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A new definition of rough paths on manifolds (*)

Youness Boutaib (1) and Terry Lyons (2)

Abstract. — Smooth manifolds are not the suitable context for trying to generalize the concept of rough paths on a manifold. Indeed, when one is working with smooth maps instead of Lipschitz maps and trying to solve a rough differential equation, one loses the quantitative estimates controlling the convergence of the Picard sequence. Moreover, even with a definition of rough paths in smooth manifolds, ordinary and rough differential equations can only be solved locally in such case. In this paper, we first recall the foundations of the Lipschitz geometry, introduced in [8], along with the main findings that encompass the classical theory of rough paths in Banach spaces. Then we give what we believe to be a minimal framework for defining rough paths on a manifold that is both less rigid than the classical one and emphasized on the local behaviour of rough paths. We end by explaining how this same idea can be used to define any notion of coloured paths on a manifold.

Résumé. — Les variétés régulières ne sont pas bien adaptées à la généralisation du concept des chemins rugueux aux variétés. En effet, quand on travaille avec des applications régulières plutôt que des applications Lipschitz pour résoudre une équation différentielle rugueuse, on perd les estimations quantitatives qui contrôlent la convergence des itérations de Picard. De plus, étant donne une définition de chemins rugueux sur variétés, on ne peut en général résoudre des équations différentielles ordinaires ou rugueuses que de manière locale. Dans cet article, on rappelle d'abord les fondations de la géométrie différentielle Lipschitz, introduite dans [8], ainsi que les principaux résultats qui généralisent ceux de la théorie classique des chemins rugueux dans les espaces de Banach. Ensuite on donne un cadre minimal pour la définition des chemins rugueux sur une variété qui est moins rigide que la précédente et qui met l'accent sur le comportement local des chemins rugueux. Finalement, on explique comment ces idées peuvent être appliquées pour généraliser la définition de tout chemin coloré à une variété.

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1. Introduction

The theory of rough paths (Lyons, [28]) and its variations (e.g. Gubinelli, [19]) generalise Young's integration theory in a way that it separates the probabilistic and deterministic parts of the strongly probabilistic Itô calculus, using only the variation of paths as a way of measuring their smoothness. A crowning achievement of the theory is understanding that, as far as ordinary differential equations are concerned, a path should not be defined by its graph but rather be identified as a choice of its signature (that is, the sequence of its iterated integrals). For example, the signature of a Brownian motion could be calculated using either Itô's or Stratonovich's calculus and it is this choice that leads one to solve a stochastic differential equation driven by a Brownian motion either in the sense of Itô's calculus or Stratonovich's. More generally, the theory of rough paths provides us with a deterministic calculus constructed in a way that it does not depend intrinsically on how a signature is defined but rather on common algebraic (that can be summerized by a Hopf algebra structure) and analytical properties that all signatures are expected to satisfy. These works have, in particular, enriched the toolbox of stochastic analysis with deterministic -path by path- results and widened its scope to rougher paths than the Brownian motion. The underlying philosophy of the theory also opened the door for solving a certain class of Stochastic Partial Differential Equations that require making sense of classically ill-defined products of distributions. This was carried incrementally through the development of alternative rough path theories: branched rough paths [20], para-controlled calculus [21] and regularity structures [22] (see [16] for a succinct exposition that goes from the theory of rough paths to that of regularity structures.) In addition to the natural applications that come with stochastic analyis, the theory of rough paths highlighted the role of signatures as highly efficient transforms of paths [5, 23], which led to their exploitation in recent works in machine learning (the literature on the subject being abundant, we cite [2, 11, 18] as varied use-case examples).

The need for such calculus on manifolds arises naturally from both the points of view of pure and applied mathematics (see for example the introductions to [8, 24]). Indeed, several physical systems and theoretical constructions arise as solutions in geometric contexts to ordinary differential equations driven by non-smooth paths. As in the Euclidean case, several applications come from stochastic analysis and extensions of rough path theory to manifolds would provide deterministic tools for the understading of approximation, continuity and regularity properties of solutions to stochastic differential equations (SDEs) on manifolds (see e.g. [15, 25]). A simple illustration of such equations is the construction of the Brownian motion on the unitary group (which is a Riemannian manifold) as the solution to a diffusion equation driven by a Brownian motion on its Lie algebra. In turn, this is related to the two-dimensional Yang–Mills theory and the free unitary Brownian motion (we refer the reader for example to [4, 13, 30] and the references therein for more details on this specific subject). These manifold-valued SDEs driven by vector-space-valued noise remain the most studied types of diffusions on manifolds both in theory and in practice (for example in finance [31] or for attitude estimation [33]). However, solving them in the rough analysis framework (and the consequences thereof) requires solving a Rough Differential Equation as a fixed point problem on the product of the vector space and the manifold, which itself is a “pure” manifold.

That being said, the literature on rough paths on manifolds is however still very limited compared to its counterpart in the Euclidean setting or even to that of stochastic analysis on manifolds. The main attempts to generalise these notions to manifolds are due to the seminal work of Cass, Litterer and Lyons [8] in the framework of what is called a Lipschitz manifold and Driver and Semko in [14] (for paths controlled by rough paths on Riemannian manifolds). We also refer to Cass, Driver and Litterer in [7] (for weakly geometric rough paths on submanifolds embedded in the Euclidean space) and Bailleul [3].

The aim of the present paper is twofold: to give an alternative definition that simplifies but also complements the understanding of manifold-valued rough paths according to [8] and to give a general methodology of defining similar non-canonical lifts of paths on manifolds. Our exposition is structured in the following manner: Section 2 reviews the key ingredients in the theory of rough paths and sets up the notations and conventions used in the rest of the article while Section 3 recalls parts of the theory laid in [8] that will be of use to check the consistency of our results with the classical theory. In Section 4, we show several results emphasising the local nature of rough paths which we use subsequently in Section 5 to define our new notion of rough paths on manifolds. As in [8], we avoid putting too much structure on the manifold we work on or exactly mimicking the construction of rough paths on the Euclidean space in the non-linear framework of a manifold. Finally, as there currently exist many variants of the theory of rough paths (each serving a well-defined purpose) and in order to emphasise the simple ideas we used to translate the notion of geometric rough paths to manifolds and to explain how this can be done without any further particular considerations of the class of manifolds one is working on (for example, one does not need the manifold to be Riemannian to be able to measure the smoothness of paths), we give in Section 6 a general recipe expressed in the language of category theory to motivate the introduction of local Lipschitz manifolds in the preceding sections as a natural framework and explain how one may more generally use the same approach to define a notion of *coloured paths* on manifolds.

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2. Review of key elements in the theory of rough paths

We start by setting up some notations and conventions and recalling the definitions and results that will be necessary to us in the rest of this work, one of which will be the extension of the notion of Lipschitz (Hölder) maps. As rough paths have become in the past few years a widely popular and familiar subject, we will strive to keep this section short and refer the reader to the literature on (geometric) rough paths, for instance [17, 27, 28, 29].

2.1. Signatures of paths

2.1.1. The concept of the p-variation

Given $p \geqslant 1$ and a path $x : [0,T] \to E$ taking values in a normed vector space $(E, \|\cdot\|)$, we denote by $\|x\|_{p,[0,T]}$ the p-variation of x over $[0,T]$. When $p = 1$, we say that the path x has bounded variation.

Assume $(E, \|\cdot\|)$ is a Banach space. The set $\mathcal{V}^p([0,T],E)$ of all continuous paths from $[0,T]$ to E that have a finite p-variation over $[0,T]$ is a Banach space when endowed with the p-variation norm ([12, 29]):

$$\|\cdot\|_{\mathcal{V}^p([0,T],E)} : x \longmapsto \|x\|_{p,[0,T]} + \sup_{t\in[0,T]} \|x_t\|$$

Moreover, we have the following natural embedding:

$$\forall\, q \geqslant p \geqslant 1 : \quad \mathcal{V}^p([0,T],E) \subseteq \mathcal{V}^q([0,T],E)$$

The manipulation of p-variations is often made easier by the introduction of controls. For a compact interval J, we will denote by Δ_J the simplex of all pairs $(s,t) \in J^2$ such that $s \leqslant t$.

Definition 2.1. — *A function $\omega : \Delta_{[0,T]} \to \mathbb{R}_+$ is said to be a control if it has the following properties:*

- *ω is continuous.*
- *ω is super-additive i.e. $\omega(s,u) + \omega(u,t) \leqslant \omega(s,t)$, $\forall\ 0 \leqslant s \leqslant u \leqslant t \leqslant T$.*

To every path x of finite p-variation, we can associate a "natural" control ω given by $\omega(s,t) = \|x\|^p_{p,[s,t]}$. Conversely, a control ω bounds the p^{th} power of the increments of a continuous path x, i.e:

$$\forall\ (s,t) \in \Delta_{[0,T]} : \quad \|x_t - x_s\|^p \leqslant \omega(s,t),$$

if and only if x has a finite p-variation. In this case, one also has $\|x\|^p_{p,[s,t]} \leqslant \omega(s,t)$ for all pairs $(s,t) \in \Delta_{[0,T]}$ and we say that the p-variation of x is controlled by ω (cf. [12, 27, 29]).

2.1.2. The tensor algebra

Given a vector space E, for every integer n, $E^{\otimes n}$ denotes the space of homogeneous tensors of E of degree n (with the convention $E^{\otimes 0} = \mathbb{R}$). The set of formal series of tensors of E is denoted by $T((E))$ (see [32] for an exhaustive exposition or [29, Chapter 2] for the part of the theory that will be in use in the rest of this work). For an integer $m \geqslant 0$, the truncated tensor algebra of order m of E is denoted by $T^{(m)}(E)$ while the canonical homomorphism $T((E)) \to T^{(m)}(E)$ is denoted by π_m. A permutation $\sigma \in \mathcal{S}_m$ acts linearly on $E^{\otimes m}$ by the following:

$$\forall\ x_1, x_2, \ldots, x_m \in E : \quad \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(m)}$$

Finally, we will only consider norms $\|\cdot\|$ on $T((E))$ that are admissible, i.e. that satisfy the two following conditions:

(1) $\forall\ n \in \mathbb{N}^*$, $\forall\ \sigma \in \mathcal{S}_n$, $\forall\ x \in E^{\otimes n}$: $\|\sigma x\| = \|x\|$.
(2) $\forall\ n, m \in \mathbb{N}^*$, $\forall\ x \in E^{\otimes n}$, $\forall\ y \in E^{\otimes m}$: $\|x \otimes y\| \leqslant \|x\|\|y\|$.

A short discussion on certain basic properties of norms on tensor product spaces and their implications on the analysis can be found for example in [6].

2.1.3. The signature of a path

To fix the notations, we now proceed to the formal definition of the signature, which is a well studied subject since K.T. Chen's work in the late fifties (see e.g. [9, 10]):

Definition 2.2. — *Let E be a Banach space and $T \geqslant 0$. Let $x : [0,T] \to E$ be a path of bounded variation. For $(s,t) \in \Delta_{[0,T]}$, we define the following sequence of iterated integrals by induction (well-defined by Stieltjes' integration theory):*

$$\begin{cases} S^0(x)_{(s,t)} = 1 \\ S^n(x)_{(s,t)} = \int_{[s,t]} S^{n-1}(x)_{(s,u)} \otimes \mathrm{d}x_u, \quad \forall\, n \in \mathbb{N}^* \end{cases}$$

For every pair $(s,t) \in \Delta_{[0,T]}$, the sequence $(S^n(x)_{(s,t)})_{n\in\mathbb{N}}$, simply denoted $S(x)_{(s,t)}$, is called the signature of x over $[s,t]$. For $N \in \mathbb{N}^*$, $(S^n(x)_{(s,t)})_{n\leqslant N}$, simply denoted $S_N(x)_{(s,t)}$, is called the truncated signature of x over $[s,t]$ of degree N.

Let us note that signatures satisfy several key algebraic and analytic properties. We do not recall these here as those that are of interest to us are shared within the larger class of rough paths that we will introduce next.

2.2. Rough Paths

The theory of rough paths generalizes the concept of signatures to more irregular paths and provides the tools to solving differential equations driven by these without having to build a whole new theory of integration for each one of them (as in Itô's calculus). The concept of rough paths finds its source in the signature and the main analytic and algebraic properties that it satisfies. The space of geometric rough paths is indeed simply defined as the completion of that of signatures of paths with bounded variation under a suitably chosen metric similar to the p-variation metric for paths introduced in Subsection 2.1.1. We introduce here the basic definitions, notations and results that will be extensively used in the subsequent sections.

