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Brilliant families of K3 surfaces: Twistor spaces, Brauer groups, and Noether–Lefschetz loci

Brilliant families of K3 surfaces: Twistor spaces, Brauer groups, and Noether–Lefschetz loci [(*)]

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Abstract. — We describe the Hodge theory of brilliant families of K3 surfaces. Their characteristic feature is a close link between the Hodge structures of any two fibres over points in the Noether–Lefschetz locus. Twistor deformations, the analytic Tate–Šafarevič group, and certain one-dimensional Shimura special cycles are covered by the theory. In this setting, the Brauer group is viewed as the Noether–Lefschetz locus of the Brauer family or as the specialization of the Noether–Lefschetz loci in a family of approaching twistor spaces. Passing from one algebraic twistor fibre to another, which by construction is a transcendental operation, is here viewed as first deforming along the more algebraic Brauer family and then along a family of algebraic K3 surfaces.

Résumé. — On étudie la théorie de Hodge des familles brillantes des surfaces K3. Deux fibres dans le lieu de Noether–Lefschetz d'une telle famille ont des structures de Hodge très similaires. Les déformations de twisteurs, le groupe de Tate–Šafarevič et certaines courbes de Shimura donnent des exemples de telles familles. Dans ce cadre le groupe de Brauer apparaît comme lieu de Noether–Lefschetz de la famille de Brauer et aussi comme spécialisation des lieux de Noether–Lefschetz des espaces de twisteurs. Le passage transcendent d'une fibre algébrique à une autre dans l'espace de twisteur est vu comme composition de deux déformations du caractère plus algébriques.

This note is concerned with the Hodge theory of brilliant families of K3 surfaces. A connected smooth holomorphic family $\mathcal{S} \longrightarrow C$ of K3 surfaces is called a *brilliant deformation* of the fibre $S = \mathcal{S}_0$ if there exists a class $\ell \in H^{1,1}(S, \mathbb{Z})$ such that under parallel transport one has

$$H^{2,0}(\mathcal{S}_t) \subset H^{2,0}(S) \oplus H^{0,2}(S) \oplus \mathbb{C} \cdot \ell$$

with ℓ not contained in $H^{2,0}(\mathcal{S}_t) \oplus H^{0,2}(\mathcal{S}_t)$, see Sections 2.1 and 3.1 for details.

The starting point is the Hodge theory of twistor spaces studied in [8], but more algebraic constructions like Brauer and Tate–Šafarevič groups as well as special cycles of K3 surfaces are covered by the theory. The Hodge theoretic approach adopted here unifies these rather different geometric notions and explains their interactions. We start out by describing three particular examples of brilliant families and then state the main results in the geometric context.

0.1. Examples of brilliant families include the following well-known constructions:

(i) The *Dwork pencil* $\mathcal{S} \longrightarrow \mathbb{A}^1 \setminus \{t \mid t^4 = 1\}$ of quartic surfaces defined by $\sum_{i=0}^{3} x_i^4 - 4t \prod_{i=0}^{3} x_i$, the special fibre $S = \mathcal{S}_0$ of which is the Fermat quartic.

(ii) The *Brauer* (or *Tate–Šafarevič*) *family* $\mathcal{S} \longrightarrow \mathbb{C} \simeq H^{0,2}(S)$ parameterizing all complex elliptic K3 surfaces with a fixed elliptic K3 surface $S \longrightarrow \mathbb{P}^1$ as relative Jacobian.

(iii) The *twistor family* $\mathcal{S} \longrightarrow \mathbb{P}^1$ associated with a polarized K3 surface (S, L).

A priori, these examples have not much in common. For example, the Dwork pencil is an algebraic family, while for the Brauer and the twistor family most of the fibres are not projective. However, as we shall explain, the Hodge theory describing the three families is similar in that $H^{2,0}(\mathcal{S}_t)$ varies in $H^{2,0}(S) \oplus H^{0,2}(S) \oplus \mathbb{C} \cdot \ell$ for a certain algebraic class $\ell \in H^{1,1}(S, \mathbb{Z})$. The three examples above correspond to $(\ell.\ell) < 0$ (Dwork), $(\ell.\ell) = 0$ (Brauer), and $(\ell.\ell) > 0$ (twistor).

