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*Bivariables and Vénéreau polynomials*

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**Bivariables and Vénéreau polynomials** (\*)JÉRÉMY BLANC <sup>(1)</sup> AND PIERRE-MARIE POLONI <sup>(2)</sup>


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**ABSTRACT.** — We study a family of polynomials introduced by Daigle and Freudenburg, which contains the famous Vénéreau polynomials and defines  $\mathbb{A}^2$ -fibrations over  $\mathbb{A}^2$ . According to the Dolgachev–Weisfeiler conjecture, every such fibration should have the structure of a locally trivial  $\mathbb{A}^2$ -bundle over  $\mathbb{A}^2$ . We follow an idea of Kaliman and Zaidenberg to show that these fibrations are locally trivial  $\mathbb{A}^2$ -bundles over the punctured plane, all of the same specific form  $X_f$ , depending on an element  $f \in k[a^{\pm 1}, b^{\pm 1}][x]$ . We then introduce the notion of bivariables and show that the set of bivariables is in bijection with the set of locally trivial bundles  $X_f$  that are trivial. This allows us to give another proof of Lewis’s result stating that the second Vénéreau polynomial is a variable and also to trivialise other elements of the family  $X_f$ . We hope that the terminology and methods developed here may lead to future study of the whole family  $X_f$ .

**RÉSUMÉ.** — Nous étudions une famille de polynômes introduits par Daigle et Freudenburg, contenant les célèbres polynômes de Vénéreau, qui définit des  $\mathbb{A}^2$ -fibrations sur  $\mathbb{A}^2$ . D’après la conjecture de Dolgachev–Weisfeiler, toute fibration de ce type devrait avoir la structure d’un  $\mathbb{A}^2$ -fibré localement trivial sur  $\mathbb{A}^2$ . Suivant une idée de Kaliman et Zaidenberg, nous montrons que ces fibrations sont des  $\mathbb{A}^2$ -fibrés localement triviaux sur le plan privé de l’origine, tous de la même forme spécifique  $X_f$ , dépendant d’un élément  $f \in k[a^{\pm 1}, b^{\pm 1}][x]$ . Nous introduisons alors la notion de bivariables et démontrons que l’ensemble des bivariables est en bijection avec l’ensemble des fibrés localement triviaux  $X_f$  qui sont triviaux. Ceci nous permet de donner une autre preuve du résultat de Lewis établissant que le deuxième polynôme de Vénéreau est une variable et de trivialisier aussi d’autres éléments de la famille  $X_f$ . Nous espérons que la terminologie et les méthodes développées ici puissent mener à une étude future de toute la famille  $X_f$ .

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## 1. Introduction

Throughout this paper, we work over a fixed ground field  $\mathbf{k}$  and all algebraic varieties and morphisms are defined over it.

The Dolgachev–Weisfeiler conjecture [4, Conjecture 3.8.5] is a famous open problem in affine algebraic geometry that concerns  $\mathbb{A}^n$ -fibrations, i.e. morphisms  $X \rightarrow Y$  between affine varieties with the property that every fibre is isomorphic to the  $n$ -dimensional affine space  $\mathbb{A}^n$ .

The conjecture predicts that every such  $\mathbb{A}^n$ -fibration should have the structure of an  $\mathbb{A}^n$ -bundle (locally trivial in the Zariski topology), when the target variety  $Y$  is normal.

Recall that every  $\mathbb{A}^n$ -bundle over an affine variety is isomorphic to a vector bundle (Bass–Connell–Wright Theorem [1]) and moreover that every vector bundle over  $\mathbb{A}^m$  is trivial (Quillen–Suslin Theorem [14, 15]). Hence, the Dolgachev–Weisfeiler conjecture is often reformulated as follows in the case where the target variety is an affine space.

**CONJECTURE 1.1** (Dolgachev–Weisfeiler conjecture). — *Every  $\mathbb{A}^n$ -fibration  $X \rightarrow \mathbb{A}^m$  is isomorphic to the trivial fibration  $\mathbb{A}^m \times \mathbb{A}^n \rightarrow \mathbb{A}^m$ .*

This was proven to be true for  $(n, m) = (1, 1)$  in [4, Proposition 3.7], when  $n = 1$  and  $m$  is arbitrary in [11], and when  $(n, m) = (2, 1)$  in [9]. See also [12] for a recent proof of these results. All other cases remain wide open.<sup>(1)</sup>

In this paper, we focus on the case  $n = m = 2$  and study a family of  $\mathbb{A}^2$ -fibrations introduced by Daigle and Freudenburg in [3]. These fibrations are of the form

$$\pi_{P,n} : \mathbb{A}^4 \rightarrow \mathbb{A}^2, (x, y, z, u) \mapsto (x, v_{P,n}),$$

where  $n$  denotes a positive integer,  $P(z) \in \mathbf{k}[z]$  a polynomial of degree at least two and where  $v_{P,n} \in \mathbf{k}[x, y, z, u]$  is given by

$$v_{P,n} = y + x^n (xz + y(yu + P(z))).$$

The special case where  $n = 1$  and  $P(z) = z^2 + z$  corresponds to an old example due to Bhatwadekar and Dutta [2] whereas the polynomials  $v_{z^2,n}$  are the famous polynomials introduced in the PhD thesis of Vénéreau [16].

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<sup>(1)</sup> We should also mention here that the proof of the Dolgachev–Weisfeiler conjecture in the case where  $n = 2$ , announced in [5], turned out to be incorrect, because of a gap in the proof of [5, Lemma 3].

Let us recall the notion of *A-variable* of a ring  $B$ . We say that an element  $f \in B$  is an *A-variable* of  $B$ , if  $B = A[X_1, \dots, X_k]$  is a polynomial ring over the commutative ring  $A$  and if there exists an automorphism of  $B$  that fixes  $A$  and sends  $f$  onto one of the indeterminates. With this notion, we can reformulate the above problem in algebraic terms as follows. The  $\mathbb{A}^2$ -fibration  $\pi_{P,n}$  is a (locally) trivial  $\mathbb{A}^2$ -bundle if and only if  $v_{P,n}$  is a  $\mathbf{k}[x]$ -variable of  $\mathbf{k}[x, y, z, u]$ .

In his PhD Thesis, Vénéreau showed that  $v_{z^2,n}$  is a  $\mathbf{k}[x]$ -variable of the ring  $\mathbf{k}[x, y, z, u] = \mathbf{k}[x][y, z, u]$  for each  $n \geq 3$ . More recently, Lewis [13] succeeded to prove that  $v_{z^2,2}$  is also a  $\mathbf{k}[x]$ -variable. Nevertheless, it is still unknown whether  $v_{z^2,1}$  is a  $\mathbf{k}[x]$ -variable, or even if it is a  $\mathbf{k}$ -variable of  $\mathbf{k}[x, y, z, u]$ . Note that the latter question, which is weaker a priori, corresponds to the Dolgachev–Weisfeiler conjecture applied to the  $\mathbb{A}^3$ -fibration given by  $\mathbb{A}^4 \rightarrow \mathbb{A}^1$ ,  $(x, y, z, u) \mapsto v_{z^2,1}$ . On the other hand, one can prove that every polynomial  $v_{P,n}$  is a 1-stable  $\mathbf{k}[x]$ -variable (see Proposition 2.3).

Let us fix from now on some coordinates  $a, b$  on  $\mathbb{A}^2$ . In [10], Kaliman and Zaidenberg introduced an interesting strategy to study Vénéreau polynomials based on the fact that the morphisms  $\pi_{z^2,n}$  are trivial  $\mathbb{A}^2$ -bundles over the two open subsets  $U_a = \mathbb{A}^2 \setminus \{a = 0\}$  and  $U_b = \mathbb{A}^2 \setminus \{b = 0\}$ . Using this idea, they could reprove that  $\pi_{z^2,n}$  is isomorphic to the trivial fibration for every  $n \geq 3$ . Nevertheless, they couldn't push this technique further at that time and were not able to treat the case of  $v_{z^2,n}$  with  $n = 1$  or  $n = 2$ . In the present paper, we give Kaliman and Zaidenberg's strategy another try and want to apply it to the more general polynomials  $v_{P,n}$ . For this, we will introduce the notion of bivariates. The fact that the second Vénéreau polynomial  $v_{z^2,2}$ , and more generally all  $v_{P,2}$  with  $P$  of degree 2, are  $\mathbf{k}[x]$ -variables will follow quite simply from this new point of view.

We shall indeed prove in Section 2 that every map  $\pi_{P,n}$  is a trivial  $\mathbb{A}^2$ -bundle over  $U_a$  and over  $U_b$ . The preimage of the origin being also isomorphic to  $\mathbb{A}^2$  (since it is given by the equations  $x = y = 0$ ), this implies that every  $\pi_{P,n}$  is an  $\mathbb{A}^2$ -fibration and therefore that it should have, according to the Dolgachev–Weisfeiler conjecture, the structure of an  $\mathbb{A}^2$ -bundle. To check that this is the case, one needs either to find a neighbourhood of the origin over which the fibration is trivial, or to show that  $\pi_{P,n}$  defines the trivial fibration over the punctured plane  $\mathbb{A}_*^2 = \mathbb{A}^2 \setminus \{(0, 0)\} = U_a \cup U_b$  (see Lemma 3.1).

Having this question in mind, we will consider the following family of locally trivial  $\mathbb{A}^2$ -bundles over  $\mathbb{A}_*^2$ , obtained by gluing trivial bundles over  $U_a$  and  $U_b$  along their intersection  $U_{ab} = U_a \cap U_b = \mathbb{A}^2 \setminus \{ab = 0\}$ .

DEFINITION 1.2. — Given an element  $f = f(a, b, x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ , we denote by  $X_f$  the variety obtained by gluing together the affine varieties  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$  by means of the transition function

$$\begin{aligned} U_{ab} \times \mathbb{A}^2 &\longrightarrow U_{ab} \times \mathbb{A}^2 \\ ((a, b), (x, y)) &\longmapsto ((a, b), (x, y + f(a, b, x))). \end{aligned}$$

We denote by  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  the  $\mathbb{A}^2$ -bundle over the punctured affine plane  $\mathbb{A}_*^2 = U_a \cup U_b$  given by the projection onto the first factor.

In Section 2, we will prove the following result, which shows that one can reduce the study of the morphisms  $\pi_{P,n}$  to the study of a special family of varieties  $X_f$ .

THEOREM 1.3. — For each  $n \geq 1$  and each  $P \in \mathbf{k}[z]$ , the restriction of the  $\mathbb{A}^2$ -fibration  $\pi_{P,n}$  over the punctured plane  $\mathbb{A}_*^2$  is a locally trivial  $\mathbb{A}^2$ -bundle isomorphic to  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  with

$$f = \frac{x}{ab^2} - \frac{1}{ab^m} \cdot \frac{b^m - (a^n x)^m}{b - a^n x} P\left(\frac{x}{a}\right) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x],$$

where  $m$  is any integer such that  $mn > \deg(P)$ .

In the special case where  $P = x^2$ , if we choose the smallest  $m$  such that  $mn > 2$ , we get that the fibration associated to the  $n$ -th Vénereau polynomial is isomorphic to the fibration  $\rho_{f_n}$ , where

$$f_n = \begin{cases} \frac{x}{ab^2} - \frac{x^2}{a^3b} & \text{when } n \geq 3, \\ \frac{x}{ab^2} - \frac{x^2}{a^3b} - \frac{x^3}{ab^2} & \text{when } n = 2, \\ \frac{x}{ab^2} - \frac{x^2}{a^3b} - \frac{x^3}{a^2b^2} - \frac{x^4}{ab^3} & \text{when } n = 1 \end{cases}$$

(see Example 2.5). We recover here at once the formulas that were first computed in [10, Proposition 2].

As the transition functions of the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  fix the coordinate  $x$ , the variety  $X_f$  can also be naturally seen as an  $\mathbb{A}^1$ -bundle  $X_f \rightarrow \mathbb{A}_*^2 \times \mathbb{A}^1$  via the projection onto the first three coordinates. On the contrary to the difficult question on the existence of an isomorphism as  $\mathbb{A}^2$ -bundles, it is straightforward to decide whether two varieties  $X_f$  and  $X_g$  are isomorphic as  $\mathbb{A}^1$ -bundles (see Lemma 3.3). As we will explain in Section 3, the triviality of an  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is related to the notion of *bivariates* which we introduce in Definition 3.5. A bivariate is a polynomial in  $\mathbf{k}[a, b][x, y]$  that becomes a variable being seen as an element of  $\mathbf{k}[a^{\pm 1}, b][x, y]$  as well as when seen as an element of  $\mathbf{k}[a, b^{\pm 1}][x, y]$ . Since the group  $\text{Aut}_{\mathbf{k}[a, b]}(\mathbf{k}[a, b][x, y])$  naturally acts on the set of bivariates, we may consider bivariates up to the action of this group (see Definition 3.11). This

leads us to a natural correspondence between bivariables and  $\mathbb{A}^2$ -bundles  $X_f$  that are trivial. More precisely, we shall establish the following result.

