ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Johannes Schäfer

Existence and uniqueness of S^1-invariant Kähler–Ricci solitons

Tome XXXII, nº 1 (2023), p. 15–53.

https://doi.org/10.5802/afst.1726

Publication membre du centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXII, nº 1, 2023
pp. 15-53

Existence and uniqueness of S^1-invariant Kähler–Ricci solitons (*)

Johannes Schäfer [1]

Abstract. — We use the momentum construction for S^1-invariant Kähler metrics as developed by Hwang–Singer to construct new examples of steady Kähler–Ricci solitons. We also prove that these solitons are unique in their Kähler class, provided the vector field and the asymptotic behaviour are fixed.

Résumé. — Nous utilisons la construction des métriques kähleriennes S^1-invariantes de Hwang–Singer pour construire des nouveaux exemples de solitons de Kähler–Ricci. Nous montrons en outre que ces solitons sont uniques dans leur classe kählerienne, si le champ de vecteurs et les asymptotiques sont fixes.

1. Introduction

A *steady Kähler–Ricci soliton* is a Kähler manifold (M, g) whose Kähler form ω satisfies

$$\mathrm{Ric}(\omega) = -\mathcal{L}_X \omega \tag{1.1}$$

for some vector field X which is the real part of a holomorphic vector field. Solutions to (1.1) are natural generalizations of Ricci-flat metrics and arise as self-similar solutions to Ricci flow.

(*) Reçu le 15 février 2020, accepté le 29 décembre 2020.

Keywords: Kähler geometry, steady solitons, S^1-invariance.

2020 *Mathematics Subject Classification:* 53C55, 53C25, 53C21, 32L05.

(1) Universität Bonn, Mathematisches Institut, Endenicher Allee 60, 53115 Bonn, Germany — jschafer@math.uni-bonn.de

The author is financially supported by the graduate school “IMPRS on Moduli Spaces” of the Max-Planck-Institute for Mathematics in Bonn and would like to thank his advisor, Prof. Ursula Hamenstädt, for her encouragement as well as helpful discussions. Moreover, the author is grateful to Prof. Hans-Joachim Hein for his interest in this work and his comments on earlier versions of this article.

Article proposé par Vincent Guedj.

If the vector field X is non-zero, the manifold must be non-compact [19]. In general, there is no classification for steady Kähler–Ricci solitons available and only few examples are known. Even if a manifold admits a Kähler–Ricci soliton, it is not understood which subset of the Kähler cone contains further examples of Ricci solitons. It is also not clear, how many solitons there are in each Kähler class.

All known examples with $X \neq 0$ are divided into two classes. One class contains explicitly constructed solutions by using ODE methods ([4, 6, 10, 11, 12, 15, 21, 24, 26]), while the other examples are obtained by using PDE gluing methods ([2]). The explicit examples are constructed on Euclidean space or on holomorphic vector bundles over Kähler manifolds, while the gluing method produces solitons on certain crepant resolutions of orbifolds $\mathbb{C}^n/G$.

In this article, we use the momentum construction introduced by Hwang–Singer [18] to find new examples of steady Kähler–Ricci solitons. More precisely, we prove the following theorem.

Theorem 1.1. — *Let $\pi : K_M \to (M, g_M)$ be the canonical line bundle over a compact Kähler manifold. Assume that the Ricci form of g_M is positive semi-definite and has constant eigenvalues with respect to g_M. Then K_M admits a 1-parameter family of complete steady Kähler–Ricci solitons in the Kähler class $[\pi^*\omega_M]$.*

Theorem 1.1 generalises results obtained in [4, 6, 10, 11, 24, 26]. The main difference is that we do *not* assume (M, g_M) to be a Kähler–Einstein Fano manifold, but only require that $\mathrm{Ric}(\omega_M)$ has constant eigenvalues.

Under the same assumption, Hwang–Singer [18] used Calabi's ansatz to construct Kähler–Einstein metrics on line bundles. They observed that the constancy of eigenvalues is sufficient to reduce the Kähler–Einstein equation to a single ODE, which is linear after applying a certain transformation. We prove Theorem 1.1 by adapting their construction to the case of steady Kähler–Ricci solitons.

Theorem 1.1 produces new examples if the base M is a flag variety. More concretely, consider the canonical bundle over $M = \mathbb{P}(T^*\mathbb{CP}^n)$, the projectivization of the cotangent bundle $T^*\mathbb{CP}^n$. Previously, it was only known that compactly supported Kähler classes admit steady solitons ([10, 24, 26]), whereas Theorem 1.1 shows they sweep out the *entire* Kähler cone.

Another interesting feature of Hwang–Singer's construction is that it can also be applied to certain vector bundles of rank $\geqslant 2$. Then we obtain a result analogue to Theorem 1.1.

Theorem 1.2. — *Let $\pi : E \to D$ be a holomorphic vector bundle of rank m over a compact Kähler manifold (D, ω_D). Assume that E admits a Hermitian metric h such that the corresponding curvature form γ of the tautological bundle $(\mathcal{O}_{\mathbb{P}(E)}(-1), h)$ is negative semi-definite and has constant eigenvalues with respect to the Kähler metric $\omega_M = p^*\omega_D - \gamma$, where $p : M = \mathbb{P}(E) \to D$ is the natural projection. Additionally, suppose that*

$$\mathrm{Ric}(\omega_M) = -m\gamma. \tag{1.2}$$

Then E admits a 1-parameter family of complete steady Kähler–Ricci solitons in the class $[\pi^*\omega_D]$.

This can be applied to certain sums of line bundles and again, if the base is a flag variety, it constructs steady solitons in each Kähler class, generalising results in [21] and [10, Theorem 4.20].

Given a Kähler–Ricci soliton, it is an interesting question whether or not it is unique in its Kähler class. It is natural to fix a vector field for this question because there can be families of solitons as in Theorem 1.1 and 1.2 for instance. In general, this question seems to be largely open.

In the special case of Ricci-flat Kähler metrics, the question of uniqueness is studied under additional assumptions on the asymptotic behaviour of the metric ([9, 14, 16, 20]). For example, asymptotically conical Ricci-flat metrics are unique in their Kähler class [9].

In the different setting of solitons with $X \neq 0$, there are only few results such as [2]. Assuming that two steady solitons ω_1, ω_2 with the same vector field are related by $\omega_1 = \omega_2 + \sqrt{-1}\partial\bar{\partial}u$, [2, Proposition 1.2] shows that $\omega_1 = \omega_2$ provided u and its derivatives tend to zero at infinity.

In this work, we extend the previous result for the metrics constructed in Theorem 1.1 and 1.2.

Theorem 1.3. — *Let $E \to D$ be a holomorphic vector bundle satisfying the assumptions in Theorem 1.1 or 1.2 and denote the steady Kähler–Ricci solitons constructed in Theorem 1.1 or 1.2 by ω_φ. Suppose that ω is a Kähler–Ricci soliton on E with the same vector field as ω_φ such that $[\omega] = [\omega_\varphi] \in H^2(E)$. If moreover $\omega_\varphi - \omega \in C^\infty_{-\delta}(\Lambda^2 T^*E)$ for some $\delta > 2$, then $\omega_\varphi = \omega$.*

We reduce the proof of Theorem 1.3 to [2][Proposition 1.2] by proving a $\partial\bar{\partial}$-Lemma with controlled growth. Assuming that $\omega_\varphi - \omega$ is asymptotic to zero, in a suitable sense, we show that there exists a smooth function u such that $\omega_\varphi - \omega = \sqrt{-1}\partial\bar{\partial}u$ and $u \in C^\infty_{-\delta+2}(E)$, i.e. u and all its derivatives tend to zero because $2 - \delta < 0$.

The strategy for finding such a function u is analogue to [9, Section 3]. The main point is proving that all harmonic 1-forms of a certain growth behaviour are identically zero which requires non-negative Ricci curvature. We will see that this is indeed true for the metrics ω_φ constructed in Theorem 1.1 and 1.2.

This article is structured as follows. In Section 2, we recall Hwang–Singer's construction of Kähler metrics and prove Theorem 1.1. For proving Theorem 1.2, we have to make some adjustments which are explained in Section 3. The metrics are studied more closely in Section 4. Here we observe in particular that the curvature of these metrics is bounded and that the Ricci curvature is non-negative. Then, in Section 5, we prove Theorem 1.3 by studying the Laplace operator and harmonic 1-forms of the metrics constructed in Theorem 1.1 and 1.2. After this paper was submitted for publication, Conlon and Deruelle [8] posted a preprint on the arXiv containing a new existence result for steady Kähler–Ricci solitons. There is some overlap between their main result [8, Theorem A] and our Theorems 1.1 and 1.2, compare Remarks 2.6 and 3.4 below.

2. Calabi's Ansatz for line bundles

Hwang–Singer's construction combines Calabi's ansatz with ideas from symplectic geometry ([18]). If $\pi : (L, h) \to (M, \omega_M)$ denotes a Hermitian holomorphic line bundle over a Kähler manifold, then Calabi's idea ([3]) was to search for Kähler metrics of the form

$$\pi^*\omega_M + \sqrt{-1}\partial\bar{\partial}f(t). \tag{2.1}$$

Here, t denotes the logarithm of the fibre-wise norm function induced by h and f is a convex function of one variable. Instead of describing the metric (2.1) in terms of the potential f, Hwang–Singer introduced a new variable $\tau = \tau(t)$ and a function $\varphi = \varphi(\tau) : (0, \infty) \to \mathbb{R}_+$ which is related to the Legendre transformation F of f by $\varphi = 1/F''$. In particular, φ determines the metric (2.1) uniquely.

Assuming that the curvature form of h has constant eigenvalues, we will see in this section that the non-linear Kähler–Ricci soliton equation (1.1) is equivalent to a single, linear ODE in the function φ, which can be solved explicitly. This leads to a proof of Theorem 1.1. Additionally, we discuss the main examples to which Theorem 1.1 applies.

2.1. Notation and set-up

We begin by briefly recalling Calabi's construction of Kähler metrics in the special case of the canonical bundle. We follow the presentation in [18, Section 2].

Let (M^n, ω_M) be a Kähler manifold of complex dimension n and equip its canonical line bundle $\pi : K_M \to M$ with the Hermitian metric h induced by ω_M. Let γ be the curvature form of h and assume that $-\gamma \geqslant 0$, i.e. γ is negative semi-definite. Recall that γ is given by

$$\gamma = -\sqrt{-1}\partial\bar{\partial}\log h(s, \bar{s}) = -\operatorname{Ric}(\omega_M),$$

where $s : U \to K_M$ is a local holomorphic section of K_M and $\operatorname{Ric}(\omega_M)$ denotes the Ricci form of ω_M. We introduce the radial function $r : K_M \to \mathbb{R}_{\geqslant 0}$ defined by $r(v) = \sqrt{h(v, \overline{v})}$ and outside the zero section, we define a new function $t : K_M \setminus M \to \mathbb{R}$ by $t = 2\log r$. The pullback $\pi^*\gamma$ is a $\partial\bar{\partial}$-exact form on $K_M \setminus M$ and satisfies

$$\pi^*\gamma = -\sqrt{-1}\partial\bar{\partial}t. \tag{2.2}$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$\lim_{t\to-\infty} f'(t) = 0 \quad \text{and} \quad f'' > 0. \tag{2.3}$$

Then Calabi's Ansatz searches for Kähler metrics ω of the form

$$\omega = \pi^*\omega_M + \sqrt{-1}\partial\bar{\partial}f(t) = \pi^*\omega_M - f'(t)\pi^*\gamma + f''(t)\sqrt{-1}\partial t \wedge \bar{\partial}t. \tag{2.4}$$

Note that ω is defined on $K_M \setminus M$, the canonical bundle with the zero section removed, and it is positive since we assumed $-\gamma \geqslant 0$ and (2.3). Depending on the behaviour of $f(t)$ as $t \to \pm\infty$, ω can be extended to all of K_M and define a complete metric. When this can happen is explained in the next subsection. We conclude this subsection by describing the Calabi metric ω in terms of the Legendre transformation of its potential f, which is well-defined since f is convex by (2.3). We now briefly recall this transformation. Let $I = \operatorname{Im} f' \subset \mathbb{R}_+$ be the image of f' and define the new variable $\tau := f'(t) \in I$. We write $I = (0, \tau_2)$, which means that

$$\lim_{s\to-\infty} \tau(s) = \lim_{t\to-\infty} f'(t) = 0, \quad \lim_{s\to+\infty} \tau(s) = \lim_{t\to+\infty} f'(t) = \tau_2.$$

We point out that in general $\tau_2 \leqslant +\infty$, but in the case considered in subsequent sections, we have in fact that $\tau_2 = +\infty$. The Legendre transform $F : I \to \mathbb{R}$ is defined by the formula

$$f(t) + F(\tau) = t\tau.$$

One can check that F is also strictly convex, so that we can define a new function $\varphi : I \to \mathbb{R}_+$ by

$$\varphi(\tau) = \frac{1}{F''(\tau)}.$$

Then we obtain the following relations

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = f''(t) = \varphi(\tau), \quad f'''(t) = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \varphi'(\tau)\varphi(\tau). \tag{2.5}$$

In particular, (2.3) translates into

$$\varphi > 0 \quad \text{on } I = (0, \tau_2). \tag{2.6}$$

We can then express the metric ω obtained from Calabi's construction (2.4) as

$$\omega = \pi^*\omega_M - \tau\pi^*\gamma + \frac{1}{\varphi(\tau)}\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau \tag{2.7}$$

by using equations (2.4) and (2.5). The function φ is called the momentum profile of ω. We note that it is possible to reconstruct the Kähler potential f of ω from its momentum profile by

$$f(t) = \int_0^{\tau(t)} \frac{x\mathrm{d}x}{\varphi(x)}. \tag{2.8}$$

Hence, the Kähler metric given by Calabi's Ansatz (2.4) is uniquely determined by its momentum profile. We emphasize this by writing $\omega = \omega_\varphi$.

2.1.1. Completeness of ω_φ

The Kähler metric $\omega = \omega_\varphi$ given by (2.7) is a priori only defined on $K_M \setminus M$ and is in general not complete. Whether or not ω_φ extends across the zero section to a complete metric is determined by the behaviour of the momentum profile φ toward the endpoints of $I = (0, \tau_2)$. This is well-understood and there is the following well-known proposition, whose proof can be found in [18, Section 2] or [12, Section 6], for example.

Proposition 2.1. — *Let ω_φ be given by* (2.7)*. Suppose the profile $\varphi : I \to \mathbb{R}$ has a zero of integer order at each endpoint of $I = (0, \tau_2)$. Then ω_φ extends across the zero section if and only if $\varphi(0) = 0$ and $\varphi'(0) = 1$. In this case, the resulting metric on K_M is complete if and only if at the upper endpoint τ_2, one of the following conditions* (i) *and* (ii) *holds:*

(i) *The endpoint τ_2 is finite and φ vanishes at least to second order.*
(ii) *The endpoint τ_2 is infinite and φ grows at most quadratically.*

Remark 2.2. — Note that [18, Proposition 2.3] is identical with Proposition 2.1, except that Hwang and Singer require $\varphi'(0) = 2$ instead of $\varphi'(0) = 1$. This is due to the fact that our Kähler potential f is *twice* the potential function used by Hwang and Singer; compare (2.4) with [18, (1.1)].

