ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

Corentin Audiard

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Tome XXX, nº 5 (2021), p. 1139–1183.

https://doi.org/10.5802/afst.1696

Annales de la faculté des sciences de Toulouse Volume XXX, n° 5, 2021
pp. 1139-1183

On the time of existence of solutions of the Euler–Korteweg system (*)

Corentin Audiard (1)

Abstract. — The Euler–Korteweg system is a dispersive perturbation of the usual compressible Euler equations. In dimension at least three, under a natural stability condition on the pressure, the author proved with B. Haspot that the Cauchy problem is globally well-posed for small, smooth, irrotational initial data. As a continuation of this work, we prove that if the initial velocity has a small rotational part, there exists a lower bound on the time of existence that depends only on some norm of this rotational part. In the zero vorticity limit we recover the previous global well-posedness result.

Independently of this analysis, we also provide (in a special case) a simple example of solution that blows up in finite time.

Résumé. — Le système d'Euler–Korteweg est une perturbation dispersive du système d'Euler compressible classique. En dimension 3 et plus, sous une condition naturelle de stabilité de la pression, l'auteur a prouvé avec B. Haspot la nature globalement bien posée du problème pour des données initiales petites et irrotationnelles. On continue ici ce travail en considérant le cas de données avec une petite composante rotationnelle, on prouve une borne inférieure sur le temps d'existence qui ne dépend que de cette composante. Dans la limite irrotationnelle on retrouve le résultat précédent d'existence globale.

Indépendamment, on construit dans un cas particulier des solutions devenant singulières en temps fini.

(*) Reçu le 14 janvier 2020, accepté le 29 mai 2020.

Keywords: Euler equations, capillarity, time of existence, dispersion.

(1) Sorbonne Université, CNRS, Université de Paris, Laboratoire Jacques-Louis Lions, 75005, Paris, France

The author was supported by the french National Research Agency project NABUCO, grant ANR 17-CE40-0025.

Article proposé par Isabelle Gallagher.

1. Introduction

The Cauchy problem for the Euler–Korteweg system reads

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u) &= 0,\\ \partial_t u+u\cdot\nabla u+\nabla g(\rho) &= \nabla\big(K\Delta\rho+\frac{1}{2}K'(\rho)|\nabla\rho|^2\big),\quad (x,t)\in\mathbb{R}^d\times\mathbb{R}^+.\\ (\rho,u)|_{t=0} &= (\rho_0,u_0).\end{cases}\tag{1.1}$$

g is the pressure, K the capillary coefficient, a smooth function $\mathbb{R}^{+*}\to\mathbb{R}^{+*}$. It appears in the litterature in various contexts depending on K. The case where K is a constant has been largely investigated, see the seminal paper [12], and corresponds to capillary fluids. The important case where K is proportional to $1/\rho$ corresponds to quantum fluids. In this case the equations are formally equivalent to the nonlinear Schrödinger equation

$$i\partial_t\psi+\Delta\psi=\frac{g(|\psi|^2)}{2}\psi,\tag{1.2}$$

through the so called Madelung transform $\psi=\sqrt{\rho}e^{i\varphi/2},\ \nabla\varphi=u$. It is worth pointing out that even for a smooth solution of NLS the map $(\psi\to(\rho,u)$ is not well defined if ψ cancels, e.g. in the presence of vortices.

The main result on local well-posedness for the general Euler–Korteweg system is due to Benzoni–Danchin–Descombes [8], we shall use the following (slightly simpler) version:

Theorem 1.1 ([8]). — *For $(\rho_0-\alpha,u_0)\in\mathcal{H}^s(\mathbb{R}^d)$, $s>d/2+1$, with $\mathcal{H}^s:=H^{s+1}\times H^s$, there exists a unique solution $(\rho,u)\in(\alpha+C_tH^{s+1})\times C_tH^s$ to* (1.1)*, and it exists on $[0,T]$ if the following two conditions are satisfied*

(1) $\inf_{\mathbb{R}^d\times[0,T]}\rho(x,t)>0$,
(2) $\displaystyle\int_0^T\|\Delta\rho(s)\|_\infty+\|\nabla u(s)\|_\infty\mathrm{d}s.$

The original proof also shows that the time of existence of the solutions is of order at least $\ln(1/\|(\rho_0-\alpha,u_0)\|_{H^{s+1}\times H^s})$. This rather small lower bound is due to the absence of assumptions on the pressure term which can cause exponentially growing instabilities. For stable pressure terms, this result was more recently sharpened by Benzoni and Chiron [6] who obtained the natural time $O(1/\|(\rho_0-\alpha,u_0)\|_{H^{s+1}\times H^s}$.

In irrotational settings, the author proved with B. Haspot [4] that small irrotational initial data lead to a global solution under the stability condition $g'(\alpha)>0$. This standard condition is equivalent to the hyperbolicity of the Euler equations at $(\alpha,0)$. The main focus of this paper is to describe more accurately the time of existence for small data that have a non zero rotational part.

We denote $\mathbb{Q} = \Delta^{-1}\nabla\,\mathrm{div}$ the projector on potential vector fields, $\mathbb{P} = I - \mathbb{Q}$ the projector on solenoidal vector fields. In this paper, we prove the following informally stated theorem (see Theorems 4.1 and 5.1 for the precise statements):

Theorem 1.2. — *Let $d \geqslant 3$, $\alpha > 0$ a positive constant such that $g'(\alpha) > 0$. For some function spaces X, Y, Z, if $\|\rho_0 - \alpha\|_X$, $\|u_0\|_Y$, $\|\mathbb{P}u_0\|_Z$ are small enough, then there exists $c(d,\alpha) > 0$ such that the time of existence of the solution to* (1.1) *is bounded from below by $c/\|\mathbb{P}u_0\|_Z$.*

Note that in the special case $\mathbb{P}u_0 = 0$, we recover the global well-posedness result from [4]. We should point out immediately that since this paper is a continuation of [4], the proof in the case of dimension 3 is not self-contained and relies on several arguments from the previous paper.

Before commenting on the proof and sharpness of this result, let us give a bit more background on the well-posedness theory of the Euler–Korteweg system.

Weak solutions. In the case of the quantum Navier–Stokes equations (K proportional to $1/\rho$ and addition of a viscosity term) the existence of global weak solutions has been obtained under various assumptions. An important breakthrough was obtained by Bresch et al. [12], introducing what is now called the Bresh–Desjardins entropy, a key a priori estimate to construct global weak solutions by compactness methods.

The inviscid case is more intricate. As the existence of global strong solutions to the Schrödinger equation (1.2) with a large range of nonlinearities is well-known, Antonelli–Marcati [2] managed to use the formal equivalence with (1.1) to construct global weak solutions, the main difficulty being to give a meaning to the Madelung transform in the vacuum region where ρ cancels. See also the review paper [14] for a simplified proof. Relative entropy methods have since been developed [13, 18] that should eventually lead to the existence of global weak solutions for more general capillary coefficients K. Noticeably, these methods allow solutions with vorticity.

Strong solutions. Theorem 1.1 from [8] is the first well-posedness result in very general settings, an important idea due to Frédéric Coquel was to use a reformulation of the equations as a quasi-linear degenerate Schrödinger equation for which energy estimates in arbitrary high Sobolev spaces can be derived.

For quantum hydrodynamics ($K = 1/\rho$) in the long wave regime with irrotational velocity, the time interval of existence was improved by Béthuel–Danchin–Smets [11] thanks to the use of Strichartz estimates and the

Madelung transform. This approach is not tractable to the general case of system (1.1). Note however that the long wave limit for general K, g was recently studied by Benzoni and Chiron [6], the authors prove convergence to more classical equations such as Burgers, KdV or KP. Their analysis does not require the solutions to be irrotational.

The analogy with the Schrödinger equation was pushed further in [4] where the authors prove the existence of global strong solutions for small irotational data in dimension at least 3. As a byproduct of the proof, such solutions behave asymptotically as solutions of the linearized system near a constant density and zero speed, i.e. they “scatter”. The strategy of proof was reminiscent of ideas developed by Gustafson, Nakanishi and Tsai [23] for the Gross–Pitaevskii equation, and more generally the method of space time resonance (see Germain–Masmoudi–Shatah [16] for a clear description) which has had prolific applications for nonlinear dispersive equations. To some extent the present paper is a continuation of such results for a mixed dispersive-transport system, instead of purely dispersive.

Traveling waves. Traveling waves are solutions that only depend on $x \cdot e - ct$ for some direction $e \in \mathbb{R}^d$ and speed c. In dimension 1, the existence of solitons (traveling waves with same limits at $\pm\infty$) and kinks (different limits at $\pm\infty$) was proved in [9] by ODE methods. A stability criterion à la Grillakis–Shatah–Strauss [20] was also exhibited. It is a stability of weak type, as it implies that the solution remains close to the soliton in a norm that does not give local well-posedness (stability “until possible blow up”). Still in dimension 1, the author proved the existence of multi-solitons type solutions, a first example of global solution in small dimension which is not an ODE solution. Finally, motivated by the scattering result [4] in dimension larger than 2, the author also proved in [3] the existence of small amplitude traveling waves in dimension 2, an obstruction to scattering.

Blow up. To the best of our knowledge, blow up for the Euler–Korteweg system is a completely open problem. The formation of vacuum for NLS equations with non zero conditions at infinity is also not completely understood. We construct in Section 6 a solution to (1.1) (quantum case $K = 1/\rho$) that blows up in finite time. The construction is very simple, it relies on the existence of smooth solutions to (1.2) such that ψ vanishes at some time and the reversibility of (1.1).

The Euler–Korteweg system with a small vorticity. To give some intuition of our approach it is useful to introduce the reformulation from [8]: set $\nabla l := w := \sqrt{K/\rho}\nabla\rho$, then for a smooth solution without vacuum (1.1) is equivalent to the extended system

$$\begin{cases} \partial_t l + u\cdot\nabla l + a\operatorname{div} u &= 0,\\ \partial_t w + \nabla(u\cdot w) + \nabla(a\operatorname{div} u) &= 0,\\ \partial_t u + u\cdot\nabla u - w\cdot\nabla w - \nabla(a\operatorname{div} w) + g'w &= 0,\end{cases} \tag{1.3}$$

with $a=\sqrt{\rho K}$ and the second equation is simply the gradient of the first one.

If u is irrotational, setting $z=u+iw$ we have using $\nabla\operatorname{div} z=\Delta z$

$$\partial_t z + ia\Delta z = \mathcal{N}(z,\nabla z,\nabla^2 z),$$

despite the fact that $\mathcal{N}$ is a highly nonlinear term, the link with the Schrödinger equation is clear. This observation is the starting point of the analysis in [4]. Note that we have included $g'(\rho)w$ in $\mathcal{N}$, which is at first order a linear term and thus should be taken into account for long time dynamics.

If u is not potential, it is natural to write $u=\mathbb{Q}u+\mathbb{P}u$ and split the potential and the solenoidal part of the last equation. Using $\mathbb{Q}u\cdot\nabla\mathbb{Q}u=\frac{1}{2}\nabla|\mathbb{Q}u|^2$, the last two equations of (1.3) rewrite

$$\begin{cases} \partial_t w + \nabla(a\operatorname{div}\mathbb{Q}u) &= -\nabla(u\cdot w),\\ \partial_t\mathbb{Q}u - \nabla(a\operatorname{div} w) + g'w &= -\mathbb{Q}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u)\\ &\qquad -\frac{1}{2}\nabla(|\mathbb{Q}u|^2-|w|^2),\\ \partial_t\mathbb{P}u + \mathbb{P}(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u) = 0. \end{cases} \tag{1.4}$$

The only important point is that the first two equations are still the same Schrödinger type equation with some nonlinear coupling with $\mathbb{P}u$, and $\mathbb{P}u$ is a kind of transport equation with forcing $\mathbb{P}(\mathbb{P}u\cdot\nabla\mathbb{Q}u)$. An essential point is that this "forcing term" also contains $\mathbb{P}u$ in factor.

One might think of this system as the coupling between an equation with some linear decay mechanism (the Schrödinger equation) and a nonlinear equation without decay. The simplest toy model that corresponds to this picture is the following ODE system

$$\begin{cases} x' = -x + x^2 + y^2,\\ y' = y(x+y), \end{cases} \tag{1.5}$$

where one should think of x as $\mathbb{Q}u+iw$, y as $\mathbb{P}u$ and the linear evolution $x'=-x$ gives decay. The proof of the following elementary property is the guideline of this paper:

Proposition 1.3. — *Assume $|x(0)|\leqslant\varepsilon$, $|y(0)|\leqslant\delta\leqslant\varepsilon$. Then for ε,δ small enough there exists $c>0$ such that the solution of* (1.5) *exists on a time interval $[0,T]$ with $T\geqslant c/\delta$.*

Proof. — We plug the ansatz

$$|x(t)| \leqslant \delta + 2\varepsilon e^{-t}, |y(t)| \leqslant 2\delta, \tag{1.6}$$

in (1.5):

$$|x(t)| \leqslant \varepsilon e^{-t} + \int_0^t e^{s-t}\big(2\delta^2 + 8\varepsilon^2 e^{-2s} + 4\delta^2\big)\,\mathrm{d}s \leqslant (\varepsilon + 8\varepsilon^2)e^{-t} + 6\delta^2,$$

$$|y(t)| \leqslant \delta + \int_0^t 2\delta(\delta + 2\varepsilon e^{-s} + 2\delta)\,\mathrm{d}s \leqslant \delta(1 + 6\delta t + 4\varepsilon).$$

For $\varepsilon, \delta \leqslant 1/16$, $t \leqslant 1/(12\delta)$ we get

$$|x(t)| \leqslant \frac{3}{2}\varepsilon e^{-t} + \frac{3}{8}\delta,\ \ |y(t)| \leqslant \frac{7}{4}\delta,$$

so that a standard continuation argument ensures that the solution exists on $[0, 1/(12\delta)]$ and (1.6) is true on this interval. □

Of course, some difficulties arise in our case: first due to the quasi-linear nature of the problem, loss of derivatives are bound to arise. This is handled by a method well-understood since the work of Klainerman–Ponce [25], where one mixes dispersive (decay) estimates with high order energy estimates (see for example the introduction of [4] for a short description).

The second difficulty is more significant and is due to some lack of integrability of the decay. Basically, we have $\|e^{it\Delta}\|_{L^p\to L^{p'}} \lesssim 1/t^{d(1/2-1/p)}$, which is weaker as the dimension decreases. Again, it was identified in [25] that this is not an issue for quasi-linear Schrödinger equations if $d \geqslant 5$, but the case $d < 5$ requires much more intricate (and recent) methods.

There has been an extremely abundant activity on global well-posedness for quasi-linear dispersive equations over the last decade. The method of space-time resonances initiated by Germain–Masmoudi–Shatah[16] and Gustafson–Nakanishi–Tsai [23] led to numerous improvements and applications to various problems such as the Euler–Poisson system [21], the water waves with or without surface tension [1, 15]…

The issue of long time existence for coupled dispersive-transport equations is more scarce. To our knowledge, the only published result is the pioneering result of Ionescu and Lie for the Euler–Maxwell system [24], see also the preprint of D. Ginsberg [19] in the context of water waves. The spirit of the proofs in these references seems close to Proposition 1.3, despite the considerable technical difficulties that surround it.

It is worth pointing out that the time of existence is quite natural: it is related to the time of existence for $y' = y^2$, which is $1/y(0)$. It should be understood that the finite time of existence is due to the lack of control of the transport equation.

Organization of the article. We define our notations, functional framework and recall some technical tools in Section 2. Section 3 is devoted to some energy estimates for (1.1). As is common for dispersive equations, the proof of Theorem 1.2 is more difficult in smaller dimensions. Here $d \geqslant 5$ is quite straightforward and is treated in Section 4 while $d = 3$ is in Section 5. The case $d = 4$ is similar to $d = 3$ but simpler, thus it is not detailed. A large part of the analysis in dimension 3 builds upon previous results from [4], and this part is not self-contained. The handling of new difficulties are detailed, but the delicate estimates for the so-called "purely dispersive" quadratic nonlinearities are redundant with those from [4] and are thus only partially carried out in Appendix B. We also construct in Section 6 an example of solution which blows up in finite time. This construction relies on the Madelung transform and the finite time formation of vacuum for the Gross–Pitaevskii equation.

Appendix A details an energy estimate from Section 3 which is more or less contained in the arguments from [8] and [6].

2. Notations and functional spaces

Constants and inequalities. We will denote by C a constant used in the bootstrap argument of Sections 4 and 5, it remains the same in the section. Constants that are allowed to change from line to line are rather denoted $C_1, C_2 \cdots$

We write $a \lesssim b$ when there exists C_1 such that $a \leqslant C_1 b$, with C_1 a "constant" that depends in a clear way on the various parameters of the problem.

Functional spaces. $L^p(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d), W^{k,2} = H^k(\mathbb{R}^d), \dot{W}^{k,p}$ are the Lebesgue, Sobolev and homogeneous Sobolev spaces. $L^{p,q} = [L^{p_1}, L^{p_2}]_{\theta,q}$, $1/p = (1-\theta)/p_1 + \theta/p_2$, is the Lorentz interpolation space see [10].

$\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class, $\mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distribution.

