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Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients (*)

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Abstract. — In this work, we are interested in the homogenization of time-harmonic Maxwell's equations in a composite medium with periodically distributed small inclusions of a negative material. Here a negative material is a material modelled by negative permittivity and permeability. Due to the sign-changing coefficients in the equations, it is not straightforward to obtain uniform energy estimates to apply the usual homogenization techniques. The goal of this article is to explain how to proceed in this context. The analysis of Maxwell's equations is based on a precise study of two associated scalar problems: one involving the sign-changing permittivity with Dirichlet boundary conditions, another involving the sign-changing permeability with Neumann boundary conditions. For both problems, we obtain a criterion on the physical parameters ensuring uniform invertibility of the corresponding operators as the size of the inclusions tends to zero. In the process, we explain the link existing with the so-called Neumann–Poincaré operator complementing the existing literature on this topic. Then we use the results obtained for the scalar problems to derive uniform energy estimates for Maxwell's system. At this stage, an additional difficulty comes from the fact that Maxwell's equations are also sign-indefinite due to the term involving the frequency. To cope with it, we establish some sort of uniform compactness result.

Résumé. — Dans ce travail, nous nous intéressons à l'homogénéisation des équations de Maxwell harmoniques dans un milieu composite contenant une distribution

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périodique de petites inclusions de matériau négatif. On désigne ici par matériau négatif un matériau décrit par une permittivité et une perméabilité négatives. En raison du changement de signe des coefficients intervenant dans les équations, il n'est pas évident d'obtenir des estimations d'énergie uniformes et d'appliquer les techniques d'homogénéisation classiques. Le but de cet article est d'indiquer comment on peut néanmoins procéder dans ce contexte. L'analyse des équations de Maxwell est basée sur une étude précise de deux problèmes scalaires : l'un faisant intervenir la permittivité changeant de signe avec des conditions aux limites de Dirichlet, et l'autre la perméabilité changeant de signe avec des conditions aux limites de Neumann. Pour chacun de ces deux problèmes, on obtient un critère portant sur les paramètres physiques garantissant l'inversibilité uniforme des opérateurs associés lorsque la taille des inclusions tend vers zéro. Incidemment, nous expliquons le lien existant avec l'opérateur de Neumann–Poincaré, complétant la littérature existant sur le sujet. Les résultats obtenus pour les problèmes scalaires sont ensuite utilisés pour obtenir des estimations d'énergie uniforme pour le système de Maxwell. A ce stade, il faut contourner une difficulté supplémentaire liée au caractère indéfini induit par le terme fréquentiel. Ceci est réalisé en établissant un résultat de type compacité uniforme.

1. Introduction

Negative index materials (also called left-handed materials) are artificially structured composite materials whose dielectric permittivity ε and magnetic permeability μ are simultaneously negative in some frequency ranges [40]. In the last two decades, these metamaterials have been the subject of a large number of studies in physics and engineering due to their potential use for several existing applications [23] such as sub-wavelength imaging and focusing, cloaking, sensing or data storage.

Besides their practical applications, negative index materials are also interesting from a mathematical point of view. The reason is that they are usually used in applications jointly with classical (positive) materials so that their mathematical modelling leads to consider operators with coefficients whose sign changes in the domain of interest. Establishing well-posedness results for such problems requires to develop a specific theory and this has been investigated by several authors. In particular, it has been shown in [10] (see also the recent work [33]) that Maxwell's equations with sign-changing electromagnetic coefficients ε, μ are uniquely solvable (except for a discrete set of frequencies) when two associated scalar problems, one involving ε with Dirichlet boundary conditions, another involving μ with Neumann boundary conditions, are well-posed.

The goal of this article is to study the homogenization process for time harmonic Maxwell's equations in the presence of $\delta-$periodically distributed inclusions of negative material embedded into a dielectric material (see Figure 2.1 for a typical configuration). The main objective is to clarify if the homogenization process is doable in this context and if so, to determine whether the corresponding homogenized material behaves like a positive or negative material as δ tends to zero. For scalar problems, the first homogenization results have been obtained in [18] using the T-coercivity approach of [12]. More precisely, it is proved therein that for negative contrasts close to 0 (the contrast being defined here as the ratio between the interior and exterior values, see (2.1)), the scalar problem with Dirichlet boundary conditions can be homogenized. In other words, it is proved that under this assumption on the contrast, the solution of the problem in the composite material is well-defined for δ small enough (this is not obvious due to the loss of coercivity due to the sign-changing coefficient) and that it two-scale converges (see Definition 5.1 below) to the solution of a well-posed problem set in a homogeneous material. These results have been extended in [14], through the analysis of the spectrum of the Neumann–Poincaré operator. In particular, the authors show that the homogenization process is possible provided the contrast between the two media (defined using the same convention as above) belongs to $(-\infty; -1/\alpha)$ or $(-\alpha; 0)$, $\alpha > 0$ (see Remark 3.1 below). The proof of this result is based on an elegant continuity argument (see [14, Corollary 5.1]). However, it does not provide a precise value for α.

The paper is organized as follows. Section 2 provides the mathematical setting of the problem and necessary notation. Before studying Maxwell's system, we collect in Section 3 some useful results concerning two associated scalar problems, a Dirichlet and a Neumann one. In particular, we prove the uniform invertibility of these operators as δ tends to zero, for small or large values of the contrast, i.e. for contrasts in $(-\infty; -1/m) \cup (-1/M; 0)$, with $0 < m < M$ (see Subsections 3.1 to 3.3). A variational characterization of the bounds m and M is also obtained (see (3.35)). Next, inspired by [14], we discuss in Section 3.4 the connection with the Neumann–Poincaré operator and the optimality of the obtained conditions. In Section 4, we study the cell problems appearing in the homogenization of Maxwell's equations. We prove that they are well-posed under the same assumptions as the scalar problems investigated in Section 3. This allows us to define homogenized tensors and we show that they are positive definite under the same assumption on the contrasts, that is for contrasts in $(-\infty; -1/m) \cup (-1/M; 0)$. This is also an improvement of the results obtained in [14] and [18]. In Section 5, we finally tackle the homogenization process for Maxwell's equations with sign-changing coefficients. Combining results from [21] and [43] obtained for classical (positive) electromagnetic materials, we first derive in Section 5.1 a homogenization result under a uniform energy estimate condition. At this stage, the sign-changing of the physical parameters does not play any role.

Related to this part of the work, let us mention the seminal book [6] as well as [5, 46] for the study of the time-dependent Maxwell equations. For the time harmonic case, we refer to [3, 6, 17, 19, 24, 28, 39, 43, 47, 48]. Then, in Section 5.2, we establish the needed uniform energy estimates for Maxwell's equations. This is done by using the results obtained for the scalar problems as well as the T -coercivity approach presented in [10] and a uniform compactness property. The final homogenization result for Maxwell's system with sign-changing coefficients is stated in Theorem 5.6. For the reader's convenience, the list of functional spaces used throughout the paper is collected in the Appendix.

2. Setting of the problem

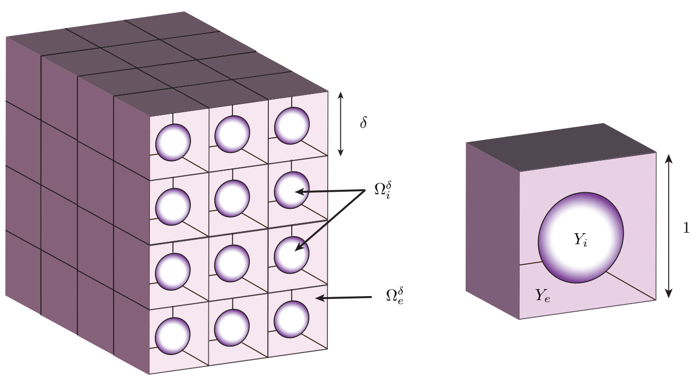

Figure 2.1. Example of a periodic material and the corresponding reference cell Y.

Let Ω be an open, connected and bounded subset of R^3 with a Lipschitz-continuous boundary $\partial\Omega$. Once and for all, we make the following assumption:

Assumption. — *The domain Ω is simply connected and $\partial\Omega$ is connected.*

When this assumption is not satisfied, the analysis below must be adapted (see some preliminary ideas in [10, Section 8.2]). We consider a situation where Ω is filled with a composite electromagnetic material constituted of periodically distributed inhomogeneous cells of small size $\delta > 0$. More precisely, let $Y = (0;1)^3$ denote the reference cell and assume that Y contains two materials:

- a metamaterial with negative dielectric permittivity $\varepsilon_i < 0$ and magnetic permeability $\mu_i < 0$ located inside a connected domain $Y_i \subset Y$ with Lipschitz boundary ∂Y_i such that $\overline{Y_i} \subset Y$;
- a dielectric material with positive dielectric permittivity $\varepsilon_e > 0$ and magnetic permeability $\mu_e > 0$ filling the region $Y_e := Y \setminus \overline{Y_i}$.

We emphasize that the assumption $\overline{Y_i} \subset Y$ is important. When the inclusion Y_i meets the boundary of the cell ∂Y, phenomena different from the ones described below can appear. We refer the reader to [14, Appendix A] for more details concerning the scalar problem in this case. To simplify the presentation, we assume that ε_i, ε_e, μ_i and μ_e are constant. However, we could also consider physical parameters which are elements of $\mathrm{L}^\infty(\Omega, \mathsf{R}^{3\times 3})$, the variational techniques we use below would work in a similar way. In our analysis, the following dielectric and magnetic contrasts

$$\kappa_\varepsilon := \frac{\varepsilon_i}{\varepsilon_e} < 0, \qquad \kappa_\mu := \frac{\mu_i}{\mu_e} < 0 \tag{2.1}$$

will play a key role. Let us stress that the four constants ε_e, ε_i, μ_e, μ_i are fixed once for all in the article. And when we make assumptions on the contrasts in the statements below (see in particular the final Theorem 5.6), they must be understood as "Assume that ε_e, ε_i, μ_e, μ_i are such that κ_ε, $\kappa_\mu, \ldots$". We define on the reference cell the two real-valued functions $\varepsilon, \mu \in \mathrm{L}^\infty(Y)$ such that

$$\varepsilon(y) = \varepsilon_e \, \mathbb{1}_{Y_e}(y) + \varepsilon_i \, \mathbb{1}_{Y_i}(y), \qquad \mu(y) = \mu_e \, \mathbb{1}_{Y_e}(y) + \mu_i \, \mathbb{1}_{Y_i}(y), \tag{2.2}$$

where for a set S, $\mathbb{1}_S(\,\cdot\,)$ stands for the indicator function of S. For any $\delta > 0$ and any integer vector $k \in \mathsf{Z}^3$, we define the shifted and scaled sets Y^δ_{ik}, Y^δ_{ek}, Y^δ_k such that

$$\begin{aligned} Y^\delta_{ik} &:= \{x \in \mathsf{R}^3 \,|\, (x-k)/\delta \in Y_i\} \\ Y^\delta_{ek} &:= \{x \in \mathsf{R}^3 \,|\, (x-k)/\delta \in Y_e\} \\ Y^\delta_{k} &:= \{x \in \mathsf{R}^3 \,|\, (x-k)/\delta \in Y\}. \end{aligned} \tag{2.3}$$

We denote by K^δ the set of $k \in \mathsf{Z}^3$ such that $Y^\delta_k \subset \Omega$. We assume that the metamaterial fills the region

$$\Omega^\delta_i := \bigcup_{k \in K^\delta} Y^\delta_{ik},$$

while the complementary set in Ω

$$\Omega^\delta_e = \Omega \setminus \overline{\Omega^\delta_i}$$

is occupied by the dielectric. We denote by Ω^δ the interior of $\bigcup_{k\in K^\delta} \overline{Y^\delta_k}$ and we set $\mathcal{U}^\delta := \Omega \setminus \overline{\Omega^\delta}$. We define the macroscopic dielectric permittivity ε^δ

and the magnetic permeability μ^δ on Ω such that

$$\varepsilon^\delta(x) = \varepsilon_e\, \mathbb{1}_{\Omega_e^\delta}(x) + \varepsilon_i\, \mathbb{1}_{\Omega_i^\delta}(x), \qquad \mu^\delta(x) = \mu_e\, \mathbb{1}_{\Omega_e^\delta}(x) + \mu_i\, \mathbb{1}_{\Omega_i^\delta}(x). \quad (2.4)$$

For a given frequency $\omega \neq 0$ ($\omega \in \mathsf{R}$), we study time harmonic Maxwell's equations

$$\mathbf{curl}\, \boldsymbol{E}^\delta - i\omega\, \mu^\delta \boldsymbol{H}^\delta = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{H}^\delta + i\omega\, \varepsilon^\delta \boldsymbol{E}^\delta = \boldsymbol{J} \ \text{ in } \Omega. \qquad (2.5)$$

Above $\boldsymbol{E}^\delta$ and $\boldsymbol{H}^\delta$ are respectively the electric and magnetic components of the electromagnetic field. The source term $\boldsymbol{J}$ is the current density. We suppose that the medium Ω is surrounded by a perfect conductor and we impose the boundary conditions

$$\boldsymbol{E}^\delta \times n = 0 \quad \text{and} \quad \mu^\delta \boldsymbol{H}^\delta \cdot n = 0 \ \text{ on } \partial\Omega, \qquad (2.6)$$

where n denotes the unit outward normal vector field to $\partial\Omega$. For an introduction to the mathematical setting of Maxwell's equations, we refer the reader to the classical monographs by Monk [30] or Nédélec [31]. We introduce some functional spaces classically used in the study of Maxwell's equations, namely, for $\xi \in \mathrm{L}^\infty(\Omega)$,

$$\begin{aligned}
\mathbf{L}^2(\Omega) &:= (\mathrm{L}^2(\Omega))^3\\
\mathbf{H}(\mathbf{curl}) &:= \{\boldsymbol{H} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\, \boldsymbol{H} \in \mathbf{L}^2(\Omega)\}\\
\mathbf{H}_N(\mathbf{curl}) &:= \{\boldsymbol{E} \in \mathbf{H}(\mathbf{curl}) \mid \boldsymbol{E} \times n = 0 \text{ on } \partial\Omega\}\\
\mathbf{V}_T(\xi) &:= \{\boldsymbol{H} \in \mathbf{H}(\mathbf{curl}) \mid \mathrm{div}(\xi \boldsymbol{H}) = 0,\ \xi \boldsymbol{H} \cdot n = 0 \text{ on } \partial\Omega\},\\
\mathbf{V}_N(\xi) &:= \{\boldsymbol{E} \in \mathbf{H}(\mathbf{curl}) \mid \mathrm{div}(\xi \boldsymbol{E}) = 0,\ \boldsymbol{E} \times n = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

For an open set $\mathcal{O} \subset \mathsf{R}^3$, the inner products in $\mathrm{L}^2(\mathcal{O})$ and $\mathbf{L}^2(\mathcal{O})$ are denoted indistinctly by $(\,\cdot\,,\cdot\,)_{\mathcal{O}}$ and the corresponding norm by $\|\cdot\|_{\mathcal{O}}$. To simplify, in $\mathrm{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, we just denote $(\,\cdot\,,\cdot\,)$ and $\|\cdot\|$. The space $\mathbf{H}(\mathbf{curl})$ and its subspaces $\mathbf{H}_N(\mathbf{curl})$, $\mathbf{V}_N(\xi)$, $\mathbf{V}_T(\xi)$ are endowed with the inner product

$$(\,\cdot\,,\cdot\,)_{\mathbf{curl}} := (\,\cdot\,,\cdot\,) + (\mathbf{curl}\,\cdot\,, \mathbf{curl}\,\cdot\,),$$

and the corresponding norm is denoted $\|\cdot\|_{\mathbf{curl}}$. We have the classical Green's formula for the curl operator (see for instance [30, Theorem 3.1]):

$$(\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v}) - (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v}) = 0, \qquad \forall\, \boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}),\ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}).$$

Let us recall a well-known property for the particular spaces $\mathbf{V}_T(1)$ and $\mathbf{V}_N(1)$ (cf. [4, 44]).

Proposition 2.1. — *The embeddings of $\mathbf{V}_T(1)$ in $\mathbf{L}^2(\Omega)$ and of $\mathbf{V}_N(1)$ in $\mathbf{L}^2(\Omega)$ are compact. Moreover, there is a constant $C > 0$ such that*

$$\|\boldsymbol{u}\| \leqslant C\, \|\mathbf{curl}\, \boldsymbol{u}\|, \qquad \forall\, \boldsymbol{u} \in \mathbf{V}_T(1) \cup \mathbf{V}_N(1).$$

Therefore, in $\mathbf{V}_T(1)$ and in $\mathbf{V}_N(1)$, $\|\mathbf{curl}\cdot\|_\Omega$ is a norm which is equivalent to $\|\cdot\|_{\mathbf{curl}}$.

Classically, one proves that if $(\boldsymbol{E}^\delta, \boldsymbol{H}^\delta)$ satisfies (2.5)–(2.6), then $\boldsymbol{E}^\delta$ and $\boldsymbol{H}^\delta$ are respectively solutions of the problems

$$\begin{cases} \text{Find } \boldsymbol{E}^\delta \in \mathbf{H}(\mathbf{curl}) \text{ such that:} & \\ \mathbf{curl}((\mu^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{E}^\delta) - \omega^2\varepsilon^\delta \boldsymbol{E}^\delta = i\omega \boldsymbol{J} & \text{in } \Omega \\ \boldsymbol{E}^\delta \times n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

$$\begin{cases} \text{Find } \boldsymbol{H}^\delta \in \mathbf{H}(\mathbf{curl}) \text{ such that:} & \\ \mathbf{curl}((\varepsilon^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{H}^\delta) - \omega^2\mu^\delta \boldsymbol{H}^\delta = \mathbf{curl}((\varepsilon^\delta)^{-1}\boldsymbol{J}) & \text{in } \Omega \\ \mu^\delta \boldsymbol{H}^\delta \cdot n = 0 & \text{on } \partial\Omega \\ (\varepsilon^\delta)^{-1}(\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J}) \times n = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.8}$$

We emphasize that in (2.7), (2.8), the boundary conditions are the usual ones one should impose to be able to prove well-posedness of the systems. In the following, we will focus our attention on the problem (2.7) for the electric field. The analysis for the magnetic field is quite similar. The variational formulation of (2.7) writes

$$(\mathsf{P}^\delta) \quad \begin{cases} \text{Find } \boldsymbol{E}^\delta \in \mathbf{H}_N(\mathbf{curl}) \text{ such that for all } \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}): \\ ((\mu^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{E}^\delta, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon^\delta \boldsymbol{E}^\delta, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}'). \end{cases} \tag{2.9}$$

Before studying the behaviour of solutions of (P^δ) as δ tends to zero, we must clarify the properties of this problem for a fixed $\delta > 0$. With the Riesz representation theorem, define the linear and continuous operator $\mathsf{A}_N^\delta(\omega)$: $\mathbf{H}_N(\mathbf{curl}) \to \mathbf{H}_N(\mathbf{curl})$ such that for all $\omega \in \mathsf{C}$,

$$(\mathsf{A}_N^\delta(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} = ((\mu^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon^\delta \boldsymbol{E}, \boldsymbol{E}'), \qquad \forall\, \boldsymbol{E}, \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}). \tag{2.10}$$

The features of $\mathsf{A}_N^\delta(\omega)$ are strongly related to the ones of two scalar operators that we define now. Set

$$\mathrm{H}^1_0(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega) \,|\, \varphi = 0 \text{ on } \partial\Omega\}$$
$$\mathrm{H}^1_\#(\Omega) := \left\{\varphi \in \mathrm{H}^1(\Omega) \,\middle|\, \int_\Omega \varphi\,\mathrm{d}x = 0\right\}.$$

In $\mathrm{H}^1_0(\Omega)$ and in $\mathrm{H}^1_\#(\Omega)$ (since Ω is connected), $\|\nabla \cdot\|$ is a norm which is equivalent to the usual norm of $\mathrm{H}^1(\Omega)$. We define the two linear and continuous operators $A^\delta_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ and $B^\delta_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ such that

$$(\nabla(A^\delta_\varepsilon\varphi), \nabla\varphi') = (\varepsilon^\delta\nabla\varphi, \nabla\varphi'), \quad \forall\, \varphi, \varphi' \in \mathrm{H}^1_0(\Omega)$$
$$(\nabla(B^\delta_\mu\varphi), \nabla\varphi') = (\mu^\delta\nabla\varphi, \nabla\varphi'), \quad \forall\, \varphi, \varphi' \in \mathrm{H}^1_\#(\Omega).$$

With these notations, Theorem 6.1 of [10] writes as follows.