2.2.1. Multiplicative functionals

The appropriate higher order generalisation of the linearity of the integral is expressed in terms of multiplicative functionals:

Definition 2.3 ([28, 29]). — *Let E be a normed vector space and $T \geqslant 0$. Let X be a map on $\Delta_{[0,T]}$ with values in $T((E))$ (respectively in $T^{(n)}(E)$, with $n \in \mathbb{N}^*$). X is said to be a multiplicative functional (resp. a multiplicative functional of degree n) if the following holds:*

(1) *X is continuous.*

(2) $\forall\, t \in [0,T] \quad X_{(t,t)} = \mathbf{1}$.
(3) *X is multiplicative:* $\forall\, 0 \leqslant s \leqslant u \leqslant t \leqslant T \quad X_{(s,t)} = X_{(s,u)} \otimes X_{(u,t)}$.

For example, for every path $x \in \mathcal{V}^1([0,T],E)$, $S(x)$ is a multiplicative functional and for every $n \in \mathbb{N}^*$, $S_n(x)$ is a multiplicative functional of degree n ([9, 10, 29]).

Remark 2.4. — When no confusion is possible, we may use the term multiplicative functional with no reference to its degree being finite or not.

Notation 2.5. — X^i will denote the component of X of degree i.

2.2.2. p-variation metric and rough paths

We generalise now the notion of p-variation:

DEFINITION 2.6 ([28, 29]). — *Let E be a normed vector space and $T \geqslant 0$. Let $p \geqslant 1$. Let X be a map on $\Delta_{[0,T]}$ with values in $T((E))$ (respectively in $T^{(n)}(E)$, with $n \in \mathbb{N}^*$) and ω be a control over $[0,T]$. X is said to have a finite p-variation over $[0,T]$ controlled by ω if:*

$$\forall\, i \in \mathbb{N}^* (\textit{resp.}\ \ \forall\, i \in [\![1,n]\!]),\ \forall\, 0 \leqslant s \leqslant t \leqslant T: \quad \|X^i_{(s,t)}\| \leqslant \frac{\omega(s,t)^{\frac{i}{p}}}{\beta_p \left(\frac{i}{p}\right)!}$$

where we write $x!$ for $\Gamma(x+1)$, with Γ being the usual extension of the factorial (the Gamma function) and:

$$\beta_p = p\left(1 + \sum_{k=1}^{\infty} \left(\frac{2}{k}\right)^{\frac{[p]+1}{p}}\right)$$

If there exists a control such that the previous properties holds, we may say that X has a finite p-variation over $[0,T]$ without mentioning the control. We denote by $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{([p])}(E))$ the set of continuous paths defined over $\Delta_{[0,T]}$ with values in $T^{([p])}(E)$ and that have finite p-variation.

For example, the signature of a path of bounded variation has finite 1-variation [26].

Remark 2.7. — One can also easily note that for $1 \leqslant q \leqslant p$, a multiplicative functional of finite q-variation is of finite p-variation.

Lyons' extension theorem states that a multiplicative functional of finite p-variation is uniquely determined by its terms of degree less than or equal to $[p]$:

Theorem 2.8 (Extension theorem [28, 29]). — *Let $p \geqslant 1$ and $n \in \mathbb{N}^* \cup \{\infty\}$ such that $n \geqslant [p]$. Let E be a Banach space. Let X be a multiplicative functional of degree $[p]$ in E that has a finite p-variation over $[0,T]$ controlled by a control function ω. There exists a unique multiplicative functional $\widetilde{X}$ of degree n that has a finite p-variation over $[0,T]$ and such that $\pi_{[p]}(X) = \pi_{[p]}(\widetilde{X})$. Furthermore, the p-variation of $\widetilde{X}$ is also controlled by ω.*

As a consequence, the signature of a path of bounded variation is then the only multiplicative functional with finite 1-variation whose component of degree 1 corresponds to the increments of said path.

We now have all the ingredients to recall the definition of rough paths:

Definition 2.9 (Rough Paths [28, 29]). — *Let $p \geqslant 1$ and E be a Banach space. A p-rough path is a multiplicative functional of degree $[p]$ that has finite p-variation. The set of all p-rough paths over $[0,T]$ with values in E will be denoted by $\Omega_p([0,T];E)$, while that of p-rough paths over sub-segments of $[0,T]$ will be denoted by $\Omega_p^{[0,T]}(E)$.*

Definition 2.10. — *We define on $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{([p])}(E))$ the p-variation metric, denoted by $\widetilde{d}_p$, by the following:*

$$\widetilde{d}_p(X,Y) = \max_{0 \leqslant i \leqslant [p]} \sup_{D \in \mathcal{D}_{[0,T]}} \left(\sum_D \|X^i_{(t_j,t_{j+1})} - Y^i_{(t_j,t_{j+1})}\|^{\frac{p}{i}} \right)^{\frac{1}{p}}$$

for all $X, Y \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{([p])}(E))$. (For a subdivision $D = (t_j)_{0 \leqslant j \leqslant n}$, we write $\sum_D a_{(t_j,t_{j+1})} := \sum_{j=0}^{n-1} a_{(t_j,t_{j+1})}$.)

2.2.3. Geometric p-rough paths

We introduce now geometric rough paths, a central notion in our paper. For more details, see [29] or [28] (or [17] for an extensive study of the subject in the finite-dimensional case motivated with examples from stochastic analysis).

Definition 2.11. — *Let $T \geqslant 0$ and E be a real Banach space. We define the set of E-valued geometric p-rough paths over $[0,T]$, denoted by $G\Omega_p([0,T];E)$, to be the closure of the set $\{S_{[p]}(x) \,|\, x \in \mathcal{V}^1([0,T],E)\}$ in the p-variation metric. The set of E-valued geometric p-rough paths over sub-segments of $[0,T]$ will be denoted $G\Omega_p^{[0,T]}(E)$.*

We recall in particular that any geometric p-rough path is a p-rough path and is then a multiplicative functional that has finite p-variation.

Similar to the notion of concatenation of paths with values in vector spaces, we can also define the concatenation of functionals taking their values in the truncated tensor algebra:

Definition 2.12. — *Let $n \in \mathbb{N}^*$. Let E be a vector space. Let $s, u, t \in \mathbb{R}$ such that $s \leqslant u \leqslant t$. Let X (resp. Y) be a functional defined on $\Delta_{[s,u]}$ (resp. on $\Delta_{[u,t]}$) with values in $T^{(n)}(E)$. We define the concatenation of X and Y, denoted $X * Y$, to be the functional over $\Delta_{[s,t]}$ defined as follows; for $(a, b) \in \Delta_{[s,t]}$:*

$$(X * Y)_{(a,b)} = \begin{cases} X_{(a,b)}, & \text{if } b \leqslant u \\ X_{(a,u)} \otimes Y_{(a,u)}, & \text{if } a \leqslant u \leqslant b \\ Y_{(a,b)}, & \text{if } u \leqslant a \end{cases}$$

The following theorem is straight-forward:

Theorem 2.13. — *Let $p \geqslant 1$. Let E be a Banach space. Let $s, u, t \in \mathbb{R}$ such that $s \leqslant u \leqslant t$. Let X (resp. Y) be a functional defined on $\Delta_{[s,u]}$ (resp. on $\Delta_{[u,t]}$) with values in $T^{([p])}(E)$. Then:*

- *If X and Y are multiplicative functionals, then $X * Y$ is a multiplicative functional;*
- *If X and Y have finite p-variation, then $X * Y$ has finite p-variation;*
- *If X and Y are geometric p-rough paths, then $X * Y$ is a geometric p-rough path.*

It will be necessary to us to attach a starting point to our geometric rough paths as we will be mostly dealing with integrals of rough paths and rough paths on manifolds; both of which require a starting point. In the next few sections, a geometric rough path in a Banach space E will be a pair (x, X), where X is a geometric rough path in the sense of Definition 2.11 and $x \in E$ is called the starting point. Hence, we identify the space of geometric rough paths with starting points with the Cartesian product $E \times G\Omega_p^{[0,T]}(E)$.

On $E \times G\Omega_p^{[0,T]}(E)$ we define a metric d_p as the product metric of $\widetilde{d_p}$ and the norm on E, i.e.:

$$d_p((x, X), (y, Y)) = \max(\|x - y\|, \widetilde{d_p}(X, Y))$$

In other situations, it will be more convenient to attach a trace to a rough path instead of a starting point (i.e. a path which increments correspond the element of degree 1 in said rough path).

Definition 2.14. — *Let E be a Banach space and $p \geqslant 1$. For an open subset U of E, a local geometric p-rough path in U is a triple (x, X, J) such that:*

- *J is a compact interval.*
- *x is a U-valued path over J.*
- *X is an E-valued geometric p-rough path over J which trace is x, i.e. for all $(s,t) \in \Delta_J$, $X^1_{s,t} = x_t - x_s$.*

The set of local geometric p-rough paths in U defined over compact sub-intervals of I will be denoted $G\Omega_p^I(U;E)$, that of local geometric p-rough paths in U defined over a compact interval J will be denoted $G\Omega_p(J;U;E)$.

Remark 2.15. — The trace of a geometric p-rough path is trivially a path of finite p-variation.

Remark 2.16. — It goes without saying that rough paths with starting points easily identify with local rough paths. Indeed, if J is of the form, say, $[0,T]$, we will identify the local rough path (x,X) (omitting the interval J when no confusion is possible) with the rough path with starting point $(x(0),X)$.

2.2.4. The integral of Lipschitz one-forms along geometric p-rough paths

When trying to make sense of integrals of one-forms along rough paths, it is very important (for example in regard to solving a differential equation) to be able to control the smoothness (in terms of variation) of the image of the path under the one-form. It appears that Lipschitz maps, first introduced by H. Whitney in [35] and later studied by E. Stein in [34], are the appropriate type of maps to use in this context (for a basic study of these maps including all the results below, see for example [6]):

Definition 2.17. — *Let $n \in \mathbb{N}$ and $0 < \varepsilon \leqslant 1$. Let E and F be two normed vector spaces and U be a subset of E. Let $f^0 : U \to F$ be a map and for every $k \in [\![1,n]\!]$, let $f^k : U \to \mathcal{L}_s(E^{\otimes k}, F)$ be a map with values in the space of the symmetric k-linear mappings from E to F.*

For $k \in [\![0,n]\!]$, the map $R_k : E \times E \to \mathcal{L}(E^{\otimes k}, F)$ defined by:

$\forall\, x,y \in U, \forall\, v \in E^{\otimes k}$:

$$f^k(x)(v) = \sum_{j=k}^{n} f^j(y)\left(\frac{v \otimes (x-y)^{\otimes (j-k)}}{(j-k)!}\right) + R_k(x,y)(v)$$

is called the remainder of order k associated to $f = (f^0, f^1, \ldots, f^n)$.

The collection $f = (f^0, f^1, \ldots, f^n)$ is said to be Lipschitz of degree $n+\varepsilon$ on U (or in short a Lip-$(n+\varepsilon)$ *map) if there exists a constant M such that for all $k \in [\![0,n]\!]$, $x,y \in U$ and $v_1, \ldots, v_k \in E$:*

(1) $\|f^k(x)(v_1 \otimes \cdots \otimes v_k)\| \leqslant M\|v_1 \otimes \cdots \otimes v_k\|$;
(2) $\|R_k(x,y)(v_1 \otimes \cdots \otimes v_k)\| \leqslant M\|x-y\|^{n+\varepsilon-k}\|v_1 \otimes \cdots \otimes v_k\|$.

The smallest constant M for which the properties above hold is called the Lip-$(n+\varepsilon)$*-norm of f and is denoted by $\|f\|_{\text{Lip-}(n+\varepsilon)}$. The set of all* Lip-$(n+\varepsilon)$ *maps defined on U with values in F will be denoted* $\text{Lip}(n+\varepsilon, U, F)$.

Remark 2.18. — Let us stress that the above definition is purely quantitative and makes sense even on discrete sets. On any non-empty open subset of U, $f^1, \ldots, f^n$ are the successive derivatives of f^0. However, these maps are not necessarily uniquely determined by f^0 on an arbitrary set U. Keeping this in mind, if $f^0 : U \to F$ is a map such that there exist $f^1, \ldots, f^n$ such that $(f^0, f^1, \ldots, f^n)$ is Lip-$(n+\varepsilon)$, we will often say that f^0 is Lip-$(n+\varepsilon)$ with no mention of $f^1, \ldots, f^n$.

