1$  (Giaquinta, Modica and Souček [40, Corollary 1 in Volume 2, Section 6.2.2], Davila and Ignat [34], Ignat [45]),  $\mathcal{N} = \mathbb{R}P^k$ (Bedford [10], Ignat and Lamy [47]) and more generally, if the fundamental group of  $\mathcal{N}, \pi_1(\mathcal{N})$ , is abelian [28]. We prove a lifting result for maps  $u \in BV(\Omega, \mathcal{N})$  without assuming that  $\pi_1(\mathcal{N})$  is abelian. Examples of closed manifolds with non-abelian fundamental group are obtained by taking the quotient of SO(3), the set of rotations of  $\mathbb{R}^3$ , by the symmetry group of a regular, convex polyhedron. The elements of this quotient space describe the possible orientations of the given polyhedron in  $\mathbb{R}^3$ . Manifolds of this form appear in variational problems, arising from applications of different kinds. For instance, in material science, they appear in models for ordered materials, such as biaxial nematics (see e.g. [55]). In numerical analysis, they are found in Ginzburg–Landau functionals with applications to mesh generation, via the so-called cross-field algorithms (see e.g. [31]).

By Nash's theorem [61], we can embed isometrically both  $\mathscr{N}$  and  $\mathscr{E}$  into Euclidean spaces,  $\mathscr{N} \subseteq \mathbb{R}^m$ ,  $\mathscr{E} \subseteq \mathbb{R}^\ell$ . Moreover, since  $\mathscr{N}$ ,  $\mathscr{E}$  are complete Riemannian manifolds, we can choose the embeddings so that the images of  $\mathscr{N}$ ,  $\mathscr{E}$  are closed subsets of  $\mathbb{R}^m$ ,  $\mathbb{R}^\ell$ , respectively [60]. From now on, we will identify  $\mathscr{N}$ ,  $\mathscr{E}$  with their closed Euclidean embeddings. Given an open set  $\Omega \subseteq \mathbb{R}^d$ , we define  $\mathrm{BV}(\Omega, \mathscr{N})$  as the set of maps  $u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$  that satisfy the pointwise constraint  $u(x) \in \mathscr{N}$  for a.e.  $x \in \Omega$ . We also define  $\mathrm{SBV}(\Omega, \mathscr{N})$  as the set of maps  $u \in \mathrm{BV}(\Omega, \mathscr{N})$  such that the distributional derivative Du (taken in the sense of  $\mathrm{BV}(\Omega, \mathbb{R}^m)$ ) has no Cantor part. We define  $BV(\Omega, \mathscr{E})$ ,  $SBV(\Omega, \mathscr{E})$  in a similar fashion. We write  $|\mu|(\Omega)$  to denote the total variation of a vector-valued Radon measure  $\mu$  on  $\Omega$ . The main result in [29] then reads

THEOREM 2.5 ([29, Theorem 1]). — Let  $\mathscr{N} \subseteq \mathbb{R}^m$  be a smooth, compact, connected manifold without boundary. Let  $\Omega \subseteq \mathbb{R}^d$  be a smooth, bounded domain with  $d \ge 1$ . Then, any  $u \in BV(\Omega, \mathscr{N})$  has a lifting  $v \in BV(\Omega, \mathscr{E})$  that satisfies

$$\begin{aligned} |\mathrm{D}v|(\Omega) &\leq C_{\Omega,\mathcal{N}} |\mathrm{D}u|(\Omega) \\ \|v\|_{L^{1}(\Omega)} &\leq C_{\Omega,\mathcal{N}} (|\mathrm{D}u|(\Omega)+1), \end{aligned}$$

where the constant  $C_{\Omega, \mathcal{N}}$  depends only on  $\Omega$  and (the given Euclidean embleding of)  $\mathcal{N}$ . Moreover, if  $u \in \text{SBV}(\Omega, \mathcal{N})$  and  $v \in \text{BV}(\Omega, \mathscr{E})$  is a lifting of u, then  $v \in \text{SBV}(\Omega, \mathscr{E})$ .

Theorem 2.5 implies, in particular, that a map  $u \in W^{1,p}(\Omega, \mathscr{N})$  with  $p \ge 1$  has a lifting  $v \in \text{SBV}(\Omega, \mathscr{E})$ , thus proving Bethuel and Chiron's conjecture in [16].

The construction given in Section 2.3 depends on the assumption that the group  $\pi_1(\mathcal{N})$  is abelian, see (H), because the theory of flat chains requires the coefficient group to be abelian. In constrast, we do not assume that  $\pi_1(\mathcal{N})$  is abelian in Theorem 2.5 and as a consequence, we may not be able to define  $\mathbf{S}_y(u)$ , not even for  $u \in W^{1,1}(\Omega, \mathscr{N})$ . Nevertheless, the proof of Theorem 2.5 is very much inspired by the arguments we have described in Section 2.3. Given  $u \in BV(\Omega, \mathcal{N})$ , we first approximate u with a sequence of *piecewise-affine* maps  $u_i: \Omega \to \mathbb{R}^m$ . (In this case, we prefer to use piecewise-affine approximations instead of smooth ones just because this simplifies some technical points in the proof.) Using the retraction  $\rho$ , and choosing suitable constants  $y_j \in \mathbb{R}^m$  with sufficiently small modulus, we project  $u_j - y_j$  onto  $\mathcal{N}$ , so to define a map  $\rho \circ (u_j - y_j) \colon \Omega \to \mathcal{N}$  with polyhedral singularities of dimension d-2 at most. Then, by combining standard topological results with the properties of  $\rho$ , we construct a suitable lifting  $v_i: \Omega \to \mathscr{E}$  of  $\varrho \circ (u_i - y_i)$ . The lifting  $v_i$  may jump on a polyhedral set of dimension d-1, but is locally Lipschitz continuous out of its jump set. Moreover, the total variation of  $\nabla v_i$  can be bounded from above, essentially by adapting the proof of Theorem 2.1. Therefore, we can pass to the (weak) limit in the  $v_i$ 's and obtain the desired lifting v for u.

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