Theorem 2.2. — *Assume that the scalar operators* $A_\varepsilon^\delta : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ *and* $B_\mu^\delta : \mathrm{H}_\#^1(\Omega) \to \mathrm{H}_\#^1(\Omega)$ *are isomorphisms. Then* $\mathsf{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}) \to \mathbf{H}_N(\mathbf{curl})$ *is an isomorphism for all* $\omega \in \mathsf{C} \setminus \mathsf{S}$ *where* S *is a discrete set with no accumulation point.*

Note that in this statement, the set S depends on the contrasts κ_ε, κ_μ but also on the geometry and hence on δ. In the next section, we give conditions ensuring that A_ε^δ and B_μ^δ are isomorphisms.

3. Uniform invertibility of the two scalar problems

We shall say that the operators $A_\varepsilon^\delta : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ and $B_\mu^\delta : \mathrm{H}_\#^1(\Omega) \to \mathrm{H}_\#^1(\Omega)$ are uniformly invertible as δ tends to zero if there is $\delta_0 > 0$ such that A_ε^δ, B_μ^δ are invertible for all $\delta \in (0;\delta_0]$ together with the estimate

$$\|(A_\varepsilon^\delta)^{-1}\| + \|(B_\mu^\delta)^{-1}\| \leqslant C,$$

where $C > 0$ is a constant which is independent of $\delta \in (0;\delta_0]$. In this section, our goal is to find criteria on κ_ε, κ_μ guaranteeing the uniform invertibility of A_ε^δ, B_μ^δ. The uniform invertibility of A_ε^δ has been considered in the articles [14, 18]. Below we combine the approaches presented in these two articles and we adapt the analysis in order to obtain a criterion ensuring the uniform invertibility of B_μ^δ.

Remark 3.1. — The result of uniform invertibility of [14, Theorem 5.2] is based on the result of Theorem 4.3 of the same article. However, its domain of validity is not completely satisfactory because the constant m defined in Theorem 4.3 is in fact equal to zero. This has been corrected by the authors and a new proof can be found in the erratum [13].

3.1. First δ-dependent criteria

In a pedagogical aim, we first derive some criteria ensuring the invertibility of A_ε^δ, B_μ^δ that are valid only for fixed δ, and hence which are not uniform.

3.1.1. Criterion of invertibility for the operator A_ε^δ

In order to get a criterion on the contrast κ_ε ensuring that $A_\varepsilon^\delta : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ is an isomorphism, we start by presenting a well-chosen decomposition of the space $\mathrm{H}_0^1(\Omega)$ which has been introduced in [14]. We recall that $\mathrm{H}_0^1(\Omega)$ is endowed with the inner product $(\nabla\cdot, \nabla\cdot)$.

LEMMA 3.2. — *We have the decomposition* $\mathrm{H}^1_0(\Omega) = \mathcal{H}^\delta_D \overset{\perp}{\oplus} \mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i)$ *where* $\mathcal{H}^\delta_D := \{\varphi \in \mathrm{H}^1_0(\Omega) \,|\, \Delta\varphi = 0 \text{ in } \Omega^\delta_e \cup \Omega^\delta_i\}$.

Remark 3.3. — The index $_D$ in the notation $\mathcal{H}^\delta_D$ stands for Dirichlet and refers to the homogeneous Dirichlet boundary condition imposed on $\partial\Omega$ to the elements of $\mathcal{H}^\delta_D$. We emphasize that the functions of $\mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i)$ vanish on $\partial\Omega^\delta_i$.

Proof. — Let φ be a given element of $\mathrm{H}^1_0(\Omega)$. Introduce $\widetilde{\varphi} \in \mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i)$ the function such that $\Delta\widetilde{\varphi} = \Delta\varphi$ in $\Omega^\delta_e \cup \Omega^\delta_i$. Then we have $\varphi = (\varphi - \widetilde{\varphi}) + \widetilde{\varphi}$ and clearly $\varphi - \widetilde{\varphi} \in \mathcal{H}^\delta_D$. Now if φ_1 and φ_2 are elements of $\mathcal{H}^\delta_D$ and $\mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i)$, a direct integration by parts gives

$$(\nabla\varphi_1, \nabla\varphi_2) = \int_{\Omega^\delta_e \cup \Omega^\delta_i} \Delta\varphi_1 \varphi_2 \,\mathrm{d}x + \int_{\partial\Omega^\delta_i} \frac{\partial\varphi_{1i}}{\partial n_i} \varphi_2 \,\mathrm{d}\sigma + \int_{\partial\Omega^\delta_i} \frac{\partial\varphi_{1e}}{\partial n_e} \varphi_2 \,\mathrm{d}\sigma = 0.$$

Here and below, $n_e = -n_i$ stands for the unit normal vector to $\partial\Omega^\delta_i$ pointing to Ω^δ_i. Moreover for $x \in \partial\Omega^\delta_i$, $\partial_n\varphi_i(x) = \lim_{t\to 0^+} \nabla\varphi(x - tn_i)\cdot n(x)$ and $\partial_n\varphi_e(x) = \lim_{t\to 0^+} \nabla\varphi(x - tn_e)\cdot n(x)$. This gives the desired result.

In what follows, some particular elements of $\mathcal{H}^\delta_D$ will play a key role. For $k \in K^\delta$, define the function $\varphi^k_D \in \mathcal{H}^\delta_D$ such that

$$\varphi^k_D = \begin{cases} 1 & \text{in } Y^\delta_{ik} \\ 0 & \text{in } Y^\delta_{ik'} \quad \text{for } k' \neq k. \end{cases} \tag{3.1}$$

Then set

$$\widehat{\mathcal{H}}^\delta_D := \{\varphi \in \mathcal{H}^\delta_D \,|\, (\nabla\varphi, \nabla\varphi^k_D) = 0, \quad \forall\, k \in K^\delta\} \tag{3.2}$$

so that we have, as in [14, Proposition 3.2],

$$\mathrm{H}^1_0(\Omega) = \widehat{\mathcal{H}}^\delta_D \overset{\perp}{\oplus} \operatorname{span}_{k\in K^\delta}\{\varphi^k_D\} \overset{\perp}{\oplus} \mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i). \tag{3.3}$$

Finally, we define the constants

$$m^\delta_D := \inf_{\varphi \in \hat{\mathcal{H}}^\delta_D \setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega^\delta_i}}{\|\nabla\varphi\|^2_{\Omega^\delta_e}}, \qquad M^\delta_D := \sup_{\varphi \in \hat{\mathcal{H}}^\delta_D \setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega^\delta_i}}{\|\nabla\varphi\|^2_{\Omega^\delta_e}}. \tag{3.4}$$

Before proceeding, let us discuss a few features of the constants m^δ_D, M^δ_D. First, observe that the functions φ^k_D satisfy

$$\|\nabla\varphi^k_D\|^2_{\Omega^\delta_i} = 0 \qquad \text{and} \qquad \|\nabla\varphi^k_D\|^2_{\Omega^\delta_e} \neq 0.$$

As a consequence, the infimum of (3.4) considered over $\mathcal{H}^\delta_D \setminus \{0\}$ is zero. On the other hand, the next lemma guarantees that the supremum of (3.4) considered over $\mathcal{H}^\delta_D \setminus \{0\}$ coincides with M^δ_D.

Lemma 3.4. — *The constant M_D^δ defined in* (3.4) *satisfies*

$$M_D^\delta = \sup_{\varphi\in\mathcal{H}_D^\delta\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}. \tag{3.5}$$

Proof. — Since $\widehat{\mathcal{H}}_D^\delta \subset \mathcal{H}_D^\delta$, clearly we have

$$M_D^\delta \leqslant \sup_{\varphi\in\mathcal{H}_D^\delta\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}. \tag{3.6}$$

Now we establish the other inequality. If $\varphi \in \mathcal{H}_D^\delta \setminus \{0\}$, we have the decomposition $\varphi = \widehat{\varphi} + \Phi$ with $\widehat{\varphi} \in \widehat{\mathcal{H}}_D^\delta$ and $\Phi \in \mathrm{span}_{k\in K^\delta}\{\varphi_D^k\}$. Since Φ is constant in each of the Y_{ik}^δ, $k \in K^\delta$, there holds

$$\|\nabla\varphi\|^2_{\Omega_i^\delta} = \|\nabla\widehat{\varphi}\|^2_{\Omega_i^\delta}. \tag{3.7}$$

As a consequence, if $\widehat{\varphi} \equiv 0$, then $0 = \|\nabla\varphi\|^2_{\Omega_i^\delta}/\|\nabla\varphi\|^2_{\Omega_e^\delta} \leqslant M_D^\delta$. If $\widehat{\varphi} \not\equiv 0$, from (3.7) and the identity $\|\nabla\varphi\|^2_\Omega = \|\nabla\widehat{\varphi}\|^2_\Omega + \|\nabla\Phi\|^2_\Omega$ (see (3.2)), we deduce that

$$\|\nabla\varphi\|^2_{\Omega_e^\delta} = \|\nabla\widehat{\varphi}\|^2_{\Omega_e^\delta} + \|\nabla\Phi\|^2_{\Omega_e^\delta} \geqslant \|\nabla\widehat{\varphi}\|^2_{\Omega_e^\delta}.$$

This implies

$$M_D^\delta \geqslant \frac{\|\nabla\widehat{\varphi}\|^2_{\Omega_i^\delta}}{\|\nabla\widehat{\varphi}\|^2_{\Omega_e^\delta}} \geqslant \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}. \tag{3.8}$$

Taking the supremum over all $\varphi \in \mathcal{H}_D^\delta \setminus \{0\}$ in (3.8), we deduce that (3.6) is also true with "$\leqslant$" replaced by "$\geqslant$". This shows (3.5).

Finally, we prove the following additional result.

Lemma 3.5. — *The constants m_D^δ, M_D^δ satisfy $0 < m_D^\delta \leqslant M_D^\delta < +\infty$.*

Proof. — By definition of m_D^δ, M_D^δ, clearly we have $m_D^\delta \leqslant M_D^\delta$. On the other hand, working by contradiction, thanks to the orthogonality conditions imposed to the elements of $\widehat{\mathcal{H}}_D^\delta$, one can show the Poincaré–Wirtinger inequality

$$\exists\, C^\delta > 0 \quad \text{such that} \quad \|\varphi\|_{\Omega_i^\delta} \leqslant C^\delta \|\nabla\varphi\|_{\Omega_i^\delta}, \qquad \forall\, \varphi \in \widehat{\mathcal{H}}_D^\delta. \tag{3.9}$$

For $\varphi \in \widehat{\mathcal{H}}_D^\delta$, since there holds $\Delta\varphi = 0$ in Ω_i^δ, from (3.9), we obtain the estimate

$$\|\nabla\varphi\|_{\Omega_i^\delta} \leqslant C^\delta \|\varphi\|_{\mathrm{H}^{1/2}(\partial\Omega_i^\delta)}.$$

Here the constant C^δ may change from one line to another. Then the continuity of the trace from $\mathrm{H}^1(\Omega_e^\delta)$ into $\mathrm{H}^{1/2}(\partial\Omega_i^\delta)$ yields the existence of a constant $C_1^\delta > 0$ such that

$$\|\nabla\varphi\|_{\Omega_i^\delta} \leqslant C_2^\delta \|\nabla\varphi\|_{\Omega_e^\delta}, \qquad \forall\, \varphi \in \widehat{\mathcal{H}}_D^\delta. \tag{3.10}$$

Similarly, using the continuity of the trace from $\mathrm{H}^1(\Omega_i^\delta)$ into $\mathrm{H}^{1/2}(\partial\Omega_i^\delta)$, we obtain that there is $C_2^\delta > 0$ such that

$$\|\nabla\varphi\|_{\Omega_e^\delta} \leqslant C_2^\delta \|\nabla\varphi\|_{\Omega_i^\delta}, \qquad \forall\, \varphi \in \widehat{\mathcal{H}}_D^\delta. \tag{3.11}$$

Estimates (3.10) and (3.11) allow one to conclude to the result of the lemma.

After these considerations, we can now establish the following criterion concerning the invertibility of A_ε^δ. To proceed, we work with the T-coercivity approach introduced in [12] (see also [20]). We emphasize however that we work with a different operator T allowing us to obtain a sharper result.

Proposition 3.6. — *Assume that* $\kappa_\varepsilon \in (-\infty; -1/m_D^\delta) \cup (-1/M_D^\delta; 0)$ *where* m_D^δ *and* M_D^δ *are defined in* (3.4). *Then* $A_\varepsilon^\delta : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ *is an isomorphism.*

Proof. — Define the operator $\mathsf{T}_D^+ : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ such that for $\varphi = \widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}$ with $\widehat{\varphi}_h \in \widehat{\mathcal{H}}_D^\delta$, $\Phi_h \in \mathrm{span}_{k\in K^\delta}\{\varphi_D^k\}$ and $\widetilde{\varphi} \in \mathrm{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$, there holds

$$\mathsf{T}_D^+\varphi = \begin{cases} \widehat{\varphi}_h + \Phi_h + \widetilde{\varphi} & \text{in } \Omega_e^\delta \\ \widehat{\varphi}_h + \Phi_h - \widetilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \tag{3.12}$$

Note that since $\widetilde{\varphi} = 0$ on $\partial\Omega_i^\delta$, the operator T_D^+ is indeed valued in $\mathrm{H}_0^1(\Omega)$. Moreover we have $\mathsf{T}_D^+ \circ \mathsf{T}_D^+ = \mathrm{Id}$ which shows that T_D^+ is an isomorphism of $\mathrm{H}_0^1(\Omega)$. For all $\varphi \in \mathrm{H}_0^1(\Omega)$, we find

$$\begin{aligned}(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^+\varphi)), \nabla\varphi) \\ = \varepsilon_e\Big(\nabla(\widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}), \nabla(\widehat{\varphi}_h + \Phi_h + \widetilde{\varphi})\Big)_{\Omega_e^\delta} + \varepsilon_i(\nabla(\widehat{\varphi}_h - \widetilde{\varphi}), \\ \nabla(\widehat{\varphi}_h + \widetilde{\varphi})\Big)_{\Omega_i^\delta}.\end{aligned} \tag{3.13}$$

Integrating by parts and using that $\widetilde{\varphi} = 0$ on $\partial\Omega_e^\delta \cup \partial\Omega_i^\delta$, we get

$$(\nabla\widehat{\varphi}_h, \nabla\widetilde{\varphi})_{\Omega_e^\delta} = (\nabla\Phi_h, \nabla\widetilde{\varphi})_{\Omega_e^\delta} = (\nabla\widehat{\varphi}_h, \nabla\widetilde{\varphi})_{\Omega_i^\delta} = 0. \tag{3.14}$$

Besides, using again that Φ_h is constant in each of the Y_{ik}^δ, from the orthogonal decomposition (3.3), we infer that

$$(\nabla\widehat{\varphi}_h, \nabla\Phi_h)_{\Omega_e^\delta} = (\nabla\widehat{\varphi}_h, \nabla\Phi_h) = 0. \tag{3.15}$$

Inserting (3.14), (3.15) in (3.13), we obtain

$$\begin{aligned}(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^+\varphi)), \nabla\varphi) \\ = (\varepsilon\nabla\widehat{\varphi}_h, \nabla\widehat{\varphi}_h) + (\varepsilon_e\nabla\Phi_h, \nabla\Phi_h)_{\Omega_e^\delta} + (|\varepsilon|\nabla\widetilde{\varphi}, \nabla\widetilde{\varphi}).\end{aligned} \tag{3.16}$$

For the first term of the right hand side of (3.16), we can write

$$\begin{aligned}(\varepsilon\nabla\widehat{\varphi}_h,\nabla\widehat{\varphi}_h) &= \varepsilon_e\|\nabla\widehat{\varphi}_h\|^2_{\Omega_e^\delta} - |\varepsilon_i|\,\|\nabla\widehat{\varphi}_h\|^2_{\Omega_i^\delta}\\ &\geqslant (\varepsilon_e - |\varepsilon_i|\,M_D^\delta)\|\nabla\widehat{\varphi}_h\|^2_{\Omega_e^\delta}\\ &\geqslant \frac{1}{2}(\varepsilon_e - |\varepsilon_i|\,M_D^\delta)(\|\nabla\widehat{\varphi}_h\|^2_{\Omega_e^\delta} + (M_D^\delta)^{-1}\|\nabla\widehat{\varphi}_h\|^2_{\Omega_i^\delta}).\end{aligned} \tag{3.17}$$

Using this estimate in (3.16), we deduce that when $\varepsilon_e > |\varepsilon_i| M_D^\delta \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/M_D^\delta$, the bilinear form $(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^+\cdot)),\nabla\cdot)$ is coercive in $\mathrm{H}_0^1(\Omega)$ (note that Lemma 3.5 guarantees that $M_D^\delta < +\infty$). With the Lax–Milgram theorem, we infer that when $\kappa_\varepsilon > -1/M_D^\delta$, the operator $A_\varepsilon^\delta \circ \mathsf{T}_D^+$ is an isomorphism of $\mathrm{H}_0^1(\Omega)$ and so is A_ε^δ.

To address the case $\kappa_\varepsilon < -1/m_D^\delta$, let us work with the operator $\mathsf{T}_D^- : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ such that

$$\mathsf{T}_D^-\varphi = \begin{cases} -\widehat{\varphi}_h + \Phi_h + \widetilde{\varphi} & \text{in } \Omega_e^\delta\\ -\widehat{\varphi}_h + \Phi_h - \widetilde{\varphi} & \text{in } \Omega_i^\delta.\end{cases} \tag{3.18}$$

We also have $\mathsf{T}_D^- \circ \mathsf{T}_D^- = \mathrm{Id}$ which guarantees that T_D^- is an isomorphism of $\mathrm{H}_0^1(\Omega)$. For all $\varphi \in \mathrm{H}_0^1(\Omega)$, we find

$$\begin{aligned}&(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^-\varphi)),\nabla\varphi)\\ &\qquad = -(\varepsilon\nabla\widehat{\varphi}_h,\nabla\widehat{\varphi}_h) + (\varepsilon_e\nabla\Phi_h,\nabla\Phi_h)_{\Omega_e^\delta} + (|\varepsilon|\nabla\widetilde{\varphi},\nabla\widetilde{\varphi}).\end{aligned} \tag{3.19}$$

This time, we can write

$$-(\varepsilon\nabla\widehat{\varphi}_h,\nabla\widehat{\varphi}_h) = -\varepsilon_e\|\nabla\widehat{\varphi}_h\|^2_{\Omega_e^\delta} + |\varepsilon_i|\,\|\nabla\widehat{\varphi}_h\|^2_{\Omega_i^\delta} \geqslant (-\varepsilon_e + |\varepsilon_i|\,m_D^\delta)\|\nabla\widehat{\varphi}_h\|^2_{\Omega_e^\delta}$$

that is

$$-(\varepsilon\nabla\widehat{\varphi}_h,\nabla\widehat{\varphi}_h) \geqslant \frac{1}{2}(-\varepsilon_e + |\varepsilon_i|\,m_D^\delta)(\|\nabla\widehat{\varphi}_h\|^2_{\Omega_e^\delta} + (M_D^\delta)^{-1}\|\nabla\widehat{\varphi}_h\|^2_{\Omega_i^\delta}). \tag{3.20}$$

As a consequence, we see from (3.19) that when $|\varepsilon_i| m_D^\delta > \varepsilon_e$, or equivalently for $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e < -1/m_D^\delta$, the bilinear form $(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^-\cdot)),\nabla\cdot)$ is coercive in $\mathrm{H}_0^1(\Omega)$ (here we used once again the result of Lemma 3.5 ensuring that $0 < m_D^\delta \leqslant M_D^\delta < +\infty$). We can conclude as above that when $\kappa_\varepsilon < -1/m_D^\delta$, the operator A_ε^δ is an isomorphism of $\mathrm{H}_0^1(\Omega)$.