Theorem 2.19 ([28, 29]). — *Let $\gamma, p \in \mathbb{R}$ such that $\gamma > p \geqslant 1$. Let E and F be two Banach spaces. Let $\alpha : E \to \mathcal{L}_c(E,F)$ be a* Lip-$(\gamma-1)$ *one-form. There exists a unique continuous map:*

$$I_\alpha : (E \times G\Omega_p^{[0,T]}(E), d_p) \longrightarrow (G\Omega_p^{[0,T]}(F), \widetilde{d}_p)$$

that extends Young's integration theory, i.e.

$$\forall\, x \in \mathcal{V}^1([0,T],E) : \quad I_\alpha(x_0, S_{[p]}(x)) = S_{[p]}\left(\int \alpha(x)\mathrm{d}x\right).$$

For a geometric p-rough path with a starting point (x_0, X) or its corresponding local rough path (x, X), we denote:

$$I_\alpha(x_0, X) = \int \alpha(x_0, X)\mathrm{d}X = \int \alpha(x, X)\mathrm{d}X$$

3. The Lipschitz geometry

In this section, we review two of the findings of [8] that are most relevant to our work in this paper: the Lipschitz structure and geometric rough paths on manifolds.

3.1. Lipschitz structures

We first recall the general definitions of Lipschitz manifolds and Lipschitz maps and one-forms on them.

Definition 3.1 ([8]). — *Let $\gamma > 0$. Let $n \in \mathbb{N}^*$ and let M be an n-topological manifold. Let I be a countable set and, for every $i \in I$, U_i be an open subset of M and $\phi_i : M \to \mathbb{R}^n$ be a compactly supported map such that its restriction on U_i defines a homeomorphism. We say that $((\phi_i, U_i))_{i\in I}$ is a Lipschitz-γ atlas if the following properties are satisfied:*

- *$(U_i)_{i\in I}$ is a pre-compact locally finite cover of M;*
- *For every $i \in I$: $\phi_i(U_i) = B(0,1)$;*
- *There exists $\delta \in (0,1)$, such that $(U_i^\delta)_{i\in I}$ covers M, where, for every $i \in I$:*

$$U_i^\delta = {\phi_i^{-1}}_{|U_i}(B(0,1-\delta));$$

- *There exists $L > 0$, such that, for every $i, j \in I$, $\phi_j \circ (\phi_{i|U_i})^{-1} : B(0,1) \to \mathbb{R}^n$ is Lipschitz-γ and $\|\phi_j \circ (\phi_{i|U_i})^{-1}\|_{\text{Lip-}\gamma} \leqslant L$.*

With the constants above, we will say that M is a Lipschitz-γ manifold with constants (δ, L).

Example 3.2. — As one would expect, finite-dimensional vector spaces can be endowed with a natural Lipschitz-γ manifold structure of any degree $\gamma \geqslant 1$.

Proof. — Indeed, let $\gamma \geqslant 1$. Let V be a finite dimensional space and let $(e_1, \ldots, e_n)$ be a basis for V. Let φ be a Lip-γ extension on V of $Id_{B(0,1)}$ with support in $B(0,2)$. For $x \in V$, let φ_x be the map: $y \mapsto \varphi(y-x)$. Then $(\varphi_x, B(x,1))_{x\in I}$ is a Lip-γ atlas on V, where:

$$I = \left\{ \sum_{i=1}^{n} \frac{k_i}{2} e_i \;\middle|\; k_1, \ldots, k_n \in \mathbb{Z} \right\} \qquad \square$$

Example 3.3 ([8][(1)]). — Let $n \in \mathbb{N}^*$ and $0 < \varepsilon \leqslant 1$. A $\mathcal{C}^{n+1}$ structure on a compact manifold induces a natural Lip-$(n+\varepsilon)$ structure.

Definition 3.4. — *Let $\gamma_0 \geqslant 1$ and $d \in \mathbb{N}^*$. Let M be a d-dimensional* Lip-γ_0 *manifold with an atlas $\{(\phi_i, U_i), i \in I\}$ and E be a normed vector space.*

- *A map $f : M \to E$ is said to be* Lip-γ, *for $\gamma \leqslant \gamma_0$, if there exists a constant C such that, for every $i \in I$, $f \circ {\phi_i}^{-1}_{|U_i} : B(0,1) \to E$ is* Lip-γ *with a Lipschitz norm at most C. The smallest constant C for which this property holds is called the* Lip-γ *norm of f and is denoted by $\|f\|_{\text{Lip-}\gamma}$.*

(1) This is a slightly stronger result than in [8] but its proof is practically the same.

- *An E-valued one-form α on M is said to be* Lip-γ, *for $\gamma \leqslant \gamma_0 - 1$, if there exists a constant C such that, for every $i \in I$:*
$$(\phi_i{}^{-1}_{|U_i})^* \alpha : B(0,1) \to \mathcal{L}(\mathbb{R}^d, E)$$
is Lip-γ *with a Lipschitz norm at most C. The smallest constant C for which this property holds is called the* Lip-γ *norm of α and is denoted by $\|\alpha\|_{\text{Lip-}\gamma}$.*

Remark 3.5. — An important property that is omitted in [8] and that we underline in the above definition is that, on a Lip-γ_0 manifold, it does not make sense geometrically to define Lip-γ maps or Lip-$(\gamma-1)$ one-forms if $\gamma > \gamma_0$. Indeed, with the notations of Definition 3.4, if for $i \in I$, $f \circ \phi_i{}^{-1}_{|U_i} : B(0,1) \to E$ is Lip-γ (where $\gamma > \gamma_0$), then, based only on the definition of a Lip-γ_0 atlas, one cannot show that, for $j \in I$, the restriction of $f \circ \phi_j{}^{-1}_{|U_j \cap U_i} : B(0,1) \to E$ on any non-trivial subset of $U_j \cap U_i$ is smoother than Lip-γ_0. The same principle applies to defining Lipschitz one-forms. A more rigorous way to state the above is obtained by translating it in the language of equivalent Lipschitz structures (see [8] for a definition).

3.2. Rough paths on a manifold

In this subsection, we recall the definition of rough paths on manifolds as presented in [8]. As one does not have a natural notion of linearity and iterated integrals on a manifold, one has to consider a different approach than the one arising from studying the p-variational properties of paths and signatures. As we will hint to later, integrals of Lipschitz one-forms along a rough paths do characterize said path; this is the idea behind the elegant definition of a rough path given in [8]. Additionally, in the absence of a natural translation, a rough path on a manifold comes attached with a starting point.

Definition 3.6 ([8]). — *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geqslant 1$ and $T \geqslant 0$. Let M be a* Lip-γ_0 *manifold and $x \in M$. $\mathbb{X}$ is a geometric p-rough path over $[0,T]$ on M starting at x if, for every $\gamma \in \mathbb{R}$ such that $\gamma_0 \geqslant \gamma > p$ and every Banach space E, the following conditions are satisfied:*

(1) $\mathbb{X}$ *maps* Lip-$(\gamma-1)$ *E-valued one-forms on M to E-valued geometric p-rough paths (in the classical sense).*
(2) *There exists a control ω such that for every E-valued* Lip-$(\gamma-1)$ *one-form α on M, $\mathbb{X}(\alpha)$ is controlled by $\|\alpha\|_{\text{Lip-}(\gamma-1)}\omega$, i.e.:*

$$\forall\, (s,t) \in \Delta_{[0,T]}, \forall\, i \in [\![1,[p]]\!] : \quad \|\mathbb{X}(\alpha)^i_{(s,t)}\| \leqslant \frac{(\|\alpha\|_{\text{Lip-}(\gamma-1)}\omega(s,t))^{i/p}}{\beta_p(i/p)!}$$

(3) *(Chain rule) For every Banach space F and every compactly supported* Lip-γ *map* $\psi : M \to F$ *and every E-valued* Lip-$(\gamma - 1)$ *one-form α on F we have:*

$$\mathbb{X}(\psi^*\alpha) = \int \alpha(\psi(x), \mathbb{X}(\psi_*))\mathrm{d}\mathbb{X}(\psi_*)$$

Contrary to the classical framework, in the context of manifolds, one does not need to make a difference between rough paths and geometric rough paths (as only the latter are defined). Consequently, we drop the word "geometric" when talking about geometric rough paths on manifolds. Moreover, in the classical sense, geometric rough paths are uniquely determined by the values of the integrals of compactly supported one-forms along them. In order to make the correspondance one-to-one between the concepts of classical geometric rough paths (on finite-dimensional spaces) and rough paths on the same space when endowed with its canonical Lipschitz structure, the following equivalence relation is introduced:

Definition 3.7 ([8]). — *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geqslant 1$. Let M be a* Lip-γ_0 *manifold. We say that two p-rough paths $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ on M are equivalent, and we write $\mathbb{X} \sim \widetilde{\mathbb{X}}$, if they have the same starting point and if, for every $\gamma \in \mathbb{R}$ such that $\gamma_0 \geqslant \gamma > p$ and for every* Lip-$(\gamma - 1)$ *compactly supported Banach space valued one-form α on M, we have $\mathbb{X}(\alpha) = \widetilde{\mathbb{X}}(\alpha)$.*

Let us note at this point that there exists a one-to-one correspondance between rough paths in the classical sense and in a Lip-γ manifold, when said manifold is a finite-dimensional vector space V (see [8] for the precise statement and proof). More precisely, to each "classical" geometric p-rough path (with a starting point) (x, X) on V over is associated a unique p-rough path $\mathbb{X}$ on V in the manifold sense in the following way: for every Banach space-valued Lip-$(\gamma - 1)$ compactly supported one-form α on V, one has:

$$\mathbb{X}(\alpha) = \int \alpha(x, X)\mathrm{d}X.$$

When working on a manifold, one has to ensure that certain properties are invariant by the change of charts (or, in other words, by local reparametrisation). For this reason, one defines the pushforward of rough paths by conveniently chosen maps:

Lemma 3.8 ([8]). — *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geqslant 1$. Let M and N be* Lip-γ_0 *manifolds and $f : M \to N$ be a map such that there exists a constant C_f such that, for all $\gamma \in (p, \gamma_0]$ and every* Lip-$(\gamma - 1)$ *Banach space valued one-form α on N, we have:*

$$\|f^*\alpha\|_{\text{Lip-}(\gamma-1)} \leqslant C_f \|\alpha\|_{\text{Lip-}(\gamma-1)}$$

Then f induces a pushforward map f_* from p-rough paths on M to p-rough paths on N defined as follows: for every p-rough path $\mathbb{X}$ over $[0,T]$ on M starting at x, $f_*\mathbb{X}$ starts at $f(x)$ and for every Lip-$(\gamma-1)$ *Banach space valued one form α on N, where $\gamma \in (p,\gamma_0]$, $f_*\mathbb{X}(\alpha)$ is given by $f_*\mathbb{X}(\alpha) = \mathbb{X}(f^*\alpha)$.*

The next proposition shows that there exists a particular class of Lipschitz maps that induce pushforwards of rough paths. In particular, one can ascertain that the pushforwards of rough paths by coordinate maps or transition maps are also rough paths:

Proposition 3.9 ([8][2]). — *Let $\gamma_0 \geqslant \gamma > 1$ and let M be a* Lip-γ_0 *manifold and W be a normed vector space. Let $f : M \to W$ be a* Lip-γ *map and α be a* Lip-$(\gamma-1)$ *Banach space valued one-form on W (or defined on a subset of W containing $f(M)$). Then $f^*\alpha$ is* Lip-$(\gamma-1)$ *and there exists a constant C_γ depending only on γ such that:*

$$\|f^*\alpha\|_{\text{Lip-}(\gamma-1)} \leqslant C_\gamma \|\alpha\|_{\text{Lip-}(\gamma-1)} \|f\|_{\text{Lip-}\gamma} \max(\|f\|_{\text{Lip-}\gamma}^{\gamma-1}, 1)$$

Like in the classical case, there exists a notion of concatenation of manifold-valued rough paths. In the context of manifolds, this is important on its own since one usually works locally on coordinate domains to solve ordinary or rough differential equations before attempting to make sense of a global solution via a concatenation procedure:

Definition 3.10 ([8]). — *Let $\gamma_0 > p \geqslant 1$ and $s \leqslant u \leqslant t$. Let M be a* Lip-γ_0 *manifold. Let $\mathbb{X}$ (respectively $\mathbb{Y}$) be a p-rough path on M over $[s,u]$ (resp. $[u,t]$). We define the concatenation of $\mathbb{X}$ and $\mathbb{Y}$, denoted by $\mathbb{X} * \mathbb{Y}$, to be the functional over $[s,t]$ mapping every Banach space-valued* Lip-$(\gamma-1)$ *compactly supported one-form α (for every $\gamma_0 \geqslant \gamma > p$) to the classical rough path $\mathbb{X}(\alpha) * \mathbb{Y}(\alpha)$.*

Unlike the classical case, the concatenation of two rough paths on a manifold is not necessarily a rough path. This is due to the fact that rough paths on a manifold come attached with a starting point and that we have no natural notion of translation. Therefore, for this concatenation to be a rough path, we have to make sure that the two rough paths in question have starting and "ending" points that agree in the following sense:

Definition 3.11. — *Let $\gamma_0 > p \geqslant 1$ and $s \leqslant t$. Let M be a* Lip-γ_0 *manifold. Let $\mathbb{X}$ be a p-rough path on M over $[s,t]$ with starting point x and let $y \in M$. We say that $\mathbb{X}$ has an end point consistent with y if for*

[2] Following [6], we correct the exponent in the inequality compared to the result that appeared in [8] and drop furthermore the dependence on the dimension of the manifold. Said inequality can also be made sharper using improved estimates in [6].

every Banach space-valued compactly supported Lip-γ *map* f *on* M *(for every* $\gamma_0 \geqslant \gamma > p$*), we have:*

$$f(x) + \mathbb{X}(f_*)^1_{s,t} = f(y)$$

In this case, one can check the consistency condition for the concatenation of two rough paths and prove that it is also a rough path:

Proposition 3.12 ([8]). — *Let* $\gamma_0 > p \geqslant 1$ *and* $s \leqslant u \leqslant t$*. Let* M *be a* Lip-γ_0 *manifold. Let* $\mathbb{X}$ *(respectively* $\mathbb{Y}$*) be a p-rough path on* M *over* $[s,u]$ *(resp.* $[u,t]$*) with starting point* x *(resp.* y*). We assume that* $\mathbb{X}$ *has an end point consistent with the starting point of* $\mathbb{Y}$*. Then* $\mathbb{X} * \mathbb{Y}$ *defines a p-rough path on* M *over* $[s,t]$ *with* x *as a starting point.*

Well-defined concatenations of rough paths constitute an associative operation:

Lemma 3.13 ([8]). — *Let* $\gamma_0 > p \geqslant 1$ *and* M *be a* Lip-γ_0 *manifold. Let* $\mathbb{X}$*,* $\mathbb{Y}$ *and* $\mathbb{Z}$ *be p-rough paths on* M *such that:*

- $\mathbb{X}$ *has an end point consistent with the starting point of* $\mathbb{Y}$*;*
- $\mathbb{Y}$ *has an end point consistent with the starting point of* $\mathbb{Z}$*.*

Then:

- $\mathbb{X}$ *has an end point consistent with the starting point of* $\mathbb{Y} * \mathbb{Z}$*;*
- $\mathbb{X} * \mathbb{Y}$ *has an end point consistent with the starting point of* $\mathbb{Z}$*;*
- $\mathbb{X} * (\mathbb{Y} * \mathbb{Z}) = (\mathbb{X} * \mathbb{Y}) * \mathbb{Z}$

In this case, we denote $\mathbb{X} * \mathbb{Y} * \mathbb{Z} := \mathbb{X} * (\mathbb{Y} * \mathbb{Z})$*.*

Since this notion of rough paths does not attach, for the moment, an underlying path on the manifold to the rough path in question, the support of the latter is defined based on the images of one-forms supported in all possible open sets:

Definition 3.14 ([8]). — *Let* $\gamma_0 > p \geqslant 1$ *and* M *be a* Lip-γ_0 *manifold. Let* $\mathbb{X}$ *be a p-rough path on* M *with starting point* x*.*

(1) *For an open subset* U *of* M*, we say that* $\mathbb{X}$ *misses* U *if, for every* $\gamma_0 \geqslant \gamma > p$ *and every Banach space-valued compactly supported* Lip-$(\gamma - 1)$ *one-form* α *on* M *with support in* U*, we have* $\mathbb{X}(\alpha) = 0$*.*
(2) *We define the support of the rough path* $\mathbb{X}$ *as the closed set:*

$$\operatorname{supp}(\mathbb{X}) := \{x\} \cup \left(M - \bigcup_{\mathbb{X} \textit{ misses } U} U \right)$$

Naturally, this notion of support is consistent with the definition of support for a classical rough path:

Proposition 3.15 ([8]). — *Let $\gamma_0, p \in \mathbb{R}$ such that $\gamma_0 > p \geqslant 1$ and $T > 0$. Let V be a finite dimensional vector space endowed with its canonical structure of a* Lip-γ_0 *manifold. Let (x, X) be a geometric p-rough path on V over $[0, T]$ (in the classical sense), and let $\mathbb{X}$ be the p-rough path associated to it in the manifold sense, i.e. for every Banach space-valued* Lip-$(\gamma - 1)$ *compactly supported one-form α on V: $\mathbb{X}(\alpha) = \int \alpha(x, X)\mathrm{d}X$. Then:*

$$\operatorname{supp}(\mathbb{X}) = \left\{x + X^1_{0,t},\, t \in [0, T]\right\}$$

Like in the classical case, the support of a rough path on a manifold can be shown to be compact:

Theorem 3.16 ([8]). — *Let $\gamma_0 > p \geqslant 1$. Let M be a* Lip-γ_0 *manifold and $\mathbb{X}$ be a p-rough path on M. Then the support of $\mathbb{X}$ is compact.*

The claim of the previous theorem can be shown by writing a rough path on a manifold as the concatenation of rough paths that have their support contained in the domain of only one chart at a time as described in the following theorem. This paper will generalise this decomposition of rough paths on manifolds into a collection of classical "local" rough paths that are compatible in a suitable sense.

Theorem 3.17 ([8]). — *Let $\gamma_0 > p \geqslant 1$ and $T \geqslant 0$. Let M be a* Lip-γ_0 *manifold. Let $\mathbb{X}$ be a p-rough path on M defined over an interval $[0, T]$. Then there exists a subdivision $D = (s_i)_{0\leqslant i\leqslant N}$ of $[0, T]$ and a collection $(\mathbb{X}^i, x_i, (\phi_i, U_i))_{1\leqslant i\leqslant N}$ such that, for every $i \in [\![1, N]\!]$:*

1. *$\mathbb{X}^i$ is a p-rough path on M defined over $[s_{i-1}, s_i]$ with starting point x_i.*
2. *(if $i < N$) x_{i+1} is consistent with the endpoint of $\mathbb{X}^i$.*
3. *(ϕ_i, U_i) is a* Lip-γ_0 *chart on M such that*

$$\operatorname{supp}(\mathbb{X}^i) \subseteq U_i$$

and such that for every Banach space-valued Lip-$(\gamma - 1)$ *one-form α defined on M (where $\gamma_0 \geqslant \gamma > p$) and compactly supported* Lip-$(\gamma - 1)$ *one-form ξ_α defined on $\mathbb{R}^d$ which agrees with $(\phi_{i\,|U_i}^{-1})^*(\alpha)$ on $B(0, 1)$, we have:*

$$\mathbb{X}^i(\alpha) = ((\phi_i)_*\mathbb{X})_{[s_{i-1}, s_i]}(\xi_\alpha)$$

4. $\mathbb{X} = \mathbb{X}^1 * \cdots * \mathbb{X}^N$.

$(\mathbb{X}^i, x_i, (\phi_i, U_i))_{1\leqslant i\leqslant N}$ is called a localising sequence for $\mathbb{X}$.

4. Local character of rough paths

4.1. Intervals

We briefly present here some elementary statements on covers of intervals which will be of use when studying rough paths locally.

Definition 4.1. — *Let M be a topological space. Let J be a subset of M and $(K_i)_{i\in I}$ be a collection of subsets of M.*

(1) *$(K_i)_{i\in I}$ is said to be locally finite if for each $x \in M$ there exists a neighbourhood U_x of x such that at most finitely many K_i's have non-empty intersection with U_x.*
(2) *$(K_i)_{i\in I}$ is said to cover J (or is a cover of J) if $J \subseteq \bigcup_{i\in I} K_i$.*
(3) *$(K_i)_{i\in I}$ is called a compact cover of J if it is locally finite, covers J and for $i \in I$, K_i is compact subset of J.*

The proof of the following lemma is straightforward:

Lemma 4.2. — *Each interval J of $\mathbb{R}$ admits a compact cover.*

Lemma 4.3. — *Let M be a topological space, J be a compact subset of M and $(K_i)_{i\in I}$ be a locally finite collection of compact sets that cover J such that all the K_i's intersect J. Then I is finite.*

Proof. — For every $x \in J$, let U_x be an open neighbourhood of x that intersects only finitely many of the K_i's. Then $(U_x)_{x\in J}$ is an open cover of J. As J is compact, there exists a finite subset J_0 of J such that $\mathcal{J}_0 = \{U_x; x \in J_0\}$ covers J. As every K_i intersects J, and hence an element in the finite set $\mathcal{J}_0$, and since every element of $\mathcal{J}_0$ intersects only finitely many of the K_i's, I must be finite. □

Corollary 4.4. — *Let J be an interval and $(K_i)_{i\in I}$ be a compact cover for J. Then I is countable. Moreover, if J is compact, then I is finite.*

Proof. — Let $(J_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of compact intervals (in the sense of inclusion) such that $J = \bigcup_n J_n$. Let $n \in \mathbb{N}$ and define:

$$I_n = \{i \in I; \quad K_i \cap J_n \neq \varnothing\}$$

Then $(K_i)_{i\in I_n}$ is a locally finite collection of compact sets that cover J_n such that all the K_i's intersect J_n. By Lemma 4.3, I_n is finite. Note now that $(I_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite subsets of I. Moreover, $I = \bigcup_n I_n$. Hence, I is countable.

The case when J is compact is covered by Lemma 4.3. □

Lemma 4.5. — *If $(K_i)_{i\in I}$ is a compact cover of an interval J, and if for each $i \in I$, $(K_j)_{j\in I_i}$ is a compact cover of K_i then $\{K_j \mid j \in I_i, i \in I\}$ is also a compact cover of J.*

Proof. — First note that:

$$\bigcup_{i\in I, j\in I_i} K_j = \bigcup_{i\in I}\left(\bigcup_{j\in I_i} K_j\right) = \bigcup_{i\in I} K_i = J$$

The K_j's ($j \in \bigcup_I I_i$) are all compact subsets of J. Let $x \in J$. Let $\mathcal{V}_{x,J}$ be a neighbourhood of x in J that intersects finitely many K_i's ($i \in I$). As I_i is finite for every $i \in I$ (Corollary 4.4), then $\mathcal{V}_{x,J}$ intersects finitely many K_j's ($j \in \bigcup_I I_i$). □

Definition 4.6. — *Let $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ be two collections of sets. We say that $(V_j)_{j\in J}$ is a refinement of $(U_i)_{i\in I}$ if, for every $j \in J$, there exists $i \in I$ such that $V_j \subseteq U_i$.*

Lemma 4.7. — *Let $\mathcal{O} = (O_i)_{i\in I}$ be a cover of an interval J by open sets and $\mathcal{K} = (K_h)_{h\in H}$ be a compact cover for J. Then there exists a compact cover for J that is a refinement of both $\mathcal{O}$ and $\mathcal{K}$.*

Proof. — Let $h \in H$ and $x \in K_h$. As $x \in J$, then there exists $i_x \in I$ and $\alpha_x > 0$ such that $(x-\alpha_x, x+\alpha_x) \subseteq O_{i_x}$. As K_h is compact and $((x-\alpha_x/2, x+\alpha_x/2))_{x\in K_h}$ covers K_h, then there exists a finite subset $P \subseteq K_h$ such that $([x-\alpha_x/2, x+\alpha_x/2])_{x\in P}$ covers K_h. Define $I_x = [x-\alpha_x/2, x+\alpha_x/2] \cap K_h$ for every $x \in P$. Then, for every $x \in P$, I_x is a compact subset of K_h that is contained in O_{i_x}. Furthermore, $(I_x)_{x\in P}$ is a compact cover of K_h. We conclude using Lemma 4.5. □

