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Francisco Nicolás and Pierre Py

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Irrational pencils and Betti numbers (*)

Francisco Nicolás (1) and Pierre Py (2)

Abstract. — We study irrational pencils with isolated critical points on compact aspherical complex manifolds. We prove that if the set of critical points is nonempty, the homology of the kernel of the morphism induced by the pencil on fundamental groups is not finitely generated. This generalizes a result by Dimca, Papadima and Suciu. By considering self-products of the Cartwright–Steger surface, this allows us to build new examples of smooth projective varieties whose fundamental group has a non-finitely generated homology.

Résumé. — Nous étudions les pinceaux irrationnels à points critiques isolés sur les variétés complexes compactes et asphériques. Nous prouvons que si un tel pinceau possède au moins un point critique, alors l'homologie du noyau du morphisme induit entre groupes fondamentaux n'est pas de type fini. Ceci généralise un résultat de Dimca, Papadima et Suciu. En considérant le produit de plusieurs copies de la surface de Cartwright–Steger, ceci nous permet de donner de nouveaux exemples de variétés projectives lisses dont le groupe fondamental a un groupe d'homologie qui n'est pas de type fini.

1. Introduction

1.1. Finiteness properties and fundamental groups of smooth projective varieties

Recall that a group G is of type $\mathscr{F}_\ell$, for some integer ℓ, if there exists a classifying space for G (i.e. a $K(G,1)$) which is a CW-complex with finite ℓ-skeleton. This condition was introduced by Wall [24]. For finitely presented

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(1) IRMA, Université de Strasbourg, 67084 Strasbourg, France — f.nicolascardona@gmail.com
(2) IRMA, Université de Strasbourg, 67084 Strasbourg, France — ppy@math.unistra.fr
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groups, this is equivalent to the property of being of type FP_ℓ (see [7] for the definition of this last property). Given a group G, one can ask whether it admits a classifying space which is a finite complex or whether it has property $\mathscr{F}_\ell$ for some ℓ. We refer to [1, 2, 5, 20] for important works on these notions.

In the context of the study of fundamental groups of smooth projective varieties, called *projective groups* in what follows, Kollár asked in [14, §0.3.1] whether a projective group is always commensurable (up to finite kernels) to a group admitting a classifying space which is a quasi-projective variety. Since any quasi-projective variety has the homotopy type of a finite complex [10, p. 27], a positive answer to this question would imply that any projective group is commensurable to a group having a finite classifying space. However, a negative answer to Kollár's question was given by Dimca, Papadima and Suciu in [11]. To describe their results let us introduce some notations.

Notation. — Throughout this text, X will be a (connected) compact complex manifold of complex dimension $n \geqslant 2$ and S will be a closed Riemann surface of positive genus. A surjective holomorphic map with connected fibers $f : X \to S$ is called an *irrational pencil.* For such a map, we will always denote by Λ the kernel of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S)$.

In [11] the authors proved the following two results.

Theorem 1.1 (see [11, Theorem C]). — *If $n \geqslant 3$ and if $f : X \to S$ is an irrational pencil with isolated critical points, then the fundamental group of a smooth fiber of f embeds into that of X and coincides with the kernel Λ of the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S)$.*

Theorem 1.2 (see [11, §2]). — *Let $X = \Sigma_1 \times \cdots \times \Sigma_n$ be a direct product of n Riemann surfaces of genus greater than 1 and let S have genus 1. If $f : X \to S$ is an irrational pencil with isolated critical points then the group $H_n(\Lambda, \mathbb{Q})$ is not finitely generated.*

Remark 1.3. — If we use the group structure on S, it is easy to see that any irrational pencil $f : X \to S$ as in Theorem 1.2 is the sum of holomorphic maps $f_i : \Sigma_i \to S$, i.e.

$$f(p_1, \dots, p_n) = f_1(p_1) + \cdots + f_n(p_n).$$

If all the f_i's are nonconstant and $n \geqslant 2$, f has connected fibers if and only if it is π_1-surjective, see Lemma 2.1 in [17]. The set of critical points of f is the product of the critical sets of the f_i's. Hence f has isolated critical points if and only if all the f_i's are nonconstant.