If there is no ambiguity we drop the $(\mathbb{R}^d)$ reference. In our settings, ρ is one derivative more regular than u, therefore we define

$$\mathcal{H}^n = H^{n+1} \times H^n, \ \mathcal{W}^{k,p} = W^{k+1,p} \times W^{k,p}.$$

We recall the Sobolev embeddings

$$\forall, kp < d, \ \dot{W}^{k,p} \hookrightarrow L^q, \ \frac{1}{q} = \frac{1}{p} - \frac{k}{d}, \ \forall \ kp > d, \ W^{k,p} \hookrightarrow C^0 \cap L^\infty,$$

the tame product estimate for $p,q,r>1$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$

$$\|uv\|_{W^{k,r}}\lesssim \|u\|_{L^p}\|v\|_{W^{k,q}}+\|u\|_{W^{k,q}}\|v\|_{L^p}, \tag{2.1}$$

and the composition rule, for F smooth, $F(0)=0$,

$$\forall\ u\in W^{k,p}\cap L^\infty, \|F(u)\|_{W^{k,p}}\leqslant C(\|u\|_\infty)\|u\|_{W^{k,p}} \tag{2.2}$$

Fourier and bilinear Fourier multiplier. The Fourier transform of $f\in\mathcal{S}'(\mathbb{R}^d)$ is denoted $\widehat{f}$ or $\mathcal{F}(f)$. A Fourier multiplier of symbol $m(\xi)$ with moderate growth acts on $\mathcal{S}$

$$m(D)f=\mathcal{F}^{-1}(m(\xi)\widehat{f}(\xi)),$$

this extends naturally to matrix valued symbols. A multiplier denoted $m(-\Delta)$ is the multiplier of symbol $m(|\xi|^2)$.

The Mihlin–Hörmander theorem (see [10]) states that for M large enough, if for any multi-index α with $|\alpha|\leqslant M$, $|\nabla^\alpha m|\lesssim |\xi|^{-|\alpha|}$, then m acts continuously on L^p, $1<p<\infty$.

A bilinear Fourier multiplier of symbol $B(\eta,\xi-\eta)$ acts on $\mathcal{S}^2$

$$\begin{aligned}B[f,g]&=\mathcal{F}^{-1}\bigg(\int_{\mathbb{R}^d}B(\eta,\xi-\eta)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\mathrm{d}\eta\bigg)\\&=\mathcal{F}^{-1}\bigg(\int_{\mathbb{R}^d}B(\zeta-\eta,\zeta)\widehat{f}(\zeta-\eta)\widehat{g}(\zeta)\mathrm{d}\eta\bigg).\end{aligned}$$

The Coifman–Meyer [26] theorem states that if $|\nabla^k B|\lesssim 1/(|\xi|+|\eta|)^k$ for sufficiently many k, then B is continuous $L^p\times L^q\to L^r$, $1/p+1+q=1/r$, $p>1,\ q,r\leqslant\infty$.

We denote $\nabla_\xi B$ the bilinear multiplier of symbol $\nabla_\xi B(\eta,\xi-\eta)$, and similarly for $\nabla_\eta B$.

Potential and solenoidal fields. Potential fields v are vector fields of the form $v=\nabla f,\ f:\ \mathbb{R}^d\to\mathbb{C}$, they satisfy

$$\mathrm{curl}(v)=(\partial_i v_j-\partial_j v_i)_{1\leqslant i,j\leqslant d}=0.$$

Solenoidal fields satisfy $\mathrm{div}(v)=\sum\partial_i v_i=0$. Any vector field is the sum of a potential and a solenoidal fields.

The projector on potential vector fields is the Fourier multiplier $\mathbb{Q}=\Delta^{-1}\nabla\,\mathrm{div}$, the projector on solenoidal vector fields is $\mathbb{P}=I-\mathbb{Q}$. According to Mihlin–Hörmander multiplier theorem, $\mathbb{P}$ and $\mathbb{Q}$ act continuously $L^p\to L^p$, $1<p<\infty$, and in the related Sobolev spaces.

Reformulation of the equations. We denote $r=\rho-\alpha$, $r_0=\rho_0-\alpha$, $w:=\nabla l:=\sqrt{K/\rho}\nabla\rho$. According to [8], if $(r_0,u_0)\in\mathcal{H}^N$ with $N>d/2+1$, there exists a unique local solution to (1.1) such that $(\rho-\alpha,u)\in C_t\mathcal{H}^N$. For N large enough the solution is smooth so it is equivalent to work on the extended formulation (1.3).

Normalization. Under the stability condition $g'(\alpha)>0$, up to a change of variables, we can assume

$$\begin{cases}\alpha &=1,\\ a(1) &=1,\\ g'(1)&=2>0.\end{cases}\tag{2.3}$$

Equations (1.4) read

$$\begin{cases}\partial_t w+\Delta\mathbb{Q}u &=\nabla\big((1-a)\operatorname{div}\mathbb{Q}u-u\cdot w\big),\\ \partial_t\mathbb{Q}u+\mathcal{N}(\mathbb{Q}u,\mathbb{P}u,w)+(-\Delta+2)w &=\nabla\big((a-1)\operatorname{div}w\big)+(2-g')w,\\ \partial_t\mathbb{P}u+\mathbb{P}(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u) &=0.\end{cases}\tag{2.4}$$

with $\mathcal{N}(\mathbb{Q}u,\mathbb{P}u,w)=\mathbb{Q}(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u)+\frac{1}{2}\nabla(|\mathbb{Q}u|^2-|w|^2)$.
Set $U=\sqrt{\frac{-\Delta}{2-\Delta}}$, $H=\sqrt{-\Delta(2-\Delta)}$, then $\psi:=\mathbb{Q}u+iU^{-1}w$ satisfies

$$\begin{cases}\partial_t\psi-iH\psi &=\mathcal{N}_1(\psi,\mathbb{P}u),\\ \partial_t\mathbb{P}u+\mathbb{P}(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u)&=0.\end{cases}\tag{2.5}$$

with

$$\mathcal{N}_1=\nabla((a-1)\operatorname{div}w)+(2-g')w+iU^{-1}\nabla\big((1-a)\operatorname{div}\mathbb{Q}u-u\cdot w\big)-\mathcal{N}.\tag{2.6}$$

Note that U^{-1} is singular, but we have for $1<p<\infty$

$$\|U^{-1}w\|_{L^p}\sim\|\rho\|_{W^{1,p}},$$

therefore using the composition rule (2.2), at least when $\|(\rho-1,\mathbb{Q}u)\|_{\mathcal{W}^{k,p}}\ll 1$ and k is large enough

$$\|\psi\|_{W^{k,p}}\sim\|(\rho-1,\mathbb{Q}u)\|_{\mathcal{W}^{k,p}}.\tag{2.7}$$

Dispersive estimates. Dispersion estimates for the semi-group e^{itH} are due to Gustafson, Nakanishi and Tsai in [22], a version in Lorentz spaces follows from real interpolation as pointed out in [23].

Theorem 2.1 ([22, 23]). — *For $2\leqslant p\leqslant\infty$, $s\in\mathbb{R}^+$, $U=\sqrt{-\Delta/(2-\Delta)}$, we have*

$$\|e^{itH}\varphi\|_{W^{s,p}}\lesssim\frac{\|U^{(d-2)(1/2-1/p)}\varphi\|_{W^{s,p'}}}{t^{d(1/2-1/p)}},\tag{2.8}$$

and for $2\leqslant p<\infty$

$$\|e^{itH}\varphi\|_{L^{p,2}}\lesssim\frac{\|U^{(d-2)(1/2-1/p)}\varphi\|_{L^{p',2}}}{t^{d(1/2-1/p)}}.\tag{2.9}$$

Remark 2.2. — The estimates from [22] actually involve Besov spaces $B^s_{p,2}$ instead of $W^{s,p}$, and are slightly better than (2.8) due to the embedding $B^s_{p,2} \subset W^{s,p}$, $B^s_{p',2} \supset W^{s,p'}$ (see [10, Chapter 6]).

3. Energy estimates

High total energy estimate. The following energy estimate bounds all components of the solution (ρ, u).

PROPOSITION 3.1. — *We recall the notation $r = \rho - 1$. For $(r_0, u_0) \in \mathcal{H}^N(\mathbb{R}^d)$, $N > d/2 + 1$, $\|r\|_{W^{2,\infty}} < \delta$ for some $\delta > 0$,*

$$\|(r,u)(t)\|_{\mathcal{H}^N} \leqslant C\|(r_0,u_0)\|_{\mathcal{H}^N} \exp\left(\int_0^t C\|(r,u)\|_{\mathcal{W}^{1,\infty}} \mathrm{d}s\right),$$

with $C = C(\|(r,u)\|_{L^\infty\mathcal{H}^N})$ a locally bounded function.

The proof, not new, is postponed for completeness in Appendix A.

Low transport energy estimate.

PROPOSITION 3.2. — *Let $\mathbb{P}u$ satisfy*

$$\partial_t \mathbb{P}u + \mathbb{P}(u \cdot \nabla \mathbb{P}u + \mathbb{P}u \cdot \nabla \mathbb{Q}u) = 0,$$

then for $p, q > 1,\ k \in \mathbb{N},\ 2k > d/q + 1$ we have the a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{P}u\|_{W^{2k,p}} \lesssim (\|\mathbb{P}u\|_{W^{2k,q}} + \|\mathbb{Q}u\|_{W^{2k,q}})\|\mathbb{P}u\|_{W^{2k,p}}. \tag{3.1}$$

Energy estimates for transport type equations are standard, see e.g. the textbook [5, chapter 3]. Note however that the "transport" term is $\mathbb{P}(u\cdot\nabla\mathbb{P}u)$ rather than $u\cdot\nabla\mathbb{P}u$, moreover there seems to be a loss of derivatives on $\mathbb{Q}u$ in the forcing term $\mathbb{P}u\cdot\nabla\mathbb{Q}u$, therefore we include a short self-contained proof.

Proof. — Set $P_k = \Delta^k\mathbb{P}u$, then $\Delta^k\mathbb{P} = \Delta^k - \Delta^{k-1}\nabla\operatorname{div}$ is a differential operator of order $2k$ so that

$$\partial_t P_k + (u \cdot \nabla P_k) = R_k(\mathbb{P}u, \mathbb{Q}u),$$

and since $\Delta^k\mathbb{P}\mathbb{Q} = 0$ we have

$$R_k = -[\Delta^k\mathbb{P}, u\cdot\nabla]\mathbb{P}u - \Delta^k\mathbb{P}(\mathbb{P}u\cdot\nabla\mathbb{Q}u) = -[\Delta^k\mathbb{P}, u\cdot\nabla]u - [\Delta^k\mathbb{P}, \mathbb{P}u\cdot\nabla]\mathbb{Q}u.$$

We take the scalar product with $|P_k|^{p-2}P_k$ and integrate in space to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P_k\|_p^p \lesssim \|\mathrm{div}(u)\|_\infty\|P_k\|_p^p + \left\|R_k|P_k|^{p-2}P_k\right\|_1.$$

Since $W^{2k,q}\subset W^{1,\infty}$, we are left to estimate terms of the form $\|\partial^\alpha\mathbb{P}u\cdot\partial^\beta v|P_k|^{p-1}\|_1$ with v a placeholder for $\mathbb{P}u$ or $\mathbb{Q}u$, $|\alpha|+|\beta|=2k+1$. For $1/p_1+1/p_2=1/p$ we have

$$\begin{aligned}\|\partial^\alpha\mathbb{P}u\cdot\partial^\beta v|P_k|^{p-1}\|_1&\lesssim\|\mathbb{P}u\|_{W^{|\alpha|,p_1}}\|v\|_{W^{|\beta|,p_2}}\|\mathbb{P}u\|^{p-1}_{W^{2k,p}}\\&\lesssim\|\mathbb{P}u\|_{W^{2k,p}}\|v\|_{W^{|\beta|,p_2}}\|\mathbb{P}u\|^{p-1}_{W^{2k,p}},\end{aligned}$$

provided $1/p-(2k-|\alpha|)/d\leqslant 1/p_1\leqslant 1/p$, which is equivalent to

$$0\leqslant\frac{1}{p_2}\leqslant\frac{2k-|\alpha|}{d}=\frac{|\beta|-1}{d}.$$

On the other hand we have $W^{2k,q}\subset W^{|\beta|,p_2}$ is satisfied provided $\frac{1}{q}-\frac{2k-|\beta|}{d}\leqslant\frac{1}{p_2}$, the two conditions on p_2 lead to $1/q<(2k-1)/d$ which is the assumption. We conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P_k\|_p^p\lesssim(\|\mathbb{P}u\|_{W^{2k,q}}+\|\mathbb{Q}u\|_{W^{2k,q}})\|\mathbb{P}u\|^p_{W^{2k,p}}.\tag{3.2}$$

Taking the L^p norm in (4.3) and using the continuity of $\mathbb{P}:\ L^p\to L^p$ directly gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{P}u\|_p&\lesssim(\|\mathbb{Q}u\|_\infty+\|\mathbb{P}u\|_\infty)\|\mathbb{P}u\|_{W^{1,p}}\\&\lesssim(\|\mathbb{Q}u\|_{W^{2k,q}}+\|\mathbb{P}u\|_{W^{2k,q}})\|\mathbb{P}u\|_{W^{2k,p}}.\end{aligned}\tag{3.3}$$

Summing (3.2) and (3.3) ends the proof. □

4. Well-posedness for $d\geqslant 5$

The main result of this section is the following:

Theorem 4.1. — *Under assumptions* (2.3)*, for* $d\geqslant 5$*, there exists* $(\varepsilon_0,c,N,k)\in(\mathbb{R}^{+*})^2\times\mathbb{N}^2$ *such that for* $\varepsilon\leqslant\varepsilon_0$, $\delta\leqslant\varepsilon$*, if*

$$\|(\rho_0-\alpha,u_0)\|_{\mathcal{H}^N\cap\mathcal{W}^{k,4/3}}\leqslant\varepsilon,\quad\|\mathbb{P}u_0\|_{W^{k,4}}\leqslant\delta,$$

then the solution of (1.1) *exists on* $[0,T]$ *with*

$$T\geqslant\frac{c}{\delta}.$$

We recall that the system satisfied by $\psi=\mathbb{Q}u+iU^{-1}w$ and $\mathbb{P}u$ is (see (2.5))

$$\begin{cases}\partial_t\psi-iH\psi&=\mathcal{N}_1(\psi,\mathbb{P}u),\\ \partial_t\mathbb{P}u+\mathbb{P}(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u)&=0.\end{cases}\tag{4.1}$$

The bootstrap argument. We shall prove the following property: for c, ε small enough, there exists $C > 0$ such that for $t \leqslant c/\delta$, if we have the estimates

$$\begin{aligned}\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} &\leqslant C\varepsilon,\\ \|\psi\|_{W^{k,4}} &\leqslant C\delta + \frac{C\varepsilon}{(1+t)^{d/4}},\\ \|\mathbb{P}u\|_{W^{k,4}} &\leqslant C\delta,\end{aligned}$$

that we respectively name total energy, dispersive estimate and transport energy, then

$$\begin{aligned}\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} &\leqslant C\varepsilon/2,\\ \|\psi\|_{W^{k,4}} &\leqslant C\delta/2 + \frac{C\varepsilon}{2(1+t)^{d/4}},\\ \|\mathbb{P}u\|_{W^{k,4}} &\leqslant C\delta/2.\end{aligned}$$

From now on, C is only used for the constant of the bootstrap argument, while other constants are labelled as $C_1, C_2, \dots$ and can change from line to line.

The energy estimate. Since $\|\psi\|_{H^N} \sim \|(\rho - 1, \mathbb{Q}u)\|_{\mathcal{H}^N}$, the energy estimate of Proposition 3.1 implies for $k > d/4 + 1$

$$\begin{aligned}\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} &\leqslant C_1\|z_0\|_{H^N} \exp\left(C_2 \int_0^t \|\psi\|_{W^{k,4}} + \|\mathbb{P}u\|_{W^{k,4}} \mathrm{d}s\right)\\ &\leqslant C_1\varepsilon \exp\big(C_2 C\big(2\delta t + \varepsilon/(d/4-1)\big)\big).\end{aligned}$$

Take $C \geqslant 2C_1 e^1$, for $t \leqslant c\delta$, ε, c small enough (depending on C) we have

$$\|\psi\|_{H^N} + \|\mathbb{P}u\|_{H^N} \leqslant C_1 e^1 \varepsilon \leqslant C\varepsilon/2. \tag{4.2}$$

The transport energy estimate. We apply Proposition 3.2 with $p = q = 4$, k even, $4k > d$, $t \leqslant c/\delta$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{P}u\|_{W^{k,4}} &\lesssim (\|\mathbb{P}u\|_{W^{k,4}} + \|\mathbb{Q}u\|_{W^{k,4}})\|\mathbb{P}u\|_{W^{k,4}}\\ &\leqslant \left(C\delta + C\delta + \frac{C\varepsilon}{(1+t)^{d/4}}\right)C\delta \qquad (4.3)\\ \Longrightarrow \|\mathbb{P}u\|_{W^{k,4}} &\leqslant \delta\left(1 + 2C_1C^2c + \frac{C^2C_1\varepsilon}{d/4-1}\right) \leqslant 2\delta < C\delta/2,\end{aligned}$$

for c, ε small enough, $C > 4$.

The dispersive estimate. The first equation in (2.5) rewrites

$$\psi(t) = e^{itH}\psi_0 + \int_0^t e^{i(t-s)H}\mathcal{N}_1(\psi, \mathbb{P}u)\,\mathrm{d}s,$$

The linear evolution $e^{itH}\psi_0$ is estimated with the dispersive estimate (2.8) and Sobolev embeddings

$$\|e^{itH}\psi_0\|_{W^{k,4}} \lesssim \min(\|\psi_0\|_{W^{k,4/3}}/t^{d/4}, \|\psi_0\|_{H^{k+d/4}}) \lesssim \frac{\varepsilon}{(1+t)^{d/4}}. \tag{4.4}$$

The structure of the nonlinearity does not matter here, the only important points are

(1) The presence of U^{-1} in $U^{-1}\nabla\big((1-a)\operatorname{div}\mathbb{Q}u)\big)$ is not an issue since $U^{-1}\nabla = \sqrt{2-\Delta}\nabla/|\nabla|$ is the composition of a smooth Fourier multiplier and the Riesz multiplier,
(2) All nonlinear terms are at least quadratic, and involve derivatives of order at most 2.

