

ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

JUANYONG WANG

On the Itaka Conjecture $C_{n,m}$ for Kähler Fibre Spaces

Tome XXX, n° 4 (2021), p. 813–897.

<https://doi.org/10.5802/afst.1690>

© Université Paul Sabatier, Toulouse, 2021.

L'accès aux articles de la revue « Annales de la faculté des sciences de Toulouse Mathématiques » (<http://afst.centre-mersenne.org/>) implique l'accord avec les conditions générales d'utilisation (<http://afst.centre-mersenne.org/legal/>). Les articles sont publiés sous la licence CC-BY 4.0.



Publication membre du centre
Mersenne pour l'édition scientifique ouverte
<http://www.centre-mersenne.org/>



On the Iitaka Conjecture $C_{n,m}$ for Kähler Fibre Spaces ^(*)

JUANYONG WANG ⁽¹⁾

ABSTRACT. — By applying the positivity theorem of direct images and a pluricanonical version of the structure theorem on the cohomology jumping loci à la Green–Lazarsfeld–Simpson, we show that the klt Kähler version of the Iitaka conjecture $C_{n,m}$ (Ueno, 1975) for $f : X \rightarrow Y$ (surjective morphism between compact Kähler manifolds with connected general fibre) holds true when the determinant of the direct image of some power of the relative canonical bundle is big on Y or when Y is a complex torus. These generalize the corresponding results of Viehweg (1983) and of Cao–Păun (2017) respectively. We further generalize the later case to the geometric orbifold setting, i.e. prove that $C_{n,m}^{\text{orbifold}}$ (Campana, 2004) holds when Y is a complex torus.

RÉSUMÉ. — En appliquant la positivité des images directes et une version pluricanonique du théorème de structure des lieux de saut cohomologique à la Green–Lazarsfeld–Simpson, nous démontrons que la version klt kählérienne de la conjecture d’Iitaka $C_{n,m}$ (Ueno, 1975) pour $f : X \rightarrow Y$ (morphisme surjectif entre variétés kählériennes compactes à fibre générale connexe) est vraie si le déterminant de l’image directe d’une certaine puissance du fibré canonique relative est gros sur Y ou si Y est un tore complexe. Ceci généralisent les résultats correspondants de Viehweg (1983) et de Cao–Păun (2017) respectivement. De plus nous généralisons le deuxième résultat ci-dessus au cadre des orbifolds géométriques, c-à-d., nous démontrons que $C_{n,m}^{\text{orbifold}}$ (Campana, 2004) est vraie quand Y est un tore complexe.

Introduction

Throughout the article, a complex variety signifies an irreducible reduced complex analytic space, a(n) (analytic) fibre space signifies a proper surjective morphism between complex varieties whose fibres are connected; in particular, an algebraic fibre space is a fibre space which is projective. A \mathbb{Q} -line

(*) Reçu le 16 juillet 2019, accepté le 24 janvier 2020.

(1) Academy of Mathematics and Systems Science, Chinese Academy of Sciences — juanyong.wang@amss.ac.cn

Article proposé par Vincent Guedj.

bundle on a complex variety X means an element of $\text{Pic}(X) \otimes \mathbb{Q}$ (cf. also [72, Lecture 1, §8.3, Definition 8.6]) and we use “+” to denote the tensor product of two line bundles. In the article we use the terminology “(analytic) Zariski open subset”, which means an open subset of a complex variety whose complement is a closed analytic subvariety. Let X be a compact complex variety and let L be a \mathbb{Q} -line bundle on X , recall that the Iitaka–Kodaira dimension of L , denoted by $\kappa(X, L)$, is the maximum dimension of the image of \bar{X} via the meromorphic mappings $\bar{X} \dashrightarrow \mathbb{P}^n(\bar{X}, \nu L^{\otimes m})$ defined by the linear series $| \nu L^{\otimes m} |$ for $m \in \mathbb{Z}_{>0}$ sufficiently large and divisible (if $| \nu L^{\otimes m} | = \emptyset$ for all $m \in \mathbb{Z}_{>0}$ then we say that $\kappa(X, L) = -\infty$), where $\nu : \bar{X} \rightarrow X$ is the normalization of X . In particular, the Kodaira dimension of a compact complex variety X , denoted by $\kappa(X)$, is the Iitaka–Kodaira dimension of the canonical bundle of any smooth model of X , and $\kappa(X)$ is known to be the most important bimeromorphic invariant of X .

The Iitaka conjecture $C_{n,m}$, in its original form, predicts the superadditivity of the Kodaira dimension with respect to algebraic fibre spaces (cf. [71, §11.5, Conjecture C_n , pp. 132–133]); more precisely, for $f : X \rightarrow Y$ a surjective morphism between normal projective varieties whose general fibre F is connected, the conjecture $C_{n,m}$ predicts that

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

This conjecture is intimately related to the study of birational classification of complex algebraic varieties (the Minimal Model Program, abbr. MMP). According to the philosophy of MMP, the conjecture $C_{n,m}$ is naturally generalized to the log version, usually called $C_{n,m}^{\text{log}}$; Moreover, Frédéric Campana further generalize $C_{n,m}$ to the setting of geometric orbifolds, called $C_{n,m}^{\text{orb}}$, which is formulated in [17, Conjecture 4.1] and in [18, Conjecture 6.1]. In addition, by taking into consideration the variation of the fibre space, Eckart Viehweg also propose a stronger version of the $C_{n,m}$, called $C_{n,m}^+$.

As shown in [53] (resp. [51]), the conjecture $C_{n,m}$ (resp. $C_{n,m}^+$) can be regarded as the consequence of the famous Minimal Model Conjecture and the Abundance Conjecture; moreover, in virtue of the superadditivity of Nakayama’s numerical dimensions (cf. [64, §V.4.a, 4.1.Theorem (1), pp. 220–221]), the conjecture $C_{n,m}^{\text{log}}$ follows from the so-called generalized Abundance Conjecture (for \mathbb{Q} -divisors), cf. [36, Remark 1.8].

Although initially stated for projective varieties, the conjecture $C_{n,m}$, as well as the MMP and the Abundance, are considered as still hold for complex varieties in the Fujiki class \mathcal{C} (cf. [17, 19, 36, 37, 48]); nevertheless they do not hold true in general for non-Kähler complex varieties, cf. [71, Remark 15.3, p. 187] for an counterexample. The objective of this article is to prove the klt Kähler version of $C_{n,m}^{\text{log}}$ in two important special cases

and further generalize the second one to the geometric orbifold setting. Let us remark that since the Kodaira dimension as well as the klt/lc property is invariant under taking log-resolutions, hence for simplicity we will state our main results always for Kähler manifolds, but one can easily see that it remains true for normal complex varieties in the Fujiki class \mathcal{C} . Now let us state our main theorem:

MAIN THEOREM A. — *Let $f : X \rightarrow Y$ be a fibre space between compact Kähler manifolds with general fibre denoted by F . And let \mathcal{L} be an effective \mathbb{Q} -divisor on X such that (X, \mathcal{L}) is Kawamata log terminal (abbr. klt). Suppose that one of following conditions is verified:*

- (I) *there is an integer $m > 0$ such that $m \mathcal{L}$ is an integral divisor and that the determinant line bundle $\det f^*(K_{X/Y} \otimes \mathcal{O}_X(m \mathcal{L}))$ is big on Y ;*
- (II) *Y is a complex torus.*

Then

$$\kappa(X, K_X + \mathcal{L}) > \kappa(F, K_F + \mathcal{L}|_F) + \kappa(Y),$$

where $\mathcal{L}|_F := \mathcal{L}|_F$.

Part (I) of Main Theorem A generalizes [74, Theorem II], which is intimately related to $C_{n,m}^+$ (cf. [74] for more details; this article, however, will not pursue in this direction); while Part (II) generalizes [25, Theorem 1.1] and it will be further generalized to the setting of geometric orbifolds, in other word, we will prove $C_{n,m}^{\text{orb}}$ for f when Y is a complex torus. Moreover, by following the same strategy of the proof of Part (I), we recover the result that klt Kähler version of $C_{n,m}^{\text{log}}$ holds for $f : (X, \mathcal{L}) \rightarrow Y$ when Y is of general type, which generalizes [49, Theorem 3]; we also further generalize this result to the geometric orbifold setting. Let us remark that the general (log canonical) version of $C_{n,m}^{\text{orb}}$ for Y general type (in the orbifold sense) has already been proved in [17]; the proof is based on a weak positivity result for direct images of twisted pluricanonical bundles, for which [17] only proves the projective case, and gives some hints for the Kähler case; it is established in this generality in [36].

Now let us explain the strategy of the proof of Main Theorem A. Generally speaking, as in the mainstream of works on $C_{n,m}$ (among others, [25, 36, 37, 49, 50, 74]), our proof is based on the positivity of relative pluricanonical bundles and of their direct images. Before getting into details let us first recall some definitions: a fibre space $f : X \rightarrow Y$ is called a *Kähler fibre space* if locally over Y , X is a Kähler variety (cf. [48, Definition 2.2]).

The key ingredient of the proof of Part (I) of Main Theorem A is the positivity of the relative m -Bergman kernel metric for Kähler fibre spaces,

which is proved by Junyan Cao in [23] by applying the Ohsawa–Takegoshi extension theorem with optimal estimation for Kähler fibre spaces (cf. Theorem 2.1) also obtained in [23], and states as follows (cf. Theorem 2.3):

Let $f : X \rightarrow Y$ be a Kähler fibre space between complex manifolds and let (L, h_L) be holomorphic line bundle on X endowed with a singular Hermitian metric whose curvature current is positive. Suppose that on the general fibre of f there exists a section of $K_{X/Y}^m \otimes L$ satisfying the $L^{2/m}$ -integrability condition for some m , then the m -Bergman kernel metric $h_{X/Y,L}^{(m)}$ on $K_{X/Y}^m \otimes L$ has positive curvature current.

With the help of this positivity result, Main Theorem A (I), as well as the klt Kähler version of $C_{n,m}^{\log}$ for general type bases can both be deduced from (a global version of) the Ohsawa–Takegoshi type extension Theorem 2.2 as follows:

- First by the useful Lemma 3.1, we can reduce the proof of the addition formula to that of the non-vanishing of the (twisted) relative pluricanonical bundle, up to adding an ample line bundle from the base.
- If Y is of general type in the orbifold sense, the non-vanishing result mentioned above follows easily from the Ohsawa–Takegoshi type extension Theorem 2.2 in contrast to the proof in [17, 36, 74], where such non-vanishing results are deduced from the weak positivity of the direct images. Let us remark that: by generalizing the weak positivity theorem for f Kähler fibre space and for \log canonical, the general (log canonical) version is proved in [17, 36].
- In the situation of Part (I) of our Main Theorem A, the proof of this non-vanishing result follows the same strategy, but requires an extra effort to establish a comparison theorem between the determinant of the direct image and the canonical bundle of X , see Theorem 3.4, which is a Kähler version of [25, Theorem 3.13].

The analytic proof given above does not explicitly involve any positivity result of direct images while it has the drawback of not being able to tackle the log canonical case.

Now we turn to the proof of Part (II) of Main Theorem A, for which we follow step by step the same argument in [25]. Our proof is based on the positivity of the canonical L^2 metric on direct images sheaves (cf. Theorem 2.6):

Let $f : X \rightarrow Y$ be a Kähler fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X endowed with a semi-positively curved singular Hermitian metric. Then the canonical L^2 metric $g_{X/Y,L}$ on the direct image sheaf $f_*(K_{X/Y} \otimes L \otimes \mathcal{J}(h_L))$

is a semi-positively curved singular Hermitian metric which satisfies the L^2 extension property.

The main strategy for the proof of the above positivity result is already implicitly comprised in [45], and the result is explicitly shown in [33] by proving a more general positivity theorem for singular Finsler metrics on direct images. In fact, this result is a consequence of the Ohsawa–Takegoshi extension theorem with optimal estimations obtained in [44] and generalized to Kähler case by [23] (cf. [78] for an alternative proof); the new feature is the L^2 extension property, which generalizes the well-known property of \mathcal{O} that a L^2 holomorphic function extends across any analytic subset (compare this with the “minimal extension property” in [45, Definition 20.1]). By combining the above positivity result of the canonical L^2 metric on direct images with the positivity of the relative m -Bergman kernel metric and by using the explicit construction of the relative m -Bergman kernel metric to get rid of the multiplier ideal (as in [25, §4, p. 367]), we obtain the following positivity theorem for direct images of twisted pluricanonical bundles, which serves as a key ingredient of the proof of Main Theorem A (II):

THEOREM B. — *Let $f : X \rightarrow Y$ a Kähler fibre space with X and Y complex manifolds. Let Σ be an effective \mathbb{Q} -divisor on X such that the pair (X, Σ) is klt. For any integer $m > 0$ such that $m\Sigma$ is an integral divisor, the torsion free sheaf*

$$F_{m, \Sigma} := f_* \left(K_{X/Y}^m \otimes \mathcal{O}_X(m\Sigma) \right)$$

admits a canonical semi-positively curved singular Hermitian metric $g_{X/Y}^{(m)}$, which satisfies the L^2 extension property.

Historically, the study of the positivity of direct images of (twisted) (pluri)canonical bundle(s) is initiated by the works of Griffiths on the variation of Hodge structures in the 60s, and is pursued by Fujita in [37] and by Kawamata in [49]; afterwards the study splits into two (related and complementary) main streams: the Hodge-theoretical aspect is further developed by Viehweg in the framework of weak positivity by algebro-geometric methods, while the curvature aspect is exploited by Berndtsson, Păun and Takayama (among others) by complex-analytic methods and by introducing the notion of (semi-positively curved) singular Hermitian metrics. The results mentioned above follow the philosophy of the later stream. Let us remark that for a torsion free sheaf on a (quasi-)projective variety, the existence of a semi-positively curved singular Hermitian metric implies the weak positivity, while the reciprocal implication is not known yet (it is in fact a singular version of Griffiths’s conjecture). The advantage to have a such metric is that: in case that the determinant line bundle is trivial, one can

further deduce, by using the L^2 extension property, that this torsion free sheaf is a Hermitian flat vector bundle (cf. Theorem 1.13), in which way we obtain a stronger regularity of the direct images and our proof of Main Theorem A, like [25], leans on this.

As a corollary of Theorem B, one finds that the induced metric $\det g_{X/Y}^{(m)}$, on the determinant bundle $\det F_m$, has positive curvature current. Now let $Y = T$ be a complex torus; by an induction argument we can further assume that T is a simple torus, that is, containing no non-trivial subtori. Then by a structure theorem for pseudo-effective line bundles on complex tori [25, Theorem 3.3] we have the following dichotomy according the sign of $\det F_m$, :

- there is a integer $m > 0$ sufficiently large and divisible such that $\det F_m$ is ample;
- for every m sufficiently large and divisible, $\det F_m$ is numerically trivial.

Apparently the first case fall into the situation of Part (I) of our Main Theorem A. Hence we only need to tackle the second case, where one can use the L^2 extension property to further conclude that $(F_m, g_{X/Y}^{(m)})$ is a Hermitian flat vector bundle. Furthermore, by a standard argument dated to Kawamata, we are reduced to the case $\kappa(X, K_X + \lfloor F \rfloor) \leq 0$, i.e. it is enough to prove that $\kappa(F, K_F + \lfloor F \rfloor) > 1$ implies $\kappa(X, K_X + \lfloor F \rfloor) > 1$. This reduction relies on the following a log Kähler version of [49, Theorem 1], which follows from [17, Theorem 4.2] or [36, Theorem 1.7] (or Theorem 3.2 for the klt case):

THEOREM C. — *Let X be a compact Kähler manifold. Suppose that there is an effective \mathbb{Q} -divisor $\lfloor F \rfloor$ on X such that $(X, \lfloor F \rfloor)$ is log canonical and that $\kappa(X, K_X + \lfloor F \rfloor) = 0$ (i.e. X is bimeromorphically log Calabi–Yau). Then the Albanese map $\text{alb}_X : X \rightarrow \text{Alb}_X$ of X is a fibre space.*

The proof of this theorem will be given in Section 4, it is similar to that of [49]. In fact, when $\lfloor F \rfloor = 0$ and X projective, the theorem is proved in [49]; for $\lfloor F \rfloor = 0$ and X Kähler a proof is also sketched in [49, Theorem 24], but does not contain enough details. In virtue of [36, Theorem 1.7] (or Theorem 3.2 for the klt case) one can easily obtain Theorem C by following the strategies of [49], and it is exactly in this way our proof in Section 4 proceeds. Let us remark that a similar result with $\lfloor F \rfloor = 0$ for special varieties in the sense of Campana is also stated in [17] where the proof is sketched based on [49].

Now we are reduced to show that $\kappa(F, K_F + \lfloor F \rfloor) > 1$ implies that $\kappa(X, K_X + \lfloor F \rfloor) > 1$. F_m being Hermitian flat, it is given by a unitary representation ρ_m of the fundamental group of T ; $\pi_1(T)$ being Abelian,

this representation is decomposed into 1-dimensional sub-representations. If the image of ρ_m is finite, then one can use the parallel transport to extend pluricanonical sections on F to X ; if the image of ρ_m is infinite, then a fortiori $\kappa(X, K_X + \epsilon) > 1$ by the following pluricanonical klt Kähler version of the structure theorem on cohomology jumping loci à la Green–Lazarsfeld–Simpson (cf. [42, 70]), which consist of another key ingredient of the proof of Main Theorem A (II):

THEOREM D. — *Let $g : X \rightarrow Y$ be a morphism between compact Kähler manifolds. Let ϵ be an effective \mathbb{Q} -divisor on X such that (X, ϵ) is a klt pair. Then for every $m > 0$ such that $m\epsilon$ is an integral divisor and for every $k > 0$, the cohomology jumping locus*

$$V_k^0(g(K_X^m \otimes \mathcal{O}_X(m\epsilon))) := \{\rho \in \text{Pic}^0(X) \mid h^0(Y, g(K_X^m \otimes \mathcal{O}_X(m\epsilon)) \otimes \rho) > k\}$$

is a finite union of torsion translates of subtori in $\text{Pic}^0(Y)$.

The study of cohomology jumping loci is initiated by the works of Green–Lazarsfeld [41, 42] which assure that the components of cohomology jumping loci are translates of subtori, and is further developed by Carlos Simpson in [70], where he proves that these translates are torsion translates. Recently, the main result of [70] is generalized by Botong Wang to the Kähler case in [77], where he treats the case $g = \text{id}_X$, $m = 1$ and $\epsilon = 0$ and this is the starting point of our proof of Theorem D. In fact, when $g = \text{id}_X$ and X projective, the proof of the theorem is already implicitly comprised in [20] although they only explicitly state and prove a result corresponding to our Corollary 5.9 with X smooth projective and (X, ϵ) log canonical by using [70]; we thus follow the strategy in [20] to deduce Theorem D from the basic case treated in [77, Corollary 1.4]. Notice that [77] and hence our Theorem D require that X is “globally” Kähler; by contrast, Theorem B holds for any Kähler fibre space (X is only assumed to be locally Kähler over Y). Let us remark that in the hypothesis of $C_{n,m}^{\text{log}}$ it is essential to suppose that X is globally Kähler, in fact [71, Remark 15.3, p. 187] provides an example of a Kähler fibre space for which $C_{n,m}$ does not hold.

Let us explain how to finish the proof of Part (II) of Main Theorem A from Theorem D. By following the argument in [21] one easily deduces from Theorem D (cf. Corollary 5.9):

- $K_X + \epsilon$ is the most effective \mathbb{Q} -line bundle in its numerical class.
- If $\kappa(X, K_X + \epsilon) = \kappa(X, K_X + \epsilon + L) = 0$ for some numerically trivial (\mathbb{Q} -)line bundle L , then L is a torsion point in $\text{Pic}^0(X)$.

Now the proof of Main Theorem A (II) can be finished as follows: if $\text{Im}(\rho_m)$ is infinite, by the decomposition of F_m , one sees that $K_X + \Delta$ has non-negative Kodaira dimension up to twisting a non-torsion numerically trivial (\mathbb{Q} -)line bundle, hence the first point above shows that $\kappa(X, K_X + \Delta) > 0$; moreover, if $\kappa(X, K_X + \Delta) = 0$ then the second point will lead to a contradiction, hence a fortiori $\kappa(X, K_X + \Delta) > 1$, thus we finish the proof of Main Theorem A. As a by-product of the first point above, we can prove the Kähler version of the (generalized) log Abundance Conjecture in the case of numerical dimension zero (cf. Theorem 5.11) by using the divisorial Zariski decomposition obtained in [9] (cf. [9, Definition 3.7]).

Let us remark that one can follow the same strategies in [25, §5] to prove more generally that the conjecture $C_{n,m}^{\log}$ is true if $\det F_m$ is numerically trivial for some $m \in \mathbb{Z}_{>0}$ (i.e. the Kähler version of [25, Theorem 5.6]) by using the remarkable result of Zuo in [79, Corollary 1]. In this article, however, we will not further pursue in this direction.

Finally by using an induction argument and by applying the results already obtained we generalize Part (II) of Main Theorem A to the geometric orbifold setting:

THEOREM E. — *Let $f : X \rightarrow T$ be a fibre space with X compact Kähler manifold and T complex torus and let F be the general fibre of f . Let Δ be an effective \mathbb{Q} -divisor on X such that (X, Δ) is klt. Then*

$$\kappa(X, K_X + \Delta) > \kappa(F, K_F) + \kappa(T, B_f).$$

where $K_F := K_X|_F$ and B_f denotes the branching divisor on T w.r.t. f and Δ .

In the theorem above, the branching divisor is defined as following: for any analytic fibre space $f : (X, \Delta) \rightarrow Y$ between compact complex manifolds with Δ an effective \mathbb{Q} -divisor on X , the branching divisor B_f (with respect to f and Δ) is defined as the most effective \mathbb{Q} -divisor on Y such that $f^* B_f \leq R_f$ modulo exceptional divisors, where the ramification divisor (w.r.t. f and Δ) is defined as $R_f := R_f + \Delta$ and

$$R_f := \sum_{f(W) \text{ is a divisor on } Y} (\text{Ram}_W(f) - 1)W$$

with $\text{Ram}_W(f)$ denoting the ramification (in codimension 1) index of f along W . Precisely, assume the singular locus of f is contained in a (reduced) divisor Δ_Y on Y and write

$$\Delta_Y = \sum_{i \in I} b_i W_i,$$

where W_i are prime divisors on X , then for $i \in I^{\text{div}}$ where

$$I^{\text{div}} := \text{set of indices } i \in I \text{ such that } f(W_i) \text{ is a divisor on } Y,$$

we have $b_i = \text{Ram}_{W_i}(f)$ and thus

$$f = \sum_{i \in I^{\text{div}}} (b_i - 1)W_i.$$

Let us remark that the above definition of B_f coincides with [17, Definition 1.29] (orbifold base) when f is lc on X , cf. Section 7.

The organization of the article is as follows. In Section 1 we recall some preliminary results which may be of independent interest; especially, the definition of semi-positively curved singular Hermitian metrics and that of the L^2 extension property are formulated in Section 1.2. Section 2 is dedicated to the construction of the m -Bergman kernel metric on the adjoint line bundle and of the canonical L^2 metric on direct images as well as the proof of Theorem B. The Part (I) of Main Theorem A and the conjecture $C_{n,m}^{\text{log}}$ for general type bases are established in Section 3.1 and Section 3.2 respectively. And the proof of Theorem C is done in Section 4. The general definition of cohomology jumping loci for coherent sheaves, as well as the proof of Theorem D will be given in Section 5, where Corollary 5.9 and Theorem 5.11 are proved in Section 5.3. In Section 6 we complete the proof of the Part (II) of our Main Theorem A. And finally the geometric orbifold version Theorem E is established in Section 7.

Acknowledgements

The author owes a lot to his thesis advisors Sébastien Boucksom and Junyan Cao, who have given him enormous help to accomplish this work as well as to polish the present article. He is also grateful to Daniel Barlet, Nero Budur, Frédéric Campana, Benoit Claudon, Andreas Höring, Sándor Kovács, Mihnea Popa and Chenyang Xu for helpful discussions about this work. Also the author would like to take this opportunity to acknowledge the support he has benefited from the ANR project “GRACK” during the preparation of the present article. Last but not least, the author would like to express his gratitude to the anonymous referee for the constructive criticism and excellent suggestions, which help to substantially improve the presentation of this work and to greatly simplify the proof of several results.

1. Preliminary Results

In this section, we collect miscellaneous results which not only serve our main purpose but also are of independent interest.

1.1. An Analytic Geometry Toolkit

In this subsection we state some auxiliary results which are well-known in algebraic geometry, but whose analytic versions, as far as we know, have not yet been well formulated in literatures; we will not give the detailed proofs but instead indicate how to get rid of the algebraicity hypothesis.

(A). A Covering Lemma. First we state a covering lemma which allow us to reduce problems on pluricanonical bundles to the case of the canonical bundle.

LEMMA 1.1. — *Let X be compact complex manifold. and let L be a line bundle on X such that $\kappa(X, L) > 0$. Suppose that there exists an integer $m > 0$ such that there exists an effective divisor $D \in |L^m|$ whose support is SNC. Then there is a compact complex manifold V admitting a surjective generically finite projective morphism $f : V \rightarrow X$ such that the direct image of K_V admits a direct decomposition:*

$$f_* K_V = \bigoplus_{i=0}^{m-1} K_X \otimes L^i \otimes \mathcal{O}_X\left(-\left\lfloor \frac{i}{m} D \right\rfloor\right).$$

The construction of f is done by taking a cyclic cover along D followed by a desingularization. This construction is standard. However, there are three main ingredients in this construction that need to be clarified:

- (a) The construction of cyclic covers: cf. [59, §4.1.B, pp. 242–243, vol.I] and [56, §2.9, p. 9], which can be easily generalized to the analytic case.
- (b) Viehweg’s results on rational singularities in [73]:
 - (b1) A finite ramified cover over a smooth projective variety with the cover space being normal and the branching locus being a SNC divisor, has quotient singularities ([73, Lemma 2]); in this case, the singularity is toroidal, and the result is standard from [58].
 - (b2) A quotient singularity is a rational singularity ([73, Proposition 1]). This follows from Kempf’s criterion on rationality of singularities (cf. [58, §I.3, condition (d)(e) pp. 50–51]), which is essentially an analytic result.

- (c) A duality theorem for canonical sheaves (the canonical sheaf of a complex variety is defined as the $(-d)$ -th cohomology of the dualizing complex, where d denotes the dimension of the complex variety) on singular complex varieties, which can be proved by applying [67] or [2] combined with a spectral sequence argument.

Remark 1.2. — For later use, we remark that the point (b2) above can be further generalized to higher relative dimension by a local computation as in [74, Lemma 3.6] and by [58]: for $f : X \rightarrow Y$ be a proper flat morphism between complex manifolds such that the singular locus $\text{Sing } Y$ is a smooth divisor and the preimage $f^{-1}(\text{Sing } Y)$ is a *reduced* SNC divisor, then for any surjective morphism $\phi : Y' \rightarrow Y$ with Y' smooth, the fibre product $X \times_Y Y'$ has (at most) rational singularities. Cf. also [47, 3.13.Lemma].

(B). The Negativity Lemma. The negativity lemma is an important tool in the study of the classification theory of complex varieties. It is already well known in the algebraic case, cf. [57, Lemma 3.39, p. 102-103]. By following the strategy of [12, Proposition 2.12] one can prove the following analytic version (for the convenience of the readers, we provide a proof in Appendix A):

LEMMA 1.3 (Negativity Lemma). — *Let $h : Z \rightarrow Y$ be a proper bimeromorphic morphism between normal complex varieties. Let B be a Cartier divisor on Z such that $-B$ is h -nef. Then B is effective if and only if $h_* B$ is effective.*

(C). A Flattening Lemma. In order to prove Part (I) of Main Theorem A we need the following auxiliary result, which is an analytic version of [74, Lemma 7.3]:

LEMMA 1.4. — *Let $p : V \rightarrow W$ a morphism of complex manifolds, then there exists a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{\pi_V} & V \\ p \downarrow & & \downarrow p \\ W & \xrightarrow{\pi_W} & W \end{array}$$

with V and W complex manifolds, the morphisms π_W and π_V projective and bimeromorphic such that the morphism p verifies the following property: every p -exceptional (i.e. $\text{codim}_W p^{-1}(D) > 2$) divisor D of V is π_V -exceptional (i.e. $\text{codim}_V(\pi_V^{-1}(D)) > 2$). In addition, we can further assume that

- (a) π_W is an isomorphism over W_0 , the (analytic) Zariski open subset of W over which p is smooth;
- (b) π_V is an isomorphism over $p^{-1}W_0$;
- (c) $\mu^{-1} := \pi_W^{-1}(W \setminus W_0)$ and $p^{-1} \mu^{-1}$ are divisors of SNC support.

Proof. — This is simply a consequence of [46, Flattening Theorem].

In the sense of [17], the lemma above shows that any fibre space admits a (higher) bimeromorphic model which is *neat* and *prepared* (cf. [17, §1.1.3]). Moreover, Lemma 1.4 is well behaved with respect to klt/lc pairs, as implies the following fact:

LEMMA 1.5. — *Let X be a complex variety and μ^{-1} an effective \mathbb{Q} -divisor on X such that the pair (X, μ^{-1}) is klt (resp. lc). For any log resolution $\mu : X \dashrightarrow X$ of (X, μ^{-1}) , there is an effective \mathbb{Q} -divisor μ^{-1} over X with SNC support such that the pair (X, μ^{-1}) is also klt (resp. lc) and that $\mu^{-1} = \mu^{-1}$. Moreover we have $\kappa(X, K_X + \mu^{-1}) = \kappa(X, K_X + \mu^{-1})$.*

Proof. — This is well known to experts of MMP, we nevertheless give a proof for the convenience of the readers. The pair (X, μ^{-1}) being klt, we can write (an isomorphism of \mathbb{Q} -line bundles):

$$K_X + \mu^{-1} = \sum_{a_i < 0} a_i E_i + \mu^{-1} (K_X + \mu^{-1}) + \sum_{a_i > 0} a_i E_i, \tag{1.1}$$

where the E_i 's are μ -exceptional prime divisors and

$$a_i := a(E_i, X, \mu^{-1})$$

denotes the discrepancy of E_i with respect to the pair (X, μ^{-1}) . Put

$$\mu^{-1} := \mu^{-1} - \sum_{a_i < 0} a_i E_i,$$

then μ^{-1} is an effective \mathbb{Q} -divisor with SNC support and $\mu^{-1} = \mu^{-1}$. The hypothesis that (X, μ^{-1}) is klt (resp. lc) implies that $a_i > -1$ (resp. $a_i > -1$) for every i and that the coefficients of prime components in μ^{-1} are < 1 (resp. ≤ 1), hence the coefficients of the prime components in μ^{-1} are all < 1 (resp. ≤ 1). By [57, Corollary 2.31(3), p. 53] the pair (X, μ^{-1}) is klt (resp. lc). The equality $\kappa(X, K_X + \mu^{-1}) = \kappa(X, K_X + \mu^{-1})$ results from [28, Lemma 7.11, p. 175] and (1.1).