Combining Theorems 1.1 and 1.2, Dimca, Papadima and Suciu answered negatively Kollár's question. Indeed in the situation of Theorem 1.2 the group Λ, which is the fundamental group of a smooth fiber of f if $n \geqslant 3$, cannot be of type FP_n as the group $H_n(\Lambda, \mathbb{Z})$ is not finitely generated. The property of being of type FP_n is invariant by the commensurability relation [2, 11]. Therefore, no group commensurable to Λ can have a finite classifying space.

Building on the work [11], further examples of fundamental groups of smooth projective varieties with exotic finiteness properties were constructed and studied by Llosa Isenrich [17, 18] and by Bridson and Llosa Isenrich [6]. Note however that all the examples studied in [6, 17, 18] are either subgroups of direct products of surface groups or extensions of such subgroups as in [6]. The purpose of this note is to provide further examples of projective groups violating the $\mathscr{F}_\ell$ condition for some ℓ and which are not isomorphic to subgroups of direct products of surface groups (see Proposition 3.1), as well as to prove that the conclusion of Theorem 1.2 holds in much greater generality.

1.2. Irrational pencils and Betti numbers

Let $f : X \to S$ be an irrational pencil with $\dim_{\mathbb{C}} X = n \geqslant 2$. As before Λ denotes the kernel of the morphism $f_* : \pi_1(X) \to \pi_1(S)$. We assume that the critical points of f are isolated and that f is not a submersion; its critical set is then a nonempty finite set. Let $\widehat{X} \to X$ be the covering space such that $\pi_1(\widehat{X}) \simeq \Lambda$. Our main results are the following:

Theorem A. — *The homology group $H_n(\widehat{X}, \mathbb{Q})$ is infinite dimensional.*

Theorem B. — *If X is aspherical, the group $H_n(\Lambda, \mathbb{Q})$ is infinite dimensional. In particular Λ is not of type* FP_n.

Note that if X is aspherical, the space $\widehat{X}$ is a $K(\Lambda, 1)$, hence Theorem B follows from Theorem A. The very short proof of Theorem A will be given in Section 2.

When $n \geqslant 3$, the group Λ is isomorphic to the fundamental group of a generic fiber of f, thanks to Theorem 1.1. Hence, when $n \geqslant 3$ and X is aspherical and projective, Λ is a projective group with exotic finiteness properties. We now discuss the relation of our results with the earlier works [3, 11, 13, 17, 18].

To study the topology of the covering space $\widehat{X}$, we follow the same method as in the articles [11, 13], which can be seen as a complex analog of Bestvina and Brady's work [1]. Namely, denoting by $\widehat{S}$ the universal cover of S, we consider a lift $\widehat{f} : \widehat{X} \to \widehat{S}$ of f and analyze the topology of $\widehat{X}$ by viewing it as the union $\bigcup_{k\geqslant 1} g^{-1}(D_k)$ where $(D_k)_{k\geqslant 1}$ is an increasing union of disks in $\widehat{S}$ and $g : \widehat{X} \to \widehat{S}$ is either the map $\widehat{f}$ or a small perturbation of it.

In the special case when X is a product of Riemann surfaces and S has genus 1, Theorem B reduces to Theorem 1.2 above. Dimca, Papadima and Suciu's proof was based on the notion of *characteristic varieties*. Our proof however is more direct and applies in full generality: X can be any aspherical complex manifold and the genus of S need not be equal to 1.[(1)]

When X is a product of Riemann surfaces, the fact that Λ is not of type FP_n can be deduced from the work of Bridson, Howie, Miller and Short [5]. See [17, 18] for further results which rely on properties of subgroups of direct products of surface groups.

Under the same assumptions as in Theorem B, the article [3] proves the weaker result that the group Λ is not of type FP see [7, §VIII.6] for the definition of this property and Theorem 7.3 in [3] for this result.

When $n = 2$, Kapovich [13] has proved that if $f : X \to S$ is an irrational pencil which is not a submersion, with no multiple fiber and with X aspherical, then the group $\Lambda = \ker(f_*)$ is not finitely presented (without assuming that f has isolated critical points). In the case of maps with isolated critical points, Theorem B gives a slight strengthening of Kapovich's result since a finitely presented group must have finitely generated second homology group. Our proof is very close in spirit to the one in [13].