If the metric ω_φ extends to the total space of K_M, we would like to identify its de Rham cohomology class. Since we assumed (2.6), i.e. $I = (0, \tau_2)$, it follows immediately that $[\omega_\varphi] = [\pi^*\omega_M] \in H^2(K_M)$. We refer to the class $[\pi^*\omega_M]$ as the Kähler class of ω_φ.

More generally, we define a Kähler class on K_M simply to be a class in $H^2(K_M)$ containing positive $(1,1)$ forms and the Kähler cone is the set of all such Kähler classes. Using this definition, the projection map $\pi^* : H^2(M) \to H^2(K_M)$ identifies the Kähler cone of the compact base M with the Kähler cone of K_M. Indeed, given a Kähler form on K_M, its restriction to M clearly is a Kähler form on M. Conversely, given a Kähler form ω_M on M, Calabi's Ansatz always produces a positive $(1,1)$ form in the class $[\pi^*\omega_M]$, for example consider ω_φ with $\varphi(\tau) = \tau$, which extends to K_M by Proposition 2.1.

2.1.2. The Ricci form

In this paragraph, we provide a description of the Ricci-form of ω_φ. The computations can be found, for example, in [18, Section 2.1]. Denote the Kähler metric of ω_M by g_M and the curvature form of (K_M, h) by γ. It gives rise to an endomorphism $B : T^{1,0}M \to T^{1,0}M$ of the holomorphic tangent bundle, which is locally defined by $B := g_M^{-1}\gamma = g_M^{\bar{k}i}\gamma_{j\bar{k}}$. As in Theorem 1.1, we assume from now on that the eigenvalues of B are constant over M. This condition is sufficient to reduce the soliton equation (1.1) to an ODE. These conditions guarantee that the function $Q : I \times M \to \mathbb{R}_+$ defined by

$$Q = \det\left(g_M^{-1}(\omega_M - \tau\gamma)\right) = \det\left(\mathrm{Id} - \tau B\right) \tag{2.9}$$

only depends on the parameter τ, i.e. is constant over M. Also observe that Q is a positive function because $-\gamma \geqslant 0$ and $\tau \geqslant 0$. Q naturally appears in the computation of $\mathrm{Ric}(\omega_\varphi)$. Indeed, the Ricci form is given by

$$\mathrm{Ric}(\omega_\varphi) = \pi^* \mathrm{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}\pi^*\gamma - \frac{1}{\varphi}\left(\frac{(\varphi Q)'}{Q}\right)' \sqrt{-1}\partial\tau \wedge \bar{\partial}\tau, \tag{2.10}$$

see [18, (2.14)].

2.2. Reduction to an ODE

We use the previously derived formula for the Ricci curvature to show that the Kähler–Ricci soliton equation is equivalent to an ODE in the function $\varphi(\tau)$. Our presentation is similar to [12, Section 4].

By definition, the soliton vector field X must be the real part of a holomorphic vectorfield, i.e. $\mathcal{L}_X J = 0$. On the line bundle K_M, there is a natural choice for X, which we now describe. K_M admits a holomorphic $\mathbb{C}^*$-action by fibre-wise multiplication and the corresponding holomorphic vector field Z is given by $Z = z_0 \frac{\partial}{\partial z_0}$, where z_0 denotes the fibre coordinate of K_M. In terms of the radial function t defined at the beginning of this section, we can write Z as

$$Z = \operatorname{Re} Z + \sqrt{-1} \operatorname{Im} Z = \frac{\partial}{\partial t} - \sqrt{-1} J \frac{\partial}{\partial t}. \tag{2.11}$$

So it is natural to set $X := \mu \operatorname{Re} Z = \mu \frac{\partial}{\partial t}$ for some constant $0 \neq \mu \in \mathbb{R}$. Before deriving the ODE, we need to calculate the following Lie-derivative:

$$\mathcal{L}_X \omega_\varphi = \mathrm{d}(\iota_X \omega_\varphi) = \sqrt{-1}\partial\bar{\partial}(\mathcal{L}_X f)(t) = \mu\sqrt{-1}\partial\bar{\partial} f'(t). \tag{2.12}$$

Here, we used $2\sqrt{-1}\partial\bar{\partial} = \mathrm{d}J\mathrm{d}$ and $\mathcal{L}_X J = 0$ to obtain the second equality. We shall write out equation (2.12) in terms of fibre and base direction, as we did for the Ricci-form in (2.10):

$$-\mathcal{L}_X \omega_\varphi = \mu\varphi(\tau)\pi^*\gamma - \mu\frac{\varphi'}{\varphi}(\tau)\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau. \tag{2.13}$$

Now we are in position to see by comparing (2.10) and (2.13) that the soliton equation (1.1) for ω_φ is equivalent to the following two equations

$$\operatorname{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}(\tau)\gamma = \mu\varphi(\tau)\gamma \tag{2.14}$$

$$\left(\frac{(\varphi Q)'}{Q}\right)'(\tau) = \mu\varphi'(\tau). \tag{2.15}$$

Since $\operatorname{Ric}(\omega_M) = -\gamma$, we see that differentiating (2.14) gives (2.15), so that we proved the following Lemma:

Lemma 2.3. — *Suppose that ω_φ is a Kähler metric with momentum profile φ. Then (1.1) with $X = \mu\frac{\partial}{\partial t}$ is equivalent to the following equation:*

$$\varphi'(\tau) + \left(\frac{Q'}{Q}(\tau) - \mu\right)\varphi(\tau) = 1 \tag{2.16}$$

For the rest of this paragraph, we study the solution φ to Equation (2.16). This is a linear ODE of the form $y' + p(x)y = q(x)$, which has an explicit one-parameter family of solutions given by

$$y = \exp\left(-\int p(x)\mathrm{d}x\right)\left(\int q(x)\exp\left(\int p(x)\mathrm{d}x\right)\mathrm{d}x + K\right). \tag{2.17}$$

Applying (2.17) to (2.16), we have

$$\varphi(\tau) = \frac{e^{\mu\tau}}{Q(\tau)}\left(\int_0^\tau e^{-\mu x}Q(x)\mathrm{d}x + K\right), \tag{2.18}$$

where $K \geqslant 0$ is determined by the initial value $\lim_{\tau\to 0}\varphi(\tau)$. Justified by (ii) of Proposition 2.1, we will assume that $K = 0$.

One can compute the integral (2.16) explicitly in terms of the coefficients $b_j \geqslant 0$ of the polynomial $Q(\tau) = \det(\mathrm{Id} - \tau B) = b_k\tau^k + b_{k-1}\tau^{k-1} + \cdots + b_0$. Note that the degree k of Q could be less than n since B is allowed to have zero eigenvalues. In fact, it is straight forward to see that

$$\varphi(\tau) = \nu(0)\frac{e^{\mu\tau}}{Q(\tau)} - \frac{\nu(\tau)}{Q(\tau)}, \tag{2.19}$$

where ν is given by

$$\nu(\tau) = \sum_{j=0}^{k}\sum_{l=0}^{j} b_j\frac{j!}{l!}\frac{\tau^l}{\mu^{j+1-l}}. \tag{2.20}$$

We point out that the explicit expression for ν is not relevant, but rather that it has the form

$$\varphi(\tau) = \nu(0)\frac{e^{\mu\tau}}{Q(\tau)} + \frac{(-b_k/\mu)\tau^k + R_{k-1}(\tau)}{Q(\tau)} \tag{2.21}$$

for a polynomial R_{k-1} of degree $k-1$. Hence, we found an explicit solution for the soliton ODE (2.16). Also note that φ is defined on $[0, +\infty)$ since $Q(0) > 0$. Moreover, φ is clearly positive on $(0, +\infty)$.

With these observations, we can now finish the proof of Theorem 1.1.

2.3. Proof of Theorem 1.1

Let $K_M \to (M, g_M)$ be the canonical bundle whose semi-negative curvature form $\gamma = -\operatorname{Ric}(\omega_M)$ has constant eigenvalues w.r.t. g_M. Suppose $\varphi : (0, +\infty) \to \mathbb{R}$ is given by (2.18) with $K = 0$ and, as before, let ω_φ be defined by (2.7). Since $\varphi(\tau) > 0$ for all $\tau > 0$, ω_φ defines a Kähler metric and hence is a steady Kähler–Ricci soliton by Lemma 2.3. We note that these metrics can only be complete if $\mu < 0$. This can be proven similarly to [11, Lemma 5.1].

Hence we assume $\mu < 0$. From (2.21), we have the following asymptotic behaviour for large τ:

$$\varphi(\tau) = -\frac{1}{\mu} + O(1/\tau). \tag{2.22}$$

Also recall that φ and the potential f are related by

$$\frac{\mathrm{d}f'}{\mathrm{d}t}(t) = \varphi(f'(t)). \tag{2.23}$$

Using (2.23) together with (2.22), we conclude that the corresponding potential $f(t)$ is indeed defined for all $t \in \mathbb{R}$, i.e. ω_φ is defined on $K_M \setminus M$.

It remains to check that ω_φ extends across the zero section and defines a complete metric as $t \to +\infty$. By the first part of Proposition 2.1, ω_φ extends provided $\varphi(0) = 0$ and $\varphi'(0) = 1$. Since we assumed $K = 0$ in (2.18), we have $\varphi(0) = 0$. Plugging this into (2.16) gives $\varphi'(0) = 1$, as desired. The completeness as $t \to +\infty$ follows immediately from the asymptotic expansion (2.22) and (ii) of Proposition 2.1.

2.4. Examples

Theorem 1.1 immediately recovers all known examples of steady Kähler–Ricci solitons on the total space of line bundles ([4, 6, 10, 24, 26]). In these cases, the base is a product of Kähler–Einstein manifolds and the considered Kähler classes are represented by convex combinations of Kähler–Einstein metrics on each factor. If the base manifold is a flag variety, Theorem 1.1 produces examples, which have not been mentioned before. In this case, steady solitons sweep out the entire Kähler cone.

Example 2.4 (Products). — Let (M_i, ω_i), $i = 1, \dots, r$ be Kähler–Einstein manifolds with non-negative scalar curvature and denote their canonical bundles by $K_{M_i} \to M_i$. We consider the bundle

$$K_M = p_1^* K_{M_1} \otimes \cdots \otimes p_r^* K_{M_r} \to M := M_1 \times \cdots \times M_r,$$

where $p_i : M \to M_i$ is the projection. Then Theorem 1.1 applies and gives a complete steady soliton in each Kähler class of the form $\sum_{i=1}^r \alpha_i [p_i^* \omega_i] \in H^2(M)$ with $\alpha_i > 0$.

The case $r = 1$ was first considered in [4] and [6] for $M = \mathbb{CP}^n$ and in [24] for a general Kähler–Einstein Fano manifold. For $r > 1$, these solitons are found in [10, Theorem 4.20].

Example 2.5 (Flag varieties). — Let G be a complex semisimple Lie group, $P \subset G$ a parabolic subgroup and $K \subset G$ a maximal compact subgroup. Then K acts transitively on the flag manifold $M = G/P$. It is well-known that M admits a K-invariant complex structure so that its anti canonical bundle is ample, compare [1, Chapter 8] for example. The previously mentioned results only produce solitons on the canonical bundle K_M whose Kähler class is a multiple of $[\pi^* c_1(M)]$. In general, however, $H^2(M)$ is not spanned by $[c_1(M)]$. We claim that *every* Kähler class admits a steady Kähler–Ricci soliton. Indeed, every Kähler class on M admits a K-invariant Kähler form ω_K whose Ricci form $\mathrm{Ric}(\omega_K)$ is also K-invariant. This means that the eigenfunctions of $\mathrm{Ric}(\omega_K)$ w.r.t. ω_K must be K-invariant and hence constant since K acts transitively on M. So Theorem 1.1 can be applied and proves the existence of a steady soliton in the class $[\pi^*\omega_K]$.

Remark 2.6. — The new metrics in Example 2.5 can also be obtained from the recent result [8, Theorem A], which was posted after this paper was uploaded to the arXiv.

3. Calabi metrics on vector bundles

Given a vector bundle $E \to D$, Hwang–Singer's idea was to apply their construction to the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$, the projectivization of E ([18, Section 3.2]). In this section, we explain the changes which are necessary to prove Theorem 1.2 and provide some examples.

The main difference is that one has to choose a new background metric on $\mathbb{P}(E)$, with respect to which the eigenvalues of the curvature form are computed. Then the discussion of the previous section can be applied and again, the soliton equation (1.1) reduces to a simple ODE. In this new setting, however, the function Q defined by (2.9) will have zeros at $\tau = 0$, so there are some details which have to be checked.

3.1. Constructing a Kähler metric

As in [18, Section 3.2], we explain how to adapt the machinery from the previous section to the tautological line bundle.

Let $\pi : E \to (D, \omega_D)$ be a holomorphic vector bundle of rank $m \geqslant 2$ equipped with an Hermitian metric h and assume that the Kähler manifold D has complex dimension d. As in the case of line bundles, we define r :

$E \to \mathbb{R}_{\geqslant 0}$ to be the radial function induced by h and let $t = \log r^2$. Then Calabi's Ansatz has the form

$$\omega = \pi^*\omega_D + \sqrt{-1}\partial\bar{\partial}f(t).$$

By construction, the projectivization of E is naturally a fibre bundle $p : \mathbb{P}(E) \to D$, with fibre isomorphic to $\mathbb{CP}^{m-1}$. Recall that the natural map $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset p^*E \to E$ identifies $\mathcal{O}_{\mathbb{P}(E)}(-1) \setminus \mathbb{P}(E) \cong E \setminus D$. By abuse of notation, we denote the bundle projection of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ also by π, so that we have a commuting diagram

$$\begin{tikzcd} L := \mathcal{O}_{\mathbb{P}(E)}(-1) \arrow[r] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ M := \mathbb{P}(E) \arrow[r, "p"] & D \end{tikzcd} \tag{3.1}$$

In the notation from the previous section, let us denote the complex dimension of M by n, i.e. $n = d+m-1$. Via the natural identification $L\backslash M \cong E\backslash D$, h induces a Hermitian metric on L, which we also simply denote by h. Hence, we can view r as a function on L and, if γ is the curvature form of (L, h), we have as before $\pi^*\gamma = -\sqrt{-1}\partial\bar{\partial}t$ with $t = \log r^2$. We again assume that $-\gamma \geqslant 0$. Then we are looking for metrics of the form

$$\omega_\varphi = \pi^*\omega_D - f'(t)\pi^*\gamma + f''(t)\sqrt{-1}\partial t \wedge \bar{\partial}t \tag{3.2}$$

where we require that $f : \mathbb{R} \to \mathbb{R}$ satisfies (2.3) to obtain a positive form. As before, we set $\tau := f'(t)$ and define $\varphi : (0, \tau_2) \to \mathbb{R}_+$ by (2.5), so that it also satisfies (2.6). Hence, ω_φ can also be expressed as in (2.7).