Proposition 4.8. — *Given any cover of a compact interval J by open sets $(O_i)_{i\in I}$, there exists a (finite) subdivision $(a_j)_{0\leqslant j\leqslant n}$ of J such that:*

$$\forall\, j \in [\![0, n-1]\!],\ \exists\, i \in I \text{ such that: } [a_j, a_{j+1}] \subseteq O_i$$

Proof. — By Lebesgue's number lemma, let $\delta > 0$ such that for every subset A of J of diameter less than δ there exists $i \in I$, such that $A \subseteq O_i$. It suffices to take any subdivision of J of mesh less than δ to conclude the proof. □

Proposition 4.9. — *Given any cover of a compact interval J by compact intervals $(K_i)_{i\in I}$, there exists a (finite) subdivision $(a_j)_{0\leqslant j\leqslant n}$ of J such that:*

$$\forall\, j \in [\![0, n-1]\!],\ \exists\, i \in I \text{ such that: } [a_j, a_{j+1}] \subseteq K_i$$

Proof. — Without loss of generality, we may assume that $J = [0,1]$. For every $x \in [0,1)$, we claim that:

$$\exists\, \varepsilon_x > 0, \quad \exists\, i \in I : [x, x+\varepsilon_x] \subset K_i$$

Indeed, assume the converse is true. Then there exists $x \in [0,1)$ such that:

$$\forall\, n \in \mathbb{N}^*, \quad \forall\, i \in I, \quad \exists\, x_{i,n} \in \left[x, x + \frac{1}{n}\right] - K_i$$

Define $I_0 = \{i \in I, x \in K_i\}$. Then necessarily $I_0 \subsetneq I$. Indeed, if $I_0 = I$, we let i_0 be such that $1 \in K_{i_0}$ and then, by convexity of K_{i_0}, we have $[x,1] \subset K_{i_0}$, which contradicts our assumption. Let $i^* \in I_0$. For all $n \in \mathbb{N}^*$, let x_n be an element of $[x, x + \frac{1}{n}] - K_{i^*}$. Then, for all $n \in \mathbb{N}^*$ and $i \in I_0$, $x_n \neq x$ (since $x \in K_{i^*}$) and $x_n \notin K_i$ (by the same convexity argument used above). Hence, (x_n) is a sequence in the compact set $\bigcup_{I-I_0} K_i$ converging to x, which leads to a contradiction. Therefore, the claim is true. For every $i \in I$, we write $K_i = [s_i, t_i]$. Let $i_0 \in I$ such that K_{i_0} contains a neighbourhood of 0 with non-empty interior and for every $i \in I$ such that $t_i \neq 1$, let $r_i \in I$ be such that there exists $\varepsilon_i > 0$ such that $[t_i, t_i + \varepsilon_i] \subset K_{r_i}$. We define $a_0 = 0$ and $a_1 = t_{i_0}$. Then $a_0 < a_1$ and $[a_0, a_1] \subset K_{i_0}$. We define the rest of the subdivision in a recursive way: for $q \geqslant 1$, given $a_q = t_{q^*}$, if $a_q = 1$, then we are done, otherwise we set $a_{q+1} = t_{r_{q^*}}$ (we have then $a_q < a_{q+1}$ and $[a_q, a_{q+1}] \subset K_{r_{q^*}}$). It is clear that this procedure converges in a finite number of steps and produces the desired subdivision. □

4.2. Locally Lipschitz maps

Definition 4.10. — *Let $\gamma > 0$. Let E and F be two normed vector spaces, U be a subset of E and $f : U \to F$ be a map. We say that f is locally Lipschitz-γ if, for every $x \in U$, there exists a neighborhood $\mathcal{V}_{x,U}$ of x in U such that $f_{|\mathcal{V}_{x,U}}$ is Lipschitz-γ. The set of all locally* Lip-γ *maps defined on U with values in F will be denoted* $\mathrm{Lip}_{\mathrm{loc}}(\gamma, U, F)$.

Example 4.11. — Lipschitz-γ maps are obviously locally Lipschitz-γ. Continuous linear and polynomial maps are locally Lipschitz (of any degree).

Remark 4.12. — One of the main reasons for introducing locally Lipschitz maps here instead of working with Lipschitz maps that are classical in the setting of geometric rough paths is to be able to use linear maps (such as the identity maps which are pivotal in the definition of categories) which are not Lipschitz in general(3) . Another possible solution to this issue is the use of almost Lipschitz maps introduced in [6]. The results below generalise automatically to this class of maps.

We recall here two important results on Lipschitz maps. They can be found for example in [6] and [8]:

(3) a usual workaround in analysis is to use suitable Whitney extensions.

Theorem 4.13 ([4]). — *Let E, F and G be three normed vector spaces. Let U be a subset of E and V be a subset of F. Let $\gamma \geqslant 1$. We assume that $(E^{\otimes k})_{k\geqslant 1}$ and $(F^{\otimes k})_{k\geqslant 1}$ are endowed with norms satisfying the projective property. Let $f : U \to F$ and $g : V \to G$ be two* Lip-γ *maps such that $f(U) \subseteq V$. Then $g \circ f$ is* Lip-γ *and there exists a constant C_γ (depending only on γ) such that:*

$$\|g \circ f\|_{\text{Lip-}\gamma} \leqslant C_\gamma \|g\|_{\text{Lip-}\gamma} \max(\|f\|^{\gamma}_{\text{Lip-}\gamma}, 1)$$

Lemma 4.14. — *Let $n \in \mathbb{N}$, $0 < \varepsilon \leqslant 1$ and $C \geqslant 0$. Let E and F be two normed vector spaces and U be an open subset of E. Let $f : U \to F$ be a map and for every $k \in [\![1, n]\!]$, let $f^k : U \to \mathcal{L}(E^{\otimes k}, F)$ be a map with values in the space of the symmetric k-linear mappings from E to F. We consider the two following assertions:*

- (A1) *$(f, f^1, \dots, f^n)$ is* Lip-$(n + \varepsilon)$ *and $\|f\|_{\text{Lip-}(n+\varepsilon)} \leqslant C$.*
- (A2) *f is n times differentiable, with $f^1, \dots, f^n$ being its successive derivatives. $\|f\|_\infty$, $\|f^1\|_\infty$, …, $\|f^n\|_\infty$ are upper-bounded by C and for all $x, y \in U : \|f^n(x) - f^n(y)\| \leqslant C\|x - y\|^\varepsilon$.*

Then (A1) $\Rightarrow$ (A2). *If furthermore U is convex then* (A1) $\Leftrightarrow$ (A2)*.*

Using for example Lemma 4.14, one can easily show the following result:

Proposition 4.15. — *Let $n \in \mathbb{N}^*$. Let E and F be two normed vector spaces, U be an open subset of E and $f : U \to F$ be a map of class $\mathcal{C}^n$. Then f is locally* Lip-n.

Unlike the notion of Lipschitzness, local Lipschitzness can easily be defined recursively on open sets:

Lemma 4.16. — *Let $\gamma > 1$. Let E and F be two normed vector spaces, U be an open subset of E and $f : U \to F$ be a differentiable map. Then its derivative* df *is locally Lipschitz-$(\gamma - 1)$ if and only if f is locally Lipschitz-γ.*

Proof. — Notice that, on every ball $B(x, \alpha) \subseteq U$ on which df is Lipschitz-$(\gamma - 1)$, one can use for example the fundamental theorem of calculus to bound f and we deduce that the restriction of f on this set is Lipschitz-γ using Lemma 4.14. The converse is obvious. □

Following Theorem 4.13, local Lipschitzness is conserved under composition:

Proposition 4.17. — *Let $\gamma \in [1, +\infty[$. Let E, F and G be normed vector spaces, U be a subset of E and V be a subset of F. Let $f : U \to F$*

[4] see [6] for a slightly improved estimate.

and $g : V \to G$ be two locally Lipschitz-γ maps such that $f(U) \subseteq V$. Then $g \circ f$ is locally Lipschitz-γ.

Before we carry on, we need the following embedding theorem for which the complete statement and proof can be found for example in [6].

Theorem 4.18. — *Let $\gamma, \gamma' > 0$ such that $\gamma' \leqslant \gamma$. Let E and F be two normed vector spaces and U be a subset of E. Let $f : U \to F$ be a* Lip-γ *map. Then f is* Lip-γ' *and there exists a constant $M_{\gamma,\gamma'}$ (depending only on γ and γ') such that $\|f\|_{\text{Lip-}\gamma'} \leqslant M_{\gamma,\gamma'} \|f\|_{\text{Lip-}\gamma}$.*

Locally Lipschitz maps conserve the smoothness (in the sense of variation) of paths:

Theorem 4.19. — *Let $p, \gamma \in [1, +\infty[$. Let E and F be two normed vector spaces, U be a subset of E and J a compact interval. Let $f : U \to F$ be a locally* Lip-γ *map over U and $x : J \to U$ a path with finite p-variation. Then $f \circ x : J \to F$ is of finite p-variation.*

Proof. — For every $t \in J$, let B_t be an open neighborhood of x_t in U such that $f_{|B_t}$ is Lipschitz-1 (following Theorem 4.18); denote its Lip-1 norm by M_t. Then $(x^{-1}(B_t))_{t \in J}$ is an open cover of J. Let $(a_i)_{0 \leqslant i \leqslant n}$ be a subdivision of J such that for all $i \in [\![0, n-1]\!]$, there exists $t_i \in J$ such that $[a_i, a_{i+1}] \subseteq x^{-1}(B_{t_i})$ (Proposition 4.8) and denote $M = \max_{1 \leqslant i \leqslant n} M_{t_i}$.

Let ω be a control of the p-variation of x over J. Let $s, u \in J$ and let $q, r \in [\![0, n]\!]$ such that $a_q \leqslant s \leqslant \cdots \leqslant u \leqslant a_r$. Since f is Lip-1 on each of the B_{t_i}'s, for $i \in [\![0, n-1]\!]$, with a norm less than M:

$$\begin{aligned}
&\|f(x_s) - f(x_u)\| \\
&\leqslant \|f(x_s) - f(x_{a_{q+1}})\| + \sum_{k=q+1}^{r-2} \|f(x_{a_k}) - f(x_{a_{k+1}})\| + \|f(x_{a_{r-1}}) - f(x_u)\| \\
&\leqslant M\left(\|x_s - x_{a_{q+1}}\| + \sum_{k=q+1}^{r-2} \|x_{a_k} - x_{a_{k+1}}\| + \|x_{a_{r-1}} - x_u\| \right) \\
&\leqslant M\left(\omega(s, a_{q+1})^{1/p} + \sum_{k=q+1}^{r-2} \omega(a_k, a_{k+1})^{1/p} + \omega(a_{r-1}, u)^{1/p} \right)
\end{aligned}$$

which, using the super-additivity of ω and Jensen's inequality, gives the control:

$$\|f(x_s) - f(x_u)\|^p \leqslant M^p n^{p-1} \omega(s, u)$$

Therefore, $f \circ x$ is of finite p-variation (M and n do not depend on s or u). □

4.3. A study of some properties of rough paths

Lemma 4.20. — *Let E be a vector space and J be a compact interval. Let X and Y be two E-valued multiplicative functionals on J. Let $(K_i)_{i\in I}$ be a compact cover for J by compact intervals such that, for all $i \in I$, $X_{|K_i}$ and $Y_{|K_i}$ are equal. Then X and Y are equal on J.*

Proof. — Let $(a, b) \in \Delta_J$. Then $(K_i \cap [a, b])_{i\in S}$, where $S = \{i \in I \mid K_i \cap [a, b] \neq \varnothing\}$, is a compact cover for $[a, b]$ by compact intervals. Let $(a_j)_{0\leqslant j\leqslant n}$ be a subdivision of $[a, b]$ such that for all $j \in [\![0, n-1]\!]$, there exists $i \in S$ such that $[a_j, a_{j+1}] \subseteq K_i$ (Proposition 4.9). Then, by assumption: $\forall\ j \in [\![0, n-1]\!]$ $X_{a_j,a_{j+1}} = Y_{a_j,a_{j+1}}$. Therefore:

$$X_{a_0,a_1} \otimes \cdots \otimes X_{a_{n-1},a_n} = Y_{a_0,a_1} \otimes \cdots \otimes Y_{a_{n-1},a_n}$$

Which, by using the multiplicativity of X and Y, gives: $X_{a,b} = Y_{a,b}$. Since this holds for all $(a, b) \in \Delta_J$, then $X = Y$. □

Lemma 4.21. — *Let E be a vector space and J be a compact interval. Let $(K_i)_{i\in I}$ be a compact cover for J by compact intervals. Let $(X_i)_{i\in I}$ be a collection of E-valued multiplicative functionals such that, for all $i \in I$, X_i is defined over K_i and for $i, j \in I$ such that $K_i \cap K_j \neq \varnothing$, we have $X_{i|K_i\cap K_j} = X_{j|K_i\cap K_j}$. Then there exists a unique multiplicative functional X defined over J such that, for all $i \in I$, $X_{|K_i} = X_i$.*

Proof. — Let $(a_j)_{0\leqslant j\leqslant n}$ be a subdivision of J and $(i_j)_{0\leqslant j\leqslant n}$ be a finite sequence of elements of I such that for all $j \in [\![0, n-1]\!]$, we have $[a_j, a_{j+1}] \subseteq K_{i_j}$. Let $(s, t) \in \Delta_J$ and let $q, r \in [\![1, n-1]\!]$ be such that:

$$a_{q-1} \leqslant s \leqslant a_q \leqslant \cdots \leqslant a_r \leqslant t \leqslant a_{r+1}$$

We set:

$$X_{s,t} = X_{i_{q-1}}(s, a_q) \otimes \cdots \otimes X_{i_r}(a_r, t)$$

It is an easy exercise to check that X defines a multiplicative functional satisfying the requirements of the statements. The uniqueness of such functional is a consequence of Lemma 4.20. □

Before we make sure that the integral of a geometric rough path against a sufficiently smooth locally Lipschitz one-form is indeed well defined, we need the following lemma quantifying the distance over an interval in the p-variation topology between two rough paths given their distances on a subdivision of that interval.