1.3. Self-products of the Cartwright–Steger surface

It is of course interesting to look for more examples of projective (or closed Kähler) manifolds endowed with an irrational pencil to which one can apply Theorems A and B. One way to build new examples is to use the *Cartwright–Steger surface*. Recall that this surface is a smooth compact complex surface which is a quotient of the unit ball B of $\mathbb{C}^2$. We will denote it by Y in what follows. Hence there exists a torsion-free cocompact lattice $\Gamma < \mathrm{PU}(2,1)$ such that $Y = B/\Gamma$. The surface Y is characterized (up to changing the sign of the complex structure) by the fact that its Euler characteristic is equal to 3 and its first Betti number is equal to 2. It was discovered in [9] in the

(1) On the other hand the article [11] also studies the finiteness properties of arbitrary normal coabelian subgroups of direct products of fundamental groups of closed surfaces, not necessarily coming from irrational pencils.

context of the classification of fake projective planes. It was further studied by several authors, see for instance [8, 21, 22].

We now denote by $h : Y \to E$ the Albanese map of Y, whose target is an elliptic curve since $b_1(Y) = 2$. Cartwright, Koziarz and Yeung [8] have proved that the map h has isolated critical points and Koziarz and Yeung later proved that these critical points are nondegenerate [15]. We can thus consider the product Y^b of Y with itself b times and the map

$$h + \cdots + h : Y^b \longrightarrow E. \tag{1.1}$$

This provides natural examples to which one can apply Theorem A. We discuss these examples in Section 3. Denoting by $\Gamma < \mathrm{PU}(2,1)$ the fundamental group of the Cartwright–Steger surface, our construction together with Theorem B immediately implies:

Theorem C. — *The direct product of b copies of Γ contains a coabelian normal subgroup N which is of type* FP_{2b-1} *but satisfies that $H_{2b}(N,\mathbb{Q})$ is infinite dimensional.*

The group N appearing above is the kernel of the morphism on fundamental groups induced by the map (1.1). The fact that N is of type FP_{2b-1} will be explained at the end of Section 2, in Remark 2.2. For a description of the lattice Γ, we refer the reader to [8].

2. Growth of the n-th Betti number

We prove here Theorem A. We let $f : X \to S$ and $\widehat{X} \to X$ be as in Section 1.2. Let $\widehat{S}$ be the universal cover of S and $\widehat{f} : \widehat{X} \to \widehat{S}$ be a lift of f. The surface $\widehat{S}$ is topologically a plane. We now make the following:

Observation. — There exists a C^∞ map $\widehat{f}_0 : \widehat{X} \to \widehat{S}$, C^∞-close to $\widehat{f}$, which is holomorphic in a neighborhood of its critical set and such that each critical point is nondegenerate.

This observation is well-known, see [19]. To prove the existence of such a map $\widehat{f}_0$ we perturb $\widehat{f}$ in a neighborhood of each degenerate critical point. We first identify $\widehat{S}$ with $\mathbb{C}$ or the unit disc of $\mathbb{C}$. Let $q \in \widehat{X}$ be a critical point of $\widehat{f}$. We pick a neighborhood U_q of q that we identify with an open ball $B(0,\varepsilon)$ centered at the origin of $\mathbb{C}^n$ via the choice of some holomorphic coordinates. So we can consider the map $\widehat{f} : B(0,\varepsilon) \simeq U_q \to \widehat{S} \subset \mathbb{C}$. Let ℓ_q be a linear form on $\mathbb{C}^n$ which is a regular value of the map $d\widehat{f} : B(0,\varepsilon) \to (\mathbb{C}^n)^*$ and which is small enough (if q is nondegenerate, we take $\ell_q = 0$). Then the map $\widehat{f}-\ell_q : B(0,\varepsilon) \to \widehat{S} \subset \mathbb{C}$ still takes values in $\widehat{S}$ and has finitely many critical points which are all nondegenerate. The number of its critical points is exactly the Milnor number of the critical point q for the map $\widehat{f}$. If ℓ_q is small enough, we can assume that all critical points lie in the ball $B(0,\frac{\varepsilon}{2})$ (see Appendix B in [19], in particular the remark on page 113, or [12, §5.4] for more details). In the following we write $V_q = B(0,\frac{\varepsilon}{2})$. We perform the previous construction for each point of the critical set $\mathrm{Crit}(\widehat{f})$ of $\widehat{f}$, assuming that the open sets

$$(U_q)_{q\in\mathrm{Crit}(\hat{f})}$$

are disjoint. One builds the map $\widehat{f}_0$ by declaring that $\widehat{f}_0$ is equal to $\widehat{f}-\ell_q$ on V_q, to $\widehat{f}$ outside the union of the open sets $(U_q)_{q\in\mathrm{Crit}(\hat{f})}$ and to a deformation between $\widehat{f}$ and $\widehat{f}-\ell_q$ on $U_q - V_q$. This can be performed in such a way that every critical point of $\widehat{f}_0$ is contained in one of the V_q's. Although this is not necessary, we also observe that all this can be done in an equivariant way for the action of $\pi_1(X)$, so that the map $\widehat{f}_0$ descends to a map $f_0 : X \to S$ homotopic to f. This implies the statement of the observation.