We only detail the estimate of $\mathbb{Q}(\mathbb{Q}u\cdot\nabla\mathbb{P}u)$ as the others can be done in a similar (simpler) way. Using the dispersion estimate and Sobolev embedding

$$\begin{aligned}&\left\|\int_0^t e^{i(t-s)H}\mathbb{Q}(\mathbb{Q}u\cdot\nabla\mathbb{P}u)\mathrm{d}s\right\|_{W^{k,4}}\\ &\qquad\lesssim \int_0^{t-1}\frac{\|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{W^{k,4/3}}}{(t-s)^{d/4}}\mathrm{d}s + \int_{t-1}^t \|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{H^{k+d/4}}\mathrm{d}s.\end{aligned}$$

The product rule (2.1) gives for $N > k+1+d/2$

$$\begin{aligned}\|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{W^{k,4/3}} &\lesssim \|\mathbb{Q}u\|_{L^4}\|\mathbb{P}u\|_{H^{k+1}} + \|\mathbb{Q}u\|_{W^{k,4}}\|\mathbb{P}u\|_{H^1}\\ &\leqslant 2C^2\left(\delta + \frac{\varepsilon}{(1+s)^{d/4}}\right)\varepsilon,\\ \|\mathbb{Q}u\cdot\nabla\mathbb{P}u\|_{H^{k+d/4}} &\lesssim \|\mathbb{Q}u\|_{W^{k+d/4,4}}\|\mathbb{P}u\|_{W^{1,4}} + \|\mathbb{Q}u\|_{L^4}\|\mathbb{P}u\|_{W^{k+1+d/4,4}}\\ &\lesssim \|\mathbb{Q}u\|_{H^N}\|\mathbb{P}u\|_{W^{k,4}} + \|\mathbb{Q}u\|_{W^{k,4}}\|\mathbb{P}u\|_{H^N}\\ &\leqslant C^2\varepsilon\delta + C^2\varepsilon\left(\delta + \frac{\varepsilon}{(1+s)^{d/4}}\right).\end{aligned}$$

The bootstrap assumption directly gives

$$\begin{aligned}&\left\|\int_0^t e^{i(t-s)H}\mathbb{Q}(\mathbb{Q}u\cdot\nabla\mathbb{P}u)\mathrm{d}s\right\|_{W^{k,4}}\\ &\leqslant C_1C^2\left(\delta\varepsilon + \varepsilon^2\int_0^{t-1}\frac{1}{(1+s)^{d/4}(t-s)^{d/4}}\mathrm{d}s\right) + C_1C^2\varepsilon\left(\delta + \frac{\varepsilon}{(1+t)^{d/4}}\right)\\ &\leqslant C_2C^2\varepsilon\left(\delta + \frac{\varepsilon}{(1+t)^{d/4}}\right).\end{aligned}$$

We conclude by using (4.4), for C large enough, ε small enough

$$\begin{aligned}\|z(t)\|_{W^{k,4}} &\leqslant \frac{C_0\varepsilon}{(1+t)^{d/4}} + C_1C^2\varepsilon\Big(\delta + \frac{\varepsilon}{(1+t)^{d/4}}\Big)\\ &\leqslant \frac{C}{2}\Big(\delta + \frac{\varepsilon}{(1+t)^{d/4}}\Big).\end{aligned} \tag{4.5}$$

End of proof. Putting together (4.2), (4.3) and (4.5), we see that as long as the solution exists and $t \leqslant c/\delta$, $\|z\|_{\mathcal{H}^N}$ remains small and ρ remains bounded away from 0. According to the blow up criterion the solution exists at least for $t \leqslant c/\delta$.

5. Well-posedness for $d = 3, 4$

This section is similar to the previous one but is significantly more technical. The low dimension version of Theorem 4.1 reads

Theorem 5.1. — *Under assumptions 2.3, for $d = 3, 4$, there exists $(\varepsilon, c, N, k) \in (\mathbb{R}^{+*})^2 \times \mathbb{N}^2$, $p > \frac{2d}{d-2}$ such that for $\delta \leqslant \varepsilon$, if*

$$\|(r,u_0)\|_{\mathcal{H}^N\cap \mathcal{W}^{k,p'}} + \||x|(r_0,\mathbb{Q}u_0)\|_{L^2} \leqslant \varepsilon,\ \|\mathbb{P}u_0\|_{W^{k,p'}\cap W^{k,p}} + \||x|\mathbb{P}u_0\|_{L^2} \leqslant \delta,$$

then the solution of (1.1) *exists on* $[0,T]$ *with*

$$T \geqslant \frac{c}{\delta}.$$

Remark 5.2. — Unlike $d \geqslant 5$, one can not directly use the dispersive estimate to get closed bounds. This approach works for cubic and higher order nonlinearities, but not for quadratic terms. Therefore the emphasis is put here on how to control quadratic terms, while the analysis of higher order terms is much less detailed. We label such terms as "*cubic*" and they are generically denoted R. The fact that they include loss of derivatives is unimportant.

For $\psi:\ [0,T] \times \mathbb{R}^d \to \mathbb{C}^d$, and C a constant to choose later, we use the following notations:

$$\begin{aligned}\|\psi\|_{X(t)} &= \max\big(\|\psi(t)\|_{H^N} + \|xe^{-itH}\psi\|_{L^2}, (1+t)^{3(1/2-1/p)}(\|\psi\|_{W^{k,p}} - C\delta)\big),\\ \|\psi\|_{X_T} &= \sup_{[0,T]} \|\psi\|_{X(t)}.\end{aligned}$$

For simplicity of notations, we only consider the (most difficult) case $d = 3$.

5.1. Preparation of the equations

We recall that the extended system on $\psi = \mathbb{Q}u + iU^{-1}w$ is

$$\begin{cases} \partial_t\psi - iH\psi & = \mathcal{N}_1(\psi, \mathbb{P}u) + R, \text{ R cubic}, \\ \partial_t\mathbb{P}u + \mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) & = 0, \end{cases} \tag{5.1}$$

$$\begin{aligned} \mathcal{N}_1 = \nabla\big((1-a)\operatorname{div} w\big) + (2-g')w - \frac{1}{2}\nabla(|\mathbb{Q}u|^2 - |w|^2) \\ + iU^{-1}\nabla\big((1-a)\operatorname{div}\mathbb{Q}u - \mathbb{Q}u\cdot w\big) \\ - \mathbb{Q}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) - iU^{-1}\nabla\big(\mathbb{P}u\cdot w\big). \end{aligned}$$

The first line of the nonlinearity $\mathcal{N}_1$ depends only on the dispersive variable ψ (purely dispersive terms) while the second line contains dispersive-transport and purely transport terms.

In order to apply the method of space-time resonances, it is useful that the Fourier transform of the purely dispersive nonlinear terms cancels at 0. As such, the real part $\nabla((1-a)\operatorname{div} w) + (2-g')w + \frac{1}{2}\nabla(|\mathbb{Q}u|^2 - |w|^2)$ is well prepared, but not the imaginary part $U^{-1}\nabla((1-a)\operatorname{div}\mathbb{Q}u)$. We refer to the discussion at the beginning of Section 5 in [4] for a more detailed motivation.

We use the following normal form transform:

Lemma 5.3. — *For*

$$w_1 = w - \nabla\big(B[w,w] - B[\mathbb{Q}u, \mathbb{Q}u]\big).$$

with B the bilinear Fourier multiplier of symbol $\frac{a'(1)-1}{2(2+|\eta|^2)+|\xi-\eta|^2)}$. Then w_1 satisfies

$$\partial_t w_1 + \Delta\mathbb{Q}u = \nabla\operatorname{div}((1-a)\mathbb{Q}u) + \nabla R, \tag{5.2}$$

where R contains cubic and higher order nonlinearities in $\mathbb{Q}u, \mathbb{P}u, l$.

Moreover, for any $T > 0$ the map $\psi = \mathbb{Q}u + iU^{-1}w \to \mathbb{Q}u + iU^{-1}w_1$ is bi-lipschitz on a neighbourhood of 0 in X_T, it is also bi-Lipschitz near 0 for the norm $\|\psi_0\|_{H^N\cap W^{k,p'}} + \||x|\psi_0\|_{L^2}$.

Proof. — According to (2.4) w satisfies

$$\begin{aligned} \partial_t w + \Delta\mathbb{Q}u &= \nabla\big((1-a)\operatorname{div}\mathbb{Q}u\big) - \nabla(u\cdot w) \\ &= \nabla\operatorname{div}\big((1-a)\mathbb{Q}u\big) + \nabla\big(\nabla a\cdot\mathbb{Q}u\big) - \nabla(\mathbb{Q}u\cdot w) - \nabla(\mathbb{P}u\cdot w) \\ &= \nabla\operatorname{div}\big((1-a)\mathbb{Q}u\big) + \nabla\big((a'(1)-1)w\cdot\mathbb{Q}u\big) - \nabla(\mathbb{P}u\cdot w) + \nabla R, \end{aligned}$$

with $R = \big((\nabla a - a'(1)w)\cdot \mathbb{Q}u\big)$ a cubic term. Then w_1 satisfies

$$\begin{aligned}\partial_t w_1 + \Delta\mathbb{Q}u &= \nabla \operatorname{div}\big((1-a)\mathbb{Q}u\big) - \nabla(\mathbb{P}u\cdot w) + \nabla R_1\\ &\quad + \nabla\Big((a'(1)-1)w\cdot\mathbb{Q}u + 2B[w,\Delta\mathbb{Q}u] + 2B[(\Delta-2)w,\mathbb{Q}u]\Big)\\ &= \nabla \operatorname{div}\big((1-a)\mathbb{Q}u\big) - \nabla(\mathbb{P}u\cdot w) + \nabla R_1,\end{aligned}$$

by construction of B, and $R_1 = R - 2B[\partial_t w + \Delta\mathbb{Q}u, w] + 2B[\mathbb{Q}u, \partial_t\mathbb{Q}u + (2-\Delta)w]$ is cubic. The fact that $w\to w_1$ is bi-Lipschitz is Propositions 5.4 and 5.5 in [4]. $\square$

Final form of the equations. We define $b(\mathbb{Q}u,w) = B[w,w] - B[\mathbb{Q}u,\mathbb{Q}u]$ so that $w = w_1 + \nabla b(\mathbb{Q}u,w) = w - \nabla b(\mathbb{Q}u,w_1) + \nabla R$, R cubic. The new system on $\Psi = \mathbb{Q}u + iU^{-1}w_1$ and $\mathbb{P}u$ is

$$\begin{cases}\partial_t\Psi - iH\Psi = \nabla\Big((\Delta-2)b + (a-1)\operatorname{div} w_1 - \frac{1}{2}\big(|\mathbb{Q}u|^2 - |w_1|^2\big)\Big)\\ \qquad\qquad + (2-g')w_1 + iU^{-1}\nabla\operatorname{div}\big((1-a)\mathbb{Q}u\big)\\ \qquad\qquad - iU^{-1}\nabla(\mathbb{P}u\cdot w_1) - \mathbb{Q}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u) + R,\\ \partial_t\mathbb{P}u = -\mathbb{P}(u\cdot\nabla\mathbb{P}u + \mathbb{P}u\cdot\nabla\mathbb{Q}u),\end{cases} \tag{5.3}$$

with R containing cubic terms.

Remark 5.4. — We emphasize that in the PDE (5.2) satisfied by w_1 the cubic terms are of the form ∇R. Hence even after applying U^{-1} to obtain the final system (5.3), the cubic terms $U^{-1}\nabla R$ are still not singular.

Remark 5.5. — An important consequence of Lemma 5.3 is that it suffices to estimate Ψ instead of ψ, and the smallness of ψ_0 implies the smallness of Ψ_0.

According to the remark above, it is sufficient to prove the following:

Theorem 5.6. — *Under assumptions* (2.3)*, there exists* $\varepsilon, c, N, k \in (\mathbb{R}^{+*})^2\times\mathbb{N}^2$*,* $p > 2d/(d-2)$ *such that for* $\delta \leqslant \varepsilon$*, if*

$$\begin{aligned}\|\Psi_0\|_{H^N\cap W^{k,p'}} + \||x|\Psi_0\|_{L^2} + \|\mathbb{P}u_0\|_{H^N} &\leqslant \varepsilon,\\ \|\mathbb{P}u_0\|_{W^{k,p}\cap W^{k,p'}} + \||x|\mathbb{P}u_0\|_{L^2} &\leqslant \delta,\end{aligned}$$

then the solution of (5.3) *exists on* $[0,T]$*,* $T\geqslant c/\delta$ *and* $\|\Psi\|_{X_T}\lesssim\varepsilon$*.*

This result implies Theorem 5.1.

5.2. The bootstrap argument

A priori estimates. The aim of this section and the next one is to prove that for c,ε small enough, there exists $C>0$ such that for $t\leqslant c/\delta$, if we have the following estimates

$$\begin{cases} \|\Psi\|_{H^N}+\|\mathbb{P}u\|_{H^N}\leqslant C\varepsilon & \text{(total energy)},\\ \||x|e^{-itH}\Psi\|_{L^2}\leqslant C\varepsilon, & \\ \qquad \|\Psi\|_{W^{k,p}}\leqslant C\delta+\frac{C\varepsilon}{(1+t)^{3(1/2-1/p)}} & \text{(dispersive estimates)},\\ \|\mathbb{P}u\|_{W^{k,p}\cap W^{k,p'}}+\||x|\mathbb{P}u\|_{L^2}\leqslant C\delta & \text{(transport energy)}, \end{cases} \tag{5.4}$$

then the same estimates hold with $C/2$ instead of C. Since $p>6$, $3(1/2-1/p)>1$ and we introduce the convenient notation

$$1+\gamma:=3\left(\frac{1}{2}-\frac{1}{p}\right). \tag{5.5}$$

Remark 5.7. — We point out that the bootstrap argument is slightly different from the one for Theorem 4.1. Indeed in large dimension, we can propagate the a priori bounds (up to multiplicative constants independent of ε,δ) on a time c/δ while for $d=3,4$ the proof implies $c=O(\varepsilon)$. In other words, if $\|\Psi_0\|_{H^N}\leqslant\varepsilon'<\varepsilon$ it is not clear if $\|\Psi(t)\|_{H^N}\lesssim\varepsilon'$ on $[0,c/\delta]$ with c independent of ε', see Remark 5.8 for technical details.

The dispersive estimates are significantly more difficult than for $d\geqslant 5$ and are detailed in Section 5.3.

The energy estimate. This is the same argument as for $d\geqslant 5$, from Proposition 3.1 and using $3(1/2-1/p)>1$ (integrability of the decay)

$$\|\Psi\|_{H^N}+\|\mathbb{P}u\|_{H^N}\lesssim C_1\varepsilon\exp\left(C_2C\left(2\delta t+\frac{\varepsilon}{\gamma}\right)\right),$$

so that for C large enough, ε,c small enough, $t\leqslant c/\delta$

$$\|\Psi\|_{H^N}+\|\mathbb{P}u\|_{H^N}\leqslant C\varepsilon/2. \tag{5.6}$$

The transport energy estimate. The $W^{k,q}$ estimate is a consequence of Proposition 3.2 as for $d\geqslant 5$ with indices (p,p): for k even large enough, c,ε small enough

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{P}u\|_{W^{k,p}}\leqslant C_1(\|\mathbb{P}u\|_{W^{k,p}}+\|\mathbb{Q}u\|_{W^{k,p}})\|\mathbb{P}u\|_{W^{k,p}}\lesssim C_1C^2\left(\delta+\frac{\varepsilon}{t^{1+\gamma}}\right)\delta$$
$$\Longrightarrow\|\mathbb{P}u\|_{W^{k,p}}\leqslant\delta+C_1C^2\delta(c+C_2\varepsilon)\leqslant\frac{C}{2}\delta. \tag{5.7}$$

Applying again Proposition 3.2 with indices p',p gives

$$\|\mathbb{P}u\|_{W^{k,p'}}\leqslant\delta+C_1\int_0^t(\|\mathbb{P}u\|_{W^{k,p}}+\|\mathbb{Q}u\|_{W^{k,p}})\|\mathbb{P}u\|_{W^{k,p'}}\mathrm{d}s\leqslant\frac{C\delta}{2}.$$

For the weighted estimate we follow a similar energy method. First multiply the equation on $\mathbb{P}u$ by x_j:

$$\begin{aligned}\partial_t(x_j\mathbb{P}u) &+ x_j\mathbb{P}(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u)\\ &= \partial_t(x_j\mathbb{P}u)+\mathbb{P}\big(u\cdot\nabla(x_j\mathbb{P}u)\\ &\qquad + x_j\mathbb{P}u\cdot\nabla\mathbb{Q}u\big)+[x_j,\mathbb{P}](u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u)-\mathbb{P}\big(u_j\mathbb{P}u\big)\\ &= 0.\end{aligned}$$

The operator $[x_j,\mathbb{P}]$ is the Fourier multiplier of symbol $i\partial_{\xi_j}\mathbb{P}(\xi)$ which is dominated by $1/|\xi|$ therefore it is bounded $\dot{H}^{-1}\mapsto L^2$. From the embedding $\dot{H}^1\subset L^6$, $[x_j,\mathbb{P}]$ is bounded $L^{6/5}\to L^2$. We deduce the following bound

$$\begin{aligned}\left|\int_{\mathbb{R}^d}[x_j,\mathbb{P}](u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u)\cdot(x_j\mathbb{P}u)\mathrm{d}x\right|&\\ \lesssim \|u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u\|_{L^{6/5}}\|x_j\mathbb{P}u\|_{L^2}&\\ \lesssim \|\mathbb{P}u\|_{W^{k,6/5}}\big(\|\mathbb{P}u\|_{W^{k,p}}+\|\mathbb{Q}u\|_{W^{k,p}}\big)\|x_j\mathbb{P}u\|_{L^2}.&\end{aligned}$$