THEOREM 1.4. —

- (1) Every bivariable  $\omega \in \mathbf{k}[a, b][x, y]$  trivialises an  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  for some  $f \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ .

More precisely, given a bivariable  $\omega \in \mathbf{k}[a, b][x, y]$ , there exist elements  $\tau_a, \tau_b$  and  $f(x)$  of  $\mathbf{k}[a^{\pm 1}, b][x, y]$ ,  $\mathbf{k}[a, b^{\pm 1}][x, y]$  and  $\mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ , respectively, such that  $\mathbf{k}[a^{\pm 1}, b][\omega, \tau_a] = \mathbf{k}[a^{\pm 1}, b][x, y]$ ,  $\mathbf{k}[a, b^{\pm 1}][\omega, \tau_b] = \mathbf{k}[a, b^{\pm 1}][x, y]$  and  $\tau_a = \tau_b + f(\omega)$ . Moreover, the variety  $X_f$  trivialised by  $\omega$  is uniquely defined up to isomorphism of  $\mathbb{A}^1$ -bundles.

- (2) The map that associates to a bivariable  $\omega$  a variety  $X_f$  trivialised by  $\omega$  induces a bijection

$$\left\{ \begin{array}{l} \text{bivariables up} \\ \text{to the action of} \\ \text{Aut}_{\mathbf{k}[a, b]}(\mathbf{k}[a, b][x, y]) \end{array} \right\} \xrightarrow{\kappa} \left\{ \begin{array}{l} \text{Varieties } X_f \text{ which are} \\ \text{trivial } \mathbb{A}^2\text{-bundles} \\ \text{up to isomorphisms} \\ \text{of } \mathbb{A}^1\text{-bundles} \end{array} \right\}$$

$$\omega \qquad \longmapsto \qquad X_f$$

According to the above bijection, the varieties  $X_f$  that define trivial  $\mathbb{A}^1$ -bundles correspond to the set of  $\mathbf{k}[a, b]$ -variables of  $\mathbf{k}[a, b, x, y]$ , i.e. to the set of trivial bivariables (see Example 3.12). The polynomials of the form  $a^m x + b^n y$  with  $m, n \geq 1$  are easy examples of non-trivial bivariables. They correspond to the varieties  $X_f$  with  $f = \frac{x}{a^m b^n}$  (see Example 3.13).

In Section 4, we describe a procedure to construct new bivariables starting with a given one. Indeed, if  $\tau_a, \tau_b$  and  $f(x)$  are given as in the first assertion in Theorem 1.4 and if  $m, n \geq 1$  are positive integers such that  $a^m b^n f(x) \in \mathbf{k}[a, b, x, y]$ , then we can define, for all polynomials  $Q \in \mathbf{k}[a, b][x]$ , new bivariables  $\tilde{\omega}$  and  $\hat{\omega}$  by  $\hat{\omega} = \omega + aQ(a^m \tau_a)$  and  $\tilde{\omega} = \omega + bQ(b^n \tau_b)$  (see Proposition 4.2).

Applying this procedure to the bivariable  $ax + by^2$ , we obtain new bivariables of the form  $ax + b^2y + bP(x)$ ,  $P(x) \in \mathbf{k}[x]$ , which correspond to the  $\mathbb{A}^2$ -bundles over  $\mathbb{A}_*^2$  associated with the fibrations  $\pi_{P, n}$  when  $n > \deg(P)$  (see Example 4.4). Therefore, this gives a simple proof that the polynomials  $v_{P, n}$  are  $\mathbf{k}[x]$ -variables for each  $n > \deg(P)$ , and in particular that the  $n$ -th Vénéreau polynomials are  $\mathbf{k}[x]$ -variables for all  $n \geq 3$  (see Remark 4.5).

In the case where  $\deg(P) = 2$  and  $\text{char}(\mathbf{k}) \neq 2$ , we can go one step further: Applying the procedure to  $\omega = ax + b^2y + bP(x)$ , we now obtain the new bivariable  $\omega + \frac{a^5}{2c}(\frac{y}{a} + \frac{P(x)}{ab} - \frac{1}{ab}P(\frac{\omega}{a}))$  which corresponds to the  $\mathbb{A}^2$ -bundle associated with the fibration  $\pi_{P, 2}$  (see Lemma 4.6). This proves that

the polynomials  $v_{P,2}$  are  $\mathbf{k}[x]$ -variables when  $\deg(P) = 2$  and  $\text{char}(\mathbf{k}) \neq 2$ . In particular, this generalises and gives a different proof for Lewis's result stating that the second Vénéreau polynomial  $v_{2^2,2}$  is a  $\mathbf{k}[x]$ -variable.

Although the bundles  $\rho_{f_3}$  and  $\rho_{f_2}$  were easily trivialised with this technique, we were unfortunately not able to go further and couldn't trivialise the bundle  $\rho_{f_1}$  associated with the first Vénéreau polynomial. Nonetheless, we can simplify it and show that it is equivalent to a bundle with the transition function of degree 3 in  $x$  (see Example 4.10) and also to another which is still of degree 4 but has only three summands (see Example 4.11). Surprisingly, we are able to trivialise the bundle  $\rho_g$ , where the function

$$g = \frac{x}{ab^2} - \frac{x^2}{a^3b} - \frac{x^3}{a^2b^2} - \frac{5}{4} \cdot \frac{x^4}{ab^3}$$

differs from  $f_1$  only by the coefficient  $\frac{5}{4}$  in its last summand (see Example 4.9).

In Section 5, we strengthen the result that the morphisms  $\pi_{P,n} : \mathbb{A}^4 \rightarrow \mathbb{A}^2$  are  $\mathbb{A}^2$ -bundles over  $\mathbb{A}_*^2$ , by proving the stronger fact that  $\pi_{P,n}$  yields a locally trivial  $\mathbb{A}^2$ -bundle  $\widehat{\pi}_{P,n} : \widehat{\mathbb{A}}^4 \rightarrow \widehat{\mathbb{A}}^2$ , where  $\widehat{\mathbb{A}}^2$  and  $\widehat{\mathbb{A}}^4$  are respectively obtained from  $\mathbb{A}^2$  and  $\mathbb{A}^4$  by blowing-up  $(0,0)$  and the surface  $\pi_{P,n}^{-1}((0,0))$  given by  $x = y = 0$  (see Theorem 5.1). Proving that  $\pi_{P,n}$  is a locally trivial  $\mathbb{A}^2$ -bundle is then equivalent to prove that  $\widehat{\pi}_{P,n}$  is trivial, or to prove that  $\widehat{\pi}_{P,n}$  is trivial on a neighbourhood of the exceptional curve of  $\widehat{\mathbb{A}}^2$ .

In Section 6, we study the family of varieties  $X_f$  and their (non necessarily trivial) associated bundles  $\rho_f : X_f \rightarrow \mathbb{A}_*^2$ . In the case where the denominator of  $f$  has degree at most 1 in either  $a$  or  $b$ , we give a direct criterion to decide whether  $\rho_f$  is a trivial  $\mathbb{A}^2$ -bundle or not.

## 2. Local triviality over the punctured plane

The aim of this section is to prove Theorem 1.3. We shall indeed show that the restriction of every  $\mathbb{A}^2$ -fibration  $\pi_{P,n}$  over the punctured plane  $\mathbb{A}_*^2$  is a locally trivial  $\mathbb{A}^2$ -bundle which is moreover isomorphic to a bundle of the form  $\rho_f : X_f \rightarrow \mathbb{A}_*^2$  for some explicit  $f$  depending on  $P$  and  $n$ . We will also prove that each polynomial  $v_{P,n}$  is a so-called stable variable.

The following lemma already tells us that  $v_{P,n}$  is a  $\mathbf{k}[x, x^{-1}]$ -variable of the ring  $\mathbf{k}[x, x^{-1}][y, z, u]$ , or equivalently that the restriction of  $\pi_{P,n}$  to  $U_a$  is a trivial  $\mathbb{A}^2$ -bundle.

LEMMA 2.1. — *The rational map  $\varphi_{P,n}: \mathbb{A}^4 \rightarrow \mathbb{A}^4$  defined by*

$$\varphi_{P,n}(x, y, z, u) = \left( x, v_{P,n}, xz + y(uy + P(z)), \frac{u}{x} - \frac{P(z + \frac{y(yu + P(z))}{x}) - P(z)}{xy} \right)$$

*has Jacobian determinant 1 and restricts to an automorphism of the complement  $\mathbb{A}^4 \setminus \{x = 0\}$  of the hyperplane defined by the equation  $x = 0$ .*

*Proof.* — One first checks by a straightforward computation that  $\varphi_{P,n}$  is equal to the composition

$$\varphi_{P,n} = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$$

of the following four birational transformations of  $\mathbb{A}^4$ :

$$\begin{aligned} \varphi_4: (x, y, z, u) &\longmapsto (x, y + x^n z, z, u), \\ \varphi_3: (x, y, z, u) &\longmapsto (x, y, z, y^{-1}(u - x^{-1}P(x^{-1}z))), \\ \varphi_2: (x, y, z, u) &\longmapsto (x, y, xz + yu, x^{-1}u), \\ \varphi_1: (x, y, z, u) &\longmapsto (x, y, z, yu + P(z)). \end{aligned}$$

Since the Jacobian determinants of  $\varphi_1, \varphi_2, \varphi_3, \varphi_4$  are equal to 1,  $1/y$ , 1,  $y$ , respectively and since all these maps fix  $y$ , it follows that the Jacobian determinant of  $\varphi_{P,n}$  is equal to 1. Moreover, since all components of  $\varphi_{P,n}$  belong to  $\mathbf{k}[x^{\pm 1}, y, z, u]$  (remark that the numerator of last component's last summand is indeed divisible by  $y$ ), all components of its inverse belong to  $\mathbf{k}[x^{\pm 1}, y, z, u]$  as well.  $\square$

As an immediate consequence, we get the following statement.

COROLLARY 2.2. — *Every map  $\pi_{P,n}: \mathbb{A}^4 \rightarrow \mathbb{A}^2$  is an  $\mathbb{A}^2$ -fibration and restricts to a trivial  $\mathbb{A}^2$ -bundle over the open set  $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$ .*

Another worth mentioning consequence of Lemma 2.1 is the fact that  $v_{P,n}$  is a 1-stable variable. This can be shown by a general argument due to El Kahoui and Ouali [8] (see also [7]).

PROPOSITION 2.3. — *Every polynomial  $v_{P,n}$  is a  $\mathbf{k}[x]$ -variable of  $\mathbf{k}[x, y, z, u, t]$ , where  $t$  denotes a new indeterminate.*

*Proof.* — We shall construct an automorphism  $\Psi \in \text{Aut}_{\mathbf{k}[x]}(\mathbf{k}[x, y, z, u, t])$  which maps  $v_{P,n}$  onto

$$w_s = v_{P,n} + x^s t = y + x^s t + x^n(xz + y^2 u + yP(z)),$$

where  $s$  denotes a suitable positive integer. The proposition will follow, since it is easy to check (see for example Lemma 4.1 below), that  $w_s$  is a  $\mathbf{k}[x, z, u]$ -variable of the ring  $\mathbf{k}[x, z, u][y, t]$ .

The construction of  $\Psi$  involves the rational map  $\varphi = \varphi_{P,n}$  defined in Lemma 2.1. Since  $\varphi$  restricts to an automorphism outside the hyperplane  $x = 0$ , its comorphism  $\varphi^*: \mathbf{k}[x^{\pm 1}, y, z, u] \rightarrow \mathbf{k}[x^{\pm 1}, y, z, u]$ ,  $Q \mapsto Q \circ \varphi$ , is a  $\mathbf{k}[x^{\pm 1}]$ -automorphism of the ring  $\mathbf{k}[x^{\pm 1}, y, z, u]$ . We denote by  $F$  the extension of  $\varphi^*$  as a  $\mathbf{k}[x^{\pm 1}, t]$ -automorphism of  $\mathbf{k}[x^{\pm 1}, y, z, u, t]$ . Note that  $F(y) = v_{P,n}$ .

For each  $\xi \in \mathbf{k}[x^{\pm 1}, t]$ , we denote by  $H_\xi$  the  $\mathbf{k}[x^{\pm 1}, z, u, t]$ -automorphism of  $\mathbf{k}[x^{\pm 1}, y, z, u, t]$  defined by  $H_\xi(y) = y + \xi$ , and by  $\Phi_\xi$  the  $\mathbf{k}[x^{\pm 1}, t]$ -automorphism of  $\mathbf{k}[x^{\pm 1}, y, z, u, t]$  defined by  $\Phi_\xi = F \circ H_\xi \circ F^{-1}$ . Note that  $\Phi_\xi(v_{P,n}) = v_{P,n} + \xi$  by construction.