Since all the discussion below is topological, we will now work with the map $\widehat{f}_0$ instead of $\widehat{f}$. As in [11], we study the topology of $\widehat{X}$ by viewing it as the increasing union of a well-chosen sequence of compact subsets. We write

$$\widehat{S} = \bigcup_{k\geqslant 1} D_k$$

where each D_k is homeomorphic to a closed disk, $D_k \subset \mathrm{Int}(D_{k+1})$, no critical value of $\widehat{f}_0$ is contained in the boundary of D_k, $D_{k+1} - D_k$ contains exactly one critical value for $k \geqslant 1$ and D_1 contains no critical value. We set $\widehat{X}_k := \widehat{f}_0^{-1}(D_k)$. Hence $\widehat{X}_1$ retracts onto a smooth fiber of $\widehat{f}_0$. For $k \geqslant 2$, the topology of the space $\widehat{X}_k$ is described thanks to the following proposition.

Proposition 2.1. — *The space $\widehat{X}_{k+1}$ has the homotopy type of a space obtained from $\widehat{X}_k$ by gluing to it a finite number $m_k > 0$ of n-dimensional cells.*

This proposition is well-known (see e.g. Lemma 3.3 in [11] for a related statement, although in that lemma the authors work with the original map $\widehat{f}$ instead of our perturbation $\widehat{f}_0$). We briefly sketch its proof below, for the reader's convenience.

Proof. — Let $c : [0,1] \to \mathrm{Int}(D_{k+1})$ be an embedded arc going from a boundary point $c(0)$ of D_k to the unique critical value contained in $D_{k+1} - D_k$. We assume that $c(t) \notin D_k$ for $t > 0$. Let D^* be a small disk centered at $c(1)$ and contained in $\mathrm{Int}(D_{k+1} \setminus D_k)$ (see Figure 2.1).

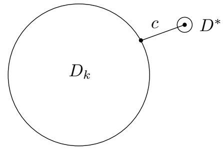

Figure 2.1. The disks D_k and D^*

Since the restriction of $\widehat{f}_0$ to the preimage of the set of regular values is a locally trivial fibration, $\widehat{X}_{k+1}$ deformation retracts onto $\widehat{X}_k \cup \widehat{f}_0^{-1}(c([0,1]) \cup D^*)$. Let $m_k > 0$ be the number of critical points in the level $\widehat{f}_0^{-1}(c(1))$. Let $x_1, \ldots, x_{m_k}$ be the corresponding critical points. We fix a regular value $t = c(1-\delta)$ (for a small $\delta > 0$) of $\widehat{f}_0$ close enough to $c(1)$. According to the theory of Lefschetz fibrations (see [23, §14.2.2]) one can find m_k disjoint $(n-1)$-dimensional spheres

$$S_1 \sqcup \ldots \sqcup S_{m_k} \subset \widehat{f}_0^{-1}(t)$$

(each sphere S_j being contained in an arbitrary small neighborhood of x_j fixed in advance) such that $\widehat{f}_0^{-1}(D^*)$ has the homotopy type of the space obtained from $\widehat{f}_0^{-1}(t)$ by gluing an n-dimensional ball to each of the spheres S_j. By fixing a trivialization of the fibration

$$\widehat{f}_0 : \widehat{f}_0^{-1}(c([0,1))) \longrightarrow c([0,1)),$$

each sphere S_j can be identified to a sphere $S_j^* \subset \widehat{f}_0^{-1}(c(0))$ in such a way that the $(S_j^*)_{1 \leqslant j \leqslant m_k}$ are disjoint. Hence $\widehat{X}_{k+1}$ retracts onto the space obtained from $\widehat{X}_k$ by gluing a ball to each of the spheres S_j^*. This proves the result. □

Proposition 2.1 immediately implies the following:

Observation. — The sequence $b_{n-1}(\widehat{X}_k) = \dim_{\mathbb{Q}} H_{n-1}(\widehat{X}_k, \mathbb{Q})$ is decreasing with k.