Using an integration by parts and the boundedness of the multiplier $\nabla[x_j,\mathbb{P}]$

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\mathbb{P}\big(u\cdot\nabla(x_j\mathbb{P}u)\big)\cdot x_j\mathbb{P}u\mathrm{d}x\right|\\ &=\left|\int_{\mathbb{R}^d}\big(u\cdot\nabla(x_j\mathbb{P}u)\big)\cdot([\mathbb{P},x_j]\mathbb{P}u+x_j\mathbb{P}u)\mathrm{d}x\right|\\ &=\left|\int_{\mathbb{R}^d}-\frac{\operatorname{div}\mathbb{Q}u}{2}\big(|x_j\mathbb{P}u|^2+2(x_j\mathbb{P}u)\cdot[\mathbb{P},x_j]\mathbb{P}u\big)-x_j\mathbb{P}u\cdot\big(u\cdot\nabla([\mathbb{P},x_j]\mathbb{P}u)\big)\mathrm{d}x\right|\\ &\lesssim\|\mathbb{Q}u\|_{W^{k,p}}(\|x_j\mathbb{P}u\|_{L^2}+\|\mathbb{P}u\|_{L^{6/5}})\|x_j\mathbb{P}u\|_{L^2}\\ &\qquad+(\|\mathbb{P}u\|_{W^{k,p}}+\|\mathbb{Q}u\|_{W^{k,p}})\|\mathbb{P}u\|_{L^2}\|x_j\mathbb{P}u\|_{L^2}.\end{aligned}$$

Similarly

$$\begin{aligned}\left|\int_{\mathbb{R}^d}(x_j\mathbb{P}u\cdot\nabla\mathbb{Q}u)\cdot(x_j\mathbb{P}u)\mathrm{d}x\right|&\leqslant\|\mathbb{Q}u\|_{W^{k,p}}\|x_j\mathbb{P}u\|_{L^2}^2,\\ \left|\int_{\mathbb{R}^d}\mathbb{P}(u_j\mathbb{P}u)\cdot x_j\mathbb{P}u\mathrm{d}x\right|&\lesssim\big(\|\mathbb{P}u\|_{W^{k,p}}+\|\mathbb{Q}u\|_{W^{k,p}}\big)\|\mathbb{P}u\|_{L^2}\|x_j\mathbb{P}u\|_{L^2}.\end{aligned}$$

From these estimates and since $p'<6/5$ we deduce

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|x_j\mathbb{P}u\|_{L^2}^2\right|\leqslant C\|x_j\mathbb{P}u\|_{L^2}\big(\|\mathbb{Q}u\|_{W^{k,p}}+\|\mathbb{P}u\|_{W^{k,p}\cap W^{k,p'}}+\|x_j\mathbb{P}u\|_{L^2}\big)^2,$$

which readily yields by integration in time and the bootstrap assumption (5.4)

$$\|x_j\mathbb{P}u\|_{L^2}\leqslant\delta+C_1C^2(\delta t+\varepsilon)\delta\leqslant\frac{C\delta}{2}. \tag{5.8}$$

5.3. The dispersive estimates

We start from (5.3) that reads $\partial_t\Psi = iH\psi + \mathcal{D}(\Psi) + \mathcal{T}(\Psi,\mathbb{P}u) + R$, with $\mathcal{D}$ the first two lines of nonlinear terms (quadratic dispersive terms), $\mathcal{T}$ the third line (dispersive-transport, and transport-transport) and R cubic. Equivalently

$$\Psi(t) = e^{itH}\Psi_0 + \int_0^t e^{i(t-s)H}\big(\mathcal{D}(\Psi) + \mathcal{T}(\Psi,\mathbb{P}u) + R\big)(s)\,\mathrm{d}s.$$

The linear part is not difficult to control:

$$\big\||x|e^{-itH}e^{itH}\Psi_0\big\|_{L^2} = \big\||x|\Psi_0\big\|_{L^2}, \tag{5.9}$$

$$\big\|e^{itH}\Psi_0\big\|_{W^{k,p}} \lesssim \frac{\|\Psi_0\|_{H^N\cap W^{k,p'}}}{(1+t)^{1+\gamma}}. \tag{5.10}$$

The terms in $\mathcal{D}$ and $\mathcal{T}$ are not estimated exactly similarly. Basically the control of $\mathcal{D}$ is quite difficult, but amounts to a straightforward modification of the estimates in [4], while $\mathcal{T}$ is new but a bit easier to control. For completeness, the key arguments to estimate $\mathcal{D}$ are sketched in the Appendix B.

The nonlinearity $\mathcal{T}$ contains four terms that are all very similar. For conciseness we only detail how to estimate $U^{-1}\nabla(\mathbb{P}u\cdot w_1)$, which contains all the difficulties of the other terms plus a singular factor U^{-1}. Finally, R contains cubic terms easier to control. To fix ideas, we shall also bound the cubic term $U^{-1}\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u),\mathbb{Q}u]$ that appears in the proof of Lemma 5.3).

Weighted bounds.

Quadratic term. We bound $x_je^{-itH}\int_0^t e^{i(t-s)H}U^{-1}\nabla(\mathbb{P}u\cdot\Psi)\mathrm{d}s$. Since $w_1 = U(\Psi-\overline{\Psi})/2$, we have

$$U^{-1}\nabla(\mathbb{P}u\cdot w_1) = U^{-1}\nabla\left(\mathbb{P}u\cdot U\frac{\Psi-\overline{\Psi}}{2}\right),$$

so we define $\varphi = U\Psi$.

We define $m(\xi,s) = -\partial_{\xi_j}(\xi U^{-1}(\xi)e^{-isH(\xi)})$, using Fourier transform we see

$$\begin{aligned}&x_je^{-itH}\int_0^t U^{-1}\nabla(\mathbb{P}u\cdot\varphi)\mathrm{d}s\\&= \mathcal{F}^{-1}\bigg(\int_0^t\int_{\mathbb{R}^d} m(\xi,s)\widehat{\mathbb{P}u}(\xi-\eta)\cdot\widehat{\varphi}(\eta)\mathrm{d}\eta\mathrm{d}s\bigg) + \int_0^t e^{-isH}U^{-1}\nabla(x_j\mathbb{P}u\cdot\varphi)\mathrm{d}s.\end{aligned}$$

We have $m = \partial_{\xi_j}(\xi U^{-1})e^{-isH} + i\xi U^{-1}e^{-isH}s\partial_{\xi_j}H := m_1 + m_2$. From elementary computations $m_1 = m_3/|\xi|$ with m_3 a bounded multiplier, therefore it is continuous $L^{6/5}\to L^2$ and

$$\begin{aligned}\left\|\int_0^t m_1(D)(\mathbb{P}u\cdot\varphi)\mathrm{d}s\right\|_{L^2} &\lesssim \int_0^t \|\mathbb{P}u\cdot\varphi\|_{L^{6/5}}\mathrm{d}s\\ &\lesssim \int_0^t \|\mathbb{P}u\|_{W^{k,p'}}\|\varphi\|_{W^{k,p}}\mathrm{d}s.\end{aligned}\tag{5.11}$$

Similarly $m_2(\xi,s)\lesssim s(1+|\xi|)^2$ so

$$\begin{aligned}\left\|\int_0^t m_2(D)(\mathbb{P}u\cdot\varphi)\mathrm{d}s\right\|_{L^2} &\lesssim \int_0^t s\|\mathbb{P}u\cdot\varphi\|_{H^2}\mathrm{d}s\\ &\lesssim \int_0^t s\|\mathbb{P}u\|_{H^2}\|\varphi\|_{W^{k,p}}\mathrm{d}s.\end{aligned}\tag{5.12}$$

Next we use a frequency truncation $\chi(D)$, with $\chi\in C_c^\infty,\ \chi\equiv 1$ near 0, and split

$$\int_0^t e^{-isH}U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)\mathrm{d}s=\int_0^t e^{-isH}(\chi+1-\chi)U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)\mathrm{d}s.$$

The low frequency part is estimated using the boundedness of $\chi U^{-1}\nabla$: $L^2\to L^2$

$$\begin{aligned}\left\|\int_0^t e^{-isH}\chi U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)\mathrm{d}s\right\|_2 &\lesssim \int_0^t \|x_j\mathbb{P}u\cdot\varphi\|_2\mathrm{d}s\\ &\lesssim \int_0^t \|x_j\mathbb{P}u\|_2\|\varphi\|_{W^{k,p}}\mathrm{d}s.\end{aligned}\tag{5.13}$$

For the high frequency part, we use that $(1-\chi)U^{-1}$ is a bounded multiplier, the identity

$$\begin{aligned}\nabla(x_j\mathbb{P}u\cdot\varphi)=(\mathbb{P}u\cdot\varphi)e_j+(\nabla\varphi)\cdot(x_j\mathbb{P}u)&\\ +\nabla(\mathbb{P}u)\cdot\big([x_j,Ue^{isH}]e^{-isH}\Psi+Ue^{isH}(x_je^{-isH}\Psi)\big)&,\end{aligned}$$

and the symbol bound $|[x_j,Ue^{isH}]|(\xi)\lesssim(1+s)(1+|\xi|)$, so

$$\begin{aligned}&\left\|\int_0^t e^{-isH}(1-\chi)U^{-1}\nabla\big(x_j\mathbb{P}u\cdot\varphi\big)\mathrm{d}s\right\|_2\\ &\quad\lesssim\int_0^t (\|\mathbb{P}u\|_2+\|x_j\mathbb{P}u\|_2)\|\varphi\|_{W^{k,p}}\\ &\qquad\qquad+\|\mathbb{P}u\|_{H^1}(1+s)\|\Psi\|_{W^{k,p}}+\|\mathbb{P}u\|_{W^{k,p}}\|x_je^{-isH}\Psi\|_2\mathrm{d}s.\end{aligned}\tag{5.14}$$

From estimates (5.11)–(5.14) and the bootstrap assumptions (5.4) we get for c, ε small enough, $t \leqslant c/\delta$

$$\begin{aligned} &\left\| x_j e^{-itH} \int_0^t e^{i(t-s)H} U^{-1} \nabla(\mathbb{P}u \cdot U\Psi) \mathrm{d}s \right\|_2 \\ &\qquad \lesssim C^2 \int_0^t (1+s)\delta \left(\delta + \frac{\varepsilon}{(1+s)^{3(1/2-1/p)}} \right) + \delta\varepsilon \mathrm{d}s \\ &\qquad \lesssim C^2(c^2 + c\varepsilon). \end{aligned} \tag{5.15}$$

Remark 5.8. — The weighted estimate is the only point in the proof where we need $c \lesssim \varepsilon$ (or more accurately $c^2 \ll \varepsilon$). This is due to the commutator identity $[x_j, e^{-isH}] = is\partial_{\xi_j} H$ which causes a strong loss of decay in the estimate (5.12). It is most likely that a more careful treatment of such estimates would lead to a weaker smallness condition on c.

Cubic term. From similar computations, we end up estimating terms among which

$$I = \int_0^t e^{-isH} U^{-1} \nabla B[\mathbb{Q}(u \cdot \nabla \mathbb{P}u), e^{isH} x_j e^{-isH} \mathbb{Q}u] \mathrm{d}\eta, \tag{5.16}$$

$$\int_0^t e^{-isH} s(\partial_{\xi_j} H) U^{-1} \nabla B[\mathbb{Q}(u \cdot \nabla \mathbb{P}u), \mathbb{Q}u] \mathrm{d}s, \tag{5.17}$$

are two significant examples.

For (5.16), since the symbol of B is $(a'(1)-1)/(2(2+|\eta|^2+|\xi-\eta|^2)))$ we may use the boundedness of the bilinear multiplier of symbol ξB

$$\begin{aligned} \|I\|_2 &\lesssim \int_0^t \|u \cdot \nabla \mathbb{P}u\|_\infty \|x_j e^{-isH} \mathbb{Q}u\|_2 \mathrm{d}s \\ &\lesssim \int_0^t \|u\|_{W^{k,p}} \|\mathbb{P}u\|_{W^{k,p}} \|x_j e^{-isH} \mathbb{Q}u\|_2 \mathrm{d}s \\ &\lesssim \int_0^t C^3 \left(\delta + \frac{\varepsilon}{(1+s)^{3(1/2-1/p)}} \right)^2 \varepsilon \mathrm{d}s \\ &\lesssim C^3(c\delta + \varepsilon^2)\varepsilon, \end{aligned} \tag{5.18}$$

similarly for (5.17)

$$\begin{aligned}\Big\|\int_0^t e^{-isH}s(\partial_j H)U^{-1}\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u),\mathbb{Q}u]\mathrm{d}s\Big\|_2&\\ \lesssim \int_0^t s\|u\|_{W^{k,p}}\|\mathbb{Q}u\|_{W^{k,p}}\|\mathbb{P}u\|_{W^{k,p}\cap W^{k,p'}}\mathrm{d}s&\\ \lesssim \int_0^t C^3 s\Big(\delta+\frac{\varepsilon}{(1+s)^{3(1/2-1/p)}}\Big)^3\mathrm{d}s&\\ \lesssim C^3(c^2\delta+\varepsilon^3).&\end{aligned}\tag{5.19}$$

(5.18) and (5.19) are clearly more than enough to close the weighted estimate.

Closing the bound. The estimates (5.15) (quadratic), and (5.18), (5.19) (cubic) lead to

$$\Big\|x_je^{-itH}\int_0^t e^{i(t-s)H}(\mathcal{T}(\psi,\mathbb{P}u)+R)\mathrm{d}s\Big\|_{L^2}\leqslant C^2C_1(c^2+c\varepsilon).\tag{5.20}$$

A similar bound can be obtained for the purely dispersive terms

$$\Big\|x_je^{-itH}\int_0^t e^{i(t-s)H}(\mathcal{D}(\psi,\mathbb{P}u))\mathrm{d}s\Big\|_{L^2}\leqslant C^2C_1(c^2+c\varepsilon).\tag{5.21}$$

See Appendix B for a sketch of argument and [4] for more details. Combining (5.9), (5.20), (5.21) we find

$$\|xe^{-itH}\Psi(t)\|_{L^2}\leqslant C_1\varepsilon+C^2C_1(c^2+c\varepsilon),\tag{5.22}$$

which gives the first part of the dispersive estimate by choosing C large enough, $c^2\ll\varepsilon$ small enough.

Bounds in $W^{k,p}$. As previously, the computations are done "up to choosing k, N larger".

Quadratic term. We focus on $U^{-1}\nabla(\mathbb{P}u\cdot U\Psi):=U^{-1}\nabla(\mathbb{P}u\cdot\varphi)$. We can assume $t\geqslant 2$, indeed for $t\leqslant 2$ by Sobolev's embedding and for N large enough

$$\begin{aligned}\Big\|\int_0^t e^{i(-s)H}U^{-1}\nabla(\mathbb{P}u\cdot\varphi)\mathrm{d}s\Big\|_{W^{k,p}}&\lesssim\int_0^t\|\nabla(\mathbb{P}u\cdot\varphi)\|_{H^{N-1}}\mathrm{d}s\\&\lesssim\|\mathbb{P}u\|_{L^\infty([0,2],H^N)}\|\varphi\|_{L^\infty([0,2],H^N)}\\&\leqslant C\varepsilon^2.\end{aligned}$$

We recall $p>6$, $3(1/2-1/p):=1+\gamma>1$. Minkowski's inequality and the dispersive estimate imply

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)H}U^{-1}\nabla(\mathbb{P}u\cdot\varphi)\mathrm{d}s\right\|_{W^{k,p}}&\\ \lesssim\int_0^{t-1}&\frac{\|\mathbb{P}u\cdot\varphi\|_{W^{k+1,p'}}}{(t-s)^{1+\gamma}}\mathrm{d}s+\int_{t-1}^t\|\mathbb{P}u\cdot\varphi\|_{H^{k+2}}\mathrm{d}s\\ \lesssim\int_0^{t-1}&\frac{\|\mathbb{P}u\|_{W^{k,p'}}\|\varphi\|_{H^N}+\|\mathbb{P}u\|_{\dot W^{k+1,q}}\|\varphi\|_{W^{k,p}}}{(t-s)^{1+\gamma}}\mathrm{d}s\\ &+\int_{t-1}^t\|\mathbb{P}u\|_{W^{k,p}}\|\varphi\|_{H^N}+\|\mathbb{P}u\|_{H^N}\|\varphi\|_{W^{k,p}}\mathrm{d}s,\end{aligned}$$

with $1/q=1-2/p$. We use Gagliardo–Nirenberg's inequality[(1)],

$$\|\mathbb{P}u\|_{\dot W^{k+1,q}}\lesssim\|\mathbb{P}u\|_{H^N}^a\|\mathbb{P}u\|_{L^{p'}}^{1-a},\ a=\frac{k+3/2}{N+1}.$$

From the crude bound $\|\mathbb{P}u\|_{H^N}^a\|\mathbb{P}u\|_{L^{p'}}^{1-a}\lesssim\varepsilon$, we have for $t\geqslant 2$, N large enough

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)H}U^{-1}\nabla(\mathbb{P}u\cdot\varphi)\mathrm{d}s\right\|_{W^{k,p}}&\\ \leqslant C_1C^2\int_0^{t-1}&\frac{\delta\varepsilon}{(t-s)^{1+\gamma}}+\frac{\varepsilon^2}{(1+s)^{1+\gamma}(t-s)^{1+\gamma}}\\ &+C_1C^2\left(\delta\varepsilon+\frac{\varepsilon^2}{(1+t)^{1+\gamma}}\right)\\ \leqslant C_1C^2&\left(\delta\varepsilon+\frac{\varepsilon^2}{(1+t)^{1+\gamma}}\right).\end{aligned}\tag{5.23}$$

Cubic term. As for the quadratic terms, we split the integral on $[0,t-1]\cup[t-1,t]$. The integral on $[t-1,t]$ is easily controlled, for the other part, using again the boundedness of ∇B, and choosing $1/q=1/p'-1/2$

$$\begin{aligned}&\left\|\int_0^{t-1}e^{i(t-s)H}U^{-1}\nabla B[\mathbb{Q}(u\cdot\nabla\mathbb{P}u),\mathbb{Q}u]\mathrm{d}s\right\|_{W^{k,p}}\\ &\lesssim\int_0^{t-1}\frac{\|u\cdot\nabla\mathbb{P}u\|_{W^{k,q}}\|\mathbb{Q}u\|_{H^N}}{(t-s)^{1+\gamma}}\mathrm{d}s\lesssim\int_0^{t-1}\frac{\|u\|_{W^{k,p}}\|\mathbb{P}u\|_{H^N}}{(t-s)^{1+\gamma}}\|\mathbb{Q}u\|_{H^N}\mathrm{d}s\\ &\lesssim C^3\varepsilon^2\left(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\right).\end{aligned}$$

(1) Provided $(k+1)/N\leqslant(k+3/2)/(N+1)$, which is true for N large enough.