Observe that all elements  $\Phi_\xi(y), \Phi_\xi(z), \Phi_\xi(u), \Phi_\xi(t) \in \mathbf{k}[x^{\pm 1}, y, z, u, t]$  depend polynomially on  $\xi$ . Moreover, since  $\Phi_0$  is the identity, their coefficients of degree 0 in  $\xi$  are  $y, z, u, t$ , respectively. Therefore, choosing  $\xi = x^s \cdot t$  for a large enough integer  $s$ , we obtain that  $\Phi_{x^s \cdot t}(\mathbf{k}[y, z, u, t]) \subseteq \mathbf{k}[x, y, z, u, t]$ . Since  $\Phi_{x^s \cdot t}$  is of Jacobian determinant 1, this implies that  $\Phi_{x^s \cdot t}$  restricts to a  $\mathbf{k}[x]$ -automorphism of  $\mathbf{k}[x, y, z, u, t]$ . Recall that  $\Phi_{x^s \cdot t}(v_{P,n}) = w_s$  by construction. This concludes the proof.  $\square$

We now proceed with the proof of Theorem 1.3.

*Proof of Theorem 1.3.* — Let  $m$  be an integer such that  $mn > \deg(P)$  and denote by  $F_{P,n,m}$  the birational map of  $\mathbb{A}^4$  defined by

$$\begin{aligned} F_{P,n,m}(x, y, z, u) &= \left( x, y, z, u - \frac{z}{xy^2} + \frac{1}{xy^m} \cdot \frac{y^m - (x^n z)^m}{y - x^n z} P\left(\frac{z}{x}\right) \right) \\ &= (x, y, z, u - f_{P,n,m}(x, y, z)). \end{aligned}$$

What we actually only need to prove is that the composition  $F_{P,n,m} \circ \varphi_{P,n}$ , where  $\varphi_{P,n}$  is given as in Lemma 2.1, restricts to an isomorphism between  $\mathbb{A}^4 \setminus \{v_{P,n} = 0\}$  and  $\mathbb{A}^4 \setminus \{y = 0\}$ . For simplicity, let us denote  $v = v_{P,n}$ ,  $\varphi = \varphi_{P,n}$  and  $F = F_{P,n,m}$ .

Note that the first two components of  $\varphi$  and  $F \circ \varphi$  are equal to  $x$  and  $v$ , respectively. Let us denote their common third component by

$$\omega = \frac{v - y}{x^n} = xz + y(uy + P(z)).$$

In order to prove the proposition, we only need to show that the last component of  $F \circ \varphi$  is an element of  $\mathbf{k}[x, y, z, u, v^{-1}]$ . Indeed, since  $F \circ \varphi$  is a birational map of  $\mathbb{A}^4$  whose second component is equal to  $v$ , this will imply that  $F \circ \varphi$  induces an isomorphism from  $\mathbb{A}^4 \setminus \{v = 0\}$  to  $\mathbb{A}^4 \setminus \{y = 0\}$ .

We remark that the last component of  $F$  is an element of  $\mathbf{k}[x^{\pm 1}, y^{\pm 1}, z, u]$ . Therefore, since the last component of

$$\varphi(x, y, z, u) = \left( x, v, \omega, \frac{1}{x}u + \frac{P(z) - P(\omega/x)}{xy} \right)$$

is in fact an element of  $\mathbf{k}[x^{\pm 1}, y, u, z]$ , we only need to show that  $x$  does not appear in the denominator of the last component of  $F \circ \varphi$ .

This component is equal to

$$\begin{aligned} (F \circ \varphi)^*(u) &= \varphi^* \left( u - \frac{z}{xy^2} + \frac{1}{xy^m} \cdot \frac{y^m - (x^n z)^m}{y - x^n z} P\left(\frac{z}{x}\right) \right) \\ &= \frac{1}{x}u + \frac{P(z) - P(\omega/x)}{xy} - \frac{\omega}{xv^2} + \frac{1}{xv^m} \frac{v^m - (x^n \omega)^m}{y} P(\omega/x) \\ &= \frac{1}{x}u + \frac{P(z)}{xy} - \frac{\omega}{xv^2} - \frac{1}{xv^m} \frac{(x^n \omega)^m}{y} P(\omega/x) \\ &= \frac{1}{xyv^2} \cdot (uv^2y + v^2P(z) - \omega y) - \frac{\omega^m}{yv^m} \cdot x^{mn-1} P(\omega/x). \end{aligned}$$

Since we took  $m$  such that  $mn > \deg(P)$ , we only need to check that

$$uv^2y + v^2P(z) - \omega y \equiv 0 \pmod{x}.$$

For this, we use the fact that  $v \equiv y$  and  $\omega \equiv y(uy + P(z))$  modulo  $(x)$  and find

$$uv^2y + v^2P(z) - \omega y \equiv uy^3 + y^2P(z) - y^2(uy + P(z)) \equiv 0 \pmod{x}. \quad \square$$

*Remark 2.4.* — As pointed out to us by one of the referees, the fact that  $\omega = xz + y(uy + P(z))$  is linear in  $u$  seems to be important for the proof of Theorem 1.3. On the other hand, one may easily generalise the construction in the proof of Lemma 2.1 by replacing the map  $\varphi_2$  with any map  $(x, y, z, u) \mapsto (x, y, xz + Q(x, y, u), x^{-1}u)$  where  $Q \in \mathbf{k}[x, y, u]$  is not necessarily of degree one in  $u$ . Doing so, we obtain  $\mathbb{A}^2$ -fibrations

$$\mathbb{A}^4 \longrightarrow \mathbb{A}^2, \quad (x, y, z, u) \longmapsto (x, y + x^n(xz + Q(x, y, yu + P(z))))$$

that restrict to trivial  $\mathbb{A}^2$ -bundles over the open set  $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$ . According to the Dolgachev–Weisfeiler conjecture, all these fibrations should be trivial  $\mathbb{A}^2$ -bundles. However, we don't know at the moment whether they are locally trivial over the punctured plane.

*Example 2.5.* — Recall that the  $n$ -th Vénéreau polynomial is defined by

$$V_n = v_{z^2, n} = y + x^n(xz + y^2u + yz^2).$$

By Theorem 1.3 above, the map  $\mathbb{A}^4 \setminus \mathbb{A}^2 \rightarrow \mathbb{A}_*^2, (x, y, z, u) \mapsto (x, V_n)$  is an  $\mathbb{A}^2$ -bundle isomorphic to  $\rho_{f_n}: X_{f_n} \rightarrow \mathbb{A}_*^2$ , where

$$f_n = \frac{x}{ab^2} - \frac{1}{ab^m} \cdot \frac{b^m - (a^n x)^m}{b - a^n x} \left(\frac{x}{a}\right)^2,$$

with  $m$  such that  $mn > 2$ . If  $n \geq 3$ , we can choose  $m = 1$  and we get

$$\begin{aligned} f_n &= \frac{x}{ab^2} - \frac{1}{ab^1} \cdot \frac{b^1 - (a^n x)^1}{b - a^n x} \cdot \frac{x^2}{a^2} \\ &= \frac{x}{ab^2} - \frac{x^2}{a^3 b} \quad \text{when } n \geq 3. \end{aligned}$$

For  $n = 2$ , we can choose  $m = 2$  and get

$$\begin{aligned} f_2 &= \frac{x}{ab^2} - \frac{1}{ab^2} \cdot \frac{b^2 - (a^2 x)^2}{b - a^2 x} \cdot \frac{x^2}{a^2} \\ &= \frac{x}{ab^2} - \frac{1}{ab^2} \cdot (b + a^2 x) \cdot \frac{x^2}{a^2} \\ &= \frac{x}{ab^2} - \frac{x^2}{a^3 b} - \frac{x^3}{ab^2}. \end{aligned}$$

For  $n = 1$ , we choose  $m = 3$  and get

$$\begin{aligned} f_1 &= \frac{x}{ab^2} - \frac{1}{ab^3} \cdot \frac{b^3 - (ax)^3}{b - ax} \cdot \frac{x^2}{a^2} \\ &= \frac{x}{ab^2} - \frac{1}{ab^3} \cdot (b^2 + axb + a^2 x^2) \cdot \frac{x^2}{a^2} \\ &= \frac{x}{ab^2} - \frac{x^2}{a^3 b} - \frac{x^3}{a^2 b^2} - \frac{x^4}{ab^3}. \end{aligned}$$

These functions are exactly those computed by Kaliman and Zaidenberg in [10, Proposition 2].

*Example 2.6.* — We now consider the simple cases where  $n > \deg(P)$  and  $n = \deg(P)$ .

- (1) If  $n > \deg(P)$ , then we can choose  $m = 1$  in Theorem 1.3. Hence, the restriction of  $\pi_{P,n}$  to  $\mathbb{A}_*^2$  is isomorphic to  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  with

$$\begin{aligned} f(x) &= \frac{x}{ab^2} - \frac{1}{ab^1} \cdot \frac{b^1 - (a^n x)^1}{b - a^n x} P\left(\frac{x}{a}\right) \\ &= \frac{x}{ab^2} - \frac{1}{ab} P\left(\frac{x}{a}\right). \end{aligned}$$

- (2) If  $n = \deg(P)$ , then we can choose  $m = 2$  and the map  $(x, y, z, u) \mapsto (x, v_{P,n})$  has the structure of an  $\mathbb{A}^2$ -bundle over  $\mathbb{A}_*^2$  isomorphic to  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  with

$$\begin{aligned} f(x) &= \frac{x}{ab^2} - \frac{1}{ab^2} \cdot \frac{b^2 - (a^n x)^2}{b - a^n x} P\left(\frac{x}{a}\right) \\ &= \frac{x}{ab^2} - \frac{1}{ab^2} \cdot (b + a^n x) P\left(\frac{x}{a}\right) \\ &= \frac{x}{ab^2} - \frac{1}{ab} P\left(\frac{x}{a}\right) - \frac{a^{n-1}}{b^2} x P\left(\frac{x}{a}\right). \end{aligned}$$

### 3. Bivariables and their relationship with trivial $\mathbb{A}^1$ -bundles – the proof of Theorem 1.4

We start by giving an easy result which was already noticed in [10].

LEMMA 3.1. — *Let  $n \geq 1$  and let  $P \in \mathbf{k}[z]$ . Then, the following statements are equivalent:*

- (1) *The polynomial  $v_{P,n}$  is a  $\mathbf{k}[x]$ -variable of  $\mathbf{k}[x, y, z, u]$ .*
- (2) *The morphism  $\pi_{P,n}: \mathbb{A}^4 \rightarrow \mathbb{A}^2$  is a trivial  $\mathbb{A}^2$ -bundle.*
- (3) *There exists a neighbourhood of the origin in  $\mathbb{A}^2$  over which  $\pi_{P,n}$  is a trivial  $\mathbb{A}^2$ -bundle.*
- (4) *The restriction of the  $\mathbb{A}^2$ -fibration  $\pi_{P,n}$  over the punctured plane  $\mathbb{A}_*^2$  is a trivial  $\mathbb{A}^2$ -bundle.*

*Proof.* — Asking that  $\pi_{P,n}: \mathbb{A}^4 \rightarrow \mathbb{A}^2$  is a trivial  $\mathbb{A}^2$ -bundle means exactly that there exist  $r, s \in \mathbf{k}[x, y, z, u]$  such that the map

$$\mathbb{A}^4 \longrightarrow \mathbb{A}^2 \times \mathbb{A}^2 = \mathbb{A}^4, \quad (x, y, z, u) \longmapsto (x, v_{P,n}, r(x, y, z, u), s(x, y, z, u))$$

is an isomorphism, i.e. such that the equality  $\mathbf{k}[x, y, z, u] = \mathbf{k}[x, v_{P,n}, r, s]$  holds. Hence, the statements (1) and (2) of the lemma are equivalent.

If the morphism  $\pi_{P,n}: \mathbb{A}^4 \rightarrow \mathbb{A}^2$  is a trivial  $\mathbb{A}^2$ -bundle, then its restriction to every subset of  $\mathbb{A}^2$  is trivial. Therefore, Assertion (2) implies both (3) and (4). It remains to prove the converse implications.

By Theorem 1.3, the map  $\pi_{P,n}$  defines a locally trivial  $\mathbb{A}^2$ -bundle over the punctured plane. Hence, if assertion (3) is true, then  $\pi_{P,n}$  is a locally trivial  $\mathbb{A}^2$ -bundle over the whole  $\mathbb{A}^2$ , which is isomorphic to a vector bundle by Bass–Connell–Wright Theorem [1] and is furthermore trivial by Quillen–Suslin Theorem [14, 15]. This shows that Assertion (3) implies (2).