3.1.2. Criterion of invertibility for the operator B^δ_μ

Now we show similar results for the operator $B^\delta_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$. First, define the space

$$\mathrm{H}^1_\diamond(\Omega) := \left\{ \varphi \in \mathrm{H}^1(\Omega) \,\middle|\, \int_{\partial\Omega^\delta_i} \varphi \, \mathrm{d}\sigma = 0 \right\}.$$

Lemma 3.7. — *We have the decomposition* $\mathrm{H}^1_\diamond(\Omega) = \mathcal{H}^\delta_N \overset{\perp}{\oplus} \mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega)$ *where* $\mathcal{H}^\delta_N := \{\varphi \in \mathrm{H}^1_\diamond(\Omega) \,|\, \Delta\varphi = 0$ *in* $\Omega^\delta_e \cup \Omega^\delta_i$, $\partial_n\varphi = 0$ *on* $\partial\Omega\}$ *and* $\mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega) \,|\, \varphi = 0$ *on* $\partial\Omega^\delta_i\} \subset \mathrm{H}^1_\diamond(\Omega)$.

Remark 3.8. — This time, the index $_N$ in the notation $\mathcal{H}^\delta_N$ stands for Neumann and refers to the homogeneous Neumann boundary condition imposed on $\partial\Omega$ to the elements of $\mathcal{H}^\delta_N$.

Proof. — For φ given in $\mathrm{H}^1_\diamond(\Omega)$, introduce $\widetilde{\varphi} \in \mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega)$ the function such that

$$(\nabla\widetilde{\varphi}, \nabla\varphi') = (\nabla\varphi, \nabla\varphi'), \qquad \forall\, \varphi' \in \mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega).$$

Note that since the Poincaré inequality holds in the space $\mathrm{H}^1_{0,\partial\Omega^\delta_i}(\Omega)$, the Lax–Milgram theorem indeed guarantees that this variational problem admits a unique solution. Then we have $\varphi = (\varphi - \widetilde{\varphi}) + \widetilde{\varphi}$ and one can check that $\varphi - \widetilde{\varphi}$ belongs to $\mathcal{H}^\delta_N$. Finally if φ_1 and φ_2 are elements of $\mathcal{H}^\delta_N$ and $\mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega)$, a direct integration by parts gives $(\nabla\varphi_1, \nabla\varphi_2) = 0$.

In what follows, some particular elements of $\mathcal{H}^\delta_N$ will play a key role. Let k_0 be an arbitrary given element of K^δ and for $k \in K^\delta \setminus \{k_0\}$, define the function $\varphi^k_N \in \mathcal{H}^\delta_N$ such that

$$\varphi^k_N = \begin{cases} 1 & \text{in } Y^\delta_{ik} \\ -1 & \text{in } Y^\delta_{ik_0} \\ 0 & \text{in } Y^\delta_{ik'} \quad \text{for } k' \in K^\delta \setminus \{k_0, k\}. \end{cases}$$

Then set

$$\widehat{\mathcal{H}}^\delta_N := \{\varphi \in \mathcal{H}^\delta_N \,|\, (\nabla\varphi, \nabla\varphi^k_N) = 0, \quad \forall\, k \in K^\delta \setminus \{k_0\}\}$$

so that we have

$$\mathrm{H}^1_\diamond(\Omega) = \widehat{\mathcal{H}}^\delta_N \overset{\perp}{\oplus} \mathrm{span}_{k\in K^\delta\setminus\{k_0\}}\{\varphi^k_N\} \overset{\perp}{\oplus} \mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega).$$

We emphasize that the choice of k_0 above does not affect this decomposition. We simply consider one particular basis for the space $\mathrm{span}_{k\in K^\delta\setminus\{k_0\}}\{\varphi^k_N\}$.

Finally, we define the constants

$$m_N^\delta := \inf_{\varphi\in\hat{\mathcal{H}}_N^\delta\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}, \qquad M_N^\delta := \sup_{\varphi\in\hat{\mathcal{H}}_N^\delta\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}. \tag{3.21}$$

Working as in the proof of Lemma 3.5, in particular establishing by contradiction the Poincaré–Wirtinger inequality

$$\exists\, C^\delta > 0 \quad \text{such that} \quad \|\varphi\|_{\Omega_i^\delta} \leqslant C^\delta\|\nabla\varphi\|_{\Omega_i^\delta}, \qquad \forall\,\varphi\in\widehat{\mathcal{H}}_N^\delta,$$

one can show that there holds $0 < m_N^\delta \leqslant M_N^\delta < +\infty$. As in (3.4), the functions φ_N^k satisfy

$$\|\nabla\varphi_N^k\|^2_{\Omega_i^\delta} = 0 \qquad \text{and} \qquad \|\nabla\varphi_N^k\|^2_{\Omega_e^\delta} \neq 0$$

so that the infimum of (3.21) considered over $\mathcal{H}_N^\delta\setminus\{0\}$ is zero. Working exactly as in the proof of Lemma 3.4, we get the following result.

Lemma 3.9. — *The constant M_N^δ defined in* (3.21) *satisfies*

$$M_N^\delta = \sup_{\varphi\in\mathcal{H}_N^\delta\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}. \tag{3.22}$$

Now, we give our criterion of invertibility for the operator B_μ^δ.

Proposition 3.10. — *Assume that $\kappa_\mu \in (-\infty;-1/m_N^\delta)\cup(-1/M_N^\delta;0)$ where m_N^δ and M_N^δ are defined in* (3.21)*. Then $B_\mu^\delta : \mathrm{H}^1_\#(\Omega)\to\mathrm{H}^1_\#(\Omega)$ is an isomorphism.*

Proof. — Introduce the mappings $\ell_\diamond : \mathrm{H}^1_\#(\Omega)\to\mathrm{H}^1_\diamond(\Omega)$ and $\ell_\# : \mathrm{H}^1_\diamond(\Omega)\to\mathrm{H}^1_\#(\Omega)$ such that

$$\ell_\diamond(\varphi) = \varphi - \frac{1}{|\partial\Omega_i^\delta|}\int_{\partial\Omega_i^\delta}\varphi\,\mathrm{d}\sigma, \qquad \ell_\#(\varphi) = \varphi - \frac{1}{|\Omega|}\int_\Omega\varphi\,\mathrm{d}x.$$

Here and in what follows, for an open set $\mathcal{O}\subset\mathsf{R}^3$, we denote by $|\mathcal{O}| = \int_{\mathcal{O}} 1\,\mathrm{d}x$ and $|\partial\mathcal{O}| = \int_{\partial\mathcal{O}} 1\,\mathrm{d}\sigma$. Then define the operators $\widetilde{\mathsf{T}}_N^\pm : \mathrm{H}^1_\diamond(\Omega)\to\mathrm{H}^1_\diamond(\Omega)$ such that for $\varphi = \widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}$ with $\widehat{\varphi}_h\in\widehat{\mathcal{H}}_N^\delta$, $\Phi_h\in\operatorname{span}_{k\in K^\delta\setminus\{k_0\}}\{\varphi_N^k\}$ and $\widetilde{\varphi}\in\mathrm{H}^1_{0,\,\partial\Omega_i^\delta}(\Omega)$, there holds

$$\widetilde{\mathsf{T}}_N^\pm\varphi = \begin{cases} \pm\widehat{\varphi}_h + \Phi_h + \widetilde{\varphi} & \text{in } \Omega_e^\delta \\ \pm\widehat{\varphi}_h + \Phi_h - \widetilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \tag{3.23}$$

Finally, we define the operators

$$\mathsf{T}_N^\pm := \ell_\#\circ\widetilde{\mathsf{T}}_N^\pm\circ\ell_\diamond.$$

For $\psi \in \mathrm{H}^1_{\#}(\Omega)$, we set $\varphi := \ell_\diamond(\psi) \in \mathrm{H}^1_\diamond(\Omega)$ and we use the notation $\varphi = \widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}$ with $\widehat{\varphi}_h \in \widehat{\mathcal{H}}^\delta_N$, $\Phi_h \in \mathrm{span}_{k \in K^\delta \setminus \{k_0\}}\{\varphi^k_N\}$ and $\widetilde{\varphi} \in \mathrm{H}^1_{0,\,\partial\Omega^\delta_i}(\Omega)$. Observing that $\nabla\varphi = \nabla\psi$ (φ and ψ differ from each other by an additive constant) and working as in (3.16), we find

$$\begin{aligned}
&(\nabla(B^\delta_\mu(\mathsf{T}^\pm_N\psi)), \nabla\psi) \\
&\qquad = \pm(\mu\nabla\widehat{\varphi}_h, \nabla\widehat{\varphi}_h) + (\mu_e\nabla\Phi_h, \nabla\Phi_h)_{\Omega^\delta_e} + (|\mu|\nabla\widetilde{\varphi}, \nabla\widetilde{\varphi}).
\end{aligned} \tag{3.24}$$

For the first term of the right hand side of (3.24), we can write

$$\begin{aligned}
(\mu\nabla\widehat{\varphi}_h, \nabla\widehat{\varphi}_h) &= \mu_e\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_e} - |\mu_i|\,\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_i} \\
&\geqslant (\mu_e - |\mu_i|\,M^\delta_N)\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_e} \\
&\geqslant \frac{1}{2}(\mu_e - |\mu_i|\,M^\delta_N)(\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_e} + (M^\delta_N)^{-1}\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_i})
\end{aligned}$$

and

$$\begin{aligned}
-(\mu\nabla\widehat{\varphi}_h, \nabla\widehat{\varphi}_h) &= -\mu_e\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_e} + |\mu_i|\,\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_i} \\
&\geqslant (-\mu_e + |\mu_i|\,m^\delta_N)\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_e} \\
&\geqslant \frac{1}{2}(-\mu_e + |\mu_i|\,m^\delta_N)(\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_e} + (M^\delta_N)^{-1}\|\nabla\widehat{\varphi}_h\|^2_{\Omega^\delta_i}).
\end{aligned} \tag{3.25}$$

Using again that $\nabla\varphi = \nabla\psi$, we deduce from the first estimate of (3.24) that when $\mu_e > |\mu_i| M^\delta_N \Leftrightarrow \kappa_\mu = \mu_i/\mu_e > -1/M^\delta_N$, the bilinear form $(\nabla(B^\delta_\mu(\mathsf{T}^+_N\cdot)), \nabla\cdot)$ is coercive in $\mathrm{H}^1_{\#}(\Omega)$. With the Lax–Milgram theorem, we infer that when $\kappa_\mu > -1/M^\delta_N$, the operator $B^\delta_\mu \circ \mathsf{T}^+_N$ is an isomorphism of $\mathrm{H}^1_{\#}(\Omega)$. Since B^δ_μ is selfadjoint (because it is bounded and symmetric), this implies that B^δ_μ is an isomorphism. Working similarly with T^-_N, from (3.25) one finds that when $|\mu_i| m^\delta_N > \mu_e \Leftrightarrow \kappa_\mu = \mu_i/\mu_e < -1/m^\delta_N$, the operator B^δ_μ is an isomorphism. Note that with additional few lines, one can check that we have $\mathsf{T}^\pm_N \circ \mathsf{T}^\pm_N = \mathrm{Id}$.

3.2. Comparison between the criteria of invertibility

In this section, we compare the constants involved in the criteria ensuring the invertibility of the operators A^δ_ε (Dirichlet) and B^δ_μ (Neumann).

Proposition 3.11. — *For all $\delta > 0$, the constants m^δ_D, M^δ_D defined in* (3.4) *and the constants m^δ_N, M^δ_N defined in* (3.21) *satisfy*

$$m^\delta_D \leqslant m^\delta_N \qquad \textit{and} \qquad M^\delta_D \leqslant M^\delta_N. \tag{3.26}$$

Proof. — We start by proving the second inequality of (3.26). Let φ be an element of $\widehat{\mathcal{H}}_D^\delta \setminus \{0\}$. Define the function $\zeta \in \mathcal{H}_N^\delta$ such that $\zeta = \varphi - c$ on $\partial\Omega_i^\delta$ where $c = |\partial\Omega_i^\delta|^{-1} \int_{\partial\Omega_i^\delta} \varphi \, \mathrm{d}\sigma$. In other words, ζ is the function such that $\Delta\zeta = 0$ in $\Omega_e^\delta \cup \Omega_i^\delta$, $\zeta = \varphi - c$ on $\partial\Omega_i^\delta$ and $\partial_n\zeta = 0$ on $\partial\Omega$. Note that necessarily, there holds $\zeta \not\equiv 0$. Then we have $\zeta = \varphi - c$ in Ω_i^δ and so

$$\|\nabla\zeta\|_{\Omega_i^\delta} = \|\nabla\varphi\|_{\Omega_i^\delta}. \tag{3.27}$$

On the other hand, integrating by parts, we find

$$\begin{aligned}(\nabla\zeta, \nabla(\zeta - \varphi))_{\Omega_e^\delta} &= (\nabla\zeta, \nabla(\zeta - (\varphi - c)))_{\Omega_e^\delta} \\ &= \int_{\partial\Omega} \frac{\partial\zeta}{\partial n}(\zeta - (\varphi - c)) \, \mathrm{d}\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\zeta_e}{\partial n_e}(\zeta - (\varphi - c)) \, \mathrm{d}\sigma \\ &= 0.\end{aligned}$$

We deduce that

$$\|\nabla\zeta\|^2_{\Omega_e^\delta} \leqslant \|\nabla\varphi\|^2_{\Omega_e^\delta}. \tag{3.28}$$

Gathering (3.27), (3.28) and using Lemma 3.9, we infer that

$$\frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}} \leqslant \frac{\|\nabla\zeta\|^2_{\Omega_i^\delta}}{\|\nabla\zeta\|^2_{\Omega_e^\delta}} \leqslant M_N^\delta. \tag{3.29}$$

Taking the supremum over all $\varphi \in \widehat{\mathcal{H}}_D^\delta \setminus \{0\}$ in (3.29), we obtain that $M_D^\delta \leqslant M_N^\delta$.

Now we show the first inequality of (3.26). Let φ be an element of $\widehat{\mathcal{H}}_N^\delta \setminus \{0\}$. Define the function $\zeta \in \mathcal{H}_D^\delta$ such that $\zeta = \varphi$ on $\partial\Omega_i^\delta$. In particular, we have $\Delta\zeta = 0$ in $\Omega_e^\delta \cup \Omega_i^\delta$ and $\zeta = 0$ on $\partial\Omega$. Then decompose ζ as $\zeta = \widehat{\zeta} + Z$ with $\widehat{\zeta} \in \widehat{\mathcal{H}}_D^\delta$ and $Z \in \mathrm{span}_{k\in K^\delta}\{\varphi_D^k\}$. Since Z is constant in each of the Y_{ik}^δ, $k \in K^\delta$, we have

$$\|\nabla\widehat{\zeta}\|_{\Omega_i^\delta} = \|\nabla\varphi\|_{\Omega_i^\delta}. \tag{3.30}$$

On the other hand, integrating by parts, we find

$$\begin{aligned}(\nabla\varphi, \nabla(\widehat{\zeta} - \varphi))_{\Omega_e^\delta} &= \int_{\partial\Omega} \frac{\partial\varphi}{\partial n}(\widehat{\zeta} - \varphi) \, \mathrm{d}\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\varphi_e}{\partial n_e}(\widehat{\zeta} - \varphi) \, \mathrm{d}\sigma \\ &= -\int_{\partial\Omega_i^\delta} \frac{\partial\varphi_e}{\partial n_e} Z \, \mathrm{d}\sigma.\end{aligned} \tag{3.31}$$

Since the function φ is in $\widehat{\mathcal{H}}_N^\delta$, for all $k \in K^\delta \setminus \{k_0\}$, we have $(\nabla\varphi, \nabla\varphi_N^k) = 0$. Integrating by parts, this implies

$$\int_{\partial Y_{ik}^\delta} \frac{\partial\varphi_e}{\partial n_e} \, \mathrm{d}\sigma = \int_{\partial Y_{ik_0}^\delta} \frac{\partial\varphi_e}{\partial n_e} \, \mathrm{d}\sigma.$$

But we also have $\int_{\partial\Omega_i^\delta} \partial_{n_e}\varphi_e \,\mathrm{d}\sigma = 0$. As a consequence, we must have, for all $k \in K^\delta$,

$$\int_{\partial Y_{ik}^\delta} \frac{\partial\varphi_e}{\partial n_e}\,\mathrm{d}\sigma = 0.$$

Since Z is constant on each of the ∂Y_{ik}^δ, we deduce that the terms of the equalities of (3.31) are equal to zero. Hence, there holds

$$\|\nabla\varphi\|^2_{\Omega_e^\delta} \leqslant \|\nabla\widehat{\zeta}\|^2_{\Omega_e^\delta}. \tag{3.32}$$

Gathering (3.30) and (3.32) leads to

$$m_D^\delta \leqslant \frac{\|\nabla\widehat{\zeta}\|^2_{\Omega_i^\delta}}{\|\nabla\widehat{\zeta}\|^2_{\Omega_e^\delta}} \leqslant \frac{\|\nabla\varphi\|^2_{\Omega_i^\delta}}{\|\nabla\varphi\|^2_{\Omega_e^\delta}}. \tag{3.33}$$

Taking the infimum over all $\varphi \in \widehat{\mathcal{H}}_N^\delta \setminus \{0\}$ in (3.33), we obtain that $m_D^\delta \leqslant m_N^\delta$.

3.3. Uniform criterion of invertibility

The bounds on the contrasts κ_ε, κ_μ that we obtained in Propositions 3.6, 3.10 which ensure the invertibility of the scalar operators A_ε^δ and B_μ^δ, depend on δ. In this paragraph, we wish to get bounds which are uniform with respect to δ.

Introduce the Hilbert spaces of functions defined in the reference cell Y

$$\begin{aligned}\mathcal{H}_0 &:= \{\varphi \in \mathrm{H}^1_0(Y) \,|\, \Delta\varphi = 0 \text{ in } Y_e \cup Y_i\}\\ \mathcal{H}_\diamond &:= \{\varphi \in \mathrm{H}^1_\diamond(Y) \,|\, \Delta\varphi = 0 \text{ in } Y_e \cup Y_i\}\end{aligned}$$

where $\mathrm{H}^1_\diamond(Y) := \{\varphi \in \mathrm{H}^1(Y) \,|\, \int_{\partial Y_i} \varphi\,\mathrm{d}\sigma = 0\}$. Define the function $\varphi_D \in \mathcal{H}_0$ such that $\varphi_D = 1$ in Y_i and set

$$\begin{aligned}\widehat{\mathcal{H}}_0 &:= \{\varphi \in \mathcal{H}_0 \,|\, (\nabla\varphi, \nabla\varphi_D) = 0\}\\ \widehat{\mathcal{H}}_\diamond &:= \{\varphi \in \mathcal{H}_\diamond \,|\, \partial_n\varphi = 0 \text{ on } \partial Y\}.\end{aligned} \tag{3.34}$$

Then we introduce the constants

$$m := \inf_{\varphi\in\hat{\mathcal{H}}_0\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}}, \qquad M := \sup_{\varphi\in\hat{\mathcal{H}}_\diamond\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}}. \tag{3.35}$$

We emphasize that m and M are independent of δ.

Lemma 3.12. — *The constant M defined in* (3.35) *satisfies*

$$M = \sup_{\varphi \in \mathcal{H}_\diamond \setminus \{0\}} \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}} \tag{3.36}$$

(here the sup is considered over $\mathcal{H}_\diamond \setminus \{0\}$ and not $\widehat{\mathcal{H}}_\diamond \setminus \{0\}$).

Proof. — Since there holds $\widehat{\mathcal{H}}_\diamond \subset \mathcal{H}_\diamond$, it suffices to show that

$$\sup_{\varphi \in \mathcal{H}_\diamond \setminus \{0\}} \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}} \leqslant M. \tag{3.37}$$

Let φ be a non zero element of $\mathcal{H}_\diamond$. We have the decomposition $\varphi = \widehat{\varphi} + (\varphi - \widehat{\varphi})$ where $\widehat{\varphi} \in \widehat{\mathcal{H}}_\diamond$ is the function such that $\widehat{\varphi} = \varphi$ in Y_i, $\Delta\widehat{\varphi} = 0$ in Y_e, $\widehat{\varphi} = \varphi$ on ∂Y_i and $\partial_n \widehat{\varphi} = 0$ on ∂Y. Observing that $\|\nabla\varphi\|^2_{Y_i} = \|\nabla\widehat{\varphi}\|^2_{Y_i}$ and that

$$\|\nabla\varphi\|^2_{Y_e} = \|\nabla\widehat{\varphi}\|^2_{Y_e} + \|\nabla(\varphi - \widehat{\varphi})\|^2_{Y_e} \geqslant \|\nabla\widehat{\varphi}\|^2_{Y_e},$$

we can write

$$\|\nabla\varphi\|^2_{Y_i} = \|\nabla\widehat{\varphi}\|^2_{Y_i} \leqslant M\,\|\nabla\widehat{\varphi}\|^2_{Y_e} \leqslant M\,\|\nabla\varphi\|^2_{Y_e}.$$

Taking the supremum over all $\varphi \in \mathcal{H}_\diamond \setminus \{0\}$ leads to (3.37).