Let k_0 be a large enough integer such that the sequence

$$(b_{n-1}(\widehat{X}_k))_{k \geqslant k_0}$$

is constant. We will now prove that the sequence $(b_n(\widehat{X}_k))_{k \geqslant k_0}$ is strictly increasing with k and that each map

$$H_n(\widehat{X}_k, \mathbb{Q}) \longrightarrow H_n(\widehat{X}_{k+1}, \mathbb{Q}) \tag{2.1}$$

induced by the inclusion $\widehat{X}_k \hookrightarrow \widehat{X}_{k+1}$ is injective for $k \geqslant k_0$. This immediately implies Theorem A since the group $H_n(\widehat{X}, \mathbb{Q})$ is the direct limit of the $H_n(\widehat{X}_k, \mathbb{Q})$'s.

We use the notations from the proof of Proposition 2.1. Let $k \geqslant k_0$. We know that $\widehat{X}_{k+1}$ has the homotopy type of a space W_{k+1} obtained by gluing a ball to each of the spheres

$$S_1^* \sqcup \ldots \sqcup S_{m_k}^* \subset \widehat{X}_k.$$

We write:

$$W_{k+1} = \widehat{X}_k \cup B_1 \cup \ldots \cup B_{m_k} \tag{2.2}$$

where each B_j is homeomorphic to an n-dimensional ball, $B_j \cap B_l = \emptyset$ for $l \neq j$ and $\widehat{X}_k \cap B_j$ is equal to the boundary of B_j (or to the sphere S_j^* depending on whether one views it inside $\widehat{X}_k$ or B_j). Since the inclusion of $\widehat{X}_k$ into $\widehat{X}_{k+1}$ induces an isomorphism on $(n-1)$-dimensional homology groups, the same occurs for each inclusion $\widehat{X}_k \hookrightarrow W_{k+1}$. We now apply the Mayer–Vietoris exact sequence to the decomposition of W_{k+1} given in (2.2).

We obtain (all homology groups being with $\mathbb{Q}$ coefficients):

$$\begin{aligned} H_n(\textstyle\bigsqcup_{j=1}^{m_k} \partial B_j) \longrightarrow H_n(\widehat{X}_k) \oplus H_n(\textstyle\bigsqcup_{j=1}^{m_k} B_j) \longrightarrow H_n(W_{k+1}) \\ \longrightarrow H_{n-1}(\textstyle\bigsqcup_{j=1}^{m_k} \partial B_j) \longrightarrow H_{n-1}(\widehat{X}_k) \oplus H_{n-1}(\textstyle\bigsqcup_{j=1}^{m_k} B_j) \\ \longrightarrow H_{n-1}(W_{k+1}). \end{aligned} \tag{2.3}$$

Since the groups $H_n(\bigsqcup_{j=1}^{m_k} \partial B_j)$, $H_n(\bigsqcup_{j=1}^{m_k} B_j)$ and $H_{n-1}(\bigsqcup_{j=1}^{m_k} B_j)$ are zero we obtain:

$$\begin{aligned} \{0\} \longrightarrow H_n(\widehat{X}_k) \longrightarrow H_n(W_{k+1}) \longrightarrow H_{n-1}(\textstyle\bigsqcup_{j=1}^{m_k} \partial B_j) \\ \longrightarrow H_{n-1}(\widehat{X}_k) \longrightarrow H_{n-1}(W_{k+1}). \end{aligned} \tag{2.4}$$

The last arrow on the right being an isomorphism, this implies that the following sequence is exact:

$$\{0\} \longrightarrow H_n(\widehat{X}_k) \longrightarrow H_n(W_{k+1}) \longrightarrow H_{n-1}(\textstyle\bigsqcup_{j=1}^{m_k} \partial B_j) \longrightarrow \{0\}. \tag{2.5}$$

This implies that each inclusion $\widehat{X}_k \hookrightarrow W_{k+1}$ (and hence the inclusion $\widehat{X}_k \hookrightarrow \widehat{X}_{k+1}$) induces an injective map on $H_n(\,\cdot\,, \mathbb{Q})$ and that $b_n(\widehat{X}_k) = b_n(\widehat{X}_{k+1}) + m_k$. Note that $m_k > 0$. This is the desired result and concludes the proof of Theorem A.