Lemma 4.22. — *Let E be a Banach space, $p \in [1, +\infty[$ and $T \geqslant 0$. Let Y and Z be two continuous maps from $\Delta_{[0,T]}$ to $T^{[p]}(E)$ with finite p-variation. Given a subdivision $\mathcal{D} = (s_i)_{0\leqslant i\leqslant r}$ of $[0, T]$, then we have:*

$$\widetilde{d}_p(Y, Z) \leqslant 3^{1-1/p} \max_{1\leqslant j\leqslant [p]} \left\| (\widetilde{d}_p^{[0,s_1]}(Y, Z), \cdots, \widetilde{d}_p^{[s_{r-1},T]}(Y, Z)) \right\|_{l_j}$$

where, for a finite sequence $x = (x_1, x_2, \dots, x_n)$ and $j > 0$, we define:

$$\|x\|_{l_j} = \left(\sum_{i=1}^{n} |x_i|^j\right)^{1/j}$$

Proof. — Let $j \in [\![1, [p]]\!]$. For $(s,u) \in \Delta_{[0,T]}$, define $V_{s,u} = Y^j_{s,u} - Z^j_{s,u}$. Let $\Delta = (t_i)_{0 \leqslant i \leqslant q}$ be a subdivision of $[0,T]$. Let $i \in [\![0, q-1]\!]$. Define the subdivision $\mathcal{D} \cap [t_i, t_{i+1}]$ of $[t_i, t_{i+1}]$ to be $(m^i_l) := (t_i, s_{\tilde{r}_i}, \dots, s_{\tilde{r}_i + n_i}, t_{i+1})$, where $\widetilde{r}_i$ and n_i are such that (if they exist) $s_{\tilde{r}_i - 1} < t_i \leqslant s_{\tilde{r}_i}$ and $s_{\tilde{r}_i + n_i} \leqslant t_{i+1} < s_{\tilde{r}_i + n_i + 1}$. Then we have:

$$\begin{aligned}
&\|V_{t_i,t_{i+1}}\|^{p/j} \\
&\leqslant \left(\|V_{t_i, s_{\tilde{r}_i}}\| + \sum_{l=\tilde{r}_i}^{\tilde{r}_i + n_i} \widetilde{d}_p^{[s_l, s_{l+1}]}(Y,Z)^j + \|V_{s_{\tilde{r}_i + n_i}, t_{i+1}}\|\right)^{p/j} \\
&\leqslant 3^{p/j-1}\left(\|V_{t_i, s_{\tilde{r}_i}}\|^{p/j} + \left(\sum_{l=\tilde{r}_i}^{\tilde{r}_i + n_i} \widetilde{d}_p^{[s_l, s_{l+1}]}(Y,Z)^j\right)^{p/j} + \|V_{s_{\tilde{r}_i + n_i}, t_{i+1}}\|^{p/j}\right)
\end{aligned}$$

Using the inequality $a^\alpha + b^\alpha \leqslant (a+b)^\alpha$, for $a, b \geqslant 0$ and $\alpha \geqslant 1$ after summing over all i's, we get:

$$\begin{aligned}
\sum_{\Delta} \|V_{t_i,t_{i+1}}\|^{p/j} &\leqslant 3^{p/j-1}\left(\sum_{i=0}^{r-1} \widetilde{d}_p^{[s_i, s_{i+1}]}(Y,Z)^j\right)^{p/j} \\
&\leqslant 3^{p/j-1} \|(\widetilde{d}_p^{[0,s_1]}(Y,Z), \cdots, \widetilde{d}_p^{[s_{r-1},T]}(Y,Z))\|^p_{l_j}
\end{aligned}$$

which gives the result. □

We can now show that the integral of a locally Lipschitz one-form along geometric rough paths is, as expected, well defined and continuous when varying the path.

Theorem 4.23. — *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$ and $T \geqslant 0$. Let E and F be two Banach spaces and U be an open subset of E. There exists a unique map:*

$$\mathcal{J} : \mathrm{Lip}_{\mathrm{loc}}(\gamma - 1, U, \mathcal{L}(E,F)) \times G\Omega_p^{[0,T]}(U; E) \longrightarrow G\Omega_p^{[0,T]}(F)$$

such that if $\alpha : U \to \mathcal{L}(E,F)$ is a Lip-$(\gamma-1)$ one-form and (x, X) is a local geometric p-rough path in U defined over an interval $[s,t] \subseteq [0,T]$ then:

$$\mathcal{J}(\alpha, (x,X)) = \int \alpha(x,X) \mathrm{d}X$$

Moreover, for a locally Lip-$(\gamma-1)$ *one-form* $\alpha : U \to \mathcal{L}(E,F)$ *and a subinterval* $[s,t] \subseteq [0,T]$*, the map:*

$$\begin{array}{rccc} \mathcal{J}_\alpha : & (G\Omega_p([s,t];U;E), d_p) & \longrightarrow & (G\Omega_p([s,t];F), \widetilde{d}_p) \\ & (x,X) & \longmapsto & \mathcal{J}(\alpha,(x,X)) \end{array}$$

is continuous.

Proof. — Let $\alpha : U \to \mathcal{L}(E,F)$ be a locally Lip-$(\gamma-1)$ one-form and $(x,X) \in G\Omega_p^{[0,T]}(U)$ defined over a compact interval J. Let $s,t \in \Delta_J$. For $u \in [s,t]$, let O_u be an open neighbourhood of x_u in U such that $\alpha_{|O_u}$ is Lip-$(\gamma-1)$. As $(x^{-1}(O_u))_{s\leqslant u\leqslant t}$ is an open covering of $[s,u]$, let $(s_i)_{0\leqslant i\leqslant n}$ be a subdivision of $[s,t]$ such that for all $i \in [\![0,n-1]\!]$, there exists $u_i \in [s,t]$ such that $[s_i,s_{i+1}] \subset x^{-1}(O_{u_i})$.

Let $i \in [\![0,n-1]\!]$. The rough path $(x,X)_{|[s_i,s_{i+1}]}$ takes its values in O_{u_i} on which α is Lip-$(\gamma-1)$. Therefore, the geometric p-rough path $\int \alpha(x(s_i),X)\mathrm{d}X$ is well defined over $[s_i,s_{i+1}]$. By Lemma 4.21, there exists a unique geometric p-rough path $\mathcal{J}(\alpha,(x,X))$ defined over $[s,u]$ which agrees with it. This proves the existence of the map $\mathcal{J}$. The uniqueness is shown following the same reasoning and decomposition.

We will show now the continuity of the map $\mathcal{J}_\alpha$. Let $(x,X) \in G\Omega_p([0,T];U;E)$. For $u \in [0,T]$, let $\eta_u > 0$ be such that $B(x_u,\eta_u) \subseteq U$ and $\alpha_{|B(x_u,\eta_u)}$ is Lip-$(\gamma-1)$. As before, let $(s_i)_{0\leqslant i\leqslant n}$ be a subdivision of $[0,T]$ such that for all $i \in [\![0,n-1]\!]$, there exists $u_i \in [0,T]$ such that $x([s_i,s_{i+1}]) \subset B(x_{u_i},\eta_{u_i}/2)$ and denote $\eta = \min_i \eta_{u_i}$. Let $\varepsilon > 0$. Let $\beta > 0$ be such that for all $i \in [\![0,n-1]\!]$ and $(z,Z) \in G\Omega_p([s_i,s_{i+1}];B(x_{u_i},\eta_{u_i});E)$, if:

$$d_p((x(s_i),X_{|[s_i,s_{i+1}]}),(z(s_i),Z)) \leqslant \beta$$

then:

$$\widetilde{d}_p^{[s_i,s_{i+1}]}\left(\int \alpha(x(s_i),X_{|[s_i,s_{i+1}]})\mathrm{d}X, \int \alpha(z(s_i),Z)\mathrm{d}Z\right) \leqslant \varepsilon$$

Let $(y,Y) \in G\Omega_p([0,T];U;E)$ such that $d_p((x(0),X),(y(0),Y)) \leqslant \min(\beta,\eta)/4$. For $u \in [0,T]$, we have then:

$$\|x(u)-y(u)\| \leqslant \|x(0)-y(0)\| + \|X^1_{0,u} - Y^1_{0,u}\| \leqslant \min(\beta,\eta)/2$$

Let $i \in [\![0,n-1]\!]$. For $u \in [s_i,s_{i+1}]$, both x_u and y_u are in $B(x_{u_i},\eta_{u_i})$, hence

$$\widetilde{d}_p^{[s_i,s_{i+1}]}(\mathcal{J}_\alpha(x,X),\mathcal{J}_\alpha(y,Y)) \leqslant \varepsilon$$

By Lemma 4.22, we deduce that:

$$\widetilde{d}_p(\mathcal{J}_\alpha(x,X),\mathcal{J}_\alpha(y,Y)) \leqslant 3^{1-1/p} n\varepsilon$$

from which we deduce the continuity of $\mathcal{J}_\alpha$ at (x,X). $\square$

We conclude this subsection by showing that integration along locally Lipschitz one-forms is consistent with Stieltjes' classical integration theory. This will be explicitly needed in the following section.

Lemma 4.24. — *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let E and F be two Banach spaces and U be an open subset of E. and J an interval. Let $f : U \to F$ be a locally* Lip-γ *map over U and $x : J \to U$ a path with bounded variation. Then:*

$$(f(x), \int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x), J) = (f(x), S_{[p]}(f(x)), J)$$

Proof. — First note that $S_{[p]}(f(x))$ is well-defined since $f(x)$ has bounded variation by Theorem 4.19. Let $s, t \in \Delta_J$:

(1) $S_{[p]}(f(x))$ has finite 1-variation and by definition $S_{[p]}(f(x))^1_{u,v} = f(x_v) - f(x_u)$ for all $(u, v) \in \Delta_{[s,t]}$.
(2) As $\mathrm{d}f$ is continuous and x is of bounded variation, then

$$\int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x)$$

has finite 1-variation and is simply equal to (the signature of) the Stieltjes integral $\int \mathrm{d}f(x)\mathrm{d}x$. Therefore, for all $(u, v) \in \Delta_{[s,t]}$:

$$\left(\int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x)\right)^1_{u,v} = \int_u^v \mathrm{d}f(x)\mathrm{d}x = f(x_v) - f(x_u).$$

Two multiplicative functionals that have finite 1-variation and which terms of the 1^{st} degree agree are equal by Theorem 2.8. Therefore, the sought identity stands. □

5. Local rough paths on manifolds

5.1. A new definition of rough paths on manifolds

Based on our findings so far, we are now able to give a minimal approach for defining rough paths on a manifold.

Definition 5.1. — *Let $\gamma \geqslant 1$. Let M be a topological manifold and $\mathcal{A}$ an atlas on M. We say that $(M, \mathcal{A})$ is a locally* Lip-γ *n-manifold if for any two charts (U, ϕ) and (V, ψ) in $\mathcal{A}$ such that $U \cap V \neq \varnothing$, the map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}^n$ is locally* Lip-γ.