Under Assertion (4), there exist  $r, s \in \mathbf{k}(x, y, z, u)$  such that the map

$$\begin{aligned} \mathbb{A}^4 \setminus \{(x = v_{P,n} = 0)\} &\longrightarrow \mathbb{A}_*^2 \times \mathbb{A}^2, \\ (x, y, z, u) &\longmapsto (x, v_{P,n}, r(x, y, z, u), s(x, y, z, u)) \end{aligned}$$

is an isomorphism. Since the locus where  $x = v_{P,n} = 0$  has codimension 2 in  $\mathbb{A}^4$  (as it is in fact the surface defined by the equations  $x = y = 0$  in  $\mathbb{A}^4$ ), the ring of regular functions on  $\mathbb{A}^4 \setminus \{(x = v_{P,n} = 0)\} = \mathbb{A}_*^2 \times \mathbb{A}^2$  is equal to the whole ring  $\mathbf{k}[x, y, z, u]$ . Hence, the map above induces an automorphism of  $\mathbf{k}[x, y, z, u]$ . In particular, we have that  $r, s \in \mathbf{k}[x, y, z, u]$  and  $\mathbf{k}[x, y, z, u] = \mathbf{k}[x, v_{P,n}, r, s]$ . This proves that (4) implies (2).  $\square$

By Theorem 1.3, the four conditions of Lemma 3.1 are equivalent to the fact that the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$ , where  $f \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$  is explicitly given in the statement of Theorem 1.3, is a trivial  $\mathbb{A}^2$ -bundle.

We recall the notation  $U_a = \mathbb{A}^2 \setminus \{a = 0\}$ ,  $U_b = \mathbb{A}^2 \setminus \{b = 0\}$  and  $U_{ab} = U_a \cap U_b = \mathbb{A}^2 \setminus \{ab = 0\}$ . Let us denote by  $G_a$ ,  $G_b$  and  $G_{ab}$  the automorphism groups of the  $\mathbb{A}^2$ -bundles  $U_a \times \mathbb{A}^2$ ,  $U_b \times \mathbb{A}^2$  and  $U_{ab} \times \mathbb{A}^2$ , respectively. In particular, remark that  $G_a$  and  $G_b$  are subgroups of  $G_{ab}$ . In the sequel, we will abuse notation and denote an element

$$((a, b), (x, y)) \longmapsto ((a, b), (F(a, b, x, y), G(a, b, x, y)))$$

by  $(F(a, b, x, y), G(a, b, x, y))$  or simply by  $(F(x, y), G(x, y))$ .

LEMMA 3.2. — *Let  $f(x), g(x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ . Then, the two  $\mathbb{A}^2$ -bundles  $\rho_f$  and  $\rho_g$  are isomorphic as  $\mathbb{A}^2$ -bundles if and only if there exist  $\alpha \in G_a$  and  $\beta \in G_b$  such that*

$$\alpha \circ (x, y + f(x)) \circ \beta^{-1} = (x, y + g(x)).$$

*In particular, the  $\mathbb{A}^2$ -bundle  $\rho_f$  is isomorphic to the trivial bundle if and only if there exist  $\alpha \in G_a$  and  $\beta \in G_b$  such that*

$$\alpha \circ \beta^{-1} = (x, y + f(x)).$$

*Proof.* — Recall that the  $\mathbb{A}^2$ -bundles  $\rho_f$  and  $\rho_g$  are constructed by gluing  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$  via the transition functions  $(x, y + f(x)) \in G_{ab}$  and  $(x, y + g(x)) \in G_{ab}$ , respectively. Hence, they are isomorphic as  $\mathbb{A}^2$ -bundles if and only if one can find automorphisms of the  $\mathbb{A}^2$ -bundles  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$  that are compatible with the gluing. This gives the result.  $\square$

Investigating on the conditions of Lemma 3.2 for a specific example (for instance in the case corresponding to the first Vénéreau polynomial) is not a simple task. On the contrary, when we consider  $X_f$  and  $X_g$  as  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2 \times \mathbb{A}^1$  (via the projection onto the first three coordinates), it becomes easy to decide whether two bundles  $X_f$  and  $X_g$  are equivalent as  $\mathbb{A}^1$ -bundles.

LEMMA 3.3. — *For all  $f, g \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ , the following conditions are equivalent.*

- (1) *The varieties  $X_f$  and  $X_g$  are isomorphic as  $\mathbb{A}^1$ -bundles.*
- (2) *There exist  $\tau_a \in \mathbf{k}[a^{\pm 1}, b][x, y]$  and  $\tau_b \in \mathbf{k}[a, b^{\pm 1}][x, y]$  such that  $\alpha = (x, \tau_a) \in G_a$ ,  $\beta = (x, \tau_b) \in G_b$  and*

$$\alpha \circ (x, y + f(x)) \circ \beta^{-1} = (x, y + g(x)).$$

- (3) *There exist  $r_a \in \mathbf{k}[a^{\pm 1}, b][x]$ ,  $r_b \in \mathbf{k}[a, b^{\pm 1}][x]$  and  $\lambda \in \mathbf{k}^*$  such that*

$$g(x) = \lambda f(x) + r_a(x) + r_b(x).$$

*Proof.* — The  $\mathbb{A}^1$ -bundle structures  $X_f \rightarrow \mathbb{A}_*^2 \times \mathbb{A}^1$  and  $X_g \rightarrow \mathbb{A}_*^2 \times \mathbb{A}^1$  are given by the restriction of the projection  $\mathbb{A}_*^2 \times \mathbb{A}^2 \rightarrow \mathbb{A}_*^2 \times \mathbb{A}^1$ ,  $(a, b, x, y) \mapsto (a, b, x)$  on both charts  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$ . An isomorphism of  $\mathbb{A}^1$ -bundles between  $X_f$  and  $X_g$  is then given by  $\alpha \in \text{Aut}(U_a \times \mathbb{A}^2)$  and  $\beta \in \text{Aut}(U_b \times \mathbb{A}^2)$ , both compatible with that projection. These automorphisms  $\alpha$  and  $\beta$  must

also belong to  $G_a$  and  $G_b$ , respectively. Moreover, their first coordinate must be equal to  $x$ . This shows that Assertions (1) and (2) are equivalent.

Suppose now that  $\alpha, \beta$  are as in (2). Since the Jacobian determinants of  $\alpha$  and  $\beta$  do not vanish on  $U_a$  and  $U_b$ , respectively, they are of the form

$$\alpha = (x, \mu_a y + r_a(x)) \text{ and } \beta^{-1} = (x, \mu_b y + r_b(x))$$

for some  $\mu_a \in \mathbf{k}[a^{\pm 1}]^*$ ,  $\mu_b \in \mathbf{k}[b^{\pm 1}]^*$ ,  $r_a \in \mathbf{k}[a^{\pm 1}, b][x]$  and  $r_b \in \mathbf{k}[a, b^{\pm 1}][x]$ . The equality  $\alpha \circ (x, y + f(x)) \circ \beta^{-1} = (x, y + g(x))$  is then equivalent to

$$\mu_a \mu_b = 1, r_a(x) + \mu_a r_b(x) + \mu_a f(x) = g(x).$$

The above equalities can only occur when  $\mu_a = (\mu_b)^{-1} \in \mathbf{k}^*$ . Therefore, Assertion (2) implies Assertion (3). Finally, it is easy to check that (3) implies (2), as we can construct suitable  $\alpha$  and  $\beta^{-1}$  given  $\lambda, r_a, r_b$  such that  $g(x) = \lambda f(x) + r_a(x) + r_b(x)$ .  $\square$

*Remark 3.4.* — Suppose that  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is isomorphic to the trivial  $\mathbb{A}^2$ -bundle. Then, by Lemma 3.2, there exist  $\alpha \in G_a$  and  $\beta \in G_b$  such that  $\alpha = (x, y + f(x)) \circ \beta$ . In particular,  $\alpha$  and  $\beta$  have the same first component, which is in fact an element of  $\mathbf{k}[a, b][x, y]$ . We call such an element a bivariable.

**DEFINITION 3.5.** — *We say that an element  $\omega \in \mathbf{k}[a, b][x, y]$  is a bivariable if it both a  $\mathbf{k}[a^{\pm 1}, b]$ -variable of  $\mathbf{k}[a^{\pm 1}, b, x, y]$  and a  $\mathbf{k}[a, b^{\pm 1}]$ -variable of  $\mathbf{k}[a, b^{\pm 1}, x, y]$ .*

*Example 3.6.* — Every  $\mathbf{k}[a, b]$ -variable of  $\mathbf{k}[a, b, x, y]$  is a bivariable.

*Example 3.7.* — Let  $m, n \geq 1$  be positive integers. Then, the polynomial  $\omega = a^m x + b^n y$  is a bivariable. Indeed, choosing  $\tau_a = a^{-m} y$  and  $\tau_b = -b^{-n} x$ , we can define  $\alpha = (\omega, \tau_a) \in G_a$  and  $\beta = (\omega, \tau_b) \in G_b$ . We remark that  $\alpha \circ \beta^{-1} = (x, y + f(x))$  with  $f(x) = \frac{x}{a^m b^n}$ .

As explained above, we can associate a bivariable to every trivial bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$ . This motivates the following definition.

**DEFINITION 3.8.** — *We say that a bivariable  $\omega \in \mathbf{k}[a, b][x, y]$  trivialises the bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$ ,  $f(x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ , if there exist elements  $\tau_a$  and  $\tau_b$  in  $\mathbf{k}[a^{\pm 1}, b][x, y]$  and  $\mathbf{k}[a, b^{\pm 1}][x, y]$ , respectively, such that  $\mathbf{k}[a^{\pm 1}, b][\omega, \tau_a] = \mathbf{k}[a^{\pm 1}, b][x, y]$ ,  $\mathbf{k}[a, b^{\pm 1}][\omega, \tau_b] = \mathbf{k}[a, b^{\pm 1}][x, y]$  and  $\tau_a = \tau_b + f(\omega)$ .*

*Remark 3.9.* — If a bivariable  $\omega \in \mathbf{k}[a, b][x, y]$  trivialises a bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  and if  $\tau_a, \tau_b$  and  $f$  are as in Definition 3.8, then

$$\alpha \circ \beta^{-1} = (x, y + f(x)),$$

where  $\alpha = (\omega, \tau_a) \in G_a$  and  $\beta = (\omega, \tau_b) \in G_b$ . In particular,  $\rho_f$  is isomorphic to the trivial  $\mathbb{A}^2$ -bundle.

LEMMA 3.10. — *Every bivariable  $\omega \in \mathbf{k}[a, b][x, y]$  trivialises a bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  for some  $f(x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ .*

*Proof.* — By definition, an element  $\omega \in \mathbf{k}[a, b][x, y]$  is a bivariable if and only if there exist  $\tau_a \in \mathbf{k}[a^{\pm 1}, b][x, y]$  and  $\tau_b \in \mathbf{k}[a, b^{\pm 1}][x, y]$  such that

$$\mathbf{k}[a^{\pm 1}, b][\omega, \tau_a] = \mathbf{k}[a^{\pm 1}, b] \text{ and } \mathbf{k}[a, b^{\pm 1}][\omega, \tau_b] = \mathbf{k}[a, b^{\pm 1}][x, y],$$

i.e. such that  $\alpha = (\omega, \tau_a) \in G_a$  and  $\beta = (\omega, \tau_b) \in G_b$ . The Jacobian determinant of  $\alpha$  is an element of  $\mathbf{k}[a^{\pm 1}, b]$  which does not vanish on  $U_a \times \mathbb{A}^2$ . Hence,  $\text{Jac}(\alpha) \in \mathbf{k}[a^{\pm 1}] \setminus \{0\}$ . Substituting  $\tau_a$  with  $\frac{\tau_a}{\text{Jac}(\alpha)}$ , we may thus assume that  $\text{Jac}(\alpha) = 1$ . Similarly, we may assume that  $\text{Jac}(\beta) = 1$ . Moreover, since the first components of  $\alpha$  and  $\beta$  are both equal to  $\omega$ , the first component of the composition  $\alpha \circ \beta^{-1} \in G_{ab}$  is equal to  $x$ . As the Jacobian determinant of  $\alpha \circ \beta^{-1}$  is equal to 1, it follows that  $\alpha \circ \beta^{-1} = (x, y + f(x))$  for some  $f \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ .

Finally, the equality  $\alpha = (x, y + f(x)) \circ \beta$  implies that  $\tau_a = \tau_b + f(\omega)$  as desired.  $\square$

The group of  $\mathbf{k}[a, b]$ -automorphisms of the ring  $\mathbf{k}[a, b][x, y]$  naturally acts on the set of bivariables. Indeed, if  $\omega \in \mathbf{k}[a, b][x, y]$  is a bivariable and if  $g \in \text{Aut}_{\mathbf{k}[a, b]}(\mathbf{k}[a, b][x, y])$  is an automorphism, then the element  $g(\omega)$  is again a bivariable. Therefore, we may consider bivariables up to the action of the group  $\text{Aut}_{\mathbf{k}[a, b]}(\mathbf{k}[a, b][x, y])$  and introduce the following definition.