Closing the bound. From (5.23) and the cubic estimates above we deduce

$$\Big\| \int_0^t e^{i(t-s)H}(\mathcal{T}(\psi, \mathbb{P}u) + R)\mathrm{d}s \Big\|_{W^{k,p}} \leqslant C^2 C_1 \varepsilon \Big(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}} \Big). \tag{5.24}$$

Again, we refer to Appendix B for a similar bound of the purely dispersive terms, and we conclude using also (5.10)

$$\begin{aligned} \|\Psi(t)\|_{W^{k,p}} &\leqslant \frac{C_1\varepsilon}{(1+t)^{1+\gamma}} + C^2 C_1 \varepsilon \Big(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}} \Big) \\ &\leqslant \frac{C}{2} \Big(\delta + \frac{\varepsilon}{(1+t)^{1+\gamma}} \Big). \end{aligned} \tag{5.25}$$

End of proof. As for Theorem 4.1, we close the bootstrap argument thanks to the energy estimate (5.6), the transport energy estimates (5.7), (5.8), and the dispersive estimates (5.22), (5.25).

6. An example of blow up

We consider in this section the special case of quantum fluids, where K is proportional to $1/\rho$. More precisely, if ψ is a smooth solution of

$$i\partial_t \psi + \Delta \psi = \frac{g(|\psi|^2)\psi}{2}, \tag{6.1}$$

that does not cancel, the so-called Madelung transform $\psi = \sqrt{\rho} e^{i\phi/2}$, $u = \nabla \phi$ is well defined and (ρ, u) satisfy

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) &= 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) &= 2\nabla\Big(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\Big). \end{cases} \tag{6.2}$$

As pointed out in the review article [14], the Madelung transform is a major tool to study nonlinear Schrödinger equations with non zero boundary conditions at infinity, with the (technical but important) drawback that it becomes singular in presence of vacuum, i.e. when ρ vanishes. We construct here an example of solution such that vacuum appears in finite time. Actually, thanks to reversibility(2) it is equivalent to construct a solution with vacuum at $t = 0$ and no vacuum for $t > 0$ and this is the property that we prove.

(2) The map $\psi(t) \to: \overline{\psi}(-t)$ leaves the solution set invariant, or equivalently $(\rho, u)(t) \to (\rho, -u)(-t)$

Proposition 6.1. — *Let ψ_0 real valued such that*

$$1-\psi_0 \in \cap_{s\geqslant 0} H^s(\mathbb{R}^d),\ \psi_0>0 \text{ on } \mathbb{R}^d\setminus\{0\},\ \psi_0(0)=0,\ \Delta\psi_0(0)\neq 0.$$

Then there exists a local solution to (6.1) *with* $\psi|_{t=0}=\psi_0$*, and* $T>0$ *such that* $|\psi|(x,t)>0$ *on* $]0,T]\times\mathbb{R}^d$*.*

Consequently, there exists a solution to (6.2) *that blows up in finite time.*

Proof. — Since $1-\psi_0$ is in $\cap_{s\geqslant 0}H^s$, the existence of a smooth solution to (6.1) is a consequence of the standard theory for NLS equations. From direct computations

$$\partial_t|\psi|^2 = -2\,\mathrm{Im}(\overline{\psi}\Delta\psi), \tag{6.3}$$

$$\partial_t^2|\psi|^2 = 2|\Delta\psi|^2 - \mathrm{Re}\left(g\overline{\psi}\Delta\psi+\overline{\psi}\Delta^2\psi-\overline{\psi}\Delta(g\psi)\right). \tag{6.4}$$

Since ψ_0 is real valued with $\psi_0(0)=0$, we deduce

$$\forall\ x\in\mathbb{R}^d,\ \partial_t|\psi(x,0)|^2=0, \partial_t^2|\psi|^2(0,0)=2|\Delta\psi_0(0)|^2>0. \tag{6.5}$$

By continuity, there exists $\alpha>0$ such that $\partial_t^2|\psi(x,t)|^2\geqslant\alpha$ on a neighbourhood U of $(x,t)=(0,0)$, we deduce by Taylor expansion

$$\forall\ (x,t)\in U,\ |\psi(x,t)|^2=|\psi(x,0)|^2+\int_0^t(t-s)\partial_t^2|\psi(x,s)|^2\mathrm{d}s\geqslant \alpha t^2/2.$$

Now by continuity, for $(x,t)\in U^c$, t small enough, $\psi(x,t)$ does not vanish hence for some $T>0$, $0<t\leqslant T$ small enough, $|\psi(\,\cdot\,,t)|>0$ on $\mathbb{R}^d$. Starting with initial data $\psi(\,\cdot\,,T)$ and going backwards in time provides a solution of (6.1) that cancels at $(x,t)=(0,0)$ in finite time. The (inverse) Madelung transform $\psi\to(\rho,u)=\left(|\psi|^2,\ \mathrm{Im}(\frac{\overline{\psi}\nabla\psi}{|\psi|^2})\right)$ then gives a solution of (6.2) smooth on $]0,T]$, but which blows up at $t=0$. Indeed define $X(t)$ as the flow associated to u, $X'(t)=u(t,X(t))$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,X(t))=-\rho\,\mathrm{div}\,u,$$

hence $\rho(t,X(t))=\rho_0(X(0))e^{-\int_0^t\mathrm{div}\,u\mathrm{d}s}$, the cancellation of ρ implies $\|u\|_{L^1([0,T],W^{1,\infty})}=\infty$. □

Remark 6.2. — While in the framework of Gross–Pitaevskii vacuum is often associated to vorticity, there is no such link here because the initial data ψ_0 is real non negative, thus with constant phase.

Appendix A. The total energy estimate

This section is devoted to the proof of Proposition 3.1. This is essentially a variation on the estimates in [8], that we include here for self-containedness.

We define $z = u + iw$ so that according to (1.3), (ρ, z) satisfy

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) & = 0, \\ \partial_t z + u \cdot \nabla z + i \nabla z \cdot w + i \nabla(a \operatorname{div} z) + \nabla(g(\rho)) & = 0. \end{cases} \tag{A.1}$$

A direct energy method where one takes the scalar product of the second equation with z and integrates causes loss of derivatives due to the term $i\nabla z \cdot w$. The remedy is done in two times: first use a gauge $\varphi_n(\rho)$ and derive an energy bound $\frac{\mathrm{d}}{\mathrm{d}t} \int |\mathbb{Q}(\varphi_n \Delta^n z)|^2 \mathrm{d}x$ for $n \in \mathbb{N}$, this estimate contains a loss of derivatives, but an other gauge estimate on $\mathbb{P}(\phi_n \Delta^n z)$ for an appropriate choice of ϕ_n compensates exactly the loss.

In what follows, R stands for a nonlinear term (quadratic of higher) that contains only derivatives of z of order at most $2n$, and such that $I_R := \int_{\mathbb{R}^d} R \mathrm{d}x$ is dominated by $\|(r, u)\|_{\mathcal{W}^{1,\infty}} \|(r, u)\|^2_{\mathcal{H}^{2n}}$.

We will need the following lemma:

Lemma A.1 ([7, Lemma 3.1]). — *For $Z \in C^1(\mathbb{R}^d, \mathbb{C}),\ W \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, with limit 0 at infinity,*

$$2i \operatorname{Im} \int_{\mathbb{R}^d} Z^* \cdot \nabla_0 Z \cdot W \mathrm{d}x = \int Z^* \cdot \operatorname{curl} W \cdot Z \mathrm{d}x.$$

In particular, if W is a gradient, the integral is 0.

Equation on $\varphi_n(\rho)\Delta^n z$. We recall that $a = \sqrt{\rho K}$, $w = \sqrt{K/\rho}\nabla\rho$, and start from

$$\partial_t z + u \cdot \nabla z + i \nabla z \cdot w + i \nabla(a \operatorname{div} z) + g' w = 0.$$

Apply $\varphi_n \Delta^n$ together with the commutator identity

$$\nabla(\Delta^n(a \operatorname{div} z)) = \nabla(a \operatorname{div} \Delta^n z) + \nabla\big((2n\nabla a) \cdot \nabla \operatorname{div} \Delta^{n-1} z\big) + R,$$

and

$$\varphi_n \nabla(a \operatorname{div} \Delta^n z) = \nabla(a \operatorname{div}(\varphi_n \Delta^n z)) - a(\nabla\varphi_n) \operatorname{div} \Delta^n z - a \nabla \Delta^n z \cdot \nabla\varphi_n + R,$$

$$\begin{aligned} \partial_t(\varphi_n \Delta^n z) + u \cdot \nabla(\varphi_n \Delta^n z) + i\varphi_n \nabla(\Delta^n z) \cdot w + i\nabla(a \operatorname{div}(\varphi_n \Delta^n z)) \\ + g'\varphi_n \Delta^n w + 2in\varphi_n \nabla(\nabla a \cdot \nabla \operatorname{div} \Delta^{n-1} z) \\ - ia(\nabla\varphi_n) \operatorname{div} \Delta^n z - ia\nabla\Delta^n z \cdot \nabla\varphi_n = R, \end{aligned}$$

so using $\operatorname{div} = \operatorname{div} \circ \mathbb{Q}$ and $\nabla \operatorname{div} \Delta^{n-1} z = \Delta^n \mathbb{Q} z$

$$\begin{aligned} \partial_t(\varphi_n \Delta^n z) + u \cdot \nabla(\varphi_n \Delta^n z) + i\nabla(a \operatorname{div}(\varphi_n \Delta^n z)) + g'\varphi_n \Delta^n w \\ + i\varphi_n \nabla(\Delta^n z) \cdot w + 2in\varphi_n \nabla(\Delta^n \mathbb{Q} z) \cdot \nabla a \\ - ia(\nabla\varphi_n) \operatorname{div} \Delta^n \mathbb{Q} z - ia\nabla\Delta^n z \cdot \nabla\varphi_n = R. \end{aligned} \tag{A.2}$$

The loss of derivative is caused by the left hand side of the second line. For $\varphi_n = a^n\sqrt{\rho}$, and denoting $\nabla_0 := \nabla - \mathrm{I}\,\mathrm{div}$, we find

$$\begin{aligned}
&a^n\sqrt{\rho}\nabla(\Delta^n z)\cdot w + 2na^n\sqrt{\rho}\nabla(\Delta^n \mathbb{Q}z)\cdot\nabla a\\
&\qquad - a\nabla(a^n\sqrt{\rho})\,\mathrm{div}\,\Delta^n\mathbb{Q}z - a\nabla\Delta^n z\cdot\nabla(a^n\sqrt{\rho})\\
&= \frac{a^{n+1}}{\sqrt{\rho}}\nabla(\Delta^n z)\cdot\nabla\rho + 2na^n a'\sqrt{\rho}\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla\rho\\
&\qquad - \left(na^n a'\sqrt{\rho} + \frac{a^{n+1}}{2\sqrt{\rho}}\right)\mathrm{div}(\Delta^n\mathbb{Q}z)\nabla\rho\\
&\qquad - \left(na^n a'\sqrt{\rho} + \frac{a^{n+1}}{2\sqrt{\rho}}\right)\nabla\Delta^n z\cdot\nabla\rho\\
&= \left(na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}}\right)\nabla_0\Delta^n\mathbb{Q}z\cdot\nabla\rho\\
&\qquad + \left(-na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}}\right)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho.
\end{aligned} \tag{A.3}$$

For the first term of the right hand side, we write $na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}} = \varphi_n(na' + a/(2\rho))$, commute φ_n with ∇_0, then we use that for $n \geqslant 1$, $\mathbb{Q}\Delta^n$ is a differential operator of order $2n$, so

$$\varphi_n\mathbb{Q}\Delta^n\cdot = \mathbb{Q}\Delta^n(\varphi_n\cdot) + [\varphi_n, \mathbb{Q}\Delta^n]\cdot = \mathbb{Q}\big(\varphi_n\Delta^n\cdot\big) + P\cdot \tag{A.4}$$

with P a differential operator of order $2n-1$. Therefore

$$\left(na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}}\right)\nabla_0\Delta^n\mathbb{Q}z\cdot\nabla\rho = \left(na' + \frac{a}{2\rho}\right)\nabla_0\mathbb{Q}(\varphi_n\Delta^n z)\cdot\nabla\rho + R. \tag{A.5}$$

Plugging (A.3) and (A.5) in (A.2) we get

$$\begin{aligned}
&\partial_t(\varphi_n\Delta^n z) + u\cdot\nabla(\varphi_n\Delta^n z) + g'\varphi_n\Delta^n w + i\nabla(a\,\mathrm{div}(\varphi_n\Delta^n z))\\
&\qquad = -i\left(na' + \frac{a}{2\rho}\right)\nabla_0\mathbb{Q}(\varphi_n\Delta^n z)\cdot\nabla\rho\\
&\qquad\quad - i\left(-na^n\sqrt{\rho}a' + \frac{a^{n+1}}{2\sqrt{\rho}}\right)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho + R.
\end{aligned} \tag{A.6}$$

Note that $g'\varphi_n\Delta^n w = g'(1)\varphi_n(1)\Delta^n w + R = 2\Delta^n w + R$ is without loss of derivatives but contains a linear term that can not be neglected for long time dynamics.

Energy estimate for $\mathbb{Q}\varphi_n\Delta^n z$. Take the scalar product of (A.6) with $\mathbb{Q}\varphi_n\Delta^n z$, integrate in space, and use Lemma A.1

$$\begin{aligned}
&\frac{\mathrm{d}}{2\mathrm{d}t}\int |\mathbb{Q}\varphi_n\Delta^n z|^2\mathrm{d}x + \mathrm{Re}\int \big(u\cdot\nabla(\varphi_n\Delta^n z)+2\Delta^n w\big)\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}\mathrm{d}x\\
&\qquad = \mathrm{Im}\int \overline{\mathbb{Q}\varphi_n\Delta^n z}\cdot\bigg[\Big(na'+\frac{a}{2\rho}\Big)\nabla_0\mathbb{Q}(\varphi_n\Delta^n z)\\
&\qquad\qquad\qquad + \Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\bigg]\cdot\nabla\rho\mathrm{d}x + I_R\\
&\qquad = \mathrm{Im}\int \overline{\mathbb{Q}\varphi_n\Delta^n z}\cdot\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho\mathrm{d}x + I_R,
\end{aligned}$$

where $I_R=\int R\cdot\mathbb{Q}(\varphi_n\Delta^n z)\mathrm{d}x$, for more details on the generic estimate I_R we refer to [4].