Example 5.2. —

- A Lip-γ manifold is a locally Lip-γ manifold. In particular, finite-dimensional vector spaces are locally Lip-γ manifolds.
- Let $n \in \mathbb{N}^*$. A $\mathcal{C}^n$ manifold is a locally Lip-n manifold.

Notation 5.3. — For any open subsets U and V of E and a map $f \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, U, V)$, we denote by f_* the following map:

$$\begin{array}{rccc} f_*: & G\Omega_p(U) & \longrightarrow & G\Omega_p(V) \\ & (x, X, J) & \longmapsto & (f(x), \int \mathrm{d}f(x, X)\mathrm{d}X, J) \end{array}$$

Definition 5.4. — *Let $n \in \mathbb{N}^*$ and $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a locally* Lip-γ *n-manifold. A local p-rough path on M over a compact interval J is a collection $(x_i, X_i, J_i, (\phi_i, U_i))_{i\in I}$ of p-rough paths on $\mathbb{R}^n$ satisfying the following conditions:*

- *$(J_i)_{i\in I}$ is a compact cover for J with compact intervals;*
- *For every $i \in I$, (ϕ_i, U_i) is a locally* Lip-γ *chart on M.*
- *$\forall\, i \in I: \ (x_i, X_i, J_i) \in G\Omega_p(\phi_i(U_i))$;*
- (Consistency condition) *If $i, k \in I$ such that $J_i \cap J_k \neq \varnothing$, then we have:*

$$(\phi_k \circ \phi_i^{-1})_*(x_i, X_i, J_i \cap J_k) = (x_k, X_k, J_i \cap J_k)$$

Remark 5.5. — For this definition to be geometrically sound; for example stable under the change of atlas or charts by equivalent ones, one needs to show that the lift of transition maps $(\phi_k \circ \phi_i^{-1}) \mapsto (\phi_k \circ \phi_i^{-1})_*$ is functorial. As our main aim is to study rough paths on manifolds rather than locally Lipschitz manifolds, we choose to carry on with our constructions in the current section and delay presenting and proving such geometric consideration until Section 6.

We identify similar local rough paths in the following way:

Definition 5.6. — *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a locally* Lip-γ *manifold and J a compact interval. Two local p-rough paths $A = (x_i, X_i, J_i, (\phi_i, U_i))_{i\in I}$ and $B = (x_i, X_i, J_i, (\phi_i, U_i))_{i\in K}$ on M over J are said to be equivalent if $A \cup B$ is also a local p-rough path.*

This, of course, defines an equivalence relation. We will henceforth only consider the equivalence classes associated to this relation.

We can now easily define the lift of a manifold-valued path into a (local) rough path:

Definition 5.7. — *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a locally* Lip-γ *manifold. A local p-rough path $(x_i, X_i, J_i, (\phi_i, U_i))_{i\in I}$ on a compact interval J is said to be a p-rough path extension for the path $x : J \to M$ if the following holds:*

$$\forall\, i \in I: \quad x(J_i) \subseteq U_i \quad \text{and} \quad x_i = \phi_i \circ x_{|J_i}$$

If such a rough path exists, we say then that x admits a p-rough path extension.

Following Definition 5.6, we will consider that two rough path extensions of a given path to be the same if their union is also a local rough path extension of that path.

5.2. Consistency with previous definitions

We make sure that our notion of rough path is compatible with the classical one (Definition 3.6):

Lemma 5.8. — *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a Lipschitz-γ manifold and J a compact interval. Let $(x_i, X_i, J_i, (\phi_i, U_i))_{i\in I}$ be a local p-rough path on M over J. Then there exists a unique p-rough path $\mathbb{X}$ on M such that:*

$$\forall\, i \in I: (\phi_i)_*(\mathbb{X})_{|J_i} = (x_i, X_i)$$

Proof. — By Lemma 4.21 and the definition of a rough path on a manifold, it is clear that if such a rough path exists, it is necessarily unique. Let $(a_j)_{0\leqslant j\leqslant n}$ be a subdivision of J and $(i_j)_{0\leqslant j\leqslant n}$ be a finite sequence of elements of I such that for all $j \in [\![0, n-1]\!]$, we have $[a_j, a_{j+1}] \subseteq J_{i_j}$. For $j \in [\![0, n-1]\!]$, we define the rough path:

$$(y_j, Y_j) = (x_{i_j\,|[a_j,a_{j+1}]}, X_{i_j\,|[a_j,a_{j+1}]}) \in G\Omega_p(U_{i_j})$$

It is trivial then that $(y_j, Y_j, [a_j, a_{j+1}], (\phi_{i_j}, U_{i_j}))_{0\leqslant j\leqslant n}$ is equivalent to the local p-rough path $(x_i, X_i, J_i, (\phi_i, U_i))_{i\in I}$. For $j \in [\![0, n-1]\!]$, we define the p-rough path $Z_j = (\phi_{i_j}^{-1})_*(y_j(a_j), Y_j)$ on M (with starting point denoted z_j). Let $\mathbb{X} = Z_1 * \cdots Z_{n-1}$. One can check that the latter concatenation is indeed well-defined as we have, for every $j \in [\![0, n-1]\!]$ and f Banach space-valued compactly supported Lip-γ map on M:

$$\begin{aligned} Z_j(f_*)^1_{a_j,a_{j+1}} &= Y_j((f \circ \phi_{i_j}^{-1})_*)^1_{a_j,a_{j+1}} \\ &= f \circ \phi_{i_j}^{-1}(y_j(a_{j+1})) - f \circ \phi_{i_j}^{-1}(y_j(a_j)) \\ &= f(z_{j+1}) - f(z_j) \end{aligned}$$

By construction $\mathbb{X}$ satisfies the required conditions. □

This identification is in fact one-to-one:

Theorem 5.9. — *Let $p, \gamma \in [1, +\infty[$ such that $\gamma > p$. Let M be a Lipschitz-γ manifold. There is a one-to-one mapping between p-rough paths on M and local p-rough paths on M.*

Proof. — Without loss of generality, we consider rough paths defined over the unit interval $J = [0, 1]$. Let $\mathbb{X}$ be a p-rough path in M over $[0, 1]$ with starting point x. Following Theorem 3.17, let $(\mathbb{X}^i, x_i, (\phi_i, U_i))_{1 \leqslant i \leqslant N}$ be a localising sequence for $\mathbb{X}$ adapted to a subdivision $(s_i)_{0 \leqslant i \leqslant N}$ of $[0, 1]$. For $i \in [\![1, N]\!]$, $(\phi_i)_*(\mathbb{X}^i)$ identifies with a local geometric p-rough path in $\phi_i(U_i)$ over $[s_{i-1}, s_i]$ that we denote (y_i, Y_i). We define the collection $Y = (y_i, Y_i, [s_{i-1}, s_i], (\phi_i, U_i))_{1 \leqslant i \leqslant N}$. For $i \in [\![1, N-1]\!]$, we have, first by the identification between (y_i, Y_i) and $(\mathbb{X}^i, x_i)$, then the consistency of the endpoint of $\mathbb{X}^i$ with the starting point of $\mathbb{X}^{i+1}$:

$$\begin{aligned}(\phi_{i+1} \circ \phi_i^{-1})_*(y_i, Y_i, \{s_i\}) &= (\phi_{i+1} \circ \phi_i^{-1})_*(y_i(s_i), 1, \{s_i\}) \\ &= ((\phi_{i+1} \circ \phi_i^{-1})(y_i(s_i)), 1, \{s_i\}) \\ &= ((\phi_{i+1} \circ \phi_i^{-1})(\phi_i(x_i) + (\mathbb{X}^i(\phi_i^{-1})_*)^1_{s_{i-1}, s_i}, 1, \{s_i\}) \\ &= (\phi_i(x_{i+1}), 1, \{s_i\}) \\ &= (y_{i+1}, Y_{i+1}, \{s_i\})\end{aligned}$$

This proves that Y is indeed a local p-rough path on M over $[0, 1]$. Lemma 5.8 shows that the mapping $\mathbb{X} \mapsto Y$ constructed above is onto and one-to-one. □

6. Coloured paths on manifolds

The constructions and the procedure followed above are a mere illustration of a general recipe that can be used to define any notion of *colouring* already existing on the Euclidean space to a manifold, assuming that we can find a suitable functorial rule. The lift of a path into a rough path can be indeed seen as a colouring: an extra bit of information that cannot necessarily be learned by looking at the base path only (the Brownian motion as an example). In this concluding section, we aim to showcase a general methodology expressed in the language of category theory that first enables one to identify a suitable notion of manifold on which one can build such coloured paths and gives then the definition of such paths.

6.1. Elements from category theory

The goal of this subsection is to recall the notion of functors which summarises in a single word many of the properties satisfied by the integrationof (local) rough paths along (locally) Lipschitz maps. We refer for example to [1] for the definitions below which may also be gathered in [24].

Definition 6.1 (Category). — *A category $\mathcal{C}$ is a triple*

$$((U_i)_{i\in I}, (\hom(U_i,U_j))_{i,j\in I}, (\psi_{i,j,k})_{i,j,k\in I})$$

satisfying the three following axioms:

(1) *For every $i, j \in I$, $\hom(U_i,U_j)$ is a set.*
(2) *For every $i, j, k \in I$, $\psi_{i,j,k}$ is a mapping from $\hom(U_i,U_j) \times \hom(U_j,U_k)$ into $\hom(U_i,U_k)$.*
(3) (Associativity) *For every i, j, k, $l \in I$, $f \in \hom(U_i,U_j)$, $g \in \hom(U_j,U_k)$ and $h \in \hom(U_k,U_l)$:*

$$\psi_{i,k,l}(\psi_{i,j,k}(f,g),h) = \psi_{i,j,l}(f,\psi_{j,k,l}(g,h))$$

(4) (Existence of identities) *For every $i \in I$, there exists $\mathrm{id}_{U_i} \in \hom(U_i,U_i)$ such that, for all $j \in I$, $g \in \hom(U_i,U_j)$ and $h \in \hom(U_j,U_i)$, we have:*

$$\psi_{i,i,j}(\mathrm{id}_{U_i},g) = g \quad \textit{and} \quad \psi_{j,i,i}(h,\mathrm{id}_{U_i}) = h$$

In this case, for all $i, j, k \in I$, U_i is called an object of $\mathcal{C}$; an element in $\hom(U_i,U_j)$ (also denoted by $\hom_{\mathcal{C}}(U_i,U_j)$) is called either an arrow, a morphism or a homomorphism in $\mathcal{C}$. The collections $(U_i)_{i\in I}$ and $(\hom_{\mathcal{C}}(U_i,U_j))_{i,j\in I}$ are respectively denoted $\mathrm{ob}(\mathcal{C})$ and $\hom(\mathcal{C})$. The binary operation $\psi_{i,j,k}$ is called a composition of morphisms and is simply denoted by $\circ$. i.e. for $i, j, k \in I$, $f \in \hom(U_i,U_j)$ and $g \in \hom(U_j,U_k)$, $\psi_{i,j,k}(f,g)$ is denoted $g \circ f$.

Remark 6.2. — If $\mathcal{C} = ((U_i)_{i\in I}, (\hom_{\mathcal{C}}(U_i,U_j))_{i,j\in I}, \circ)$ is a category, and with the notations of the previous definition, the axioms of associativity and the existence of identities can be rewritten in the following way:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

and

$$g \circ \mathrm{id}_{U_i} = g \quad \text{and} \quad \mathrm{id}_{U_i} \circ h = h$$

Example 6.3. —

- By taking a family of sets considered as objects and all maps between these sets considered as arrows and the composition of maps as binary operations we obtain a category, usually called the category of sets.

- Let $(G_i)_{i\in I}$ be a family of groups. For every $i, j \in I$, let $\hom(G_i, G_j)$ be the set of all group homomorphisms from G_i to G_j. Then $\mathcal{C} = ((G_i)_{i\in I}, (\hom(G_i, G_j))_{i,j\in I}, \circ)$ is a category (called category of groups).
- Let $(M_i)_{i\in I}$ be a family of topological spaces (respectively smooth manifolds). For every $i, j \in I$, let $\hom(M_i, M_j)$ be the set of all continuous maps (resp. smooth maps) from M_i to M_j. Then $\mathcal{C} = ((M_i)_{i\in I}, (\hom(M_i, M_j))_{i,j\in I}, \circ)$ is a category.