For the computation of Ricci curvature below, we need to choose a background Kähler metric ω_M on M. Define

$$\omega_M = p^*\omega_D - \gamma, \tag{3.3}$$

which is clearly positive in base direction of the fibration $p : \mathbb{P}(E) \to D$. To see that ω_M is positive in fibre direction, we note that $-\gamma$ restricts to the Fubini-Study metric on each fibre $\cong \mathbb{CP}^{m-1}$.

The Ricci form

The calculation is in principle the same as in the line bundle case, but the polynomial Q does have zeros. Let $B = g_M^{-1}\gamma$ be the curvature endomorphism of γ, where g_M is the metric with Kähler form given by (3.3) and assume that the eigenvalues of B are constant over M. Then we define a function Q by

$$Q = \det(g_M^{-1}(p^*\omega_D - \tau\gamma)), \tag{3.4}$$

which can be viewed as a function $Q : (0, \tau_2) \to \mathbb{R}_{\geqslant 0}$. Indeed, we can write

$$g_M^{-1}\left(p^*\omega_D - \tau\gamma\right) = g_M^{-1}\left(\omega_M - (\tau-1)\gamma\right) = \mathrm{Id} - (\tau-1)B, \tag{3.5}$$

so that Q is constant over M, i.e. it only depends on τ. If $\beta_1, \dots, \beta_n$ are the eigenvalues of B, we must have $\beta_{d+1} = \cdots = \beta_n = -1$ by the definition of ω_M and $\beta_1, \dots, \beta_d \leqslant 0$ by assumption. From (3.5), we conclude that Q is given by

$$Q(\tau) = \tau^{n-d} \prod_{j=1}^{d} (1 + \beta_j - \tau\beta_j) = \tau^{n-d}\widehat{Q}(\tau), \tag{3.6}$$

for some polynomial $\widehat{Q}$. Since $p^*\omega_D$ is positive in base direction, we conclude from (3.5) that $1 + \beta_j > 0$ for all $j = 1, \dots d$. Hence, $\widehat{Q}(0) > 0$ and Q has a zero at $\tau = 0$ of order $n - d = m - 1$.

As in (2.10), one can find the following expression for the Ricci form:

$$\mathrm{Ric}(\omega_\varphi) = \pi^* \mathrm{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}\pi^*\gamma - \frac{1}{\varphi}\left(\frac{(\varphi Q)'}{Q}\right)' \partial\tau \wedge \bar{\partial}\tau. \tag{3.7}$$

3.2. The ODE

The natural $\mathbb{C}^*$-action on E by biholomorphisms induces a holomorphic vector field Z. On $L \setminus M$, which is the tautological bundle with the zero section removed, the real part of Z is given by $\mathrm{Re}\, Z = \partial/\partial t$, so we are looking for Ricci solitons with vector field $X = \mu\partial/\partial t$. Again, we find

$$-\mathcal{L}_X\omega_\varphi = -\mu\sqrt{-1}\partial\bar{\partial}f'(t) = \mu\varphi(\tau)\pi^*\gamma - \mu\frac{\varphi'}{\varphi}(\tau)\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau. \tag{3.8}$$

Combining (3.2) with (3.7) and (3.8), one can check that the soliton equation (1.1) is equivalent to

$$\mathrm{Ric}(\omega_M) = c\gamma \tag{3.9}$$

$$\varphi'(\tau) + \left(\frac{Q'}{Q}(\tau) - \mu\right)\varphi(\tau) = -c \tag{3.10}$$

for some integration constant $c \in \mathbb{R}$. In fact, we must have $c = -m$ since the first Chern class of $M = \mathbb{P}(E)$ is given by

$$c_1(M) = -mc_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) + p^*c_1(E) + p^*c_1(D). \tag{3.11}$$

Equation (3.10) has the same form as (2.16), but with a different Q. Hence, the solution φ is given by

$$\varphi(\tau) = \frac{e^{\mu\tau}}{Q(\tau)}\left(\int_0^\tau me^{-\mu x}Q(x)\mathrm{d}x\right), \tag{3.12}$$

if we assume the integration constant to be zero.

We end this section by studying the solution φ. Let us write $Q(\tau) = b_{k+n-d}\tau^{k+n-d} + \cdots + b_{n-d}\tau^{n-d}$ with coefficients $b_j \geqslant 0$ for $j = 1+n-d, \ldots, k+n-d$ and $b_{n-d} = \widehat{Q}(0) > 0$. Adapting (2.19) and (2.20) to this case, we obtain

$$\varphi(\tau) = \nu(0)\frac{e^{\mu\tau}}{Q(\tau)} - \frac{\nu(\tau)}{Q(\tau)} \tag{3.13}$$

as well as

$$\nu(\tau) = m\sum_{j=n-d}^{k+n-d}\sum_{l=0}^{j} b_j \frac{j!}{l!}\frac{\tau^l}{\mu^{j+1-l}}. \tag{3.14}$$

A priori, φ given by (3.12) is defined on the interval $(0, +\infty)$ and because $Q(0) = 0$ one needs to check that φ and its derivatives have a limit as $\tau \to 0$. To see that this is the case, note that we can rewrite (3.14) as

$$\nu(0)e^{\mu\tau} - \nu(\tau) = m\sum_{j=n-d}^{k+n-d} b_j \frac{j!}{\mu^{j+1}}\sum_{l=j+1}^{\infty}\frac{(\mu\tau)^l}{l!},$$

i.e. $\tau^{-(n-d)}(\nu(0)e^{\mu\tau} - \nu(\tau))$ tends to zero as $\tau \to 0$. Since Q vanishes of order $n-d$ at $\tau = 0$, we then deduce from (3.13) that $\lim_{\tau\to 0}\varphi = 0$. Similarly, it follows that all derivatives of φ have a limit as $\tau \to 0$.

3.3. Proof of Theorem 1.2

The proof is now analogue to Section 2.3. The only part that might a priori be different is the extension of ω_φ to a complete metric on E. However, one can check that Proposition 2.1 also applies to the vector bundle case, see [18, Lemma 3.7].

As before, one can check that φ has the behaviour required by Proposition 2.1. Indeed, one can compute that $\varphi(0) = 0$ and $\varphi'(0) = 1$, as desired. Sending $\tau \to +\infty$, we conclude the following asymptotic expansion from (3.12) and (3.13)

$$\varphi(\tau) = -\frac{m}{\mu} + O(1/\tau), \tag{3.15}$$

and so we obtain a complete metric on the total space E.

3.4. Examples

We briefly discuss three different situations to which Theorem 1.2 applies. New examples of steady solitons are given in Example 3.2.

Example 3.1 (Complex plane). — We let D be a single point and $E \cong \mathbb{C}^n$ be the trivial bundle over D. Let h be the Euclidean metric on E, so that $\omega_M = -\gamma$ is the Fubini-Study metric on $M = \mathbb{CP}^{m-1}$. This is the situation first studied in [4].

Example 3.2 (Sum of line bundles). — Let (D, ω_D) be a Kähler–Einstein Fano manifold of Fano index m. Define $L := K_D^{1/m}$ and consider the m-fold sum of L with itself, i.e. $E = L \otimes \mathbb{C}^m$. Then we have $M = \mathbb{P}(E) = \mathbb{CP}^{m-1} \times D$ and

$$\mathcal{O}_{\mathbb{P}(\mathbb{E})}(-1) = p_1^* \mathcal{O}_{\mathbb{CP}^{m-1}}(-1) \otimes p_2^* L,$$

where p_1, p_2 denote the projections onto the first and second factor of M, respectively. Let ω_{FS} be the Fubini-Study metric on $\mathbb{CP}^{m-1}$, so that $\gamma = -p_1^* \omega_{FS} - 1/m p_2^* \operatorname{Ric}(\omega_D)$ is the curvature form of $\mathcal{O}_{\mathbb{P}(\mathbb{E})}(-1)$, and define $\omega_M = p_2^* \omega_D - \gamma$. Then we clearly have

$$\operatorname{Ric}(\omega_M) = m p_1^* \omega_{FS} + p_2^* \operatorname{Ric}(\omega_D) = -m\gamma, \tag{3.16}$$

since ω_D is Kähler–Einstein. Moreover, the eigenvalues of $\operatorname{Ric}(\omega_M)$ w.r.t. ω_M are constant, so that Theorem 1.2 can be applied. These examples of steady solitons are obtained in [21, Theorem 2.1] and [10, Theorem 4.20].

If the base $D = G/P$ is a flag manifold for G a complex semisimple Lie group and $P \subset G$ a parabolic subgroup, one can find steady solitons in *every* Kähler class, similarly as in Example 2.5.

To see this, assume that ω_D represents a given Kähler class (not necessarily the first Chern class of D). We can pick ω_D to be K-invariant, where $K \subset G$ is a maximal compact subgroup. Since $\operatorname{Ric}(\omega_D)$ is also K-invariant, the form $-\gamma = p_1^* \omega_{FS} + 1/m p_2^* \operatorname{Ric}(\omega_D)$ is invariant under the diagonal action of $SU(m) \times K$ and also positive.

We claim that $\operatorname{Ric}(-\gamma) = -m\gamma$. By [1, Theorem 8.2], we know that there exists a $SU(m) \times K$-invariant Kähler–Einstein metric $\omega_{\mathrm{KE}} \in c_1(\mathbb{CP}^{m-1} \times D)$. Also recall that the Ricci forms of *all* $SU(m) \times K$-invariant Kähler metrics agree, i.e. $\operatorname{Ric}(-\gamma) = \operatorname{Ric}(\omega_{\mathrm{KE}})$. Since $-m\gamma$ and ω_{KE} are in the same Kähler class, we deduce from the uniqueness part of Calabi's conjecture that $-m\gamma = \omega_{\mathrm{KE}} = \operatorname{Ric}(-\gamma)$.

As the form $\omega_M = p_2^*\omega_D - \gamma$ is also invariant under $SU(m) \times K$, we conclude

$$\mathrm{Ric}(\omega_M) = \mathrm{Ric}(-\gamma) = -m\gamma,$$

and hence the assumptions in Theorem 1.2 are satisfied.

Example 3.3 (Cotangent bundle of $\mathbb{CP}^d$*).* — Let $D = \mathbb{CP}^d$ be projective space equipped with the Fubini-Study metric and consider $E = T^*\mathbb{CP}^d$, the cotangent bundle of $\mathbb{CP}^d$. E is naturally a $SU(d+1)$-homogeneous vector bundle, where the fibre action is given by the coadjoint action of $SU(d+1)$ on its Lie algebra. Since $\mathbb{CP}^d$ is a rank 1 symmetric space, the induced action of $SU(d+1)$ on $M = \mathbb{P}(E)$ is transitive. Verifying the assumptions of Theorem 1.2 is now similar to the previous Example. These steady solitons on $T^*\mathbb{CP}^d$ are of cohomogeneity one and are contained in [10, Section 5].

Remark 3.4. — All the previous examples can also be constructed from [8, Theorem A].

4. Properties of ω_φ

In this short section, we study curvature properties of the previously constructed metric ω_φ. We show that ω_φ has bounded curvature and that its Ricci curvature is non-negative. Moreover, we obtain estimates on the growth of the function f and its derivatives.

Recall that $f = f(t)$ is the Kähler potential of ω_φ as defined in (2.4) and $\varphi = \varphi(\tau)$ is its momentum profile, see (2.7). If ω_φ is a steady Kähler–Ricci soliton constructed in Theorem 1.1 or Theorem 1.2, then φ satisfies (2.16) or (3.10), respectively. This ODE is in turn determined by the polynomial $Q = Q(\tau)$ defined by either (2.9) or (3.4). The statements in this section mainly reduce to understanding Q and how it effects the asymptotic behaviour of φ, compare (2.19) or (3.13) depending on the rank of the underlying vector bundle.

We begin by considering the Ricci curvature of ω_φ. More precisely, we prove the following theorem, which we need in the subsequent section. It generalises the observation made in [26, Case 7].

Theorem 4.1. — *The complete steady Kähler–Ricci solitons constructed in Theorem 1.1 and 1.2 have non-negative Ricci curvature. Moreover, if the curvature form $-\gamma$ is positive definite, then the Ricci curvature is positive away from the zero section.*

Proof. — First, we consider the solitons constructed on line bundles in Theorem 1.1. Let ω_φ be the Kähler metric given by (2.7) with φ satisfying (2.16) and $\varphi(0)=0$. Recall that the Ricci curvature is given by

$$\operatorname{Ric}(\omega_\varphi) = -\mathcal{L}_X(\omega_\varphi) = \mu\varphi(\tau)\pi^*\gamma - \mu\frac{\varphi'}{\varphi}\sqrt{-1}\partial\tau\wedge\bar\partial\tau.$$

Since $\varphi(0)=0$, $\varphi>0$ on $(0,\infty)$, and $\mu\gamma\geqslant 0$, we only need to show that $\varphi'>0$. To see that this is the case, we define a function

$$H(\tau) := \frac{Q^2}{Q'-\mu Q}e^{-\mu\tau} - \int_0^\tau e^{-\mu x}Q(x)\mathrm{d}x. \tag{4.1}$$

Using the ODE (2.16), it is straight forward to prove that $\varphi'\geqslant 0$ iff $H\geqslant 0$. As $H(0)>0$ for Q given by (2.9), we are done if we can show that $H'\geqslant 0$. From the definition of H, we compute

$$H'(\tau) = e^{-\mu\tau}\frac{Q}{(Q'-\mu Q)^2}\left((Q')^2 - QQ''\right), \tag{4.2}$$

so that $H'\geqslant 0$ if and only if $(Q')^2-QQ''\geqslant 0$. The later condition can be checked easily starting from the explicit expression for Q. Indeed, let $\beta_1,\dots,\beta_n$ be the eigenvalues of the endomorphism $B=g_M^{-1}\gamma : T^{1,0}M\to T^{1,0}M$, and write

$$Q(\tau) = \det(\operatorname{Id} - \tau B) = \prod_{j=1}^n (1-\beta_j\tau). \tag{4.3}$$

Then we have

$$\frac{(Q')^2 - QQ''}{Q^2} = \sum_{j=1}^n \frac{\beta_j^2}{(1-\beta_j\tau)^2} \geqslant 0, \tag{4.4}$$

as required. For the second statement, it suffices to observe that $\varphi'(\tau)>0$ if and only if $(Q')^2-QQ''>0$, which is certainly true if $\gamma<0$. This proves Theorem 4.1 for line bundles. The arguments for the metrics in Theorem 1.2 are analogous. It also reduces to showing that $(Q')^2-QQ''\geqslant 0$, where Q is this time given by (3.6). $\square$

Note that the non-negativity of Ricci curvature can also be expressed in terms of the potential function f. In particular, we have the following

Corollary 4.2. — *Let ω_φ be a steady Kähler–Ricci soliton constructed in Theorem 1.1 or 1.2 and let $f=f(t)$ be defined by* (2.4) *or* (3.2), *respectively. Then f'' is monotone increasing.*

Proof. — Recall from (2.12) or (3.8) that we have

$$-\mathcal{L}_X\omega_\varphi = -\mu\sqrt{-1}\partial\bar\partial f'(t) = \mu f''(t)\pi^*\gamma - \mu f'''(t)\sqrt{-1}\partial t\wedge\bar\partial t,$$

and so $\mathrm{Ric}(\omega_\varphi) = -\mathcal{L}_X\omega_\varphi$ can only be non-negative if $f'''(t) \geqslant 0$ since $\mu < 0$. Thus, Theorem 4.1 implies that f'' is increasing. □

We end this section by pointing out some growth properties of the potential function f.