1.2. Griffiths Semi-positive Singular Hermitian Metrics on Vector Bundles / Torsion Free Sheaves

In this subsection we will recall the notion of Griffiths semi-positively curved singular Hermitian metrics on vector bundles / torsion free sheaves.

Cf. [24, 2.1 et 2.2] for a generalization of this semi-positivity notion. Let us fix X a complex manifold.

DEFINITION 1.6. — *Let E be holomorphic vector bundle on X . A (Griffiths) semi-positively curved singular Hermitian metric h on E is given by a measurable family of Hermitian functions on each fibre of E , such that for every (holomorphic) local section $s \in H^0(U, E^*)$ of the dual bundle E^* , the function $\log |\sigma|_h^2$ is psh on U . The vector bundle E is said semi-positively curved if it admits a semi-positively curved singular metric.*

Remark 1.7. — This definition implies that h is bounded almost everywhere, moreover, fix any smooth Hermitian metric h_0 on E , then as a consequence of [65, 2.10.Remark, 2.18.Remark] the singular metric h is locally uniformly bounded from below by $C \cdot h_0$ for some constant $C > 0$.

The semi-positivity of singular Hermitian metrics is preserved by tensor products, pull-back by proper surjective morphisms, and by generically surjective morphisms of vector bundles (thus by symmetric products and wedge products), cf. [40, II.B.4] and [65, 2.14.Lemma, 2.15.Lemma]. Moreover one has the following extension theorem for semi-positively curved singular Hermitian metrics:

PROPOSITION 1.8 (cf. [24, 2.4.Proposition]). — *Let E be a holomorphic vector bundle on X . Suppose that there is a (analytic) Zariski open subset $X_0 = ?$ of X and a semi-positively curved singular Hermitian metric h on $E|_{X_0}$. Then h extends to a semi-positive singular Hermitian metric on E if one the following two conditions is verified:*

- (1) $\text{codim}(X \setminus X_0) > 2$;
- (2) h is locally uniformly bounded below by a constant $C > 0$ on X_0 with respect to some smooth Hermitian metric on E .

In virtue of Proposition 1.8 and [55, Corollary 5.5.15, p. 147] one can extend Definition 1.6 to torsion free sheaves:

DEFINITION 1.9. — *Let X be a complex manifold and let F be a torsion free sheaf on X . By [55, Corollary 5.5.15, p. 147], F is locally free in codimension 1. A semi-positively curved singular Hermitian metric h on F is a semi-positively curved singular Hermitian metric on $F|_U$ for some (analytic) Zariski open subset U such that $\text{codim}_X U > 2$ and $F|_U$ locally free. The torsion free sheaf F is said semi-positively curved if it admits a semi-positively curved singular Hermitian metric.*

Remark 1.10. — The notion of semi-positively curved metric on torsion free sheaves can lead to some unexpected pathology, e.g. the ideal sheaf I_Z of an analytic subset Z of codimension > 2 admits a natural semi-positively

curved singular Hermitian metric. In order to exclude such pathology we introduce in Definition 1.12 below the notion of “ L^2 -extension property”.

Let F and h as in Definition 1.9 above, then h induces a semi-positively curved singular Hermitian metric $\det h$ on the line bundle $\det F$ where the determinant bundle $\det F$ is defined as

$$\det F := \left(\bigwedge^r F \right)$$

with $r = \text{rk } F$ and $(\cdot) = (\cdot)$ denotes the reflexive hull (cf. [55, §5.6, pp. 149–154]).

We end this subsection by two regularity theorems:

THEOREM 1.11. — *Let (E, h) be a holomorphic vector bundle on X equipped with a semi-positively curved singular Hermitian metric h . Suppose that the metric $\det h$ is locally bounded from above, then the coefficients of the Chern connection form θ_E (defined by the equation $h\theta_E = \partial h$) are L^2_{loc} on U , and in consequence the total curvature current $\chi_h(E)$ of E is well defined and semi-positive in the sense of Griffiths, which can be locally written as $\chi_h(E) = \bar{\partial}\theta_E$. In particular, if the curvature current $\chi_{\det h}$ vanishes, then (E, h) is Hermitian flat.*

For the proof, see [68, Theorem 1.6] and [25, 2.7.Theorem] (cf. also [65, 2.25.Theorem, 2.26.Corollary]). Heuristically, this is a higher rank version of the well known fact (the line bundle case) that if a psh function ϕ is L^2_{loc} , then ϕ is L^2_{loc} and the last statement results from the ellipticity of the Laplacian $\partial\bar{\partial}$.

In the sequel we introduce the notion of “ L^2 -extension property”, which is simply an analogue of the property of \mathcal{O} that every L^2 holomorphic function extends. It helps to exclude certain unexpected pathology as mentioned in Remark 1.10, e.g. the natural semi-positively curved (generically flat) singular Hermitian metric on the ideal sheaf \mathcal{I}_Z of an analytic subset Z of codimension > 2 does not satisfy the L^2 extension property.

DEFINITION 1.12. — *Let F be a torsion free sheaf on X equipped with a singular Hermitian metric h . h is said to satisfy the “ L^2 -extension property” if for any open subset $U \subset X$, for any $Z \Subset U$ analytic subset of U such that F is locally free over $U \setminus Z$ and for any section $\sigma \in H^0(U \setminus Z, F)$ such that*

$$\int_{U \setminus Z} |\sigma|_h^2 d\mu < +\infty,$$

the section σ extends (uniquely) to a section $\bar{\sigma} \in H^0(U, F)$.

This propriety is particularly useful when we consider a torsion free sheaf whose determinant bundle is numerically trivial. In fact, we have the following:

THEOREM 1.13. — *Let X be a connected complex manifold and let F be a torsion free sheaf of rank r on X equipped with a semi-positively curved singular Hermitian metric h . Suppose that*

- (1) $\det F$ is numerically trivial, i.e. $c_1(\det F) = c_1(F) = 0$;
- (2) h satisfies the L^2 -extension property as in Definition 1.12.

Then (F, h) is a Hermitian flat vector bundle.

Proof. — The proof is essentially analogous to that of [25, Theorem 5.2]. Since h is semi-positively curved, the metric $\det h$ on $\det F$ is semi-positively curved, thus the curvature current $\det h(\det F)$ is positive; but $\det F$ is numerically trivial, hence a fortiori $\det h(\det F) = 0$. Then by Theorem 1.11, $(F|_{X_{\mathcal{F}}}, h|_{X_{\mathcal{F}}})$ is a Hermitian flat vector bundle (i.e. $h|_{X_{\mathcal{F}}}$ is a smooth Hermitian metric $F|_{X_{\mathcal{F}}}$ whose curvature vanishes) where $X_{\mathcal{F}}$ denotes the locally free locus of F . By [55, Proposition 1.4.21, p. 13] the Hermitian flat vector bundle $(F|_{X_{\mathcal{F}}}, h|_{X_{\mathcal{F}}})$ is defined by a representation

$$\pi_1(X_{\mathcal{F}}) \rightarrow \mathrm{U}(r),$$

$\pi_1(X_{\mathcal{F}})$ being isomorphic to $\pi_1(X)$, this extends to a representation

$$\pi_1(X) \rightarrow \mathrm{U}(r),$$

which gives rise to a Hermitian vector bundle (E, h_E) of rank r on X . Then by construction we have an isometry

$$\phi : F|_{X_{\mathcal{F}}} \rightarrow E|_{X_{\mathcal{F}}}.$$

By reflexivity of $\mathrm{Hom}_{\mathcal{O}_X}(F, E)$ this extends to an injection of sheaves $F \hookrightarrow E$ which we still denote by ϕ . It remains to show that ϕ is surjective. The problem being local, we can assume that X is a small open ball, so that E is trivial. Now take $u \in \mathrm{H}^0(X, E)$ a holomorphic section of E , since h_E is a flat metric (hence smooth), $|u|_{h_E, z}$ is finite for every $z \in X$. $\phi|_{X_{\mathcal{F}}}$ being an isometry, there exists a section $v_0 \in \mathrm{H}^0(X_{\mathcal{F}}, F)$ such that $i(v_0) = u|_{X_{\mathcal{F}}}$ and $|v_0|_{h, z} = |u|_{h_E, z} < +\infty$ for all $z \in X_{\mathcal{F}}$. But (F, h) satisfies the L^2 extension property, v_0 extends to a section $v \in \mathrm{H}^0(X, F)$, thus $\phi(v) = u$, implying the surjectivity of ϕ .

Let us remark that the condition on the L^2 extension property is indispensable in the theorem above. For example, as mentioned above, the ideal sheaf \mathcal{I}_Z of an analytic subset Z of codimension > 2 admits a natural semi-positively curved singular Hermitian metric $h_{\mathcal{I}_Z}$, which equals to the flat metric of \mathcal{O} on $X \setminus Z$. The determinant of \mathcal{I}_Z is trivial, but definitely \mathcal{I}_Z

is not a (Hermitian flat) vector bundle. Notice that $(\mathcal{L}|_Z, h|_Z)$ does not satisfy the L^2 extension property: let B be a small ball in X meeting Z , then non-zero constant functions on $B \setminus Z$ (which are L^2) cannot extend across Z .

1.3. Reflexive Hull of the Direct Image of Line Bundles

In this subsection we will prove the following theorem, which is nothing but an analytic version of [64, III.5.10.Lemma, pp. 107–108]. The proof of the theorem is not essentially different from that in [64]; except that, for the analytic case, one has to modify the arguments, especially in the Step 2 below, so that one can avoid the usage of the relative Zariski decomposition (which is not known in analytic case; even in the algebraic case, it is only established in some special cases in [64] and it does not hold in general due to a counterexample in [60]).

THEOREM 1.14. — *Let $\pi : X \rightarrow S$ be a proper surjective morphism between normal complex varieties, and let L be a π -effective (i.e. $\pi_* L = 0$) line bundle on X . Then there is an effective π -exceptional (i.e. $\text{codim}_S \pi(E) > 2$) Weil divisor E such that for any $k \in \mathbb{Z}_{>0}$ one has*

$$[\pi_* (L^{\otimes k})] \subset \pi_* [L^{\otimes k} \otimes \mathcal{O}_X(kE)]. \quad (1.2)$$

Intuitively the theorem means that the vertical poles of the sections of $L^{\otimes k}$ are linearly bounded. The proof of Theorem 1.14 proceeds in five steps:

Step 0. First let us remark that we can always assume that X is smooth by taking a desingularization by the following observation

LEMMA 1.15. — *Let $h : Z \rightarrow Y$ a bimeromorphic morphism between normal complex varieties. Then for every Weil divisor D on Z , we have an inclusion*

$$h_* \mathcal{O}_Z(D) \subset \mathcal{O}_Y(h_* D).$$

Proof. — Since h is an isomorphism over a(n) (analytic) Zariski open subset of codimension > 2 in Y , $h_* \mathcal{O}_Z(D)$ and $\mathcal{O}_Y(h_* D)$ are isomorphic in codimension 1; $h_* \mathcal{O}_Z(D)$ being torsion free and $\mathcal{O}_Y(h_* D)$ reflexive, we have (noting that on a normal complex variety reflexive sheaves are determined in codimension 1):

$$h_* \mathcal{O}_Z(D) \subset (h_* \mathcal{O}_Z(D)) \otimes \mathcal{O}_Y(h_* D).$$

In fact, assume that Theorem 1.14 holds for X smooth, let us prove that it holds in general. To this end, let $\mu : X' \rightarrow X$ be a desingularization of X , then by our assumption, there is an effective divisor E on X' such that

$$[\pi_* (\mu_* L^{\otimes k})] \subset (\pi_*) [\mu_* L^{\otimes k} \otimes \mathcal{O}_{X'}(kE)],$$

hence by Lemma 1.15 and the projection formula we have an inclusion

$$\begin{aligned} [\pi(L^k)] &= \pi(L^k \otimes_{\mu} \mathcal{O}_X(kE)) \subset \pi(L^k \otimes_{\mathcal{O}_X} \mathcal{O}_X(kE)) \\ &\subset [\pi(L^k \otimes_{\mathcal{O}_X} \mathcal{O}_X(kE))] \end{aligned}$$

where $E := \mu^* E$. Since the inclusion is an isomorphism in codimension 1, it is in fact an equality. Consequently, we always assume that X is smooth in the sequel.

Step 1. By the coherence of the reflexive hull $(\pi^* L)$ there is an π -exceptional divisor E making the equation (1.2) holds for $k = 1$ (and thus one can choose E such that (1.2) holds for a finite number of k).

Step 2. In virtue of Step 1 we are able to prove the reflexivity criterion below:

PROPOSITION 1.16 (Reflexivity Criterion). — *Let $\pi : X \rightarrow S$ and L as in Theorem 1.14. Suppose that for every effective π -exceptional divisor G , there is a component G_i of G such that $[L \otimes_{\mathcal{O}_X}(G_i)]$ is not π -pseudoeffective, then $\pi^* L$ is reflexive on S .*

Let us recall the notion of relative pseudoeffectivity for \mathbb{Q} -line bundles / Cartier divisors in the analytic setting: Let $p : V \rightarrow W$ a proper surjective morphism of analytic varieties and let L be a \mathbb{Q} -line bundle on V , then L is said to be p -pseudoeffective if its pull-back L/\tilde{F} is pseudoeffective (cf. [29, §6.A, (6.2) Definition, p. 47]) where \tilde{F} denotes a desingularization of the general fibre F of p . A \mathbb{Q} -Cartier divisor D on V is said to be p -pseudoeffective if its associated \mathbb{Q} -line bundle $\mathcal{O}_X(D)$ is so. Before going to the proof let us first prove the following auxiliary lemma:

LEMMA 1.17. — *Let $\pi : X \rightarrow S$ and L as in Proposition 1.16, then for any effective π -exceptional divisor B on X , one has:*

$$\pi^* L \otimes_{\mathcal{O}_X} \pi^* [L \otimes_{\mathcal{O}_X}(B)] \quad (1.3)$$

Proof. — B is effective, one can write

$$B = \sum_{i=1}^r b_i B_i,$$

with $b_i \in \mathbb{Z}_{>0}$ and $r \in \mathbb{N}$ ($r = 0$ simply means that $B = 0$). Note

$$b := \sum_{i=1}^r b_i.$$

Now let us prove (1.3) by induction on b :

By our hypothesis on L (the condition in Proposition 1.16), $i \in \{1, \dots, r\}$ such that $[L \otimes \mathcal{O}_X(B)]/_{B_i}$ is non- $\pi|_{B_i}$ -pseudoeffective, thus

$$(\pi|_{B_i})_* [L \otimes \mathcal{O}_X(B)]/_{B_i} = 0.$$

Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-B_i) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{B_i} \rightarrow 0.$$

By tensoring with $L \otimes \mathcal{O}_X(B)$ and applying the functor π_* on gets

$$0 \rightarrow \pi_* [L \otimes \mathcal{O}_X(B - B_i)] \rightarrow \pi_* [L \otimes \mathcal{O}_X(B)] \rightarrow (\pi|_{B_i})_* [L \otimes \mathcal{O}_X(B)]/_{B_i} = 0$$

hence $\pi_* [L \otimes \mathcal{O}_X(B - B_i)] \cong \pi_* [L \otimes \mathcal{O}_X(B)]$. Apply the induction hypothesis we obtain that $\pi_* [L \otimes \mathcal{O}_X(B - B_i)] \cong \pi_* L$, which proves the isomorphism (1.3).

Now return to the proof of the Reflexivity Criterion 1.16:

Proof of Proposition 1.16. — We will show in the sequel that it suffices to prove Lemma 1.17 below. In fact, by Step 1 there is an effective π -exceptional E , such that

$$(\pi_* L) \otimes \pi_* [L \otimes \mathcal{O}_X(E)];$$

Apply Lemma 1.17 to E and we obtain:

$$\pi_* L \otimes \pi_* [L \otimes \mathcal{O}_X(E)] \cong (\pi_* L) \otimes \pi_* [L \otimes \mathcal{O}_X(E)],$$

hence $\pi_* L$ is reflexive.

Step 3. In this step, we prove that in the situation of Theorem 1.14 there exists a π -exceptional divisor which is not relatively pseudoeffective on each component of $\text{Exc}(\pi)$. More precisely we will show:

PROPOSITION 1.18. — *For any $\pi : X \rightarrow S$ as in Theorem 1.14, there is an effective π -exceptional divisor E such that for any π -exceptional prime divisor F , $E|_F$ is not $\pi|_F$ -exceptional.*

The proof is the same as that in [64, III.5.10.Lemma, pp. 107–108]. For the convenience of the readers, we provide the details in Appendix B.

Step 4. Let $\pi : X \rightarrow S$ and L a π -effective line bundle on X as in Theorem 1.14. The problem begins local, one can replace X (resp. S) by a neighbourhood of a compact in X (resp. in S); in particular the set of π -exceptional prime divisors, denoted by $E_{\text{exc}}(\pi)$, is a finite set, and thus we can write:

$$E_{\text{exc}}(\pi) = \{F_1, \dots, F_t\}$$

By Step 3 an effective π -exceptional divisor E such that $E|_i$ is non- $\pi|_i$ -pseudoeffective. In the sequel we will deduce Theorem 1.14 from the Reflexivity Criterion 1.16:

1. — E being π -exceptional effective, we can write

$$E = \sum_{i=1}^t a_i \cdot \mathcal{O}_i, \quad a_i \in \mathbb{Z}_{>0}.$$

We claim that the a_i 's are all strictly positive. Otherwise, there exists a j such that $a_j = 0$, implying that $\mathcal{O}_j \subset \text{Supp}(E)$, then $E|_j$ is an effective divisor, in particular it is $\pi|_j$ -pseudoeffective, contradicting the hypothesis on E .

2. — Moreover claim that there is a $b \in \mathbb{Z}_{>0}$ such that $\beta > b, \beta \in \mathbb{Q}_{>0}$, $(L + \beta E)|_i$ is a \mathbb{Q} -line bundle which is non- $\pi|_i$ -pseudoeffective for all $i = 1, 2, \dots, t$. Otherwise there is a sequence of positive rational numbers $\beta_n \rightarrow +\infty$ such that for every n , $(L + \beta_n E)|_{i_n}$ is a $\pi|_{i_n}$ -pseudoeffective \mathbb{Q} -line bundle for some i_n . In general i_n should depend on n , but $E_{\text{ex}}(\pi)$ being a finite set, there must exist an i appearing an infinity of times in the sequence $(i_n)_{n>0}$, thus up to taking a subsequence one can suppose that there exists an i such that $(L + \beta_n E)|_i$ is a $\pi|_i$ -pseudoeffective \mathbb{Q} -line bundle for every n . Hence

$$\left(E + \frac{1}{\beta_n} L \right) \Big|_i$$

is an $\pi|_i$ -pseudo-effective \mathbb{Q} -line bundle for every n . This implies (by letting $\beta_n \rightarrow +\infty$) that $E|_i$ is $\pi|_i$ -pseudoeffective, contradicting to the point 1 above.

3. — Let us set

$$L_k = L^k \otimes \mathcal{O}_X(kbE),$$

then in order to prove Theorem 1.14 we only need to show that $\pi^* L_k$ is reflexive. In fact, since S is normal, and since $\pi^* (L^k)$ and $\pi^* L_k$ are isomorphic outside an analytic subset of codimension > 2 , therefore as soon as $\pi^* L_k$ is reflexive, we get immediately

$$\pi^* L_k \cong [\pi^* (L^k)]^{\otimes b}.$$

In the sequel we will prove that $\pi^* L_k$ is reflexive in virtue of Proposition 1.16. It suffices to check that L_k satisfies the conditions in Proposition 1.16: let G be an π -exceptional effective divisor, then there is a minimal $c \in \mathbb{Q}_{>0}$ such that $cE > G$. In fact, if we write

$$G = \sum_{i=1}^t g_i \cdot \mathcal{O}_i,$$

then we can take

$$c = \max_{i=1, \dots, t} \left\{ \frac{g_i}{a_i} \right\}.$$

In particular, by the minimality of c there exists an i such that $\text{Supp}(cE - G)|_i$ is $\pi|_i$ -pseudo-effective. However by the point 2 above, the \mathbb{Q} -line bundle

$$(L_k + G)|_i + (cE - G)|_i = k \left[L + \left(b + \frac{c}{k} \right) E \right]|_i$$

is non- $\pi|_i$ -pseudoeffective, hence a fortiori the line bundle $(L_k + G)|_i$ is not $\pi|_i$ -pseudoeffective. Therefore L_k satisfies the conditions in Proposition 1.16, thus L_k is reflexive.

2. Positivity of Relative Pluricanonical Bundles and of their Direct Images

Let $f : X \rightarrow Y$ be a Kähler fibre space between complex manifolds, that is, a proper surjective morphism with connected fibres (an analytic fibre space) such that locally over Y the complex manifold X is Kähler (cf. the definitions in the Introduction). Let (L, h_L) be a line bundle on X equipped with a singular Hermitian metric h_L whose curvature current $\Theta_{h_L}(L)$ is positive. The main purpose of this section is to establish the positivity result for the L -twisted relative pluricanonical bundles and their direct images mentioned in the Introduction (cf. Theorem 2.3 and Theorem 2.6). To this end, we will explain the construction of the relative m -Bergman kernel metric $h_{X/Y, L}^{(m)}$ on $K_{X/Y}^m \otimes L$ and of the canonical L^2 metric $g_{X/Y, L}$ on the direct image sheaf $f_*(K_{X/Y}^m \otimes L \otimes \mathcal{J}(h_L))$.

Let us recall briefly the history of the study of these canonical metrics. Initially, the case with h_L a smooth metric and f smooth is considered in [3], where the positivity of $f_*(K_{X/Y} \otimes L)$ is proved by an explicit calculation of the curvature; as a simple consequence, one deduces the positivity of the relative Bergman kernel metric (with $m = 1$), cf. [5, §1, p. 348]. In the more general case where f is projective but not necessarily smooth and $f_*(K_{X/Y} \otimes L)$ is locally free, the positivity of $f_*(K_{X/Y} \otimes L)$ is proved in [5, Theorem 3.5] based on the work of Berndtsson; this result, is in turn used in [5, Corollary 4.2] to prove the positivity of the relative m -Bergman kernel metric under the assumption that the direct image sheaf $f_*(K_{X/Y}^m \otimes L)$ is locally free. In [66], these positivity results are established for f projective with the locally freeness conditions for direct images removed: it is made clear that the positivity of the relative m -Bergman kernel metric can be regarded as a result of the Ohsawa–Takegoshi extension Theorem with the

optimal estimate, and thus can be obtained independent of the positivity of direct images; while the proof of the positivity of $f(K_{X/Y} \otimes L)$ is based on [5] and is done by a semistable reduction plus an explicit calculation. A little later, it is realized that the positivity of the canonical metric is also a consequence of the Ohsawa–Takegoshi extension theorem with the optimal estimate, as is explained in [45]. Therefore in order to obtain a Kähler version of this theorem, all one needs is to generalize the Ohsawa–Takegoshi extension theorem to the Kähler case. Thanks to [23], this result is established and the positivity of the relative m -Bergman kernel metric is also proved in [23] as a corollary; in consequence, by virtue of the main result in [23] one can follow the same arguments in [45] to demonstrate the positivity of the canonical L^2 metric $g_{X/Y,L}$ for f Kähler fibre space. Recently we are informed that this result is established in [33] by following the strategy of [45] and by a more general positivity theorem for singular Finsler metrics on direct images. For the convenience of the readers, we will nevertheless provide some details of the proof in Section 2.3.

2.1. Ohsawa–Takegoshi Extension Theorems

As is explained above, the key point of the proof of Theorem 2.6, like many other results in complex geometry, is the Ohsawa–Takegoshi extension theorem. In this subsection we will state theorems of Ohsawa–Takegoshi type for Kähler fibre spaces in the following two forms:

Local Version. For a Kähler fibre space whose base is an open ball in some \mathbb{C}^d , we have the following extension theorem of Ohsawa–Takegoshi type with optimal estimation:

THEOREM 2.1 (higher dimensional version of [23, Theorem 1.1 (Corollary 3.1)]). — *Let $p : X \rightarrow B$ be a analytic (Kähler) fibre space with X a Kähler manifold and $B \subset \mathbb{C}^d$ the open ball of centre 0 and of radius R . Let (L, h_L) be a holomorphic line bundle on X equipped with h_L a singular Hermitian metric such that the curvature current of h_L is positive. Suppose that $X_0 := p^{-1}(0)$ is a smooth fibre of p , and that h_{L/X_0} is not identically $+\infty$. Then for any holomorphic section $f \in H^0(X_0, K_{X_0} \otimes L|_{X_0} \otimes (h_{L/X_0}))$, there exists a section $F \in H^0(X, K_X \otimes L)$ such that $F|_{X_0} = f$ and*

$$\frac{1}{\mu(B)} \int_X |F|^2 e^{-\phi_L} \leq \int_{X_0} |f|^2 e^{-\phi_L},$$

where $\mu(B)$ denotes the Lebesgue measure of B .

Proof. — We obtain the theorem by applying [23, Theorem 1.3] to the fibre space $X \xrightarrow{p} B$ with $E = p^* \mathcal{O}_B^d$, $v = p^* t$ where $t = (t_1, \dots, t_d)$ and

t_i 's are standard coordinates of \mathbb{C}^d , $A = 2d \log R$, $c_A(t) = 1$, and by letting $\epsilon > 0$ (cf. also [44, §4.2, Lemma 4.14]). In particular, when $d = 1$ one recovers [23, Theorem 1.1 (Corollary 3.1)].

Global Version. In many cases, one needs a global version of Ohsawa–Takegoshi extension theorem for Kähler fibre spaces over projective bases; in this case, one cannot obtain an optimal estimation, but one still has an surjection of section spaces up to a twisting by an ample line bundle from the base, along with a weaker estimation on the L^2 norm. In fact we have the following:

THEOREM 2.2 (Kähler version of [34, Corollary 2.10]). — *Let Y be a smooth projective variety of dimension d and let $f : X \rightarrow Y$ be a surjective morphism between compact Kähler manifolds with connected fibres. Let (A_Y, y) be any pair with A_Y ample line bundle on Y and $y \in Y_0$ (where Y_0 denotes the smooth locus of f), such that the Seshadri constant*

$$\epsilon(A_Y, y) > \dim Y = d.$$

By [59, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I] such A_Y exists. Let (L, h_L) be any holomorphic line bundle on X equipped with a singular Hermitian metric h_L whose curvature current is positive, such that $h_L|_{X_y} \in \mathcal{H}^+(X_y, K_{X_y} \otimes L|_{X_y} \otimes (h_L|_{X_y}))$, there is a section $\sigma \in \mathcal{H}^0(X, K_X \otimes L \otimes f^ A_Y)$ such that $\sigma|_{X_y} = u$ with an L^2 estimate independent of L .*

For the proof, refer to [34, Corollary 2.10]. Just remark that: in [34] this theorem is only stated for f a projective morphism. The above Kähler version holds because the proof of [34, Corollary 2.10] depends only on [31, (2.8) Theorem] (cf. also [34, Theorem 2.9]), which is valid for any pseudoconvex Kähler manifold.

2.2. Positivity of the Relative m -Bergman Kernel Metric

Let $f : X \rightarrow Y$ be an analytic fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L with curvature current $\Theta_{h_L}(L) > 0$. Set $n = \dim X$, $d = \dim Y$ and $e = \dim X - \dim Y = n - d$. Let us recall the construction of the relative m -Bergman kernel metric on $K_{X/Y}^m \otimes L$. We will follow [25, §2.1] and [23, §3.2]; for more details, cf. [5, §1]. Let Y_0 be the (analytic) Zariski open subset of Y over which f is smooth. Let $x \in f^{-1}Y_0$ and let z_1, \dots, z_{d+e} the local coordinates near x ; write $y = f(x) \in Y_0$ and let t_1, \dots, t_d be local coordinates near y such that $z_{j+e} = f^* t_j$. Suppose in addition that over the

coordinate neighbourhood of x (resp. of y) chosen as above the line bundle L as well as the canonical bundles of X are trivial (resp. the canonical bundle of Y is trivial).

Suppose that $f(K_{X/Y}^m \otimes L) = 0$. We define a $L^{2/m}$ -Finsler norm on $H^0(X_y, K_{X_y}^m \otimes L|_{X_y})$ by taking the integral over the fibre

$$u \Big|_{\frac{2}{m}, y, L} := \int_{X_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L}, \tag{2.1}$$

where ϕ_L denotes the local weight of the metric h_L (we authorize this to be $+$, which is the case when $h_L|_{X_y} = +$). In addition, we denote by F_u the coefficient of $(dz_1 \cdots dz_{d+e})^m$ in the local expression of $u = f(dt_1 \cdots dt_d)^m$. Then local weight $\phi_{X/Y, L}^{(m)}$ of the relative m -Bergman kernel metric $h_{X/Y, L}^{(m)}$ is given by

$$e^{\phi_{X/Y, L}^{(m)}(x)} = \sup_{u \in \frac{2}{m}, y, L \in 1} |F_u(x)|^2. \tag{2.2}$$

Let us remark that if $h_L|_{X_y} = +$, (2.2) is equal to 0 by convention and thus $\phi_{X/Y, L}(x) = -$. The metric $h_{X/Y, L}^{(m)} = e^{-\phi_{X/Y, L}^{(m)}}$ can also be described in an intrinsic way as follows: for $\xi \in (K_{X/Y}^{(-m)} \otimes L^{-1})_x$, we have

$$|\xi|_{h_{X/Y, L}^{(m)}, x} = \sup_{u \in \frac{2}{m}, y, L \in 1} |\xi(u(x))|.$$

Suppose in the sequel of this subsection that f is a Kähler fibre space with X and Y complex manifolds. By using the Ohsawa–Takegoshi extension theorem with optimal estimate (cf. Theorem 2.1) Junyan Cao proved in [23] that the relative m -Bergman kernel metric constructed above is semipositively curved (since the construction is local over Y , the Kähler hypothesis on X and Y in the original statement of [23, Theorem 3.5] is in fact not necessary, and can be replaced by the hypothesis that f is a Kähler fibre space, in which case X is only assumed to be Kähler locally over Y):

THEOREM 2.3 ([23, Theorem 1.2 (Theorem 3.5)]). — *Let $f : X \rightarrow Y$ be a Kähler fibre space with X and Y complex manifolds and (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L whose curvature current is positive. Let m be a positive integer. Suppose that for a general point $y_0 \in Y$ there exists a non-zero section $u \in H^0(X_{y_0}, K_{X_{y_0}}^m \otimes L|_{X_{y_0}})$ satisfying*

$$\int_{X_{y_0}} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L} < + \infty,$$

then the curvature current of the relative m -Bergman kernel metric $h_{X/Y,L}^{(m)}$ is positive. More precisely, there is an (analytic) Zariski open subset of $f^{-1}Y_0$ (cf. Remark 2.4 below) such that the local weight $\phi_{X/Y,L}^{(m)}$ of the metric $h_{X/Y,L}^{(m)}$ defined above is a psh function uniformly bounded from above, thus it admits a unique (psh) extension on X .