Lemma 3.13. — *For all $\delta > 0$, we have the relations*

$$m \leqslant m^\delta_D \leqslant m^\delta_N \qquad \textit{and} \qquad M^\delta_D \leqslant M^\delta_N \leqslant M, \tag{3.38}$$

where m^δ_D, M^δ_D are defined in (3.4)*, m^δ_N, M^δ_N are defined in* (3.21) *and m, M are defined in* (3.35)*.*

Proof. — From Proposition 3.11, we know that we have $m^\delta_D \leqslant m^\delta_N$ and $M^\delta_D \leqslant M^\delta_N$. Now we show that we have $M^\delta_N \leqslant M$. Let φ be a non zero element of $\widehat{\mathcal{H}}^\delta_N$. For all $k \in K^\delta$, we define the function $\varphi^\delta_k \in \mathrm{H}^1(Y)$ such that $\varphi^\delta_k(y) = \varphi(\delta(k+y))$ for $y \in Y$ and we set $c_k := |\partial Y|^{-1} \int_{\partial Y} \varphi^\delta_k \,\mathrm{d}\sigma$. Since $\varphi^\delta_k - c_k \in \mathcal{H}_\diamond$, using Lemma 3.12, we can write

$$\begin{aligned} \|\nabla\varphi\|^2_{Y^\delta_{ik}} = \delta\|\nabla\varphi^\delta_k\|^2_{Y_i} = \delta\|\nabla(\varphi^\delta_k - c_k)\|^2_{Y_i} &\leqslant \delta\, M\,\|\nabla(\varphi^\delta_k - c_k)\|^2_{Y_e} \\ &\leqslant \delta\, M\,\|\nabla\varphi^\delta_k\|^2_{Y_e} \leqslant M\,\|\nabla\varphi\|^2_{Y^\delta_{ek}}. \end{aligned} \tag{3.39}$$

Summing these estimates over all $k \in K^\delta$, we get (recall that $\mathcal{U}^\delta = \Omega \setminus \overline{\Omega^\delta}$)

$$\|\nabla\varphi\|^2_{\Omega^\delta_i} \leqslant M\,\|\nabla\varphi\|^2_{\Omega^\delta_e \setminus \overline{\mathcal{U}^\delta}} \leqslant M\,\|\nabla\varphi\|^2_{\Omega^\delta_e}. \tag{3.40}$$

Taking the supremum in (3.40) over all $\varphi \in \widehat{\mathcal{H}}^\delta_N$, we deduce that $M^\delta_N \leqslant M$.

To establish (3.38), it remains to show that $m \leqslant m_D^\delta$. For φ given in $\widehat{\mathcal{H}}_D^\delta$, introduce the function $v \in \mathrm{H}_0^1(\Omega)$ such that for all $k \in K^\delta$,

$$\left\{\begin{array}{rl} v = \varphi & \text{in } Y_{ik}^\delta \\ \Delta v = 0 & \text{in } Y_{ek}^\delta \\ v = 0 & \text{on } \partial Y_{ek}^\delta. \end{array}\right.$$

We also impose $v = 0$ in $\mathcal{U}^\delta = \Omega \setminus \overline{\Omega^\delta}$. For all $k \in K^\delta$, define the function $\phi_D^k \in \mathrm{H}_0^1(\Omega)$ such that

$$\left\{\begin{array}{rl} \phi_D^k = 1 & \text{in } Y_{ik}^\delta \\ \Delta \phi_D^k = 0 & \text{in } Y_{ek}^\delta \\ \phi_D^k = 0 & \text{in } \Omega \setminus Y_k^\delta. \end{array}\right.$$

Then set

$$\widetilde{v} := v - \sum_{k \in K^\delta} a_k \phi_D^k \qquad \text{with } a_k := (\nabla v, \nabla \phi_D^k)/\|\nabla \phi_D^k\|^2.$$

Integrating by parts, we find

$$\begin{aligned}(\nabla\varphi, \nabla(\varphi - \widetilde{v}))_{\Omega_e^\delta} &= \int_{\partial\Omega} \frac{\partial\varphi}{\partial n}(\varphi - \widetilde{v})\,\mathrm{d}\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\varphi_e}{\partial n_e}(\varphi - \widetilde{v})\,\mathrm{d}\sigma \\ &= \sum_{k \in K^\delta} a_k \int_{\partial Y_{ik}^\delta} \frac{\partial\varphi_e}{\partial n_e} \phi_D^k\,\mathrm{d}\sigma. \end{aligned} \tag{3.41}$$

Since the function φ is in $\widehat{\mathcal{H}}_D^\delta$, for all $k \in K^\delta$, we have $(\nabla\varphi, \nabla\varphi_D^k) = 0$. Integrating by parts, this implies

$$\int_{\partial Y_{ik}^\delta} \frac{\partial\varphi_e}{\partial n_e}\,\mathrm{d}\sigma = 0.$$

Using that ϕ_D^k is constant on the $\partial Y_{ik'}^\delta$, we deduce from (3.41) that $(\nabla\varphi, \nabla(\varphi - \widetilde{v}))_{\Omega_e^\delta} = 0$. Hence, we have

$$\|\nabla\varphi\|_{\Omega_e^\delta}^2 \leqslant \|\nabla\widetilde{v}\|_{\Omega_e^\delta}^2 = \sum_{k \in K^\delta} \|\nabla\widetilde{v}\|_{Y_{ek}^\delta}^2. \tag{3.42}$$

For $k \in K^\delta$, define the function $\widetilde{v}_k^\delta \in \mathrm{H}^1(Y)$ such that $\widetilde{v}_k^\delta(y) = \widetilde{v}(\delta(k+y))$ for $y \in Y$. Observe that we have $\widetilde{v}_k^\delta \in \widehat{\mathcal{H}}_0$ so that we can write

$$\begin{aligned}\|\nabla\widetilde{v}\|_{Y_{ek}^\delta}^2 = \delta\|\nabla\widetilde{v}_k^\delta\|_{Y_e}^2 &\leqslant \delta\, m^{-1}\,\|\nabla\widetilde{v}_k^\delta\|_{Y_i}^2 \\ &\leqslant \delta\, m^{-1}\,\|\nabla\varphi_k^\delta\|_{Y_i}^2 \leqslant m^{-1}\,\|\nabla\varphi\|_{Y_{ik}^\delta}^2. \end{aligned} \tag{3.43}$$

As a consequence, inserting (3.43) in (3.42), we obtain

$$\|\nabla\varphi\|_{\Omega_e^\delta}^2 \leqslant m^{-1}\,\|\nabla\varphi\|_{\Omega_i^\delta}^2. \tag{3.44}$$

Taking the infimum in (3.44) over all $\varphi \in \widehat{\mathcal{H}}_D^\delta$, we deduce that $m \leqslant m_D^\delta$.

Finally, we deduce a criterion of uniform invertibility for the operators A^δ_ε and B^δ_μ.

THEOREM 3.14. — *Let m, M be the constants defined in* (3.35).

When $\kappa_\varepsilon \in (-\infty;-1/m)\cup(-1/M;0)$, $A^\delta_\varepsilon : \mathrm{H}^1_0(\Omega)\to\mathrm{H}^1_0(\Omega)$ is uniformly invertible as $\delta\to 0$.

When $\kappa_\mu \in (-\infty;-1/m)\cup(-1/M;0)$, $B^\delta_\mu : \mathrm{H}^1_\#(\Omega)\to\mathrm{H}^1_\#(\Omega)$ is uniformly invertible as $\delta\to 0$.

Proof. — Let us show the result for A^δ_ε, the proof is completely similar for B^δ_μ. From the decomposition of the space $\mathrm{H}^1_0(\Omega)$ in (3.3), one observes that the operators $\mathsf{T}^\pm_D = (\mathsf{T}^\pm_D)^{-1}$ defined in (3.12) and (3.18) are uniformly continuous. From the estimate (3.17) (resp. (3.20)) together with the result of Lemma 3.13, one infers that as $\delta\to 0$, $(\nabla(A^\delta_\varepsilon(\mathsf{T}^+_D\cdot)),\nabla\cdot)$ (resp. $(\nabla(A^\delta_\varepsilon(\mathsf{T}^-_D\cdot)),\nabla\cdot)$) is uniformly coercive in $\mathrm{H}^1_0(\Omega)$ when $\kappa_\varepsilon > -1/M$ (resp. when $\kappa_\varepsilon < -1/m$). Since A^δ_ε is also uniformly continuous, this is enough to guarantee that A^δ_ε is uniformly invertible as δ tends to zero.

3.4. Optimality of the criterion and connection to the Neumann–Poincaré operator

Let us discuss the criterion we have obtained above. We focus our attention on the analysis for the operator A^δ_ε, similar comments can be made for the operator B^δ_μ. We assume in this paragraph that ∂Y_i, and so $\partial\Omega^\delta_i$, is of class $\mathcal{C}^2$. Note that this assumption is important to ensure that the spectrum of Problem (3.46) below is discrete. It has been proved in [9] that in this case, A^δ_ε is Fredholm of index zero when $\kappa_\varepsilon \neq -1$. Therefore when $\kappa_\varepsilon \neq -1$, the operator A^δ_ε is an isomorphism if and only if it is injective. As it has been observed in different works (see in particular [14]), and as we recall below, the question of the injectivity of A^δ_ε is directly linked to the spectrum of the so-called Neumann–Poincaré operator. The latter has been widely studied when Ω is the whole space R^d. For this problem, among the references, let us cite [1, 15, 16, 25, 26, 27, 29, 35, 36, 37, 38]. Below, we use a symmetrization argument similar to the one used in [29]. We work with Dirichlet-to-Neumann maps following the approach of [25].

3.4.1. Spectrum of the Neumann–Poincaré operator

Set $\Sigma^\delta := \partial\Omega^\delta_i$ and introduce the two Dirichlet-to-Neumann operators $\Lambda_e : \mathrm{H}^{1/2}(\Sigma^\delta)\to\mathrm{H}^{-1/2}(\Sigma^\delta)$, $\Lambda_i : \mathrm{H}^{1/2}(\Sigma^\delta)\to\mathrm{H}^{-1/2}(\Sigma^\delta)$ such that for all $\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta)$, we have $\Lambda_e\varphi = \partial_{n_e}u_e$, $\Lambda_i\varphi = \partial_{n_i}u_i$ where u_e, u_i solve respectively the problems

$$\begin{cases} \Delta u_e = 0 & \text{in } \Omega_e^\delta \\ u_e = 0 & \text{on } \partial\Omega \\ u_e = \varphi & \text{on } \Sigma^\delta \end{cases} \qquad \begin{cases} \Delta u_i = 0 & \text{in } \Omega_i^\delta \\ u_i = \varphi & \text{on } \Sigma^\delta. \end{cases} \tag{3.45}$$

Define also the lifting operator $\mathcal{R} : \mathrm{H}^{1/2}(\Sigma^\delta) \to \mathrm{H}^1_0(\Omega)$ such that $\mathcal{R}\varphi = u_e$ in Ω_e^δ, $\mathcal{R}\varphi = u_i$ in Ω_i^δ, where u_e, u_i are the solutions to (3.45).

If u belongs to $\ker A_\varepsilon^\delta \setminus \{0\}$, then $\varphi := u|_{\Sigma^\delta} \in \mathrm{H}^{1/2}(\Sigma^\delta) \setminus \{0\}$ satisfies $\Lambda_e\varphi = -\kappa_\varepsilon\Lambda_i\varphi$. By a straightforward computation, we find that the pair (α, φ), with $\alpha := (\kappa_\varepsilon + 1)/(\kappa_\varepsilon - 1) \in (-1; 1)$, is a solution to the generalized eigenvalue problem

$$\begin{cases} \text{Find } (\alpha, \varphi) \in \mathsf{R} \times (\mathrm{H}^{1/2}(\Sigma^\delta) \setminus \{0\}) \text{ such that:} \\ \Lambda_-\varphi = \alpha\Lambda_+\varphi \end{cases} \tag{3.46}$$

with $\Lambda_\pm := \Lambda_e \pm \Lambda_i$. Reciprocally, assume that (α, φ) is a solution to (3.46) with $\alpha \in (-1; 1)$. Then, $\mathcal{R}\varphi \in \mathrm{H}^1_0(\Omega)$ is an element of $\ker A_\varepsilon^\delta \setminus \{0\}$ for $\kappa_\varepsilon = (\alpha + 1)/(\alpha - 1) \in (-\infty; 0)$. This shows that it is sufficient to determine the eigenvalues of problem (3.46) to study the injectivity of A_ε^δ. Note that the spectrum of (3.46) coincides with the spectrum of the so called Neumann–Poincaré operator studied for example in [29].

Theorem 3.15. — *The spectrum of the generalized eigenvalue problem* (3.46) *is discrete and coincides with two sequences of real numbers*

$$-1 < \alpha_1^- \leqslant \alpha_2^- \leqslant \cdots \leqslant 0$$

and

$$1 = \alpha_1^+ = \cdots = \alpha^+_{\mathrm{card}(K^\delta)} > \alpha^+_{\mathrm{card}(K^\delta)+1} \geqslant \cdots \geqslant 0$$

such that $\lim_{n\to+\infty} \alpha_n^\pm = 0$. *Here* $\mathrm{card}(K^\delta)$ *is the cardinal of the set* K^δ *defined after* (2.3).

Proof. — First, we show that $\Lambda_+ : \mathrm{H}^{1/2}(\Sigma^\delta) \to \mathrm{H}^{-1/2}(\Sigma^\delta)$ is an isomorphism. Consider some $\psi \in \mathrm{H}^{-1/2}(\Sigma^\delta)$. If $\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta)$ verifies $\Lambda_+\varphi = \psi$, then $\mathcal{R}\varphi$ is a solution to

$$\begin{cases} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that} \\ (\nabla u, \nabla v) = \langle \psi, v\rangle_{\Sigma^\delta}, \qquad \forall\, v \in \mathrm{H}^1_0(\Omega). \end{cases} \tag{3.47}$$

Reciprocally, assume that u is a solution to (3.47). Then the function $\varphi := u|_{\Sigma^\delta}$ satisfies $\Lambda_+\varphi = \psi$. According to the Lax–Milgram theorem, Problem (3.47) admits a unique solution for all $\psi \in \mathrm{H}^{-1/2}(\Sigma^\delta)$. We infer that $\Lambda_+ : \mathrm{H}^{1/2}(\Sigma^\delta) \to \mathrm{H}^{-1/2}(\Sigma^\delta)$ is indeed an isomorphism.

Now, remarking that Λ_e, Λ_i have the same principal symbol and using standard arguments of pseudo-differential operators theory (work as in the proof of [26, Theorem 1]), we can show that $\Lambda_- = \Lambda_e - \Lambda_i : \mathrm{H}^{1/2}(\Sigma^\delta) \to \mathrm{H}^{-1/2}(\Sigma^\delta)$ is compact. We emphasize that the assumption of smoothness of Σ^δ here is important.

Using the Riesz representation theorem, define the operator

$$K : \mathrm{H}^{1/2}(\Sigma^\delta) \to \mathrm{H}^{1/2}(\Sigma^\delta)$$

such that

$$(K\varphi, \varphi')_{\Sigma^\delta} = \langle \Lambda_- \varphi, \varphi' \rangle_{\Sigma^\delta} \qquad \text{for all } \varphi, \varphi' \in \mathrm{H}^{1/2}(\Sigma^\delta). \tag{3.48}$$

Here, we use the notation $(\,\cdot\,,\cdot\,)_{\Sigma^\delta} := \langle \Lambda_+ \cdot\,,\cdot\,\rangle_{\Sigma^\delta}$. Note that according to the features of Λ_+, the latter form is an inner product in $\mathrm{H}^{1/2}(\Sigma^\delta)$ equivalent to the usual one. Remark that (α, φ) is an eigenpair for (3.46) if and only if we have $K\varphi = \alpha\varphi$. But due to the properties of Λ_-, K is a selfadjoint and compact operator. Therefore, the spectrum of (3.46) coincides with a sequence of eigenvalues which accumulate at zero. We can use the *min-max* principle (see [45, Chapter 3]) to characterize these eigenvalues. We have

$$\alpha_1^+ = \sup_{\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta)\setminus\{0\}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \tag{3.49}$$

By the *min-max* principle, we know that this sup is attained for some φ_1^+. By induction, for $k > 2$, we define

$$\alpha_k^+ = \sup_{\substack{\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta)\setminus\{0\}, \\ \varphi \perp \{\varphi_1^+, \dots, \varphi_{k-1}^+\}}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \tag{3.50}$$

Here, if φ, φ' are two elements of $\mathrm{H}^{1/2}(\Sigma^\delta)$, we write $\varphi \perp \varphi'$ when $(\varphi, \varphi')_{\Sigma^\delta} = \langle \Lambda_+ \varphi, \varphi' \rangle_{\Sigma^\delta} = (\nabla(\mathcal{R}\varphi), \nabla(\mathcal{R}\varphi')) = 0$. Similarly, we define

$$\alpha_1^- = \inf_{\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta)\setminus\{0\}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}, \tag{3.51}$$

and, by induction, for $k > 2$,

$$\alpha_k^- = \inf_{\substack{\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta)\setminus\{0\}, \\ \varphi \perp \{\varphi_1^-, \dots, \varphi_{k-1}^-\}}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \tag{3.52}$$

Observing that for all $\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta) \setminus \{0\}$ we have

$$\frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}} = \frac{1-a}{1+a}, \qquad \text{with } a = \langle \Lambda_i \varphi, \varphi \rangle_{\Sigma^\delta} / \langle \Lambda_e \varphi, \varphi \rangle_{\Sigma^\delta} > 0,$$

we deduce that there holds $\alpha_k^\pm \in [-1; 1]$ for all $k \in \mathsf{N}^* := \{1, 2, \dots\}$. Taking $\varphi = \varphi_D^k|_{\Sigma^\delta}$ with φ_D^k defined in (3.1), we find $a = 0$ and consequently $\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta} / \langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta} = 1$. This allows one to prove that $\alpha_1^+ = \dots = \alpha_{\mathrm{card}(K^\delta)}^+ = 1$. Now, if $\alpha_{\mathrm{card}(K^\delta)+1}^+ = 1$, then there is $\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta) \setminus \{0\}$

such that $\langle \Lambda_i \varphi, \varphi \rangle_{\Sigma^\delta} = 0$ and $\mathcal{R}\varphi \in \widehat{\mathcal{H}}_D^\delta \setminus \{0\}$. This is impossible and therefore there holds $\alpha^+_{\mathrm{card}(K^\delta)+1} < 1$. Similarly, if $\alpha_1^- = -1$, then there exists $\varphi \in \mathrm{H}^{1/2}(\Sigma^\delta) \setminus \{0\}$ such that $\langle \Lambda_e \varphi, \varphi \rangle_{\Sigma^\delta} = 0$. This can not happen, which implies that $\alpha_1^- > -1$.

3.4.2. Optimality of the invertibility conditions

From the discussion preceding the statement of Theorem 3.15, we deduce the following result.

THEOREM 3.16. — *For $\kappa_\varepsilon \in (-\infty; 0) \setminus \{-1\}$, the operator $A^\delta_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism if and only if*

$$\kappa_\varepsilon \notin \left\{ \frac{\alpha_k^- + 1}{\alpha_k^- - 1},\ k > 1 \right\} \cup \left\{ \frac{\alpha_k^+ + 1}{\alpha_k^+ - 1},\ k > \mathrm{card}(K^\delta) + 1 \right\},$$

where the $\alpha_k^\pm$ are defined in (3.50)–(3.52).