Lemma 4.3. — *Let ω_φ be a steady Kähler–Ricci soliton constructed in Theorem 1.1 or 1.2 and let $f = f(t)$ be related to φ by* (2.8). *Then there is a constant $C > 0$ such that for all $t \geqslant C$, we have*

$$C^{-1} \leqslant f''(t) \leqslant C \quad \text{and} \quad C^{-1}t \leqslant f'(t) \leqslant Ct. \tag{4.5}$$

Moreover, for all $j \in \mathbb{N}_0$ and $t \geqslant C$

$$C^{-1}(1+f'(t))^{-j} \leqslant |f^{(2+j)}(t)| \leqslant C(1+f'(t))^{-j}. \tag{4.6}$$

Proof. — First note that the bound on $f''(t)$ in (4.5) implies the bound on $f'(t)$ after integrating the parameter t, so we only need to find $C > 0$ such that

$$C^{-1} \leqslant f''(t) \leqslant C \tag{4.7}$$

for all $t \geqslant C$. Translating the problem into bounding $\varphi(\tau)$, we recall from (2.5) that

$$\tau = \tau(t) = f'(t) \quad \text{and} \quad \varphi(\tau(t)) = f''(t). \tag{4.8}$$

Since $f'(t)$ is positive and increasing, we can choose a $C \geqslant 1$ such that the following estimate

$$\tau(t) = f'(t) \geqslant C^{-1} \tag{4.9}$$

holds for all $t \geqslant C$. Then we recall the asymptotic expansion (3.15)

$$\varphi(\tau(t)) = -\frac{m}{\mu} + O(1/\tau(t))$$

with $\mu < 0$ implying that $\varphi(\tau(t))$ is uniformly bounded from above because of (4.9). Together with (4.8), this proves the upper bound for $f''(t)$ in (4.7). For the lower bound, note that $f''(t) > 0$ is increasing and thus is bounded from below by some positive constant if $t \geqslant C$. Inequality (4.7) now follows, and so does (4.5).

Next, consider the case $j > 0$, i.e. we estimate $f^{(2+j)}(t)$. Differentiating (4.8) and using the chain rule, we see that

$$f'''(t) = \varphi'(\tau(t))\frac{\mathrm{d}\tau}{\mathrm{d}t}(t) = \varphi'(\tau(t)) \cdot f''(t).$$

Taking further derivatives of this equation, we conclude that $f^{(2+j)}$ can be written as

$$f^{(2+j)} = \sum_\alpha c_\alpha \cdot \varphi^{(\alpha_1)} \cdot \ldots \cdot \varphi^{(\alpha_i)} \cdot (f'')^j, \tag{4.10}$$

where the sum is over all multi-indices α with $\alpha_1 + \ldots + \alpha_i = j$ and c_α are constants only depending on the multi-index α. Since $f''(t)$ satisfies (4.7), it is sufficient to estimate derivatives of φ. In fact, we have for all $\beta \in \mathbb{N}$ that

$$C^{-1}\tau^{-\beta} \leqslant |\varphi^{(\beta)}(\tau)| \leqslant C\tau^{-\beta}, \tag{4.11}$$

because $\varphi(\tau)$ behaves asymptotically like a rational function of the form P/Q with polynomials $P(\tau), Q(\tau)$ having the same degree, see (2.21). Substituting $\tau(t) = f'(t)$ in (4.11) and combining the resulting estimate with (4.10), we finally obtain (4.6) as desired. □

The important point about Lemma 4.3 is estimate (4.5), i.e. that $f''(t)$ behaves like a constant and $f'(t)$ growths roughly like the function t in the limit $t \to \infty$. This will be crucial in the next section because we want to understand the asymptotic geometry of ω_φ.

Another interesting consequence of Lemma 4.3 is that the metrics ω_φ have bounded curvature and positive injectivity radius.

Lemma 4.4. — *The curvature tensor of the steady solitons constructed in Theorem 1.1 and 1.2 is uniformly bounded and each of these metrics has positive injectivity radius.*

Proof. — It is straight forward to see that the first claim reduces to bounding $f''(t), f'''(t)$ and $f^{(4)}(t)$, where φ and f are related by (2.8), so we focus on the second one. According to [7, Theorem 4.7], the lower bound on the injectivity radius follows if we can bound the volume of all unit balls uniformly from below. For this, recall that the function t identifies $E\backslash D \cong \mathbb{R}\times S$, where S is the S^1-bundle associated to $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$, see (3.1). Under this identification, the metric g_φ admits the following decomposition on $\mathbb{R} \times S$

$$g_\varphi = f''(t)\left(\mathrm{d}t^2 + (J\mathrm{d}t)^2\right) + f'(t)\pi^*\widehat{g} + \pi^* g_D, \tag{4.12}$$

where J denotes the complex structure on E, and $\widehat{g}$, g_D are the (2,0) tensors associated to $-\gamma$, ω_D, respectively. By compactness of D, we only have to consider the set $\{t \gg 1\}$, on which g_φ is uniformly equivalent to the metric

$$g_t := \mathrm{d}t^2 + (J\mathrm{d}t)^2 + t\pi^*\widehat{g} + \pi^* g_D, \tag{4.13}$$

compare Lemma 4.3. Let us further denote $g_{S^1} := (J\mathrm{d}t)^2$ and rescale g_t by some fixed constant so that the diameter $\mathrm{diam}(S, g_{S^1})$ of each S^1-fibre satisfies $\mathrm{diam}(S, g_{S^1}) = 1/4$. It then suffices to bound the volume of unit balls w.r.t. g_t on the set $\{t \gg 1\}$ uniformly from below.

Let $x \in E$ with $t(x) \gg 1$ and denote the unit ball of g_t around x by $B_{g_t}(x,1)$. We introduce families of metrics on M and S by declaring

$$g_{M,\tau} := \tau\widehat{g} + p^* g_D$$
$$g_{S,\tau} := g_{S^1} + \pi^* g_{M,\tau}$$

for each $\tau \geqslant 1$, where $p : M \to D$ is the projection as in (3.1) and $\pi : S \to M$. In particular, the projection π becomes a Riemannian submersion $\pi : (S, g_{S,\tau}) \to (M, g_{M,\tau})$. Using this notation and writing $x = (t(x), y)$, we obtain the following inclusion

$$B := [t(x) - 1/2, t(x)] \times B_{g_{S,t(x)}}(y, 1/2) \subset B_{g_t}(x, 1).$$

This is an immediate consequence of the decomposition (4.13) together with the fact that for all $p \in B$ we have $t(p) \leqslant t(x)$ and $g_{S,\tau_0} \leqslant g_{S,\tau_1}$ for all $\tau_0 \leqslant \tau_1$. Before estimating the g_t-volume $\mathrm{Vol}_{g_t}(B_{g_t}(x,1))$ of the unit ball $B_{g_t}(x,1)$, we observe that

$$g_{S,t(x)-1/2} = g_{S,t(x)} - \frac{1}{2}\pi^*\widehat{g} \geqslant g_{S,t(x)} - \frac{1}{2}(t(x)-1)\pi^*\widehat{g} \geqslant \frac{1}{2} g_{S,t(x)} \tag{4.14}$$

provided $t(x) \geqslant 2$. Using the inclusion $B \subset B_{g_t}(x,1)$ then implies that

$$\begin{aligned}
\mathrm{Vol}_{g_t}(B_{g_t}(x,1)) &\geqslant \mathrm{Vol}_{g_t}(B) \\
&\geqslant \frac{1}{2} \cdot \mathrm{Vol}_{g_{S,t(x)-1/2}}(B_{g_{S,t(x)}}(y,1/2)) \\
&\geqslant 2^{-\frac{\dim_{\mathbb{R}} S}{2} - 1} \mathrm{Vol}_{g_{S,t(x)}}(B_{g_{S,t(x)}}(y,1/2)),
\end{aligned}$$

where we applied Fubini's theorem in the second line, and the last inequality follows from (4.14). Thus, it remains to bound the $g_{S,t(x)}$-volume of $B_{g_{S,t(x)}}(y,1/2)$ uniformly from below. We further reduce this volume bound to an integration on M by observing that the projection $\pi : S \to M$ satisfies

$$\pi^{-1}(B_{g_{M,t(x)}}(\pi(y), 1/4)) \subset B_{g_{S,t(x)}}(y, 1/2) \tag{4.15}$$

Indeed, given a $b \in B_{g_{M,t(x)}}(\pi(y), 1/4)$ and a length-minimizing curve $q : [0,1] \to M$ from $\pi(y)$ to b, we may lift q to a horizontal curve $\widetilde{q}$ in S from $\widetilde{q}(0) = y$ to some point $\widetilde{q}(1) \in \pi^{-1}(b)$. For any $a \in \pi^{-1}(b)$, the triangle inequality for the distance function $\mathrm{dist}_{g_{S,t(x)}}$ then yields

$$\begin{aligned}
\mathrm{dist}_{g_{S,t(x)}}(y,a) &\leqslant \mathrm{dist}_{g_{S,t(x)}}(y, \widetilde{q}(1)) + \mathrm{dist}_{g_{S,t(x)}}(\widetilde{q}(1), a) \\
&\leqslant \mathrm{dist}_{g_{M,t(x)}}(\pi(y), b) + \frac{1}{4} \\
&< \frac{1}{4} + \frac{1}{4} = \frac{1}{2},
\end{aligned}$$

where the second inequality holds since we normalised each fibre $\pi^{-1}(b)$ to be of diameter $1/4$ and the third one follows since $\pi : S \to M$ is a Riemannian submersion. Hence, we conclude that $a \in B_{g_{S,t(x)}(y,1/2)}$ as claimed.

Inclusion (4.15) yields an estimate on the $g_{S,t(x)}$-volume as follows. We write $\omega_{t(x)}$ for the Kähler form of $g_{M,t(x)}$ and χ for the characteristic function of the ball $B_{g_{S,t(x)}}(y,1/2)$, and then observe that

$$\begin{aligned}
\int_{B_{g_{S,t(x)}}(y,1/2)} (J\mathrm{d}t)\wedge\pi^*\omega_{t(x)}^{\dim_{\mathbb{C}}M} &= \int_{B_{g_{M,t(x)}}(\pi(y),1/2)} \pi_*(\chi J\mathrm{d}t)\cdot\omega_{t(x)}^{\dim_{\mathbb{C}}M}\\
&\geqslant \int_{B_{g_{M,t(x)}}(\pi(y),1/4)} \pi_*(J\mathrm{d}t)\cdot\omega_{t(x)}^{\dim_{\mathbb{C}}M}\\
&= \mathrm{Vol}_{g_{S^1}}(S^1)\cdot\int_{B_{g_{M,t(x)}}(\pi(y),1/2)} \omega_{t(x)}^{\dim_{\mathbb{C}}M}.
\end{aligned}$$

Here, $\pi_*(\chi J\mathrm{d}t)$ denotes the function on M obtained by integrating χJdt over fibres, i.e. $\pi_*(\chi J\mathrm{d}t)(b)=\int_{\pi^{-1}(b)}\chi Jdt$, so that the first equality follows from Fubini's theorem. In the second line, we used that $\chi\equiv 1$ on the set $\pi^{-1}(B_{g_{M,t(x)}}(\pi(y),1/4))$ by (4.15) and the final equation holds because the volume of each S^1-fibre is the same by (4.13). Thus, it remains to find a constant $C>0$, independent of $x=(t(x),y)$, such that

$$\mathrm{Vol}_{g_{M,t(x)}}(B_{g_{M,t(x)}}(\pi(y),1/4))\geqslant C^{-1}>0. \tag{4.16}$$

To prove this, let us first assume that $\widehat{g}$ is positive definite. Then we can find a constant $C_0>0$, only depending on the eigenvalues of $\widehat{g}$ w.r.t. g_M, such that

$$C_0^{-1}\tau g_M\leqslant g_{M,\tau}\leqslant C_0\tau g_M,$$

for all $\tau\geqslant 1$ and with $g_M:=g_{M,1}$. Since we can rescale by a fixed constant, it suffices to bound the τg_M-volume of $B_{\tau g_M}(z,1)$ from below by a constant independent of both $z\in M$ and $\tau\gg 1$. We note that

$$B_{\tau g_M}(z,1)=B_{g_M}(z,\tau^{-\frac{1}{2}})$$

and by compactness, the g_M-volume $\mathrm{Vol}_{g_M}(B_{g_M}(z,\tau^{-\frac{1}{2}}))$ is, up to some uniform constant, bounded from below by $\tau^{-\frac{\dim_{\mathbb{R}}M}{2}}$ for τ sufficiently large. This shows that $\mathrm{Vol}_{\tau g_M}(B_{\tau g_M}(z,1))=\tau^{\frac{\dim_{\mathbb{R}}M}{2}}\,\mathrm{Vol}_{g_M}(B_{g_M}(z,\tau^{-\frac{1}{2}}))$ is indeed uniformly bounded from below and (4.16) then follows.

Let us now assume that γ has at least one zero eigenvalue w.r.t. g_M. Since these eigenvalues are assumed to be constant over M, its Kernel $\mathrm{Ker}\,\gamma$ defines a proper subbundle of $T^{1,0}M$. Moreover, γ is closed, so that $\mathrm{Ker}\,\gamma$ is integrable according to Frobenius' theorem. Thus, if n is the complex dimension of M and k the number of positive eigenvalues of γ, we find a chart around each point defined on some neighborhood of the Euclidean unit ball $B(1) \subset \mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}$ around the origin such that each slice $\{z_0\} \times \mathbb{C}^{n-k}$ in $B(1)$ is an integral manifold for $\operatorname{Ker}\gamma$, i.e.

$$\gamma(v, v) > 0 \quad \text{and} \quad \gamma(v, w) = 0 \quad \text{for all } v \in T\mathbb{C}^k,\ w \in T\mathbb{C}^{n-k}.$$

By compactness, we can cover M by finitely many of such Euclidean balls $B_j(1)$ for $j = 1, \ldots, N$ and also find a constant $C_0 > 0$ such that

$$C_0^{-1} g_{\mathbb{C}^n} \leqslant g_M \leqslant C_0 g_{\mathbb{C}^n} \quad \text{on each } B_j(1),$$

where $g_{\mathbb{C}^n}$ is the Euclidean metric on $B_j(1)$ and the constant C_0 is independent of the ball $B_j(1)$. For $\tau \geqslant 1$, we also consider the following product metric

$$g_{\mathbb{C}^n,\tau} := (1+\tau) g_{\mathbb{C}^k} + g_{\mathbb{C}^{n-k}} \quad \text{on} \quad \mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}$$

Then there exists a uniform constant $C > 0$, which only depends on C_0 and the g_M-eigenvalues of γ, such that

$$C^{-1} g_{\mathbb{C}^n,\tau} \leqslant g_{M,\tau} \leqslant C g_{\mathbb{C}^n,\tau} \quad \text{on each} \quad B_j(1). \tag{4.17}$$

Let $\varepsilon > 0$ be the Lebesgue number associated to the cover $\{B_j(1)\}_{j=1,\ldots N}$ of the manifold (M, g_M), i.e. the ball $B_{g_M}(z, \varepsilon)$ is contained in $B_j(1)$ for some j. Note that since $g_M \leqslant g_{M,\tau}$, we also have $B_{g_{M,\tau}}(z, 1) \subset B_{g_M}(z, 1) \subset B_j(1)$. Additionally, we may assume that $\varepsilon < 1$, so that it suffices to bound the $g_{M,\tau}$-volume of the smaller ball $B_{g_{M,\tau}}(z, \varepsilon)$ from below because the constant $\varepsilon > 0$ is independent of both $z \in M$ and $\tau \geqslant 1$.