Remark 2.4. — Though we do not use this, let us make it precise the (analytic) Zariski open in Theorem 2.3 above. Define for every (quasi-) psh function ϕ and for every integer $m > 0$ the ideal sheaf $\mathcal{J}_m(\phi)$ by taking

$$\mathcal{J}_m(\phi)_x := \left\{ f \in \mathcal{O}_{X,x} \mid |f|^{2/m} e^{-\frac{1}{m}\phi} \in L_{\text{loc}}^1 \right\},$$

which is proved to be coherent in [23]. Then the integrability condition in Theorem 2.3 is equivalent to the non-vanishing condition that $f \in (K_{X/Y}^m \otimes \mathcal{J}_m(h_L)) = 0$. And the open subset mentioned in Theorem 2.3 can be taken to be $f^{-1}U$ where $U \subset Y_0$ is the (analytic) Zariski open subset consist of all point $t \in U$ such that

$$h^0(X_t, (K_{X/Y}^m \otimes L \otimes \mathcal{J}_m(h_L))|_{X_t}) = \text{rk } f \in (K_{X/Y}^m \otimes L \otimes \mathcal{J}_m(h_L)).$$

In particular by the Grauert’s semi-continuity theorem [71, Theorem 1.4(3), p. 6], $f \in (K_{X/Y}^m \otimes L \otimes \mathcal{J}_m(h_L))$ satisfies the base change property over U . For more details, cf. [23, Proof of Theorem 3.5] and Lemma 2.7 below.

By an explicit local calculation as in [25, Theorem 2.3] or [65, 3.33.Theorem] we obtain (in virtue of Theorem 2.3 the proof in [25] apparently does not require the projectivity of f):

PROPOSITION 2.5 (Kähler version of [25, Remark 2.5] or [65, 3.35.Remark]). — Let $f : X \rightarrow Y$ be a Kähler fibre space with X and Y complex manifolds and (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L whose curvature current is positive. Let m be a positive integer. Suppose that for a general point $y_0 \in Y$ there exists a non-zero section $u \in H^0(X_{y_0}, K_{X_{y_0}}^m \otimes L|_{X_{y_0}})$ satisfying

$$\int_{X_{y_0}} |u|^{2/m} e^{-\frac{1}{m}\phi_L} < +\infty,$$

(as in the hypothesis of Theorem 2.3). Then we have

$$h_{X/Y,L}^{(m)}(K_{X/Y}^m \otimes L) > m[\text{div}_f] \tag{2.3}$$

in the sense of currents, where the divisor div_f is defined in the Introduction. In particular, the current $h_{X/Y,L}^{(m)}(K_{X/Y}^m \otimes L)$ is singular along the multiple fibres of f in codimension 1.

Proof. — Let us remark that in [25] the proof of inequality (2.3) is only sketched for $m = 1$. For the convenience of the readers let us give a detailed proof for the general case here. Since a positive $(1, 1)$ -current extends across analytic subsets of codimension 2, it suffices to check the inequality around a general point of W_i for every $i \in I_{\text{div}}$ (so that one can assume that every W_i is smooth). Say $i = 1 \in I_{\text{div}}$, and let x be a general point of W_1 . Take a small ball B_y (of radius < 1) around $y = f(x)$ with holomorphic local coordinates $(t_j)_{j=1,\dots,d}$ and a small ball $B_x = f^{-1}(B_y)$ around x_0 with holomorphic local coordinates $(z_i)_{i=1,\dots,n}$, such that W_1 is locally defined by the equation $z_{e+1} = 0$ and that $f(W_1)$ is defined by $t_1 = 0$. Then f is locally given by the formula (up to reordering the indices):

$$(z_1, \dots, z_e, z_{e+1}, \dots, z_n) = (z_{e+1}^{b_1}, z_{e+2}, \dots, z_n).$$

Now let $y_0 \in B_y \setminus \{t_1 = 0\}$, and let $u \in H^0(X_{y_0}, K_{X_{y_0}}^m \otimes L_{X_{y_0}})$ satisfying the $L^{2/m}$ condition as in the hypothesis; up to a normalization one can suppose that $\int_{m, y_0, L} u = 1$. Then by the construction of F_u we have

$$1 = \int_{m, y_0, L} u^{\frac{2}{m}} = \int_{X_{y_0}} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\phi_L} > \int_{x \in X_{y_0}} \left| \frac{F_u}{z_{e+1}^{m(b_1-1)}} \right|^{\frac{2}{m}} d\mu_{X_{y_0}}$$

where $d\mu_{X_{y_0}}$ is the Lebesgue measure on X_{y_0} with respect to the z_i 's. Notice that

$$x \in X_{y_0} = \left\{ z_{e+1}^{b_1} = t_1(y_0), z_{e+i} = t_i(y_0), 2 \leq i \leq d \right\},$$

hence by applying the Ohsawa–Takegoshi type extension theorem [33, Theorem 4.2] (or [6, 0.2.Proposition]) to $u = x, p = (z_{e+1}, \dots, z_n)$ and $\phi = (b_1 - 1) \log |z_{e+1}|^2$, the holomorphic function F_u extends to a function G_u defined on x satisfying the following $L^{2/m}$ -integrability condition:

$$\int_x \left| \frac{G_u}{z_{e+1}^{m(b_1-1)}} \right|^{\frac{2}{m}} d\mu_X \leq \mu(B_y)$$

By valuative integrability criterion [10, Theorem 10.11] the generic Lelong number of $\log |G_u|^2$ over W_1 is strictly superior to $m(b_1 - 1)$, implying that

$$\log |G_u|^2 \leq m(b_1 - 1) \log |z_{e+1}|^2 + C_{y_0}$$

for some uniform (the section space $H^0(X_{y_0}, K_{X_{y_0}}^m \otimes L_{X_{y_0}})$ being finite-dimensional) constant C_{y_0} depending on y_0 . Hence by the construction (2.2) we have

$$\phi_{X/Y,L}^{(m)}(z) \leq m(b_1 - 1) \log |z_{e+1}|^2 + C_{f(z)};$$

by the mean-value inequality the constant $C_{f(z)}$ can be chosen locally uniform, which proves (2.3).

2.3. Positivity of the Canonical L^2 Metric on the Direct Image Sheaf

In this subsection, let $f : X \rightarrow Y$ be an analytic fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L with curvature current $\Theta_{h_L}(L) > 0$. We will show in the sequel that the canonical L^2 metric on the direct image sheaf $f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$ is semipositively curved, that is, to prove the following theorem:

THEOREM 2.6 (Kähler version of [66, Theorem 1 (b)]). — *let $f : X \rightarrow Y$ be an analytic fibre space between complex manifolds and let (L, h_L) be a holomorphic line bundle on X equipped with a singular Hermitian metric h_L with curvature current $\Theta_{h_L}(L) > 0$. Then the torsion free sheaf $f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$ admits a canonical semipositively curved singular Hermitian metric $g_{X/Y,L}$ which satisfies the L^2 extension property.*

The argument is very close to that in [45, §22-24]. For the convenience of the readers, we will nevertheless explain it in details. First recall the construction of the canonical L^2 metric on the direct image of the adjoint line bundle (twisted by the multiplier ideal). Briefly speaking, it is done as following: when $Y = \text{pt}$, then X is compact, and this is nothing other than the natural L^2 norm on $H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L))$; for the general case, we just do this construction in family.

Precisely, $g_{X/Y,L}$ is constructed as following: let Y_0 be the Zariski open of Y over which f is smooth and let $y \in Y_0$. Take a coordinate neighbourhood B of y , so that K_Y is trivial over B , then there is a nowhere vanishing holomorphic d -form η such that $K_B \cong \mathcal{O}_B \cdot \eta$. For any section $u \in H^0(B, f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L)))$, one can regard it as a morphism of \mathcal{O}_B -modules (in virtue of the projection formula)

$$u : K_B \rightarrow f_*(K_X \otimes L \otimes \mathcal{I}(h_L))/B$$

Thus we obtain a section $u(\eta) \in H^0(B, f_*(K_X \otimes L \otimes \mathcal{I}(h_L))) = H^0(f^{-1}B, K_X \otimes L \otimes \mathcal{I}(h_L))$. Locally over $f^{-1}(B \cap Y_0)$ we can write $u(\eta) = \sigma_u \cdot \eta$; whilst the choice of σ_u depends on η , its restriction to the fibre $\sigma_u|_{X_y}$ does not. The local sections $\sigma_u|_{X_y}$'s glue together to give rise to a section $\sigma_{u,y} \in H^0(X_y, K_{X_y} \otimes (L \otimes \mathcal{I}(h_L))|_{X_y})$. Then we define the canonical L^2 metric as following: for two local sections u, v of $f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$ (resp. of $f_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h_L))$), define

$$g_{X/Y,L}(u, v)(y) = (\overline{-1})^{n^2} \int_{X_y} \sigma_{u,y} \overline{\sigma_{v,y}} e^{-\phi_L}. \tag{2.4}$$

Before proving the result, let us recall the following comparison result of the restriction of the multiplier ideal of a metric to a fibre and the multiplier ideal of the restriction of the metric to a fibre:

LEMMA 2.7. — *Let $f : X \rightarrow Y$ and (L, h_L) as in the Theorem 2.6 above. Suppose that f is smooth. Then for any $y \in Y$ we have*

$$J(h_L|_{X_y}) = J(h_L)|_{X_y}.$$

Moreover, for almost every $y \in Y$ we have

$$J(h_L|_{X_y}) = J(h_L)|_{X_y}.$$

Proof. — The inclusion $J(h_L|_{X_y}) \subset J(h_L)|_{X_y}$ results from the local Ohsawa–Takegoshi extension theorem (see e.g. [8, Theorem 1]) while the equality for a.e. $y \in Y$ is simply a consequence of the Fubini’s theorem. Cf. [65, 3.29.Remark] for more details. Let us remark that the same result holds for J_m as defined in Remark 2.4.

Next let us fix some notations for later use:

Notation 2.8. — Set Y_1 the (analytic) Zariski open subset of Y_0 such that

- (i) $f(K_{X/Y} \otimes L \otimes J(h_L))$ and the quotient sheaf of $f(K_{X/Y} \otimes L)$ by $f(K_{X/Y} \otimes L \otimes J(h_L))$ are both locally free over Y_1 ;
- (ii) $f(K_{X/Y} \otimes L)$ satisfies the base change property over Y_1 , i.e. $f(K_{X/Y} \otimes L) \otimes_{\kappa(y)} H^0(X_y, K_{X_y} \otimes L|_{X_y})$ for every $y \in Y_1$. (e.g. if the function $y \mapsto h^0(X_y, K_{X_y} \otimes L|_{X_y})$ is locally constant on Y_1 , cf. [71, Theorem 1.4(3), p. 6]).

Set in addition $G_L := f(K_{X/Y} \otimes L \otimes J(h_L))$. With these notations we get immediately from Lemma 2.7 the following:

LEMMA 2.9. — *We have inclusions*

$$H^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes J(h_L|_{X_y})) \subset G_L \otimes_{\kappa(y)} f(K_{X/Y} \otimes L) \otimes_{\kappa(y)} H^0(X_y, K_{X_y} \otimes L|_{X_y})$$

for every $y \in Y_1$.

For any $y \in Y_1$, since $f(K_{X/Y} \otimes L)$ satisfies the base change property, the expression of the metric $g_{X/Y,L}$ is simpler: for $u \in G_L \otimes_{\kappa(y)} H^0(X_y, K_{X_y} \otimes L|_{X_y})$, u can be regarded as a section in $H^0(X_y, K_{X_y} \otimes (L \otimes J(h_L))|_{X_y}) \otimes H^0(X_y, K_{X_y} \otimes L|_{X_y})$, and we have

$$|u|_{g_{X/Y,L},y}^2 = \int_{X_y} |u|^2 e^{-\phi_L}. \tag{2.5}$$

In particular, $|u|_{g_{X/Y,L},y}^2(y \in Y_1)$ is finite if and only if $u \in H^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes (h_L|_{X_y}))$. Now let us prove the following result which assures that $g_{X/Y,L}$ is well-behaved:

PROPOSITION 2.10. — *The metric $g_{X/Y,L}$ defined above on $f(K_{X/Y} \otimes L \otimes (h_L))$ is measurable, and is non-degenerate and bounded almost everywhere.*

Proof. — We check successively:

(a). $g_{X/Y,L}$ is measurable. — This is surely well known to experts, but since it does not appear explicitly in the literatures we give the details for the convenience of the readers and take this chance to fix some notations for later use. Let $s \in H^0(B, G_L)$ be a local section on B with B a small ball in Y , we will show that $|s|_{g_{X/Y,L}}^2$ is a measurable function. To this end, we can assume that B is contained in Y_0 ; in addition, s can be regarded as a section in $H^0(f^{-1}B, K_{X/Y} \otimes L)$, and thus $s(y) \in H^0(X_y, K_{X_y} \otimes L|_{X_y})$; $s = f^* \eta + H^0(f^{-1}B, K_X \otimes L)$ where η is a nowhere vanishing holomorphic d -form, giving rise to a trivialization $K_B \cong \mathcal{O}_B \cdot \eta$. By definition, for any $y \in B \cap Y_1$ we have

$$|s(y)|_{g_{X/Y,L}}^2 = \int_{X_y} |s(y)|^2 e^{-\phi_L},$$

By Ehresmann's theorem (cf. for example [76, §9.1.1, Proposition 9.3, pp. 209–210]) we have a diffeomorphism $X_0 \times B \xrightarrow{\tau} f^{-1}B$ such that $\tau|_{X_0 \times \{0\}} = \text{id}_{X_0}$ where $i_y : X_0 \rightarrow X_0 \times B$ is the natural inclusion which identifies $X_0 \cong X_0 \times \{y\}$ in $X_0 \times B$. Then we can write

$$|s(y)|_{g_{X/Y,L}}^2 = \int_{X_0} G_s(y, \cdot) \text{Vol}_{X_0} \tag{2.6}$$

where Vol_{X_0} is a fixed volume form on X_0 and G_s is a function such that

$$G_s(y, \cdot) \text{Vol}_{X_0} = \left| \tau^* (s - f^* \eta)|_{X_0 \times \{y\}} \right|^2 e^{-\phi_L}. \tag{2.7}$$

ϕ_L being a psh function, the function G_s is lower semi-continuous and is well defined on $X_0 \times (B \cap Y_1)$, in particular it is measurable. Hence by Fubini's theorem, $|s|_{g_{X/Y,L}}^2$ is measurable.

(b). $g_{X/Y,L}$ is non-degenerate and bounded almost everywhere (cf. also [65, 3.29.Remark]). — First one notices that by the formula (2.5) the metric $g_{X/Y,L}$ is non-degenerate over Y_1 since ϕ_L is a psh function. In order to show that $g_{X/Y,L}$ is bounded almost everywhere, it suffices to prove that the natural inclusion

$$H^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes (h_L|_{X_y})) \subset G_L \otimes \kappa(y)$$

is an isomorphism for $y \in Y_1$ almost everywhere. This simply results from Lemma 2.7.

By virtue of Proposition 2.10, in order to prove that $g_{X/Y,L}$ defined above extends to a semipositively curved singular Hermitian metric on G_L , it remains to show: for $U \subset Y$ an open subset, and for $\alpha \in H^0(U, G_L)$ a non-zero section, $\psi_\alpha := \log |\alpha|_{g_{X/Y,L}}^2$ (a function well-defined on $U \setminus Y_0$) extends to a psh function on U . To this end, we will successively establish (by Proposition 2.10, ψ_α is — on $U \setminus Y_0$):

- (A) ψ_α is locally uniformly bounded from above on $U_1 := U \setminus Y_1$;
- (B) ψ_α is upper semi-continuous on U_1 ;
- (C) ψ_α satisfies the mean value inequality on any disc in U_1 .

In fact, the points (B) and (C) imply that ψ_α is a psh function over U_1 ; and the point (A) implies moreover that $\psi_\alpha|_{U_1}$ admits a unique psh extension to U . In addition, let us remark that up to replacing Y par U , one can suppose that α is a global section; in this case ψ_α is a function well defined over Y_0 . The proof of these three points relies on the Ohsawa–Takegoshi type extension Theorem 2.1, which permits us to extend a section on the fibre to a neighbourhood along with an L^2 estimate (in some cases we should require this estimate to be optimal).

Proof of (A). Let $y_0 \in Y$, we will prove that y_0 admits a neighbourhood on whose intersection with Y_1 the function ψ_α is uniformly upper bounded. To this end, take a small open ball B_0 of centre y_0 in Y and denote $B_1 := \frac{1}{2}B_0$, $B = B_2 := \frac{1}{4}B_0$ and $R_0 =$ radius of B_0 . We will prove in the sequel that ψ_α est uniformly upper bounded on $B \setminus Y_1$. This proceeds in two steps:

(A1). — Firstly we prove that

$$\psi_\alpha|_{B \setminus Y_1} \in \text{punctual supremum of the family} \\ \text{of functions } \{ \log |\alpha(s)|^2 \}_s \quad S_{M_0} \quad (2.8)$$

where S_{M_0} denotes the set of sections $s \in H^0(B_1, G_L) = H^0(f^{-1}B_1, K_{X/Y} \otimes L \otimes (h_L))$ satisfying the following L^2 condition:

$$\int_{f^{-1}B_1} |s|_{f^* \eta|_{B_1}}^2 e^{-\phi_L} \in \left(\frac{3}{4}\right)^d \mu(B_0) := M_0, \quad (2.9)$$

where $\mu(B_0)$ denotes the Lebesgue measure of B_0 and η a nowhere vanishing holomorphic n -form on B_0 (which gives rise to a trivialization $K_{B_0} \cong \mathcal{O}_{B_0} \cdot \eta$).

For every $y \in B \setminus Y_1$ such that $h_L|_{X_y} \neq 0$ (if $h_L|_{X_y} = 0$, then $\psi_\alpha(y) = -\infty$ and (2.8) is automatically established at y), we have

$$\psi_\alpha(y) = \log |\alpha(y)|_{g_{X/Y,L}}^2 = \sup_{u \in \mathcal{O}_{y,L} \setminus \{0\}} \log |\alpha(y)(u)|^2.$$

The set $\{u \in H^0(X_y, K_{X_y} \otimes L/X_y) \mid |u|_{y,L} \leq 1\}$ being compact, the supremum is attained by a vector $v_y \in G_L \otimes \kappa(y)$ satisfying $|v_y|_{y,L} = |v_y|_{g_{X/Y}, y} = 1$ (we denote $| \cdot |_{1,y,L} = | \cdot |_{y,L}$, compare (2.1) and (2.5)); in particular $v_y \in H^0(X_y, K_{X_y} \otimes L/X_y \otimes (h_L/X_y))$. Consider the open ball $B_y := B(y, \frac{3}{4}R_0)$ of centre y and of radius $= \frac{3}{4}R_0$. Then $B \supset B_1 \supset B_y \supset B_0$. By Theorem 2.1 we get a section $s_y \in H^0(B_y, G_L)$ such that $s_y|_{X_y} = v_y$ and satisfies the following L^2 condition:

$$\int_{f^{-1}B_y} |s_y - f|_{B_y}|^2 e^{-\phi_L} \leq \mu(B_y) \cdot |v_y|_{y,L} = \mu(B_y) = \left(\frac{3}{4}\right)^d \mu(B_0) = M_0.$$

In particular, $s_y|_{B_1}$ satisfies the condition (2.9), then $s_y|_{B_1} \in S_{M_0}$. In addition, we have

$$\psi_\alpha(y) = \left(\log |\alpha(s_y)|^2\right)(y),$$

which proves (2.8).

(A2). — By the previous step, it remains to prove that the functions $\log |\alpha(s)|^2$ ($s \in S_{M_0}$) are all uniformly upper bounded over \bar{B} by a uniform constant. In fact we can prove the following more general:

LEMMA 2.11. — For a fixed $M > 0$, define

$$S_M := \left\{ \begin{array}{l} s \in H^0(B_1, G_L) \\ = H^0(f^{-1}B_1, K_{X/Y} \otimes L \otimes (h_L)) \end{array} \mid \int_{f^{-1}B_1} |s - f|_{B_1}|^2 e^{-\phi_L} \leq M \right\},$$

then for every compact $K \subset B_1$, there exists a constant $C_K > 0$ (independent of s) such that

$$\sup_K |\alpha(s)| \leq C_K$$

for every $s \in S_M$.

Proof. — The lemma is deduced from the some well known facts about the Fréchet space structure on the cohomology spaces of coherent sheaves over complex spaces, as presented in [39, §VIII.A, pp. 234–246]. By [39, §VIII.A, 8.Theorem, pp. 239–240], for any coherent sheaf on an analytic space, we can equip its section spaces with a unique Fréchet space structure, s.t. the restriction morphisms are continuous.

- (a) By [39, §VIII.A, 7.Theorem, pp. 237–238], the section α , regarded as a morphism $G_L \rightarrow O_Y$, induces continuous map between Fréchet spaces

$$\alpha|_{B_1} : H^0(B_1, G_L) \rightarrow H^0(B_1, O_X).$$

- (b) By uniqueness, the a priori topologies on the two isomorphic spaces $H^0(B_1, G_L)$ and $H^0(f^{-1}B_1, K_{X/Y} \otimes L \otimes (h_L))$ are homeomorphic.

- (c) $S_M \cap H^0(B_1, G_L)$ is compact with respect to the Fréchet space topology. This is a result of (b) and Montel's Theorem.
- (d) By [26, §V.4.2, Proposition 2.1, pp. 165–166] the compacts in $H^0(B_1, \mathcal{O}_X)$ are closed and bounded.

By combining (a), (c) and (d) we establish the lemma.

Proof of (B). Let $y_0 \in Y_1$, and let $\{y_k\}_{k>0}$ be any sequence in Y_1 convergent to y_0 , we will prove that

$$\limsup_k \psi_\alpha(y_k) \leq \psi_\alpha(y_0).$$

The problem being local, we can replace Y by B_0 a small open ball of centre y_0 ($y_0 = 0$ in B_0) in Y . Note $R_0 :=$ the radius of B_0 and $B_i := (\frac{1}{2})^i B_0$. Since there is a subsequence of $\{\psi_\alpha(y_k)\}_{k>0}$ which converges to the limit superior of $\{\psi_\alpha(y_k)\}_{k>0}$, we can assume that the sequence $\{\psi_\alpha(y_k)\}_{k>0}$ is convergent. In addition, up to shifting the numbering of the sequence we can assume that $\{y_k\}_{k>0} \subset B_3$; we can also assume that $\psi_\alpha(y_k) = -\frac{1}{k}$, k (in particular, $h_L|_{X_{y_k}} = \frac{1}{k} + \dots$). As in the step (A1) above, there exists for every $k \in \mathbb{Z}_{>0}$ a vector $v_k \in H^0(X_{y_k}, K_{X_{y_k}} \otimes L|_{X_{y_k}} \otimes \mathcal{J}(h_L|_{X_{y_k}}))$ such that $\|v_k\|_{y_k, L} = 1$ and

$$\psi_\alpha(y_k) = \log |\alpha(y_k)(v_k)|^2.$$

Consider $B_{y_k} := B(y_k, \frac{7}{8}R_0)$ the open ball of centre y_k and of radius $\frac{7}{8}R_0$, then $B_3 \supset B_2 \supset B_1 \supset B_{y_k} \supset B_0$. Still by Theorem 2.1, we obtain a section $s_k \in H^0(B_{y_k}, G_L) = H^0(f^{-1}B_{y_k}, K_{X/Y} \otimes L \otimes \mathcal{J}(h_L))$ such that $s_k|_{X_{y_k}} = v_k$ and

$$\int_{f^{-1}B_{y_k}} |s_k|^2 e^{-\phi_L} \leq \left(\frac{7}{8}\right)^d \mu(B_0) := M_0.$$

Denote $F_k = \alpha(s_k)|_{B_1}$ and $\theta_k := \log |F_k|^2$, then F_k is a holomorphic function on B_1 and θ_k is a psh function (with analytic singularities); in addition, we have that $\psi_\alpha(y_k) = \theta_k(y_k)$. By Lemma 2.11 (taking $M = M_0$ and $K = \bar{B}_2$), there is a constant $C_{\bar{B}_2}$ independent of k such that $|F_k| \leq C_{\bar{B}_2}$ on \bar{B}_2 for every k ; in consequence, the derivatives of F_k satisfy

$$|F_k| \leq C_{\bar{B}_2} := \frac{16}{R_0} \bar{n} C_{\bar{B}_2}$$

on \bar{B}_3 (cf. [26, §V.1.2, Lemme, p. 146]). In particular, since $\{y_k\}_{k>0} \subset B_3$, we have

$$\left| \frac{|F_k(0)|}{|F_k(y_k)|} - 1 \right| \leq \frac{|F_k(0) - F_k(y_k)|}{|F_k(y_k)|} \leq C_{\bar{B}_2} |y_k - 0| \rightarrow 0 \text{ when } k \rightarrow +\infty,$$

hence we get

$$\begin{aligned} \lim_{k \rightarrow +} \theta_k(y_k) &= \lim_{k \rightarrow +} (\log /F_k(y_k)/) \\ &= \lim_{k \rightarrow +} (\log /F_k(0)/) = \lim_{k \rightarrow +} \theta_k(0) \end{aligned} \quad (2.10)$$

By definition, we have

$$|\alpha(s_k)| \circlearrowleft |\alpha|_{g_{X/Y,L}} |s_k|_{g_{X/Y,L}} = \psi_\alpha + \log \lambda_k > \theta_k,$$

where $\lambda_k := |s_k|_{g_{X/Y,L}}^2$. By passing to the limit superior we obtain (in virtue of (2.10))

$$\begin{aligned} \psi_\alpha(0) + \limsup_{k \rightarrow +} (\log \lambda_k(0)) &> \limsup_{k \rightarrow +} \theta_k(0) = \lim_{k \rightarrow +} \theta_k(0) \\ &= \lim_{k \rightarrow +} \theta_k(y_k) = \lim_{k \rightarrow +} \psi_\alpha(y_k). \end{aligned}$$

It remains thus to show

$$\limsup_{k \rightarrow +} (\log \lambda_k(0)) \circlearrowleft 0,$$

and this amounts to show (the function \log being increasing and continuous)

$$\limsup_{k \rightarrow +} \lambda_k(0) \circlearrowleft 1.$$

Now up to taking an extraction, we can assume that the sequence $\{\lambda_k(0)\}_{k>0}$ is convergent. By the compactity of S_{M_0} (the point (b) in the proof of Lemma 2.11), up to taking a subsequence, we can further assume that $\{s_k\}_{k>0}$ converges uniformly on all compacts in B_1 to a section $s \in S_{M_0}$. By (2.6) (cf. point (a) in the proof of Proposition 2.10) we have for $y \in B_1 \cap Y_1$ that

$$\begin{aligned} \lambda_k(y) &= \int_{X_0} G_{s_k}(y, \cdot) \text{Vol}_{X_0}, \\ s(y) &= \int_{X_0} G_s(y, \cdot) \text{Vol}_{X_0}. \end{aligned}$$

By (2.7) the compact convergence $\{s_k\}_{k>0}$ implies that $\{G_{s_k}\}_{k>0}$ converges uniformly over all compacts to G_s (especially over \bar{B}_3). By the point (a) in the proof of Proposition 2.10, the G_{s_k} 's as well as G_s are all lower semi-continuous functions, thus

$$\begin{aligned} G_{s_k}(0, \cdot) &\circlearrowleft \liminf_{l \rightarrow +} G_{s_k}(y_l, \cdot), \\ G_s(0, \cdot) &\circlearrowleft \liminf_{l \rightarrow +} G_s(y_l, \cdot), \end{aligned}$$

and in consequence (by a diagonal process)

$$G_s(0, \cdot) \circlearrowleft \liminf_{k \rightarrow +} G_{s_k}(y_k, \cdot).$$

Then Fatou's lemma implies that,

$$\begin{aligned} \lim_{k \rightarrow +\infty} \lambda_k(0) &= \lambda_s(0) = \int_{X_0} G_s(0, \cdot) \text{Vol}_{X_0} \leq \int_{X_0} \liminf_{k \rightarrow +\infty} G_{s_k}(y_k, \cdot) \text{Vol}_{X_0} \\ &\leq \liminf_{k \rightarrow +\infty} \int_{X_0} G_{s_k}(y_k, \cdot) = \liminf_{k \rightarrow +\infty} \lambda_k(y_k) = 1, \end{aligned}$$

which proves the result.

Proof of (C). Let D be any disc contained in Y_1 , we will prove that

$$\psi_\alpha(0) \leq \frac{1}{\mu(D)} \int_D \psi_\alpha d\mu. \tag{2.11}$$

We can assume that $Y = Y_1 = Y_0$, in particular, f is a smooth fibration. If $\psi_\alpha(0) = -\infty$, then the inequality (2.11) is automatically established; hence we can assume that $\psi_\alpha(0) > -\infty$, in particular $h_{L/X_0} + \dots$. As in the step (A1), there is a section $v \in H^0(X_0, K_{X_0} \otimes L \otimes \mathcal{J}(h_L))$ such that $\|v\|_{0,L} = 1$ and

$$\psi_\alpha(0) = \log |\alpha(0)(v)|^2.$$

Again by Theorem 2.1 we get a section $s \in H^0(X, K_X \otimes L \otimes \mathcal{J}(h_L))$ such that $s|_{X_0} = v$ and

$$\int_X |s(t)|^2 dt = \int_X |s|^2 e^{-\phi_L} \leq \mu(D).$$

In particular $(\log |\alpha(s)|^2)|_D = \psi_\alpha(0)$. By definition we have

$$|\alpha(s)| \leq |\alpha|_{g_{X/Y,L}} |s|_{g_{X/Y,L}} = e^{\psi_\alpha} |s| > |\alpha(s)|^2.$$

The function $\log |\alpha(s)|$ being psh on D , it satisfies the mean value inequality, hence we have

$$\begin{aligned} \frac{1}{\mu(D)} \int_D \psi_\alpha d\mu + \frac{1}{\mu(D)} \int_D \log |s| d\mu &> \frac{1}{\mu(D)} \int_D \log |\alpha(s)| d\mu \\ &> (\log |\alpha(s)|^2)|_D(0) = \psi_\alpha(0). \end{aligned}$$

It remains to show that

$$\int_D \log |s| d\mu \leq 0,$$

but the function $\log |s|$ being concave, this is a result of Jensen's inequality: $\log |s|$ being integrable, we have

$$\int_D \log |s| \frac{d\mu}{\mu(D)} \leq \log \left(\int_D |s| \frac{d\mu}{\mu(D)} \right) = \log 1 = 0.$$

This proves (2.11), and thus finishes the proof of the step (C). Hence $g_{X/Y,L}$ is a semipositively curved singular Hermitian metric on G_L .