Remark 2.2. — Assume that X is aspherical. Then the space $\widehat{X}$ is a $K(\Lambda, 1)$. It has the homotopy type of a smooth fiber of $\widehat{f}$ (or f) with an infinite number of n-dimensional cells attached. This observation already appears in [11] (see Corollary 5.8). This implies that Λ is of type $\mathscr{F}_{n-1}$ (hence FP_{n-1}), although it is not of type FP_n.

3. Examples built from the Cartwright–Steger surface

We resume with the notation from Section 1.3. The Cartwright–Steger surface is denoted by Y and $h : Y \to E$ is its Albanese map. Besides considering the products $Y \times \cdots \times Y$, one can also build more examples by combining the construction by Dimca, Papadima and Suciu and our construction. We fix a family of ramified covers $p_i : \Sigma_i \to E$ of the elliptic curve E $(1 \leqslant i \leqslant a)$, where each Σ_i has negative Euler characteristic. We then consider the map

$$f : \Sigma_1 \times \cdots \times \Sigma_a \times Y \times \cdots \times Y \longrightarrow E$$

(where there are $b \geqslant 1$ copies of Y) which is the sum of the p_i's and of the map h on each copy of Y. All the results until the end of this section also apply when $a = 0$, i.e. when one studies the map $f = h + \cdots + h$ as in (1.1).

The map f has a finite non-empty set of critical points and connected fibers. This last point follows from the fact that $h : Y \to E$ has connected fibers. We denote by Λ the kernel of the map induced by f on fundamental groups. Theorem 1.1 and Theorem B imply that the group $H_{a+2b}(\Lambda, \mathbb{Q})$ is not finitely generated and that Λ is projective if and only if $2b + a \geqslant 3$. The following proposition shows that the group Λ is of a different nature compared to the examples from [11, 17].

Proposition 3.1. — *No finite index subgroup of Λ embeds in a direct product of surface groups.*

By a *surface group* we mean here the fundamental group of an oriented surface of finite type (open or closed). Hence a surface group is either free or the fundamental group of a closed oriented surface. To prove Proposition 3.1, we will make use of the following theorem due to Bridson, Howie, Miller and Short [5].

Theorem 3.2. — *Let $F_1, \ldots, F_m$ be surface groups. Let G be a subgroup of the direct product $F_1 \times \cdots \times F_m$. If G is of type* FP_m*, then G is virtually isomorphic to a direct product of the form $H_1 \times \cdots \times H_k$ where $k \leqslant m$ and each H_i is a surface group. In particular G is of type* FP_∞*.*

Besides this theorem, we will also use the fact that the property of being FP_m (for $m \in \mathbb{N} \cup \{\infty\}$) is invariant under passing to a subgroup or an overgroup of finite index [7, VIII.5.1].

Proof of Proposition 3.1. — The group Λ sits inside the direct product

$$\pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y) \times \cdots \times \pi_1(Y). \tag{3.1}$$

The *factors* of this direct product are the subgroups of the form

$$\{1\} \times \cdots \times \pi_1(\Sigma_j) \times \cdots \times \{1\}$$

or

$$\{1\} \times \cdots \times \pi_1(Y) \times \cdots \times \{1\}.$$

Each factor intersects Λ nontrivially. This implies that Λ contains copies of the group $\mathbb{Z}^{a+b}$. The Gromov hyperbolicity of each factor implies that $\mathbb{Z}^{a+b+1}$ does not embed in Λ. Now suppose that a finite index subgroup Λ_1 of Λ embeds in a direct product of surface groups $F_1 \times \cdots \times F_m$. By taking m to be minimal, we may assume that $L_i = \Lambda_1 \cap F_i$ is nontrivial for $i = 1, \ldots, m$. Otherwise Λ_1 embeds in a direct product of $m-1$ surface groups. Since Λ_1 does not contain any nontrivial abelian normal subgroup, this implies that each F_i is non-abelian, hence hyperbolic. A similar argument as before then shows that Λ_1 contains copies of $\mathbb{Z}^m$ but no copy of $\mathbb{Z}^{m+1}$. Hence $m = a+b$. By Remark 2.2, the group Λ is of type FP_{2b+a-1}, hence Λ_1 is. Since $2b+a-1 \geqslant m = a+b$, Λ_1 is of type FP_m and Theorem 3.2 implies that Λ_1 is of type FP_∞. This contradicts the fact that Λ_1 is not of type FP_{a+2b}. □

Finally, we compute, in some cases, the first Betti number of the group $\Lambda = \ker(f_*)$. A similar computation appears in [18, §7], which applies to some of the examples built in [11, 17, 18].