DEFINITION 3.11. — *We say that two bivariables  $\omega_1, \omega_2 \in \mathbf{k}[a, b][x, y]$  are equivalent if there exists a  $\mathbf{k}[a, b]$ -automorphism of the ring  $\mathbf{k}[a, b][x, y]$  that maps  $\omega_1$  onto  $\omega_2$ .*

We now proceed with the proof of Theorem 1.4.

*Proof of Theorem 1.4.* —

(1). — By Lemma 3.10 and Remark 3.9, every bivariable trivialises an  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  isomorphic to the trivial bundle.

We now prove that the isomorphism class of  $X_f$ , as a  $\mathbb{A}^1$ -bundle, is uniquely determined by  $\omega$ . Suppose that a bivariable  $w$  trivialises two such bundles  $\rho_f$  and  $\rho_{\tilde{f}}$  and consider two suitable pairs  $(\tau_a, \tau_b)$  and  $(\tilde{\tau}_a, \tilde{\tau}_b)$  as in Definition 3.8. Define  $\alpha = (\omega, \tau_a)$ ,  $\tilde{\alpha} = (\omega, \tilde{\tau}_a) \in G_a$ ,  $\beta = (\omega, \tau_b)$  and  $\tilde{\beta} = (\omega, \tilde{\tau}_b) \in G_b$ . Then,  $\alpha \circ \beta^{-1} = (x, y + f(x))$  and  $\tilde{\alpha} \circ \tilde{\beta}^{-1} = (x, y + \tilde{f}(x))$ . Since

$$\alpha \circ \tilde{\alpha}^{-1} = (x, y + r_a(x)) \text{ and } \beta \circ \tilde{\beta}^{-1} = (x, y + r_b(x))$$

for some  $r_a \in \mathbf{k}[a^{\pm 1}, b][x]$  and  $r_b \in \mathbf{k}[a, b^{\pm 1}][x]$ , the two varieties  $X_f$  and  $X_{\tilde{f}}$  are isomorphic as  $\mathbb{A}^1$ -bundles by Lemma 3.3.

(2). — Suppose that two bivariables  $\omega'$  and  $\omega$  are equivalent and let  $\varphi \in \text{Aut}_{\mathbf{k}[a,b]}(\mathbf{k}[a,b][x,y])$  be such that  $\varphi(\omega) = \omega'$ . Then,  $\psi = (\varphi(x), \varphi(y)) \in G_a \cap G_b$ . Choosing  $\alpha = (\omega, \tau_a) \in G_a$  and  $\beta = (\omega, \tau_b) \in G_b$  as above, we obtain

$$\alpha' = \alpha \circ \psi = (\omega', \tau'_a) \in G_a \quad \text{and} \quad \beta' = \beta \circ \psi = (\omega', \tau'_b) \in G_b$$

for suitable elements  $\tau'_a, \tau'_b$ . Since,  $\alpha \circ \beta^{-1} = \alpha' \circ \beta'^{-1}$ , this implies that  $\omega$  and  $\omega'$  trivialise isomorphic (as  $\mathbb{A}^1$ -bundles)  $\mathbb{A}^2$ -bundles. Hence, the map  $\kappa$  of Assertion (2) is well defined. It remains to prove that this map is bijective.

The surjectivity of  $\kappa$  follows from Remark 3.4. Finally, we prove the injectivity. Consider two bivariables  $\omega$  and  $\omega'$  trivialising the same bundle  $\rho_f$ . Let  $\alpha = (\omega, \tau_a) \in G_a, \beta = (\omega, \tau_b) \in G_b, \alpha' = (\omega', \tau'_a) \in G_a$  and  $\beta' = (\omega', \tau'_b) \in G_b$  be such that  $\alpha \circ \beta^{-1} = \alpha' \circ \beta'^{-1} = (x, y + f(x))$ . Then, the element  $\psi = \alpha^{-1} \circ \alpha' = \beta^{-1} \circ \beta' \in G_a \cap G_b$  is an automorphism of  $\mathbb{A}^2 \times \mathbb{A}^2$ , whose action on  $\mathbf{k}[a,b,x,y]$  gives an automorphism  $\psi^* \in \text{Aut}_{\mathbf{k}[a,b]}(\mathbf{k}[a,b][x,y])$  sending  $\omega$  onto  $\omega'$ . This shows that  $\omega$  and  $\omega'$  are equivalent and concludes the proof.  $\square$

We finish this section by considering again two simple families of bivariables.

*Example 3.12.* — Let  $v \in \mathbf{k}[a,b,x,y]$  be a  $\mathbf{k}[a,b]$ -variable and let  $\tau \in \mathbf{k}[a,b,x,y]$  be such that  $\mathbf{k}[a,b][x,y] = \mathbf{k}[a,b][v,\tau]$ . Defining  $\tau_a = \tau_b = \tau$ , the bijection of Theorem 1.4 associates the (equivalence class) of the bivariable  $v$  to the (isomorphism class as  $\mathbb{A}^1$ -bundle) of the trivial bundle  $X_f$  with  $f = 0$ .

*Example 3.13.* — Let  $m, n \geq 1$  be positive integers and  $P \in \mathbf{k}[a,b]$ . Then, the polynomial  $\omega = a^m x + b^n y + P$  is a bivariable. Indeed, choosing  $\tau_a = a^{-m} y$  and  $\tau_b = -b^{-n} x$ , we get that  $\alpha = (\omega, \tau_a) \in G_a$  and  $\beta = (\omega, \tau_b) \in G_b$ . Then, the bijection of Theorem 1.4 sends (the class of) this bivariable onto (the class of)  $X_f$  where  $f = f(x) = \frac{x-P}{a^m b^n}$ , since  $\tau_a = \tau_b + f(\omega)$ .

#### 4. A procedure to construct bivariables

Let  $\omega \in \mathbf{k}[a,b][x,y]$  be a bivariable, and let  $\tau_a \in \mathbf{k}[a^{\pm 1}, b][x,y]$  and  $\tau_b \in \mathbf{k}[a, b^{\pm 1}][x,y]$  be such that  $\alpha = (\omega, \tau_a) \in G_a$  and  $\beta = (\omega, \tau_b) \in G_b$ . An easy way to get a new bivariable from  $\omega$  is to compose  $\alpha$  with a triangular automorphism  $\varphi = (x + P(y), y)$ , where  $P \in \mathbf{k}[a,b][x]$  should be well chosen, so that  $\tilde{\omega} = \omega + P(\tau_a)$  is also a  $G_b$ -variable. (Note that  $\tilde{\omega}$  is the first component of  $\varphi \circ \alpha$ , hence it is a  $G_a$ -variable.) In order to see that this idea works in general, we recall the following well-known result (see for example the proof of [6, Theorem 4]).

LEMMA 4.1. — *Let  $R$  be a commutative ring with unity,  $m \geq 1$  be a positive integer,  $a \in R$  be a non-zero divisor and  $f, Q \in R[X]$  be polynomials in one indeterminate with coefficient in  $R$ . Then, the polynomial*

$$v = x + aQ(a^m y + f(x))$$

*is a  $R$ -variable of  $R[x, y]$ .*

*Proof.* — Let us shortly recall, how the proof goes. The idea is to construct, working by induction on  $i$ , polynomials  $g_i(x) \in R[x]$  such that

$$f(x) - g_i(v) \in a^i R[x, y]$$

for all integers  $i$  with  $1 \leq i \leq m$ . This allows us to define a map  $\varphi \in \text{End}_R(R[x, y])$  by setting

$$\varphi(x) = v \quad \text{and} \quad \varphi(y) = y + \frac{f(x) - g_m(v)}{a^m}.$$

It turns out that  $\varphi$  is actually an automorphism of  $R[x, y]$ . We refer to [6, Theorem 4] for the details.  $\square$

We now apply the above lemma to construct new bivariables.

PROPOSITION 4.2. — *Let  $\omega \in \mathbf{k}[a, b][x, y]$  be a bivariable. According to Theorem 1.4, let  $\tau_a, \tau_b$  and  $f(x)$  be such that  $\mathbf{k}[a^{\pm 1}, b][\omega, \tau_a] = \mathbf{k}[a^{\pm 1}, b][x, y]$ ,  $\mathbf{k}[a, b^{\pm 1}][\omega, \tau_b] = \mathbf{k}[a, b^{\pm 1}][x, y]$  and  $\tau_a = \tau_b + f(\omega)$ . Suppose that  $m, n \geq 1$  are positive integers such that  $a^m b^n f(x) \in \mathbf{k}[a, b, x]$  and let  $Q \in \mathbf{k}[a, b][x]$  be a polynomial. Then, the elements*

$$\omega + aQ(a^m \tau_a) = \omega + aQ(a^m \tau_b + a^m f(\omega))$$

*and*

$$\omega + bQ(b^n \tau_b) = \omega + bQ(b^n \tau_a - b^n f(\omega))$$

*are bivariables of  $\mathbf{k}[a, b][x, y]$ .*

*Proof.* — The proof that  $\omega + bQ(b^n \tau_b)$  is a bivariable being similar (by exchanging the roles of  $a$  and  $b$ ), we only prove that

$$\widehat{\omega} = \omega + aQ(a^m \tau_a) = \omega + aQ(a^m \tau_b + a^m f(\omega))$$

is a bivariable.

On the one hand, it is a  $\mathbf{k}[a^{\pm 1}, b]$ -variable since it is the first component of the composition of the  $\mathbf{k}[a^{\pm 1}, b]$ -automorphisms defined by  $(x, y) \mapsto (x + aQ(a^m y), y)$  and by  $(x, y) \mapsto (\omega, \tau_a)$ .

On the other hand, there exists, by Lemma 4.1, a  $\mathbf{k}[a, b^{\pm 1}]$ -automorphism, say  $\varphi$ , of  $\mathbf{k}[a, b^{\pm 1}][x, y]$  whose first component is equal to  $v = x + aQ(a^m y + a^m f(x))$ . The element  $\widehat{\omega} = \omega + aQ(a^m \tau_b + a^m f(\omega))$  is therefore a  $\mathbf{k}[a, b^{\pm 1}]$ -variable, since it is equal to the first component of the composition of  $\varphi$  with the  $\mathbf{k}[a, b^{\pm 1}]$ -automorphism defined by  $(x, y) \mapsto (\omega, \tau_b)$ .  $\square$

*Remark 4.3.* — Let  $\omega$  be a bivariable associated with the data  $\tau_a, \tau_b$  and  $f(x)$  as in Theorem 1.4. Let  $\widehat{\omega}$  be a new bivariable obtained from  $\omega$  by applying Proposition 4.2. Then, one can easily compute elements  $\widehat{\tau}_a, \widehat{\tau}_b$  and  $\widehat{f}(x)$  associated with  $\widehat{\omega}$ . Indeed, in the proof of Lemma 4.1, we explain how to construct an explicit automorphism of  $R[x, y]$  whose first component is equal to  $v$ .

*Example 4.4.* — Let us consider the bivariable  $\omega = ax + b^2y$  from Example 3.7. Recall that it is indeed a bivariable, associated with  $\tau_a = \frac{y}{a}$ ,  $\tau_b = -\frac{x}{b^2}$  and  $f(x) = \frac{x}{ab^2}$ . By Proposition 4.2, the polynomial

$$\widehat{\omega} = \omega + bP(-b^2\tau_b) = ax + b^2y + bP(x)$$

is a bivariable for each  $P(x) \in \mathbf{k}[x]$ .

Furthermore, one claims that this new bivariable is associated with

$$\widehat{f}(x) = \frac{x}{ab^2} - \frac{P\left(\frac{x}{a}\right)}{ab},$$

with  $\widehat{\tau}_b = \tau_b = -\frac{x}{b^2}$  and with

$$\widehat{\tau}_a = \widehat{\tau}_b + \widehat{f}(\widehat{\omega}) = \frac{y}{a} + \frac{P(x)}{ab} - \frac{1}{ab}P\left(x + \frac{b^2y + bP(x)}{a}\right).$$

Our claims is easy to check. First, it is straightforward to see that  $\widehat{\tau}_a \in \mathbf{k}[a^{\pm 1}, b, x, y]$ . Then, since  $(\widehat{\omega}, \widehat{\tau}_b) \in G_b \subset G_{ab}$  is an automorphism of Jacobian determinant 1, it follows that the element  $(\widehat{\omega}, \widehat{\tau}_a) \in G_{ab}$ , whose both components belong to  $\mathbf{k}[a^{\pm 1}, b, x, y]$ , is also of Jacobian determinant 1, hence that it is an element of  $G_a$ .

*Remark 4.5.* — We recall that, as computed in Example 2.6, the function  $\widehat{f}(x)$  above is actually the transition function of the bundle  $\rho_{\widehat{f}}$  corresponding to the fibration  $\pi_{P,n}$  in the case where  $n > \deg(P)$ . Consequently, it follows from Theorem 1.4 and Lemma 3.1 that every polynomial  $v_{P,n}$  is a  $\mathbf{k}[x]$ -variable, when  $n > \deg(P)$ . In particular, we recover the fact that the  $n$ -th Vénéreau polynomials are  $\mathbf{k}[x]$ -variables for all  $n \geq 3$ .