The right hand side is an unavoidable loss of derivative. The second term on the left hand side rewrites with the convention of summation on repeated indices and using $\partial_j(\mathbb{Q}v)_i=\partial_i(\mathbb{Q}v)_j$

$$\begin{aligned}
&\int u_j\partial_j(\varphi_n\Delta^n z_i)\overline{(\mathbb{Q}\varphi_n\Delta^n z)}_i)\mathrm{d}x\\
&\qquad = -\int \mathrm{div}(u)\varphi_n\Delta^n z\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}+u_j\varphi_n\Delta^n z_i\partial_j(\overline{\mathbb{Q}\varphi_n\Delta^n z})_i\mathrm{d}x\\
&\qquad = -\int \mathrm{div}(u)\varphi_n\Delta^n z\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}+\frac{\mathrm{div}\,u}{2}|\mathbb{Q}\varphi_n\Delta^n z|^2\\
&\qquad\qquad\qquad + u_j(\mathbb{P}\varphi_n\Delta^n z)_i\partial_i\overline{(\mathbb{Q}\varphi_n\Delta^n z)}_j\mathrm{d}x\\
&\qquad = -\int \mathrm{div}(u)\varphi_n\Delta^n z\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}+\frac{\mathrm{div}\,u}{2}|\mathbb{Q}\varphi_n\Delta^n z|^2\\
&\qquad\qquad\qquad + (\mathbb{P}\varphi_n\Delta^n z)\cdot\nabla u\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}\mathrm{d}x\\
&\qquad = I_R.
\end{aligned}$$

To summarize

$$\begin{aligned}
&\frac{\mathrm{d}}{2\mathrm{d}t}\int |\mathbb{Q}\varphi_n\Delta^n z|^2\mathrm{d}x + \mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}\mathrm{d}x\\
&\quad = \mathrm{Im}\int \overline{\mathbb{Q}\varphi_n\Delta^n z}\cdot\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\nabla\Delta^n\mathbb{P}z\cdot\nabla\rho\mathrm{d}x + I_R.
\end{aligned}\tag{A.7}$$

Energy estimate for $\mathbb{P}(\phi_n\Delta^n z)$ and compensated loss. Let $\phi_n(\rho)$ be a second gauge. Recall that (A.2) with ϕ_n instead of φ_n reads

$$\begin{aligned}\partial_t(\phi_n\Delta^n z)+u\cdot\nabla(\phi_n\Delta^n z)+i\nabla(\phi_n\Delta^n z)\cdot w\\ +i\nabla(a\,\mathrm{div}(\phi_n\Delta^n z))+2g'\phi_n\Delta^n w+2in\phi_n\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla a\\ -ia(\nabla\phi_n)\,\mathrm{div}\,\Delta^n\mathbb{Q}z-ia\nabla\Delta^n z\cdot\nabla\phi_n=R.\end{aligned}$$

We take the scalar product with $\mathbb{P}\phi_n\Delta^n z$ and integrate in space, the first two terms are

$$\begin{aligned}\mathrm{Re}\int\big(\partial_t(\phi_n\Delta^n z)+u\cdot\nabla(\phi_n\Delta^n z)\big)\cdot\overline{(\mathbb{P}\phi_n\Delta^n z)}\\ =\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbb{P}\phi_n\Delta^n z|^2\mathrm{d}x\\ -\mathrm{Re}\int\frac{\mathrm{div}\,u}{2}|\mathbb{P}\phi_n\Delta^n z|^2-\mathbb{Q}\phi_n\Delta^n z\cdot\nabla u\cdot\mathbb{P}\phi_n\Delta^n z\,\mathrm{d}x\\ =\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbb{P}\phi_n\Delta^n z|^2\mathrm{d}x+I_R.\end{aligned}$$

Most of the other terms are actually neglectible

$$\int\overline{(\mathbb{P}\phi_n\Delta^n z)}_i\partial_i(\phi_n\Delta^n z_j)w_j\mathrm{d}x=-\int\overline{(\mathbb{P}\varphi_n\Delta^n z)}_i\phi_n\Delta^n z_j\partial_i w_j\mathrm{d}x=I_R,$$

and from the same computation

$$\int\overline{\mathbb{P}\phi_n\Delta^n z}\cdot(2in\phi_n\nabla(\Delta^n\mathbb{Q}z)\cdot\nabla a-ia\nabla\Delta^n z\cdot\nabla\phi_n)\mathrm{d}x=I_R.$$

We are only left with

$$\begin{aligned}\mathrm{Im}\int\mathrm{div}(\Delta^n\mathbb{Q}z)a\nabla\phi_n\cdot\overline{\mathbb{P}\phi_n\Delta^n z}\mathrm{d}x\\ =-\mathrm{Im}\int a(\mathbb{Q}\Delta^n z)\cdot\nabla\overline{(\mathbb{P}\phi_n\Delta^n z)}\cdot\nabla\phi_n\mathrm{d}x+I_R.\end{aligned}$$

Arguing as for (A.4) we have $\nabla(\mathbb{P}(\phi_n\Delta^n z))=\phi_n\nabla\mathbb{P}(\Delta^n z)+C$, C a commutator term without loss of derivatives, therefore

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbb{P}\phi_n\Delta^n z|^2\mathrm{d}x&=\mathrm{Im}\int a(\mathbb{Q}\Delta^n z)\nabla\overline{(\mathbb{P}\Delta^n z)}\cdot(\phi_n\nabla\phi_n)\mathrm{d}x+I_R\\ &=-\mathrm{Im}\int a\overline{(\mathbb{Q}\Delta^n z)}\nabla(\mathbb{P}\Delta^n z)\cdot(\phi_n\nabla\phi_n)\mathrm{d}x+I_R\end{aligned}\qquad\text{(A.8)}$$

We sum (A.7) and (A.8)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbb{Q}\varphi_n\Delta^n z|^2+|\mathbb{P}\phi_n\Delta^n z|^2\mathrm{d}x+\mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}\mathrm{d}x\\ =\mathrm{Im}\int\varphi_n\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\overline{\mathbb{Q}\Delta^n z}\cdot\nabla\mathbb{P}\Delta^n z\cdot\nabla\rho\mathrm{d}x\\ -\mathrm{Im}\int a\phi_n\phi_n'\overline{(\mathbb{Q}\Delta^n z)}\cdot\nabla(\mathbb{P}\Delta^n z)\cdot\nabla\rho\mathrm{d}x+I_R.\end{aligned}\tag{A.9}$$

It is now apparent that the right choice for ϕ_n is a function such that

$$a\phi_n\phi_n'=\varphi_n\Big(-na^n\sqrt{\rho}a'+\frac{a^{n+1}}{2\sqrt{\rho}}\Big)\Longleftrightarrow(\phi_n^2)'=-2na^{2n-1}\rho a'+a^{2n},$$

and which is positive close to $\rho=1$. Of course there exists such functions.

Correction of the linear drift. There only remains to cancel the "linear" term

$$\mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}\mathrm{d}x=\int 2\Delta^n\nabla\rho\cdot\mathbb{Q}\Delta^n u\mathrm{d}x+I_R.$$

We apply Δ^n to the mass conservation equation, multiply by $\Delta^n\rho$ and integrate,

$$\begin{aligned}\int\Delta^n\big(\partial_t\rho+\mathrm{div}(\rho u)\big)\Delta^n\rho\mathrm{d}x\\ =\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\Delta^n\rho)^2\mathrm{d}x+\int\rho\,\mathrm{div}(\Delta^n u)\Delta^n\rho\mathrm{d}x\\ =\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\Delta^n\rho)^2\mathrm{d}x-\int\mathbb{Q}(\Delta^n u)\nabla\Delta^n\rho\mathrm{d}x.\end{aligned}\tag{A.10}$$

Therefore adding (A.9) to two times (A.10) we obtain after using $\mathrm{Re}\int 2\Delta^n w\cdot\overline{\mathbb{Q}\varphi_n\Delta^n z}\mathrm{d}x=\int 2\Delta^n\nabla\rho\cdot\mathbb{Q}\Delta^n u\mathrm{d}x+I_R$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbb{Q}\varphi_n\Delta^n z|^2+|\mathbb{P}\phi_n\Delta^n z|^2+2|\Delta^n\rho|^2\mathrm{d}x=I_R,\tag{A.11}$$

Conclusion. By integration of (A.11) we find

$$\begin{aligned}\|\mathbb{Q}\varphi_n\Delta^n z\|_{L^2}^2+\|\mathbb{P}\phi_n\Delta^n z\|_{L^2}^2+2\|\Delta^n r\|_{L^2}^2\\ \leqslant\|\mathbb{Q}\varphi_n(\rho_0)\Delta^n z_0\|_{L^2}^2+\|\mathbb{P}\phi_n(\rho_0)\Delta^n z_0\|_{L^2}^2+2\|\Delta^n r_0\|_{L^2}^2\\ +C\int_0^t\|z\|_{\mathcal{H}^{2n}}^2\|z\|_{\mathcal{W}^{1,\infty}}\mathrm{d}s.\end{aligned}\tag{A.12}$$

Note that for $n = 0$, we may choose $\varphi_0 = \phi_0 = \sqrt{\rho}$, leading to

$$\begin{aligned}\|z\|_{\mathcal{H}^0} \sim \|\mathbb{Q}\varphi_0 z\|_{L^2}^2 + \|\mathbb{P}\phi_0 z\|_{L^2}^2 + 2\|r\|_{L^2}^2 \\ \leqslant \|\sqrt{\rho_0} z_0\|_{L^2} + 2\|r_0\|_{L^2} + C\int_0^t \|z\|_{\mathcal{H}^0}^2 \|z\|_{\mathcal{W}^{1,\infty}} \mathrm{d}s,\end{aligned}$$

(the estimate is actually a conservation of energy, see [8, Section 3.1]).

Moreover for $n \geqslant 1$, $\mathbb{Q}\varphi_n \Delta^n z = \mathbb{Q}[\varphi_n, \Delta]\Delta^{n-1} z + [\mathbb{Q}\Delta, \varphi_n]\Delta^{n-1} z + \varphi_n \mathbb{Q}\Delta^n z = R + \varphi_n \mathbb{Q}\Delta^n z$, with $\|R\|_{L^2} = O(\|r\|_{W^{2,\infty}}\|z\|_{H^{2n-1}})$ and the same observation stands for $\mathbb{P}\phi_n \Delta^n z$, thus

$$\|\mathbb{Q}\varphi_n \Delta^n z\|_{L^2} + \|\mathbb{P}\phi_n \Delta^n z\|_{L^2} = \|\Delta^n z\|_{L^2} + O(\|r\|_{W^{2,\infty}}\|z\|_{H^{2n-1}}).$$

Using (A.12) for $n = 0, N$ we conclude that for δ small enough

$$\|z(t)\|_{\mathcal{H}^{2N}}^2 \sim \|z(t)\|_{H^{2N}}^2 + \|r\|_{L^2}^2 \lesssim \|z_0\|_{\mathcal{H}^{2N}}^2 + \int_0^t \|z(s)\|_{\mathcal{H}^{2N}}^2 \|z(s)\|_{\mathcal{W}^{1,\infty}} \mathrm{d}s.$$

The conclusion follows by an application of Gronwall's lemma.

Appendix B. Control of the quadratic dispersive terms

The article of the author and B. Haspot [4] was restricted to the irrotational case. In this case, since $\mathbb{Q}u + iw = u + iw = \nabla(\phi + r)$, it was more convenient to work on $\widetilde{\Psi} = U\phi + r$. This difference causes merely a shift in regularity indices as $\|\widetilde{\Psi}\|_{H^N \times W^{k,p}} \sim \|\Psi\|_{H^{N-1} \times W^{k-1,p}}$.

The key result in [4] was the uniform bounds for $t \geqslant 0$

$$\|xe^{-itH}\widetilde{\Psi}\|_{L^2} \lesssim \varepsilon,\ \|\widetilde{\Psi}\|_{W^{k,p}} \lesssim \varepsilon/(1+t)^{1+\gamma},$$

Actually, one might observe that more precise estimates were obtained that are sufficient to our purpose: using a partition of the pase space, the quadratic nonlinearity was split in "space non resonant and time non resonant terms"

$$\mathcal{D}(\psi) = \mathcal{D}_S(\psi) + \mathcal{D}_T(\psi),$$

for which we proved "essentially" the following estimates

$$\begin{aligned}
\Big\|\int_0^t e^{i(t-s)H}\mathcal{D}_T(\Psi)\mathrm{d}s\Big\|_{W^{k,p}} &\lesssim \int_0^t \frac{\|\Psi\|_{H^N}^2\|\Psi\|_{W^{k,p}}}{(1+t-s)^{1+\gamma}}\mathrm{d}s,\\
\Big\||x|e^{-itH}\int_0^t e^{i(t-s)H}\mathcal{D}_T(\Psi)\mathrm{d}s\Big\|_{L^2} &\lesssim \int_0^t s\|\Psi\|_{H^N}\|\Psi\|_{W^{k,p}}^2\\
&\quad + \||x|e^{-isH}\Psi\|_{L^2}\|\Psi\|_{H^N}\|\Psi\|_{W^{k,p}}\mathrm{d}s,\\
\Big\|\int_0^t e^{i(t-s)H}\mathcal{D}_S(\Psi)\mathrm{d}s\Big\|_{W^{k,p}} &\lesssim \int_0^t \frac{\|\Psi\|_{H^N}\|\Psi\|_{W^{k,p}}}{(1+s)(1+t-s)^{1+\gamma}}\mathrm{d}s,\\
\Big\||x|e^{-itH}\int_0^t e^{i(t-s)H}\mathcal{D}_S(\Psi)\mathrm{d}s\Big\|_{W^{k,p}} &\lesssim \int_0^t \|\Psi\|_{H^N}\|\Psi\|_{W^{k,p}}\\
&\quad + \frac{\||x|e^{-isH}\Psi\|_{L^2}\|\Psi\|_{H^N}^{1-\gamma}\|\Psi\|_{W^{k,p}}^{\gamma}}{s}\mathrm{d}s.
\end{aligned}$$

It is easy to see that such estimates are sufficient to close our bootstrap argument. Unfortunately, it is quite painful to collect these from [4], which are never written explicitly in this form, and actually are not exactly true (one should actually split the integral as $\int_0^1+\int_1^{t-1}+\int_{t-1}^t$, which makes a precise statement even longer).

Rather than repeating the entire argument from [4] to estimate $\mathcal{D}$ with minor modifications, we choose to include only a partial proof for the $W^{k,p}$ estimate and refer entirely to [4] for the weighted estimate.

Generic nonlinearity. According to (5.3), and linearizing $a-1=a'(1)l+R$, with R quadratic in l, $2w-g'w=\nabla(2l-g'(l))=\nabla(g''(1)l^2/2+O(l^3))$, the quadratic purely dispersive nonlinearity is

$$\begin{aligned}\nabla\bigg((\Delta-2)b+a'(1)l\operatorname{div}w_1-\frac{1}{2}\big(|\mathbb{Q}u|^2-|w_1|^2\big)\bigg)&\\
-g''(1)\nabla(l^2/2)-iU^{-1}\nabla\operatorname{div}\big(a'(1)l\mathbb{Q}u\big)&.\end{aligned}\tag{B.1}$$

Following [4] we denote $\Psi^{\pm}$ as a placeholder for Ψ or $\overline{\Psi}$. Since $\mathbb{Q}u=(\Psi^++\Psi^-)/2$, $w_1=U(\Psi^+-\Psi^-)/(2i)$, $l=\Delta^{-1}U\operatorname{div}(\Psi^+-\Psi^-)/(2i)+R$, R quadratic (see the change of variables of Lemma 5.3), all quadratic nonlinearities can be written as nonlinearities in $\Psi^{\pm}$. Their precise definition does not really matter, the main point is that they all take the form

$$\nabla A[\Psi^{\pm},\Psi^{\pm}]=\nabla\mathcal{F}^{-1}\bigg(\int_{\mathbb{R}^d}\widehat{\Psi^{\pm}}\cdot A(\eta,\xi-\eta)\cdot\widehat{\Psi^{\pm}}d\eta\bigg),\tag{B.2}$$

with A a matrix valued symbol that can be for example $\frac{(|\xi|^2+2)(a'(1)-1)}{2(2+|\eta|^2+|\xi-\eta|^2)}$, $a'(1)\frac{U(\eta)\eta}{|\eta|^2}\otimes(\xi-\eta)$, $-iU^{-1}(\xi)\frac{U(\eta)\eta}{|\eta|^2}\otimes\xi\ldots$ For commodity, we denote

$\nabla A[\Psi^\pm,\Psi^\pm] = B[\Psi^\pm,\Psi^\pm]$, even though one should keep in mind that the symbol of B is a vector valued bilinear map.

The method of space time resonances. We denote $\widetilde{\Psi^\pm} = \mathcal{F}\big(e^{\mp itH}\Psi^\pm\big)$. We recall that the equation (5.3) reads

$$\partial_t\Psi - iH\Psi = \mathcal{N}(\Psi,\mathbb{P}u) = \mathcal{D}(\Psi) + \mathcal{T}(\Psi,\mathbb{P}u) + R,$$

where $\mathcal{D}$, resp. $\mathcal{T}$, correspond to the purely dispersive, resp. dispersive transport quadratic terms, and R are higher order nonlinearities. Let $B[\Psi^\pm,\Psi^\pm]$ a generic nonlinearity, the Duhamel formula leads to terms

$$\begin{aligned}\mathcal{F}&\bigg(e^{-itH}\int_0^t e^{i(t-s)H}B[\Psi^\pm,\Psi^\pm]\mathrm{d}s\bigg)\\ &\quad= \int_0^t\int_{\mathbb{R}^d} e^{-is\Omega_{\pm\pm}}\widetilde{\Psi^\pm}(\eta)\cdot B(\eta,\xi-\eta)\cdot\widetilde{\Psi^\pm}(\xi-\eta)\mathrm{d}\eta\mathrm{d}s,\end{aligned} \tag{B.3}$$

where $\Omega_{\pm\pm}(\xi,\eta) = H(\xi)\mp H(\eta)\mp H(\xi-\eta)$. The estimates do not require to distinguish the various cases $\Omega_{++},\Omega_{-+}\dots$ so we write Ω instead of $\Omega_{\pm\pm}$.

Since $\partial_s(e^{-isH}\Psi) = e^{-isH}(\mathcal{D}(\Psi)+\mathcal{T}(\Psi,\mathbb{P}u))$, an integration by parts in s "improves" the nonlinearity which becomes cubic. Similarly from the identity

$$e^{-is\Omega} = \frac{\nabla_\eta\Omega}{-is|\nabla_\eta\Omega|^2}\cdot\nabla_\eta e^{-is\Omega},$$

an integration by parts in η leads to a gain of decay of $1/s$. Of course these integrations by parts are fruitful only if Ω, $|\nabla_\eta\Omega|$ do not cancel (resp. no time resonances and no space resonances), this leads to define the space-time resonant set as $\{(\xi,\eta):\ \Omega=0\}\cap\{(\xi,\eta):\ \nabla_\eta\Omega=0\}$. The so-called method of space-time resonances consists in splitting the phase space in time non resonant and space non resonant regions and do the integration by parts accordingly.

Some difficulties are that the space-time resonant region is actually quite large, as one can check that in the case of Ω_{-+} it is $\{(\xi,\eta):\ \xi=0\}$, thus a subspace of dimension 3 in $\mathbb{R}^6$. A second issue is that the symbol $H(\xi) = |\xi|\sqrt{2+|\xi|^2}$ is similar to $\sqrt{2}|\xi|$ at low frequencies (wave-like), so that for ε,η small $\Omega_{-+}(\varepsilon\eta,\eta)\sim -3\varepsilon|\eta|^3/(2\sqrt{2})$. This third order cancellation is worse than for the Schrödinger equation, and prevents any use of the Coifman–Meyer theorem. Instead, we use the following rough multiplier lemma due to Guo and Pausader (inspired by Lemma 10.1 in [23]). The statement requires the Chemin–Lerner spaces $\widetilde{L}^\infty_\xi\dot{B}^s_{2,1,\zeta}$, see [5, Section 2.6.3] for a precise definition (which is not mandatory to read this appendix).