This can be reduced to bounding the $g_{\mathbb{C}^n,\tau}$ -volume of $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$. Indeed, this is a direct consequence of (4.17), which implies that the volume forms of $g_{\mathbb{C}^n,\tau}$ and $g_{M,\tau}$ are uniformly equivalent on $B_{g_{M,\tau}}(z, \varepsilon)$ and also that we have the inclusion

$$B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon) \subset B_{g_{M,\tau}}(z, \varepsilon).$$

For the remaining lower volume bound, observe that the following product of Euclidean balls

$$B_{g_{\mathbb{C}^k}}\left(z, \frac{1}{2} C^{-\frac{1}{2}} \varepsilon (1+\tau)^{-\frac{1}{2}}\right) \times B_{g_{\mathbb{C}^{n-k}}}\left(z, \frac{1}{2} C^{-\frac{1}{2}} \varepsilon\right) \tag{4.18}$$

is contained in $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$. Applying Fubinis' theorem to the product (4.18) and using the fact that the volume form of $g_{\mathbb{C}^n,\tau}$ is equal to $(1+\tau)^k$-times the volume form of $g_{\mathbb{C}^n}$ then yields the required lower bound on the $g_{\mathbb{C}^n,\tau}$-volume of $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$, which is independent of both $z \in M$ and $\tau \geqslant 1$. This finishes the proof. $\square$

5. Uniqueness in a Kähler class

The purpose of this section is to prove Theorem 1.3. We begin by briefly recalling notation from Sections 2 and 3 and then define the function spaces appearing in Theorem 1.3. We also explain how to reduce the proof to a $\partial\bar{\partial}$-Lemma, which is stated below (Theorem 5.4).

5.1. A $\partial\bar{\partial}$-Lemma

Throughout this section, let $\pi : E \to D$ be a rank m holomorphic vector bundle over a compact Kähler manifold (D, ω_D). The complex dimension of E (as a manifold) is denoted by $m+d$, where d is the complex dimension of D. If $m = 1$, we assume that it satisfies the conditions in Theorem 1.1, and if $m \geqslant 2$, we assume E is given as in Theorem 1.2. Also recall that we defined a radial function $r : E \to \mathbb{R}_{\geqslant 0}$ by $r(v) = \sqrt{h(v, \overline{v})}$, which vanishes along the zero section of E and we set $t := 2\log r$. Note that we can use the function t to identify E, with its zero section removed, as the product $\mathbb{R} \times S$, where S is the S^1-bundle associated to $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$, see Diagram (3.1). Under this identification, the function t on $E \setminus D$ corresponds to the projection onto the first factor of $\mathbb{R} \times S$.

Let ω_φ be the Kähler Ricci soliton constructed in Theorem 1.1 or 1.2, i.e. ω_φ is defined by (2.7) with φ satisfying (2.16) if E is a line bundle or by (3.2) and (3.10) if $m \geqslant 2$. We denote the corresponding Riemannian metric by g_φ.

On the manifold $\mathbb{R} \times S$, we can write the metric g_φ as follows. If J denotes the complex structure on E and g_D and $\widehat{g}$ are the (2,0) tensors associated to ω_D and $-\gamma$, respectively, then

$$g_\varphi = f''(t)\left(\mathrm{d}t^2 + (J\mathrm{d}t)^2\right) + f'(t)\pi^*\widehat{g} + \pi^* g_D, \tag{5.1}$$

where f can be reconstructed from φ via (2.8). We would also like to point out that we allowed $-\gamma$ to have zero-eigenvalues, i.e. $\widehat{g}$ is only semidefinite. As a consequence, the volume growth of g_φ will be determined by the zero-eigenvalues of $-\gamma$.

Before stating the main theorem of this section, we require a definition of weighted function spaces. As a weight function, we choose $w : E \to \mathbb{R}_+$ to be defined by

$$w(t) := 1 + f'(t). \tag{5.2}$$

This choice is inspired by the work of Hein [17]. Indeed, the following lemma shows that w has the same properties as the function ρ in [17, Theorem 1.6].

Lemma 5.1. — *Fix $x_0 \in E$ and denote the distance function of g_φ by $\rho(x)$. Then there exists a constant $C > 0$ such that*

$$C^{-1} w(t(x)) \leqslant (1 + \rho(x)) \leqslant C w(t(x)) \tag{5.3}$$

for all $x \in E$ with $w(t(x)) \geqslant C$. Moreover, w satisfies

$$|\nabla w| + w|\Delta w| \leqslant C, \tag{5.4}$$

where $|\cdot|, \nabla$ and Δ are associated with g_φ.

Proof. — We identify $E \setminus D \cong \mathbb{R} \times S$ and without loss of generality, we can assume $x_0 = (t_0, y_0) \in \mathbb{R} \times S$. Let $(t, y) \in \mathbb{R} \times S$ with $t_0 \leqslant t$ and consider a shortest path $q_{t,y} = (q_t, q_y) : [0,1] \to \mathbb{R} \times S$ from (t_0, y_0) to (t, y). Its length $L(q_{t,y})$ is given by

$$L(q_{t,y}) = \int_0^1 \sqrt{g_\varphi\left(\dot{q}_{t,y}(\sigma), \dot{q}_{t,y}(\sigma)\right)} \mathrm{d}\sigma.$$

Then (5.3) reduces to finding a constant $C > 0$ such that

$$C^{-1} w(t) \leqslant L(q_{t,y}) \leqslant C w(t) \tag{5.5}$$

for all $y \in S$ and all $t \geqslant C$. In fact, it is sufficient to show inequality (5.5) with $w(t)$ replaced by t since there is a $C > 0$ such that

$$C^{-1} t \leqslant w(t) \leqslant C t \tag{5.6}$$

for $t \geqslant C$, compare (4.5). Thus, we begin by choosing $C > 0$ such that (5.6) holds, and we increase $C > 0$ as we go along, if necessary. For proving the lower bound in (5.5), we estimate

$$L(q_{t,y}) \geqslant \int_0^1 \sqrt{f''(q_{t,y})} \dot{q}_t(\sigma) \mathrm{d}\sigma \geqslant \sqrt{f''(t_0)}(t - t_0),$$

as required. Before showing the upper bound, we conclude from (4.5) that

$$g_\varphi \leqslant C\left(\mathrm{d}t^2 + t g_S\right),$$

where we define $g_S := (J\mathrm{d}t)^2 + \pi^* \widehat{g} + \pi^* g_D$ with $\widehat{g}$ and g_D as in (5.1). Also observe that we can now assume q_t to be the linear path in the $\mathbb{R}$-factor, i.e. $q_t(\sigma) = \sigma(t - t_0) + t_0$. Then we obtain

$$\begin{aligned} L(q_{t,y}) &\leqslant C \int_0^1 \dot{q}_t(\sigma) \mathrm{d}\sigma + C \int_0^1 \sqrt{q_t(\sigma)} \cdot \sqrt{g_S(\dot{q_y}(\sigma), \dot{q_y}(\sigma)} \mathrm{d}\sigma \\ &\leqslant Ct + C \operatorname{diam}(S, g_S)\sqrt{t} \\ &\leqslant Ct, \end{aligned} \tag{5.7}$$

for all t sufficiently large and with $\operatorname{diam}(S, g_S)$ denoting the diameter of the compact manifold (S, g_S). Now (5.1) follows immediately. For the second claim, observe from (3.2) that we have

$$|\nabla w| = f'',$$

which is uniformly bounded according to Lemma 4.3. For bounding the Laplace operator $\Delta w = \Delta f'$, recall that on a Kähler manifold, the Laplace operator Δ satisfies $\Delta = 2\,\mathrm{tr}_{\omega_\varphi}\sqrt{-1}\partial\bar{\partial}$, where $\mathrm{tr}_{\omega_\varphi}$ denotes the trace computed w.r.t. g_φ. Then we apply (3.8) to obtain

$$\begin{aligned}\Delta f' &= 2\,\mathrm{tr}_{\omega_\varphi}\left(\sqrt{-1}\partial\bar{\partial} f'\right)\\ &= \frac{2}{\mu}\,\mathrm{tr}_{\omega_\varphi}\left(\mathcal{L}_X\omega_\varphi\right)\\ &= 2\left(\varphi\frac{Q'}{Q} + \varphi'\right),\end{aligned}$$

where the last equality holds since there is the following formula

$$\mathrm{tr}_{\omega_\varphi}(-\pi^*\gamma) = \frac{Q'}{Q},$$

see [18][(2.22)]. Using the soliton ODE (3.10), we continue

$$\begin{aligned}w\Delta w &= (1+f')\Delta f'\\ &= 2(1+\tau)\left(\varphi\frac{Q'}{Q} + \varphi'\right)\\ &= 2(1+\tau)\left(m+\mu\varphi\right),\end{aligned}$$

which is also uniformly bounded because of the asymptotic expansion (3.15). This, together with the uniform bound on $|\nabla w|$, implies (5.4). □

Lemma 5.1 ensures that our definition of weighted function spaces below coincides with the one used in [17]. These spaces are well-adapted to study the Laplace operator on a wide class of complete manifolds.

Definition 5.2. — *Let Λ^*T^*E be the exterior algebra of T^*E and consider $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$. We define $C^k_\delta(\Lambda^*T^*E)$ to be the space of k-times continuously differentiable sections η of Λ^*T^*E such that the norm*

$$\|\eta\|_{C^k_\delta} := \sum_{j=0}^{k} \sup_E |w^{j-\delta}\nabla^j\eta|$$

is finite, where w is given by (5.2) and ∇, $|\cdot|$ are associated to g_φ. We also set

$$C^\infty_\delta(\Lambda^*T^*E) := \bigcap_{k\in\mathbb{N}_0} C^k_\delta(\Lambda^*T^*E).$$

In other words, elements in $C^\infty_\delta(\Lambda^* T^* E)$ grow at most like w^δ and their l-th derivatives at most like $w^{\delta-l}$. Having introduced the necessary notation, we can now state the main result of this section.

Theorem 5.3. — *Let ω_φ be a steady Kähler–Ricci soliton constructed in Theorem 1.1 or 1.2. Assume that ω is a Kähler–Ricci soliton on E with the same vector field as ω_φ such that $[\omega] = [\omega_\varphi] \in H^2(E)$. If moreover $\omega_\varphi - \omega \in C^\infty_{-\delta}(\Lambda^2 T^* E)$ for some $\delta > 2$, then $\omega_\varphi = \omega$.*

The main part of proving Theorem 5.3 will be a $\partial\bar\partial$-Lemma, with controlled growth. In fact, we will prove

Theorem 5.4. — *Let $\delta > 2$ and $\eta \in C^\infty_{-\delta}(\Lambda^* T^* E)$ be a real $(1,1)$ form. If η is d-exact, then $\eta = \sqrt{-1}\partial\bar\partial u$ for some $u \in C^\infty_{2-\delta}(E)$.*

Assuming this result, Theorem 5.3 follows immediately.

Proof of Theorem 5.3. — By Theorem 5.4, there exists a $u \in C^\infty_{2-\delta}(E)$ such that $\omega_\varphi - \omega = \sqrt{-1}\partial\bar\partial u$. Since $2-\delta < 0$, u and all its derivatives tend to zero at infinity, so we can apply the maximum principle [2, Proposition 1.2] and conclude that $\omega_\varphi = \omega$. $\square$

The remainder of this section is devoted to proving Theorem 5.4. We follow the ideas for asymptotically conical metrics given in [9, Section 3], which rely on two main ingredients. Firstly, we need to understand solutions to Poisson's equation $\Delta u = h$ and their growth behaviour (Section 5.2). Secondly, we need to show that harmonic (1,0) forms of certain growth behaviour are identically zero (Section 5.3). The proof of Theorem 5.4 will then be finished in Section 5.4.

5.2. The Laplace Operator

We start by considering the Laplace operator Δ of the metric g_φ acting on suitably weighted Hölder spaces, which we now define.

Definition 5.5. — *Let* $\operatorname{dist}(x,y)$ *be the distance between $x, y \in E$ measured w.r.t. g_φ and denote the injectivity radius of g_φ by i_0. (Note that $i_0 > 0$ by Lemma 4.4). For $0 < \alpha < 1$ and $\delta \in \mathbb{R}$, we define a seminorm on the space of all tensor fields T on E by*

$$[T]_{C^{0,\alpha}_\delta} := \sup_{\substack{x \neq y \in E \\ \operatorname{dist}(x,y) < \frac{i_0}{2}}} \left(\min(w(x), w(y))^{-\delta} \frac{|T_x - T_y|}{\operatorname{dist}(x,y)^\alpha} \right),$$

where the norm $|\cdot|$ is induced by g_φ and the difference $T_x - T_y$ is defined by using parallel transport along the minimal geodesic from x to y. The weighted Hölder space $C^{k,\alpha}_\delta(E)$ is then defined to be the subset of all $u \in C^k_\delta(E)$ for which the norm

$$\|u\|_{C^{k,\alpha}_\delta} := \|u\|_{C^k_\delta} + [\nabla^k u]_{C^{0,\alpha}_{\delta-k-\alpha}}$$

is finite.

The Laplace operator Δ acts as

$$\Delta : C^{2,\alpha}_{2+\delta}(E) \to C^{0,\alpha}_\delta(E),$$

for any $\delta \in \mathbb{R}$ and we are interested in the surjectivity of this operator. A partial answer to this question is provided in [17]. Given $h \in C^{0,\alpha}_\delta(E)$ with $\delta < -2$, we can essentially always solve Poisson's equation $\Delta u = h$, but it is not clear how the solution u will behave as $t \to \infty$. This depends on the volume growth of g_φ, which is related to the degree k of the polynomial Q defined in (2.9) for $m = 1$ or (3.6) for $m \geqslant 2$. Alternatively, it is evident from the definition of Q that k is equal to $m + d - 1$ minus the number of zero-eigenvalues of γ. (Recall that $m + d - 1$ is the complex dimension of $\mathbb{P}(E)$.) More precisely, we will prove the following important proposition about the existence of solutions to $\Delta u = h$.