In order to finish the proof of Theorem 2.6, it remains to show that $(G_L, g_{X/Y, L})$ satisfies the L^2 extension property. To this end, take an open subset U of X and Z an analytic subset of U and, and take a local section $s \in H^0(U \setminus Z, G_L)$ satisfying the L^2 integrability condition, we will show that s extends to a section over U . The problem being local, we can replace U by a small ball B in Y (with t_1, \dots, t_d the standard coordinates). Then $s \in H^0(B \setminus Y_1, G_L) = H^0(f^{-1}Y_1, K_{X/Y} \otimes L \otimes (h_L))$ satisfies the following L^2 condition:

$$\int_B \left(|s|_{g_{X/Y, L}}^2 \right) \eta = \int_{f^{-1}B} |s|_f^2 e^{-\phi_L} < +\infty,$$

where $\eta = dt_1 \wedge \dots \wedge dt_d$ is a nowhere vanishing holomorphic d -form (giving rise to a trivialization $K_B \cong \mathcal{O}_B \cdot \eta$). Then it is an elementary consequence of Riemann extension that s extends to a section in $H^0(f^{-1}B, K_{X/Y} \otimes L \otimes (h_L)) = H^0(B, G_L)$, meaning that $(G_L, g_{X/Y, L})$ satisfies the L^2 extension property. This finishes the proof of Theorem 2.6.

2.4. Positivity of Direct Images of Twisted Pluricanonical Bundles

In this subsection, we will apply Theorem 2.3 and Theorem 2.6 to prove Theorem B, which will serve as a key ingredient in the proof of our Main Theorem A.

Proof of Theorem B. — Recall that

$$F_{m, \bullet} := f \left(K_{X/Y}^m \otimes \mathcal{O}_X(m \bullet) \right).$$

If $F_{m, \bullet} = 0$, then there is nothing to prove; hence we assume that $F_{m, \bullet} \neq 0$. Since (X, \bullet) is klt (implying that (X_y, \bullet_y) is klt for y general by [59, §9.5.D, Theorem 9.5.35, pp. 210–211, vol.II]) and $F_{m, \bullet} \neq 0$, the condition in the hypothesis of Theorem 2.3 is satisfied for $L = \mathcal{O}_X(m \bullet)$ and $h_L = h^{\bullet, m}$ where $h^{\bullet, \bullet}$ is the canonical (singular) Hermitian metric defined by the local equations of \bullet , then we obtain a singular Hermitian metric $h_{X/Y, m}^{(m)}$ over $K_{X/Y}^m \otimes \mathcal{O}_X(m \bullet)$ whose curvature current is positive. However one cannot directly apply Theorem 2.6 to obtain a semipositively curved singular Hermitian metric on $F_{m, \bullet}$. In order to overcome this difficulty, we introduce the line bundle

$$L_{m-1} = K_{X/Y}^{(m-1)} \otimes \mathcal{O}_X(m \bullet),$$

equipped with the metric

$$h_{L_{m-1}} := \left(h_{X/Y, m}^{(m)} \right)^{\frac{m-1}{m}} \cdot h^{\bullet, \bullet}.$$

Then the curvature current of $h_{L_{m-1}}$ is positive. We are now ready to apply Theorem 2.6 to $L = L_{m-1}$, except that we need to establish in addition that the natural inclusion

$$f \left(K_{X/Y} \otimes L_{m-1} \otimes \mathcal{J}(h_{L_{m-1}}) \right) \subset F_m,$$

is generically an isomorphism.

To this end, let Y_2 be the (analytic) Zariski open subset of Y_0 satisfying the conditions (i) and (ii) in the definition of Y_1 for $L = L_{m-1}$ (see Notation 2.8) and such that the pair $(X_y, \mathcal{J}(h_{L_{m-1}}))$ is klt for $y \in Y_2$ (cf. [59, §9.5.D, Theorem 9.5.35, pp. 210-211, vol.II]). By virtue of the base change property of F_m over Y_2 and Lemma 2.9, it suffices to prove that the natural inclusion

$$H^0(X_y, K_{X_y} \otimes L_{m-1}/X_y \otimes \mathcal{J}(h_{L_{m-1}}/X_y)) \subset H^0(X_y, K_{X_y} \otimes L_{m-1}/X_y) \quad (2.12)$$

is an isomorphism for $y \in Y_2$. But this results from the following Lemma 2.12.

LEMMA 2.12. — *Let $f : X \rightarrow Y$ be a Kähler fibre space between complex manifolds and let N be a \mathbb{Q} -line bundle endowed with a semipositively curved singular Hermitian metric h_N such that $\mathcal{J}(h_N/X_y) = \mathcal{O}_{X_y}$ for almost every $y \in Y_0$ (which is the case, e.g. if $\mathcal{J}(h_N) = \mathcal{O}_X$, by Lemma 2.7). If the direct image sheaf $f_*(K_{X/Y}^m \otimes N^m) = 0$, then by Theorem 2.3 one can construct the relative m -Bergman kernel metric $h_{X/Y,m,N}^{(m)}$ on $K_{X/Y}^m \otimes N^m$ whose curvature current is positive. Set*

$$N_{m-1} := K_{X/Y}^{(m-1)} \otimes N^m$$

and

$$h_{N_{m-1}} := \left(h_{X/Y,m,N}^{(m)} \right)^{\frac{m-1}{m}} \cdot h_N.$$

Then the natural inclusion

$$H^0(X_y, K_{X_y} \otimes N_{m-1}/X_y \otimes \mathcal{J}(h_{N_{m-1}}/X_y)) \subset H^0(X_y, K_{X_y} \otimes N_{m-1}/X_y)$$

is an isomorphism (or equivalently, surjective) for a.e. $y \in Y_0$.

Proof. — Let $y \in Y_0$ be a point such that $\mathcal{J}(h_N/X_y) = \mathcal{O}_{X_y}$ and let $v \in H^0(X_y, K_{X_y} \otimes N_{m-1}/X_y) = H^0(X_y, K_{X_y}^m \otimes N^m/X_y)$, then with the same notations as in Section 2.2 we can write

$$v \cdot (dt_1 \cdots dt_d)^m = F_v \cdot (dz_1 \cdots dz_n)^m.$$

Since $\mathcal{J}(h_N/X_y) = \mathcal{O}_{X_y}$, we have

$$v \frac{2}{m} \frac{1}{m,y,m,N} = \int_{X_y} |v|^{\frac{2}{m}} e^{-\phi_N} = \int_{X_y} \left(|F_v|^{\frac{2}{m}} e^{-\phi_N} \right) \text{Vol}_{X_y} < +\infty, \quad (2.13)$$

where ϕ_N denotes the local weight of the metric h_N . By (2.2) (cf. also [6, §A.2, p. 8]) the local weight $\phi_{X/Y, mN}^{(m)}$ satisfies

$$\phi_{X/Y, mN}^{(m)} = \log \left(\sup_{u \in \mathcal{O}_{X/Y, mN}} |F_u|^2 \right) > \log \left(\frac{|F_v|^2}{v^{\frac{2}{m, y, mN}}} \right),$$

and thus

$$\log |F_v|^2 \in \phi_{X/Y, mN}^{(m)} + O(1) = |F_v|^{\frac{2(m-1)}{m}} e^{-\frac{m-1}{m} \phi_{X/Y, mN}^{(m)}} \in O(1). \quad (2.14)$$

Regarded as a holomorphic n -form with values in the line bundle N_{m-1}/X_y , the section v satisfies

$$|v|^2 e^{-\phi_{N_{m-1}}} = |F_v|^2 e^{-\frac{m-1}{m} \phi_{X/Y, mN}^{(m)} - \phi_N} \cdot \text{Vol}_{X_y},$$

where $\phi_{N_{m-1}}$ denotes the local weight of the metric $h_{L_{m-1}}$, hence by (2.13) and (2.14) we have

$$\begin{aligned} v^{\frac{2}{y, N_{m-1}}} &= \int_{X_y} |v|^2 e^{-\phi_{N_{m-1}}} = \int_{X_y} \left(|F_v|^2 e^{-\frac{m-1}{m} \phi_{X/Y, mN}^{(m)} - \phi_N} \right) \text{Vol}_{X_y} \\ &= \int_{X_y} \left(|F_v|^{\frac{2}{m}} e^{-\phi_N} \right) \cdot \left(|F_v|^{\frac{2(m-1)}{m}} e^{-\frac{m-1}{m} \phi_{X/Y, mN}^{(m)}} \right) \text{Vol}_{X_y} \\ &\in C \int_{X_y} \left(|F_v|^{\frac{2}{m}} e^{-\phi_N} \right) \text{Vol}_{X_y} = C \cdot v^{\frac{2}{m, y, mN}} < +\infty, \end{aligned}$$

where C is a constant given by (2.14). Therefore $v \in H^0(X_y, K_{X_y} \otimes L/X_y \otimes (h_{N_{m-1}}/X_y))$, which proves the lemma.

By combining Theorem B and Theorem 1.13 we immediately get:

COROLLARY 2.13. — *Let $f : X \rightarrow Y$ and \mathcal{L} as in Theorem B. Suppose that the determinant of F_m is numerically trivial. Then $(F_m, \mathcal{L} \otimes g_{X/Y}^{(m)})$ is a Hermitian flat vector bundle.*

3. Log Kähler Version of Results of Kawamata and of Viehweg

In this section we will apply the Ohsawa–Takegoshi type extension Theorem 2.2 to prove Main Theorem A (I). Along the way we also give a proof of the conjecture $C_{n, m}$ over general type bases (cf. Theorem 3.2 below) which is a main ingredient in the proof of Theorem C in Section 4.

Classically the proof of Main Theorem A (I) and Theorem 3.2 is based on Viehweg’s weak positivity theorem on the direct image; here we will take a new argument which only depends on the Ohsawa–Takegoshi type extension Theorem 2.2. Precisely, Theorem 2.2 is used to ensure the effectivity of the

twisted relative canonical bundle up to adding an ample line bundle from the base, in virtue of the following auxiliary result:

LEMMA 3.1. — *Let $f : X \rightarrow Y$ be an analytic fibre space with X a normal complex variety and Y a projective variety. Let L be a holomorphic line bundle on X such that $\kappa(L) > 0$ and let A be an ample line bundle on Y . Then*

$$\kappa(X, L \otimes f^* A) = \kappa(F, L|_F) + \dim Y$$

where F denotes the general fibre of f .

Before giving the proof, let us remark that this simple but useful result has been implicitly used in the works on $C_{n,m}$, e.g. [35, 74]; it is explicitly formulated in [17, Lemma 4.9] but without proof. For the convenience of the readers, we will give the detailed proof.

Proof of Lemma 3.1. — Up to multiplying L and A_Y by a sufficiently large and divisible integer, we can assume that $H^0(X, L) = 0$ and A is very ample; we can further assume that the closure of the image of the meromorphic mapping

$$\varphi := \varphi_{L \otimes f^* A} : X \dashrightarrow \mathbb{P}V$$

with $V := H^0(X, L \otimes f^* A)$ is of dimension $\kappa(X, L \otimes f^* A)$. Up to blowing up X we can assume that φ is an analytic fibre space (cf. [71, Lemma 5.3, pp. 51–52, and Corollary 5.8, p. 57]). Then consider the sub-linear series defined by the inclusion

$$H^0(Y, A) \subset H^0(X, f^* A) \subset H^0(X, L \otimes f^* A) \subset H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)),$$

this gives rise to a meromorphic mapping

$$\mathbb{P}V \dashrightarrow \mathbb{P}H^0(Y, A).$$

On the other hand, since A is very ample, the linear series $|A|$ defines a closed embedding $i := \varphi_{|A|} : Y \hookrightarrow \mathbb{P}H^0(Y, A)$, thus we have the following “commutative” diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi := \varphi_{L \otimes f^* A}} & \mathbb{P}V \\
 \downarrow f & \searrow \varphi_{f^* A} & \downarrow \text{dashed} \\
 Y & \xrightarrow{i := \varphi_{|A|}} & \mathbb{P}H^0(Y, A)
 \end{array}$$

In particular, the general fibre G of π is contracted by f , hence we get an analytic fibre space

$$\pi|_F : F \rightarrow \text{Im}(\pi|_F),$$

whose general fibre is isomorphic to G . $\pi|_F$ is defined by the linear series $|L - f^*A|$ restricted to F , which is a sub-linear series of $|L - f^*A|_F| = |\pi^*L|_F|$, hence we have

$$\kappa(F, L|_F) > \dim \text{Im}(\pi|_F) = \dim \text{Im}(\pi) - \dim Y = \kappa(X, L - f^*A) - \dim Y.$$

In addition, by applying the easy inequality [71, Theorem 5.11, pp. 59–60] to $\pi|_F$ and $(L - f^*A_Y)|_F$ we get

$$\begin{aligned} \kappa(F, L|_F) &= \kappa(F, (L - f^*A)|_F) \\ &\leq \kappa(G, (L - f^*A)|_G) + \dim \text{Im}(\pi|_F) = \dim \text{Im}(\pi|_F), \end{aligned}$$

therefore $\kappa(X, L - f^*A) = \kappa(F, L|_F) + \dim Y$.

3.1. Kähler Version of $C_{n,m}^{\log}$ over General Type Bases

In this subsection we will apply the Ohsawa–Takegoshi type extension Theorem 2.2 to recover the result that $C_{n,m}^{\log}$ holds for fibre spaces over general type bases, i.e. to give a new proof of the following theorem:

THEOREM 3.2 (Kähler version of [49, Theorem 3], [74, Theorem III]). *Let $f : X \rightarrow Y$ be a fibre space between compact complex varieties in Fujiki class \mathcal{C} and let π be an \mathbb{Q} -effective divisor on X such that (X, π) is klt. Suppose that Y of general type (thus projective). Then*

$$\kappa(X, K_X + \pi) > \kappa(F, K_F + \pi|_F) + \dim Y,$$

where F denotes the general fibre of f and $\pi|_F := \pi|_F$.

Let us remark that by virtue of the easy inequality [71, Theorem 5.11, pp. 59–60], the inequality in the theorem is in fact an equality. In order to establish Theorem 3.2, we first prove the following lemma, which can be regarded as a (log) Kähler version of [74, Corollary 7.1]:

LEMMA 3.3. — *Let $f : X \rightarrow Y$ be an analytic fibre space with X a (compact) Kähler manifold and Y a smooth projective variety. Let π be an \mathbb{Q} -effective \mathbb{Q} -divisor on X such that the pair (X, π) is klt. Then for any ample \mathbb{Q} -line bundle A_Y on Y , we have*

$$\kappa(X, K_{X/Y} + \pi + f^*A_Y) = \kappa(F, K_F + \pi|_F) + \dim Y. \quad (3.1)$$

where F denotes the general fibre of f , and $\pi|_F := \pi|_F$.

Proof. — If $\kappa(F, K_F + F) = -$, then for any integer $\mu > 0$ sufficiently large and divisible (so that A_Y^μ is a line bundle and μ is an integral divisor) we have

$$F_{\mu,} := f \left(K_{X/Y}^\mu \otimes \mathcal{O}_X(\mu) \right) = 0,$$

thus $F_{\mu,} \otimes A_Y^\mu = 0$, and in particular

$$H^0(X, K_{X/Y}^\mu \otimes \mathcal{O}_X(\mu) \otimes f^* A_Y^\mu) = H^0(Y, F_{\mu,} \otimes A_Y^\mu) = 0,$$

therefore $\kappa(X, K_{X/Y} + F + f^* A_Y) = -$, hence the equality (3.1).

Suppose in the sequel that $\kappa(F, K_F + F) > 0$. Let m be a sufficiently large and divisible positive integer, so that A_Y^m is a line bundle, m is an integral divisor, $F_{m,} = 0$ and that there is a very ample line bundle A_Y on Y which satisfies $(A_Y)^2 \otimes A_Y^m$ and the following inequality for Seshadri constant

$$\epsilon(A_Y \otimes K_Y^{-1}, y) > \dim Y, \text{ for general } y \in Y.$$

Such m exists by [59, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]. By Theorem 2.3 the relative m -Bergman kernel metric $h_{X/Y,m}^{(m)}$ on $K_{X/Y}^m \otimes \mathcal{O}_X(m)$ is semi-positively curved. Then as in the proof of Theorem B we consider the line bundle

$$L_{m-1} := K_{X/Y}^{(m-1)} \otimes \mathcal{O}_X(m)$$

equipped with the semi-positively curved metric

$$h_{L_{m-1}} := \left(h_{X/Y,m}^{(m)} \right)^{\frac{m-1}{m}} \otimes h, \quad ,$$

where h denotes the singular Hermitian metric whose local weight is defined by the local equation of F . Then apply Theorem 2.2 to $L = L_{m-1}$ (by virtue of Lemma 2.12) and we get a surjection

$$H^0(X, K_X \otimes L_{m-1} \otimes f^*(A_Y \otimes K_Y^{-1})) \rightarrow H^0(F, K_F \otimes L_{m-1}/F),$$

i.e.

$$H^0(X, K_{X/Y}^m \otimes \mathcal{O}_X(m) \otimes f^* A_Y) \rightarrow H^0(F, K_F^m \otimes \mathcal{O}_F(m - F)),$$

which implies that

$$H^0(X, K_{X/Y}^m \otimes \mathcal{O}_X(m) \otimes f^* A_Y) = 0. \quad (3.2)$$

By (3.2) we can apply Lemma 3.1 to $L = K_{X/Y}^m \otimes \mathcal{O}_X(m) \otimes f^* A_Y$ and $A = A_Y$ and we get

$$\begin{aligned} \kappa(X, K_{X/Y} + F + f^* A_Y) &= \kappa(X, (mK_{X/Y} + mF + f^* A_Y) + f^* A_Y) \\ &= \kappa(F, (mK_{X/Y} + mF + f^* A_Y)/F) + \dim Y \\ &= \kappa(F, K_F + F) + \dim Y. \end{aligned}$$

By virtue of Lemma 3.3, one easily deduces Theorem 3.2:

Proof of Theorem 3.2. — By Lemma 1.5, up to replacing Y by a higher smooth model and up to taking a desingularization of the fibre product, we can assume that X and Y are smooth. Since Y is of general type, it is projective. Then fix an ample line bundle H on Y ; its canonical bundle K_Y being big, the Kodaira Lemma (cf. [57, Lemma 2.60, pp. 67–68]) implies that there exists an integer $b > 0$ such that $K_Y^b \otimes H^{-1}$ is effective. Now by applying Lemma 3.3 to $A_Y = \frac{1}{b}H$ we obtain

$$\begin{aligned} \kappa(X, K_X + f^*A_Y) &> \kappa(X, bK_{X/Y} + b f^*A_Y + f^*H) \\ &= \kappa(F, K_F + bA_Y) + \dim Y, \end{aligned}$$

thus we prove Theorem 3.2.

3.2. Iitaka Conjecture for Kähler Fibre Spaces with Big Determinant Bundle of the Direct Image of Relative Pluricanonical Bundles

The proof of Main Theorem A(I) is obtained by combining Lemma 3.1 and Theorem 2.2 plus the following result:

THEOREM 3.4 (Kähler version of [25, Theorem 3.4]). — *Let $f : X \rightarrow Y$ be a fibre space with X a compact Kähler manifold and Y a smooth projective variety and let F be the general fibre of f . Let L be a holomorphic \mathbb{Q} -line bundle on X equipped with a singular Hermitian metric h_L such that its curvature current $\Theta_{h_L}(L) > 0$ and that $\int (h_L) \in \mathbb{Q} \cdot \mathcal{O}_X$. Suppose that there is an integer $m > 0$ such that $L^{\otimes m}$ is a line bundle and that*

$$f_* \left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m} \right) = 0. \tag{3.3}$$

Such m exists if and only if $\kappa(F, K_F + L|_F) > 0$. Suppose that there is a SNC divisor D_Y containing $Y \setminus Y_0$ where Y_0 is the (analytic) Zariski open subset over which f is smooth, such that $f^{-1}D_Y$ has SNC support (in other word, f is prepared in the sense of [17]). Then there exists a constant $\epsilon_0 > 0$ and an f -exceptional effective \mathbb{Q} -divisor E such that the \mathbb{Q} -line bundle

$$K_{X/Y} + L + E - \epsilon_0 f_* \det f \left(K_{X/Y}^{\otimes m} \otimes L^{\otimes m} \right) \tag{3.4}$$

is pseudoeffective.

Before giving the proof, let us remark that:

Remark 3.5. — The condition (3.3) concerning the positivity of the Kodaira dimension of the general fibre does not appear in the original statement of [25, Theorem 3.14], but is indispensable. In fact, consider for example the case where $Y = \text{pt}$, X is a smooth Fano variety (or more generally a smooth uniruled projective variety) with $\kappa(X) = 0$, f is the structural morphism $X \rightarrow \text{pt}$ and $L = \mathcal{O}_X$; f being a smooth morphism, there is no f -exceptional divisors, and the direct image (space of global sections) of K_X^m is always 0, then the \mathbb{Q} -line bundle (3.4) is equal to K_X , which can never be pseudoeffective for X Fano (or uniruled projective, by [11]).

Proof of Theorem 3.4. — The proof follows the same idea as that of [25, Theorem 3.4]; in fact, the algebraicity of f (or equivalently, the algebraicity of X) is not essential in the original proof: it is only used in [25] to apply the Ohsawa–Takegoshi extension theorem and [64, III.5.10.Lemma, pp. 107–108]; as have been seen in Section 2.1 and Section 1.3 respectively, both of them can be generalized to the Kähler case. Nevertheless, the proof being highly technical, we will give more details for the convenience of the readers. Let us summarize the central idea of the proof as follows: from the natural inclusion of the determinant into the tensor product, we can construct, by the diagonal method of Viehweg, a non-zero section on $X^{(r)}$ (where $X^{(r)}$ denotes the resolution of some fibre product X^r of X over Y) of a line bundle of the form (3.4) (with X replaced by X^r and $\epsilon_0 = 1$); and then we “restrict” this section to the diagonal so that we get a section of the line bundle (3.4) on X . However one cannot deduce the effectivity of the line bundle (3.4), since the section constructed as above can vanish along the diagonal. To overcome this difficulty, we have to take a twisted approach: at the cost of tensoring by an ample divisor on Y , we can use the Ohsawa–Takegoshi extension Theorem 2.2 to extend pluricanonical forms on the general fibre F (by virtue of the condition (3.3)) to sections of the line bundle of the form (3.4) on $X^{(r)}$, then one can restrict them to the diagonal and get non-zero sections. However, these sections usually have poles, due to the singularities of f ; in order to get rid of them, one has to carefully analyse these singularities (this analysis takes up a technical part of the proof), then it turns out that the poles are supported on the non-reduced fibres in codimension 1 and hence one can use Proposition 2.5 to control them. Finally one use an approximation argument to conclude the pseudoeffectivity of the line bundle (3.4). The proof of the theorem proceeds in six steps:

(A). Analysis of singular fibres of f . In this step, we will use a standard argument to show that the (analytic Zariski) open subset of $y \in Y$ such that X_y is Gorenstein is of codimension > 2 (whilst the generic smoothness only ensure this to be analytic Zariski open). To this end, note

$$Y_f := Y_{\text{flat}} \cup Y_{F_{m,L}}$$

the (analytic) Zariski open subset over which f is flat and $F_{m,L} := f(K_{X/Y}^m \otimes L^m)$ is locally free; and denote $X_f := f^{-1}Y_f$. since X and Y are reduced, $\text{codim}_Y(Y \setminus Y_f) > 2$ (cf. [55, Corollary 5.5.15, p. 147] and [38, Example A.5.4, p. 416]). By [62, Theorem 23.4, p. 181], for every $y \in Y_f$, the fibre X_y is Gorenstein.

(B). Construction of the fibre product X^r and the canonical section. Over Y_f one has a natural morphism (injection of vector bundles)

$$\det f \left(K_{X/Y}^m \otimes L^m \right) \subset \bigotimes^r f \left(K_{X/Y}^m \otimes L^m \right), \quad (3.5)$$

where $r := \text{rk } F_{m,L}$, which gives rise to a non-trivial section of

$$\left(\bigotimes^r f \left(K_{X/Y}^m \otimes L^m \right) \right) \left(\det f \left(K_{X/Y}^m \otimes L^m \right) \right)^{-1}. \quad (3.6)$$

over Y_f . In order to get a section of a line bundle of the form (3.4), we will apply the diagonal method of Viehweg (cf. for example [75, §6.5, pp. 192–196]). Let

$$X^r := \underbrace{X \times_Y X \times_Y \dots \times_Y X}_{r \text{ times}}$$

be the r -fold fibre product of X over Y , equipped with a morphism (a Kähler fibration) $f^r : X^r \rightarrow Y$ as well as the natural projections $\text{pr}_i : X^r \rightarrow X$ to the i -th factor. Denote $X_f^r := (f^r)^{-1}Y_f$, then $f^r|_{X_f^r}$ is flat; moreover, since Y and $X_y^r = X_y \times \dots \times X_y$ are Cohen–Macaulay for every $y \in Y_f$, X_f^r is also Cohen–Macaulay (by [61, (21.C) Corollary 2, p. 154]). By the base change formula for relative canonical sheaves we see that X_f^r is Gorenstein and

$$\omega_{X^r} \otimes f^r K_Y^{-1} = \omega_{X^r/Y} \otimes \bigotimes_{i=1}^r \text{pr}_i^* K_{X/Y} \quad (3.7)$$

Note

$$L_r := \bigotimes_{i=1}^r \text{pr}_i^* L,$$

then by an induction argument, the projection formula together with the base change formula imply that (cf. [47, Lemma 3.15])

$$\bigotimes^r f \left(K_{X/Y}^m \otimes L^m \right) = (f^r) \left(\omega_{X^r/Y} \otimes L_r^m \right) \quad \text{over } Y_f.$$

In consequence, the morphism (3.5) gives rise to a non-zero section

$$\begin{aligned}
 s_0 & \quad \mathbb{H}^0 \left(X_f^r, \omega_{X^r/Y}^m \otimes L_r^m \otimes (f^r)^* \left(\det f \left(K_{X/Y}^m \otimes L^m \right) \right)^{-1} \right) \\
 & = \mathbb{H}^0 \left(Y_f, \left(\bigotimes_{i=1}^r f^* \left(K_{X/Y}^m \otimes L^m \right) \right) \otimes \left(\det f \left(K_{X/Y}^m \otimes L^m \right) \right)^{-1} \right). \quad (3.8)
 \end{aligned}$$

(C). Analysis of the singularities of X^r . Take a desingularization $\mu : X^{(r)} \rightarrow X^r$ which is an isomorphism over the smooth locus of X^r . Note $f^{(r)} := f^r \circ \mu$ and $X_f^{(r)} := \mu^{-1} X_f^r$. The natural morphism

$$\mu^* K_{X^{(r)}} \cong \omega_{X^{(r)}}, \quad (3.9)$$

which is an isomorphism over X_{rat}^r where X^r denotes the (analytic Zariski) open subset of point with rational singularities on X^r , gives rise to a meromorphic section of the line bundle (by virtue of (3.7))

$$K_{X^{(r)}/Y}^{-1} \otimes \mu^* \left(\bigotimes_{i=1}^r \text{pr}_i^* K_{X/Y} \right),$$

whose zeros and poles are contained in $X^{(r)} \setminus \mu^{-1} X_{\text{rat}}^r$. In consequence, there are two effective divisors D_1 and D_2 over $X^{(r)}$ such that $\text{Supp}(D_1), \text{Supp}(D_2) \subset X^{(r)} \setminus \mu^{-1} X_{\text{rat}}^r$ and that

$$K_{X^{(r)}/Y} \otimes \mathcal{O}_{X^{(r)}}(D_1) = \mu^* \left(\bigotimes_{i=1}^r \text{pr}_i^* K_{X/Y} \right) \otimes \mathcal{O}_{X^{(r)}}(D_2). \quad (3.10)$$

Now let us further analyse the rational singularities locus X_{rat}^r by virtue of our hypothesis on Y and $f : Y \rightarrow X$. Write

$$f^* \mathcal{O}_X = \sum_i W_i + \sum_j a_j V_j \quad (3.11)$$

with the W_i 's and V_j 's prime divisors over X and $a_i > 2$; by hypothesis,

$$W := \sum_i W_i \quad \text{et} \quad V := \sum_j V_j$$

are (reduced) SNC divisors. As is explained in Remark 1.2, the fibre product

$$\begin{aligned}
 & (X_f \setminus (V \cup f^{-1} \text{Sing}(Y)))^r \\
 & := \underbrace{(X_f \setminus (V \cup f^{-1} \text{Sing}(Y))) \times_{Y_f \setminus \text{Sing}(Y)} \cdots \times_{Y_f \setminus \text{Sing}(Y)} (X_f \setminus (V \cup f^{-1} \text{Sing}(Y)))}_{r \text{ times}}
 \end{aligned}$$

is contained in X_{rat}^r .

In consequence, both D_1 and D_2 are contained in the set D where D denotes the set of divisors D on $X^{(r)}$ such that every component of D satisfies (at least) one of the following three conditions:

- (D1) $f^{(r)}(\sigma) \cdot Y \setminus Y_f$ (in particular, σ is $f^{(r)}$ -exceptional);
- (D2) σ is pr_i μ -exceptional for some i ;
- (D3) $\text{pr}_i \mu(\sigma) = V_j$ for some i and j .