We can now start with the bivariable of Example 4.4 and apply again Proposition 4.2 to it. Doing so, we construct the following new bivariables.

LEMMA 4.6. — *Let  $P \in \mathbf{k}[X]$  be a polynomial of degree 2 with leading coefficient  $c \in \mathbf{k}^*$  and suppose that  $\text{char}(\mathbf{k}) \neq 2$ . Define  $\omega = ax + b^2y + bP(x)$ . Then, the polynomial*

$$\omega + \frac{a^5}{2c} \left( \frac{y}{a} + \frac{P(x)}{ab} - \frac{1}{ab}P\left(\frac{\omega}{a}\right) \right)$$

*is a bivariable, associated with the  $\mathbb{A}^2$ -bundle over  $\mathbb{A}_*^2$  corresponding to the fibration  $\pi_{P,2}$ .*

*Remark 4.7.* — As a corollary, we recover the fact that the second Vénéreau polynomial is a  $\mathbf{k}[x]$ -variable of  $\mathbf{k}[x, y, z, u]$  when  $\text{char}(\mathbf{k}) \neq 2$ .

*Proof of Lemma 4.6.* — We recall that  $\omega = ax + b^2y + bP(x)$  is the bivariable constructed in Example 4.4 and that it is associated with the data  $\tau_a = \frac{y}{a} + \frac{P(x)}{ab} - \frac{1}{ab}P\left(\frac{\omega}{a}\right)$ ,  $\tau_b = -\frac{x}{b^2}$  and  $f(x) = \frac{x}{ab^2} - \frac{1}{ab}P\left(\frac{x}{a}\right)$ . By Proposition 4.2, the polynomial

$$\widehat{\omega} = \omega + aQ(a^m\tau_a)$$

is a bivariable for every  $m > \deg(P)$  and every  $Q \in \mathbf{k}[a, b][x]$ . In particular, we may choose  $m = 3$  and  $Q = \frac{ax}{2c}$ . With these choices, we obtain the bivariable

$$\widehat{\omega} = \omega + \frac{a^5}{2c}\tau_a = \omega + \frac{a^5}{2c} \left( \frac{y}{a} + \frac{P(x)}{ab} - \frac{1}{ab}P\left(\frac{\omega}{a}\right) \right).$$

Now, it remains to check that this bivariable indeed corresponds to the transition function of  $\pi_{P,2}$ , i.e. that it is associated with the function

$$\widehat{f}(x) = \frac{x}{ab^2} - \frac{1}{ab}P\left(\frac{x}{a}\right) - \frac{a}{b^2}xP\left(\frac{x}{a}\right)$$

that we computed at Example 2.6.

As  $(\omega, \tau_a) \in G_a$ , we also have  $(\widehat{\omega}, \tau_a) \in G_a$  and we can let  $\widehat{\tau}_a = \tau_a$ . To conclude the proof, we only need to prove that

$$\widehat{\tau}_b = \widehat{\tau}_a - \widehat{f}(\widehat{\omega}) = \tau_a - \widehat{f}(\widehat{\omega}) = \tau_b + f(\omega) - \widehat{f}(\widehat{\omega})$$

is an element of the ring  $\mathbf{k}[a, b^{\pm 1}, x, y]$ , i.e. that it has no denominators in  $a$ . So, we need to prove that  $f(\omega) - \widehat{f}(\widehat{\omega}) \in \mathbf{k}[a, b^{\pm 1}, x, y]$ .

Let us define

$$\Delta = \omega - ax = b^2y + bP(x) \in \mathbf{k}[a, b, x, y].$$

The defining expression of  $\widehat{\omega}$  gives

$$\widehat{\omega} \equiv \omega - \frac{a^4}{2bc}P\left(\frac{\omega}{a}\right) \equiv \omega - \frac{a^4}{2bc}P\left(x + \frac{\Delta}{a}\right) \equiv \omega - a^2\frac{\Delta^2}{2b} \pmod{a^3},$$

where the congruence holds in the ring  $\mathbf{k}[a, b^{\pm 1}, x, y]$ . In other words, there exists an element  $R \in \mathbf{k}[a, b^{\pm 1}, x, y]$  such that

$$\widehat{\omega} = \omega - a^2\frac{\Delta^2}{2b} + a^3R.$$

By Taylor expansion, we then get, as  $\deg(f) = \deg(P) = 2$ , that

$$\begin{aligned}
 f(\widehat{\omega}) &= f\left(\omega - a^2 \frac{\Delta^2}{2b} + a^3 R\right) \\
 &= f(\omega) + \left(-a^2 \frac{\Delta^2}{2b} + a^3 R\right) \cdot f'(\omega) + \frac{1}{2} \left(-a^2 \frac{\Delta^2}{2b} + a^3 R\right)^2 \cdot f''(\omega) \\
 &= f(\omega) + \left(-a^2 \frac{\Delta^2}{2b} + a^3 R\right) \cdot \left(\frac{1}{ab^2} - \frac{1}{a^2 b} P'\left(\frac{\omega}{a}\right)\right) \\
 &\quad + \frac{1}{2} \left(-a^2 \frac{\Delta^2}{2b} + a^3 R\right)^2 \cdot \left(-\frac{1}{a^3 b} P''\left(\frac{\omega}{a}\right)\right) \\
 &= f(\omega) + \frac{c\Delta^3}{ab^2} + R_2,
 \end{aligned}$$

for some element  $R_2 \in \mathbf{k}[a, b^{\pm 1}, x, y]$ .

Finally, since

$$\frac{a}{b^2} \widehat{\omega} P\left(\frac{\widehat{\omega}}{a}\right) = \frac{c\Delta^3}{ab^2} + R_3$$

for some element  $R_3 \in \mathbf{k}[a, b^{\pm 1}, x, y]$ , we get that

$$\begin{aligned}
 \widehat{f}(\widehat{\omega}) - f(\omega) &= f(\widehat{\omega}) - \frac{a}{b^2} \widehat{\omega} P\left(\frac{\widehat{\omega}}{a}\right) - f(\omega) \\
 &= \frac{c\Delta^3}{ab^2} + R_2 - \frac{c\Delta^3}{ab^2} - R_3 \\
 &= R_2 - R_3 \in \mathbf{k}[a, b^{\pm 1}, x, y],
 \end{aligned}$$

as desired.  $\square$

Lemma 4.1 does not only allow us to construct bivariables, but it is also useful to study  $\mathbb{A}^2$ -bundles  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$ ,  $f \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ , that are not necessarily trivial. More precisely, given a bundle  $\rho_f$ , one can construct other bundles  $\rho_g$ , that are isomorphic to  $\rho_f$  as follows.

**PROPOSITION 4.8.** — *Let  $f_b, g_b \in \mathbf{k}[a, b^{\pm 1}][x]$ . If there exist  $m \geq 1$  and  $Q \in \mathbf{k}[a, b][x]$  such that*

$$g_b(x + aQ(f_b(x))) \equiv f_b(x) \pmod{a^m},$$

*then  $\rho_{f_b/a^m}$  and  $\rho_{g_b/a^m}$  are isomorphic  $\mathbb{A}^2$ -bundles.*

*Proof.* — Define  $v = x + aQ(a^m y + f_b(x))$ . By Lemma 4.1,  $v$  is a  $\mathbf{k}[a, b^{\pm 1}]$ -variable of  $\mathbf{k}[a, b^{\pm 1}, x, y]$ . We also define  $\alpha = (x - aQ(a^m y), y) \in G_a$  and  $\gamma = (v, y + \frac{f_b(x) - g_b(v)}{a^m}) \in G_{ab}$ . Since  $\alpha \circ (x, y + \frac{g_b(x)}{a^m}) \circ \gamma = (x, y + \frac{f_b(x)}{a^m})$ , the result will follow from Lemma 3.2 if we prove that  $\gamma$  is actually an element of  $G_b$ . As  $(x, y + f_b(x)/a^m) \in G_{ab}$  has Jacobian 1, the same holds for  $(v, y + f_b(x)/a^m) \in G_{ab}$  and for  $\gamma \in G_{ab}$ . To obtain that  $\gamma \in G_b$  as

desired, it suffices to observe that  $y + \frac{f_b(x) - g_b(v)}{a^m} \in \mathbf{k}[a^{\pm 1}, b][x, y]$ , as  $g_b(v) \equiv g_b(x + aQ(f_b(x))) \equiv f_b(x) \pmod{a^m}$ .  $\square$

Although we unfortunately did not succeed to address the case of the first Vénéreau polynomial with our techniques, Proposition 4.8 does lead to the following three results. Recall that the first Vénéreau polynomial is a  $\mathbf{k}[x]$ -variable if and only if the  $\mathbb{A}^2$ -bundle  $\rho_{f_1}: X_{f_1} \rightarrow \mathbb{A}_*^2$  is isomorphic to the trivial bundle, where

$$f_1(x) = \frac{x}{ab^2} - \frac{x^2}{a^3b} - \frac{x^3}{a^2b^2} - \frac{x^4}{ab^3}.$$

*Example 4.9.* — Suppose that  $\text{char}(\mathbf{k}) \neq 2$ . Then, one can apply Proposition 4.8 with  $Q(x) = \frac{x}{2}$  to show that the bundle  $\rho_g$  where

$$g(x) = \frac{x}{ab^2} - \frac{x^2}{a^3b} - \frac{x^3}{a^2b^2} - \frac{5}{4} \cdot \frac{x^4}{ab^3}$$

is isomorphic to the bundle  $\rho_f$  where

$$f(x) = f_3(x) = \frac{x}{ab^2} - \frac{x^2}{a^3b}.$$

Indeed, writing  $f_b = a^3f, g_b = a^3g \in \mathbf{k}[a, b^{\pm 1}][x]$ , a straightforward calculation gives  $g_b(x + aQ(f_b(x))) \equiv f_b(x) \pmod{a^3}$ .

As  $\rho_{f_3}$  is a trivial bundle, the same holds for  $\rho_g$ . Note that  $g$  differs from  $f_1$  only by the coefficient  $\frac{5}{4}$  in its last summand.

*Example 4.10.* — Assume that  $\text{char}(\mathbf{k}) \neq 2$  and that 5 is a square in  $\mathbf{k}$ . Then, one can apply Proposition 4.8 with  $Q(x) = \frac{1+\xi}{2}x$  where  $\xi = \frac{1}{\sqrt{5}}$  to show that the  $\mathbb{A}^2$ -bundles  $\rho_{f_1}$  and  $\rho_g$  are isomorphic, where

$$g(x) = \frac{x}{ab^2} - \frac{x^2}{a^3b} + \frac{\xi x^3}{a^2b^2}$$

is of degree three in  $x$ . Indeed, writing  $f_b = a^3f_1, g_b = a^3g \in \mathbf{k}[a, b^{\pm 1}][x]$ , a straightforward calculation gives  $f_b(x + aQ(g_b(x))) \equiv g_b(x) \pmod{a^3}$ .

*Example 4.11.* — Assume that  $\text{char}(\mathbf{k}) \neq 2$ . Then, one can apply Proposition 4.8 with  $Q(x) = \frac{x}{2}$  to show that the  $\mathbb{A}^2$ -bundles  $\rho_{f_1}$  and  $\rho_g$  are isomorphic, where

$$g(x) = \frac{x}{ab^2} - \frac{x^2}{a^3b} + \frac{1}{4} \frac{x^4}{ab^3}$$

has the same degree than  $f_1$  but with one summand less. Indeed, writing  $f_b = a^3f_1, g_b = a^3g \in \mathbf{k}[a, b^{\pm 1}][x]$ , a straightforward calculation gives  $f_b(x + aQ(g_b(x))) \equiv g_b(x) \pmod{a^3}$ .

### 5. Local triviality on the blow-up

By Theorem 1.3, every map

$$\pi_{P,n}: \begin{array}{ccc} \mathbb{A}^4 & \longrightarrow & \mathbb{A}^2 \\ (x, y, z, u) & \longmapsto & (x, v_{P,n}) \end{array}$$

has the structure of an  $\mathbb{A}^2$ -bundle over the punctured affine plane. In fact, a stronger fact holds: every map  $\pi_{P,n}$  yields an  $\mathbb{A}^2$ -bundle over  $\mathbb{A}^2$  blown-up at the origin.