Proposition 5.6. — *Let $\delta > 2$ and suppose $h \in C^{0,\alpha}_{-\delta}(E)$.*

(i) *If $k \leqslant 1$, assume $\int h\omega_\varphi^{m+d} = 0$ additionally. Then there exists a $u \in C^{2,\alpha}(E)$ such that $\Delta u = f$ and the integral $\int |\nabla u|^2 \omega_\varphi^{m+d}$ is finite.*
(ii) *If $k > 1$ and $2 < \delta < k + 1$, then there exists $u \in C^{2,\alpha}(E)$ such that $\Delta u = h$ and $u = O(w^{2-\delta+\varepsilon})$ as well as $|\nabla u| = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$.*

Before proceeding with its proof, we first of all need to check that we can indeed apply Hein's work [17, Theorem 1.5, 1.6], i.e. we have to verify that the metric (E, g_φ) satisfies Hein's condition SOB(β). For the sake of completeness, we recall [17, Definition 1.1] here.

Definition 5.7 ([17, Definition 1.1]). — *A Riemannian manifold (M, g) is called SOB(β) if there exists a $x_0 \in M$ and a constant $C \geqslant 1$ satisfying the following:*

(i) *The set $B(x_0, s_1) \setminus \overline{B}(x_0, s_0)$ is connected for all $s_1 > s_0 \geqslant C$,*
(ii) $\mathrm{Vol}(B(x_0, s)) \leqslant Cs^\beta$ *holds for all $s \geqslant C$,*
(iii) $\mathrm{Vol}(B(x, (1 - C^{-1})\rho(x))) \geqslant C^{-1}\rho(x)^\beta$ *holds for all $x \in M$ with $\rho(x) \geqslant C$,*

(iv) $\mathrm{Ric}_x \geqslant -C\rho(x)^{-2}$ *holds if* $\rho(x) \geqslant C$.

Here $B(x_0, s)$ *denotes the geodesic ball around* x_0, $\mathrm{Vol}(B(x_0, s))$ *its volume and* $\rho(x)$ *denotes the distance from* x *to* x_0.

As the next lemma shows, the soliton metrics (E, g_φ) constructed in Theorem 1.1 and 1.2 are $\mathrm{SOB}(k+1)$.

LEMMA 5.8. — *The metric* (E, g_φ) *is SOB(*$k+1$*), where* k *is equal to* $m+d-1$ *minus the number of zero-eigenvalues of the curvature form* γ *on* $\mathbb{P}(E)$.

Proof. — We fix $x_0 \in D \subset E$ to be a point on the zero-section of E. Thanks to Theorem 4.1, Condition (iv) in Definition 5.7 is clearly satisfied, so we focus on (i), (ii), (iii). Beginning with the volume estimates (ii) and (iii), we consider the volume form of g_φ, which is given by

$$\frac{\omega_\varphi^{m+d}}{(m+d)!} = \frac{\sqrt{-1}}{(m+d)!} f'' Q(f') \partial t \wedge \bar{\partial} t \wedge (\pi^* \omega_D - \pi^* \gamma)^{m+d-1}, \tag{5.8}$$

where the polynomial Q is defined by (2.9) if $m = 1$ or (3.4) if $m \geqslant 2$. Recall from above, that the degree of Q is equal to k as defined in Lemma 5.8. If we then choose $C \geqslant 1$ such that (4.5) is satisfied, we obtain for large $t \geqslant C$:

$$C^{-1} t^k \leqslant f''(t) Q(f'(t)) \leqslant C t^k. \tag{5.9}$$

Moreover, Lemma 5.1 implies that

$$C^{-1} t(x) \leqslant \rho(x) \leqslant C t(x) \tag{5.10}$$

if $\rho(x) \geqslant C$. In the estimates that follow, we increase $C > 0$ if necessary but it still denotes a uniform constant which only depends on the geometry of (E, g_φ) and the choice of base point x_0. For verifying (ii), let $s \geqslant C$ and observe that (5.10) implies

$$B(x_0, s) \subset B(x_0, C) \cup \{y \in E \,|\, 0 \leqslant t(y) \leqslant Cs\}.$$

Integrating over these sets and using (5.8), we obtain

$$\begin{aligned} \mathrm{Vol}(B(x_0, s)) &\leqslant C + C \int_0^{Cs} \int_S f''(t) Q(f'(t)) \mathrm{d}t \wedge \bar{\partial} t \wedge (\pi^*(\omega_D - \gamma))^{m+d-1} \\ &\leqslant C + C \int_0^{Cs} t^k \mathrm{d}t \\ &\leqslant C s^{k+1}, \end{aligned}$$

where we used (5.9) in the second line. This proves (ii) of Definition 5.7 with $\beta = k+1$. For showing (iii), the goal is to choose a new $C_0 \geqslant 1$ such that for all $x \in E$ with $\rho(x) \gg 1$ sufficient large, we have an inclusion of the form

$$B(x, (1 - C_0^{-1})\rho(x)) \supset \left\{t(y) \in \left[t(x)+1, t(x)+C_0^{-1}\rho(x) - \sqrt{\rho(x)}\right]\right\}. \tag{5.11}$$

Indeed, if (5.11) holds, we can integrate and use (5.9) to estimate

$$\begin{aligned}\mathrm{Vol}(B(x, (1 - C_0^{-1})\rho(x))) &\geqslant C_0^{-1} \int_{t(x)+1}^{t(x)+C_0^{-1}\rho(x)-\sqrt{\rho(x)}} \sigma^k \mathrm{d}\sigma \\ &\geqslant C_0^{-1}\rho(x)^{k+1},\end{aligned}$$

which is (iii) with $\beta = k + 1$ as required. Hence it remains to check inclusion (5.11). To see that this is true, we again introduce the metric $g_S := (J\mathrm{d}t)^2 + \pi^*\widehat{g} + \pi^* g_D$ on the cross-section S as in the proof of Lemma 5.1, so that

$$g_\varphi \leqslant C\left(\mathrm{d}t^2 + t g_S\right) =: g_t. \tag{5.12}$$

To estimate the distance function of g_t from above, we proceed as in (5.7). Given $x, y \in E$ with $C \leqslant t(x)$ and $t(x) \leqslant t(y)$, we consider a path $q : [0, 1] \to E$ from $q(0) = x$ to $q(1) = y$, which we write as $q = (q_1, q_2)$ under the identification $E \setminus D \cong \mathbb{R} \times S$. Furthermore, we assume that $q_1(\sigma) = \sigma(t(y) - t(x)) + t(x)$ is the linear path from $t(x)$ to $t(y)$, so that we estimate using (4.5)

$$\begin{aligned}\mathrm{dist}_{g_t}(x, y) &\leqslant C \int_0^1 \dot{q}_1(\sigma)\mathrm{d}\sigma + C \int_0^1 \sqrt{q_1(\sigma)}\sqrt{g_S(\dot{q}_2(\sigma), \dot{q}_2(\sigma))}\mathrm{d}\sigma \\ &\leqslant C(t(y) - t(x)) + C\,\mathrm{diam}(S, g_S)\left(\sqrt{t(y) - t(x)} + \sqrt{t(x)}\right)\end{aligned}$$

Together with (5.10) and (5.12), this implies

$$\mathrm{dist}_{g_\varphi}(x, y) \leqslant C\left(t(y) - t(x) + \sqrt{t(y) - t(x)}\right) + C\sqrt{\rho(x)} \tag{5.13}$$

from which we can deduce inclusion (5.11). Indeed, let $C > 0$ satisfy (5.13) and define a new constant $C_0 > 0$ by $C_0^{-1} = C^{-1}(1 - C^{-1})$. If we then assume that $\rho(x) \gg 1$ is large enough so that $C_0^{-1}\rho(x) - \sqrt{\rho(x)} > 1$, we estimate for all $y \in E$ with $t(x) + 1 \leqslant t(y) \leqslant t(x) + C_0^{-1}\rho(x) - \sqrt{\rho(x)}$:

$$\begin{aligned}\mathrm{dist}_{g_\varphi}(x, y) &\leqslant C\left(t(y) - t(x)\right) + C\sqrt{\rho(x)} \\ &\leqslant CC_0^{-1}\rho(x) - C\sqrt{\rho(x)} + C\sqrt{\rho(x)} \\ &= \left(1 - C^{-1}\right)\rho(x).\end{aligned}$$

Here we obtained the first inequality by applying $t(y) - t(x) \geqslant 1$ to (5.13) and the second inequality makes use of the upper bound on $t(y)$. This shows inclusion (5.11) and thus (iii). It remains to verify Condition (i). By compactness of D, we can choose $C > 1$ such that for all $s \geqslant C$ the ball $B(x_0, s)$ contains a tubular neighborhood of the zero section. Given $x \in E \setminus D$, we denote the complex line thorough x by $L_x \cong \mathbb{C}$. We need to understand the shape of the intersection of L_x with the set $B_{(s_0,s_1)}(x_0) := B(x_0,s_1) \setminus \overline{B}(x_0,s_0)$ for all $s_1 > s_0 \geqslant C$. First, we claim that for each $x \in B(x_0,s)$, the radial path q_{rad} in L_x from x to $0 \in L_x$ is entirely contained in the ball $B(x_0,s)$. Note that for this to be true it suffices to show that the function ρ is increasing along q_{rad}. In order to prove this, use the identification $E \setminus D \cong \mathbb{R} \times S$ and write $x = (a_1,b)$. Let $q : [0,1] \to E$ be a shortest geodesic from $q(0) = x_0$ to $q(1) = (a_1,b)$. On $E \setminus D$, we decompose $q = (q_1,q_2)$ and let us assume for the moment that $q_1(\sigma)$ is increasing in $\sigma \in [0,1]$. Given $a_0 < a_1$, we then choose a $\sigma_0 \in (0,1)$ with $q_1(\sigma_0) = a_0$ and reparameterize the path q by declaring $q_{\sigma_0}(\sigma) := (q_1(\sigma_0\sigma), q_2(\sigma))$, so that q_{σ_0} is a path from x_0 to (a_0,b). It follows from (5.1) that we have

$$g_\varphi\left(\dot q_{\sigma_0}(\sigma), \dot q_{\sigma_0}(\sigma)\right) \leqslant g_\varphi\left(\dot q(\sigma), \dot q(\sigma)\right)$$

for all $\sigma \in [0,1]$, since f'' and f' are both increasing and we assumed that $q_1(\sigma_0\sigma) \leqslant q_1(\sigma)$. Then we conclude $L(q_{\sigma_0}) \leqslant L(q)$ and thus $\rho(a_0,b) \leqslant \rho(a_1,b)$ for all $a_0 < a_1$ and $b \in S$, as we claimed. Hence, the claim holds if we show that q_1 is increasing. Recall that by definition, $q_1 = t(q)$ and clearly q_1 increases if and only if $r^2(q) = e^{t(q)}$ does, where $r : E \to \mathbb{R}_{\geqslant 0}$ is defined at the beginning of Section 5. Since x_0 lies on the zero section of E, we have $r(q(0)) = 0$ and consequently there is a $\widehat\sigma \in [0,1)$ such that

$$r(q(\sigma)) = 0 \ \text{ for all } \sigma \in [0,\widehat\sigma] \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\sigma} r^2(q(\sigma)) > 0 \quad \text{on} \ \ (\widehat\sigma, \widehat\sigma + \varepsilon)$$

for some small $\varepsilon > 0$. In particular, $\lim_{\sigma \to \hat\sigma^+} \dot q_1(\sigma) \geqslant 0$ and we only have to rule out the existence of two points $\sigma_1, \sigma_2 \in [\widehat\sigma, 1]$ with $\sigma_1 < \sigma_2$ such that

$$q_1(\sigma_1) = q_2(\sigma_2) \quad \text{ and } q_1(\sigma_1) < q_1(\sigma) \quad \text{for all } \sigma \in (\sigma_1,\sigma_2). \tag{5.14}$$

However, if this was the case, then the path q cannot be length-minimizing. Indeed, suppose that there are such numbers σ_1, σ_2 satisfying (5.14). Then we define a new path $\widetilde q$ from x_0 to x by

$$\widetilde q(\sigma) = \begin{cases} q(\sigma) & \text{if } \sigma \in [0,1] \setminus (\sigma_1,\sigma_2) \\ (q_1(\sigma_1), q_2(\sigma)) & \text{if } \sigma \in (\sigma_1,\sigma_2). \end{cases}$$

Using the decomposition (5.1) and the fact that f'' is increasing, we see that

$$L(\widetilde q) < L(q),$$

contradicting the minimality of q. It follows that q_1 must be increasing.

Now we can verify Condition (i), so consider any $s_1 > s_0 \geqslant C$. As shown in the previous paragraph, both $L_x \cap B(x_0, s_j)$ with $j = 1,2$ are star-shaped regions with center $0 \in L_x$, so the complement $L_x \cap B_{(s_0,s_1)}(x_0)$ is diffeomorphic to a genuine open annulus in $\mathbb{C}$. From this, we deduce that $B_{(s_0,s_1)}(x_0)$ is a fibre bundle over $\mathbb{P}(E)$ with annuli in $\mathbb{C}$ as fibres. In particular, $B_{(s_0,s_1)}(x_0)$ is connected because D is, finishing the proof. $\square$

Before proving Proposition 5.6, we study the spaces $C^{k,\alpha}_\delta(E)$ further. In fact, Lemma 5.1 allows us to obtain the expected embedding theorems.

Lemma 5.9 (Embeddings). — *Let $k, l \in \mathbb{N}$, $0 < \alpha_0, \alpha_1 < 1$ and $\delta_0 \leqslant \delta_1$. Then there are the following continuous embeddings:*

(i) $C^k_{\delta_0}(E) \subset C^l_{\delta_1}(E)$ *if $l \leqslant k$,*
(ii) $C^{k,\alpha_0}_{\delta_0}(E) \subset C^{l,\alpha_1}_{\delta_1}(E)$ *if $l \leqslant k$ and $\alpha_1 \leqslant \alpha_0$,*
(iii) $C^{k+1}_\delta(E) \subset C^{k,1}_\delta(E)$. *In particular, $C^\infty_\delta(E) = \bigcap_{k\in\mathbb{N}_0} C^{k,\alpha}_\delta(E)$.*

The proof of this lemma is analogue to [5, Lemma 2], so we omit it here. Now we are in a position to show Proposition 5.6.

Proof of Proposition 5.6. — Part (i) is a direct consequence of Theorem 1.5 in [17]. Indeed, (E, g_φ) is $\mathrm{SOB}(k+1)$ by Lemma 5.8, and, because of Lemma 5.1, we have that $|h| = O(\rho^{-\delta})$ with $\delta > 2$, where $\rho(x)$ denotes the distance to some fixed point x_0. Then (i) is precisely [17, Theorem 1.5].