Observing that the map $\alpha \mapsto (\alpha+1)/(\alpha-1)$ is decreasing on $(-1;1)$, we deduce in particular from Theorem 3.16 that A^δ_ε is an isomorphism for

$$\kappa_\varepsilon \in \left(-\infty;\ \frac{\alpha^+_{\mathrm{card}(K^\delta)+1} + 1}{\alpha^+_{\mathrm{card}(K^\delta)+1} - 1} \right) \cup \left(\frac{\alpha_1^- + 1}{\alpha_1^- - 1}; 0 \right). \tag{3.53}$$

But one can verify that we have

$$\frac{\alpha^+_{\mathrm{card}(K^\delta)+1} + 1}{\alpha^+_{\mathrm{card}(K^\delta)+1} - 1} = -1/m_D^\delta \qquad \text{and} \qquad \frac{\alpha_1^- + 1}{\alpha_1^- - 1} = -1/M_D^\delta$$

where m_D^δ, M_D^δ are the constants defined in (3.4). As a consequence, the invertibility condition for A^δ_ε obtained in Proposition 3.6 is the same as (3.53). This shows that the result of Proposition 3.6 is optimal in a certain sense. This is the first remark of this section.

3.4.3. Comparison with existing literature

In previous articles (see in particular [12] and [18]), authors have worked with the operator $\mathsf{T} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$\mathsf{T}\varphi = \begin{cases} \varphi & \text{in } \Omega_e^\delta \\ -\varphi + 2P\varphi & \text{in } \Omega_i^\delta \end{cases} \tag{3.54}$$

where, setting $\mathrm{H}^1_{0,\partial\Omega}(\Omega^\delta_e) := \{\varphi|_{\Omega^\delta_e},\, \varphi \in \mathrm{H}^1_0(\Omega)\}$, $P : \mathrm{H}^1_{0,\partial\Omega}(\Omega^\delta_e) \to \mathrm{H}^1(\Omega^\delta_i)$ denotes the harmonic extension operator, i.e. the operator such that $P\varphi$ solves the problem

$$\begin{cases} \Delta(P\varphi) = 0 & \text{in } \Omega^\delta_i \\ P\varphi = \varphi & \text{on } \partial\Omega^\delta_i. \end{cases} \tag{3.55}$$

We have $\mathsf{T}\circ\mathsf{T} = \mathrm{Id}$ which shows that T is an isomorphism of $\mathrm{H}^1_0(\Omega)$. On the other hand, for all $\varphi \in \mathrm{H}^1_0(\Omega)$, we find

$$(\nabla(A^\delta_\varepsilon(\mathsf{T}\varphi)), \nabla\varphi) = \varepsilon_e\|\nabla\varphi\|^2_{\Omega^\delta_e} + |\varepsilon_i|\|\nabla\varphi\|^2_{\Omega^\delta_i} + 2\varepsilon_i(\nabla(P\varphi),\nabla\varphi)_{\Omega^\delta_i}. \tag{3.56}$$

Set

$$\widetilde{M}^\delta_D := \sup_{\varphi\in\mathrm{H}^1_{0,\partial\Omega}(\Omega^\delta_e)\setminus\{0\}} \frac{\|\nabla(P\varphi)\|^2_{\Omega^\delta_i}}{\|\nabla\varphi\|^2_{\Omega^\delta_e}}. \tag{3.57}$$

Using Young's inequality, from (3.56) we infer that for all $\tau > 0$, there holds

$$|(\nabla(A^\delta_\varepsilon(\mathsf{T}\varphi)), \nabla\varphi)| \geqslant (\varepsilon_e - \tau^{-1}|\varepsilon_i|\widetilde{M}^\delta_D)\|\nabla\varphi\|^2_{\Omega^\delta_e} + |\varepsilon_i|(1-\tau)\|\nabla\varphi\|^2_{\Omega^\delta_i}. \tag{3.58}$$

As a consequence, we deduce that when $\varepsilon_e > |\varepsilon_i|\widetilde{M}^\delta_D \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/\widetilde{M}^\delta_D$, the operator A^δ_ε is an isomorphism of $\mathrm{H}^1_0(\Omega)$. Let us compare this operator T introduced in (3.54) with the T^+_D defined in (3.12). Clearly in Ω^δ_e, we have $\mathsf{T}\varphi = \mathsf{T}^+_D\varphi$. In Ω^δ_i, for $\varphi = \widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}$ with $\widehat{\varphi}_h \in \widehat{\mathcal{H}}^\delta_D$, $\Phi_h \in \mathrm{span}_{k\in K^\delta}\{\varphi^k_D\}$ and $\widetilde{\varphi} \in \mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i)$, we have

$$\mathsf{T}^+_D\varphi = \widehat{\varphi}_h + \Phi_h - \widetilde{\varphi}.$$

But one observes that

$$P\varphi = P(\widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}) = P(\widehat{\varphi}_h + \Phi_h) = \widehat{\varphi}_h + \Phi_h.$$

Therefore, we have $-\varphi + 2P\varphi = \widehat{\varphi}_h + \Phi_h - \widetilde{\varphi} = \mathsf{T}^+_D\varphi$ in Ω^δ_i which shows that the operator T defined in (3.54) coincides with T^+_D. Moreover, using Lemma 3.4, it is an exercise to prove that $\widetilde{M}^\delta_D$ is equal to the constant M^δ_D defined in (3.4). Therefore, the simple operator T in (3.54) is already very efficient. This is the second remark of this section.

3.4.4. T-coercivity operator in the general case

Finally, we explain how to construct an operator of T-coercivity for contrasts κ_ε as in the statement of Theorem 3.16, in particular for contrasts in $(-1/m^\delta_D; -1/M^\delta_D) \setminus \{-1\}$, this case being not covered by Proposition 3.6. First, we reindex the eigenvalues $\{\alpha^-_n\}_{n\geqslant 1}$, $\{\alpha^+_n\}_{n\geqslant \mathrm{card}(K^\delta)+1}$ and denote them $\{\alpha_n\}_{n\geqslant 1}$. Let (φ_n) be a family of eigenfunctions of the operator K introduced in (3.48) associated with the eigenvalues α_n. We choose them so that the functions $\mathcal{R}\varphi_n$, $n \geqslant 1$, form an orthonormal basis of $\widehat{\mathcal{H}}^\delta_D$. Now we define the operator $\mathsf{T}_D : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that for $\varphi = \widehat{\varphi}_h + \Phi_h + \widetilde{\varphi}$ with $\widehat{\varphi}_h = \sum_{n\in\mathsf{N}^*} \gamma_n \mathcal{R}\varphi_n \in \widehat{\mathcal{H}}^\delta_D$, $\Phi_h \in \mathrm{span}_{k\in K^\delta}\{\varphi^k_D\}$ and $\widetilde{\varphi} \in \mathrm{H}^1_0(\Omega^\delta_e \cup \Omega^\delta_i)$, there holds

$$\mathsf{T}_D\varphi = \begin{cases} \sum_{n\in\mathsf{N}^*} t_n\gamma_n\mathcal{R}\varphi_n + \Phi_h + \widetilde{\varphi} & \text{in } \Omega^\delta_e \\ \sum_{n\in\mathsf{N}^*} t_n\gamma_n\mathcal{R}\varphi_n + \Phi_h - \widetilde{\varphi} & \text{in } \Omega^\delta_i. \end{cases} \tag{3.59}$$

Here we take $t_n = 1$ for n such that $\kappa_\varepsilon > \kappa_n := (\alpha_n+1)/(\alpha_n-1)$ and $t_n = -1$ otherwise. The operator T_D is valued in $\mathrm{H}^1_0(\Omega)$ and we have $\mathsf{T}_D \circ \mathsf{T}_D = \mathrm{Id}$ which guarantees that T_D is an isomorphism of $\mathrm{H}^1_0(\Omega)$.

Proposition 3.17. — *Assume that $\kappa_\varepsilon \neq -1$ is such that for all $n \in \mathsf{N}^*$, we have $\kappa_\varepsilon \neq \kappa_n$ with*

$$\kappa_n = \frac{\alpha_n + 1}{\alpha_n - 1}. \tag{3.60}$$

Let $T_D : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ denote the isomorphism defined in (3.59). Then $(\nabla(A^\delta_\varepsilon(T_D\cdot)), \nabla\cdot)$ is coercive in $\mathrm{H}^1_0(\Omega)$. As a consequence, $A^\delta_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism.

Proof. — For all $\varphi \in \mathrm{H}^1_0(\Omega)$, we find

$$(\varepsilon\nabla\varphi, \nabla(\mathsf{T}_D\varphi)) = \sum_{n\in\mathsf{N}^*} t_n\,|\gamma_n|^2(\varepsilon\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) \\ + (\varepsilon_e\nabla\Phi_h, \nabla\Phi_h)_{\Omega^\delta_e} + (|\varepsilon|\nabla\widetilde{\varphi}, \nabla\widetilde{\varphi}). \tag{3.61}$$

But by the definition of the κ_n, we have, for all $n \in \mathsf{N}^*$,

$$(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega^\delta_e} = -\kappa_n(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega^\delta_i}.$$

This allows us to write

$$\begin{aligned}\sum_{n\in\mathsf{N}^*} t_n\,|\gamma_n|^2(\varepsilon\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) &= \varepsilon_e \sum_{n\in\mathsf{N}^*} t_n\,|\gamma_n|^2(\kappa_\varepsilon - \kappa_n)(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega^\delta_i} \\ &= \varepsilon_e \sum_{n\in\mathsf{N}^*} |\gamma_n|^2|\kappa_\varepsilon - \kappa_n|(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega^\delta_i}. \end{aligned} \tag{3.62}$$

Observing that we have $\|\nabla(\mathcal{R}\varphi_n)\|^2_{\Omega^\delta_i} \geqslant \inf_{m\in\mathsf{N}^*}|\kappa_m|^{-1}\|\nabla(\mathcal{R}\varphi_n)\|^2_{\Omega^\delta_e}$ (note that the sequence $(|\kappa_m|)$ is bounded), from (3.62) we obtain

$$\sum_{n\in\mathsf{N}^*} t_n\,|\gamma_n|^2(\varepsilon\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) \\ \geqslant C \inf_{n\in\mathsf{N}^*}|\kappa_\varepsilon - \kappa_n| \sum_{n\in\mathsf{N}^*} |\gamma_n|^2\|\nabla(\mathcal{R}\varphi_n)\|^2. \tag{3.63}$$

Using (3.63) into (3.61), we get $(\varepsilon\nabla\varphi, \nabla(\mathsf{T}_D\varphi)) \geqslant C \inf_{n\in\mathsf{N}^*}|\kappa_\varepsilon - \kappa_n|\,\|\nabla\varphi\|^2$ for all $\varphi \in \mathrm{H}^1_0(\Omega)$.

Remark 3.18. — In the following, we will not work with the operator T_D defined in (3.59) to investigate what happens for contrasts in $(-1/m;-1/M)\setminus\{-1\}$. The reason is that the value of the κ_n defined in (3.60) depends on δ and the operator T_D is useful to prove a result of uniform invertibility of A^δ_ε only if we know that there is a segment of $(-1/m;-1/M)\setminus\{-1\}$ of non empty interior which is uniformly free of the κ_n as δ tends to zero. It is an open question to find conditions on the geometry such that this occurs.

4. Analysis of the cell problem and properties of the homogenized tensors

In this section, we study a scalar problem set in the reference cell (supplemented with periodic boundary conditions) and the associated homogenized tensor. These quantities, which appear in the homogenization of Maxwell's equations considered in Section 5, are the same as the ones in [14] and [18], so that the results below complement and improve those obtained therein.

4.1. Cell problem

Denote by $\mathsf{C}^\infty_{\mathrm{per}}(\overline{Y})$ the subset of functions of $\mathsf{C}^\infty(\overline{Y})$ satisfying periodic boundary conditions on ∂Y. Let $\mathrm{H}^1_{\mathrm{per}}(Y)$ be the closure of $\mathsf{C}^\infty_{\mathrm{per}}(\overline{Y})$ for the norm of $\mathrm{H}^1(Y)$. Then set

$$\mathrm{H}^1_{\mathrm{per},\diamond}(Y) := \left\{\varphi\in\mathrm{H}^1_{\mathrm{per}}(Y)\,\middle|\,\int_{\partial Y_i}\varphi\,\mathrm{d}\sigma=0\right\}.$$

We endow this space with the inner product $(\nabla\cdot,\nabla\cdot)_Y$. For η equal to ε or μ as defined in (2.2), the problem we are interested in writes

$$\begin{cases}\text{Find } \varphi\in\mathrm{H}^1_{\mathrm{per},\diamond}(Y) \text{ such that:}\\ (\eta\nabla\varphi,\nabla\varphi')_Y=\ell(\varphi'), \qquad \forall\,\varphi'\in\mathrm{H}^1_{\mathrm{per},\diamond}(Y),\end{cases}\tag{4.1}$$

where ℓ is a continuous linear functional on $\mathrm{H}^1_{\mathrm{per},\diamond}(Y)$. In order to study this problem, we introduce the closed subspace of $\mathrm{H}^1_{\mathrm{per},\diamond}(Y)$

$$\mathrm{H}^1_{\mathrm{per},\,0,\,\partial Y_i}(Y) := \{\varphi\in\mathrm{H}^1_{\mathrm{per},\diamond}(Y)\,|\,\varphi=0 \text{ on } \partial Y_i\}.$$

Then we define the space $\widehat{\mathcal{H}}_\flat$ such that

$$\mathrm{H}^1_{\mathrm{per},\diamond}(Y)=\widehat{\mathcal{H}}_\flat\overset{\perp}{\oplus}\mathrm{H}^1_{\mathrm{per},\,0,\,\partial Y_i}(Y).\tag{4.2}$$

We will not look for an exact characterization of $\widehat{\mathcal{H}}_\flat$. Let us simply remark that if $\varphi\in\widehat{\mathcal{H}}_\flat$, then for all $\zeta\in\mathsf{C}^\infty_0(Y_e\cup Y_i)\subset\mathrm{H}^1_{\mathrm{per},\,0,\,\partial Y_i}(Y)$, we have

$0 = (\nabla\varphi, \nabla\zeta)_Y$. This implies that the elements of $\widehat{\mathcal{H}}_\flat$ are harmonic in $Y_e \cup Y_i$. Then we introduce the constants

$$m_\flat := \inf_{\varphi \in \hat{\mathcal{H}}_\flat \setminus \{0\}} \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}}, \qquad M_\flat := \sup_{\varphi \in \hat{\mathcal{H}}_\flat \setminus \{0\}} \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}}. \tag{4.3}$$

Theorem 4.1. — *Assume that κ_ε (resp. κ_μ) $\in (-\infty; -1/m) \cup (-1/M; 0)$ where m, M are defined in* (3.35)*. Then the problem* (4.1) *with $\eta = \varepsilon$ (resp. $\eta = \mu$) admits a unique solution which depends continuously on ℓ.*

Proof. — To set ideas, we take $\eta = \varepsilon$, the proof is the same for $\eta = \mu$. With the Riesz representation theorem, define the operator $D_\varepsilon : \mathrm{H}^1_{\mathrm{per},\diamond}(Y) \to \mathrm{H}^1_{\mathrm{per},\diamond}(Y)$ such that

$$(\nabla(D_\varepsilon\varphi), \nabla\varphi')_Y = (\varepsilon\nabla\varphi, \nabla\varphi')_Y, \qquad \forall\, \varphi, \varphi' \in \mathrm{H}^1_{\mathrm{per},\diamond}(Y). \tag{4.4}$$

Let us show that D_ε is an isomorphism when $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e \in (-\infty; -1/m) \cup (-1/M; 0)$. For $\varphi \in \mathrm{H}^1_{\mathrm{per},\diamond}(Y)$, consider the decomposition $\varphi = \varphi_h + \widetilde{\varphi}$ with $\varphi_h \in \widehat{\mathcal{H}}_\flat$ and $\widetilde{\varphi} \in \mathrm{H}^1_{\mathrm{per},\,0,\,\partial Y_i}(Y)$. With this decomposition, we define the operators $\mathsf{T}^\pm_\flat$ such that

$$\mathsf{T}^\pm_\flat\varphi = \begin{cases} \pm\varphi_h + \widetilde{\varphi} & \text{in } Y_e \\ \pm\varphi_h - \widetilde{\varphi} & \text{in } Y_i. \end{cases}$$

Working as in the proof of Proposition 3.6 with the operators $\mathsf{T}^\pm$ replaced by $\mathsf{T}^\pm_\flat$, one establishes that D_ε is an isomorphism when $\kappa_\varepsilon \in (-\infty; -1/m_\flat) \cup (-1/M_\flat; 0)$. To obtain the desired result, it remains to show that $m \leqslant m_\flat$ and $M_\flat \leqslant M$. Since $\widehat{\mathcal{H}}_\flat \subset \mathcal{H}_\diamond$, from Lemma 3.12, we clearly have $M_\flat \leqslant M$. Now let φ be an element of $\widehat{\mathcal{H}}_\flat \setminus \{0\}$. Denote $\zeta \in \mathcal{H}_0$ the function such that $\zeta = \varphi$ on ∂Y_i. The function ζ decomposes as $\zeta = \widehat{\zeta} + \alpha\,\varphi_D$ with $\widehat{\zeta} \in \widehat{\mathcal{H}}_0$ and $\alpha \in \mathbb{R}$ (φ_D is defined before (3.34)). Note that $\widehat{\zeta} \not\equiv 0$ otherwise we would have $\alpha = 0$ (because $\varphi_D = 1$ on ∂Y_i and $\int_{\partial Y_i} \zeta \,\mathrm{d}\sigma = 0$) and so $\zeta \equiv 0$. Observing that $\varphi - \widehat{\zeta} - \alpha$ is in $\mathrm{H}^1_{\mathrm{per},\,0,\,\partial Y_i}(Y)$, due to the decomposition (4.2), we can write

$$(\nabla\varphi, \nabla(\varphi - \widehat{\zeta}))_Y = (\nabla\varphi, \nabla(\varphi - \widehat{\zeta} - \alpha))_Y = 0. \tag{4.5}$$

But on the other hand, since we have $\nabla\varphi = \nabla\widehat{\zeta} = \nabla\zeta$ in Y_i, so that in particular there holds

$$\|\nabla\varphi\|^2_{Y_i} = \|\nabla\widehat{\zeta}\|^2_{Y_i}, \tag{4.6}$$

we infer from (4.5) that

$$(\nabla\varphi, \nabla(\varphi - \widehat{\zeta}))_{Y_e} = 0. \tag{4.7}$$

This implies

$$\|\nabla\varphi\|^2_{Y_e} \leqslant \|\nabla\widehat{\zeta}\|^2_{Y_e}. \tag{4.8}$$

Gathering (4.6) and (4.8), we deduce that

$$m \leq \frac{\|\nabla\widehat{\zeta}\|^2_{Y_i}}{\|\nabla\widehat{\zeta}\|^2_{Y_e}} \leq \frac{\|\nabla\varphi\|^2_{Y_i}}{\|\nabla\varphi\|^2_{Y_e}}. \tag{4.9}$$

Taking the infimum over all $\varphi \in \widehat{\mathcal{H}}_\flat\backslash\{0\}$ in (4.9), we obtain that $m \leq m_\flat$.