**THEOREM 5.1.** — *If we denote by  $\epsilon: \widehat{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$  the blow-up of  $(0, 0) \in \mathbb{A}^2$  and by  $\eta: \widehat{\mathbb{A}}^4 \rightarrow \mathbb{A}^2$  the blow-up of  $x = y = 0$ , then the pull-back of  $\pi_{P,n}$  by  $\epsilon$*

$$\widehat{\pi}_{P,n}: \mathbb{A}^4 \times_{\mathbb{A}^2} \widehat{\mathbb{A}}^2 \rightarrow \widehat{\mathbb{A}}^2$$

*is a (locally trivial)  $\mathbb{A}^2$ -bundle. Moreover,  $\mathbb{A}^4 \times_{\mathbb{A}^2} \widehat{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$  is simply the blow-up of  $x = y = 0$ .*

*Proof.* — The blow-up  $\epsilon: \widehat{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$  of the origin can be seen at the projection  $((a, b), [A : B]) \mapsto (a, b)$ , where

$$\widehat{\mathbb{A}}^2 = \{((a, b), [A : B]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid aB = bA\}.$$

We consider the two open subsets defined by  $A \neq 0$  and by  $B \neq 0$ . They correspond to  $\text{Spec}(\mathbf{k}[a, \frac{b}{a}])$  and  $\text{Spec}(\mathbf{k}[b, \frac{a}{b}])$  respectively, and their intersection is isomorphic to  $\text{Spec}(\mathbf{k}[a, b, \frac{a}{b}, \frac{b}{a}])$ . The transition function  $\alpha^{-1} \circ (x, y + f(x)) \circ \beta \in G_{ab}$ , computed in Lemma 5.22 below, is then an isomorphism over this intersection. This proves that  $\widehat{\pi}_{P,n}$  is a locally trivial  $\mathbb{A}^2$ -bundle. Moreover, as  $\widehat{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$  is the blow-up of  $(0, 0)$ , the morphism  $\mathbb{A}^4 \times_{\mathbb{A}^2} \widehat{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$  is the blow-up of  $(\pi_{P,n})^{-1}(0, 0)$ , which is the surface given by  $x = y = 0$ .  $\square$

**LEMMA 5.2.** — *Let  $n \geq 1$ ,  $P \in \mathbf{k}[z]$  and let  $\omega \in \mathbf{k}[a, b, x, y]$  be defined by  $\omega = ax + b^2y + bP(x)$ . For every  $m \geq 1$ , define*

$$f_m = \frac{x}{ab^2} - \frac{1}{ab^m} \cdot \frac{b^m - (a^n x)^m}{b - a^n x} P\left(\frac{x}{a}\right) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x].$$

*Then, the following hold:*

- (1) *The elements  $\alpha = (\omega, \frac{y}{a} + \frac{P(x) - P(a^{-1}\omega)}{ab})$  and  $\beta = (\omega, -b^{-2}x)$  belong to the groups  $G_a$  and to  $G_b$ , respectively, and they satisfy that  $\alpha^{-1} \circ (x, y + f_1(x)) \circ \beta = (x, y)$ .*
- (2) *For each  $m \geq 1$ , the components of  $\alpha^{-1} \circ (x, y + f_m(x)) \circ \beta \in G_{ab}$ , as well as the components of its inverse, all belong to the ring  $\mathbf{k}[a, b, \frac{a}{b}, \frac{b}{a}, x, y]$ .*

*Proof.* — Assertion (1) is straightforward to check. Let us denote  $T_m = (x, y + f_m(x)) \in G_{ab}$  for each  $m \geq 1$ . In particular,  $T_1 = (x, y + \frac{x}{ab^2} - \frac{1}{ab}P(\frac{x}{a})) \in G_{ab}$  and we observe that

$$\begin{aligned} (T_1)^{-1} \circ T_m &= \left( x, y + f_m(x) - \frac{x}{ab^2} + \frac{1}{ab}P\left(\frac{x}{a}\right) \right) \\ &= \left( x, y + \left( \frac{1}{ab} - \frac{1}{ab^m} \cdot \frac{b^m - (a^n x)^m}{b - a^n x} \right) P\left(\frac{x}{a}\right) \right) \\ &= \left( x, y + \left( \frac{1}{ab} - \frac{1}{ab^m} \cdot \sum_{k=0}^{m-1} b^{m-1-k} (a^n x)^k \right) P\left(\frac{x}{a}\right) \right) \\ &= \left( x, y - \frac{1}{ab} \sum_{k=1}^{m-1} \left( \frac{a^n x}{b} \right)^k \cdot P\left(\frac{x}{a}\right) \right). \end{aligned}$$

Since,  $\beta^{-1} = (-b^2y, \frac{x}{b^2} + ay - \frac{1}{b} \cdot P(-b^2y))$  and  $T_1 = \alpha \circ \beta^{-1}$ , we can now calculate

$$\begin{aligned} &\alpha^{-1} \circ T_m \circ \beta \\ &= \beta^{-1} \circ (T_1)^{-1} \circ T_m \circ \beta \\ &= \beta^{-1} \circ \left( x, y - \frac{1}{ab} \sum_{k=1}^{m-1} \left( \frac{a^n x}{b} \right)^k \cdot P\left(\frac{x}{a}\right) \right) \circ \beta \\ &= \beta^{-1} \circ \left( \omega, -\frac{x}{b^2} - \frac{1}{ab} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right) \right) \\ &= \left( x + \frac{b}{a} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right), \frac{\omega}{b^2} - \frac{ax}{b^2} - \frac{1}{b} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right) \right. \\ &\quad \left. - \frac{1}{b} P\left( x + \frac{b}{a} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right) \right) \right) \\ &= \left( x + \frac{b}{a} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right), y - \frac{1}{b} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right) \right. \\ &\quad \left. + \frac{1}{b} P(x) - \frac{1}{b} P\left( x + \frac{b}{a} \sum_{k=1}^{m-1} \left( \frac{a^n \omega}{b} \right)^k \cdot P\left(\frac{\omega}{a}\right) \right) \right) \end{aligned}$$

As  $\frac{\omega}{a} \in \mathbf{k}[a, b, \frac{a}{b}, \frac{b}{a}]$ , it is straightforward to check that the two components of the above map are contained in  $\mathbf{k}[a, b, \frac{a}{b}, \frac{b}{a}]$ . Moreover, since the Jacobian determinants of  $\alpha$ ,  $\beta$  and  $T_m$  are equal to 1, the Jacobian determinant of  $\alpha^{-1} \circ T_m \circ \beta$  is also equal to 1 and the components of its inverse also belong to the same ring.  $\square$

### 6. The varieties $X_f$ for small denominators

In this section, we study the varieties  $X_f$  introduced in Definition 1.2. One may always choose  $m, n \geq 0$  such that  $a^m b^n f(x) \in \mathbf{k}[a, b, x]$ . Our main result is Proposition 6.4 which gives a criterion to decide whether the corresponding bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is a trivial  $\mathbb{A}^2$ -bundle, in the special case where one of the two integers  $m$  or  $n$  is at most 1.

We will proceed as follows. We will first realise  $X_f$  as an open subset of an affine hypersurface  $Y \subseteq \mathbb{A}^5$  (see Lemma 6.1). Then, we will study when the morphism  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  extends to a (locally) trivial  $\mathbb{A}^2$ -bundle  $Y \rightarrow \mathbb{A}^2$  (see Lemma 6.2). Afterwards, we will compute the ring of regular functions on a variety  $X_f$  (see Proposition 6.3) and show that, in the case where  $m = 1$  or  $n = 1$ , this ring is equal to the ring of regular functions of  $Y$ . We will finally use this result to prove Proposition 6.4.

LEMMA 6.1. — *For each  $f = f(a, b, x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$  and for all integers  $m, n \geq 0$  such that  $P(a, b, x) = a^m b^n f \in \mathbf{k}[a, b, x]$ , the variety  $X_f$  admits an open embedding into the hypersurface  $Y \subset \mathbb{A}^5 = \text{Spec}(\mathbf{k}[a, b, x, u, v])$  defined by the equation*

$$a^m u - b^n v = P(a, b, x).$$

More precisely,  $X_f$  is isomorphic to  $Y \setminus \{a = b = 0\}$ , via the isomorphisms

$$\begin{aligned} \varphi: \quad U_a \times \mathbb{A}^2 &\longrightarrow Y \setminus \{a = 0\}, \\ ((a, b), (x, y)) &\longmapsto (a, b, x, b^n y + a^{-m} P(a, b, x), a^m y). \\ \psi: \quad U_b \times \mathbb{A}^2 &\longrightarrow Y \setminus \{b = 0\}, \\ ((a, b), (x, y)) &\longmapsto (a, b, x, b^n y, a^m y - b^{-n} P(a, b, x)). \end{aligned}$$

*Proof.* — We observe that the maps  $\psi, \varphi$  are isomorphisms, whose inverse maps are given by

$$\begin{aligned} \varphi^{-1}: Y \setminus \{a = 0\} &\xrightarrow{\cong} U_a \times \mathbb{A}^2, \quad (a, b, x, u, v) \longmapsto ((a, b), (x, a^{-m} v)) \\ \psi^{-1}: Y \setminus \{b = 0\} &\xrightarrow{\cong} U_b \times \mathbb{A}^2, \quad (a, b, x, u, v) \longmapsto ((a, b), (x, b^{-n} u)). \end{aligned}$$

Moreover, as we obtain the transition function of  $X_f$  via the composition

$$\begin{aligned} \psi^{-1} \circ \varphi: \quad U_{ab} \times \mathbb{A}^2 &\longrightarrow U_{ab} \times \mathbb{A}^2 \\ ((a, b), (x, y)) &\longmapsto ((a, b), (x, y + a^{-m} b^{-n} P(a, b, x))), \end{aligned}$$

we see that the two maps  $\varphi$  and  $\psi$  induce inverse isomorphisms between  $X_f$  and  $Y \setminus \{a = b = 0\}$ .  $\square$

LEMMA 6.2. — *Let  $m, n \geq 1$  and  $P \in \mathbf{k}[a, b, x]$ . Define  $Q = a^m u - b^n v - P \in \mathbf{k}[a, b, x, u, v]$  and let  $\pi: \text{Spec}(\mathbf{k}[a, b, x, u, v]/Q) \rightarrow \mathbb{A}^2$  be the morphism defined by  $(a, b, x, u, v) \mapsto (a, b)$ . Then, the following conditions are equivalent:*

- (1)  $\pi$  is a trivial  $\mathbb{A}^2$ -bundle.
- (2)  $\pi$  is a locally trivial  $\mathbb{A}^2$ -bundle.
- (3) The fibre  $\pi^{-1}((0,0))$  is isomorphic to  $\mathbb{A}^2$ .
- (4)  $P(0,0,x) \in \mathbf{k}[x]$  is of degree 1.
- (5)  $Q$  is a  $\mathbf{k}[a,b]$ -variable of  $\mathbf{k}[a,b,x,u,v]$ .

*Proof.* —

(1)  $\Rightarrow$  (2). — Because any trivial bundle is locally trivial.

(2)  $\Rightarrow$  (3). — As  $\pi$  is a locally trivial  $\mathbb{A}^2$ -bundle, the fibre over  $(0,0)$  is isomorphic to  $\mathbb{A}^2$ .

(3)  $\Rightarrow$  (4). — The fibre over  $(0,0)$  is defined by the equation  $P(0,0,x)$ . If this fibre is isomorphic to  $\mathbb{A}^2$ , the polynomial  $P(0,0,x)$  is then of degree 1.

(4)  $\Rightarrow$  (5). — We suppose that  $P(0,0,x)$  is of degree 1 and prove that  $Q$  is a  $\mathbf{k}[a,b]$ -variable of  $\mathbf{k}[a,b,x,u,v]$ . We may apply elements of  $G = \text{Aut}_{\mathbf{k}[a,b]}(\mathbf{k}[a,b,x,u,v])$ , since these automorphisms send  $\mathbf{k}[a,b]$ -variables onto  $\mathbf{k}[a,b]$ -variables. Applying such an automorphism that sends  $x$  onto  $\xi x + \mu$  for suitable  $\xi \in \mathbf{k}^*$  and  $\mu \in \mathbf{k}$ , we can suppose that  $P(0,0,x) = x$  and replace  $Q$  with

$$Q_1 = x + a^m u + b^n v + aP_1(a,b,x) + bP_2(a,b,x) \in \mathbf{k}[a,b,x,u,v],$$

where  $P_1, P_2 \in \mathbf{k}[a,b,x]$ . Now, Lemma 4.1 implies that  $x + a^m u + aP_1(a,b,x)$  is a  $\mathbf{k}[a,b,v]$ -variable of  $\mathbf{k}[a,b,v][x,u]$ . There is thus  $g \in G$  such that  $v = g(v)$  and  $x = g(x + a^m u + aP_1(a,b,x))$ . Hence,  $Q_2 = g(Q_1)$  is of the form

$$Q_2 = x + b^n v + bP_3(a,b,x,u) \in \mathbf{k}[a,b,x,u,v],$$

where  $P_3 \in \mathbf{k}[a,b,x,u]$ . In turn, again by Lemma 4.1, the element  $Q_2$  is a  $\mathbf{k}[a,b,u]$ -variable of  $\mathbf{k}[a,b,u][x,v]$  and it is thus in particular a  $\mathbf{k}[a,b]$ -variable.