(D). Extension of pluricanonical forms on $X_y^{(r)}$ by Ohsawa–Takegoshi. The section s_0 (cf. (3.8)) gives rise the section

$$\mu s_0 \in H^0\left(X_f^{(r)}, K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1) \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1}\right).$$

Since $\text{codim}_Y Y_f > 2$, the section μs_0 , regarded as a section of the torsion free sheaf (3.6) over Y_f , extends to a global section \bar{s}_0 of the reflexive hull

$$\left[\left(\bigotimes^r f\left(K_{X/Y}^m \otimes L^m\right) \otimes \left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1} \right) \otimes f^{(r)}\left(K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1) \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1} \right) \right].$$

By Theorem 1.14, there is an $f^{(r)}$ -exceptional effective divisor D_3 such that

$$\begin{aligned} & \left[f^{(r)}\left(K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1) \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1} \right) \right] \\ &= \left[f^{(r)}\left(K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1)\right) \right] \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1} \\ &= f^{(r)}\left(K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1 + D_3)\right) \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1} \\ &= f^{(r)}\left(K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1 + D_3) \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1}\right), \end{aligned}$$

hence \bar{s}_0 can be regarded as a (global) section of the line bundle

$$K_{X^{(r)}/Y}^m \otimes L_r^m \otimes \mathcal{O}_{X^{(r)}}(mD_1 + D_3) \otimes f^{(r)}\left(\det f\left(K_{X/Y}^m \otimes L^m\right)\right)^{-1}.$$

Moreover, since the torsion free sheaf (3.6) is locally free on Y_f , hence

$$f^{(r)}(\text{Supp}(D_3)) \subset Y \setminus Y_f,$$

in particular, $D_3 \subset D$. Now choose $\epsilon \in \mathbb{Q}_{>0}$ small enough such that $\sigma_0 := \epsilon \text{div}(\bar{s}_0)$ is klt on $X^{(r)}$. The \mathbb{Q} -line bundle $\mathcal{O}_{X^{(r)}}(\sigma_0)$ is equipped with a

canonical singular Hermitian metric $h_{\bar{\sigma}_0}$ whose local weight is given by

$$\phi_{\bar{\sigma}_0} = \frac{\epsilon}{2} \log |g_{\bar{\sigma}_0}|^2,$$

where $g_{\bar{\sigma}_0}$ denotes a local equation of $\text{div}(\bar{\sigma}_0)$. Denote $L_0 := \mu^* L_r \otimes_{\mathcal{O}_{X^{(r)}}} \mathcal{O}_{X^{(r)}}(-\bar{\sigma}_0)$, this \mathbb{Q} -line bundle is equipped with the singular Hermitian metric

$$h_{L_0} := h_{\bar{\sigma}_0} \otimes_{i=1}^r \mu^* \text{pr}_i^* h_{L_i}.$$

whose curvature current is positive. By strong openness [43, Theorem 1.1] for ϵ sufficiently small we have

$$\mathcal{J}(h_{L_0}) = \mathcal{J} \left(\bigotimes_{i=1}^r \mu^* \text{pr}_i^* h_{L_i} \right). \tag{3.12}$$

Since μ is supposed to be an isomorphism over Y_0 , we have $X_y^{(r)} \cong X_y \times \cdots \times X_y$ for $y \in Y_0$ (cf. Step (E1) below), then by Lemma 2.7 and [32, Theorem 2.6(i)] we have $\mathcal{J}(h_{L_0}/X_y) = \mathcal{O}_{X_y}$ for a.e. $y \in Y_0$.

Let A_Y be an ample line bundle over Y such that the line bundle $A_Y \otimes K_Y^{-1}$ is ample and that the Seshadri constant $\epsilon(A_Y \otimes K_Y^{-1}, y) > d := \dim Y$ for general $y \in Y_0$ (such A_Y exists by [59, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]). Claim that the restriction map

$$H^0(X^{(r)}, K_{X^{(r)}/Y}^k \otimes L_0^k \otimes f^{(r)*} A_Y) \rightarrow H^0(X_y^{(r)}, K_{X_y^{(r)}}^k \otimes L_0^k / X_y^{(r)}) \tag{3.13}$$

is surjective for any k sufficiently large and divisible and for every $y \in Y_0$ such that $\mathcal{J}(h_{L_0}/X_y^{(r)}) = \mathcal{O}_{X_y^{(r)}}$. In fact, $\bar{\sigma}_0$ being effective, the hypothesis (3.3) implies that

$$\begin{aligned} f^{(r)*} \left(K_{X^{(r)}/Y}^k \otimes L_0^k \right) &= f^{(r)*} \left(K_{X^{(r)}/Y}^k \otimes \mu^* L_r^k \otimes \mathcal{O}_{X^{(r)}}(k \bar{\sigma}_0) \right) \\ f^{(r)*} \left(K_{X^{(r)}/Y}^k \otimes \mu^* L_r^k \right) &= 0 \end{aligned}$$

for k sufficiently large and divisible (e.g. such that $\epsilon k > Z_{>0}$ and k divisible by m) hence the integrability condition in Theorem 2.3 is satisfied (cf. Remark 2.4). Moreover, since $\int h_{L_0}(L_0) > 0$, Theorem 2.3 implies that the k -Bergman kernel metric $h_{X^{(r)}/Y, kL_0}^{(k)}$ is semi-positively curved. Set $M_k := K_{X^{(r)}/Y}^{(k-1)} \otimes L_0^k$, equipped with a singular Hermitian metric

$$h_{M_k} := \left(h_{X^{(r)}/Y, kL_0}^{(k)} \right)^{\frac{k-1}{k}} \otimes h_{L_0}$$

whose curvature current is positive. Then by Lemma 2.12 one has

$$\begin{aligned} H^0\left(X_y^{(r)}, K_{X_y^{(r)}}^k \otimes L_0^k / X^{(r)} \otimes (h_{M_k} / X_y^{(r)})\right) \\ = H^0\left(X_y^{(r)}, K_{X_y^{(r)}}^k \otimes L_0^k / X^{(r)}\right) \end{aligned} \quad (3.14)$$

for a.e. $y \in Y_0$. Hence we can apply Theorem 2.2 to

$$K_{X^{(r)}} \otimes M_k \otimes f^{(r)}(A_Y \otimes K_Y^{-1}) = K_{X^{(r)}/Y}^k \otimes L_0^k \otimes f^{(r)}(A_Y)$$

to obtain the surjectivity of the restriction morphism (3.13) for a.e. $y \in Y_0$. Moreover, set $H_k := A_Y \otimes \det f(K_{X/Y}^m \otimes L^m)^{-\epsilon k}$, then we can rewrite (3.13) as

$$\begin{aligned} H^0\left(X^{(r)}, (K_{X^{(r)}/Y} \otimes \mu^* L_r)^{(1+\epsilon m)k} \otimes \mathcal{O}_{X^{(r)}}(\epsilon k m D_1 + \epsilon k D_3) \otimes f^{(r)}(H_k)\right) \\ \xrightarrow{\text{restriction}} H^0\left(X_y^{(r)}, (K_{X_y^{(r)}} \otimes \mu^* L_r / X_y^{(r)})^{(1+\epsilon m)k}\right) \end{aligned} \quad (3.15)$$

for a.e. $y \in Y_0$ and for k sufficiently large and divisible.

(E). Extension of pluricanonical forms on X_y via restriction to the diagonal. For general $y \in Y_0$ take a section

$$u \in H^0\left(X_y, (K_{X_y} \otimes L / X_y)^{(1+\epsilon m)k}\right)$$

with k sufficiently large and divisible, we will construct a section s in

$$H^0\left(X, (K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes \mathcal{O}_X(CkV + kE_0) \otimes f^*(H_k)^r\right),$$

for $C > 0$ a constant and E_0 an f -exceptional effective divisor, both independent of k , such that $s|_{X_y} = u^r$.

(E1). *Extending the section u to a section over $X^{(r)}$ by Step (D).* — Note

$$X_0^r := X_0 \times_{Y_0} X_0 \times_{Y_0} \dots \times_{Y_0} X_0 \rightarrow X^r,$$

then X_0^r is smooth, hence $\mu^{-1} X_0^r \xrightarrow{\mu} X_0^r$ is an isomorphism. In particular, we have

$$X_y^{(r)} \xrightarrow{\mu} X_y^r = \underbrace{X_y \times X_y \times \dots \times X_y}_{r \text{ times}}. \quad (3.16)$$

Hence u gives rise to a section

$$u^{(r)} := \mu \left(\bigotimes_{i=1}^r \text{pr}_i^* u \right) \in H^0\left(X_y^{(r)}, (K_{X_y^{(r)}} \otimes \mu^* L_r / X_y^{(r)})^{(1+\epsilon m)k}\right), \quad (3.17)$$

such that the restriction of $u^{(r)}$ to the diagonal is equal to $u^{(r)}$. Using the surjection (3.15) we obtain a section $\sigma^{(r)}$ of the line bundle

$$(K_{X^{(r)}/Y} \otimes \mu^* L_r)^{(1+\epsilon m)k} \otimes_{\mathcal{O}_{X^{(r)}}} (\epsilon k m D_1 + \epsilon k D_3) \otimes f^{(r)*} H_k, \quad (3.18)$$

such that $\sigma^{(r)}|_{X_y^{(r)}} = u^{(r)}$.

(E2). *Restricting the section $\sigma^{(r)}|_{\mu^{-1}X_0^r}$ to the diagonal.* — In order to restrict $\sigma^{(r)}|_{\mu^{-1}X_0^r}$ to the diagonal, use (3.10) to rewrite the line bundle (3.18) as follows:

$$\begin{aligned} & (K_{X^{(r)}/Y} \otimes \mu^* L_r)^{(1+\epsilon m)k} \otimes_{\mathcal{O}_{X^{(r)}}} (\epsilon k m D_1 + \epsilon k D_3) \otimes f^{(r)*} H_k \\ &= \mu^* \left(\bigotimes_{i=1}^r \text{pr}_i^*(K_{X/Y} \otimes L) \right)^{(1+\epsilon m)k} \\ & \quad \otimes_{\mathcal{O}_{X^{(r)}}} (-k D_1 + (1 + \epsilon m)k D_2 + \epsilon k D_3) \otimes f^{(r)*} H_k. \end{aligned} \quad (3.19)$$

In consequence, $\sigma^{(r)}$ can be regarded as a meromorphic section of the line bundle

$$\mu^* \left(\bigotimes_{i=1}^r \text{pr}_i^*(K_{X/Y} \otimes L) \right)^{(1+\epsilon m)k} \otimes f^{(r)*} H_k \quad (3.20)$$

whose poles are contained $\text{Supp}(D_2) \cup \text{Supp}(D_3)$. Locally, by choosing a trivialization of the line bundle (3.20), the section $\sigma^{(r)}$ can be written as a meromorphic function $F^{(r)}$ such that

$$g_{D_1}^{-k} g_{D_2}^{(1+\epsilon m)k} g_{D_3}^{\epsilon k} \cdot F^{(r)} \quad (3.21)$$

is holomorphic, where g_{D_l} is a local equation of the divisor D_l ($l = 1, 2, 3$).

By construction, $D_1, D_2, D_3 \subset D$ (in particular, D_3 is $f^{(r)}$ -exceptional), hence there exist constants C_1 et C_2 such that

$$D_l \subset C_l \cdot \mu^* \sum_{i=1}^r \text{pr}_i^* V, \quad \text{pour } l = 1, 2 \quad (3.22)$$

over $X_f^{(r)} \setminus S$ where $S \subset X^{(r)}$ denotes the union of the components in $D_1 + D_2$ which are pr_i μ -exceptional for every $i = 1, \dots, r$. By Step (D) we have

$$f^{(r)*}(\text{Supp}(D_3)) \subset Y \setminus Y_f,$$

hence locally over $X_f^{(r)} \setminus S$ the meromorphic function

$$F^{(r)} \cdot \prod_{i=1}^r ((\text{pr}_i^* \mu)^* g_V)^{C_2(1+\epsilon m)k} = F^{(r)} \cdot \prod_{i=1}^r (\text{pr}_i^* \mu)^* \left(g_V^{C_2(1+\epsilon m)k} \right)$$

is holomorphic where $g_V = \prod_j g_{V_j}$ is a local equation of V .

Note $\pi_{X,r} : X \rightarrow X^r$ the inclusion of the diagonal. Then $\text{pr}_i \circ \pi_{X,r} = \text{id}_X$ for $i = 1, \dots, r$. Since the D_l 's ($l = 1, 2, 3$) are disjoint to

$$\mu^{-1}X_{\text{rat}}^r \cup \mu^{-1}X_0^r \cup \mu^{-1}(\pi_{X,r}(X_0)),$$

then locally the meromorphic function $F^{(r)}$ is holomorphic over $\mu^{-1}X_0^r$. Therefore we can restrict $\sigma^{(r)}/\mu^{-1}X_0^r$ to the diagonal and obtain a section

$$s_1 := (\mu|_{X_0^r}^{-1} \circ \pi_{X,r}/X_0) \left(\sigma^{(r)}/\mu^{-1}X_0^r \right)$$

over X_0 of the line bundle

$$(K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes f^* H_k^{\otimes r}. \quad (3.23)$$

Locally over an open subset of X_0 trivializing the line bundle (3.23) the section s_1 is given by a holomorphic function

$$F_1 := (\mu|_{X_0^r}^{-1} \circ \pi_{X,r}/X_0) \left(F^{(r)}/\mu^{-1}X_0^r \right).$$

(E3). *Extending the section s_1 across the singular fibres of X .* — In order to extend s_1 across $f^{-1} \cup V$, one needs to know its behaviour around the W_i 's and the V_j 's; this can be done by analysing the poles along the D_l 's of $\sigma^{(r)}$, regarded as a meromorphic section of the line bundle (3.20), as we explain in the sequel:

(E3-i). — By Step (C) $(X_f \setminus (V \cup \text{Sing}(W)))^r$ is contained in X_{rat}^r , thus disjoint to the D_l 's ($l = 1, 2, 3$); regarding F_1 as a holomorphic function on $X_{f,r}(X_0)$, one has

$$\mu^* F_1 = F^{(r)}/\mu^{-1}(\pi_{X,r}(X_0)),$$

but the poles of $F^{(r)}$ are contained in $\text{Supp}(D_2) \cup \text{Supp}(D_3)$, hence the function F_1 is bounded around the $X_f \setminus (V \cup f^{-1} \text{Sing}(\cup V))$, and thus F_1 can be extended to $X_f \setminus (V \cup f^{-1} \text{Sing}(\cup V))$ by Riemann extension; moreover, by Hartogs extension, F_1 extends to a holomorphic function over $X_f \setminus V$.

(E3-ii). — In general, F_1 is not bounded around V . Nevertheless, by Step (E2) the meromorphic function

$$F^{(r)} \cdot \mu \prod_{i=1}^r \text{pr}_i \left(g_V^{C_2(1+\epsilon m)k} \right)$$

is holomorphic over $X_f^r \setminus S$. And the restriction of S to the diagonal is an analytic subset of codimension > 2 (cf. (E2) for the definition of S), hence the function

$$F_1 \cdot g_V^{C_2(1+\epsilon m)kr}$$

is bounded around a general point of $V \cup X_f$. By Riemann extension (as well as Hartogs extension) F_1 extends across $V \cup X_f$ as a holomorphic local

section of the line bundle

$$(K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV \otimes f^* H_k^{-r}),$$

where $C := C_2(1 + \epsilon m)r$ is a constant independent of k . Combining this with (E3-i) we obtain an extension of s_1 to a section over X_f :

$$\bar{s}_1 \in H^0(X_f, (K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV \otimes f^* H_k^{-r})).$$

(E3-iii). — At last, we will extend \bar{s}_1 to a global section, which provides the section s that we search for. In fact, \bar{s}_1 can be regarded as a section of the direct image sheaf

$$f_* \left((K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV \otimes f^* H_k^{-r}) \right) \quad (3.24)$$

over Y_f . But $\text{codim}_Y(Y \setminus Y_f) > 2$, hence \bar{s}_1 extends to a global section s of the reflexive hull of the (torsion free) sheaf (3.24). By Theorem 1.14, there is an f -exceptional effective divisor E_0 , independent of k , such that

$$\begin{aligned} & f_* \left((K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV \otimes f^* H_k^{-r}) \right) \\ &= f_* \left((K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV + kE_0 \otimes f^* H_k^{-r}) \right), \end{aligned}$$

hence

$$s \in H^0(X, (K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV + kE_0 \otimes f^* H_k^{-r})).$$

Moreover, by (3.16) as well as the construction of the section $u^{(r)}$ (cf. (3.17)) we have

$$s|_{X_y} = s_1|_{X_y} = (\mu)^{-r} u^{(r)} = u^{-r}.$$

This finishes (E3) and thus the Step (E).

(F). Conclusion. By the hypothesis (3.3), for any general $y \in Y$ and for any integer k sufficiently large and divisible (e.g. such that $\epsilon k \geq Z_{>0}$ and that k divisible par m), we have a non-zero section

$$u \in H^0(X_y, (K_{X_y} \otimes L|_{X_y})^{(1+\epsilon m)k}).$$

Assume further that $y \in Y_0$ and $\mathcal{J}(h_{L_0}|_{X_y^{(r)}}) = \mathcal{O}_{X_y^{(r)}}$, then by Step (E) above, we can construct a section

$$s \in H^0(X, (K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV + kE_0 \otimes f^* H_k^{-r})),$$

for C and E_0 independent of k such that $s|_{X_y} = u^{-r}$. In particular $s \neq 0$, implying that the line bundle

$$(K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X} (CkV + kE_0 \otimes f^* H_k^{-r}) \quad (3.25)$$

is effective. By writing $V = V_{\text{div}} + V_{\text{exc}}$ with V_{div} (resp. V_{exc}) the non-exceptional (resp. exceptional) part of V with respect to f , one can rewrite the line bundle (3.25) as follows:

$$\begin{aligned} (K_{X/Y} \otimes L)^{(1+\epsilon m)kr} & \otimes_{\mathcal{O}_X}(CkV + kE_0) \otimes f^* H_k^{-r} \\ & = (K_{X/Y} \otimes L)^{(1+\epsilon m)kr} \otimes_{\mathcal{O}_X}(CkV_{\text{div}} + kE_1) \otimes f^* H_k^{-r} \end{aligned}$$

where $E_1 = CV_{\text{exc}} + E_0$ is f -exceptional. In addition, the hypothesis (3.3) implies that the relative m -Bergman kernel metric $h_{X/Y,L}^{(m)}$ on $K_{X/Y} \otimes L^{-m}$ is semi-positively curved, hence by Proposition 2.5 and (3.11) the line bundle

$$K_{X/Y} \otimes L \otimes \mathcal{O}_X(-bV_{\text{div}})$$

is pseudoeffective, where $b := \min_j \{a_j - 1\}$. Therefore the \mathbb{Q} -line bundle

$$\begin{aligned} \left((1 + \epsilon m)kr + \frac{Ck}{b} \right) (K_{X/Y} \otimes L) + kE_1 + rf^* A_Y \\ - \epsilon kr f^* \det f^* (K_{X/Y} \otimes L^{-m}) \end{aligned}$$

is pseudoeffective. By letting $k \rightarrow +\infty$ and by putting

$$E := \frac{b}{(1 + \epsilon m)br + C} E_1 \quad \text{and} \quad \epsilon_0 := \frac{\epsilon br}{(1 + \epsilon m)br + C}$$

we obtain the pseudoeffectivity of the \mathbb{Q} -line bundle (3.4), thus prove Theorem 3.4.

Now turn to the proof of Main Theorem A(II). In fact one can prove a stronger result as following, whose proof is quite similar to [25, Corollary 4.1]:

THEOREM 3.6. — *Let $f : X \rightarrow Y$ be a fibre space between compact Kähler manifolds. Let Δ be an effective \mathbb{Q} -divisor on X such that (X, Δ) is klt. Suppose that there exists an integer $m > 0$ such that $m\Delta$ is an integral divisor and the determinant line bundle $\det f^* (K_{X/Y} \otimes \mathcal{O}_X(m\Delta))$ is big. Then*

$$\kappa(X, K_X + \Delta) > \kappa(Y) + \kappa(F, K_F + \Delta_F). \tag{3.26}$$

where F denotes the general fibre of f and $\Delta_F := \Delta|_F$. Moreover, if $\kappa(Y) > 0$ then we have

$$\kappa(X, K_X + \Delta) > \kappa(F, K_F + \Delta_F) + \dim Y.$$

Proof. — The key point of the proof has already been proved in Theorem 3.4, the rest is quite similar to that of Theorem 3.2. Nevertheless, in order to apply Theorem 3.4, one should be able to add an “exceptional” positivity to the pluricanonical bundle; therefore we take a diagram as in

Lemma 1.4:

$$\begin{array}{ccc} X & \xrightarrow{\pi_X} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\pi_Y} & Y, \end{array}$$

and take an effective \mathbb{Q} -divisor on X as in Lemma 1.5, so that every f -exceptional divisor is also π_X -exceptional and that (X, ν) is klt. By construction, the morphism f is smooth over $Y_0 := \pi_Y^{-1}Y_0$ where Y_0 denotes the (analytic) Zariski open subset of Y over which f is smooth; $\pi_X|_{X_0} : X_0 \rightarrow X_0$ with $X_0 := (f)^{-1}Y_0$ and $X_0 := f^{-1}Y_0$ is an isomorphism. In particular, for $y \in Y_0$, we have an isomorphism $X_y \cong X_y$ (with $y := \pi_Y(y)$) between complex manifolds, implying that $F \cong F$ where F denotes the general fibre of f ; moreover this isomorphism identifies $\nu|_F := \nu|_F$ to $\nu|_F$.

In addition, we have the following (non-trivial) morphism of base change

$$\pi_Y f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu)) \rightarrow f^*(\pi_X(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu))) \rightarrow f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu)), \quad (3.27)$$

where the first morphism is an isomorphism over Y_0 , and the second morphism is injective, which is a result of the fact that $K_Y|_{Y_0}$ is π_Y -exceptional and effective; π_Y being birational, the line bundle

$$\pi_Y \det f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu))$$

is big over Y , therefore the morphism (3.27) implies that the determinant line bundle $\det f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu))$ is also big over Y . In particular

$$f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu)) = 0. \quad (3.28)$$

Hence we can apply Theorem 3.4 to f , and we get an f -exceptional \mathbb{Q} -divisor E and $\epsilon_0 \in \mathbb{Q}_{>0}$ such that the \mathbb{Q} -line bundle

$$K_{X/Y} + \nu + E - \epsilon_0(f^*(\det f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu))))$$

is pseudoeffective. Let us fix a very ample line bundle A_Y on Y such that $A_Y \otimes K_Y^{-1}$ is ample and that the Seshadri constant $\epsilon(A_Y \otimes K_Y^{-1}, y) > \dim Y$ for general $y \in Y$ (such A_Y exists by [59, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]). Since $\det f^*(K_{X/Y}^m \otimes \mathcal{O}_X(m\nu))$ is big, the Kodaira's Lemma (cf. [57, Lemma 2.60, pp. 67–68]) implies that there exists a integer $m_1 > 0$ sufficiently large and divisible and a pseudoeffective line bundle L_0 on X such that $m_1 \nu$ and $m_1 E$ are integral divisors and that

$$K_{X/Y}^{m_1} \otimes \mathcal{O}_X(m_1(\nu + E)) = (f^*) A_Y^{2m_1} \otimes L_0.$$

And we have $L_0/F = K_F^{m_1} \otimes_{\mathcal{O}_F} (m_1 - F)$. Now L_0 being pseudoeffective, we can equip it with a singular Hermitian metric h_{L_0} whose curvature current is positive. Since X is klt, by strong openness [43, Theorem 1.1] (or [4, Theorem 1.1]) we can find $m_2 \in \mathbb{Z}_{>0}$ sufficiently large and divisible such that

$$\mathcal{J} \left(h_{L_0}^{\frac{1}{m_2}} \right) = \mathcal{O}_X.$$

Now we can endow $K_X^{m_2} \otimes_{\mathcal{O}_X} (m_2 - L_0)$ with the relative m_2 -Bergman kernel metric $h_{X/Y, m_2 - L_0}^{(m_2)}$, then by applying Lemma 2.12 to the \mathbb{Q} -line bundle $N = K_X^{m_2} \otimes_{\mathcal{O}_X} (m_2 - L_0)$ we have

$$\mathbb{H}^0(F, K_F \otimes_{\mathcal{O}_F} N_{m_2-1}/F \otimes_{\mathcal{O}_F} \mathcal{J} (h_{N_{m_2-1}|_F})) = \mathbb{H}^0(F, K_F \otimes_{\mathcal{O}_F} N_{m_2-1}/F),$$

where $N_{m_2-1} := K_X^{(m_2-1)} \otimes_{\mathcal{O}_X} (m_2 - L_0)$ equipped with singular Hermitian metric

$$h_{N_{m_2-1}} := \left(h_{X/Y, m_2 - L_0}^{(m_2)} \right)^{\frac{m_2-1}{m_2}} \cdot h_{L_0}^{\frac{1}{m_2}}.$$

Now by Theorem 2.2 we have a surjection

$$\mathbb{H}^0(X, K_X \otimes_{\mathcal{O}_X} N_{m_2-1} \otimes_{\mathcal{O}_X} (f) \otimes_{\mathcal{O}_Y} K_Y^{-1}) \rightarrow \mathbb{H}^0(F, K_F \otimes_{\mathcal{O}_F} N_{m_2-1}/F)$$

which amounts to:

$$\begin{aligned} \mathbb{H}^0(X, K_X^{m_2} \otimes_{\mathcal{O}_X} (m_2 - L_0) \otimes_{\mathcal{O}_X} (f) \otimes_{\mathcal{O}_Y} A_Y) \\ \rightarrow \mathbb{H}^0(F, K_F^{(m_1+m_2)} \otimes_{\mathcal{O}_F} ((m_1 + m_2) - F)) \end{aligned}$$

where the space on the right hand is non-vanishing by (3.28).

Then by applying Lemma 3.1 to $L = K_X^{m_2} \otimes_{\mathcal{O}_X} (m_2 - L_0) \otimes_{\mathcal{O}_X} (f) \otimes_{\mathcal{O}_Y} A_Y$ (noting that $L/F = K_F^{(m_1+m_2)} \otimes_{\mathcal{O}_F} ((m_1 + m_2) - F)$) and by [28, Lemma 7.11, p. 175] we obtain the following equality:

$$\begin{aligned} \kappa(X, K_X/Y + L) &= \kappa(X, (m_1 + m_2)(K_X/Y + L) + m_1 E) \\ &= \kappa(X, m_2 K_X/Y + m_2 L + 2(f) \otimes_{\mathcal{O}_Y} A_Y) \\ &= \kappa(X, L + (f) \otimes_{\mathcal{O}_Y} A_Y) \\ &= \kappa(F, K_F + F) + \dim Y. \end{aligned} \tag{3.29}$$

Then by Lemma 1.5 we get

$$\kappa(X, K_X/Y + L) > \kappa(F, K_F + F) + \dim Y.$$

If $\kappa(Y) = -$ then the inequality (3.26) is automatically established; otherwise, there is an integer $k > 0$ such that K_Y^k is effective, then by (3.29)

we get

$$\begin{aligned} \kappa(X, K_X + \dots) &= \kappa(X, kK_{X/Y} + k \dots + kf K_Y) \\ &> \kappa(X, kK_{X/Y} + k \dots) \\ &> \kappa(F, K_F + \dots_F) + \dim Y. \end{aligned}$$

4. Albanese Maps of Compact Kähler Manifolds of log Calabi–Yau Type

Having proved Theorem 3.2, one can follow the same argument as that in [49] to deduce Theorem C. Let us remark that in [49] a result equivalent to Theorem C with $\dots = 0$ is also stated ([49, Theorem 25]). Similar to [49] the first step of the proof of Theorem C is to obtain the following proposition, which generalize [71, Theorem 10.9, pp. 120–123] and can be regarded as an analytic version of [49, Theorem 13]:

PROPOSITION 4.1. — *Let $p : V \rightarrow T$ be a finite morphism with V a compact normal complex variety and T a complex torus. Then $\kappa(V) > 0$, and there is a subtorus S of T and a (projective) normal variety of general type W , which is finite over T/S , such that*

- (a) *there is an analytic fibre space $\phi_p : V \rightarrow W$ whose general fibre is equal to \tilde{S} , a complex torus which admits a finite étale cover $\tilde{S} \rightarrow S$ over S .*
- (b) $\kappa(W) = \kappa(V) = \dim W$;

Before showing the proposition, let us recall the following lemma, which can be proved by following the same argument as in [63] (combined with an analytic version of [1, Proposition (1.3)]):

LEMMA 4.2 (analytic version of [63, Theorem 3.1]). — *A meromorphic mapping from a complex manifold to a complex torus is always defined everywhere, thus gives rise to a morphism.*

Proof of Proposition 4.1. — By [71, Lemma 6.3, pp. 66–67], we have $\kappa(V) > \kappa(T) = 0$. Let $\nu : V \rightarrow W$ be the Iitaka fibration of V where V is smooth model lying over V and W a complex manifold. For a general point w in W , V_w and V_w are bimeromorphic and thus $\kappa(V_w) = \kappa(V_w) = 0$, where V_w is the image of V_w in V . Denote $S_w = p(V_w)$ for $w \in W$, then by [71, Theorem 10.9, pp. 120–123] we have $\kappa(S_w) > 0$; on the other hand, p being a finite morphism, [71, Lemma 6.3, pp. 66–67] implies that $\kappa(S_w) \subset \kappa(V_w) = 0$ for $w \in W$ general, hence $\kappa(S_w) = 0$ pour w general. Again by [71, Theorem 10.9, pp. 120–123], S_w is a translate of a subtorus

of T for w general (in particular, S_w is isomorphic to a complex torus for w general). Therefore $\{S_w\}_w \subset W \times T$ forms an analytic family of complex varieties over W whose general fibre is isomorphic to a complex torus; but T has only countably many subtori, hence there exists a subtorus S of T such that for very general w we have $S_w = S$. Now by (the analytic version of) [49, Lemma 14] (applied to $f = (V \rightarrow V \rightarrow T/S)$ and $g = \nu$), this implies that we have a meromorphic mapping $q : W \dashrightarrow T/S$; but W is smooth, then by Lemma 4.2 the meromorphic mapping q is everywhere defined, hence $q \circ \nu$ is equal to the composition morphism $V \rightarrow V \xrightarrow{p} T \xrightarrow{\text{quotient}} T/S$.