Definition 6.4. — *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. A functor (or a functorial rule) from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a pair of mappings* $\mathrm{ob}(\mathcal{C}_1) \to \mathrm{ob}(\mathcal{C}_2)$ *and* $\hom(\mathcal{C}_1) \to \hom(\mathcal{C}_2)$*, which we will denote by the same letter F, satisfying:*

(1) *For every object U in $\mathcal{C}_1$, $F(U)$ is an object in $\mathcal{C}_2$;*
(2) *For every two objects U and V in $\mathcal{C}_1$ and an arrow $f \in \hom_{\mathcal{C}_1}(U, V)$, $F(f)$ is in $\hom_{\mathcal{C}_2}(F(U), F(V))$;*
(3) *For all objects U, V and W in $\mathcal{C}_1$, and arrows $f \in \hom_{\mathcal{C}_1}(U, V)$ and $g \in \hom_{\mathcal{C}_1}(V, W)$ $F(g \circ f) = F(g) \circ F(f)$ and $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$.*

Example 6.5. —

- Given any category, there exists a trivial functor from this category to itself preserving all objects and arrows called the identity functor.
- Given a category of groups $\mathcal{C}_1$, let $\mathcal{C}_2$ be a category of sets whose objects are the underlying sets to the objects of $\mathcal{C}_1$ and whose arrows are the group homomorphisms in $\mathcal{C}_1$ taken as maps between the underlying sets. The natural map associating to each object and arrow in $\mathcal{C}_1$ their set-counterparts in $\mathcal{C}_2$ is a functor. Functors constructed in a similar manner are called forgetful functors.
- Let $(G_i)_{i\in I}$ be a family of Lie groups. For every $i, j \in I$, let:
  - $\hom(G_i, G_j)$ denote the set of all Lie group homomorphisms from G_i to G_j,
  - $\mathrm{Lie}(G_i)$ denote the Lie algebra of G_i,
  - $\hom(\mathrm{Lie}(G_i), \mathrm{Lie}(G_j))$ denote the set of all Lie algebra homomorphisms from $\mathrm{Lie}(G_i)$ to $\mathrm{Lie}(G_i)$.

  Then
  $$\mathcal{C}_1 := ((G_i)_{i\in I}, (\hom(G_i, G_j))_{i,j\in I}, \circ)$$
  and
  $$\mathcal{C}_2 := ((\mathrm{Lie}(G_i))_{i\in I}, (\hom(\mathrm{Lie}(G_i), \mathrm{Lie}(G_j)))_{i,j\in I}, \circ)$$
  are categories. The mapping associating to each object in $\mathcal{C}_1$ its Lie algebra and to each arrow in $\mathcal{C}_1$ its pushforward at the identity defines a functor from $\mathcal{C}_1$ to $\mathcal{C}_2$:

$$G_i \mapsto \mathrm{Lie}(G_i); \quad \varphi \in \hom(G_i, G_j) \mapsto \varphi_*(1_{G_i}) \in \hom(\mathrm{Lie}(G_i), \mathrm{Lie}(G_j))$$

6.2. A functorial rule for rough paths

Let E be a Banach space and $1 \leqslant p < \gamma$. Let $(U_i)_{i\in I}$ be a family of open subsets of E. It is obvious that

$$\mathcal{C}_1 = ((U_i)_{i\in I}, (\mathrm{Lip}_{\mathrm{loc}}(\gamma, U_i, U_j)))_{i,j\in I}, \circ)$$

is a category. For $i, j \in I$, we denote by $\hom(G\Omega_p(U_i), G\Omega_p(U_j))$ the set of maps that assign in a continuous way (in the p-rough path topology) to each rough path in U_i a rough path in U_j defined over the same compact interval. It is also straightforward that:

$$\mathcal{C}_2 = ((G\Omega_p(U_i))_{i\in I}, (\hom(G\Omega_p(U_i), G\Omega_p(U_j)))_{i,j\in I}, \circ)$$

defines a category. Finally, recall that for, any open subsets U and V of E and $f \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, U, V)$, $f_* : G\Omega_p(J; U; E) \to G\Omega_p(J; V; E)$ is continuous for any compact interval J (Theorem 4.23).

Theorem 6.6. — *The rule that assigns to every object U in $\mathcal{C}_1$ the object $G\Omega_p(U)$ and to every morphism $f \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, U, V)$, where U and V are objects in $\mathcal{C}_1$, the map f_*, is functorial.*

Proof. — For every open subsets U, V and W of E that are objects in $\mathcal{C}_1$, we need to prove the following:

(1) $(\mathrm{Id}_U)* = \mathrm{Id}_{G\Omega_p(U)}$;
(2) $\forall\, f \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, U, V), \forall\, g \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, V, W) : (g \circ f)_* = g_* \circ f_*$.

The first assertion regarding the identity map is straightforward. Let U, V and W be open subsets of E that are objects in $\mathcal{C}_1$. Let $f \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, U, V)$ and $g \in \mathrm{Lip}_{\mathrm{loc}}(\gamma, V, W)$. Let x be a U-valued path over a segment J with bounded variation. By Lemma 4.24

$$f_*(x, S_{[p]}(x), J) = (f(x), \int \mathrm{d}f(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x), J) = (f(x), S_{[p]}(f(x)), J)$$

As $f(x)$ has bounded variation (Lemma 4.19), we similarly have:

$$g_*(f(x), S_{[p]}(f(x)), J) = (g \circ f(x), S_{[p]}(g \circ f(x)), J)$$

and as $g \circ f$ is locally Lip-γ:

$$(g \circ f(x), S_{[p]}(g \circ f(x)), J) = (g \circ f(x), \int \mathrm{d}(g \circ f)(x, S_{[p]}(x))\mathrm{d}S_{[p]}(x), J)$$

Therefore $((g \circ f)_*)_{|G\Omega_1(U)} = (g_* \circ f_*)_{|G\Omega_1(U)}$. As both $(g \circ f)_*$ and $g_* \circ f_*$ are continuous in the p-variation metric (Theorem 4.23) and as $G\Omega_1(U)$ is dense in $G\Omega_p(U)$ for this metric we deduce that $(g \circ f)_* = g_* \circ f_*$. □

Remark 6.7. — As remarked earlier, we have chosen in the previous sections to focus on the definition of rough paths on suitably chosen manifolds; and that for these definitions to be geometrically sound, the lifts of Lipschitz maps need to be functorial in the sense of Theorem 6.6. This now ensures for example (among several other things) that our definition of a rough path is stable across equivalent atlases (the transformation of the Euclidean space under a diffeomorphism being the simplest illustration). These properties being of relatively common knowledge in the context of Lipschitz maps and rough paths, we highlighted them here in view of the generalised procedure we present next.

6.3. A general recipe

We can define a notion of *coloured* charts and atlas over any n-topological manifold provided that the transition maps are arrows in an appropriate category. We will call such a manifold a *coloured manifold.*

Definition 6.8. — *Let $n \in \mathbb{N}^*$. Let $\mathcal{C}$ be a category whose objects are all open subsets of $\mathbb{R}^n$ and for which inclusion and restriction maps are also arrows. Let M be a topological manifold and $\mathcal{A}$ an atlas on M. We say that $(M, \mathcal{A})$ is a coloured manifold with respect to $\mathcal{C}$ if for any two charts (U, ϕ) and (V, ψ) in $\mathcal{A}$ such that $U \cap V \neq \varnothing$, we have $\psi \circ \phi^{-1} \in \hom_C(\phi(U \cap V), \psi(U \cap V))$.*

We easily retrieve some familiar constructions of manifolds using this language:

Example 6.9. — Let $n \in \mathbb{N}^*$. Let $\mathcal{C}_1$ be the category whose objects are all open subsets of $\mathbb{R}^n$ and whose arrows are continuous maps between these sets. Topological n-dimensional manifolds are exactly the coloured manifolds with respect to $\mathcal{C}_1$.

Example 6.10. — Let $n \in \mathbb{N}^*$. Let $\mathcal{S}_1$ be the category whose objects are all open subsets of $\mathbb{R}^n$ and whose arrows are smooth maps between these sets. Smooth n-dimensional manifolds are the coloured manifolds with respect to $\mathcal{S}_1$.

Example 6.11. — Let $n \in \mathbb{N}^*$ and $\gamma \geqslant 1$. Let $\mathcal{L}_1$ be the category whose objects are all open subsets of $\mathbb{R}^n$ and whose arrows are locally Lipschitz-γ maps between these sets. Locally Lip-γ n-dimensional manifolds are the coloured manifolds with respect to $\mathcal{L}_1$.

Suppose now that we have a notion of *coloured paths* on $\mathbb{R}^n$ that have underlying base paths which we will call traces (rough paths are an example).

Denote by $T(U)$ the sets of coloured paths whose traces lie in an open subset U of $\mathbb{R}^n$. Denote by $\mathcal{C}$ a category as in Definition 6.8 and let $\widetilde{\mathcal{C}}$ be a category whose objects are the sets of coloured paths with traces in open subsets of $\mathbb{R}^n$. We assume there exists a functor from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$:

$$U \mapsto T(U) \quad ; \quad f \mapsto f_*$$

such that for each arrow f between objects U and V of $\mathcal{C}$, the arrow f_* between $T(U)$ and $T(V)$ associates to each coloured path X on U a coloured path on V whose trace is the image by f of the trace of X. We can now define coloured paths on a coloured manifold in the same manner as in Definition 5.4 and the existence of coloured path extensions for manifold-based paths as in Definition 5.7.

Definition 6.12. — *Let $n \in \mathbb{N}^*$. Let $\mathcal{C}$ be a category whose objects are all open subsets of $\mathbb{R}^n$ and for which inclusion and restriction maps are also arrows.Let M be a coloured n-manifold w.r.t. $\mathcal{C}$. A coloured path on M over a compact interval J is a collection $(X_i, J_i, (\phi_i, U_i))_{i\in I}$ of coloured paths on $\mathbb{R}^n$ satisfying the following conditions:*

- *$(J_i)_{i\in I}$ is a compact cover for J by segments;*
- *For every $i \in I$, (ϕ_i, U_i) is in the atlas of the coloured manifold M.*
- *$\forall\, i \in I: \quad X_i \in T(\phi_i(U_i))$ and X_i is defined over J_i;*
- (Consistency condition) *If $i, k \in I$ such that $J_i \cap J_k \neq \varnothing$, then we have:*
$$(\phi_k \circ \phi_i^{-1})_*(X_{i|J_i\cap J_k}) = X_{k|J_i\cap J_k}$$

A coloured path $(X_i, J_i, (\phi_i, U_i))_{i\in I}$ on an interval J is said to be a coloured path extension for the path $x : J \to M$ if the following holds:

$$\forall\, i \in I : x(J_i) \subseteq U_i \quad \textit{and} \quad \operatorname{trace}(X_i) = \phi_i \circ x_{|J_i}$$

If such a coloured path exists, we say then that x admits a coloured path extension.

Remark 6.13. — As the study of the classical examples below will show, we emphasize that the rule linking $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ need be functorial in order to have sound geometric definitions and for these definitions to make sense on their own and be consistent with the definitions of coloured paths on the Euclidean space seen now as a coloured manifold.

Example 6.14 (Example 6.9 continued). — Take $\mathcal{C}_2$ to be the category whose objects are the sets of continuous paths with values in open subsets of $\mathbb{R}^n$. The colouring map associated to an arrow f in $\mathcal{C}_1$ is a map that assigns to every continuous path x the path $f \circ x$. Then our new definition of a continuous path on M (as a coloured path) can be identified with the classical one which relies only on the topology on M: every continuous path on M in the classical sense can be seen as the concatenation of pushforwards of continuous paths on the Euclidean space that are consistent among themselves.

Example 6.15 (Example 6.10 continued). — Take $\mathcal{S}_2$ to be the category whose objects are the sets of smooth paths valued in open subsets of $\mathbb{R}^n$. The associated functorial rule from $\mathcal{S}_1$ to $\mathcal{S}_2$ is the same as above by replacing continuity with smoothness. Finally, one can see the definitions of smooth maps in the classical sense and when described as coloured paths are equivalent.

Example 6.16 (Example 6.11 continued). — Let $1 \leqslant p < \gamma$. Let $\mathcal{L}_2$ be the category whose objects are sets of local geometric p-rough paths supported in the open subsets of $\mathbb{R}^n$ and whose arrows are the set of mappings between the rough path in these objects defined over the same interval in a continuous way (in the p-rough path topology). We showed in the previous sections that rough paths can be seen as coloured paths in this setting.

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