Lemma B.1 ([21]). — *For $0 \leqslant s \leqslant d/2$, let*

$$\|B\|_{[B^s]} = \min\Big(\|B(\eta,\xi-\eta)\|_{\widetilde{L}^\infty_\xi \dot B^s_{2,1,\eta}}, \|B(\xi-\zeta,\zeta)\|_{\widetilde{L}^\infty_\xi \dot B^s_{2,1,\zeta}}\Big).$$

For q_1, q_2 such that $2 \leqslant q_2, q_1' \leqslant \frac{2d}{d-2s}$ and $\frac{1}{q_2}+\frac{1}{2}=\frac{1}{q_1}+\frac{1}{2}-\frac{s}{d}$, then

$$\|B[f,g]\|_{L^{q_1}} \lesssim \|B\|_{[B^s]}\|f\|_{L^{q_2}}\|g\|_{L^2}.$$

Moreover, for $2 \leqslant q_1, q_2, q_3 \leqslant 2d(d-2s)$, and $\frac{1}{q_3}+\frac{1}{2}-\frac{s}{d}=\frac{1}{q_1}+\frac{1}{q_2}$

$$\|B[f,g]\|_{L^{q_3}} \lesssim \|B\|_{[B^s]}\|f\|_{L^{q_1}}\|g\|_{L^{q_2}}.$$

The black box. We use the following arguments directly taken from [4]: let $(\chi^a)_{a\in 2^{\mathbb{Z}}}$ a dyadic partition of unity, $\mathrm{supp}(\chi^a) \subset \{|\xi| \sim a\}$, B a symbol associated to one of the nonlinearities. There exists a function $\Phi(\xi,\eta)$ that splits the phase space in time non resonant and space non resonant regions in the following sense: let the frequency localized symbols be

$$B^{a,b,c,T} = \Phi\chi^a(\xi)\chi^b(\eta)\chi^c(\zeta)B(\eta,\xi-\eta),\ B^{a,b,c,X} = (1-\Phi)\chi^a\chi^b\chi^c B, \quad \text{(B.4)}$$

with $\zeta = \xi-\eta$, then we have the multiplier estimates:

Lemma B.2 ([4, Lemmas 6.1, 6.2]). — *For $a,b,c \in (2^{\mathbb{Z}})^3$, $1 \leqslant j \leqslant 3$, let*

$$\mathcal{B}^{a,b,c,T} := \frac{B^{a,b,c,T}}{\Omega}, \quad \mathcal{B}^{a,b,c,X}_{1,j} = \frac{(\partial_{\eta_j}\Omega)B^{a,b,c,X}}{|\nabla_\eta \Omega|^2}, \quad \mathcal{B}^{a,b,c,X}_{2,j} = \partial_{\eta_j}\mathcal{B}^{a,b,c,X}_{1,j},$$

$m = \min(a,b,c)$, $M = \max(a,b,c)$, $l = \min(b,c)$. For $0 < s < 2$, we have

$$\text{if } M \gtrsim 1,\ \|\mathcal{B}^{a,b,c,T}\|_{[B^s]} \lesssim \frac{\langle M\rangle l^{3/2-s}}{\langle a\rangle},\ \max_j \|\mathcal{B}^{a,b,c,X}_{1,j}\|_{[B^s]} \lesssim \frac{\langle M\rangle^2 l^{3/2-s}}{\langle a\rangle},$$
$$\max_j \|\mathcal{B}^{a,b,c,X}_{2,j}\|_{[B^s]} \lesssim \frac{\langle M\rangle^2 l^{1/2-s}}{\langle a\rangle},$$

$$\text{if } M \ll 1,\ \|\mathcal{B}^{a,b,c,T}\|_{[B^s]} \lesssim M^{-s} l^{1/2-s},\ \max_j \|\mathcal{B}^{a,b,c,X}_{1,j}\|_{[B^s]} \lesssim M^{1-s} l^{3/2-s},$$
$$\max_j \|\mathcal{B}^{a,b,c,X}_{2,j}\|_{[B^s]} \lesssim M^{-s} l^{1/2-s},$$

We will also use the elementary estimate:

Lemma B.3. — *For $t > 0$*

$$\|U^{-1}\Psi\|_{L^6} \lesssim \frac{1}{t^{3/5}}(\|xe^{-itH}\Psi\|_{L^2} + \|\Psi\|_{H^1}). \quad \text{(B.5)}$$

Proof. — By interpolation and the dispersion estimate 2.1

$$\begin{aligned}\|U^{-1}\Psi\|_{L^6} &\leqslant \|U^{-1/3}\Psi\|_6^{3/5}\|U^{-2}\Psi\|_{L^6}^{2/5}\\ &\lesssim \left(\frac{\|e^{-itH}\Psi\|_{L^{6/5,2}}}{t}\right)^{3/5}\left(\|U^{-1}\Psi\|_{L^2}+\|\Psi\|_{H^1}\right)^{2/5}\\ &\lesssim \left(\frac{\|xe^{-itH}\Psi\|_2}{t}\right)^{3/5}\left(\|xe^{-itH}\Psi\|_2+\|\Psi\|_{H^1}\right)^{2/5}. \qquad \square\end{aligned}$$

Control of the purely dispersive quadratic terms in $W^{k,p}$. We first bound the time non resonant terms associated to the bilinear multiplier $B^{a,b,c,T}$, then the space non resonant terms associated to $B^{a,b,c,X}$.

Control of time non resonant terms in $W^{k,p}$**.** Integrating by parts in s, the frequency localized Duhamel terms of (B.3) lead to the following quantities

$$\begin{aligned}I^{a,b,c,T} := \int_0^t e^{i(t-s)H}\big(\mathcal{B}^{a,b,c,T}[\mathcal{N}^\pm,\psi^\pm]+\mathcal{B}^{a,b,c,T}[\psi^\pm,\mathcal{N}^\pm]\big)\mathrm{d}s&\\ -\big[e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\Psi^\pm,\Psi^\pm]\big]_0^t.&\end{aligned}\tag{B.6}$$

Consider for example $\int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{D}^\pm+\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s$, $b\lesssim a\sim c$. Compared to the irrotational case there are two novelties: the bootstrap assumptions are different, and more interestingly we have a new term $\mathcal{T}^\pm$.

We choose p,N such that $1/2-2\gamma>0$, $N-k-1/2+\gamma>0$ (this corresponds to p slightly larger than 6 and N large enough) and apply Lemma B.1 with $s=1+\gamma$. For $k_1\leqslant k$

$$\begin{aligned}&\left\|\nabla^{k_1}\int_0^{t-1}\sum_{b\lesssim a\sim c}\mathcal{B}^{a,b,c,T}[\mathcal{D}^\pm,\Psi^\pm]\mathrm{d}s\right\|_p\\ &\quad\lesssim\int_0^{t-1}\sum_{b\lesssim a\sim c\leqslant 1}\frac{ab\|\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}\|U^{-1}\mathcal{D}\|_2\|U^{-1}\Psi\|_2}{(t-s)^{1+\gamma}}\\ &\qquad+\sum_{b\lesssim a\sim c,\ c\geqslant 1}\frac{a^{-N+k_1}\|\mathcal{B}^{a,b,c,T}\|_{[B^{1+\gamma}]}\|\mathcal{D}\|_2\|\Psi\|_{H^N}}{(t-s)^{1+\gamma}}\mathrm{d}s\end{aligned}$$

$$\lesssim \int_0^{t-1} \sum_{b\lesssim a\sim c\leqslant 1} \frac{ba^{-\gamma}b^{1/2-(1+\gamma)}\|U^{-1}\mathcal{D}\|_2\|U^{-1}\Psi\|_2}{(t-s)^{1+\gamma}} + \sum_{b\lesssim a\sim c,\ c\geqslant 1} \frac{b^{3/2-(1+\gamma)}\|\mathcal{D}\|_2\|\Psi\|_{H^N}}{a^{N-k_1}(t-s)^{1+\gamma}}\mathrm{d}s$$
$$\lesssim \int_0^{t-1} \frac{\|U^{-1}\mathcal{D}\|_2(\|U^{-1}\Psi\|_2+\|\Psi\|_{H^N})}{(t-s)^{1+\gamma}}\mathrm{d}s.$$

Then $\|U^{-1}\Psi\|_2 \lesssim \|e^{-itH}|\Delta|^{-1/2}\Psi\|_2+\|\Psi\|_2 \lesssim \|e^{-itH}\Psi\|_{L^{6/5,2}}+\|\Psi\|_2 \lesssim \|xe^{-itH}\Psi\|_2+\|\Psi\|_2$, where $L^{6/5,2}$ is the Lorentz space, and we used the generalized Hölder inequality $L^{6/5,2}\times L^{3,\infty}\subset L^2$. On the other hand since ∇ is in factor of all purely dispersive quadratic nonlinearities (see equation (B.1)), it compensates the singular factor U^{-1} in $U^{-1}\mathcal{D}$, and one easily gets

$$\|U^{-1}\mathcal{D}\|_{L^2} \lesssim \|\Psi\|_{W^{2,4}}^2 \lesssim \|\Psi\|_{H^2}^{\frac{1+4\gamma}{2+2\gamma}}\|\Psi\|_{W^{2,p}}^{\frac{3}{2+2\gamma}}.$$

The bootstrap assumption gives the bound

$$\left\|\int_0^{t-1}\sum_{b\lesssim a\sim c}\mathcal{B}^{a,b,c,T}[\mathcal{D}^\pm,\Psi^\pm]\mathrm{d}s\right\|_{W^{k,p}} \lesssim C^3\int_0^{t-1}\frac{\varepsilon^{3/2}\big(\delta+\varepsilon/(1+s)^{1+\gamma}\big)^{\frac{3}{2(1+\gamma)}}}{(t-s)^{1+\gamma}}\mathrm{d}s \leqslant C^3\varepsilon^{3/2}\left(\delta+\frac{\varepsilon}{t^{1+\gamma}}\right),$$

for $\gamma\leqslant 1/2$. Like the case $d\geqslant 5$, the estimate of $\int_{t-1}^t$ is simpler, so is the estimate of $[e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\Psi^\pm,\Psi^\pm]]_0^{t-1}$. The other ranges of indices $c\lesssim a\sim b,\ a\lesssim b\sim c$ can be estimated similarly, more detailed computations can be found in [4, Section 6.1.2].

Omitting these details, to summarize,

$$\left\|\int_0^t e^{i(t-s)H}\sum_{a,b,c}\mathcal{B}^{a,b,c,T}[\mathcal{D}^\pm,\psi^\pm]-[e^{-i(t-s)H}\mathcal{B}^{a,b,c,T}[\Psi^\pm,\Psi^\pm]]_0^t\right\|_{W^{k,p}} \lesssim C^3\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\right). \tag{B.7}$$

Now for the new dispersive-transport term $\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\psi^\pm]$, for $b\lesssim a\sim c$, if $c\leqslant 1$

$$\left\|\nabla^{k_1}\int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\right\|_{L^p} \lesssim \int_0^{t-1}\frac{a^{-\gamma}b^{1/2-\gamma}}{(t-s)^{1+\gamma}}\|U^{-1}\mathcal{T}\|_2\|U^{-1}\Psi\|_2\mathrm{d}s,$$

and if $c\geqslant 1$

$$\left\|\nabla^{k_1}\int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\right\|_{L^p} \lesssim \int_0^{t-1}\frac{b^{3/2-(1+\gamma)}}{(t-s)^{1+\gamma}a^{N-k_1}}\|\mathcal{T}\|_2\|\Psi\|_{H^N}\mathrm{d}s.$$

We deduce by summation

$$\left\|\sum_{b\lesssim a\sim c}\int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\right\|_{W^{k,p}} \lesssim \int_0^{t-1}\frac{\|U^{-1}\mathcal{T}\|_2(\|U^{-1}\Psi\|_2+\|\Psi\|_{H^N})}{(t-s)^{1+\gamma}}\mathrm{d}s.$$

Unlike the nonlinearity $\mathcal{D}$, the dispersive-transport nonlinearity $\mathcal{T}$ is not well prepared for the operator U^{-1}, let us recall it is

$$\mathcal{T}=-iU^{-1}\nabla(\mathbb{P}u\cdot w_1)-\mathbb{Q}\big(u\cdot\nabla\mathbb{P}u+\mathbb{P}u\cdot\nabla\mathbb{Q}u\big).$$

Nevertheless, $\mathbb{P}u$ is better behaved thus we can simply apply the following estimates

$$\begin{aligned}\|U^{-1}\mathcal{T}\|_2\lesssim\|\mathcal{T}\|_{W^{1,6/5}}&\lesssim\|\mathbb{P}u\|_{W^{k,6/5}}(\|\Psi\|_{H^N}+\|\mathbb{P}u\|_{H^N}),\\ \|\mathcal{T}\|_2&\lesssim\|\mathbb{P}u\|_2(\|\Psi\|_{H^N}+\|\mathbb{P}u\|_{H^N}),\end{aligned}$$

combined with the bootstrap assumption (5.4) we find

$$\left\|\sum_{b\lesssim a\sim c}\int_0^{t-1} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\right\|_{W^{k,p}} \lesssim C^3\int_0^{t-1}\frac{\varepsilon^2\delta}{(t-s)^{1+\gamma}}\mathrm{d}s\lesssim C^3\varepsilon^2\delta. \tag{B.8}$$

The integral over $[t-1,t]$ is estimated in the same spirit: for $b \lesssim a \sim c \leqslant 1$

$$\begin{aligned}\Big\|\int_{t-1}^{t} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\Big\|_{W^{k,p}} &\\ &\lesssim \int_{t-1}^{t}\big\|\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\big\|_{H^{k+2}}\mathrm{d}s\\ &\lesssim \int_{t-1}^{t} ab\|\mathcal{B}^{a,b,c,T}\|_{[B^1]}\|U^{-1}\mathcal{T}\|_2\|U^{-1}\Psi\|_6\mathrm{d}s,\end{aligned}$$

for $c \geqslant 1$

$$\begin{aligned}\Big\|\int_{t-1}^{t} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\Big\|_{W^{k,p}} &\\ &\lesssim \int_{t-1}^{t}\big\|\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\big\|_{H^{k+2}}\mathrm{d}s\\ &\lesssim \int_{t-1}^{t}\frac{\|\mathcal{B}^{a,b,c,T}\|_{[B^1]}}{c^{N-k-3}}\|\mathcal{T}\|_2\|\Psi\|_{W^{N-1,6}}\mathrm{d}s\\ &\lesssim \int_{t-1}^{t}\frac{\|\mathcal{B}^{a,b,c,T}\|_{[B^1]}}{c^{N-k-3}}\|\mathcal{T}\|_2\|\Psi\|_{H^{N}}\mathrm{d}s\end{aligned}$$

As previously, we have

$$\|U^{-1}\mathcal{T}\|_2+\|\mathcal{T}\|_2 \lesssim \big(\|\mathbb{P}u\|_{W^{k,6/5}}+\|\mathbb{P}u\|_{H^k}\big)\big(\|\mathbb{P}u\|_{H^N}+\|\Psi\|_{H^N}\big)$$

and

$$\sum_{b\lesssim a\sim c\leqslant 1} ab\|\mathcal{B}^{a,b,c,T}\|_{[B^1]}+\sum_{b\lesssim a\sim c,\ c\geqslant 1}\frac{\|\mathcal{B}^{a,b,c,T}\|_{[B^1]}}{c^{N-k-2}}<\infty,$$

thus

$$\Big\|\int_{t-1}^{t}\sum_{b\lesssim a\sim c} e^{i(t-s)H}\mathcal{B}^{a,b,c,T}[\mathcal{T}^\pm,\Psi^\pm]\mathrm{d}s\Big\|_{W^{k,p}} \lesssim C^3\int_{t-1}^{t}\varepsilon^2\delta\mathrm{d}s = C^3\varepsilon^2\delta. \quad \text{(B.9)}$$

Putting together (B.7), (B.8), (B.9),

$$\Big\|\sum_{b\lesssim a\sim c} I^{a,b,c,T}\Big\|_{W^{k,p}} \lesssim C^3\varepsilon\bigg(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\bigg). \quad \text{(B.10)}$$

The other sums $\sum_{a\lesssim b\sim c}$ and $\sum_{c\lesssim a\sim b}$ are estimated similarly.

Control of space non resonant terms in $W^{k,p}$. Space non resonant terms require even less modifications from [4] compared to time non resonant terms, indeed integration by parts in η does not require to handle the new nonlinear term $\mathcal{T}$.

For the comfort of having a proof with unified notations, we repeat here a part of the arguments from [4, Section 6.1.3]. Since control for t small just follows from the H^N bounds, we focus on $t \geqslant 1$. Also we only bound the Duhamel integral over $[1, t-1]$, the integrals over $[0,1]$ and $[t-1, t]$ being easier to bound.