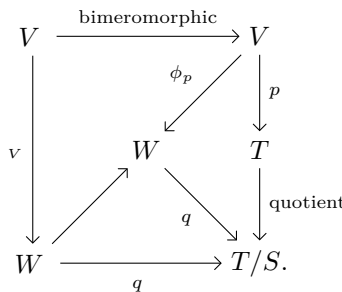
Note $W_0 = q^{-1}(S) = \text{image of } V \text{ in } T/S$. Since we have

$$\dim W = \dim V - \dim V_w = \dim p^{-1}(S) - \dim S_w = \dim W_0,$$

q is generically finite. Take a Stein factorization of $q : W \dashrightarrow T/S$ is a finite morphism and $W \rightarrow W$ an analytic fibre space; in addition, W is normal by our construction. Since q is generically finite, $W \rightarrow W$ is a fortiori bimeromorphic, in particular we have

$$\dim W = \dim W = \kappa(V). \tag{4.1}$$

By construction $q : W \dashrightarrow T/S$ also gives a Stein factorization of the proper morphism $V \xrightarrow{\nu} W \xrightarrow{q} T/S$ since $\nu^* \mathcal{O}_V = \mathcal{O}_W$; $V \rightarrow V$ being bimeromorphic morphism, the fibres of the morphism $V \rightarrow V$ are connected, hence they are contracted by $V \xrightarrow{\nu} W \rightarrow W$, by [28, §1.3, Lemma 1.15, pp. 12–13] there is a morphism $\phi_p : V \rightarrow W$ such that $q \circ \phi_p$ is equal to the morphism $V \xrightarrow{p} T \rightarrow T/S$, i.e. the following diagram is commutative:



Moreover, since $V \rightarrow V$ is bimeromorphic, Zariski’s Main Theorem (cf. [71, Corollary 1.14, p. 12]) implies that $\phi_p^* \mathcal{O}_V = \mathcal{O}_W$, hence ϕ_p is an analytic fibre space; by our construction ϕ_p and q provide a Stein factorization of the proper morphism $V \rightarrow T \rightarrow T/S$. In order to prove (a) it suffices to apply [49, Theorem 22], which is an analytic version of [54, Main Theorem]. In fact, since $\kappa(V_w) = 0$ for $w \in W$ general ($W \rightarrow W$ bimeromorphic),

[49, Theorem 22] implies that the finite surjective morphism $p/V_w : V_w \rightarrow p(V_w) = S_w$ is a finite étale cover, hence V_w is isomorphic to a (disjoint) union of copies of \tilde{S} with \tilde{S} a complex torus admitting a finite étale cover over S ; V_w being connected, we must have $V_w \cong \tilde{S}$. In other words, ϕ_p is an analytic fibre space whose general fibre equals to \tilde{S} . In order to establish (b), it remains, by virtue of (4.1), to show that W is of general type, i.e. $\kappa(W) = \dim W$. To see this, we will follow the same argument as in [71, Proof of Theorem 10.9, p. 122]. Assume by contradiction that $\kappa(W) < \dim W$, then one can apply the above argument to the finite morphism $q : W \rightarrow T/S$ and get the following commutative diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{\phi_p} & W & \xrightarrow{\phi_q} & W_1 \\
 \downarrow p & & \downarrow q & & \downarrow q_1 \\
 T & \xrightarrow{\text{quotient}} & T/S & \xrightarrow{\text{quotient}} & T/S_1,
 \end{array}$$

where $\dim W_1 = \kappa(W) < \dim W$, S_1 is a subtorus of T containing S , ϕ_q is an analytic fibre space whose general fibre is equal to \tilde{S}_1 , a complex torus admitting a finite étale cover over S_1/S , and q_1 is a finite morphism. Then $\phi_q \circ \phi_p : V \rightarrow W_1$ is an analytic fibre space whose general fibre is denoted by F . By construction F admits a finite morphism $F \rightarrow S_1$, thus F is Kähler and by (a) we have $\kappa(F) > 0$. Moreover, we have an analytic fibre space $\phi_p/F : F \rightarrow \tilde{S}_1$ whose general fibre is equal to \tilde{S} . $K_{\tilde{S}_1}$ being trivial, consider the relative Bergman kernel metric h_{F/\tilde{S}_1} on $K_F \cong K_{F/\tilde{S}_1}$ (cf. Section 2.2). Since $K_{F_t} \cong K_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}}$ is trivial for general $t \in \tilde{S}_1$, then by (2.2) and by the Riemann extension, the local weight of h_{F/\tilde{S}_1} is a constant psh function, hence $(K_F, h_{F/\tilde{S}_1})$ is an Hermitian flat line bundle. Consequently we have $\kappa(F) \leq 0$ by [71, Example 5.4.3, p. 72], hence $\kappa(F) = 0$. By the easy inequality [71, Theorem 5.11, pp. 59–60] we have

$$\kappa(V) \leq \kappa(F) + \dim W_1 = \dim W_1 < \dim W = \kappa(V),$$

which is absurd. Therefore we must have $\kappa(W) = \dim W = \kappa(V)$.

Proof of Theorem C. — Take a Stein factorization of the Albanese map of X : $f : X \rightarrow Y$ is an analytic fibre space and $p : Y \rightarrow T := \text{Alb}_X$ is a finite morphism. Then by Proposition 4.1, one can find a subtorus S of T and a projective variety Z of general type which admits a finite morphism $q : Z \rightarrow T/S$ such that there is an Kähler fibre space $\phi_p : Y \rightarrow Z$ whose

general fibre \tilde{S} is a complex torus, which is a finite étale cover over S .

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{\phi_p} & Z \\
 & \searrow \text{alb}_X & \downarrow p & \curvearrowright u & \downarrow q \\
 & & T = \text{Alb}_X & \xrightarrow{\text{quotient}} & T/S.
 \end{array}$$

Since Z is of general type, apply Theorem 3.2 as well as the easy inequality [71, Theorem 5.11, pp. 59–60] to the Kähler fibre space $f \circ \phi_p : X \rightarrow Z$ and we get:

$$0 = \kappa(X, K_X + \dots) = \kappa(X_z, K_{X_z} + \dots) + \dim Z > \dim Z,$$

where $z \in Z$ is a general point and $X_z := f^{-1}(z)$. Hence Z must be a singleton. In consequence $Y = \tilde{S}$ is a complex torus. By the universal property of the Albanese map, we obtain a unique morphism $u : T \rightarrow Y$ of complex tori, such that $u \circ \text{alb}_X = f$ (up to change the base point of alb_X); in particular, the fibres of alb_X are connected, hence alb_X is also an analytic fibre space, hence a Kähler fibre space, thus proves Theorem C. Let us remark that alb_X being an analytic fibre space, then so is p (all its fibres are connected); p is thus a fortiori an isomorphism by Zariski’s Main Theorem (cf. [71, Theorem 1.11, pp. 9–10]).

5. Pluricanonical Version of the Structure Theorem for Cohomology Jumping Loci

In this section we will prove Theorem D by combining Lemma 1.1 and the main result in [77]. First let us recall some notions: let V be a complex manifold, and let F be a coherent sheaf on V , for every $k > 0$ denote

$$V_k^i(F) := \{ \rho \in \text{Pic}^0(V) \mid h^i(V, F \otimes \rho) > k \},$$

the “ k -th jumping locus of the i -th cohomology”. With the help of the Poincaré line bundle on $V \times \text{Pic}^0(V)$, one can express this as the locus where a certain coherent sheaf (in fact, some higher direct image sheaf) of $\text{Pic}^0(V)$ has rank $> k$, hence $V_k^i(F)$ is a closed analytic subspace of $\text{Pic}^0(V)$. The study of the cohomology jumping loci is initiated by the works of Green–Lazarsfeld [41, 42] where they treat the case $F = \mathcal{O}_V$. When $F = \mathcal{O}_V(p)$ for V a smooth projective variety (resp. a compact Kähler manifold) these cohomology jumping loci are described by the result of Simpson [70] (resp. of

Wang [77]). Now let everything as in Theorem D, then as mentioned above, the case $g = \text{id}_X$, $m = 1$ and $\nu = 0$ has been proved in [77]; and in the sequel we will follow the ideas in [20, 45] to deduce Theorem D from this special case. First let us reduce to the proof of Theorem D to a “key lemma”.

Reduction to Lemma 5.1. — The idea of the proof is similar to that of [45, Theorem 10.1]. In fact, when $\nu = 0$, Theorem D is nothing other than the Kähler version of [45, Theorem 10.1]; moreover, as in [45] the theorem is proved by a Baire category theorem argument combined with the following “key lemma”:

LEMMA 5.1 (Key Lemma). — *Every irreducible component of $V_k^0(g(K_X^m \otimes \mathcal{O}_X(m\nu)))$ is a union of torsion translates of subtori in $\text{Pic}^0(Y)$.*

Assuming that Lemma 5.1 is true, let us prove Theorem D. Since $\text{Pic}^0(Y)$ is compact, the jumping locus

$$V_k^0(g(K_X^m \otimes \mathcal{O}_X(m\nu))),$$

as a closed analytic subspace of $\text{Pic}^0(Y)$, has only finite many irreducible components, thus it suffices to prove that every irreducible component of

$$V_k^0(g(K_X^m \otimes \mathcal{O}_X(m\nu)))$$

is a torsion translate of a subtorus. Let Z be a irreducible component of

$$V_k^0(g(K_X^m \otimes \mathcal{O}_X(m\nu))).$$

By Lemma 5.1, Z is a union of torsion translates of subtori. Then by the following Lemma 5.2, Theorem D is proved.

LEMMA 5.2. — *Let Z be a analytic subvariety of a complex torus T . Suppose that Z is a union of torsion translates of subtori of T . Then Z itself is a torsion translate of a subtorus of T .*

Proof. — Since T has only countably many subtori (cf. [7, Ch.1, Exercise (1-b), p. 20]) and countably many torsion points, hence the set of torsion translates of subtori is countable, then by hypothesis we can write $Z = \bigcup_n E_n$ with each E_n being a torsion translate of a subtorus of T . By the Baire category theorem (Z is (locally) compact, hence it is a Baire space: every countable union of closed subsets of empty interior is of empty interior), there is one E_n , say E_1 , which dominates Z , a fortiori $Z = E_1$.

The following two subsections will be dedicated to the proof of the “key lemma”.

Remark 5.3. — Remark that in order to prove Lemma 5.1 it suffices to show that every point of

$$V_k^0(g(K_X^m \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)))$$

is in a torsion translate of a subtorus contained in $V_k^0(g(K_X^m \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)))$. In fact, assume this to be true, and let Z be an irreducible component of

$$V_k^0(g(K_X^m \otimes_{\mathcal{O}_X} \mathcal{O}_X(m))),$$

with Z_0 be the dense (analytic Zariski) open subset of Z complementary to the other irreducible components of

$$V_k^0(g(K_X^m \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)));$$

then Z_0 is contained in a union of torsion translates of subtori: $Z_0 = \bigcup_{\lambda} E_{\lambda}$, with each $E_{\lambda} \subset Z$ being a torsion translate of a subtorus. Hence $Z = \bigcup_{\lambda} E_{\lambda}$ by the density of Z_0 . By Lemma 5.2 we get Lemma 5.1.

5.1. Result of Wang and Reduction to the Case $g = \text{id}$

In this subsection we consider the case where $m = 1$ and $i = 0$, this is also the case considered by Simpson and Wang. In particular, if $g = \text{id}$, Theorem D is proved by Botong Wang in [77]; effectively, he proves the more general:

PROPOSITION 5.4 ([77, Corollary 1.4]). — *Let V a compact Kähler manifold, then each $V_k^i(\frac{p}{V})$ is a finite union of torsion translates of subtori in $\text{Pic}^0(V)$.*

In the sequel we concentrate on the case $i = 0$, as in Theorem D. For every integer $k > 0$ and for every coherent sheaf F on X , by the projection formula we have:

$$\begin{aligned} V_k^0(g(F)) &= \{ \rho \in \text{Pic}^0(Y) \mid h^0(Y, g(F) \otimes \rho) > k \} \\ &= \{ \rho \in \text{Pic}^0(Y) \mid h^0(X, F \otimes g(\rho)) > k \} \\ &= (g^{-1})^{-1}(V_k^0(F) \otimes \text{Im } g) \end{aligned} \tag{5.1}$$

where $g : \text{Pic}^0(Y) \rightarrow \text{Pic}^0(X)$ is the morphism of complex tori given by $L \mapsto g(L)$. Then the following lemma permit us to reduce to the case $g = \text{id}$:

LEMMA 5.5. — *Let $\alpha : T_1 \rightarrow T_2$ a morphism of complex tori. Let $t \in T_2$ a torsion point and $S \subset T_2$ a subtorus. Then $\alpha^{-1}(t + S)$ is also a torsion translate of a subtorus in T_1 .*

Proof. — By [27, §1.2, Théorème 2.3, p. 7] α can be factorized as

$$T_1 \xrightarrow{\text{quotient}} T_1/(\text{Ker } \alpha)^0 \xrightarrow[\text{isogeny}]{\bar{\alpha}} T_1/\text{Ker } \alpha = \text{Im } \alpha \xleftarrow{\text{inclusion}} T_2.$$

Thus it suffices to prove the lemma in the following three cases:

- α is the quotient by a subtorus,
- α is an isogeny,
- α is the inclusion of a subtorus.

Each of these cases can be done by elementary linear algebra.

In particular we obtain immediately:

PROPOSITION 5.6. — *Let $g : X \rightarrow Y$ a morphism between compact Kähler manifolds. Then for every $k > 0$ and for every $0 \leq p \leq n$, $V_k^0(g : \mathcal{O}_X^p)$ is a finite union of torsion translates of subtori in $\text{Pic}^0(Y)$.*

5.2. Proof of the “Key Lemma”

Not turn to the demonstration of Lemma 5.1. It proceeds in four steps:

(A). Reduction to the case $g = \text{id}$. First apply the formula (5.1) to $F = K_X^m \otimes \mathcal{O}_X(m)$ and then by Lemma 5.5 we see that Lemma 5.1 is true for $V_k^0(g : (K_X^m \otimes \mathcal{O}_X(m)))$ as soon as it holds for $V_k^0(K_X^m \otimes \mathcal{O}_X(m))$. In consequence we can suppose that $g = \text{id}$ (and $X = Y$).

(B). Case $m = 1$ and $n = 0$. This is nothing other than Proposition 5.6 for $p = n$.

(C). Case $m = 1$ and $n > 0$ is of SNC support. In this step, we consider the case where $m = 1$ and D is an effective \mathbb{Q} -divisor of SNC support; in addition, we do not require D to be an integral divisor, but only assume that it is given by a line bundle L^+ , i.e. there is a line bundle L^+ , $(L^+)^{\otimes N} \otimes \mathcal{O}_X(N)$ for any $N \in \mathbb{Z}_{>0}$ which makes N an integral divisor. In this case, Lemma 5.1 can be deduced from Lemma 1.1 combined with the following auxiliary result (cf. also [77, Lemma 3.1]):

LEMMA 5.7 (analytic version of [45, Lemma 10.3]). — *Let F and G be coherent sheaves on X such that F is a direct summand of G . Then for $i \in \mathbb{N}$ and $k \in \mathbb{Z}_{>0}$, each irreducible component of $V_k^i(F)$ is also an irreducible component of $V_l^i(G)$ for some $l > k$.*

Proof. — This is simply a result of Grauert’s semi-continuity theorem (cf. [2, §III.4, Theorem 4.12(i), p. 134])

Now let L^+ be the line bundle given by (X, σ) . Since (X, σ) is a klt pair, then $\sigma = 0$. Moreover, σ being a \mathbb{Q} -divisor of SNC support, then for any N making $N\sigma$ an integral divisor, we can construct, by Lemma 1.1, a generically finite morphism $f : V \rightarrow X$ of compact Kähler manifolds such that

$$f^* K_V \cong \bigoplus_{i=0}^{N-1} K_X \otimes (L^+)^{-i} \otimes \mathcal{O}_X(-i\sigma).$$

By Lemma 5.7 each irreducible component of $V_k^0(K_X \otimes L^+)$ is also a irreducible component of a certain $V_l^0(f^* K_V)$ for some $l > 0$. Then by Step (B) (or Proposition 5.6), every irreducible component of $V_k^0(K_X \otimes L^+)$ is a torsion translate of a subtorus in $\text{Pic}^0(X)$.

(D). General case. In order to prove the general case we use a reduction to the case of Step (C). This reduction process is inspired by [20, §1.A-1.C], whose idea has already appeared in [14]. Let L be a point in

$$V_k^0(K_X^{\otimes m} \otimes \mathcal{O}_X(m\sigma)) \subset \text{Pic}^0(X),$$

we will prove in the sequel that there exists a torsion translate of a subtorus contained in

$$V_k^0(K_X^{\otimes m} \otimes \mathcal{O}_X(m\sigma))$$

which contains L . $\text{Pic}^0(X)$ being complex torus, thus divisible, then we can write $L = mL_0 = L_0^{\otimes m}$ for some $L_0 \in \text{Pic}^0(Y)$. Then we have $h^0(X, L_{m, \sigma}) > k$, where

$$L_{m, \sigma} := K_X^{\otimes m} \otimes \mathcal{O}_X(m\sigma) \otimes L_0^{\otimes m}.$$

Now take a log resolution $\mu : X \rightarrow X$ for both (X, σ) and the linear series $|L_{m, \sigma}|$. Then we can write

$$K_X^{\otimes m} \otimes \mathcal{O}_X(m\sigma) \cong \mu^*(K_X^{\otimes m} \otimes \mathcal{O}_X(m\sigma)) \otimes \mathcal{O}_X\left(\sum_{i \in I^+} ma_i E_i\right), \quad (5.2)$$

$$\mu^* |L_{m, \sigma}| = |\mu^* L_{m, \sigma}| = F_{m, \sigma} + |M_{m, \sigma}|,$$

where:

- $\{E_i \mid i \in I\}$ denotes the set of μ -exceptional prime divisors, and

$$a_i := a(E_i, X, \sigma)$$

denotes the discrepancy of E_i with respect to the pair (X, σ) ; I^+ (resp. I^-) is the set of indices i such that $a_i > 0$ (resp. $a_i < 0$).

- is an effective \mathbb{Q} -divisor on X as in the proof of Lemma 1.5, i.e.

$$:= \mu^{-1} - \sum_{i \in I^-} a_i E_i,$$

by Lemma 1.5 the pair $(X,)$ is also klt.

- $F_{m,}$ (resp. $M_{m,}$) is the fixed part (resp. mobile part) of the linear series $\mu |L_{m,}|$; by construction, $|M_{m,}|$ is base point free.

By construction (μ being a log resolution of and of $|L_{m,}|$), $m + \sum_{i \in I} E_i$ and $F_{m,} + \sum_{i \in I} E_i$ are (integral) divisors of SNC support. Let H be a general member in $|M_{m,}|$, then H has no common component either with $F_{m,}$ or with $\sum_{i \in I} E_i$ or with ; by Bertini's theorem, H is smooth (in particular H is reduced), $H + F_{m,} + \sum_{i \in I} E_i$ is of SNC support. Set

$$F_{m,} := F_{m,} + \sum_{i \in I^+} m a_i E_i,$$

$$L_{m,} := K_X^m \otimes_{\mathcal{O}_X} (m) \otimes_{\mu} L_0^m.$$

Then we have

$$L_{m,} \otimes_{\mu} L_{m,} \otimes_{\mathcal{O}_X} \left(\sum_{i \in I^+} m a_i E_i \right),$$

thus

$$|L_{m,}| = |M_{m,}| + F_{m,}.$$

By [28, Lemma 7.11, p. 175] we have

$$H^0(X, L_{m,}) = H^0(X, L_{m,}) + H^0(X, \mathcal{O}_X(M_{m,})),$$

hence $F_{m,}$ is equal to the fixed part of the linear series $|L_{m,}|$ and by construction it is of SNC support.

Put

$$\mu^{-1} := \sum_j d_j D_j, \quad d_j \in \mathbb{Q}_{>0},$$

$$b_j := \text{coefficient of } D_j \text{ in } F_{m,}, \quad j \in J,$$

$$b_i := \text{coefficient of } E_i \text{ in } F_{m,}, \quad i \in I^-,$$

and take

$$- := - \sum_j \min\left(d_j, \frac{b_j}{m}\right) D_j - \sum_{i \in I^-} \min\left(-a_i, \frac{b_i}{m}\right) E_i,$$

$$\bar{F}_{m,} := F_{m,} - \sum_j \min(m d_j, b_j) D_j - \sum_{i \in I^-} \min(-m a_i, b_i) E_i,$$

so that \bar{F}_m and F_m have no common component. We see that $\bar{F}_m \not\subset F_m$, $\bar{F}_m \not\subset F_m$. Now consider the line bundle

$$\bar{L}_m := K_X^m \otimes_{\mathcal{O}_X} (m^{-}) \otimes \mu L_0^m,$$

then the same argument as above shows that \bar{F}_m equals to the fixed part of the linear series $|\bar{L}_m|$, hence we have

$$|\bar{L}_m| = \bar{F}_m + |M_m|.$$

In addition we have

$$\begin{aligned} \bar{L}_m &= \mathcal{O}_X \left(- \left[\frac{m-1}{m} \bar{F}_m \right] \right) \\ &= K_X^m \otimes_{\mathcal{O}_X} (m^{-}) \otimes \mu L_0^m \otimes_{\mathcal{O}_X} \left(- \left[\frac{m-1}{m} (\bar{F}_m + H) \right] \right) \\ &= K_X \otimes_{\mathcal{O}_X} (+) \otimes \mu L_0 \end{aligned}$$

where the \mathbb{Q} -divisor

$$+ := - + \left\{ \frac{m-1}{m} (\bar{F}_m + H) \right\}.$$

Since H has no common component with either $-$ or \bar{F}_m , hence

$$+ = - + \left\{ \frac{m-1}{m} \bar{F}_m \right\} + \frac{m-1}{m} H;$$

but H is reduced, $-$ and \bar{F}_m have no common components, then the coefficients of the irreducible components in $+$ are all < 1 ; since $+$ is of SNC support, then [57, Corollary 2.31 (3), p. 53] implies that the pair $(X, +)$ is klt. A priori $\mathcal{O}_X(+)$ is only a \mathbb{Q} -line bundle, but by our construction $\mathcal{O}_X(+)$ is given by a line bundle

$$L^+ := \mathcal{O}_X(+)= \bar{L}_m \otimes_{\mathcal{O}_X} \left(- \left[\frac{m-1}{m} \bar{F}_m \right] \right) \otimes K_X^{-1} \otimes \mu L_0^{-1}.$$

Moreover, we have

$$\begin{aligned} h^0(X, K_X \otimes L^+ \otimes \mu L_0) &= h^0 \left(X, \bar{L}_m \otimes_{\mathcal{O}_X} \left(- \left[\frac{m-1}{m} \bar{F}_m \right] \right) \right) \\ &> h^0(X, M_m) > k, \end{aligned}$$

which means that $\mu L_0 \otimes V_k^0(K_X \otimes L^+)$. Let W an irreducible component $V_k^0(K_X \otimes L^+)$ containing μL_0 . By Step (C) W is a torsion translate of subtorus, then we can write $W = \beta_{\text{tor}} + T_0$ with β_{tor} a torsion point in $\text{Pic}^0(X)$ and T_0 a subtorus, in particular μL_0 can be written as the sum of β_{tor} and an element of T_0 , thus

$$(m-1)\mu L_0 + W = m\beta_{\text{tor}} + T_0$$

is also a torsion translate of a subtorus as $m\beta_{\text{tor}}$ is also a torsion point of $\text{Pic}^0(X)$. In addition, $(m-1)\mu L_0 + W$ contains $\mu L = m\mu L_0$ as $\mu L_0 \sim W$. It remains to see that $(m-1)\mu L_0 + W$ is contained in $V_k^0(K_X^m \otimes \mathcal{O}_X(m))$. In fact, for every $\alpha \in W$, we have (since $W \subset V_k^0(K_X \otimes L^+)$):

$$\begin{aligned} h^0(X, K_X^m \otimes \mathcal{O}_X(m)) &\subset \mu L_0^{(m-1)} \otimes \alpha \\ &> h^0(X, K_X^m \otimes \mathcal{O}_X(m^-) \otimes \mu L_0^{(m-1)} \otimes \alpha) \\ &= h^0(X, \bar{L}_m, \mu L_0^{-1} \otimes \alpha) \\ &> h^0\left(X, \bar{L}_m, \mathcal{O}_X\left(-\left\lfloor \frac{m-1}{m} \bar{F}_m \right\rfloor\right) \otimes \mu L_0^{-1} \otimes \alpha\right) \\ &= h^0(X, K_X \otimes L^+ \otimes \alpha) > k. \end{aligned}$$

Therefore $(m-1)\mu L_0 + W \subset V_k^0(K_X^m \otimes \mathcal{O}_X(m))$.

In virtue of the isomorphism (5.2) we have

$$\begin{aligned} V_k^0(K_X^m \otimes \mathcal{O}_X(m)) & \\ &= \{\rho \in \text{Pic}^0(X) \mid h^0(X, K_X^m \otimes \mathcal{O}_X(m) \otimes \rho) > k\} \\ &= \{\rho \in \text{Pic}^0(X) \mid h^0(X, \mu(K_X^m \otimes \mathcal{O}_X(m) \otimes \rho)) > k\} \\ &= \{\rho \in \text{Pic}^0(X) \mid h^0(X, K_X^m \otimes \mathcal{O}_X(m) \otimes \mu \rho) > k\} \\ &= (\mu)^{-1}(V_k^0(K_X^m \otimes \mathcal{O}_X(m)) \cap \text{Im } \mu), \end{aligned}$$

where the third equality is a consequence of [28, Lemma 7.11, p. 175]. Hence by Lemma 5.5,

$$W := (\mu)^{-1}(((m-1)\mu L_0 + W) \cap \text{Im } \mu)$$

is a torsion translate of a subtorus in $V_k^0(K_X^m \otimes \mathcal{O}_X(m))$ and $L = mL_0 \sim W$. This proves Lemma 5.1.

Remark 5.8. — If X is a smooth projective variety, then one can prove Theorem D for log canonical pair (X, \mathbb{Q}) as follows:

- First apply [15, Theorem 1.1] along with [76, Théorème 8.35(ii), p. 201] to prove Lemma 5.1 (thus also Theorem D) for $m = 1$ and a reduced SNC divisor (cf. also [52]);
- Then by [20, Lemma 2.1] and Lemma 1.1 one can deduce further Lemma 5.1 for the case of $m = 1$ and a log canonical \mathbb{Q} -divisor of SNC support, which is given by a line bundle, but is not necessarily an integral divisor;

- Finally one can follow the same argument as in Step (D) above to prove Lemma 5.1 and thus Theorem D.

As for the Kähler case, as soon as [16, Conjecture 1.2] is solved, one can prove Theorem D for log canonical pair (X, \mathbb{Q}) .

5.3. Kähler version of a result of Campana–Koziarz–Păun

Before ending this section, let us prove the following significant corollary of Theorem D, which generalizes a result of Campana, Koziarz and Păun to the Kähler case, and will be used in the proof of Main Theorem A (II). In the algebraic case, it is proved in [21, Theorem 3.1] for $\mathbb{Q} = 0$, and in [20, Theorem 0.1] for \mathbb{Q} log canonical.

COROLLARY 5.9. — *Let (X, \mathbb{Q}) a klt pair with X a Kähler manifold, and let L_0 a numerically trivial line bundle on X , i.e. $L_0 \in \text{Pic}^0(X)$. Then*

- (a) $\kappa(X, K_X + \mathbb{Q}) > \kappa(X, mK_X + m\mathbb{Q} + L_0)$, $m \in \mathbb{Z}_{>0}$. Namely, for any \mathbb{Q} -line bundle⁽¹⁾ L on X such that $c_1(L) = c_1(K_X + \mathbb{Q})$, we have $\kappa(X, K_X + \mathbb{Q}) > \kappa(X, L)$.
- (b) If there is an integer $m > 0$ such that $\kappa(X, K_X + \mathbb{Q}) = \kappa(X, mK_X + m\mathbb{Q} + L_0) = 0$, then L_0 is a torsion point in $\text{Pic}^0(X)$.

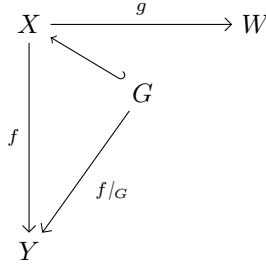
Remark 5.10. — Before entering into the proof, let us remark that one cannot omit the condition “ $\kappa(X, K_X + \mathbb{Q}) = 0$ ” in the point (b) above. For example, if (X, \mathbb{Q}) is of log general type, then for any $L_0 \in \text{Pic}^0(X)$ we always have $\kappa(X, K_X + \mathbb{Q}) = \kappa(X, mK_X + m\mathbb{Q} + L_0) = \dim X$.

Proof of Corollary 5.9. — We will follow the argument in [21] with a little simplification. First prove the point (a), the proof proceeds in three steps:

Step 1. — Reduction to the case $\kappa(X, K_X + \mathbb{Q}_X(\mathbb{Q})) \leq 0$. Assuming (a) for any klt pair (X, \mathbb{Q}) with $\kappa(X, K_X + \mathbb{Q}) \leq 0$, we will prove it for any klt pair (X, \mathbb{Q}) with $\kappa(X, K_X + \mathbb{Q}) > 0$. Let $g : X \dashrightarrow W$ the Iitaka fibration (cf. [71, §5, Theorem 5.10, p. 58]) of the \mathbb{Q} -line bundle $K_X + \mathbb{Q}$ and $f : X \dashrightarrow Y$ that of $mK_X + m\mathbb{Q} + L_0$. By Lemma 1.5 the point (a) is preserved by log resolutions of (X, \mathbb{Q}) , we can thus suppose that f and g are

⁽¹⁾ In fact, since $\text{Pic}^0(X)$ is divisible, this a priori \mathbb{Q} -line bundle L is an “authentic” line bundle.

morphisms (instead of meromorphic mappings).



By construction we have $\dim Y = \kappa(X, mK_X + m + L_0)$, $\dim W = \kappa(X, K_X +)$. Denote by F (resp. by G) the general fibre of f (resp. of g), then

$$\begin{aligned}
 \kappa(X, K_X +) &> \kappa(X, mK_X + m + L_0) && \dim W > \dim Y \\
 &&& \dim G \leq \dim F,
 \end{aligned}$$

then it suffices to prove that G is contracted by f (i.e. $f(G) = \text{pt}$). By adjunction formula the \mathbb{Q} -line bundle

$$K_G + G - (K_X +)/G$$

where $G := f/G$, hence f/G is bimeromorphically equivalent to a meromorphic mapping defined by a sub-linear series of $|K_G^{km} \otimes_{\mathbb{O}_G}(km) - L_0/G^k|$ for some k sufficiently large and divisible.⁽²⁾ Therefore it suffices to show

$$\kappa(G, mK_G + m + L_0/G) = 0.$$

But by our construction

$$\kappa(G, K_G + G) = \kappa(G, (K_X +)/G) = 0,$$

hence our assumption implies that (a) holds for the klt pair (G, G) . Since $L_0/G \in \text{Pic}^0(G)$ we have

$$\kappa(G, mK_G + m + L_0/G) \leq \kappa(G, K_G + G) = 0.$$

Step 2. — By the precedent step, we can assume that $\kappa(X, K_X +) \leq 0$. If $\kappa(X, mK_X + m + L_0) = -$, then the inequality is automatically established, hence we can assume that $\kappa(X, mK_X + m + L_0) > 0$; in addition, up to replacing m and L_0 by a multiple, we can assume that m is an integral divisor and

$$H^0(X, K_X^m \otimes_{\mathbb{O}_X}(m) - L_0) = 0.$$

For every integer $k > 0$ denote

$$r_k := h^0(X, K_X^{km} \otimes_{\mathbb{O}_X}(km) - L_0^k) > 0.$$

⁽²⁾ In the proof of [21, Theorem 3.1], it is said that f/G is equal to the Iitaka fibration of $mK_G + m + L_0/G$; but it is not true in general.