4.2. Homogenized tensors

Assume that the contrasts κ_ε and κ_μ are located in $(-\infty;-1/m) \cup (-1/M;0)$. For $\eta = \varepsilon$ or μ and $j = 1,2,3$, we define the function $\chi^\eta_j \in \mathrm{H}^1_{\mathrm{per},\diamond}(Y)$ such that

$$(\eta\nabla\chi^\eta_j, \nabla\xi)_Y = (\eta\nabla y_j, \nabla\xi)_Y, \qquad \forall\, \xi \in \mathrm{H}^1_{\mathrm{per},\diamond}(Y). \tag{4.10}$$

Note that the right hand side of (4.10) simply writes

$$(\eta\nabla y_j, \nabla\xi)_Y = \int_Y \eta\frac{\partial\xi}{\partial y_j}\,\mathrm{d}y$$

and that Theorem 4.1 ensures that the functions χ^η_j are well-defined. It is also worth noticing that by setting $\boldsymbol{\chi}^\eta := (\chi^\eta_1, \chi^\eta_2, \chi^\eta_3)^{\mathrm{T}}$, we have for all $\boldsymbol{\lambda} \in \mathsf{R}^3$:

$$(\eta\nabla(\boldsymbol{\lambda}\cdot\boldsymbol{\chi}^\eta), \nabla\xi)_Y = (\eta\nabla(\boldsymbol{\lambda}\cdot\boldsymbol{y}), \nabla\xi)_Y = \int_Y \eta\boldsymbol{\lambda}\cdot\nabla\xi\,\mathrm{d}y, \qquad \forall\,\xi \in \mathrm{H}^1_{\mathrm{per},\diamond}(Y). \tag{4.11}$$

Denoting by $\nabla\boldsymbol{\chi}^\eta$ the jacobian matrix of $\boldsymbol{\chi}^{\boldsymbol{\eta}}$:

$$\nabla\boldsymbol{\chi}^\eta = \left(\frac{\partial\chi^\eta_j}{\partial y_k}\right)_{1\leq j,k\leq 3},$$

the homogenized tensor associated with η is classically defined as the 3×3 symmetric matrix $\mathsf{H}(\eta) = (\mathsf{H}_{jk}(\eta))_{1\leq j,k\leq 3}$ given by (see, for instance, identity (6.35) in [22])

$$\mathsf{H}(\eta) = \frac{1}{|Y|}\int_Y \eta(y)\left[\mathrm{Id} - (\nabla\boldsymbol{\chi}^{\boldsymbol{\eta}})^{\mathrm{T}}\right]\,\mathrm{d}y, \tag{4.12}$$

or equivalently (see (6.37) in [22]):

$$\mathsf{H}_{jk}(\eta) = \frac{1}{|Y|}\int_Y \eta\nabla(y_j - \chi^\eta_j)\cdot\nabla(y_k - \chi^\eta_k)\,\mathrm{d}y. \tag{4.13}$$

Proposition 4.2. — *Assume that* κ_ε *(resp.* κ_μ*)* $\in (-\infty;-1/m) \cup (-1/M;0)$ *where* m, M *are defined in* (3.35)*. Then the matrix* $\mathsf{H}(\varepsilon)$ *(resp.* $\mathsf{H}(\mu)$*) is positive definite.*

Proof. — The proofs for $\mathsf{H}\,(\varepsilon)$ and $\mathsf{H}\,(\mu)$ are the same and to set ideas, we choose to work with ε. According to formula (6.44) in [22], for all $\xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathsf{R}^3$, we have

$$\mathsf{H}\,(\varepsilon)\,\xi\cdot\xi = \int_Y \varepsilon|\nabla\varphi_\xi|^2\,\mathrm{d}y$$

where the function φ_ξ is defined by

$$\varphi_\xi(y) = \sum_{j=1}^{3} \xi_j(y_j - \chi_j^\varepsilon(y)).$$

Note that if φ_ξ is constant in Y, then evaluating φ_ξ on ∂Y and using the fact that the functions χ_j^ε satisfy periodic boundary conditions, we find that $\xi = 0$ and so $\varphi_\xi \equiv 0$. Now, we assume that $\xi \neq 0$. Subtracting the mean value of the test functions on ∂Y_i, we see from (4.10) that χ_j^ε satisfy the slightly more general variational equality (the variational space is not the same as in (4.10))

$$(\varepsilon\nabla\chi_j^\varepsilon, \nabla\varphi')_Y = (\varepsilon\nabla y_j, \nabla\varphi')_Y, \qquad \forall\,\varphi' \in \mathrm{H}^1_{\mathrm{per}}(Y).$$

Taking $\varphi' \in \mathsf{C}_0^\infty(Y)$, this implies that we have

$$\mathrm{div}(\varepsilon\nabla\varphi_\xi) = 0 \;\text{ in } Y. \tag{4.14}$$

(*i*). — Introduce the function $\widehat{\varphi}_\xi$ such that

$$\widehat{\varphi}_\xi = \varphi_\xi - \frac{1}{|\partial Y_i|}\int_{\partial Y_i} \varphi_\xi\,\mathrm{d}\sigma \;\in \mathrm{H}^1_\diamond(Y).$$

From (4.14), we deduce that $\widehat{\varphi}_\xi$ is harmonic in $Y_e \cup Y_i$. Therefore, we have $\widehat{\varphi}_\xi \in \mathcal{H}_\diamond$ and from Lemma 3.12, we can write

$$\|\nabla\varphi_\xi\|^2_{Y_i} = \|\nabla\widehat{\varphi}_\xi\|^2_{Y_i} \leqslant M\,\|\nabla\widehat{\varphi}_\xi\|^2_{Y_e} = M\,\|\nabla\varphi_\xi\|^2_{Y_e}.$$

This allows us to write

$$\mathsf{H}\,(\varepsilon)\,\xi\cdot\xi = \varepsilon_e\|\nabla\varphi_\xi\|^2_{Y_e} - |\varepsilon_i|\,\|\nabla\varphi_\xi\|^2_{Y_i} \geqslant (\varepsilon_e - |\varepsilon_i|\,M)\|\nabla\varphi_\xi\|^2_{Y_e}.$$

Hence, for $\varepsilon_e > |\varepsilon_i|\,M \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/M$, the matrix $\mathsf{H}\,(\varepsilon)$ is definite-positive. Note that we have $\nabla\varphi_\xi \neq 0$ in Y_e otherwise we would have $\nabla\varphi_\xi = 0$ in Y (because $\varphi_\xi \in \mathrm{H}^1(Y)$ is harmonic in Y_e) which is impossible when $\xi \neq 0$ (see the discussion above).

(*ii*). — Now, we consider the case $\kappa_\varepsilon \in (-\infty; -1/m)$. The proof is a bit less straightforward and we divide it into two steps. Define the quadratic form $q_\varepsilon(\,\cdot\,) : \mathsf{R}^3 \to \mathsf{R}$ such that

$$q_\varepsilon(\xi) = \mathsf{H}\,(\varepsilon)\,\xi\cdot\xi.$$

Step 1. — First, we prove the following result.

Lemma 4.3. — *Assume that $\kappa_\varepsilon \in (-\infty;-1/m)$. Then the form q_ε is definite ($q_\varepsilon(\xi)=0 \Rightarrow \xi=0$).*

Proof of the lemma. — A bit more generally (this will serve in the proof of Lemma 4.4 below), assume that $\xi \in \mathbb{R}^3 \setminus \{0\}$ is such that

$$q_\varepsilon(\xi) \leqslant 0 \qquad \Longleftrightarrow \qquad \int_Y \varepsilon|\nabla\varphi_\xi|^2\,\mathrm{d}y \leqslant 0.$$

Then from identity (4.14), we infer that we must have

$$\int_{\partial Y} \varepsilon_e \frac{\partial\varphi_\xi}{\partial n}\varphi_\xi\,\mathrm{d}\sigma \leqslant 0. \tag{4.15}$$

Now, introduce $\zeta \in \mathcal{H}_0$ the function such that $\zeta = \varphi_\xi$ on ∂Y_i. The function ζ decomposes as $\zeta = \widehat{\zeta} + \alpha\,\varphi_D$ with $\widehat{\zeta} \in \widehat{\mathcal{H}}_0$ and $\alpha \in \mathbb{R}$ (φ_D is defined before (3.34)). Observe that we have $\widehat{\zeta} \not\equiv 0$. Indeed, otherwise φ_ξ would be constant in Y_i. And then (4.14) together with the unique continuation principle would imply that φ_ξ be constant in Y_e (because we would have that $\Delta\varphi_\xi = 0$ in Y_e, $\varphi_\xi = cste$ on ∂Y_i and $\partial_{n_e}\varphi_{\xi e} = 0$ on ∂Y_i) and so in Y. According to the discussion above, this is impossible when $\xi \neq 0$. Observing that $\varphi_\xi - (\widehat{\zeta}+\alpha) = 0$ on ∂Y_i, integrating by parts, we can write

$$\begin{aligned}(\nabla\varphi_\xi, \nabla(\varphi_\xi - \widehat{\zeta}))_{Y_e} &= (\nabla\varphi_\xi, \nabla(\varphi_\xi - (\widehat{\zeta}+\alpha)))_{Y_e}\\ &= \int_{\partial Y} \frac{\partial\varphi_\xi}{\partial n}\varphi_\xi\,\mathrm{d}\sigma - \int_{\partial Y}\frac{\partial\varphi_\xi}{\partial n}\alpha\,\mathrm{d}\sigma\\ &= \int_{\partial Y}\frac{\partial\varphi_\xi}{\partial n}\varphi_\xi\,\mathrm{d}\sigma \leqslant 0.\end{aligned} \tag{4.16}$$

The last equality above has been obtained using (4.15) and identity (4.14) multiplied by α. From (4.16) and the Cauchy–Schwarz inequality, we infer that

$$\|\nabla\varphi_\xi\|^2_{Y_e} \leqslant \|\nabla\widehat{\zeta}\|^2_{Y_e}.$$

Since on the other hand there holds $\nabla\varphi_\xi = \nabla\widehat{\zeta}$ in Y_i so that $\|\nabla\varphi_\xi\|^2_{Y_i} = \|\nabla\widehat{\zeta}\|^2_{Y_i}$, we deduce that

$$m \leqslant \frac{\|\nabla\widehat{\zeta}\|^2_{Y_i}}{\|\nabla\widehat{\zeta}\|^2_{Y_e}} \leqslant \frac{\|\nabla\varphi_\xi\|^2_{Y_i}}{\|\nabla\varphi_\xi\|^2_{Y_e}}. \tag{4.17}$$

But then, when $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e < -m^{-1} \Leftrightarrow \varepsilon_e < |\varepsilon_i|m$, we can write

$$\begin{aligned}q_\varepsilon(\xi) = H(\varepsilon)\,\xi\cdot\xi = \int_Y \varepsilon\nabla|\varphi_\xi|^2\,\mathrm{d}y &= \varepsilon_e\|\nabla\varphi_\xi\|^2_{Y_e} - |\varepsilon_i|\,\|\nabla\varphi_\xi\|^2_{Y_i}\\ &\leqslant (\varepsilon_e - |\varepsilon_i|m)\|\nabla\varphi_\xi\|^2_{Y_e} < 0.\end{aligned}$$

In particular we obtain a contradiction if $\xi \neq 0$ is such that $q_\varepsilon(\xi) = 0$. This proves that q_ε is definite.

From classical results concerning quadratic forms, we deduce from Lemma 4.3 that for each $\kappa_\varepsilon \in (-\infty; -1/m)$, $q_\varepsilon(\,\cdot\,)$ is either positive definite or negative definite.

Step 2. — Now consider some $\xi \in \mathsf{R}^3 \setminus \{0\}$. Corollary 5.6 of [14] or Lemma 4.4 below guarantee that $q_\varepsilon(\xi)$ is positive for κ_ϵ tending to $-\infty$. Using the fact that $\kappa_\varepsilon \mapsto q_\varepsilon(\xi)$ is continuous and that $q_\varepsilon(\,\cdot\,)$ is always definite for $\kappa_\varepsilon \in (-\infty; -1/m)$, we infer that $q_\varepsilon(\,\cdot\,)$ is positive definite for all $\kappa_\varepsilon \in (-\infty; -1/m)$. This achieves the proof of Proposition 4.2.

Below, for the sake of completeness, we present an alternative proof to Corollary 5.6 of [14] which is a bit more direct.

Lemma 4.4. — *For any given $\xi \in \mathsf{R}^3 \setminus \{0\}$, we have $q_\varepsilon(\xi) > 0$ for κ_ϵ tending to $-\infty$.*

Proof. — Impose that $\kappa_\varepsilon \in (-\infty; -1/m)$ and for $\xi \in \mathsf{R}^3 \setminus \{0\}$, assume that we have $q_\varepsilon(\xi) < 0$. Define the function

$$\check{\varphi}_\xi = \varphi_\xi - \frac{1}{|\partial Y|} \int_{\partial Y} \varphi_\xi \, \mathrm{d}\sigma.$$

From (4.14), we can write

$$\begin{aligned} |\varepsilon_i| \, \|\nabla \varphi_\xi\|^2_{Y_i} = |\varepsilon_i| \, \|\nabla \check{\varphi}_\xi\|^2_{Y_i} &= \varepsilon_e \|\nabla \check{\varphi}_\xi\|^2_{Y_e} - \int_{\partial Y} \varepsilon_e \frac{\partial \check{\varphi}_\xi}{\partial n} \check{\varphi}_\xi \, \mathrm{d}\sigma \\ &\leqslant C \, \varepsilon_e \, \|\nabla \check{\varphi}_\xi\|^2_{Y_e} = C \, \varepsilon_e \, \|\nabla \varphi_\xi\|^2_{Y_e}. \end{aligned} \tag{4.18}$$

The last inequality in (4.18) is a consequence of the continuity of the mappings $\varphi \mapsto \varphi|_{\partial Y}$ and $\varphi \mapsto \partial_n \varphi|_{\partial Y}$ from $\{\varphi \in \mathrm{H}^1(Y_e) \,|\, \Delta\varphi = 0 \text{ in } Y_e\}$ to $\mathrm{H}^{1/2}(\partial Y)$ and $\mathrm{H}^{-1/2}(\partial Y)$ respectively. Note that since the mean of $\check{\varphi}_\xi$ over ∂Y is null, a classical Poincaré type inequality allows one to prove that the H^1 norm of $\check{\varphi}_\xi$ in Y_e is controlled by $\|\nabla \check{\varphi}_\xi\|_{Y_e}$. From (4.18), we get

$$\frac{\|\nabla \varphi_\xi\|^2_{Y_i}}{\|\nabla \varphi_\xi\|^2_{Y_e}} \leqslant \frac{C}{|\kappa_\varepsilon|} \tag{4.19}$$

where $C > 0$ is independent of κ_ε. Taking the limit $\kappa_\varepsilon \to -\infty$ in (4.19), we obtain a contradiction with (4.17) (here we use that $q_\varepsilon(\xi) < 0$) because $m > 0$ is independent of κ_ε. Therefore we must have $q_\varepsilon(\xi) > 0$ for contrasts tending to $-\infty$.

4.3. Numerical illustrations

Proposition 4.2 guarantees that if $\kappa_\varepsilon,\ \kappa_\mu \in (-\infty;-1/m)\cup(-1/M;0)$, the matrices $\mathsf{H}\ (\varepsilon)$, $\mathsf{H}\ (\mu)$ are positive definite. This may seem a bit surprising and when one looks at the definition in (4.13), this is far from being obvious. The goal of this paragraph is to present some numerics to illustrate this property. To set ideas we compute $\mathsf{H}\ (\varepsilon)$ and to simplify we work in 2D. In this case, $\mathsf{H}\ (\varepsilon)$ is a 2×2 symmetric matrix. We do not expect particular differences between 2D and 3D settings. Numerically, we approximate the solutions of the problems (4.10) using a P2 finite element method. To proceed, we use the library *FreeFem++*[(1)] to compute the matrix $\mathsf{H}\ (\varepsilon)$ using formula (4.13). The mesh size is chosen equal to 0.02. Admittedly the numerical analysis of problems (4.10) is not standard because of the sign-changing ε. However in general, at least for contrasts κ_ε "not too close" to -1 when ∂Y_i is smooth, we obtain a reasonable numerical solution. We refer the reader to [8, 20, 34] for more details concerning these aspects. In Figures 4.1 and 4.2 below, we display the two real eigenvalues of $\mathsf{H}\ (\varepsilon)$ with respect to the contrast $\kappa_\varepsilon \in (-10;0)$ (we take $\varepsilon_i=-1$ and ε_e varies) for two different geometries of Y_i. For the numerics of Figure 4.1, the inclusion Y_i is an ellipse while for Figure 4.2, it is a rectangle. We emphasize that in the latter case, problem (4.10) is not well-posed in the Fredholm sense for $\kappa_\varepsilon \in (-3,-1/3)$ (see [7, 11]). As a consequence, for this range of contrasts, our numerical solutions have no sense. But for both settings, we observe that for contrasts large enough or small enough, the matrix $\mathsf{H}\ (\varepsilon)$ is positive definite as expected. Interestingly, at least in the case of the ellipse where we know that the numerical solution is meaningful except for $\kappa_\varepsilon \neq -1$, we also note that $\mathsf{H}\ (\varepsilon)$ is not positive definite for all contrasts. We emphasize however that we do not investigate these regimes in our analysis below.

5. Homogenization of Maxwell's equations

We come back to Maxwell's problem $(\mathsf{P}^{\,\delta})$ for the electric field (see (2.9)). We define the bilinear form $a_\omega^\delta(\,\cdot\,,\cdot\,)$ associated with (2.9) such that

$$a_\omega^\delta(\boldsymbol{E},\boldsymbol{E}') = ((\mu^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E}') - \omega^2(\varepsilon^\delta\boldsymbol{E},\boldsymbol{E}'),\quad \forall\ \boldsymbol{E},\boldsymbol{E}'\in \mathbf{H}_N(\mathbf{curl}).$$

Let m, M be the constants defined in (3.35). When $\kappa_\varepsilon,\ \kappa_\mu \in (-\infty;-1/m)\cup(-1/M;0)$, the matrices $\mathsf{H}\ (\varepsilon)$ and $\mathsf{H}\ (\mu)$ are well-defined according to Theorem 4.1. Moreover, according to Proposition 4.2, these matrices are positive

(1) *FreeFem++*, `http://www.freefem.org/ff++/`.

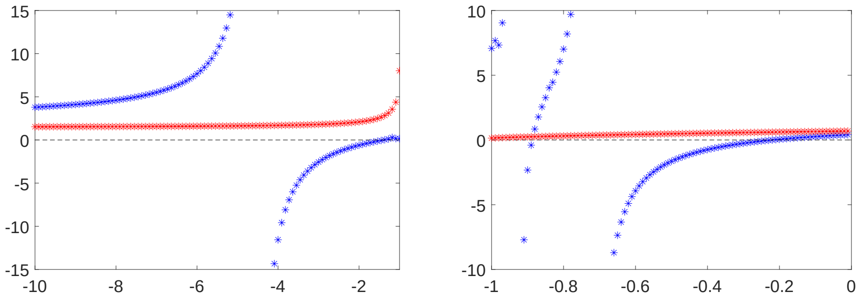

Figure 4.1. Representation of the two eigenvalues of $\mathsf{H}\,(\varepsilon)$ with respect to κ_ε varying in $(-10;-1)$ (left) and $(-1;0)$ (right). Here the inclusion Y_i coincides with the interior of the ellipse $\{(x = 0.5 + 0.4\cos\theta, y = 0.5 + 0.2\cos\theta),\ \theta \in [0;2\pi)\}$.

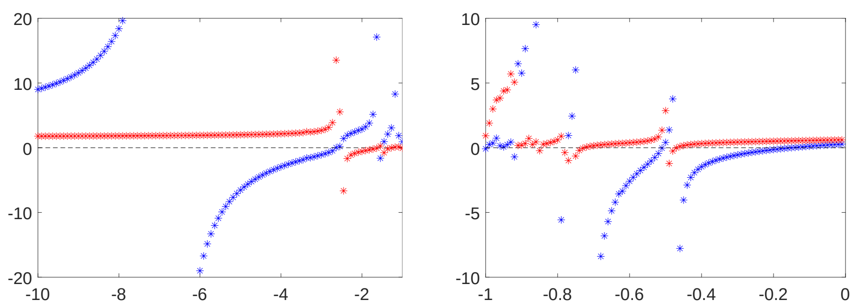

Figure 4.2. Representation of the two eigenvalues of $\mathsf{H}\,(\varepsilon)$ with respect to κ_ε varying in $(-10;-1)$ (left) and $(-1;0)$ (right). Here the inclusion Y_i coincides with the rectangle $(0.1;0.9)\times(0.3;0.7)$.

definite. Hence, we can introduce the homogenized problem

$$(\mathsf{P}^{\,\text{eff}}) \qquad \begin{cases} \text{Find } \boldsymbol{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that} \\ \mathbf{curl}(\mathsf{H}\,(\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}) - \omega^2 \mathsf{H}\,(\varepsilon)\boldsymbol{E}^{\text{eff}} = i\omega\boldsymbol{J} \end{cases} \tag{5.1}$$

whose variational formulation writes

$$\begin{cases} \text{Find } \boldsymbol{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that for all } \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}) \\ a_\omega^{\text{eff}}(\boldsymbol{E}^{\text{eff}}, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}'). \end{cases} \tag{5.2}$$

Here $a_\omega^{\text{eff}}(\,\cdot\,,\cdot\,)$ is the bilinear form defined on the space $\mathbf{H}_N(\mathbf{curl})$ such that

$$a_\omega^{\text{eff}}(\boldsymbol{E}, \boldsymbol{E}') = (\mathsf{H}\,(\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\mathsf{H}\,(\varepsilon)\boldsymbol{E}, \boldsymbol{E}').$$

It is worth noticing that the above homogenized problem (which has exactly the same form as the one obtained for classical (positive) Maxwell's equations) involves the homogenized tensors of the scalar problems studied in the previous sections. This fact will be used in a crucial way in the sequel to prove our homogenization result for Maxwell's system. Classically, one can easily prove that $(\mathsf{P}^{\,\mathrm{eff}})$ admits a unique solution for all

$$\omega^2 \in \mathsf{C} \setminus \Lambda^{\mathrm{eff}} \tag{5.3}$$

where Λ^{eff} is a discrete subset of $[0;+\infty)$.