For (ii), we note that the function w satisfies the assumption of Theorem 1.6 in [17], see Lemma 5.1. Consequently, [17, Theorem 1.6] gives a $u \in C^{2,\alpha}(E)$ such that $\Delta u = h$ and $u = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$. Then it only remains to verify the decay rate of $|\nabla u|$, which is a consequence of standard Schauder theory. Indeed, since the curvature of g_φ is bounded by Lemma 4.4, we can find $s > 0$ and $Q > 0$ such that for all $x \in E$, there is a chart Φ_x from to the Euclidean ball $B_x(s) \subset \mathbb{R}^{m+d}$ of radius s onto a neighborhood of x so that $\frac{1}{Q} g_{\mathrm{euc}} \leqslant \Phi_x^* g_\varphi \leqslant Q g_{\mathrm{euc}}$ and $\|\Phi_x^* g_\varphi\|_{C^{1,\alpha}(B_x(s))} \leqslant Q$ ([25, Theorem 4.1]). Here, g_{euc} denotes the flat metric and $\|\cdot\|_{C^{1,\alpha}(B_x(s))}$ the Euclidean Hölder norm. For simplicity, we suppress the chart Φ_x and view $B_x(s)$ as a subset of E. Also note that we can assume that s is strictly smaller than the injectivity radius of (E, g_φ). Applying the Euclidean Schauder estimates ([13, Theorem 6.2]) to the balls $B_x(s)$, we obtain that

$$
\begin{aligned}
|\nabla u|_{g_\varphi}(x) &\leqslant Q |du|_{g_{\mathrm{euc}}}(x) \\
&\leqslant Q \|u\|_{C^{2,\alpha}(B_x(s))} \\
&\leqslant Q C_0 \left(\|h\|_{C^{0,\alpha}(B_x(s))} + \|u\|_{C^0(B_x(s))} \right)
\end{aligned}
\tag{5.15}
$$

for some uniform constant $C_0 > 0$ depending only on α, s and Q. Moreover, the weight function w is chosen so that there is a uniform constant $C_1 > 0$ such that for all $x \in E$ with $t(x) \gg 1$ and all $y \in B_x(s)$, we have $\frac{1}{C_1} w(y) \leqslant w(x) \leqslant C_1 w(y)$. This follows directly from Lemma 5.1 and the fact that g_φ and g_{euc} are uniformly equivalent on $B_x(s)$. Therefore, we continue to estimate for all $x \in E$ with $t(x) \gg 1$ and all $y \in B_x(s)$:

$$u(y) \leqslant Cw(y)^{2-\delta+\varepsilon} \leqslant CC_1^{2-\delta+\varepsilon} w(x)^{2-\delta+\varepsilon},$$

i.e. $\|u\|_{C^0(B_x(s))} = O(w(x)^{2-\delta+\varepsilon})$. Similarly, we conclude that

$$\|h\|_{C^{0,\alpha}(B_x(s))} = O(w(x)^{-\delta})$$

because s is chosen strictly smaller than the injectivity radius and $h \in C^{0,\alpha}_{-\delta}(E)$. In combination with (5.15) we consequently arrive at

$$|\nabla u|_{g_\varphi} = O(w^{2-\delta+\varepsilon}),$$

as claimed. □

The issue with (ii) of Proposition 5.6 is that one can in general not conclude $u = O(w^{2-\delta})$ if $u = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$. For proving Theorem 5.4, however, we would like to conclude that indeed $u = O(w^{2-\delta})$. The following proposition gives a criterion, when this conclusion is true.

Lemma 5.10. — *Let $\delta > 0$ and suppose that $\xi \in C^\infty_{-1-\delta}(T^*E)$. If $\xi = du$ for some $u \in C^1(E)$, then there exists a constant function u_c such that $u - u_c \in C^\infty_{-\delta}(E)$. If additionally $u \to 0$ as $t \to \infty$, then $u_c \equiv 0$.*

Proof. — This lemma is proven analogously to the corresponding statement for conical metrics [22, Lemma 5.10]. First observe that we only need to find a constant u_c such that $u - u_c \in C^0_{-\delta}(E)$ because $\nabla(u - u_c) = du \in C^\infty_{-1-\delta}(T^*E)$ by assumption. We work on $E \setminus D \cong \mathbb{R} \times S$ and fix a point $(t_0, y_0) \in \mathbb{R} \times S$. Viewing S as the slice $\{0\} \times S$, we endow S with a metric g_S by restricting g_φ to S. For a different point (t, y), we let $q_{t_0,t}$ be the straight line path from (t_0, y_0) to (t, y_0) and $q_{y_0,y}$ be a path joining the points (t, y_0) and (t, y), so that its projection onto S is a length minimizing geodesic. Then we have by Stoke's theorem

$$u(t,y) - u(t_0,y_0) = \int_{q_{t_0,t}} \xi + \int_{q_{y_0,y}} \xi. \tag{5.16}$$

As in the proof of [22, Lemma 5.10 (c)], the key is to notice that the integral of ξ along the path $q_{t_0,\infty}$ is finite, where $q_{t_0,\infty}$ is the linear path from (t_0, y_0) to $(+\infty, y_0)$. Indeed, since $\xi \in C^\infty_{-1-\delta}(T^*E)$ and $\delta > 0$, we can estimate

$$\begin{aligned} \left| \int_{q_{t_0,\infty}} \xi \right| &\leqslant \int_{t_0}^{\infty} |\xi(\dot{q}_{t_0,\infty})| \, \mathrm{d}s \\ &\leqslant \|\xi\|_{C^0_{-1-\delta}} \int_{t_0}^{\infty} f'' w^{-1-\delta} \mathrm{d}s \\ &\leqslant \|\xi\|_{C^0_{-1-\delta}} \frac{w^{-\delta}(t_0)}{\delta} < +\infty, \end{aligned} \tag{5.17}$$

Splitting the integral $\int_{q_{t_0,\infty}} \xi$ into two parts, we can rewrite (5.16) as follows:

$$u(t,y) - u(t_0,y_0) - \int_{q_{t_0,\infty}} \xi = -\int_{q_{t,\infty}} \xi + \int_{q_{y_0,y}} \xi. \tag{5.18}$$

As in (5.17), it is easy to see that the right hand side of (5.18) is bounded by $w^{-\delta}(t)$. In fact, we have

$$\begin{aligned}\left|\int_{q_{y_0,y}} \xi\right| &\leqslant \int_a^b |\xi|_\varphi |\dot{q}_{y_0,y}|_\varphi \mathrm{d}s \qquad (5.19)\\ &\leqslant C\|\xi\|_{C^0_{-1-\delta}} w^{-1-\delta}(t)\sqrt{f'(t)} \int_a^b |\dot{q}_{y_0,y}|_{g_S} \mathrm{d}s\\ &\leqslant C\|\xi\|_{C^0_{-1-\delta}} w^{-\delta}(t)\,\mathrm{diam}(S,g_S),\end{aligned}$$

where $q_{y_0,y}$ is defined on the interval $[a,b]$ and $C > 0$ is some constant independent of t. Combining with (5.17), we obtain

$$\left|u(t,y) - u(t_0,y_0) - \int_{q_{t_0,\infty}} \xi\right| \leqslant \|\xi\|_{C^0_{-1-\delta}} \left(\delta^{-1} + C\,\mathrm{diam}(S,g_S)\right) w^{-\delta}(t),$$

i.e. $u - u_c \in C^0_{-\delta}(E)$ where we set

$$\begin{aligned}u_c &= u(t_0,y_0) + \int_{q_{t_0,\infty}} \xi\\ &= u(t_0,y_0) + \lim_{t\to\infty} \left(u(t,y_0) - u(t_0,y_0)\right)\\ &= \lim_{t\to\infty} u(t,y_0).\end{aligned}$$

Thus, it remains to show that u_c is indeed constant. Let q_{y_0,y_1} be a path in the slice $\{t\} \times S$ connecting two points (t,y_0) and (t,y_1). Then we have

$$u(t,y_1) - u(t,y_0) = \int_{q_{y_0,y_1}} \xi, \tag{5.20}$$

and by (5.19), the right hand side of (5.20) goes to 0 as $t \to \infty$. Hence $\lim_{t\to\infty} u(t,y_0) = \lim_{t\to\infty} u(t,y_1)$ for any $y_0, y_1 \in S$, proving the lemma. □

5.3. Vanishing of harmonic forms.

We aim at proving a vanishing theorem for harmonic (1,0)-forms on the manifold (E, g_φ). This will be needed for the $\partial\bar{\partial}$-Lemma. We start with a basic observation which is immediate from the standard Bochner formula.

LEMMA 5.11. — *Any harmonic 1-form β on (E, g_φ) such that $|\beta| \to 0$ as $t \to \infty$ must vanish identically.*

Proof. — Since $\mathrm{Ric}(\omega_\varphi)$ is non-negative by Theorem 4.1, the Bochner formula reads

$$\Delta|\beta|^2 \geqslant 0,$$

and the claim then follows from the Maximum principle. □

It becomes more interesting if we replace the asymptotic condition of β in the previous lemma by requiring that β be square-integrable instead. If β is moreover of type $(1,0)$, it is also holomorphic and it must be zero by the following Theorem.

Theorem 5.12. — *Any L^2-holomorphic (1,0)-form on (E, g_φ) is identically zero.*

Proof. — We adapt the idea behind [23, Theorem 7]. Let β be a holomorphic (1,0)-form, which is square integrable w.r.t. the metric g_φ. Then $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$, and by the Kähler identities $\Delta_d\beta = 0$, i.e. β is harmonic. Since every L^2-harmonic form on a complete manifold is closed and coclosed, we conclude $\mathrm{d}\beta = \mathrm{d}^*\beta = 0$. Observe that β and $\pi^*j^*\beta$ are in the same de-Rham cohomology class, where $\pi : E \to D$ is the projection and $j : D \to E$ is the inclusion of D as the zero section. Hence $\beta = \pi^*j^*\beta + \partial h$ for some function h. It follows immediately that $\bar{\partial}\partial h = 0$. For some $\varepsilon > 0$, consider the tube $D_\varepsilon = \{z \in E \mid r(z) \leqslant \varepsilon\}$ around the zero section. Then by Stoke's theorem, there is the following formula

$$\int_{D_\varepsilon} |\partial h|^2 = -\int_{D_\varepsilon} \langle h, \partial^*\partial h\rangle + \int_{\partial D_\varepsilon} h\iota_\nu(\partial h). \tag{5.21}$$

Here, $\nu := \frac{X}{|X|}$ denotes the outward pointing unit normal vector to ∂D_ε. As X is a real holomorphic vector field, the function $\iota_X(\partial h)$ is also holomorphic and we claim that it is in L^2. Indeed, using $\iota_X(\pi^*j^*\beta) = 0$, we observe that

$$|\iota_X(\partial h)| = |\iota_X(\beta)| \leqslant |X| \cdot |\beta|$$

so that $\iota_X(\partial h)$ is square-integrable since X is bounded and β is L^2. Hence, $\iota_X(\partial h)$ is an L^2-holomorphic function and must consequently be zero. Moreover, $2\partial^*\partial h = \Delta h = 0$ because h is pluriharmonic. Thus, ∂h vanishes identically on D_ε by (5.21). So ∂h must be zero everywhere since it is a holomorphic (1,0)-form. We conclude that $\beta = \pi^*j^*\beta$. However, a form pulled back from the base can never be in L^2, unless it vanishes identically. Indeed, let α be a 1-form on D which is non-zero at some point p. Keeping the expression (5.1) in mind, we can always estimate in a neighborhood around p

$$\langle \pi^*\alpha, \pi^*\alpha\rangle \geqslant Cw^{-1} > 0$$

for some $C > 0$ independent of t. It follows that $\int_E |\pi^*\alpha|^2 = +\infty$ since w^{-1} is not integrable. This finishes the proof. □

5.4. The $\partial\bar{\partial}$-Lemma

In this paragraph, we prove Theorem 5.4 on the manifold E analogue to [9, Theorem 3.11].

The first step is to find a primitive of η, with controlled growth. In fact, one can write down an explicit primitive for η on the product $E \setminus D \cong \mathbb{R} \times S$ and then read off its growth behaviour. This is the idea behind the next proposition.

Proposition 5.13. — *Let $\delta > 2$ and $\eta \in C^\infty_{-\delta}(\Lambda^2 T^* E)$ be a d-exact 2-form. Then $\eta = \mathrm{d}\theta$ for some $\theta \in C^\infty_{-\delta+1}(T^*E)$.*

Proof. — As in [9, Theorem 3.11], we first reduce the problem to finding a primitive for η on the product $\mathbb{R} \times S$. By assumption, there exists a 1-form ξ such that $\eta = \mathrm{d}\xi$. Let $t_1 < t_2$ and define two compact sets K_j with $j = 1, 2$ by

$$K_j = \{z \in E \mid t(z) \leqslant t_j\},$$

where we view the zero section of E to be the set $\{z \in E \mid t(z) = -\infty\}$. We pick a cut-off function χ so that $\chi \equiv 0$ on K_1 and $\chi \equiv 1$ on the complement of K_2. Then we put $\widehat{\xi} := \chi\xi$ and $\widehat{\eta} := \mathrm{d}\widehat{\xi}$. Note that if $\widehat{\eta} = \mathrm{d}\widehat{\theta}$ for some $\widehat{\theta}$, then $\theta := \xi - \widehat{\xi} + \widehat{\theta}$ satisfies

$$\mathrm{d}\theta = \mathrm{d}\xi - \mathrm{d}(\chi\xi) + \widehat{\eta} = \eta.$$

Since $\theta = \widehat{\theta}$ outside K_2, it suffices to find $\widehat{\theta} \in C^\infty_{1-\delta}(\Lambda^* E)$ with $\widehat{\eta} = \mathrm{d}\widehat{\theta}$ and $\widehat{\theta} \equiv 0$ on K_1. The following construction of $\widehat{\theta}$ can be found in the proof of [22, Proposition 5.8].