Then $L_0^k \in V_{r_k}^0(K_X^{km} \otimes \mathcal{O}_X(km)) \setminus V_1^0(K_X^{km} \otimes \mathcal{O}_X(km))$, thus by Theorem D, $L_0^k \in \beta_{\text{tor}} + T_0 \in V_{r_k}^0(K_X^{km} \otimes \mathcal{O}_X(km))$ for β_{tor} a torsion point in $\text{Pic}^0(X)$ and T_0 a subtorus; in particular, $\beta_{\text{tor}} \in V_{r_k}^0(K_X^{km} \otimes \mathcal{O}_X(km))$. Let $m_0 > 0$ an integer such that $\beta_{\text{tor}}^{m_0} \in \mathcal{O}_X$. Then

$$\begin{aligned} h^0(X, K_X^{kmm_0} \otimes \mathcal{O}_X(kmm_0)) & \\ & > h^0(X, K_X^{km} \otimes \mathcal{O}_X(km) \otimes \beta_{\text{tor}}) > r_k. \end{aligned} \quad (5.3)$$

Step 3. — By hypothesis we have $\kappa(X, K_X + \mathcal{O}_X) \leq 0$, hence (5.3) implies that $\kappa(X, K_X + \mathcal{O}_X) = 0$, which means that $r_k \leq 1$ for every $k \in \mathbb{Z}_{>0}$. Therefore $\kappa(X, mK_X + m\mathcal{O}_X + L_0) = 0$. This proves (a).

Now turn to the proof of (b): assume by contradiction that there is a line bundle $L \in \text{Pic}^0(X)$ with L non-torsion such that $\kappa(X, mK_X + m\mathcal{O}_X + L) = \kappa(X, K_X + \mathcal{O}_X) = 0$ for some $m > 0$. Up to replacing m and L by a multiple, we can assume that m is an integral divisor and that $h^0(X, K_X^m \otimes \mathcal{O}_X(m) \otimes L) = 1$, then $L \in V_1^0(K_X^m \otimes \mathcal{O}_X(m))$. By Theorem D there exists $\beta_{\text{tor}} \in \text{Pic}^0(X)_{\text{tor}}$ and T_0 a subtorus in $\text{Pic}^0(X)$ such that $L \in \beta_{\text{tor}} + T_0 \in V_1^0(K_X^m \otimes \mathcal{O}_X(m))$, then we can write $L = \beta_{\text{tor}} \otimes F$ with $F \in T_0$. By our assumption L is not a torsion point in $\text{Pic}^0(X)$, hence F cannot be trivial and thus T_0 is not reduced to a singleton. In consequence there is a (non-trivial) one-parameter subgroup $(F_t)_{t \in \mathbb{R}}$ in T_0 passing through F (by choosing an isomorphism $T_0 \cong \mathbb{C}^q / \Lambda$, we can take $F_t = t \cdot F$), then for every $t \in \mathbb{R}$, $\beta_{\text{tor}} \otimes F_t \in \beta_{\text{tor}} + T_0 \in V_1^0(K_X^m \otimes \mathcal{O}_X(m))$ hence there is a non-zero section s_t in

$$H^0(X, K_X^m \otimes \mathcal{O}_X(m) \otimes \beta_{\text{tor}} \otimes F_t).$$

We claim that:

CLAIM (*). — *There is a $t \in \mathbb{R}_{>0}$ such that the sections s_t, s_{-t} and s_0^2 are not linearly independent in $H^0(X, K_X^{2m} \otimes \mathcal{O}_X(2m) \otimes \beta_{\text{tor}}^2)$.*

In fact, this leads to a contradiction: we have immediately

$$h^0(X, K_X^{2m} \otimes \mathcal{O}_X(2m) \otimes \beta_{\text{tor}}^2) > 2,$$

which implies that

$$\kappa(X, K_X + \mathcal{O}_X) = \kappa(X, K_X^{2m} \otimes \mathcal{O}_X(2m) \otimes \beta_{\text{tor}}^2) > 1,$$

and this contradicts the hypothesis that $\kappa(X, K_X + \mathcal{O}_X) = 0$. Therefore (b) is proved.

Let us prove Claim (*). Assume by contradiction that s_t, s_{-t} and s_0^2 are linearly dependent for every $t \in \mathbb{R}$. Then $t \in \mathbb{R}$, $\text{div}(s_t) + \text{div}(s_{-t}) = 2 \text{div}(s_0)$; in particular, $\text{div}(s_t) \leq 2 \text{div}(s_0)$ for every $t \in \mathbb{R}_{>0}$. Take ϵ sufficiently small such that $t \in F_t$ is injective for $t \in]-\epsilon, \epsilon[$. By Dirichlet's

drawer principle, there are $t_1, t_2 \in]0, \epsilon[$ such that $\text{div}(s_{t_1}) = \text{div}(s_{t_2})$, hence the divisor

$$0 = \text{div}(s_{t_2}) - \text{div}(s_{t_1}) \in |F_{t_2} - F_{t_1}^{-1}|,$$

which implies that $F_{t_1} = F_{t_2}$ in $\text{Pic}^0(X)$ with $t_1, t_2 \in]0, \epsilon[$; but this contradicts our hypothesis on ϵ . This proves Claim (*).

As a by-product of Corollary 5.9(a) we obtain the following special case of the Kähler version of the (generalized) log Abundance Conjecture by using the divisorial Zariski decomposition (cf. [9, Definition 3.7]):

THEOREM 5.11. — *Let (X, ω) be a klt pair with X a compact Kähler manifold whose numerical dimension $\nu(X, K_X + \omega) = 0$, then $\kappa(X, K_X + \omega) = 0$.*

Proof. — For the definition of the numerical dimension of (non necessarily nef) \mathbb{Q} -line bundles (or cohomology classes in $H^{1,1}(X, \mathbb{R})$) over a compact Kähler manifold, cf. [29, §18.13, p. 198]. Since $\nu(K_X + \omega) = 0$, the \mathbb{Q} -line bundle $K_X + \omega$ is pseudoeffective, hence we can consider the divisorial Zariski decomposition (cf. [9, Definition 3.7] and [29, §18.12(d), p. 195]) of its first Chern class:

$$c_1(K_X + \omega) = \{N(c_1(K_X + \omega))\} + c_1(K_X + \omega).$$

By hypothesis $\nu(c_1(K_X + \omega)) = 0$, which means that $c_1(K_X + \omega) = 0$; in other word, the \mathbb{Q} -line bundle $K_X + \omega$ is numerically equivalent to the effective \mathbb{R} -divisor $N = N(c_1(K_X + \omega))$, a fortiori N is an \mathbb{Q} -divisor. Therefore by Corollary 5.9(a), we have

$$\kappa(K_X + \omega) > \kappa(N) > 0.$$

Finally by [29, §18.15, p. 199] we get $\kappa(K_X + \omega) = 0$.

6. Kähler Version of $C_{n,m}^{\text{log}}$ for Fibre Spaces over Complex Tori

In this section, we will prove our Main Theorem A. To this end, we do some reductions by an induction on the dimension of T and by applying Theorem 2.6, Theorem 3.6 and Theorem C; at last, we deduce Main Theorem A from Corollary 5.9.

6.1. Reduction to the case T is a simple torus

By an induction on $\dim T$ we can assume that T is a simple torus, i.e. admitting no non-trivial subtori. In fact, if T is not simple, take a non-trivial

subtorus $S \rightarrow T$ and denote by $q : T \rightarrow T/S$ the canonical morphism (of complex analytic Lie groups), this is a Kähler fibre space (more precisely a principle S -bundle). We obtain thus a Kähler fibre space $f = q \circ f : X \rightarrow T/S$, and then by induction hypothesis we have

$$\kappa(X, K_X + \dots) > \kappa(F, K_F + \dots),$$

where $F := f^{-1}(\cdot)$ with F the general fibre f . In addition, $f|_F : F \rightarrow S$ is also a Kähler fibre space of general fibre F over a complex torus S of dimension $< \dim T$, hence by induction hypothesis we have

$$\kappa(F, K_F + \dots) > \kappa(F, K_F + \dots),$$

thus we get

$$\kappa(X, K_X + \dots) > \kappa(F, K_F + \dots).$$

6.2. Dichotomy according to the Determinant Bundle and Reduction to the Case of Numerical Flat Direct Images

For positive integer m such that m is an integral divisor, consider the direct image

$$F_{m, \cdot} := f_* (K_X^m \otimes \mathcal{O}_X(m \cdot)) = f_* (K_{X/T}^m \otimes \mathcal{O}_X(m \cdot)).$$

If $\kappa(F, K_F + \dots) = -$ then Part (II) of Main Theorem A is automatically established; hence we can assume that $\kappa(F, K_F + \dots) > 0$. In consequence for m sufficiently divisible $F_{m, \cdot} = 0$. Let us denote by \mathcal{M} the set of positive integers m such that m is an integral divisor and that $F_{m, \cdot} = 0$, then we can suppose that $\mathcal{M} = \mathbb{Z}$, this is moreover an additive subset of \mathbb{Z} . By Theorem B, for $m \in \mathcal{M}$ the torsion free sheaf $F_{m, \cdot}$ admits a semi-positively curved metric $g_{X/T}^{(m)}$; in addition, the induced metric $\det g_{X/T}^{(m)}$ on its determinant bundle $\det F_{m, \cdot}$ is of curvature current

$$\theta_{m, \cdot} := \det g_{X/T}^{(m)} (\det F_{m, \cdot}) > 0.$$

In particular, the line bundle $\det F_{m, \cdot}$ is pseudoeffective on T for every $m \in \mathcal{M}$. By Section 6.1 we can assume that T is a simple torus, hence [22, Proposition 2.2] (cf. also [25, Theorem 3.3]) implies that we fall into the following two cases:

- Either $\theta_{m, \cdot} = 0$, in this case T is an Abelian variety equipped with $\det F_{m, \cdot}$ an ample line bundle;
- Or $\theta_{m, \cdot} = 0$, in this case $\det F_{m, \cdot}$ is a numerically trivial line bundle, and thus Corollary 2.13 implies that $(F_{m, \cdot}, g_{X/T}^{(m)})$ is a Hermitian flat vector bundle.

If there is an integer $m \in \mathcal{M}$ such that the determinant bundle $\det F_m$, is ample, then Main Theorem A (II) can be deduced by Main Theorem A (I) (which is proved in Section 3.2, cf. Theorem 3.6). Hence in order to finish the proof of Main Theorem A (II), one only need to tackle the case that the determinant bundle $\det F_m$, is numerically trivial for every $m \in \mathcal{M}$, which implies that $(F_m, g_{X/T}^{(m)})$ is a Hermitian flat vector bundle for every $m \in \mathcal{M}$.

6.3. Reduction to the case $\kappa \leq 0$

In this subsection we will demonstrate that we can reduce to the case $\kappa(X, K_X + \Delta) \leq 0$, which is an observation dating back to Kawamata, cf. [49, §3, Proof of Claim 2, pp. 256–266]. Suppose that Main Theorem A (II) holds true for klt pair (X, Δ) with $\kappa(X, K_X + \Delta) \leq 0$. Now take a klt pair (X, Δ) such that $\kappa(X, K_X + \Delta) > 1$. By Lemma 1.5, we can freely replace X by a superior bimeromorphic model (the Kodaira dimension remains unchanged), and in consequence we can suppose that the Iitaka fibration of $K_X + \Delta$ is a morphism, denoted by

$$\phi : X \rightarrow Y,$$

whose general fibre is G . Then $\dim Y = \kappa(X, K_X + \Delta) > 0$ and $\kappa(G, K_G + \Delta_G) = 0$ where $\Delta_G := \Delta|_G$. Consider

$$f|_G : G \rightarrow f(G) =: S \rightarrow T,$$

and take a Stein factorization of $f|_G$:

$$\begin{array}{ccc} G & & \\ \downarrow f|_G & \searrow & \\ & & S \\ & \swarrow & \\ & & S. \end{array}$$

Case 1: $S = T$. T being a simple torus, [71, Theorem 10.9, pp. 120–123] implies that S is of general type, then so is S by [71, Lemma 6.3, pp. 66–67]. By Theorem 3.2, for general $s \in S$ we have

$$\begin{aligned} 0 = \kappa(G, K_G + \Delta_G) &= \kappa(G_s, K_{G_s} + \Delta_{G_s}) + \dim S \\ &= \kappa(G_s, K_{G_s} + \Delta_{G_s}) + \dim S, \end{aligned}$$

where $G_s := f/G_s = G/G_s$. This forces $\dim S = \dim S = 0$, hence $f(G) =$ pt, and in consequence G is contained in F . Therefore $\phi|_F : F \rightarrow \phi(F) = Y$ is a Kähler fibre space of general fibre G , and thus by the easy inequality [71, Lemma 5.11, pp. 59–60] we obtain (noting that $G = F/G$):

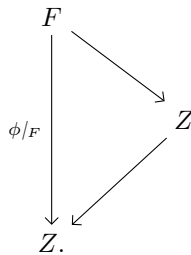
$$\begin{aligned} \kappa(F, K_F + \mathcal{O}_F) \leq \kappa(G, K_G + \mathcal{O}_G) + \dim h(F) &= \dim h(F) \\ &\leq \dim Y = \kappa(X, K_X + \mathcal{O}_X). \end{aligned}$$

Case 2: $S = T$. First we prove that $S \rightarrow T$ is a finite étale cover (thus S is also a complex torus) with the help of Theorem C. In fact, let $\text{alb}_G : G \rightarrow \text{Alb}_G$ the Albanese map of (G, y) with base point y such that $f(y) = e \in T$. By the universal property of the Albanese map we get a (unique) morphism $u : \text{Alb}_G \rightarrow T$ of complex tori (a morphism of complex analytic Lie groups) such that $f/G = u \circ \text{alb}_G$. Then by Theorem C, u factorizes through $S \rightarrow T$, and hence by [27, Théorème 2.3, p. 7] S is a complex torus and $S \rightarrow T$ is a finite morphism.

Put F_t to be the general fibre of $G \rightarrow S$, then for general $t \in T$, we have $G_t = F_t \rightarrow G$ is finite union of copies of F . Now apply our assumption to $G \rightarrow T$ ($\kappa(G, K_G + \mathcal{O}_G) = 0$) and we get

$$0 = \kappa(G, K_G + \mathcal{O}_G) > \kappa(F, K_F + \mathcal{O}_F).$$

where $F := f/F = G/F$. Furthermore, consider a Stein factorization of $\phi|_F : F \rightarrow \phi(F) =: Z \rightarrow Y$:



For $z \in Z$ general $F_z = F \rightarrow G$, hence the general fibre of the analytic fibre space $F \rightarrow Z$ is isomorphic to F . Then by the easy inequality [71, Lemma 5.11, pp. 59–60] we obtain:

$$\begin{aligned} \kappa(F, K_F + \mathcal{O}_F) \leq \kappa(F, K_F + \mathcal{O}_F) + \dim Z &\leq \dim Z = \dim Z \\ &\leq \dim Y = \kappa(X, K_X + \mathcal{O}_X). \end{aligned}$$

6.4. End of the Proof of the Main Theorem

By Section 6.2 we have that $(F_{m, \cdot}, g_{X/T}^{(m)})$ is a Hermitian flat vector bundle for every $m \in \mathcal{M}$. In other words $F_{m, \cdot}$ is constructed by a unitary representation of the fundamental group (cf. for example [55, Proposition 1.4.21, p. 13] or [30, §6, pp. 260–261])

$$\rho_m : \pi_1(T, t_0) \rightarrow \mathrm{U}(r_m)$$

where

$$r_m := \mathrm{rk} F_{m, \cdot} = h^0(F, K_F^m \otimes_{\mathcal{O}_F} (m - F)).$$

Since $\pi_1(T, t_0)$ is an Abelian group, every representation of $\pi_1(T)$ can be decomposed into (irreducible) sub-representations of rank 1, hence a decomposition of $F_{m, \cdot}$ into (numerically trivial) line bundles:

$$F_{m, \cdot} = L_1 \otimes L_2 \otimes \cdots \otimes L_{r_m}, \quad \text{with } L_i \in \mathrm{Pic}^0(T), \quad i = 1, \dots, r_m. \quad (6.1)$$

Step 1. First prove that $\mathrm{Im}(\rho_m)$ is finite for every $m \in \mathcal{M}$. In fact, suppose by contradiction that there exists $m \in \mathcal{M}$ such that $\mathrm{Im}(\rho_m)$ is infinite, hence there exists $j \in \{1, 2, \dots, r_m\}$, say $j = 1$, such that L_j is not a torsion point in $\mathrm{Pic}^0(T)$. Consider the natural inclusion $L_1 \subset F_{m, \cdot}$, which induces a non-zero section

$$H^0(T, F_{m, \cdot} \otimes L_1^{-1}) = H^0(X, K_X^m \otimes_{\mathcal{O}_X} (m - F) \otimes f^* L_1^{-1}).$$

This implies that $\kappa(X, mK_X + m - F + f^* L_1) > 0$. As $f^* L_1 \in \mathrm{Pic}^0(X)$, by Corollary 5.9(a) and Section 6.3 we have

$$\kappa(X, mK_X + m - F + f^* L_1) \leq \kappa(X, K_X + F) \leq 0,$$

hence a fortiori

$$\kappa(X, mK_X + m - F + f^* L_1) = \kappa(X, K_X + F) = 0. \quad (6.2)$$

By Corollary 5.9(b), the equality (6.2) implies that $f^* L_1$ is a torsion point in $\mathrm{Pic}^0(X)$, i.e. there is an $e > 0$ such that $f^* L_1^e \in \mathcal{O}_X$, meaning that $L_1^e \in \mathcal{O}_T$ since the morphism

$$f : \mathrm{Pic}^0(T) \rightarrow \mathrm{Pic}^0(X)$$

is injective (f being an analytic fibre space). This contradicts our supposition that L_1 is not a torsion element in $\mathrm{Pic}^0(T)$. Hence $\mathrm{Im}(\rho_m)$ is finite for each $m \in \mathcal{M}$.

Step 2. By the precedent step we see that $\text{Im}(\rho_m)$ is a finite group. Set $H_m := \text{Ker}(\rho_m)$, then H_m is normal subgroup of $\pi_1(T)$ of finite index. Hence H_m induces a finite étale cover of T . Up to passing to this finite étale cover (the Kodaira dimension is invariant under finite étale covers) we can assume that the representation ρ_m is trivial, and consequently F_m is a trivial vector bundle, then we have

$$h^0(X, K_X^m \otimes_{\mathcal{O}_X}(m)) = h^0(T, F_m) = r_m = h^0(F, K_F^m \otimes_{\mathcal{O}_F}(m - F)),$$

which implies that $\kappa(X, K_X +) = \kappa(F, K_F + - F)$.

7. Geometric Orbifold Version of the Main Results

In this last section, we will prove Theorem E, in other word, generalize Main Theorem A (II), which is established in Section 6, to the geometric orbifold setting. Along the way, we also show that $C_{n,m}^{\text{orb}}$ holds when (Y, B_f) is of log general type. Before entering into the proof of theses results, let us first clarify some definitions. Remind that for $f : X \dashrightarrow Y$ analytic fibre space between compact complex manifolds and for \mathbb{Q} -divisor on Y , the branching divisor B_f is defined as the most effective \mathbb{Q} -divisor on Y such that $f^* B_f \subset R_f$ modulo exceptional divisors (see below, cf. also the Introduction); on the other hand, in [17, Definition 1.29] Frédéric Campana defines a divisor on Y with respect to f and in the setting of geometric orbifolds, named “orbifold base”. We will see in the sequel that these two definitions coincide when $(X,)$ is lc. Let us first recall the definition of Campana:

DEFINITION 7.1. — *Let $f : X \dashrightarrow Y$ and as above such that $(X,)$ is lc. For any prime divisor G on Y , write*

$$f^* G = \sum_j \text{Ram}_{G_j}(f) G_j + (f\text{-exceptional divisor}),$$

where $J(f, G)$ is the index set of all prime divisors mapped onto G . Then the orbifold base with respect to f and is defined to be the \mathbb{Q} -divisor

$$B_{f, } := \sum_G \left(1 - \frac{1}{m(f, ; G)} \right) G$$

where the multiplicity $m(f, ; G)$ of G with respect to f and is defined to be

$$m(f, ; G) := \inf \{ \text{Ram}_{G_j}(f) m(; G_j) \mid j \in J(f, G) \}$$

with $m(; G_j) \in \mathbb{Q}_{>1}$ $\{ + \}$ satisfying

$$\text{ord}_{G_j}() = 1 - \frac{1}{m(; G_j)}.$$

Now we have:

LEMMA 7.2. — *Let $f : X \rightarrow Y$ and π as above such that (X, π) is lc. Let $B_{f, \pi}$ be the orbifold base respect to f and π in the sense of Campana, as defined in Definition 7.1 above. Then there is an f -exceptional effective \mathbb{Q} -divisor E such that the \mathbb{Q} -divisor $R_{f, \pi} + E - f^* B_{f, \pi}$ is effective; and $B_{f, \pi}$ is the most effective \mathbb{Q} -divisor on Y satisfying this property.*

Proof. — The second assertion is evident by construction of $B_{f, \pi}$. In fact, if B is a divisor on Y such that $f^* B \in R_{f, \pi}$, then for every prime divisor G on Y we have

$$\begin{aligned} \text{ord}_{G_j}(f^* B) &= \text{Ram}_{G_j}(f) \text{ord}_G(B) \in \text{ord}_{G_j}(R_{f, \pi}) \\ &= \text{Ram}_{G_j}(f) - 1 + \text{ord}_{G_j}(\pi) \\ &= \text{Ram}_{G_j}(f) - \frac{1}{m(\pi; G_j)}, \quad j \in J(f, G), \end{aligned}$$

where

$$f^* G = \sum_{j \in J(f, G)} \text{Ram}_{G_j} G_j + (f - \text{exceptional divisor});$$

this implies that

$$\text{ord}_G(B) \in 1 - \frac{1}{\text{Ram}_{G_j}(f)m(\pi; G_j)}, \quad j \in J(f, G),$$

and hence

$$\begin{aligned} \text{ord}_G(B) &\in \inf_{j \in J(f, G)} \left(1 - \frac{1}{\text{Ram}_{G_j}(f)m(\pi; G_j)} \right) \\ &= 1 - \frac{1}{\inf \{ \text{Ram}_{G_j}(f)m(\pi; G_j) \mid j \in J(f, G) \}} \\ &= \text{ord}_G(B_{f, \pi}). \end{aligned}$$

Now turn to the proof of the first assertion. To this end, it suffices to show that for any prime divisor D on X such that $f(D)$ is a divisor on Y we have

$$\text{ord}_D(R_{f, \pi}) = \text{ord}_D(\pi) + \text{ord}_D(\pi) > \text{ord}_D(f^* B_{f, \pi}). \quad (7.1)$$

Let π_Y be a (reduced) divisor containing $Y \setminus Y_0$ with $Y_0 \subset Y$ the smooth locus of f and write

$$f^* \pi_Y = \sum_{i \in I} b_i W_i,$$

then

$$f^* \pi_Y := \sum_{i \in I^{\text{div}}} (b_i - 1) W_i.$$

where I^{div} denotes the set of indices in I such that $f(W_i)$ is a divisor on Y . Now we consider separately the two cases:

Case 1: $D \not\subset \text{Supp}(B_f)$. — Then $\text{ord}_D(B_f) = 0$ and a general point of $f(D)$ is contained in Y_0 , thus

$$f(B_f) = D + (f\text{-exceptional divisor}).$$

In consequence $\text{Ram}_D(f) = 1$ and $J(f, f(D)) = \{D\}$, which implies that $m(f, B_f; f(D)) = m(B_f, D)$. Hence

$$\begin{aligned} \text{ord}_D(f(B_f)) &= \text{ord}_{f(D)}(B_f) = 1 - \frac{1}{m(B_f, D)} \\ &= \text{ord}_D(B_f) = \text{ord}_D(B_f) + \text{ord}_D(1). \end{aligned}$$

Case 2: $D \subset \text{Supp}(B_f)$. — Then $D = W_i$ for some $i \in I^{\text{div}}$. In consequence, $f(W_i) \subset \text{Supp}(f(Y))$ and

$$f(B_f) = \sum_{j \in J(f, f(W_i))} b_j W_j + (f\text{-exceptional divisor}),$$

with $J(f, f(W_i)) = \{j \in I^{\text{div}} \mid f(W_j) = f(W_i)\}$ and $\text{Ram}_{W_j}(f) = b_j$. By definition we have

$$\begin{aligned} m(f, B_f; f(W_i)) &= \inf \{b_j m(B_f, W_j) \mid j \in I^{\text{div}} \text{ and } f(W_j) = f(W_i)\} \\ &\leq b_i m(B_f, W_i). \end{aligned}$$

Hence

$$\begin{aligned} \text{ord}_{W_i}(f(B_f)) &= b_i \cdot \text{ord}_{f(W_i)}(B_f) = b_i \left(1 - \frac{1}{m(f, B_f; f(W_i))}\right) \\ &\leq 1 - \frac{1}{b_i m(B_f, W_i)} = (b_i - 1) + \left(1 - \frac{1}{m(B_f, W_i)}\right) \\ &= \text{ord}_{W_i}(B_f) + \text{ord}_{W_i}(1). \end{aligned}$$

In both cases, the inequality (7.1) is established for prime divisor D vertical w.r.t. f , hence the proof is proved.

Remark 7.3. — As a corollary of the above lemma, one sees clearly:

- $f(B_f)$ being a vertical divisor w.r.t. f (i.e. not dominating Y), it is in fact the most effective divisor on Y such that $f(B_f) \leq R_{f, \text{vert}} = B_f + \text{vert}$ where vert denotes the vertical part of B_f .
- If (X, \mathbb{Q}) is klt and $F_m := f(K_{(X, \mathbb{Q})/Y}^m) = 0$ for some m sufficiently large and divisible, one can easily deduce from Proposition 2.5 (applied to $L = \mathcal{O}_X(m \text{ horiz})$ with horiz the horizontal part of B_f) that there is an f -exceptional effective \mathbb{Q} -divisor E such

that the \mathbb{Q} -line bundle $K_f^{\text{orb}} + E$ is pseudoeffective, where the orbifold relative canonical bundle is defined (as a \mathbb{Q} -line bundle) by the formula:

$$K_f^{\text{orb}} := K_{(X, \sigma)/(Y, B_f, \sigma)} = K_{X/Y} + \sigma - f^* B_f .$$

Before proving Theorem E, let us first prove the result that the klt version of $C_{n,m}^{\text{orb}}$ holds for fibre spaces over bases of general type in the sense of geometric orbifolds:

THEOREM 7.4. — *Let $f : X \rightarrow Y$ be a surjective morphism between compact Kähler manifolds whose general fibre F is connected. Let σ be an effective \mathbb{Q} -divisor on X such that (X, σ) is klt. Suppose that (Y, B_f, σ) is of log general type. Then*

$$\kappa(X, K_X + \sigma) > \kappa(F, K_F + \sigma|_F) + \dim Y,$$

where $\sigma|_F := \sigma|_F$.

Notice that a stronger (log canonical) version of the above theorem is proved in [17] (for X projective) based on the a weak positivity theorem for direct images of twisted pluricanonical bundles. We will give here a new argument depending on the Ohsawa–Takegoshi extension theorem:

Proof of Theorem 7.4. — First, as in the proof of Theorem 3.6, by passing to a higher bimeromorphic model of f , we can assume that f is neat and prepared (in virtue of Lemma 1.4 and Lemma 1.5), that is, every f -exceptional divisor is also exceptional with respect to some bimeromorphic morphism $X \rightarrow X'$ and the singular locus of f is a (reduced) SNC divisor; in particular, for every effective f -exceptional divisor E_0 on X , we have $\kappa(X, K_X + \sigma) = \kappa(X, K_X + \sigma + E_0)$.

If $\kappa(F, K_F + \sigma|_F) = -\infty$ then there is nothing to prove, hence suppose that $\kappa(F, K_F + \sigma|_F) > 0$, this implies that there is $m > 0$ sufficiently large and divisible such that $F_{m, \sigma} := f^*(K_{(X', \sigma')/Y}) = 0$. By Remark 7.3, there is an effective f -exceptional \mathbb{Q} -divisor E such that the \mathbb{Q} -line bundle $K_f^{\text{orb}} + E$ is pseudoeffective. Since (Y, B_f, σ) is of log general type, Y is projective, one can fix a very ample line bundle A_Y on Y such that the \mathbb{Q} -line bundle $A_Y - K_Y - B_f$ is ample and that the Seshadri constant $\epsilon(A_Y - K_Y - B_f, y) > \dim Y$ for general y (such A_Y exists by [59, §5.1, Example 5.1.4, p. 270 and Example 5.1.18, p. 274, Vol.I]). Now by our hypothesis $K_Y + B_f$ is a big \mathbb{Q} -line bundle, then (up to replacing m by a multiple) we can assume that $m(K_Y + B_f) - 2A_Y$ is effective. Then we have

$$\kappa(X, K_X + \sigma) = \kappa(X, K_X + \sigma + E) > \kappa(X, mK_f^{\text{orb}} + mE + 2f^* A_Y).$$

In virtue of Lemma 3.1 it suffices to show that

$$H^0(X, (K_f^{\text{orb}})^m \otimes \mathcal{O}_X(mE) \otimes f^* A_Y) = 0,$$

which is a direct consequence of the Ohsawa–Takegoshi type extension Theorem 2.2, as we precise below:

Since X is klt, by Theorem 2.3 the relative m -Bergman kernel metric $h_{X/Y, m}^{\text{horiz}}$ on $(K_{X/Y}^m \otimes \mathcal{O}_X(m \text{horiz}))^{\text{horiz}}$ is semipositive (noting that $(\text{horiz}/F = F)$). Set

$$\begin{aligned} L_{m-1} &:= K_{X/Y}^{(m-1)} \otimes \mathcal{O}_X(m \text{horiz}), \\ L_{m-1} &:= L_{m-1} \otimes \mathcal{O}_X(mE + m \text{vert} - (m-1)f^* B_f), \end{aligned}$$

respectively equipped with the singular Hermitian metrics:

$$\begin{aligned} h_{L_{m-1}} &:= \left(h_{X/Y, m}^{\text{horiz}} \right)^{\frac{m-1}{m}} \cdot h_{\text{horiz}}, \\ h_{L_{m-1}} &:= h_{L_{m-1}} \cdot h_E^m \cdot h_{\text{vert}}^m \cdot f^* h_{B_f}^{-(m-1)}. \end{aligned}$$

where h_{horiz} , h_{vert} , h_E and h_{B_f} denote the canonical singular metrics defined by the divisors. Then by Proposition 2.5 and Lemma 7.2 the curvature current of $h_{L_{m-1}}$ satisfies

$$\begin{aligned} & h_{L_{m-1}}(L_{m-1}) \\ &= \frac{m-1}{m} \cdot h_{X/Y, m}^{\text{horiz}}(K_{X/Y}^m \otimes \mathcal{O}_X(m \text{horiz})) + [\] + (m-1)[\ \text{vert}] \\ &\quad + m[E] - (m-1)[f^* B_f] \\ &> (m-1)([f^*] + [E] + [\ \text{vert}] - [f^* B_f]) + [\] + [E] \\ &> [\] + [E] > 0. \end{aligned}$$

Moreover, since $L_{m-1}/F = L_{m-1}/F$ and $h_{L_{m-1}}/F = h_{L_{m-1}}/F$, by Lemma 2.12 the natural inclusion

$$\begin{aligned} & H^0(F, K_F \otimes L_{m-1}/F \otimes (h_{L_{m-1}}/F)) \\ &= H^0(F, K_F \otimes L_{m-1}/F \otimes (h_{L_{m-1}}/F)) \\ &\hookrightarrow H^0(F, K_F \otimes L_{m-1}/F) = H^0(F, K_F^m \otimes \mathcal{O}_F(m \text{horiz})) \end{aligned}$$

is an isomorphism. Hence by Theorem 2.2 we get a surjection

$$H^0(X, K_X \otimes L_{m-1} \otimes f^*(A_Y \otimes K_{(Y, B_f)}^{-1})) \rightarrow H^0(F, K_F^m \otimes \mathcal{O}_F(m \text{horiz})).$$

Since

$$K_X \otimes L_{m-1} \otimes f^*(A_Y \otimes K_{(Y, B_f)}^{-1}) = (K_f^{\text{orb}})^m \otimes \mathcal{O}_X(mE) \otimes f^* A_Y,$$

this proves the non-vanishing of $H^0((K_f^{\text{orb}})^m \otimes \mathcal{O}_X(mE) \otimes f^* A_Y)$.