The proof of a homogenization result for Maxwell's equations without sign-changing coefficients is by now quite classical (see for instance [6, 21, 43, 48]). It may be achieved by using, for instance, a notion of convergence specific to the periodic homogenization, namely the two-scale convergence, which was introduced by G. Nguetseng in [32] and further developed by G. Allaire [2]. Using this notion, a typical proof for such a homogenization result relies on three main ingredients. First, a uniform energy estimate is obtained for the sequence of solutions of $(\mathsf{P}^{\,\delta})$. Next, one shows that this uniformly bounded sequence has a (two-scale) limit that solves a two-scale limit problem. Finally, this limit problem is decoupled, yielding the homogenized problem which is proved to be well-posed. Due to the sign-changing coefficients and the presence of the non sign-definite L^2 term involving ω^2, proving the first ingredient is far from being obvious. In particular, the strategy proposed for instance in [21] does not apply anymore (as the spectral decomposition available in the strongly elliptic case fails). Instead, we proceed as follows. First, we prove a homogenization result for solutions of $(\mathsf{P}^{\,\delta})$ under a uniform energy estimate condition. Using this result, we prove by contradiction the needed uniform energy estimate for the solutions $(\mathsf{P}^{\,\delta})$. This leads to the main result of the paper (Theorem 5.6), namely the homogenization result for sign-changing Maxwell's equations.

5.1. Homogenization result under uniform energy estimate condition

Let $\boldsymbol{J}$ be a given field of $\mathbf{L}^2(\Omega)$. The aim of this section is to obtain a homogenization result for a sequence of functions $(\boldsymbol{E}^\delta)$ solving $(\mathsf{P}^{\,\delta})$ and satisfying the uniform energy estimate

$$\exists\, C > 0,\ \forall\, \delta \in (0;1], \qquad \|\boldsymbol{E}^\delta\|^2 + \|\mathbf{curl}\,\boldsymbol{E}^\delta\|^2 \leqslant C\,\|\boldsymbol{J}\|^2. \tag{5.4}$$

As it was already observed in [18] in the analysis of the homogenization process for the Dirichlet scalar operator A^δ_ε, the presence of sign-changing coefficients does not affect the two-scale convergence result. However, for the sake of completeness, we give here a proof of this convergence result following [6, 43, 48] and in particular [21]. We start by recalling the definition of the two-scale convergence (see [2]). Here we set $\boldsymbol{\mathsf{C}}^\infty_{\mathrm{per}}(\overline{Y}) := (\mathsf{C}^\infty_{\mathrm{per}}(\overline{Y}))^3$.

Definition 5.1. — *A sequence* $(\boldsymbol{E}^\delta)$ *in* $\mathbf{L}^2(\Omega)$ *two-scale converges to* $\boldsymbol{E}^0 \in \mathbf{L}^2(\Omega \times Y)$ *if we have*

$$\lim_{\delta\to 0}(\boldsymbol{E}^\delta, \boldsymbol{v}(\,\cdot\,,\cdot\,/\delta)\,) = \int_\Omega (\boldsymbol{E}^0(x,\cdot), \boldsymbol{v}(x,\cdot))_Y \,\mathrm{d}x$$

for all $\boldsymbol{v} \in \mathsf{C}_0^\infty(\Omega; \boldsymbol{\mathsf{C}}^\infty_{\mathrm{per}}(\overline{Y}))$. *Then we denote* $\boldsymbol{E}^\delta \xrightarrow{2s} \boldsymbol{E}^0$.

The notion of two-scale convergence is interesting due to the following compactness result (see for instance [43, Proposition 2.5]). It was first obtained by N. Wellander in [46] and then by V. Tiep Chu and V.H. Hoang in [43]. Here, $\mathbf{H}_{\mathrm{per}}(\mathbf{curl}; Y)$ denotes the closure of $\boldsymbol{\mathsf{C}}^\infty_{\mathrm{per}}(\overline{Y})$ for the norm $(\|\cdot\|_Y^2 + \|\mathbf{curl}\cdot\|_Y^2)^{1/2}$.

Proposition 5.2. — *Let* $(\boldsymbol{E}^\delta)$ *be a bounded sequence in* $\mathbf{H}(\mathbf{curl})$. *Then, there exist a sub-sequence, still denoted* $(\boldsymbol{E}^\delta)$, *and functions* $\boldsymbol{E}^{\mathrm{eff}} \in \mathbf{H}(\mathbf{curl})$, $\Theta \in \mathbf{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$, $\boldsymbol{E}^1 \in \mathrm{L}^2(\Omega; \mathbf{H}_{\mathrm{per}}(\mathbf{curl}, Y))$ *such that the following two-scale convergence results hold as* $\delta \to 0$*:*

$$\boldsymbol{E}^\delta \xrightarrow{2s} \boldsymbol{E}^{\mathrm{eff}} + \nabla_y \Theta, \qquad\qquad \mathbf{curl}\,\boldsymbol{E}^\delta \xrightarrow{2s} \mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}} + \mathbf{curl}_y\,\boldsymbol{E}^1.$$

Moreover, we also have the following weak convergence results in $\mathbf{L}^2(\Omega)$*:*

$$\boldsymbol{E}^\delta \rightharpoonup \boldsymbol{E}^{\mathrm{eff}} \quad \text{in } \mathbf{L}^2(\Omega), \qquad\qquad \mathbf{curl}\,\boldsymbol{E}^\delta \rightharpoonup \mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}} \quad \text{in } \mathbf{L}^2(\Omega).$$

We are now in position to prove the main result of this section, namely the convergence of a sequence of solutions of problem (P^δ) satisfying the energy estimate (5.4) to a solution of $(\mathsf{P}^{\mathrm{eff}})$ when $\delta \to 0$.

Proposition 5.3. — *Assume that* κ_ε, $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ *where* m, M *are defined in* (3.35). *Let* $(\boldsymbol{E}^\delta)$ *be a sequence of solutions of* (P^δ) *satisfying the uniform estimate* (5.4). *Then as* $\delta \to 0$, *we have*

$$\boldsymbol{E}^\delta \rightharpoonup \boldsymbol{E}^{\mathrm{eff}} \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{E}^\delta \rightharpoonup \mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}} \qquad \text{in } \mathbf{L}^2(\Omega)$$

where $\boldsymbol{E}^{\mathrm{eff}}$ *solves the homogenized problem* $(\mathsf{P}^{\mathrm{eff}})$.

Proof. — We take in (P^δ) (see (2.9)) a test function of the form

$$\boldsymbol{E}'(x) = \boldsymbol{\varphi}(x) + \delta\boldsymbol{\varphi}^{\mathbf{1}}\left(x, \frac{x}{\delta}\right) + \delta\nabla\left(\psi\left(x, \frac{x}{\delta}\right)\right),$$

with $\boldsymbol{\varphi} \in \boldsymbol{\mathsf{C}}_0^\infty(\Omega), \boldsymbol{\varphi}^{\mathbf{1}} \in \boldsymbol{\mathsf{C}}_0^\infty(\Omega; \boldsymbol{\mathsf{C}}^\infty_{\mathrm{per}}(Y)), \psi \in \mathsf{C}_0^\infty(\Omega; \mathsf{C}^\infty_{\mathrm{per}}(Y))$. By taking the limit as $\delta \to 0$ thanks to Proposition 5.2, we get as in [43, Proposition 2.5] the following two-scale limit problem:

$$\begin{aligned}\int_{\Omega\times Y}(\mu(y))^{-1}&\left(\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}(x)+\mathbf{curl}_y\,\boldsymbol{E}^1(x,y)\right)\\ &\cdot\left(\mathbf{curl}\,\boldsymbol{\varphi}(x)+\mathbf{curl}_y\,\boldsymbol{\varphi}^1(x,y)\right)\,\mathrm{d}x\,\mathrm{d}y\\ -\omega^2\int_{\Omega\times Y}&\varepsilon(y)\left(\boldsymbol{E}^{\text{eff}}(x)+\nabla_y\Theta(x,y)\right)\cdot(\boldsymbol{\varphi}(x)+\nabla_y\psi(x,y))\,\mathrm{d}x\,\mathrm{d}y\\ &=i\omega\int_\Omega\boldsymbol{J}\cdot\boldsymbol{\varphi}\,\mathrm{d}x+i\omega\int_{\Omega\times Y}\boldsymbol{J}(x)\cdot\nabla_y\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{5.5}$$

Since $\psi(x,\cdot)$ is $Y-$periodic, the second integral of the right hand side vanishes and hence, setting

$$\boldsymbol{R}(x,y):=(\mu(y))^{-1}\left(\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}(x)+\mathbf{curl}_y\,\boldsymbol{E}^1(x,y)\right)\tag{5.6}$$

and

$$\boldsymbol{S}(x,y):=\varepsilon(y)\left(\boldsymbol{E}^{\text{eff}}+\nabla_y\Theta\right),\tag{5.7}$$

relation (5.5) reads

$$\begin{aligned}\int_{\Omega\times Y}&\boldsymbol{R}(x,y)\cdot\left(\mathbf{curl}\,\boldsymbol{\varphi}(x)+\mathbf{curl}_y\,\boldsymbol{\varphi}^1(x,y)\right)\,\mathrm{d}x\,\mathrm{d}y\\ &-\omega^2\int_{\Omega\times Y}\boldsymbol{S}(x,y)\cdot(\boldsymbol{\varphi}(x)+\nabla_y\psi(x,y))\,\mathrm{d}x\,\mathrm{d}y=i\omega\int_\Omega\boldsymbol{J}\cdot\boldsymbol{\varphi}\,\mathrm{d}x.\end{aligned}\tag{5.8}$$

In order to prove that $\boldsymbol{E}^{\text{eff}}$ solves the homogenized problem $(\mathsf{P}^{\text{ eff}})$, it suffices to show that the two terms of the left hand side in the above equation can also be written as follows:

$$\begin{aligned}\int_{\Omega\times Y}&\boldsymbol{R}(x,y)\cdot\left(\mathbf{curl}\,\boldsymbol{\varphi}(x)+\mathbf{curl}_y\,\boldsymbol{\varphi}^1(x,y)\right)\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_\Omega(\mathsf{H}\;(\mu))^{-1}\,\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}x\end{aligned}\tag{5.9}$$

$$\int_{\Omega\times Y}\boldsymbol{S}(x,y)\cdot(\boldsymbol{\varphi}(x)+\nabla_y\psi(x,y))\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega\mathsf{H}\;(\varepsilon)\boldsymbol{E}^{\text{eff}}\cdot\boldsymbol{\varphi}\,\mathrm{d}x.\tag{5.10}$$

Indeed, once these two last relations proved, the conclusion follows immediately since problem (5.8) writes then

$$\int_\Omega(\mathsf{H}\;(\mu))^{-1}\,\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}x-\omega^2\int_\Omega\mathsf{H}\;(\varepsilon)\boldsymbol{E}^{\text{eff}}\cdot\boldsymbol{\varphi}\,\mathrm{d}x=i\omega\int_\Omega\boldsymbol{J}\cdot\boldsymbol{\varphi}\,\mathrm{d}x,$$

which is exactly the weak formulation of the homogenized problem $(\mathsf{P}^{\text{ eff}})$.

Step 1: Proof of relation (5.9). — Taking in (5.8) test functions $\boldsymbol{\varphi} = 0$ and $\psi = 0$, we obtain that

$$\int_{\Omega\times Y} \boldsymbol{R}(x,y)\cdot \mathbf{curl}_y\,\boldsymbol{\varphi}^1(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall\,\boldsymbol{\varphi}^{\mathbf{1}} \in \boldsymbol{\mathsf{C}}_0^\infty(\Omega;\boldsymbol{\mathsf{C}}_{\mathrm{per}}^\infty(Y)). \quad (5.11)$$

The above relation implies the existence of a function $\rho \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{per},\diamond}(Y))$ such that (see for instance the proof of Proposition 1.14 of [2], and more precisely the discussion following relation (1.19) therein)

$$\boldsymbol{R}(x,y) = \nabla_y\rho(x,y) + \int_Y \boldsymbol{R}(x,\widehat{y})\,\mathrm{d}\widehat{y}. \quad (5.12)$$

Now, we follow the ideas of [6] and [21]. From the definition (5.6) of $\boldsymbol{R}$ and direct calculation, one has for $\xi\in\mathrm{H}^1_{\mathrm{per}}(Y)$:

$$\begin{aligned}\int_Y \mu(y)\boldsymbol{R}(x,y)\cdot\nabla\xi(y)\,\mathrm{d}y\\ = \int_Y (\mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}}(x) + \mathbf{curl}_y\,\boldsymbol{E}^1(x,y))\cdot\nabla\xi(y)\,\mathrm{d}y = 0. \quad (5.13)\end{aligned}$$

Combining (5.12) and (5.13) we get that

$$\int_Y \mu(y)\nabla_y\rho(x,y)\cdot\nabla\xi(y)\,\mathrm{d}y = \int_Y \mu(y)\boldsymbol{\lambda}\cdot\nabla\xi(y)\,\mathrm{d}y,$$

where we have set $\boldsymbol{\lambda} = -\int_Y \boldsymbol{R}(x,y)\,\mathrm{d}y \in \mathsf{R}^3$ (here, x is fixed and can be considered as a parameter). Comparing with (4.11), we immediately obtain that $\rho = \boldsymbol{\lambda}\cdot\boldsymbol{\chi}^{\boldsymbol{\mu}} = \sum_{j=1}^3 \lambda_j\cdot\chi_j^\mu$, where $\boldsymbol{\chi}^{\boldsymbol{\mu}} = (\chi_1^\mu,\chi_2^\mu,\chi_3^\mu)^{\mathrm{T}}$ solve the cell problems (4.10) with $\eta=\mu$. Consequently, we have $\nabla_y\rho = \sum_{j=1}^3\lambda_j\cdot\nabla\chi_j^\mu = (\nabla\boldsymbol{\chi}^\mu)^{\mathrm{T}}\boldsymbol{\lambda}$, and hence

$$\boldsymbol{R}(x,y) = \nabla_y\rho(x,y) + \int_Y \boldsymbol{R}(x,\widehat{y})\,\mathrm{d}\widehat{y} = \left[\mathrm{Id} - (\nabla\boldsymbol{\chi}^{\boldsymbol{\mu}})^{\mathrm{T}}\right]\int_Y \boldsymbol{R}(x,\widehat{y})\,\mathrm{d}\widehat{y},$$

Using the above formula and expression (4.12) of $\mathsf{H}\ (\mu)$, we get that

$$\int_Y \mu(y)\boldsymbol{R}(x,y)\,\mathrm{d}y = \mathsf{H}\ (\mu)\int_Y \boldsymbol{R}(x,y)\,\mathrm{d}y.$$

But on the other hand, we also have from definition (5.6) of $\boldsymbol{R}(x,y)$ that

$$\int_Y \mu(y)\boldsymbol{R}(x,y)\,\mathrm{d}y = \int_Y (\mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}}(x) + \mathbf{curl}_y\,\boldsymbol{E}^1(x,y))\,\mathrm{d}y = \mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}}(x).$$

Since $\mathsf{H}\ (\mu)$ is positive definite for $\kappa_\mu \in (-\infty;-1/m)\cup(-1/M;0)$, we obtain by combining the last two relations, that

$$\int_Y \boldsymbol{R}(x,y)\,\mathrm{d}y = (\mathsf{H}\ (\mu))^{-1}\,\mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}}(x),$$

which also reads (due to the definition of $\boldsymbol{R}$)

$$\int_Y (\mu(y))^{-1}(\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}(x) + \mathbf{curl}_y\,\boldsymbol{E}^1(x,y))\,\mathrm{d}y = (\mathsf{H}\,(\mu))^{-1}\,\mathbf{curl}\,\boldsymbol{E}^{\text{eff}}(x).$$

The claimed relation (5.9) simply follows by multiplying the above equation by $\mathbf{curl}\,\boldsymbol{\varphi}$, integrating over Ω and adding (5.11).

Step 2: Proof of relation (5.10). — Taking $\boldsymbol{\varphi} = \boldsymbol{\varphi}^{\mathbf{1}} = 0$ in (5.8), we obtain that (since $\omega \neq 0$):

$$\int_{\Omega\times Y} \boldsymbol{S}(x,y)\cdot\nabla_y\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{5.14}$$

Since ψ is arbitrary in $\mathsf{C}_0^\infty(\Omega;\mathsf{C}^\infty_{\text{per}}(Y))$, this implies in particular that for almost every $x\in\Omega$ and for all $\xi\in\mathsf{C}^\infty_{\text{per}}(Y)$:

$$\begin{aligned}\int_Y \boldsymbol{S}(x,y)\cdot\nabla\xi(y)\,\mathrm{d}y&\\ = \int_Y \varepsilon(y)(\nabla_y\Theta(x,y) + \boldsymbol{E}^{\text{eff}}(x))\cdot\nabla\xi(y)\,\mathrm{d}x\,\mathrm{d}y &= 0.\end{aligned} \tag{5.15}$$

Hence

$$\int_Y \varepsilon(y)\nabla_y\Theta(x,y)\cdot\nabla\xi(y)\,\mathrm{d}y = \int_Y \varepsilon(y)\boldsymbol{\lambda}'\cdot\nabla\xi(y)\,\mathrm{d}y,$$

where we have set $\boldsymbol{\lambda}' = -\boldsymbol{E}^{\text{eff}}(x)\in\mathsf{R}^3$ (for a fixed value of x). Comparing the above relation with (4.11) for $\eta=\varepsilon$, we get that $\Theta(x,y) = \boldsymbol{\lambda}'\cdot\boldsymbol{\chi}^{\boldsymbol{\varepsilon}}$ and hence $\nabla_y\Theta = (\nabla\boldsymbol{\chi}^\varepsilon)^{\mathrm{T}}\boldsymbol{\lambda}' = -(\nabla\boldsymbol{\chi}^\varepsilon)^{\mathrm{T}}\boldsymbol{E}^{\text{eff}}(x)$. Using expression (4.12) of the homogenized matrix, we obtain that for every $\boldsymbol{\varphi}\in\boldsymbol{\mathsf{C}}_0^\infty(\Omega)$:

$$\begin{aligned}\int_{\Omega\times Y}\boldsymbol{S}(x,y)\cdot\boldsymbol{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\Omega\times Y}\varepsilon(y)\left(\boldsymbol{E}^{\text{eff}}(x) + \nabla_y\Theta(x,y)\right)\cdot\boldsymbol{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_\Omega \mathsf{H}\,(\varepsilon)\boldsymbol{E}^{\text{eff}}\cdot\boldsymbol{\varphi}\,\mathrm{d}x.\end{aligned} \tag{5.16}$$

Relation (5.10) follows immediately by adding (5.14) and (5.16).

5.2. Proof of the uniform energy estimate

This section is devoted to the proof of the uniform estimate (5.4) for solutions of $(\mathsf{P}^{\,\delta})$. More precisely, we have the following proposition.

Proposition 5.4. — *Assume that κ_ε, $\kappa_\mu\in(-\infty;-1/m)\cup(-1/M;0)$ where m, M are defined in* (3.35)*. Assume that $\omega^2\in\mathsf{C}\setminus\Lambda^{\text{eff}}$ where Λ^{eff} appears in* (5.3)*. Then, there exists $\delta_0>0$ such that for all $\delta\in(0;\delta_0]$, problem $(\mathsf{P}^{\,\delta})$ admits a unique solution $\boldsymbol{E}^\delta$. Moreover we have the estimate*

$$\|\boldsymbol{E}^\delta\| + \|\mathbf{curl}\,\boldsymbol{E}^\delta\| \leqslant C\,\|\boldsymbol{J}\| \tag{5.17}$$

where $C > 0$ is independent of $\delta \in (0;\delta_0]$.