For each $t \in \mathbb{R}$, there is an inclusion $i_t : \{t\} \times S \to \mathbb{R} \times S$ given by $i_t(y) = (t, y)$. Write $\widehat{\eta} = dt \wedge \widehat{\eta}_1 + \widehat{\eta}_2$, where $\widehat{\eta}_j$ are 1-parameter families of j-forms such that

$$\iota_{\frac{\partial}{\partial t}} \widehat{\eta}_j = 0 \quad \text{and} \quad i_t^* \widehat{\eta}_j = 0 \quad \text{for all } t \leqslant t_1. \tag{5.22}$$

We define a family $\widehat{\theta}_t$ with $t \in \mathbb{R}$ of 1 forms on S by

$$\widehat{\theta}_t = -\int_t^\infty i_s^*(\widehat{\eta}_1)\mathrm{d}s. \tag{5.23}$$

Then we define a 1-form $\widehat{\theta}$ on $\mathbb{R} \times S$ by requiring that

$$\iota_{\frac{\partial}{\partial t}} \widehat{\theta} = 0 \quad \text{and} \quad i_t^* \widehat{\theta} = \widehat{\theta}_t \quad \text{for all } t \in \mathbb{R}. \tag{5.24}$$

We have to prove that $\widehat{\theta}$ is well-defined, i.e. that the integral (5.23) exists. We start by looking at $|\widehat{\eta}_1|$. As $\mathrm{d}t$ and $\widehat{\eta}_1$ are orthogonal to each other, we have that

$$|\mathrm{d}t \wedge \widehat{\eta}_1| = |\mathrm{d}t||\widehat{\eta}_1| = \frac{1}{\sqrt{f''(t)}}|\widehat{\eta}_1|.$$

Using that $\mathrm{d}t \wedge \widehat{\eta}_1$ is orthogonal to $\widehat{\eta}_2$, we can estimate

$$|\widehat{\eta}_1| = \sqrt{f''(t)}|\mathrm{d}t \wedge \widehat{\eta}_1| \leqslant \sqrt{f''(t)}|\mathrm{d}t \wedge \widehat{\eta}_1 + \widehat{\eta}_2| = O(w^{-\delta}), \tag{5.25}$$

since f'' is bounded and $|\widehat{\eta}| = O(w^{-\delta})$ by assumption. To compute the integral (5.23), we work in coordinates. Let $(y_0 = t, y_1, \ldots, y_{2(m+d)-1})$ be real coordinates of $\mathbb{R} \times S$ and write $\widehat{\eta}_1 = \sum_{j\geqslant 1} \widehat{\eta}_{1,j} \mathrm{d}y_j$. Then (5.23) becomes

$$\widehat{\theta}_t = -\sum_{j\geqslant 1} \left(\int_t^\infty i_s^* \widehat{\eta}_{1,j} \mathrm{d}s \right) \mathrm{d}y_j. \tag{5.26}$$

Note that the norms $|\mathrm{d}y_j|$ may not have the same asymptotic behaviour for different values of $j = 1, \ldots, m+d-1$. In fact, it follows from (5.1) that we have

$$|\mathrm{d}y_j| = \begin{cases} O(w^{-\frac{1}{2}}) & \text{if } \pi^*\widehat{g}_{jj} > 0, \\ O(1) & \text{if } \pi^*\widehat{g}_{jj} = 0, \end{cases} \tag{5.27}$$

$$\text{and} \quad \frac{1}{|\mathrm{d}y_j|} = \begin{cases} O(w^{\frac{1}{2}}) & \text{if } \pi^*\widehat{g}_{jj} > 0, \\ O(1) & \text{if } \pi^*\widehat{g}_{jj} = 0. \end{cases} \tag{5.28}$$

As $|\widehat{\eta}| = O(w^{-\delta})$, we conclude that either $|\widehat{\eta}_{1,j}| = O(w^{-\delta+\frac{1}{2}})$ or $|\widehat{\eta}_{1,j}| = O(w^{-\delta})$ and hence, the integrals in (5.26) are all finite because we chose $-\delta + 1 < -1$.

We also observe from (5.22) that $\widehat{\theta}_t = \widehat{\theta}_s$ for all $s, t \leqslant t_1$, so $\widehat{\theta}$ extends to a smooth 1-form on E. Moreover, we can read off (5.26) that $|\widehat{\theta}| = O(w^{-\delta+1})$, i.e. $\widehat{\theta} \in C^0_{-\delta+1}(T^*E)$. It is possible to obtain estimates on derivatives of $\widehat{\theta}$ and to show that $\widehat{\theta} \in C^\infty_{-\delta+1}(T^*E)$. However, this is a long calculation which relies only on two main observations. First, we deduce from Lemma 4.3 that $|\nabla^l \mathrm{d}y_j|$ behaves asymptotically like $|\mathrm{d}y_j| w^{-l}$ for all $l \geqslant 0$ and $j = 0, \ldots, 2(m+d)-1$. Secondly, we can conclude from $\eta \in C^\infty_{-\delta}(\Lambda^* T^* E)$ that also $|\nabla^l \widehat{\eta}_1| = O(w^{-\delta-l})$. Using formula (5.26), it is then straight forward to verify $|\nabla^l \widehat{\theta}| = O(w^{-\delta-l+1})$, as claimed. We leave the details to the reader, but remark that the required estimate is similar to bounding $|\widehat{\theta}|$.

It remains to show that $\widehat{\eta} = \mathrm{d}\widehat{\theta}$ by considering its components. In fact, it is an easy calculation ([22, p. 80]) to prove that

$$\frac{\partial}{\partial t}\left(i_t^*(\widehat{\eta} - \mathrm{d}\widehat{\theta}) \right) = 0,$$

i.e. $i_s^*(\widehat{\eta} - \mathrm{d}\widehat{\theta}) = i_t^*(\widehat{\eta} - \mathrm{d}\widehat{\theta})$ for all $s, t \in \mathbb{R}$. Since $\widehat{\eta}, \widehat{\theta} \to 0$ as $t \to \infty$, we conclude that $i_t^*(\widehat{\eta} - \mathrm{d}\widehat{\theta}) = 0$ for any $t \in \mathbb{R}$. Moreover, it is shown in [22, p. 80] that

$$\iota_{\frac{\partial}{\partial t}} \widehat{\eta} = \iota_{\frac{\partial}{\partial t}} \mathrm{d}\widehat{\theta},$$

and hence $\widehat{\eta} = \mathrm{d}\widehat{\theta}$ as we claimed. □

Proof of Theorem 5.4. — The strategy is the same as for the proof of [9, Theorem 3.11]. We start with some basic observations. By Proposition 5.13, there exists a $\theta \in C^{\infty}_{1-\delta}(\Lambda^* E)$ such that $\mathrm{d}\theta = \eta$. Since η is real, θ will also be a real form, i.e. $\theta^{1,0} = \overline{\theta^{0,1}}$ if $\theta = \theta^{1,0} + \theta^{0,1}$ is the decomposition into types. Moreover, η is of type (1,1), so we must have that $\partial\theta^{1,0} = \bar{\partial}\theta^{0,1} = 0$. If ∂^* denotes the formal dual of ∂ (w.r.t. the L^2-metric induced by g_φ), then $\partial^*\theta^{1,0} \in C^{\infty}_{-\delta}(E)$. We would like to find a solution u to the equation $\Delta u = \partial^*\theta^{1,0}$, whose growth we can control. There are two cases to consider, corresponding to part (i) and (ii) of Proposition 5.6. First, we consider the case where the degree k of the polynomial Q is greater or equal to 2. By (ii) of Proposition 5.6, there exists $u \in C^{2,\alpha}(E)$ such that $\Delta u = \partial^*\theta^{1,0}$ and $|u| + |\nabla u| = O(w^{2-\delta+\varepsilon})$. It follows that $\partial^*\left(\partial u - \theta^{1,0}\right) = \partial\left(\partial u - \theta^{1,0}\right) = 0$, and hence the 1-form $\partial u - \theta^{1,0}$ is harmonic by the Kähler identities. Choosing $\varepsilon > 0$ small enough, we can assume that $2 - \delta + \varepsilon < 0$ and hence we see from $|\nabla u| = O(w^{2-\delta+\varepsilon})$ and $\theta \in C^{\infty}_{1-\delta}(\Lambda^* E)$ that

$$|\partial u - \theta^{1,0}| \leqslant |\mathrm{d}u| + |\theta| \to 0,$$

as $t \to \infty$. Then Lemma 5.11 implies $\partial u - \theta^{1,0} = 0$ and consequently,

$$\eta = \mathrm{d}\theta = \partial\theta^{0,1} + \bar{\partial}\theta^{1,0} = \partial\bar{\partial}\bar{u} + \bar{\partial}\partial u = -2\sqrt{-1}\partial\bar{\partial}\,\mathrm{Im}\, u,$$

where $\mathrm{Im}\, u$ is the imaginary part of u. It remains to show that $\mathrm{Im}\, u \in C^{\infty}_{2-\delta}(E)$ as opposed to only $\mathrm{Im}\, u \in C^{2,\alpha}(E)$ and $\mathrm{Im}\, u = O(w^{2-\delta+\varepsilon})$. As we can choose $\varepsilon > 0$ so that $2 - \delta + \varepsilon < 0$, this improvement of the decay rate, however, follows immediately from Proposition 5.10 if we can show $\mathrm{d}\,\mathrm{Im}\, u \in C^{\infty}_{1-\delta}(\Lambda^* E)$. This last condition is clearly satisfied since $\theta^{1,0}, \theta^{0,1} \in C^{\infty}_{1-\delta}(\Lambda^* E)$ and $\theta^{1,0} - \theta^{0,1} = \partial u - \overline{\partial u} = d\,\mathrm{Re}\, u + \sqrt{-1} d\,\mathrm{Im}\, u$. This settles the first case. For the second case, assume that $k \leqslant 1$. We want to use (i) of Proposition 5.6 to solve $\Delta u = \partial^*\theta^{1,0}$. This time, however, we only know that the solution u satisfies $\int |\nabla u|^2 \omega_\varphi^{m+d} < +\infty$, and not necessarily that u decays towards infinity. So the idea is to use the vanishing Theorem 5.12 instead. Before applying Proposition 5.6 (i), we need to verify that $\int \partial^*\theta^{1,0}\omega_\varphi^{m+d}$ is zero. For any $t_0 \in \mathbb{R}$, define $K_{t_0} = \{z \in E \mid t(z) \leqslant t_0\}$ and consider the integral

$$\int_{K_{t_0}} \partial^*\theta^{1,0}\omega_\varphi^{m+d} = \int_{K_{t_0}} d * \theta^{1,0} = \int_{\{t_0\}\times S} *\theta^{1,0}, \tag{5.29}$$

where we used Stoke's for the last equality. If we equip the slice $\{t_0\} \times S$ with the restriction of g_φ and denote the corresponding volume by $\mathrm{Vol}(\{t_0\} \times S)$, then we can estimate

$$\left| \int_{\{t_0\}\times S} *\theta^{1,0} \right| \leqslant \mathrm{Vol}(\{t_0\} \times S) \sup_{\{t_0\}\times S} |\theta| = O(w^{k+1-\delta}(t_0)),$$

since $|\theta| = O(w^{1-\delta})$ and $\mathrm{Vol}(\{t_0\} \times S) = O(w^k)$. It follows that the right hand side of (5.29) goes to zero as $t_0 \to +\infty$, as we assumed $k \leqslant 1$ and $\delta > 2$. Hence $\int \partial^* \theta^{1,0} \omega_\varphi^{m+d} = 0$, as claimed. So we find a $u \in C^{2,\alpha}(E)$ such that $\Delta u = \partial^* \theta^{1,0}$ and $\int |\nabla u|^2 \omega_\varphi^{m+d}$ is finite. In particular, the 1-form $\beta = \theta^{1,0} - \partial u$ is harmonic. Also note that

$$|\theta| \omega_\varphi^{m+d} = O(w^{2-2\delta+k})$$

with $2 - 2\delta + k < -1$, so that θ is in L^2, and thus β is L^2 as well. It follows that $d\beta = d^*\beta = 0$, and in particular, β is an L^2-holomorphic (1,0)-form. Hence it must be identically zero by Theorem 5.12, i.e. $\theta^{1,0} = \partial u$. The rest of the proof is now analogous to the first case. □

Bibliography

[1] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 10, Springer, 1987.

[2] O. Biquard & H. Macbeth, "Steady Kähler-Ricci solitons on crepant resolutions of finite quotients of $\mathbb{C}^n$", `https://arxiv.org/abs/1711.02019`, 2017.

[3] E. Calabi, "Métriques kählériennes et fibrés holomorphes", *Ann. Sci. Éc. Norm. Supér.* **12** (1979), no. 2, p. 269-294.

[4] H.-D. Cao, "Existence of gradient Kähler–Ricci solitons", in *Elliptic and parabolic methods in geometry*, A K Peters, 1996, p. 1-16.

[5] A. Chaljub-Simon & Y. Choquet-Bruhat, "Problèmes elliptiques du second ordre sur une variété euclidienne à l'infini", *Ann. Fac. Sci. Toulouse, Math.* **1** (1979), no. 1, p. 9-25.

[6] T. Chave & G. Valent, "On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties", *Nucl. Phys., B* **478** (1996), no. 3, p. 758-778.

[7] J. Cheeger, M. Gromov & M. Taylor, "Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds", *J. Differ. Geom.* **17** (1982), no. 1, p. 15-53.

[8] R. J. Conlon & A. Deruelle, "Steady gradient Kähler-Ricci solitons on crepant resolutions of Calabi-Yau cones", `https://arxiv.org/abs/2006.03100`, 2020.

[9] R. J. Conlon & H.-J. Hein, "Asymptotically conical Calabi–Yau manifolds, I", *Duke Math. J.* **162** (2013), no. 15, p. 2855-2902.

[10] A. S. Dancer & M. Y. Wang, "On Ricci solitons of cohomogeneity one", *Ann. Global Anal. Geom.* **39** (2011), no. 3, p. 259-292.

[11] M. Feldman, T. Ilmanen & D. Knopf, "Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons", *J. Differ. Geom.* **65** (2003), no. 2, p. 169-209.

[12] A. Futaki & M.-T. Wang, "Constructing Kähler–Ricci solitons from Sasaki–Einstein manifolds", *Asian J. Math.* **15** (2011), no. 1, p. 33-52.

[13] D. Gilbarg & N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer, 2001, reprint of the 1998 edition.

[14] R. Goto, "Calabi–Yau structures and Einstein–Sasakian structures on crepant resolutions of isolated singularities", *J. Math. Soc. Japan* **64** (2012), no. 3, p. 1005-1052.

[15] R. S. Hamilton, "The Ricci flow on surfaces", in *Mathematics and general relativity (Santa Cruz, CA, 1986)*, Contemporary Mathematics, vol. 71, American Mathematical Society, 1988, p. 237-262.

[16] M. Haskins, H.-J. Hein & J. Nordström, "Asymptotically cylindrical Calabi–Yau manifolds", *J. Differ. Geom.* **101** (2015), no. 2, p. 213-265.

[17] H.-J. Hein, "Weighted Sobolev inequalities under lower Ricci curvature bounds", *Proc. Am. Math. Soc.* **139** (2011), no. 8, p. 2943-2955.

[18] A. D. Hwang & M. A. Singer, "A momentum construction for circle-invariant Kähler metrics", *Trans. Am. Math. Soc.* **354** (2002), no. 6, p. 2285-2325.

[19] T. Ivey, "Ricci solitons on compact three-manifolds", *Differ. Geom. Appl.* **3** (1993), no. 4, p. 301-307.

[20] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, 2000.

[21] C. Li, "On rotationally symmetric Kähler–Ricci solitons", `https://arxiv.org/abs/1004.4049`, 2010.

[22] S. P. Marshall, "Deformations of special Lagrangian submanifolds", PhD Thesis, University of Oxford, 2002.

[23] O. Munteanu & J. Wang, "Kähler manifolds with real holomorphic vector fields", *Math. Ann.* **363** (2015), no. 3-4, p. 893-911.

[24] H. Pedersen, C. Tønnesen-Friedman & G. Valent, "Quasi-Einstein Kähler metrics", *Lett. Math. Phys.* **50** (1999), no. 3, p. 229-241.

[25] P. Petersen, "Convergence theorems in Riemannian geometry", in *Comparison geometry*, Mathematical Sciences Research Institute Publications, vol. 30, Cambridge University Press, 1997, p. 167-202.

[26] B. Yang, "A characterization of noncompact Koiso-type solitons", *Int. J. Math.* **23** (2012), no. 05, article no. 1250054 (13 pages).