Frequency splitting. — The weighted term $x_j e^{-itH}\Psi$ is only in L^2, thus following [17] we use a frequency truncation with a threshold frequency depending on t. Let $\theta \in C_c^\infty(\mathbb{R}^+)$, $\theta|_{[0,1]} = 1$, $\operatorname{supp}(\theta) \subset [0,2]$, $\Theta(t) = \theta(\frac{|D|}{t^\nu})$, $\nu > 0$ small to choose later. For any quadratic term $B[\Psi^\pm, \Psi^\pm]$, we write

$$B[\Psi^\pm, \Psi^\pm] = \overbrace{B[(1-\Theta(t))\Psi^\pm, \Psi^\pm] + B[\Theta(t)\Psi^\pm, (1-\Theta)(t)\Psi^\pm]}^{\text{high frequencies}} + \overbrace{B[\Theta(t)\Psi^\pm, \Theta(t)\Psi^\pm]}^{\text{low frequencies}}.$$

High frequencies. — Using the dispersion estimate of Theorem 2.1, product estimates and Sobolev embedding we have for $\frac{1}{p_1} = \frac{1+\gamma}{3}$ and for any quadratic term $B[\Psi^\pm, \Psi^\pm]$:

$$\begin{aligned} &\left\| \int_1^{t-1} e^{i(t-s)H}\big(B[(1-\Theta(t))\Psi^\pm, \Psi^\pm] + B[\Theta(t)z, (1-\Theta)(t)\Psi^\pm]\big)\mathrm{d}s \right\|_{W^{k,p}} \\ &\qquad \leqslant \int_1^{t-1} \frac{1}{(t-s)^{1+\gamma}} \|\Psi\|_{W^{k+2,p_1}} \|(1-\Theta(s))\Psi\|_{H^{k+2}}\mathrm{d}s \qquad \text{(B.11)} \\ &\qquad \leqslant \int_1^{t-1} \frac{1}{(t-s)^{1+\gamma}} \|\Psi\|_{H^N}^2 \frac{1}{s^{\nu(N-2-k)}}\mathrm{d}s, \end{aligned}$$

choosing N large enough so that $\nu(N-2-k) \geqslant 1+\gamma$, we obtain a bound $C_1 C^2 \varepsilon^2 / t^{1+\gamma}$.

Low frequencies. — We estimate now the quadratic terms associated to $B^{a,b,c,X}[\Theta\Psi^\pm, \Theta\Psi^\pm]$:

$$\begin{aligned} &\mathcal{F}I^{a,b,c,X} \\ &:= e^{itH(\xi)} \int_1^{t-1} \int_{\mathbb{R}^N} \Big((e^{-is\Omega} B^{a,b,c,X}(\eta, \xi-\eta) \widetilde{\Theta\Psi^\pm}(s,\eta) \widetilde{\Theta\Psi^\pm}(s, \xi-\eta) \Big) \mathrm{d}\eta \mathrm{d}s, \end{aligned}$$

with $\Omega = H(\xi) \mp H(\eta) \mp H(\xi-\eta)$. Using $e^{-is\Omega} = \frac{i\nabla_\eta \Omega}{s|\nabla_\eta \Omega|^2} \cdot \nabla_\eta e^{-is\Omega}$ and denoting $R_j = \frac{\partial_j}{|\nabla|}$ the Riesz operators, $\Theta'(t) := \theta'(\frac{|D|}{t^\nu})$, $J_j = e^{itH} x_j e^{-itH}$, an integration by part in η gives:

$$\begin{aligned}
&I^{a,b,c,X}\\
&= \mathcal{F}^{-1}\int_1^{t-1}\frac{e^{itH(\xi)}}{is}\int_{\mathbb{R}^N} e^{-is\Omega(\xi,\eta)}\partial_{\eta_j}\\
&\qquad\qquad \Big(\mathcal{B}_{1,j}^{a,b,c,X}(\eta,\xi-\eta)\Big[\Theta\widetilde{\Psi^\pm}(\eta),\Theta\widetilde{\Psi^\pm}(\xi-\eta)\Big]\Big)\mathrm{d}\eta\mathrm{d}s\\
&= -\sum_{j=1}^3\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\Big(\mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)(J_j\Psi)^\pm,\Theta(s)\Psi^\pm]\\
&\qquad - \mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)(J_jz)^\pm]+\mathcal{B}_{2,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)\Psi^\pm]\Big)\mathrm{d}s\\
&\quad -\sum_{j=1}^3\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\Big(\mathcal{B}_{1,j}^{a,b,c,X}\Big[\frac{1}{s^\nu}R_j\,\Theta'(s)\Psi^\pm,\Theta(s)\Psi^\pm\Big]\\
&\qquad\qquad - \mathcal{B}_{1,j}^{a,b,c,X}\Big[\Theta(s)\Psi^\pm,\frac{1}{s^\nu}R_j\Theta'(s)\Psi^\pm\Big]\Big)\mathrm{d}s.
\end{aligned}\tag{B.12}$$

where we recall:

$$\mathcal{B}_{1,j}^{a,b,c,X}=\frac{\partial_{\eta_j}\Omega}{|\nabla_\eta\Omega|^2}B^{a,b,c,X},\quad \mathcal{B}_{2,j}^{a,b,c,X}=\partial_{\eta_j}\mathcal{B}_{1,j}^{a,b,c,X}.$$

We now use Lemma B.2 to bound the first term of (B.12). As for time non resonant terms, we distinguish $b\lesssim c\sim a,\ c\lesssim c\lesssim a\sim b$ and $a\lesssim b\sim c$.

Estimates for quadratic terms involving $\mathcal{B}_{1,j}^{a,b,c,X}$**.** In the case $c\lesssim a\sim b$, let $\varepsilon_1>0$ to be fixed later. Using Minkowski's inequality, dispersion and the rough multiplier Theorem B.1 with $s=1+\varepsilon_1$, $\frac{1}{q}=1/2+(\gamma-\varepsilon_1)/3$, $s=4/3$, $\frac{1}{q_1}=7/18+\gamma/3$ for $a\geqslant 1$, $0\leqslant k_1\leqslant k$ we obtain

$$\begin{aligned}
&\sum_j\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{c\lesssim a\sim b}\mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)(J_j\Psi)^\pm,\Theta(s)\Psi^\pm]\mathrm{d}s\Big\|_{L^p}\\
&\lesssim\sum_j\int_1^{t-1}\frac{1}{s(t-s)^{1+3\varepsilon}}\Big(\sum_{c\lesssim a\sim b\leqslant 1}\|\mathcal{B}_{1,j}^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|\Theta(s)J_j\Psi\|_{L^2}\|\Theta(s)\Psi]\|_{L^q}\\
&\qquad+\sum_{c\lesssim a\sim b,\ 1\lesssim a\lesssim s^\nu}a^k\|\mathcal{B}_{1,j}^{a,b,c,X}\|_{[B^{4/3}]}\|\Theta(s)J_j\Psi\|_{L^2}\|\Theta(s)]\|_{L^{q_1}}\Big)\mathrm{d}s
\end{aligned}$$

$$\lesssim \sum_j \int_1^{t-1} \frac{1}{s(t-s)^{1+3\varepsilon}} \Bigg(\sum_{a\lesssim 1} \sum_{c\lesssim a\sim b} \|\mathcal{B}_{1,j}^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]} \|\Theta(s) J_j \Psi\|_{L^2} \|\Theta(s)\Psi]\|_{L^q}$$

$$+ \sum_{1\lesssim a\lesssim s^\nu} a^k \sum_{c\lesssim a\sim b} \|\mathcal{B}_{1,j}^{a,b,c,X}\|_{[B^{4/3}]} \|\Theta(s) J_j \Psi\|_{L^2} \|\Theta(s)\Psi]\|_{L^{q_1}} \Bigg) \mathrm{d}s$$

Using Lemma B.2 and interpolation we have for $\varepsilon_1 < 1/4$ and $\varepsilon_1 - \gamma > 0$,

$$\sum_j \sum_{a\lesssim 1} \sum_{c\lesssim a\sim b} \|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]} \lesssim \sum_{a\lesssim 1} a^{1-(1+\varepsilon_1)} \sum_{c\lesssim a} c^{\frac{3}{2}-(1+\varepsilon_1)} \lesssim 1,$$

$$\|\Psi(s)\|_{L^q} \lesssim \|\Psi(s)\|_{L^p}^{\frac{\varepsilon_1-\gamma}{1+\gamma}} \|\Psi(s)\|_{L^2}^{1-\frac{\epsilon_1-\gamma}{1+\gamma}} \lesssim C\varepsilon^{1-\frac{\epsilon_1-\gamma}{1+\gamma}} \left(\delta + \frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{\varepsilon_1-\gamma}{1+\gamma}}.$$

In high frequencies we have:

$$\sum_{1\lesssim a\lesssim s^\nu} a^k \sum_{c\lesssim a\sim b} \frac{\langle M\rangle^2 c^{3/2-4/3}}{\langle a\rangle} \lesssim s^{\nu(k+7/6)},$$

$$\|\Psi(s)\|_{L^{q_1}} \lesssim \varepsilon^{\frac{2+6\gamma}{3+3\gamma}} \left(\delta + \frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{1-3\gamma}{3+3\gamma}}.$$

Finally we conclude that if $\min\big(\varepsilon_1 - 2\gamma, 1/3 - 2\gamma - \nu(k+7/6)\big) \geqslant 0$ (this choice is possible provided γ and ν are small enough):

$$\begin{aligned} \sum_j \Bigg\| \int_1^{t-1} \frac{1}{s} e^{-i(t-s)H} \sum_{c\lesssim a\sim b} \mathcal{B}_{1,j}^{a,b,c,X}[\Theta(s)(J_j\Psi)^\pm, \Theta(s)\Psi^\pm]\mathrm{d}s \Bigg\|_{W^{k,p}} \\ \lesssim \int_1^{t-1} \frac{C^2\varepsilon}{s(t-s)^{1+\gamma}} \varepsilon^{1-\frac{\varepsilon_1-\gamma}{1+\gamma}} \left(\delta + \frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{\varepsilon_1-\gamma}{1+\gamma}} \\ + \frac{C^2\varepsilon s^{\nu(k+7/6)}}{s(t-s)^{1+\gamma}} \varepsilon^{\frac{2+6\gamma}{3+3\gamma}} \left(\delta + \frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{1-3\gamma}{3+3\gamma}} \mathrm{d}s \\ \lesssim \frac{C^2\varepsilon^2}{t^{1+\gamma}} + \frac{C^2\varepsilon^{2-\frac{\varepsilon_1-\gamma}{1+\gamma}}\delta^{\frac{\varepsilon_1-\gamma}{1+\gamma}}}{t} + \frac{C^2\varepsilon^{\frac{5+9\gamma}{3+3\gamma}}\delta^{\frac{1-3\gamma}{3+3\gamma}}}{t^{1-\nu(k+7/6)}} \\ \lesssim \frac{C^2\varepsilon^2}{t^{1+\gamma}} + C^2\varepsilon\delta. \end{aligned} \tag{B.13}$$

The cases $b \lesssim c \sim a$, $a \lesssim c \sim b$ are very similar. The second term in (B.12) is symmetric to the previous computation and the terms

$$\Bigg\| \nabla^{k_1} \int_1^{t-1} \frac{1}{s} e^{i(t-s)H} \Bigg(\mathcal{B}_{1,j}^{a,b,c,X}\left[\frac{1}{s^\nu} R_j \Theta'(s)\Psi^\pm, \Theta(s)\Psi^\pm\right] - \mathcal{B}_1^{a,b,c,X}\left[\Theta(s)\Psi^\pm, \frac{1}{s^\nu} R_j\Theta'(s)\Psi^\pm\right] \Bigg) \mathrm{d}s \Bigg\|_{L^p},$$

are simpler since there is no weighted term Jz involved.

Estimates for quadratic terms involving $\mathcal{B}_{2,j}^{a,b,c,X}$**.** The last term to consider is

$$\left\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{a,b,c}\mathcal{B}_2^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)\Psi^\pm]\mathrm{d}s\right\|_{L^p}.$$

Let us focus on the case $b\lesssim a\sim c$. We use the same indices as in the previous section: $s=1+\varepsilon_1$, $\frac{1}{q}=1/2+(\gamma-\varepsilon_1)/3$, $\frac{1}{q_1}=7/18+\gamma/3$,

$$\begin{aligned}
&\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_{2,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)\Psi^\pm]\mathrm{d}s\Big\|_{L^p}\\
&\lesssim\int_1^{t-1}\frac{1}{s(t-s)^{1+\gamma}}\Bigg(\sum_{a\leqslant 1}\sum_{b\lesssim a\sim c}U(b)U(c)\big\|\mathcal{B}_{2,j}^{a,b,c,X}\big\|_{[B^{1+\varepsilon_1}]}\\
&\qquad\qquad\|U^{-1}\Theta(s)\Psi\|_{L^2}\|U^{-1}\Theta(s)\Psi]\|_{L^q} \qquad \text{(B.14)}\\
&\qquad+\sum_{1\leqslant a\lesssim s^\nu}a^k\sum_{b\lesssim a\sim c}\frac{U(b)}{\langle c\rangle^k}\|\mathcal{B}_{2,j}^{a,b,c,X}\|_{[B^{4/3}]}\\
&\qquad\qquad\|U^{-1}\Theta(s)\Psi\|_{L^2}\|\langle\nabla\rangle^k\Theta(s)\Psi]\|_{L^{q_1}}\Bigg)\mathrm{d}s.
\end{aligned}$$

According to Lemma B.2, we can bound the first sum if $\varepsilon_1<1/4$:

$$\sum_{a\leqslant 1}\sum_{b\lesssim c\sim a}U(b)U(c)\|\mathcal{B}_{2,j}^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\lesssim\sum_{a\leqslant 1}\sum_{b\lesssim c\sim a}b^{1/2-\varepsilon_1}a^{-\varepsilon_1}\lesssim 1,$$

and according to Lemma B.3 and the bootstrap assumption (5.4)

$$\begin{aligned}
\|U^{-1}\Psi(s)\|_{L^2}&\lesssim\|\Psi\|_X,\\
\|U^{-1}\Psi(s)\|_{L^q}&\lesssim\|U^{-1}\Psi\|_{L^2}^{1-\varepsilon_1+\gamma}\|U^{-1}\Psi\|_{L^6}^{\varepsilon_1-\gamma}\\
&\lesssim\frac{\|xe^{-itH}\Psi\|_2^{1-\varepsilon_1+\gamma}\big(\|xe^{-itH}\Psi\|_2+\|\Psi\|_{H^1}\big)^{\varepsilon_1-\gamma}}{s^{\frac{3(\varepsilon_1-\gamma)}{5}}}\\
&\lesssim\frac{C\varepsilon}{s^{\frac{3(\varepsilon_1-\gamma)}{5}}}.
\end{aligned}$$

Now for $M\gtrsim 1$

$$\sum_{1\leqslant a\lesssim s^\nu}a^k\sum_{b\lesssim c\sim a}\frac{U(b)\langle M\rangle^2 b^{1/2-4/3}}{\langle a\rangle\langle c\rangle^k}\lesssim\sum_{1\leqslant a\lesssim s^\nu}a\lesssim s^\nu,$$

$$\|\Psi(s)\|_{W^{k,q_1}}\lesssim\varepsilon^{\frac{2+6\gamma}{3+3\gamma}}\left(\delta+\frac{\varepsilon}{(1+s)^{1+\gamma}}\right)^{\frac{1-3\gamma}{3+3\gamma}}.$$

We plug these estimates in (B.14) and from the same computations as for (B.13) we find that if $\min\big(3(\varepsilon_1-\gamma)/5, 1/3-\gamma-\nu\big) \geqslant \gamma$,

$$\begin{aligned}&\left\|\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{b\lesssim c\sim a}\mathcal{B}_{2,j}^{a,b,c,X}[\Theta(s)\Psi^\pm,\Theta(s)\Psi^\pm]\mathrm{d}s\right\|_{W^{k,p}}\\
&\lesssim\int_1^{t-1}\frac{C^2\varepsilon^2}{(t-s)^{1+\gamma}s^{1+\frac{3(\varepsilon_1-\gamma)}{5}}}+\frac{C^2\varepsilon^{1+\frac{2+6\gamma}{3+3\gamma}}}{(t-s)^{1+\gamma}}\left(\delta+\frac{\varepsilon}{(t-s)^{1+\gamma}}\right)^{\frac{1-3\gamma}{3+3\gamma}}\mathrm{d}s \qquad \text{(B.15)}\\
&\lesssim\frac{C^2\varepsilon^2}{t^{1+\gamma}}+C^2\varepsilon\delta.\end{aligned}$$

The two other cases $c\lesssim a\sim b$ and $a\lesssim b\sim c$ can be treated in a similar way.

From (B.11), (B.13), (B.15), quadratic space non resonant terms are bounded by

$$\left\|\int_0^t\sum_{a,b,c}e^{i(t-s)H}B^{a,b,c,X}[\Psi^\pm,\Psi^\pm]\mathrm{d}s\right\|_{W^{k,p}}\lesssim C^2\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\right). \qquad \text{(B.16)}$$

Conclusion. From (B.10) and (B.16), we have

$$\left\|\int_0^1 e^{i(t-s)H}\mathcal{D}(\Psi)\mathrm{d}s\right\|_{W^{k,p}}\lesssim C^2\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\right). \qquad \text{(B.17)}$$

Higher order (cubic and quartic) terms are easier to control, see [4, Section 5.2]. To conclude

$$\begin{aligned}&\left\|e^{itH}\Psi_0+\int_0^t e^{i(t-s)H}\mathcal{N}(\Psi,\mathbb{P}u)\mathrm{d}s\right\|_{W^{k,p}}\\
&\qquad\leqslant\frac{C_1\varepsilon}{(1+t)^{1+\gamma}}+C_1C^2\varepsilon\left(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}},\right),\end{aligned} \qquad \text{(B.18)}$$

with C_1 independent of C, ε, δ. Hence for C large enough, ε small enough we have as expected

$$\|\Psi\|_{W^{k,p}}\leqslant\frac{C}{2}\left(\delta+\frac{\varepsilon}{(1+t)^{1+\gamma}}\right).$$

Control of the purely quadratic terms in the weighted norm. We refer to [4, Section 6.2], which can be applied with the same "routine" modifications as for the $W^{k,p}$ estimates.

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