Proof. — When κ_ε, $\kappa_\mu \in (-\infty;-1/m) \cup (-1/M;0)$, according to Theorem 3.14, we know that $A^\delta_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ and $B^\delta_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ are isomorphisms. From the Theorem 6.1 of [10], we infer that $\mathsf{A}^\delta_N(\omega) : \mathbf{H}_N(\mathbf{curl}) \to \mathbf{H}_N(\mathbf{curl})$ is an isomorphism if it is injective. Therefore, we have to prove that $\mathsf{A}^\delta_N(\omega)$ is injective for δ small enough. To proceed we work by contradiction. Slightly more generally, for a given $\boldsymbol{J} \in \mathbf{L}^2(\Omega)$, assume that there is a sequence of values of δ denoted $(\delta_k)_{k\in\mathsf{N}}$, with $\delta_k \to 0$, such that if we set $\varepsilon_k := \varepsilon^{\delta_k}$, $\mu_k := \mu^{\delta_k}$, $\boldsymbol{E}_k := \boldsymbol{E}^{\delta_k} \in \mathbf{H}_N(\mathbf{curl})$, we have for all $\boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl})$

$$a^k_\omega(\boldsymbol{E}_k, \boldsymbol{E}') := ((\mu_k)^{-1}\,\mathbf{curl}\,\boldsymbol{E}_k, \mathbf{curl}\,\boldsymbol{E}') - \omega^2\left(\varepsilon_k \boldsymbol{E}_k, \boldsymbol{E}'\right) = i\omega\left(\boldsymbol{J}, \boldsymbol{E}'\right),$$

as well as

$$\|\boldsymbol{E}_k\|^2 + \|\mathbf{curl}\,\boldsymbol{E}_k\|^2 > k.$$

Then set

$$\widetilde{\boldsymbol{E}}_k := \boldsymbol{E}_k/(\|\boldsymbol{E}_k\|^2 + \|\mathbf{curl}\,\boldsymbol{E}_k\|^2)$$
$$\text{and}\quad \widetilde{\boldsymbol{J}}_k := \boldsymbol{J}/(\|\boldsymbol{E}_k\|^2 + \|\mathbf{curl}\,\boldsymbol{E}_k\|^2).$$

We have

$$a^k_\omega(\widetilde{\boldsymbol{E}}_k, \boldsymbol{E}') = i\omega(\widetilde{\boldsymbol{J}}_k, \boldsymbol{E}'), \qquad \forall\, \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}) \tag{5.18}$$

and

$$\|\widetilde{\boldsymbol{E}}_k\|^2 + \|\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k\|^2 = 1, \qquad \lim_{k\to+\infty} \|\widetilde{\boldsymbol{J}}_k\| = 0.$$

Since $(\widetilde{\boldsymbol{E}}_k)$ is bounded in $\mathbf{H}_N(\mathbf{curl})$, we can extract a subsequence, still denoted $(\widetilde{\boldsymbol{E}}_k)$, such that $(\widetilde{\boldsymbol{E}}_k)$ converges weakly in $\mathbf{H}_N(\mathbf{curl})$ to some $\boldsymbol{E}_0 \in \mathbf{H}_N(\mathbf{curl})$. Thanks to Proposition 5.3, we can pass to the limit in (5.18) to get

$$a^{\mathrm{eff}}_\omega(\boldsymbol{E}_0, \boldsymbol{E}') = 0, \qquad \forall\, \boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}). \tag{5.19}$$

Since $\omega^2 \in \mathsf{C} \setminus \Lambda^{\mathrm{eff}}$, this implies that $\boldsymbol{E}_0 = 0$. In order to obtain a contradiction, it remains to show that $(\widetilde{\boldsymbol{E}}_k)$ strongly converges to zero in $\mathbf{H}_N(\mathbf{curl})$. To proceed, we have to establish some sort of compactness result using the fact that when $\omega \neq 0$, we have $\mathrm{div}(\varepsilon_k \widetilde{\boldsymbol{E}}_k) = 0$ in Ω which implies that $\widetilde{\boldsymbol{E}}_k \in \mathbf{V}_N(\varepsilon_k)$. For each $k > 1$, from Theorem 5.1 of [10], we know that when $\kappa_\varepsilon \in (-\infty;-1/m) \cup (-1/M;0)$, $\mathbf{V}_N(\varepsilon_k)$ is compactly embedded in $\mathbf{L}^2(\Omega)$. But here we need some uniform result with respect to k. To proceed, we will take in (5.18) a well-chosen test function. Let us mention that a similar difficulty appears in the justification of the approximation of Maxwell's equations with finite elements methods, the mesh size h replacing the parameter δ (see [30, Section 7.3.2] and the references therein). First, introduce the unique function $\psi_k \in \mathrm{H}^1_\#(\Omega)$ such that

$$(\mu_k \nabla\psi_k, \nabla\psi') = (\mu_k\ \mathbf{curl}\,\widetilde{\boldsymbol{E}}_k, \nabla\psi'), \qquad \forall\, \psi' \in \mathrm{H}^1_\#(\Omega).$$

When $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$, from Theorem 3.14, we know that ψ_k is well-defined. Moreover, we have $\|\nabla\psi_k\| \leqslant C\,\|\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k\| \leqslant C$ where $C > 0$ is independent of δ (note that (μ_k) is a bounded sequence of functions of $\mathrm{L}^\infty(\Omega)$ and we have $\|\mu_k\|_{\mathrm{L}^\infty(\Omega)} = \max(\mu_e, |\mu_i|)$ for all $k \in \mathsf{N}$). Then $\mu_k\,(\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k - \nabla\psi_k)$ is divergence free in Ω and satisfies $\mu_k(\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k - \nabla\psi_k)\cdot n = 0$ on $\partial\Omega$. From [4, Theorem 3.17], we know that there exists a unique $\mathsf{P}_k\widetilde{\boldsymbol{E}}_k \in \mathbf{V}_N(1)$ such that

$$\mathbf{curl}(\mathsf{P}_k\widetilde{\boldsymbol{E}}_k) = \mu_k(\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k - \nabla\psi_k). \tag{5.20}$$

Since in $\mathbf{V}_N(1)$, $\|\mathbf{curl}\,\cdot\|_\Omega$ is a norm which is equivalent to $\|\cdot\|_{\mathbf{curl}}$ (Proposition 2.1), we infer that (P_k) is a sequence of operators which are uniformly bounded from $\mathbf{H}_N(\mathbf{curl})$ to $\mathbf{V}_N(1)$. Testing in (5.18) with $\boldsymbol{E}' = \mathsf{P}_k\widetilde{\boldsymbol{E}}_k$, using (5.20) and integrating by parts, we get

$$\begin{aligned} i\omega(\widetilde{\boldsymbol{J}}_k, \mathsf{P}_k\widetilde{\boldsymbol{E}}_k) + \omega^2(\varepsilon_k\widetilde{\boldsymbol{E}}_k, \mathsf{P}_k\widetilde{\boldsymbol{E}}_k) &= ((\mu_k)^{-1}\,\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k, \mathbf{curl}(\mathsf{P}_k\widetilde{\boldsymbol{E}}_k)) \\ &= (\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k, \mathbf{curl}\,\widetilde{\boldsymbol{E}}_k - \nabla\psi_k) \end{aligned}$$

that is

$$i\omega(\widetilde{\boldsymbol{J}}_k, \mathsf{P}_k\widetilde{\boldsymbol{E}}_k) + \omega^2(\varepsilon_k\widetilde{\boldsymbol{E}}_k, \mathsf{P}_k\widetilde{\boldsymbol{E}}_k) = \|\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k\|^2. \tag{5.21}$$

Using that $\mathsf{P}_k : \mathbf{H}_N(\mathbf{curl}) \to \mathbf{V}_N(1)$ are uniformly bounded, $(\widetilde{\boldsymbol{E}}_k)$ converges weakly to zero in $\mathbf{H}_N(\mathbf{curl})$ and $\mathbf{V}_N(1)$ is compactly embedded in $\mathbf{L}^2(\Omega)$ (Proposition 2.1), we deduce that we can extract a subsequence, still denoted $(\widetilde{\boldsymbol{E}}_k)$, such that $(\mathsf{P}_k\widetilde{\boldsymbol{E}}_k)$ converges strongly to zero in $\mathbf{L}^2(\Omega)$. Then from (5.21), we deduce that the sequence $(\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k)$ converges strongly to zero in $\mathbf{L}^2(\Omega)$. Using the result of Proposition 5.5 below which guarantees that $\|\widetilde{\boldsymbol{E}}_k\| \leqslant C\,\|\mathbf{curl}\,\widetilde{\boldsymbol{E}}_k\|$ with some $C > 0$ which is independent of k, we deduce that $(\widetilde{\boldsymbol{E}}_k)$ converges to zero in $\mathbf{H}_N(\mathbf{curl})$. This contradicts the initial assumption. As a consequence, taking first $\boldsymbol{J} = 0$ above, we deduce that $(\mathscr{P}^{\,\delta})$ is injective and so uniquely solvable for δ small enough. Then for a given non zero $\boldsymbol{J} \in \mathbf{L}^2(\Omega)$, the above lines imply the uniform estimate (5.17).

Proposition 5.5. — *Assume that* $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ *where* m, M *are defined in* (3.35)*. Then there is a constant* $C > 0$ *independent of* δ *such that*

$$\|\boldsymbol{E}\| \leqslant C\,\|\mathbf{curl}\,\boldsymbol{E}\|, \qquad \forall\, \boldsymbol{E} \in \mathbf{V}_N(\varepsilon^\delta). \tag{5.22}$$

Proof. — If $\boldsymbol{E} \in \mathbf{V}_N(\varepsilon^\delta)$, according to [4, Theorem 3.12], we know that there is a unique $\boldsymbol{u} \in \mathbf{V}_T(1)$ such that $\boldsymbol{E} = (\varepsilon^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{u}$. Then integrating by parts, we find

$$((\varepsilon^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{u}') = (\mathbf{curl}\,\boldsymbol{E}, \boldsymbol{u}'), \qquad \forall\, \boldsymbol{u}' \in \mathbf{V}_T(1). \tag{5.23}$$

Introduce the function $\varphi \in \mathrm{H}^1_0(\Omega)$ such that

$$(\varepsilon^\delta\nabla\varphi, \nabla\varphi') = (\varepsilon^\delta\,\mathbf{curl}\,\boldsymbol{u}, \nabla\varphi'), \qquad \forall\, \varphi' \in \mathrm{H}^1_0(\Omega).$$

Since $\kappa_\varepsilon \in (-\infty;-1/m) \cup (-1/M;0)$, from Theorem 3.14, we know that φ is well-defined. Moreover, we have $\|\nabla\varphi\| \leqslant C\,\|\mathbf{curl}\,\boldsymbol{u}\|$ where $C>0$ is independent of δ (note that $\|\varepsilon^\delta\|_{\mathrm{L}^\infty(\Omega)} = \max(\varepsilon_e, |\varepsilon_i|)$ for all $\delta>0$). Then $\varepsilon^\delta(\mathbf{curl}\,\boldsymbol{u} - \nabla\varphi)$ is divergence free in Ω and again from [4, Theorem 3.12], we know that there is a unique $\mathsf{T}\boldsymbol{u} \in \mathbf{V}_T(1)$ such that $\mathbf{curl}(\mathsf{T}\boldsymbol{u}) = \varepsilon^\delta(\mathbf{curl}\,\boldsymbol{u} - \nabla\varphi)$. Since in $\mathbf{V}_T(1)$, $\|\mathbf{curl}\cdot\|_\Omega$ is a norm which is equivalent to $\|\cdot\|_{\mathbf{curl}}$ (Proposition 2.1), we infer that $\mathsf{T} : \mathbf{V}_T(1) \to \mathbf{V}_T(1)$ is a uniformly bounded operator. Choosing $\boldsymbol{u}' = \mathsf{T}\boldsymbol{u}$ in (5.23) and integrating by parts, we obtain

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{E}, \mathsf{T}\boldsymbol{u}) = ((\varepsilon^\delta)^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}(\mathsf{T}\boldsymbol{u}))& \\ = \|\mathbf{curl}\,\boldsymbol{u}\|^2 - (\mathbf{curl}\,\boldsymbol{u}, \nabla\varphi) = \|\mathbf{curl}\,\boldsymbol{u}\|^2.&\end{aligned}$$

Using the Cauchy–Schwarz inequality, this gives $\|\mathbf{curl}\,\boldsymbol{u}\| \leqslant C\,\|\mathbf{curl}\,\boldsymbol{E}\|$ where $C>0$ is independent of δ. This yields the desired estimate (5.22).

5.3. Final result

Gathering Propositions 5.3 and 5.4, we can state the final result of this article.

Theorem 5.6. — *Assume that* $\kappa_\varepsilon, \kappa_\mu \in (-\infty;-1/m)\cup(-1/M;0)$ *where* m, M *are defined in* (3.35). *Assume that* $\omega \in \mathbb{C} \setminus \Lambda^{\mathrm{eff}}$ *where* Λ^{eff} *appears in* (5.3). *Then, there exists* $\delta_0 > 0$ *such that for* $\delta \in (0, \delta_0]$, *the solution* $\boldsymbol{E}^\delta$ *of problem* (P^{δ}), *which is well-defined according to Proposition 5.4, satisfies*

$$\boldsymbol{E}^\delta \rightharpoonup \boldsymbol{E}^{\mathrm{eff}} \quad \textit{and} \quad \mathbf{curl}\,\boldsymbol{E}^\delta \rightharpoonup \mathbf{curl}\,\boldsymbol{E}^{\mathrm{eff}} \qquad \textit{weakly in } \mathbf{L}^2(\Omega)$$

where $\boldsymbol{E}^{\mathrm{eff}}$ *is the unique solution of problem* $(\mathsf{P}^{\mathrm{eff}})$ *given by* (5.1).

Let us conclude this paper with two comments. Firstly, in this work, we only prove weak convergence results. Strong convergence results (using correctors) for Maxwell's equations with positive materials have been obtained in [41, 42]. It would be interesting to understand if we can adapt the approach proposed in these two articles to our setting. Secondly, the obtained bounds for the contrasts (involving m and M) to ensure the homogenization process are probably not optimal. Improving them would require a sharp analysis of the asymptotic behavior of the critical contrasts given by (3.60) as δ tends to zero (see Remark 3.18). Is it possible that the two scalar problems with Dirichlet and Neumann boundary conditions be uniformly well-posed as δ tends to zero, even when some cell problems have a non zero kernel or when the homogenized tensors are not positive definite? This has still to be clarified.

Appendix. Table of notation for the functional spaces

For the reader's convenience, we list below the main functional spaces used throughout the paper:

$$
\begin{aligned}
\boldsymbol{C}_0^\infty(\Omega) &:= (C_0^\infty(\Omega))^3\\
\boldsymbol{C}_{\mathrm{per}}^\infty(\overline{Y}) &:= (C_{\mathrm{per}}^\infty(\overline{Y}))^3\\
\mathrm{H}_0^1(\Omega) &:= \{\varphi\in\mathrm{H}^1(\Omega)\,|\,\varphi=0 \text{ on } \partial\Omega\}\\
\mathrm{H}_\#^1(\Omega) &:= \{\varphi\in\mathrm{H}^1(\Omega)\,|\,\textstyle\int_\Omega\varphi\,\mathrm{d}x=0\}\\
\mathcal{H}_D^\delta &:= \{\varphi\in\mathrm{H}_0^1(\Omega)\,|\,\Delta\varphi=0 \text{ in } \Omega_e^\delta\cup\Omega_i^\delta\}\\
\widehat{\mathcal{H}}_D^\delta &:= \{\varphi\in\mathcal{H}_D^\delta\,|\,(\nabla\varphi,\nabla\varphi_D^k)=0,\quad\forall\,k\in K^\delta\}\\
\mathrm{H}_\diamond^1(\Omega) &:= \{\varphi\in\mathrm{H}^1(\Omega)\,|\,\textstyle\int_{\partial\Omega_i^\delta}\varphi\,\mathrm{d}\sigma=0\}\\
\mathcal{H}_N^\delta &:= \{\varphi\in\mathrm{H}_\diamond^1(\Omega)\,|\,\Delta\varphi=0 \text{ in } \Omega_e^\delta\cup\Omega_i^\delta,\ \partial_n\varphi=0 \text{ on } \partial\Omega\}\\
\widehat{\mathcal{H}}_N^\delta &:= \{\varphi\in\mathcal{H}_N^\delta\,|\,(\nabla\varphi,\nabla\varphi_N^k)=0,\quad\forall\,k\in K^\delta\setminus\{k_0\}\}\\
\mathrm{H}_{0,\,\partial\Omega_i^\delta}^1(\Omega) &:= \{\varphi\in\mathrm{H}^1(\Omega)\,|\,\varphi=0 \text{ on } \partial\Omega_i^\delta\}\\
\mathrm{H}_{0,\partial\Omega}^1(\Omega_e^\delta) &:= \{\varphi|_{\Omega_e^\delta},\ \varphi\in\mathrm{H}_0^1(\Omega)\}\\
\mathrm{H}_\diamond^1(Y) &:= \{\varphi\in\mathrm{H}^1(Y)\,|\,\textstyle\int_{\partial Y_i}\varphi\,\mathrm{d}\sigma=0\}\\
\mathcal{H}_0 &:= \{\varphi\in\mathrm{H}_0^1(Y)\,|\,\Delta\varphi=0 \text{ in } Y_e\cup Y_i\}\\
\mathcal{H}_\diamond &:= \{\varphi\in\mathrm{H}_\diamond^1(Y)\,|\,\Delta\varphi=0 \text{ in } Y_e\cup Y_i\}\\
\widehat{\mathcal{H}}_0 &:= \{\varphi\in\mathcal{H}_0\,|\,(\nabla\varphi,\nabla\varphi_D)=0\}\\
\widehat{\mathcal{H}}_\diamond &:= \{\varphi\in\mathcal{H}_\diamond\,|\,\partial_n\varphi=0 \text{ on } \partial Y\}\\
\mathrm{H}_{\mathrm{per}}^1(Y) &:= \text{Closure of } \boldsymbol{C}_{\mathrm{per}}^\infty(\overline{Y}) \text{ for the norm of } \mathrm{H}^1(Y)\\
\mathrm{H}_{\mathrm{per},\diamond}^1(Y) &:= \{\varphi\in\mathrm{H}_{\mathrm{per}}^1(Y)\,|\,\textstyle\int_{\partial Y_i}\varphi\,\mathrm{d}\sigma=0\}\\
\mathrm{H}_{\mathrm{per},\,0,\,\partial Y_i}^1(Y) &:= \{\varphi\in\mathrm{H}_{\mathrm{per},\diamond}^1(Y)\,|\,\varphi=0 \text{ on } \partial Y_i\}\\
\widehat{\mathcal{H}}_\flat &:= \text{Orthogonal complement of } \mathrm{H}_{\mathrm{per},\,0,\,\partial Y_i}^1(Y) \text{ in } \mathrm{H}_{\mathrm{per},\diamond}^1(Y)\\
\mathbf{H}(\mathbf{curl}) &:= \{\boldsymbol{H}\in\mathbf{L}^2(\Omega)\mid\mathbf{curl}\,\boldsymbol{H}\in\mathbf{L}^2(\Omega)\}\\
\mathbf{H}_N(\mathbf{curl}) &:= \{\boldsymbol{E}\in\mathbf{H}(\mathbf{curl})\mid\boldsymbol{E}\times n=0 \text{ on } \partial\Omega\}\\
\mathbf{H}_{\mathrm{per}}(\mathbf{curl};Y) &:= \text{Closure of } \boldsymbol{C}_{\mathrm{per}}^\infty(\overline{Y}) \text{ for the norm } (\|\cdot\|_Y^2+\|\mathbf{curl}\cdot\|_Y^2)^{1/2}\\
\mathbf{L}^2(\Omega) &:= (\mathrm{L}^2(\Omega))^3\\
\mathbf{V}_T(\xi) &:= \{\boldsymbol{H}\in\mathbf{H}(\mathbf{curl})\mid\mathrm{div}(\xi\boldsymbol{H})=0,\ \xi\boldsymbol{H}\cdot n=0 \text{ on } \partial\Omega\}\\
\mathbf{V}_N(\xi) &:= \{\boldsymbol{E}\in\mathbf{H}(\mathbf{curl})\mid\mathrm{div}(\xi\boldsymbol{E})=0,\ \boldsymbol{E}\times n=0 \text{ on } \partial\Omega\}.
\end{aligned}
$$

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