Finally, let us turn to the proof of Theorem E:

Proof of Theorem E. — Let us proceed by an induction on $\dim T$. If $B_{f_i} = 0$, then Theorem E is reduced to the Part (II) of Main Theorem A. Hence we assume that $B_{f_i} \neq 0$. Then by [22, Proposition 2.2], there is a subtorus S of T of dimension $< \dim T$ and an ample \mathbb{Q} -divisor H on $A : T/S$ such that $\pi^* H = B_{f_i}$ with $\pi : T \rightarrow A = T/S$ the quotient map.

Now let $f = \pi \circ f : X \rightarrow A$, which is a fibre space with general fibre F . Then $f|_F : F \rightarrow S$ is a fibre space with general fibre F . We have $B_{f|_F, F} > (B_{f_i})|_S$, as one can easily check: for every component G of $(B_{f_i})|_S$, it arises from a prime divisor of X , hence $B_{f|_F, F}$ has the same vanishing order over G . This is enough for our use; we nevertheless remark that we have in fact the equality $B_{f|_F, F} = (B_{f_i})|_S$ since every component of $B_{f|_F, F}$ must arise from a divisor on X : in fact, every component of $B_{f|_F, F}$ is either the image of a component of $f = f|_F$ or the image of a component of $f|_F = (f)|_F$ (we have the equality if we choose S to be a general translate). Now the induction hypothesis gives:

$$\kappa(F, K_F + f|_F) > \kappa(F, K_F + f) + \kappa(S, (B_{f_i})|_S).$$

Furthermore, since $\kappa(S, (B_{f_i})|_S) > 0$, we have

$$\kappa(F, K_F + f) > \kappa(F, K_F + f). \tag{7.2}$$

We claim that

$$\kappa(X, K_X + f) > \kappa(F, K_F + f) + \dim A. \tag{7.3}$$

If $\kappa(F, K_F + f) = -\infty$, then (7.3) evidently holds. Hence we can assume that $\kappa(F, K_F + f) > 0$. In this case, for m sufficiently large and divisible,

$$H^0(F, K_F^m \otimes \mathcal{O}_F(m \cdot f)) = 0.$$

Since (X, f) is klt, (F, f) is klt, then by Theorem 2.3 we can construct the relative $2m$ -Bergman kernel metric $h_{X/A, 2m}^{(2m)}$ on $K_{X/A}^{2m}$ $\otimes_{\mathcal{O}_X} (2m \cdot \text{horiz})$. Now put

$$L := K_X^{(2m-1)} \otimes_{\mathcal{O}_X} (2m \cdot f + 2mE - m(f) \cdot H)$$

equipped with the singular Hermitian metric

$$h_L := \left(h_{X/A, 2m}^{(2m)} \right)^{\frac{2m-1}{2m}} \cdot h_{\text{horiz}} \cdot h_{\text{vert}}^{2m} \cdot h_E^{2m} \cdot (f) \cdot h_H^{-m},$$

where E is an f -exceptional effective divisor as in Lemma 7.2 and h_{horiz} , h_{vert} , h_E and h_H are the canonical singular metrics defined by the divisors.

Then by Proposition 2.5 and Lemma 7.2 the curvature current of h_L satisfies

$$\begin{aligned}
 h_L(L) &= \frac{2m-1}{2m} h_{X/A, \text{horiz}}^{(2m)} \left(K_{X/A}^{2m} \circlearrowleft_X(2m \text{ horiz}) \right) + [\text{horiz}] \\
 &\quad + 2m[\text{vert}] + 2m[E] - m[(f) H] \\
 &> (2m-1)[f] + [\text{horiz}] + 2m[\text{vert}] + 2m[E] - m[f B_f,] \\
 &= [] + [E] + (m-1)([f] + [\text{vert}] + [E]) \\
 &\quad + m([f] + [\text{vert}] + [E] - [f B_f,]) \\
 &> [] + [E] + (m-1)([f] + [\text{vert}] + [E]) > 0.
 \end{aligned}$$

Since $h_L|_F = h_{L_{2m-1}}|_F$, where $L_{2m-1} := K_X^{(2m-1)} \circlearrowleft_X(2m \text{ horiz})$ equipped with the singular metric

$$h_{L_{2m-1}} := \left(h_{X/A, \text{horiz}}^{(2m)} \right)^{\frac{2m-1}{2m}} h_{\text{horiz}},$$

then by Lemma 2.12 we see that the natural inclusion

$$f \left(K_{X/A} \otimes L \otimes \mathcal{J}(h_L) \right) \subset f \left(K_{X/A} \otimes L \right)$$

is generically an isomorphism, hence by Theorem 2.6 the canonical L^2 metric on

$$f \left(K_{X/A} \otimes L \right) = f \left(K_{X/A}^{2m} \circlearrowleft_X(2m + 2mE) \right) \otimes H^{-m}$$

is semi-positively curved. In particular its determinant is pseudoeffective, which implies that $\det f \left(K_{X/A}^{2m} \circlearrowleft_X(2m + 2mE) \right)$ is big on A . Since $f \left(K_{X/A}^{2m} \circlearrowleft_X(2m + 2mE) \right)$ and $f \left(K_{X/A}^{2m} \circlearrowleft_X(2m) \right)$ are equal in codimension 1, hence

$$\det f \left(K_{X/A}^{2m} \circlearrowleft_X(2m + 2mE) \right) = \det f \left(K_{X/A}^{2m} \circlearrowleft_X(2m) \right),$$

implying that $\det f \left(K_{X/A}^{2m} \circlearrowleft_X(2m) \right)$ is big on A . Since $\kappa(A) = 0$, (7.3) results from Theorem 3.6.

At last, by combining (7.2) and (7.3) with the easy inequality [71, Theorem 5.11, pp. 59–60] (applied to $\pi : T \rightarrow A$) we obtain:

$$\begin{aligned}
 \kappa(X, \otimes K_X) &> \kappa(F, K_F \otimes \otimes F) + \dim A \\
 &> \kappa(F, K_F \otimes \otimes F) + \kappa(S, (B_f,)/S) + \dim A \\
 &> \kappa(F, K_F \otimes \otimes F) + \kappa(T, B_f,).
 \end{aligned}$$

Appendix A. Proof of the Negativity Lemma 1.3

In this appendix we are engaged to prove the Negativity Lemma 1.3 in Section 1.1. Let us recall the statement: $h : Z \rightarrow Y$ being a proper

bimeromorphic morphism between normal complex varieties and B being a Cartier divisor on Z such that $-B$ is h -nef, we will prove that B is effective if and only if $h B$ is effective. First notice that if B is effective, then $h B$ is effective; hence it remains to show that $h B$ is effective $\implies B$ is effective. To this end we proceed in three steps:

(A). Reduction to the case where h is a sequence of blow-ups with smooth centres. For any proper bimeromorphic morphism $f : Z \rightarrow X$, B is effective $\iff f^* B$ is effective; moreover, if we note $h = h \circ f$, then $h f^* B = h B$ and $-f^* B$ is h -nef. This observation gives us the flexibility to replace Z with a higher bimeromorphic model. In particular, by Chow's Lemma ([46, Corollary 2]) we can suppose that h is projective. In addition, by Hironaka's construction in [46] we see that h is in fact the blow-up of an analytic subspace (a coherent ideal) of X (cf. [46, Definition 4.1]); hence by Hironaka's resolution of singularities, we can take a principalization h of this ideal, which is constructed by a sequence of blow-ups with smooth centres, by the universal property of blow-ups, h dominates h . Cf. also [13, Lemma 4.1]. Now up to replacing h with h , we can assume that h is a locally finite (over Y) sequence of blow-ups with smooth centres; moreover the problem being local over Y , one can further assume that h is a finite sequence. In particular, (e.g. by an induction on the number of blow-ups contained in h) there exists an effective Cartier divisor h -exceptional divisor A such that $-A$ is h -ample.

(B). Reduction to the case where $-B$ is h -ample by an approximation argument. In this step we use an approximation argument to reduce to the case where $-B$ is h -ample. To this end, assume that the lemma is true for h -anti-ample divisors. By Step (A), one gets an h -exceptional divisor A such that $-A$ is h -ample. Since $h A = 0$, our assumption implies that A is effective. For every $m > 0$, the Cartier divisor $-mB - A$ is h -ample; in addition, $h(mB + A) = mh B > 0$, hence by our assumption, $mB + A$ is effective. By arguing coefficients by coefficients and by letting m tend to $+\infty$ we obtain that B is effective.⁽³⁾

(C). The case where $-B$ is h -ample. By the reduction procedures (A) and (B), we can suppose that h is projective and that B is a Cartier divisor on Z such that $-B$ is h -ample. Since $-B$ is h -ample, then for any $m \gg 0$, the Cartier divisor $-mB$ is relatively globally generated, i.e. we have

⁽³⁾ Let us remark that in many cases, when there is no ambiguity, by saying that “a divisor is effective” we mean that it is *linearly equivalent* to an effective divisor; but in the statement of the Negativity Lemma 1.3 we take “effectivity” in its strict sense.

an surjection

$$h^* h^* \mathcal{O}_Z(-mB) \rightarrow \mathcal{O}_Z(-mB).$$

In particular, $\mathcal{O}_Z(-mB) = h^{-1} \mathfrak{a}_m \cdot \mathcal{O}_Y$ where $\mathfrak{a}_m = h^* \mathcal{O}_Z(-mB)$ fractional ideal on Y (i.e. a torsion free subsheaf of rank 1 of \mathcal{M}_Y the sheaf of germs of meromorphic functions on Y) since h is bimeromorphic. It remains to see that \mathfrak{a}_m is an authentic ideal. To this end it suffices to consider the inclusion (by hypothesis $h^* B$ is effective)

$$\mathfrak{a}_m = h^* \mathcal{O}_Z(-mB) \rightarrow \mathcal{O}_Y(-mh^* B) \rightarrow \mathcal{O}_Y,$$

where the inclusion $h^* \mathcal{O}_Z(-mB) \rightarrow \mathcal{O}_Y(-mh^* B)$ above results from Lemma 1.15.

Appendix B. Proof of Proposition 1.18

In this appendix, we give the detailed proof of Proposition 1.18 which serves to complete the proof of Theorem 1.14. Let $X \rightarrow S$ be a surjective morphism between complex varieties with X smooth and S normal, we will show that there is an effective π -exceptional divisor E such that for any π -exceptional prime divisor σ , E/σ is not π/σ -pseudoeffective.

The starting point of the proof is the following observation: if π is flat, then $\pi^* L$ is always reflexive. Consider thus a flattening of π (cf. [46], or for the algebraic case, [69, §4.1, Theorem 1, p. 26]): let $\nu : S' \rightarrow S$ be a projective bimeromorphic morphism (a sequence of blow-ups with smooth centres) which flattens π and let X' be the normalization of the main component of $X \times_S S'$ equipped with morphisms $X' \xrightarrow{\mu} X$ and $X' \xrightarrow{\phi} S'$ (μ is projective and ϕ is equidimensional).

By the construction of ν , there is a π -exceptional effective (Cartier) divisor E such that $\pi^* L - E$ is ν -ample. Consider the divisor $E := \mu^* (\phi^* L - E')$. Then E is effective since $\phi^* L - E'$ is effective; E is Cartier since X is smooth. Moreover, $\pi^* L - E$ is ν -ample, hence $-\phi^* L + E'$ is μ -nef: in fact, let C be a curve contracted by μ , then $\phi^* C$ (which is, by definition, a curve on S' if C is not contracted by μ , or is equal to 0 otherwise) is contracted by ν since $\pi^* \mu = \nu^* \phi$, hence by the projection formula we get

$$(-\phi^* L + E') \cdot C = (-\pi^* L + E) \cdot \phi^* C > 0,$$

μ being projective, this implies that $-\phi^* L + E'$ is μ -nef; then so is $\mu^* E - \phi^* L$. Now since

$$\mu^* (\mu^* E - \phi^* L) = E - E = 0,$$

then we have $\mu^* E - \phi^* L \leq 0$ by the Negativity Lemma 1.3.

Assume by contradiction that there exists a π -exceptional prime divisor such that E/ν is π/ν -pseudoeffective and denote

$\mu :=$ the strict transformation of X by μ^{-1} .

$$\begin{array}{ccc}
 & \xrightarrow{\mu/\nu} & \\
 \downarrow \phi/\nu & & \downarrow \pi/\nu \\
 X & \xrightarrow{\mu} & X \\
 \downarrow \phi & & \downarrow \pi \\
 S & \xrightarrow{\nu} & S \\
 \downarrow \phi(\zeta) & & \downarrow \pi(\zeta) \\
 \phi(\zeta) & \xrightarrow{\nu/\phi(\zeta)} & \pi(\zeta)
 \end{array}$$

Then $\mu E/\nu$ is $(\pi/\mu)/\nu$ -pseudoeffective, hence ϕ/ν is $(\nu/\phi)/\nu$ -pseudoeffective since $\mu E \subset \phi$. On the other hand, by our construction ν is ν -ample, then $\nu/\phi(\zeta)$ is $\nu/\phi(\zeta)$ -ample, and thus

$$-\phi/\nu(\zeta) = (\phi/\nu(\zeta)) - \nu/\phi(\zeta)$$

is $(\nu/\phi)/\nu$ -nef. Therefore $-\phi/\nu(\zeta)$ is $(\nu/\phi)/\nu$ -numerically trivial, which implies that $\nu/\phi(\zeta)$ is $\nu/\phi(\zeta)$ -numerically trivial. But $\nu/\phi(\zeta)$ is $\nu/\phi(\zeta)$ -ample, this cannot happen unless $\nu/\phi(\zeta) : \phi(\zeta) \rightarrow \pi(\zeta)$ is finite. We will show in the sequel that $\nu/\phi(\zeta)$ is never finite:

Since ϕ is the composition of a finite morphism (normalization) followed by a flat morphism, ϕ is equidimensional; in particular, $\phi(\zeta)$ is Weil divisor on S . Moreover, $\nu(\phi(\zeta)) = \pi/\mu(\zeta) = \pi(\zeta)$ is of codimension > 2 , hence $\phi(\zeta)$ is ν -exceptional; in particular, the general fibre of the morphism $\nu/\phi(\zeta) : \phi(\zeta) \rightarrow \pi(\zeta)$ is of dimension > 1 . Thus we prove the proposition.

Bibliography

- [1] M. ARTIN, “Néron Models”, in *Arithmetic Geometry*, Springer, 1986, p. 213-230.
- [2] C. BĂNICĂ & O. STĂNĂȘILĂ, *Algebraic Methods in the Global Theory of Complex Spaces*, John Wiley & Sons, 1976, revised English version of *Metode algebrice în teoria global a spațiilor complexe*, Editura academiiei, București, 1974.
- [3] B. BERNDTSSON, “Curvature of Vector Bundles Associated to Holomorphic Fibrations”, *Ann. Math.* **169** (2009), no. 2, p. 531-560.
- [4] ———, “The Openness Conjecture for Plurisubharmonic Functions”, <https://arxiv.org/abs/1305.5781>, 2013.

- [5] B. BERNDTSSON & M. PÄUN, “Bergman Kernels and the Pseudoeffectivity of Relative Canonical Bundles”, *Duke Math. J.* **145** (2008), no. 2, p. 341-378.
- [6] ———, “Bergman Kernels and the Subadjunction”, <https://arxiv.org/abs/1002.4145>, 2010.
- [7] C. BIRKENHAKE & H. LANG, *Complex Abelian Varieties*, Grundlehren der Mathematischen Wissenschaften, vol. 302, Springer, 2004.
- [8] Z. BŁOCKI, “Saita Conjecture and the Ohsawa-Takegoshi Extension Theorem”, *Invent. Math.* **193** (2013), no. 1, p. 149-158.
- [9] S. BOUCKSOM, “Divisorial Zariski Decomposition on Compact Complex Manifolds”, *Ann. Sci. Éc. Norm. Supér.* **37** (2004), no. 4, p. 45-67.
- [10] ———, “Singularities of Plurisubharmonic Functions and Multiplier Ideals”, 2016, Notes of Course, <http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf>.
- [11] S. BOUCKSOM, J.-P. DEMAILLY, M. PÄUN & T. PETERNELL, “The Pseudo-effective Cone of a Compact Kähler Manifold and Varieties of Negative Kodaira Dimension”, *J. Algebr. Geom.* **22** (2013), no. 2, p. 201-248.
- [12] S. BOUCKSOM, T. D. FERNEX & C. FAVRE, “The Volume of an Isolated Singularity”, *Duke Math. J.* **161** (2012), no. 8, p. 1455-1520.
- [13] S. BOUCKSOM & M. JONSSON, “Tropical and non-Archimedean Limits of Degenerating Families of Volume Forms”, *J. Éc. Polytech., Math.* **4** (2017), p. 87-139.
- [14] N. BUDUR, “Unitary Local Systems, Multiplier Ideals, and Polynomial Periodicity of Hodge Numbers”, *Adv. Math.* **221** (2009), no. 1, p. 217-250.
- [15] N. BUDUR & B. WANG, “Cohomology Jump Loci of Quasi-projective Varieties”, *Ann. Sci. Éc. Norm. Supér.* **48** (2015), no. 1, p. 227-236.
- [16] ———, “Cohomology Jump Loci of Quasi-compact Kähler Manifolds”, <https://arxiv.org/abs/1702.02186>, 2017.
- [17] F. CAMPANA, “Orbifolds, Special Varieties and Classification Theory”, *Ann. Inst. Fourier* **54** (2004), no. 3, p. 499-665.
- [18] ———, “Orbifolles géométriques spéciales et classification biméromorphe des variétés kählériennes compactes”, Prépublication IECN <https://hal.archives-ouvertes.fr/hal-00356763>, 2009.
- [19] F. CAMPANA, A. HÖRING & T. PETERNELL, “Abundance for Kähler Threefolds”, *Ann. Sci. Éc. Norm. Supér.* **49** (2016), no. 4, p. 971-1025.
- [20] F. CAMPANA, V. KOZIARZ & M. PÄUN, “Numerical Character of the Effectivity of Adjoint Line Bundles”, *Ann. Inst. Fourier* **62** (2012), no. 1, p. 107-119.
- [21] F. CAMPANA & T. PETERNELL, “Geometric Stability of the Cotangent Bundle and the Universal Cover of a Projective Manifold”, *Bull. Soc. Math. Fr.* **139** (2011), no. 1, p. 41-74.
- [22] J. CAO, “On the Approximation of Kähler Manifolds by Algebraic Varieties”, *Math. Ann.* **363** (2015), no. 1-2, p. 393-422.
- [23] ———, “Ohsawa-Takegoshi Extension Theorem for Compact Kähler Manifolds and Applications”, in *Complex and Symplectic Geometry*, Springer INdAM Series, vol. 21, Springer, 2017, p. 19-38.
- [24] J. CAO & A. HÖRING, “A Decomposition Theorem for Projective Manifolds with Nef Anticanonical Bundle”, to appear in *J. Algebr. Geom.*, <https://arxiv.org/abs/1706.08814>, 2017.
- [25] J. CAO & M. PÄUN, “Kodaira Dimension of Algebraic Fibre Space over Abelian Varieties”, *Invent. Math.* **207** (2017), no. 1, p. 345-387.
- [26] H. CARTAN, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Herman, éditeur des sciences et des arts, 1961.

- [27] O. DEBARRE, *Tores et variétés abéliennes complexes*, Contributions in Mathematical and Computational Sciences, vol. 6, Société Mathématique de France; EDP Sciences, 1999.
- [28] ———, *Higher-Dimensional Algebraic Geometry*, Universitext, Springer, 2001.
- [29] J.-P. DEMAILLY, *Analytic Methods in Algebraic Geometry*, Surveys of Modern Mathematics, vol. 1, Higher Education Press; International Press, 2010.
- [30] ———, “Complex Analytic and Differential Geometry”, OpenContent Book <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>, 2012.
- [31] ———, “Extension of Holomorphic Functions Defined on non Reduced Analytic Subvarieties”, <https://arxiv.org/abs/1510.05230>, 2015.
- [32] J.-P. DEMAILLY, L. EIN & R. LAZARSEFELD, “A Subadditivity Property of Multiplier Ideals”, *Mich. Math. J.* **48** (2000), no. 1, p. 137-156.
- [33] F. DENG, Z. WANG, L. ZHANG & X. ZHOU, “New Characterizations of Plurisubharmonic Functions and Positivity of Direct Image Sheaves”, <https://arxiv.org/abs/1809.10371>, 2018.
- [34] Y. DENG, “Applications of the Ohsawa–Takegoshi Extension Theorem to Direct Image Problems”, <https://arxiv.org/abs/1703.07279>, 2017.
- [35] H. ESNAULT, “Classification des variétés de dimension 3 et plus”, in *Séminaire Bourbaki n°23 (1981)*, Lecture Notes in Mathematics, vol. 901, Springer, 1981, exposé 568.
- [36] O. FUJINO, “Notes on the Weak Positivity Theorem”, in *Algebraic Varieties and Automorphism Groups, July 7-11, 2014*, Advances Studies in Pure Mathematics, vol. 75, Mathematical Society of Japan, 2017, p. 73-118.
- [37] T. FUJITA, “On Kähler Fibre Spaces over Curves”, *J. Math. Soc. Japan* **30** (1978), no. 4, p. 779-794.
- [38] W. FULTON, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 2, Springer, 1984.
- [39] H. GRAUERT & R. REMMERT, “Plurisubharmonische Funktionen in komplexen Räumen”, *Math. Z.* **65** (1956), p. 175-194.
- [40] M. GREEN & P. GRIFFITHS, “Positivity of Vector Bundles and Hodge Theory”, <https://arxiv.org/abs/1803.07405>, 2018.
- [41] M. GREEN & R. LAZARSEFELD, “Deformation Theory, Generic Vanishing Theorems, and some Conjectures of Enriques, Cantanese and Beauville”, *Invent. Math.* **90** (1987), no. 2, p. 389-407.
- [42] ———, “Higher Obstructions to Deforming Cohomology Groups of Line Bundles”, *J. Am. Math. Soc.* **4** (1991), no. 1, p. 87-103.
- [43] Q. GUAN & X. ZHOU, “A Proof of Demailly’s Strong Openness Conjecture”, *Ann. Math.* **182** (2015), no. 2, p. 605-616.
- [44] ———, “A Solution of an L^2 Extension Problem with an Optimal Estimate and Applications”, *Ann. Math.* **181** (2015), no. 3, p. 1139-1208.
- [45] C. HACON, M. POPA & C. SCHNELL, “Algebraic Fibre Spaces over Abelian Varieties: Around a Recent Theorem by Cao and Păun”, in *Local and Global Methods in Algebraic Geometry*, Contemporary Mathematics, vol. 712, American Mathematical Society, 2018, p. 143-195.
- [46] H. HIRONAKA, “Flattening Theorem in Complex Analytic Geometry”, *Am. J. Math.* **97** (1975), no. 2, p. 503-547.
- [47] A. HÖRING, “Positivity of Direct Image Sheaves. A Geometric Point of View”, *Enseign. Math.* **56** (2010), no. 1-2, p. 87-142.
- [48] A. HÖRING & T. PETERNELL, “Minimal Models for Kähler Threefolds”, *Invent. Math.* **203** (2016), no. 1, p. 217-264.

- [49] Y. KAWAMATA, “Characterization of Abelian Varieties”, *Compos. Math.* **43** (1981), no. 2, p. 253-276.
- [50] ———, “Kodaira Dimension of Algebraic Fibre Spaces over Curves”, *Invent. Math.* **66** (1982), no. 1, p. 57-71.
- [51] ———, “Minimal Models and the Kodaira Dimension of Algebraic Fibre Spaces”, *J. Reine Angew. Math.* **1985** (1985), no. 363, p. 1-46.
- [52] ———, “On the Abundance Theorem in the Case of Numerical Kodaira Dimension Zero”, *Am. J. Math.* **135** (2013), no. 1, p. 115-124.
- [53] Y. KAWAMATA, K. MATSUDA & K. MATSUKI, “Introduction to the Minimal Model Problem”, in *Algebraic Geometry, Sendai, 1985*, Advanced Studies in Pure Mathematics, vol. 10, Mathematical Society of Japan, 1987, p. 283-360.
- [54] Y. KAWAMATA & E. VIEHWEG, “On a Characterization of an Abelian Variety in the Classification Theory of Algebraic Varieties”, *Compos. Math.* **41** (1980), no. 3, p. 355-359.
- [55] S. KOBAYASHI, *Differential Geometry of Complex Vector Bundles*, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, 1987.
- [56] J. KOLLÁR, “Singularities of Pairs”, in *Algebraic Geometry Santa Cruz 1995*, Proceedings of Symposia in Pure Mathematics, vol. 62, American Mathematical Society, 1997, p. 221-287.
- [57] J. KOLLÁR & S. MORI, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.
- [58] G. R. KUMPF, F. F. KNUDSEN, D. B. MUMFORD & B. SAINT-DONAT, *Toroidal Embeddings*, Lecture Notes in Mathematics, vol. 339, Springer, 1973.
- [59] R. LAZARSFELD, *Positivity in Algebraic Geometry I & II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48 & 49, Springer, 2004.
- [60] J. LESIEUTRE, “A Pathology of Asymptotic Multiplicity in the Relative Setting”, *Math. Res. Lett.* **23** (2016), no. 5, p. 1433-1451.
- [61] H. MATSUMURA, *Commutative Algebra*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., 1970.
- [62] ———, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, 1989.
- [63] J. S. MILNE, “Abelian Varieties”, in *Arithmetic Geometry*, Springer, 1986, p. 103-150.
- [64] N. NAKAYAMA, *Zariski-decomposition and Abundance*, MSJ Memoirs, Mathematical Society of Japan, 2004.
- [65] M. PĂUN, “Singular Hermitian Metrics and Positivity of Direct Images of Pluricanonical Bundles”, 2016, Survey <https://arxiv.org/abs/1606.00174>.
- [66] M. PĂUN & S. TAKAYAMA, “Positivity of Twisted Relative Pluricanonical Bundles and Their Direct Images”, *J. Algebr. Geom.* **27** (2018), no. 2, p. 211-272.
- [67] J.-P. RAMIS & G. RUGET, “Complexes dualisant et théorèmes de dualité en géométrie analytique complexe”, *Publ. Math., Inst. Hautes Étud. Sci.* **38** (1970), p. 77-91.
- [68] H. RAUFI, “Singular Hermitian Metrics on Holomorphic Vector Bundles”, *Ark. Mat.* **53** (2015), no. 2, p. 359-382.
- [69] M. RAYNUAD, “Flat Modules in Algebraic Geometry”, *Compos. Math.* **24** (1972), no. 1, p. 11-31.
- [70] C. SIMPSON, “Subspaces of Moduli Spaces of Rank one Local Systems”, *Ann. Sci. Éc. Norm. Supér.* **26** (1993), no. 3, p. 361-401.
- [71] K. UENO, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Mathematics, vol. 439, Springer, 1975.
- [72] D. VAROLIN, “Three Variations on a Theme in Complex Analytic Geometry”, in *Analytic And Algebraic Geometry: Common Problems, Different Methods*, IAS/Park City

- Mathematics Series, vol. 17, American Mathematical Society; Institute for Advanced Studies, 2010, p. 183-295.
- [73] E. VIEHWEG, “Rational Singularities of Higher Dimensional Schemes”, *Proc. Am. Math. Soc.* **63** (1977), no. 1, p. 6-8.
 - [74] ———, “Weak Positivity and the Additivity of the Kodaira Dimension for Certain Fibre Spaces”, in *Algebraic Varieties and Analytic Varieties*, Advanced Studies in Pure Mathematics, vol. 1, Mathematical Society of Japan, 1983, p. 329-353.
 - [75] ———, *Quasi-projective Moduli for Polarized Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 30, Springer, 1995.
 - [76] C. VOISIN, *Théorie de Hodge et géométrie algébrique complexe*, Contributions in Mathematical and Computational Sciences, vol. 10, Société Mathématique de France; EDP Sciences, 2002.
 - [77] B. WANG, “Torsion Points on the Cohomology Jump Loci of Compact Kähler Manifolds”, *Math. Res. Lett.* **23** (2016), no. 2, p. 545-563.
 - [78] X. ZHOU & L. ZHU, “An optimal L^2 Extension Theorem on Weakly Pseudoconvex Kähler Manifolds”, *J. Di er. Geom.* **110** (2018), no. 1, p. 135-186.
 - [79] K. ZUO, “Kodaira Dimension and Chern Hyperbolicity of the Shafarevich Maps for Representations of π_1 of Compact Kähler Manifolds”, *J. Reine Angew. Math.* **472** (1996), no. 2, p. 139-156.