(5)  $\Rightarrow$  (1). — If  $Q$  is a  $\mathbf{k}[a,b]$ -variable of  $\mathbf{k}[a,b,x,u,v]$ , there exists a  $\mathbf{k}[a,b]$ -automorphism of  $\mathbf{k}[a,b,x,u,v]$  which sends  $Q$  onto  $x$ . This isomorphism trivialises the  $\mathbb{A}^2$ -bundle  $\pi$ .  $\square$

PROPOSITION 6.3. — For each  $f = f(a,b,x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$ , the following hold:

- (1) The ring  $\mathcal{O}(X_f)$  of regular functions on  $X_f$  is given on the two charts  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$  by  $R_a$  and  $R_b$ , respectively, where

$$R_a = \mathbf{k}[a^{\pm 1}, b, x, y] \cap \mathbf{k}[a, b^{\pm 1}, x, y + f],$$

$$R_b = \mathbf{k}[a, b^{\pm 1}, x, y] \cap \mathbf{k}[a^{\pm 1}, b, x, y - f].$$

- (2) For all integers  $m, n \geq 0$  such that  $P = a^m b^n f \in \mathbf{k}[a,b,x]$ , we have  $\mathbf{k}[a,b,x, a^m y, b^n y + a^{-m} P] \subseteq R_a$  and  $\mathbf{k}[a,b,x, b^n y, a^m y - b^{-n} P] \subseteq R_b$ .

Moreover, the first inclusion is an equality if and only if the second is also an equality.

- (3) If  $P(0, 0, x) \neq 0$  and  $m = 1$  or  $n = 1$ , the inclusions in (2) are equalities.
- (4) Under the assumption that the inclusions in (2) are equalities, we have that  $X_f$  is a trivial  $\mathbb{A}^1$ -bundle if and only if  $P(0, 0, x) = P|_{a=b=0}$  is of degree 1.

*Proof.* — Denote by  $\psi: U_a \times \mathbb{A}^2 \dashrightarrow U_b \times \mathbb{A}^2$  the birational map  $((a, b), (x, y)) \mapsto ((a, b), (x, y + f(a, b, x)))$  and by  $\varphi$  its inverse.

(1). — A regular function on  $X_f$  is a function that is regular on  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$ . Regular functions on  $U_a \times \mathbb{A}^2$  correspond to the ring  $\mathcal{O}(U_a \times \mathbb{A}^2) \cap \psi^*(\mathcal{O}(U_b \times \mathbb{A}^2))$ . Similarly, regular functions on  $U_b \times \mathbb{A}^2$  correspond to  $\mathcal{O}(U_b \times \mathbb{A}^2) \cap \varphi^*(\mathcal{O}(U_a \times \mathbb{A}^2))$ . As  $\mathcal{O}(U_a \times \mathbb{A}^2) = \mathbf{k}[a^{\pm 1}, b, x, y]$  and  $\mathcal{O}(U_b \times \mathbb{A}^2) = \mathbf{k}[a, b^{\pm 1}, x, y]$ , Assertion (1) follows.

(2). — As  $P = a^m b^n f \in \mathbf{k}[a, b, x]$ , we have  $b^{-n}P = a^m f \in \mathbf{k}[a, b^{\pm 1}, x]$  and  $a^{-m}P = b^m f \in \mathbf{k}[a^{\pm 1}, b, x]$ . Hence,  $b^n(y + f) = b^n y + a^{-m}P \in R_a$  and  $a^m(y - f) = a^m y - b^{-n}P \in R_b$ . This shows that the two inclusions in Assertion (2) hold. Moreover, since  $\varphi^*(b^n y + a^{-m}P) = b^n y$  and  $\psi^*(a^m y - b^{-n}P) = a^m y$ , it follows these inclusions are either both strict or both an equality.

(3). — We now assume that  $m = 1$  and  $P(0, 0, x) \neq 0$ . This implies that  $P(0, b, x) \in \mathbf{k}[b, x] \setminus b\mathbf{k}[b, x]$ . We want to prove that  $R_a \subseteq \mathbf{k}[a, b, x, ay, b^n y + a^{-1}P]$ . Writing  $P = P(0, b, x) + aS$  for some suitable  $S \in \mathbf{k}[a, b, x]$ , we obtain that  $R_a = \mathbf{k}[a^{\pm 1}, b, x, y] \cap \mathbf{k}[a, b^{\pm 1}, x, y + \frac{P_0}{ab^n}]$  and  $\mathbf{k}[a, b, x, ay, b^n y + a^{-1}P] = \mathbf{k}[a, b, x, ay, b^n y + a^{-1}P_0]$ . So, we may assume that  $P = P(0, b, x) \in \mathbf{k}[b, x] \setminus b\mathbf{k}[b, x]$ .

Let us replace  $y$  by  $\frac{y}{a}$  and define  $u = \frac{b^n y + P}{a} \in \mathbf{k}[a^{\pm 1}, b, x, y]$ . Doing so, we now need to prove that any element  $w \in \mathbf{k}[a^{\pm 1}, b, x, y] \cap \mathbf{k}[a, b^{\pm 1}, x, u]$  belongs to the ring  $\mathbf{k}[a, b, x, y, u]$ .

We write  $w = \frac{W}{b^s}$  with  $W \in \mathbf{k}[a, b, x, u] \subseteq \mathbf{k}[a, b, x, y, u]$  and  $s \geq 0$ . Using the fact that  $au = b^n y + P$ , we can rewrite  $W$  as

$$W = \sum_{i=1}^d u^i Q_i + R,$$

where  $d \geq 1$ ,  $Q_i \in \mathbf{k}[b, x, y]$  for each  $i$  and  $R \in \mathbf{k}[a, b, x, y]$ .

If  $s = 0$ , then  $w = W \in \mathbf{k}[a, b, x, y, u]$  and we have nothing to prove. Therefore, we assume that  $s > 0$ . This implies that  $W \equiv 0 \pmod{b}$ , since  $w = \frac{W}{b^s} \in \mathbf{k}[a^{\pm 1}, b, x, y]$ . Denoting by  $\widehat{P} \in \mathbf{k}[x]$ ,  $\widehat{Q}_i \in \mathbf{k}[x, y]$ ,  $\widehat{R} \in \mathbf{k}[a, x, y]$

the elements such that  $\widehat{P} \equiv P, \widehat{Q}_i \equiv Q_i, \widehat{R} \equiv R \pmod{b}$ , gives us the equality  $0 = \sum_{i=1}^d \binom{\widehat{P}}{a}^i \widehat{Q}_i + \widehat{R}$ . Multiplying this equality by  $a^d$ , we then obtain that

$$0 = \sum_{i=1}^d \widehat{P}^i a^{d-i} \widehat{Q}_i + a^d \widehat{R} = \sum_{i=0}^{d-1} \widehat{P}^{d-i} \widehat{Q}_{d-i} a^i + a^d \widehat{R} \in \mathbf{k}[a, x, y] = \mathbf{k}[x, y][a].$$

Note that  $\widehat{P} \neq 0$ , since we assumed that  $P \notin b \cdot \mathbf{k}[b, x]$ . Therefore, since all coefficients of  $a^i$  with  $i < d$  are equal to 0, we get  $\widehat{Q}_i = 0$  for all  $i = 1, \dots, d$ , and that  $\widehat{R} = 0$ .

This means that  $W$  is divisible by  $b$  and we can thus write  $w = \frac{W}{b^s} = \frac{W'}{b^{s-1}}$  with  $W' \in \mathbf{k}[a, b, x, y, u]$ . Arguing as before, we eventually conclude that  $w \in \mathbf{k}[a, b, x, y, u]$ , as desired.

The case where  $n = 1$  and  $P(0, 0, x) \neq 0$  is similar, when exchanging the roles of  $a$  and  $b$ .

(4). — We now assume that the inclusions of 2 are equalities, and prove that  $X_f$  is a trivial  $\mathbb{A}^1$ -bundle if and only if  $P(0, 0, x) = P|_{a=b=0}$  is of degree 1.

The equalities of 2 imply that the ring  $\mathcal{O}(X_f)$  of regular functions on  $X_f$  is given on the two charts  $U_a \times \mathbb{A}^2$  and  $U_b \times \mathbb{A}^2$  by

$$\mathbf{k}[a, b, x, a^m y, b^n y + a^{-m} P] \text{ and } \mathbf{k}[a, b, x, b^n y, a^m y - b^{-n} P],$$

respectively. In particular, considering the open embedding  $X_f \hookrightarrow Y$  defined in Lemma 6.1, the ring of regular functions on  $X_f$  is the restriction of the ring of regular functions on  $Y$ . Moreover, the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is the restriction of  $\pi: Y \rightarrow \mathbb{A}^2$  given by  $(a, b, x, u, v) \rightarrow (a, b)$  (see Lemma 6.1).

If  $P(0, 0, x)$  is of degree 1, then the morphism  $\pi: Y \rightarrow \mathbb{A}^2$ , is a trivial  $\mathbb{A}^2$ -bundle (Lemma 6.2), so the restriction  $\rho_f$  is also a trivial  $\mathbb{A}^2$ -bundle.

Conversely, we now assume that  $\rho_f$  is a trivial  $\mathbb{A}^2$ -bundle and prove that  $P(0, 0, x)$  is of degree 1. Let  $\chi: \mathbb{A}^2 \times \mathbb{A}_*^2 \xrightarrow{\simeq} X_f$  be an isomorphism such that  $\rho_f \circ \chi$  is the projection onto the second factor. Then  $\chi$  extends to a birational map  $\mathbb{A}^4 \dashrightarrow Y \subseteq \mathbb{A}^5$  between two affine varieties, which induces an isomorphism between their regular rings, since  $\mathcal{O}(X_f) = \mathcal{O}(Y)|_{X_f}$  and  $\mathcal{O}(\mathbb{A}^2 \times \mathbb{A}_*^2) = \mathcal{O}(\mathbb{A}^4)|_{\mathbb{A}^2 \times \mathbb{A}_*^2}$ . Thus,  $\chi$  is in fact an isomorphism  $\mathbb{A}^2 \xrightarrow{\simeq} Y$ . By Lemma 6.1, it sends  $\mathbb{A}^2 \times (0, 0)$  onto  $Y \setminus X_f = Y \cap \{a = b = 0\} = \pi^{-1}(0, 0)$ . Since  $\pi^{-1}(0, 0) \simeq \mathbb{A}^2$ , we can now conclude, by Lemma 6.2, that the polynomial  $P(0, 0, x)$  is of degree 1.  $\square$

PROPOSITION 6.4. — Let  $f = f(a, b, x) \in \mathbf{k}[a^{\pm 1}, b^{\pm 1}][x]$  and let  $m, n \geq 0$  be such that  $P(a, b, x) = a^m b^n f \in \mathbf{k}[a, b, x]$ .

- (1) If  $m = 0$  or  $n = 0$ , then the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is trivial.
- (2) Suppose that  $P(0, 0, x) \neq 0$ , i.e. that  $m$  and  $n$  are minimal with the property that  $P(a, b, x) = a^m b^n f \in \mathbf{k}[a, b, x]$ . If  $m = 1$  or  $n = 1$ , then the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is trivial if and only if  $P(0, 0, x)$  is a polynomial in  $\mathbf{k}[x]$  of degree one.

*Proof.* —

(1). — If  $m = 0$  or  $n = 0$ , then  $f \in \mathbf{k}[a, b^{\pm 1}][x]$  or  $f \in \mathbf{k}[a^{\pm 1}, b][x]$ . This implies that the corresponding  $\mathbb{A}^1$ -bundle  $X_f \rightarrow \mathbb{A}_*^2 \times \mathbb{A}^1$  is trivial (see Lemma 3.3). In particular,  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  is a trivial  $\mathbb{A}^2$ -bundle.

(2). — This follows directly from the assertions (3) and (4) in Proposition 6.3.  $\square$

*Remark 6.5.* — If  $m, n \geq 2$ , then the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  can be trivial, even if the polynomial  $P(0, 0, x)$  is not of degree one. This occurs for example in the special case where  $f(a, b, x) = a^{-3} b^{-2} (a^2 x - b x^2)$ , which corresponds to the third Vénéreau polynomial. The next example is another instance of this phenomena when  $n = m = 2$ .

*Example 6.6.* — Let  $P(a, b, x) = (a + b)x$  and  $f(a, b, x) = a^{-2} b^{-2} \times P(a, b, x) = a^{-2} b^{-2} (a + b)x$ . Then, the  $\mathbb{A}^2$ -bundle  $\rho_f: X_f \rightarrow \mathbb{A}_*^2$  corresponds to the bivariable  $\omega = a^2 x + (b - a)y$  and is thus trivial. Indeed, a straightforward calculation gives  $\alpha \circ \beta^{-1} = (x, y + f(a, b, x))$  where

$$\alpha = (\omega, a^{-2}y) \in G_a \quad \text{and} \quad \beta = (\omega, b^{-2}y - b^{-2}(a + b)x) \in G_b.$$

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