Proposition 3.3. — *Assume that $a+2b \geqslant 3$. Assume furthermore that $b \geqslant 2$ or that $b = 1$ and that the map $\pi_1(\Sigma_j) \to \pi_1(E)$ is surjective for some $j \in \{1, \ldots, a\}$. Then the first Betti number of Λ is equal to:*

$$b_1(\Sigma_1 \times \cdots \Sigma_a \times Y^b) - 2.$$

Proof of Proposition 3.3. — We consider the surjective homomorphism

$$\Lambda \longrightarrow \pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1}$$

obtained by considering the inclusion of Λ in the direct product (3.1) and by projecting onto the first $a+b-1$ factors. Its kernel N consists of elements of the form

$$(1, \ldots, 1, g) \in \pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y) \times \cdots \times \pi_1(Y)$$

where $g \in \ker(h_*)$; it is isomorphic to $\ker(h_*)$. Hence we have the following exact sequence:

$$0 \longrightarrow N \longrightarrow \Lambda \longrightarrow \pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1} \longrightarrow 0.$$

It induces the following short exact sequence (see [7, VII.6], all homology groups are taken with $\mathbb{Z}$ coefficients):

$$H_1(N)_\Lambda \longrightarrow H_1(\Lambda) \longrightarrow H_1(\pi_1(\Sigma_1) \times \cdots \times \pi_1(\Sigma_a) \times \pi_1(Y)^{b-1}) \longrightarrow 0. \quad (3.2)$$

Here $H_1(N)_\Lambda$ is the group of coinvariants of $H_1(N)$ for the Λ-action. It is isomorphic to the quotient of N by the group $[N,\Lambda]$ generated by commutators of elements of Λ and of N. Note that if $x=(1,\dots,1,g)\in N$ and $y=(y_1,\dots,y_a,h_1,\dots,h_b)\in\Lambda$ then

$$xyx^{-1}y^{-1}=(1,\dots,1,gh_bg^{-1}h_b^{-1}).$$

Hence when we identify N with $\ker(h_*)$, $[N,\Lambda]$ is identified with the group $[\ker(h_*),\pi_1(Y)]$ (we are using here that $b\geqslant 2$ or that one of the $\pi_1(\Sigma_j)$ surjects onto $\pi_1(E)$). In particular the groups $H_1(N)_\Lambda$ and $H_1(\ker(h_*))_{\pi_1(Y)}$ are isomorphic. Now the short exact sequence

$$0\longrightarrow\ker(h_*)\longrightarrow\pi_1(Y)\longrightarrow\mathbb{Z}^2\longrightarrow 0$$

induces the short exact sequence (see [7, VII.6] again):

$$H_2(\mathbb{Z}^2)\longrightarrow H_1(\ker(h_*))_{\pi_1(Y)}\longrightarrow H_1(Y)\longrightarrow H_1(\mathbb{Z}^2)\longrightarrow 0.$$

Since the map $H_1(Y)\otimes\mathbb{Q}\to H_1(\mathbb{Z}^2)\otimes\mathbb{Q}$ is an isomorphism, we obtain that $H_1(\ker(h_*))_{\pi_1(Y)}\otimes\mathbb{Q}$ has dimension at most 1. Hence $H_1(N)_\Lambda\otimes\mathbb{Q}$ has dimension at most 1. Since Λ is Kähler and hence has even first Betti number, this implies that the first arrow in (3.2) has finite image. Hence $H_1(\Lambda)\otimes\mathbb{Q}$ and $H_1(\pi_1(\Sigma_1)\times\cdots\times\pi_1(\Sigma_a)\times\pi_1(Y)^{b-1})\otimes\mathbb{Q}$ are isomorphic. This gives the desired result. $\square$

Remark 3.4. — The topology of the generic fiber of our map f (mainly its homotopy groups up to dimension $a+2b-1$) can be described thanks to Theorem 5.2 and Corollary 5.4 from [11]. Indeed all the hypotheses required in loc. cit. are met in our context, except possibly for the hypothesis on resonance varieties. However, that hypothesis is only used in [11] to prove that the group $H_{a+2b}(\Lambda,\mathbb{Q})$ is not finitely generated, a result that follows from our Theorem B.

Remark 3.5. — By considering the case where $a\in\{0,1\}$, our construction provides for each $n\geqslant 2$ an example of a CAT(0) group G containing a subgroup of type FP_{n-1} but not FP_n and such that G does not contain free Abelian subgroups of rank greater than $\lfloor\frac{n+1}{2}\rfloor$. See [4, 16] for related results and motivation. The article [16] produces other examples with a smaller bound on the rank of Abelian subgroups. More precisely, for each positive integer n, Kropholler gives in [16] an example of a group of type $\mathscr{F}_{n-1}$ but not of type $\mathscr{F}_n$, which does not contain free Abelian groups of rank greater than $\lceil\frac{n}{3}\rceil$ and which is a subgroup of a CAT(0) group.

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Bibliography

[1] M. Bestvina & N. Brady, "Morse theory and finiteness properties of groups", *Invent. Math.* **129** (1997), no. 3, p. 445-470.

[2] R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes, Queen Mary College, 1976.

[3] I. Biswas, M. Mj & D. Pancholi, "Homotopical height", *Int. J. Math.* **25** (2014), no. 13, article no. 1450123 (43 pages).

[4] N. Brady, "Branched coverings of cubical complexes and subgroups of hyperbolic groups", *J. Lond. Math. Soc.* **60** (1999), no. 2, p. 461-480.

[5] M. R. Bridson, J. Howie, C. F. Miller, III & H. Short, "The subgroups of direct products of surface groups", *Geom. Dedicata* **92** (2002), p. 95-103.

[6] M. R. Bridson & C. Llosa Isenrich, "Kodaira fibrations, Kähler groups, and finiteness properties", *Trans. Am. Math. Soc.* **372** (2019), no. 8, p. 5869-5890.

[7] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, 1982.

[8] D. I. Cartwright, V. Koziarz & S.-K. Yeung, "On the Cartwright-Steger surface", *J. Algebr. Geom.* **26** (2017), no. 4, p. 655-689.

[9] D. I. Cartwright & T. Steger, "Enumeration of the 50 fake projective planes", *C. R. Math. Acad. Sci. Paris* **348** (2010), no. 1, p. 1-2.

[10] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer, 1992.

[11] A. Dimca, Ş. Papadima & A. I. Suciu, "Non-finiteness properties of fundamental groups of smooth projective varieties", *J. Reine Angew. Math.* **629** (2009), p. 89-105.

[12] W. Ebeling, *Functions of several complex variables and their singularities*, Graduate Studies in Mathematics, vol. 83, American Mathematical Society, 2007.

[13] M. Kapovich, "On normal subgroups in the fundamental groups of complex surfaces", `https://arxiv.org/abs/math/9808085`, 1998.

[14] J. Kollár, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, 1995.

[15] V. Koziarz & S.-K. Yeung, "Stability of the Albanese fibration on the Cartwright-Steger surface", *Taiwanese J. Math.* **25** (2021), no. 2, p. 251-256.

[16] R. Kropholler, "Almost hyperbolic groups with almost finitely presented subgroups", `https://arxiv.org/abs/1802.01658`, 2018.

[17] C. Llosa Isenrich, "Branched covers of elliptic curves and Kähler groups with exotic finiteness properties", *Ann. Inst. Fourier* **69** (2019), no. 1, p. 335-363.

[18] ———, "Kähler groups and subdirect products of surface groups", *Geom. Topol.* **24** (2020), no. 2, p. 971-1017.

[19] J. W. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, vol. 61, University of Tokyo Press, 1968.

[20] J. R. Stallings, "A finitely presented group whose 3-dimensional integral homology is not finitely generated", *Am. J. Math.* **85** (1963), p. 541-543.

[21] M. Stover, "On general type surfaces with $q = 1$ and $c_2 = 3p_g$", *Manuscr. Math.* **159** (2019), no. 1, p. 171-182.
[22] S. Vidussi, "The slope of surfaces with Albanese dimension one", *Math. Proc. Camb. Philos. Soc.* **167** (2019), no. 2, p. 355-360.
[23] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés (Paris), vol. 10, Société Mathématique de France, 2002.
[24] C. T. C. Wall, "Finiteness conditions for CW-complexes. I", *Ann. Math. (2)* **81** (1965), p. 56-69.