Example 0.1. — To an elliptic K3 surface $S \longrightarrow \mathbb{P}^1$ with a section one can associate brilliant families of all three types, choosing ℓ as the fibre class of the elliptic fibration, as an ample class, or as any class of negative square, e.g. the class of a component of a reducible fibre.
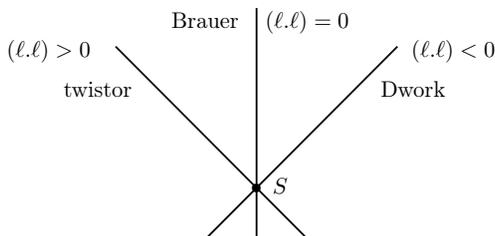
A similar picture can be drawn for non-elliptic K3 surfaces by working with higher-dimensional moduli spaces of sheaves on K3 surfaces, see Section 3.2.

0.2. The first result is the observation that the main result in [8] generalizes from twistor spaces associated to K3 surfaces with complex multiplication to general brilliant deformations of K3 surfaces with complex multiplication.

THEOREM 0.2. — *Assume* $\mathcal{S} \longrightarrow C$ *is a brilliant deformation of a K3 surface* S *with complex multiplication. Then any algebraic fibre* $\mathcal{S}_t$ *with* $\rho(\mathcal{S}_t) = \rho(\mathcal{S})$ *has complex multiplication as well and the maximal totally real subfields of the Hodge endomorphism rings of the rational transcendental lattices* $T(S) \otimes \mathbb{Q}$ *and* $T(\mathcal{S}_t) \otimes \mathbb{Q}$ *are isomorphic.*

Beware that except for scalar multiplications, Hodge endomorphisms contained in the maximal totally real subfield do not deform sideways in a brilliant family, i.e. they are typically not realized as Hodge endomorphisms of the transcendental lattice of a fibre with $\rho(\mathcal{S}_t) < \rho(S)$.

We think of the countable set of all algebraic fibres $\mathcal{S}_t$, $t \in C$, with $\rho(\mathcal{S}_t) = \rho(S)$ as the Noether–Lefschetz locus $\mathrm{NL}(\mathcal{S}/C) \subset C$. Then the result can be rephrased by saying that the maximal totally real subfield of $\mathrm{End}_{\mathrm{Hdg}}(T(S) \otimes \mathbb{Q})$ is reproduced as a field of Hodge endomorphisms of the transcendental lattice of the fibre over any point in the Noether–Lefschetz locus.

Viewing endomorphisms of $T(S) \otimes \mathbb{Q}$ as Hodge classes on $S \times S$, the assertion becomes more geometric. However, for $(\ell.\ell) \neq 0$ we are currently lacking a geometric explanation for the reappearance of these algebraic classes over all points in the Noether–Lefschetz locus, see Section 1.2.

0.3. The Hodge theoretic characterization of brilliant families reveals that in some appropriate sense the Noether–Lefschetz locus $\mathrm{NL}(\mathcal{S}/\mathbb{P}^1)$ of the twistor family associated with (S, L) specializes to the Brauer group $\mathrm{Br}(S)$. This affirms and refines the point of view, suggested by F. Charles and further advocated by D. Bragg and M. Lieblich [2], that the Brauer group of a supersingular K3 surface should be viewed as the analogue of the twistor base in positive characteristic. We can use Hodge theory to make this more precise in characteristic zero as follows.

THEOREM 0.3. — *Consider an elliptic K3 surface* S *with polarizations* L_s *approaching the fibre class of an elliptic fibration of* S. *Then the twistor families* $\mathcal{S}(s) \longrightarrow \mathbb{P}^1$ *associated with* (S, L_s) *restricted to the upper hemisphere of* $S^2 \simeq \mathbb{P}^1$ *specialize to the Brauer family* $\mathcal{S} \longrightarrow C$. *Furthermore,*

the Noether–Lefschetz loci $\mathrm{NL}(\mathcal{S}(s)/\mathbb{P}^1)$ specialize to $\mathrm{Br}(S)$ up to the free abelian group $H^2(S,\mathbb{Z})/\mathrm{NS}(S)$.
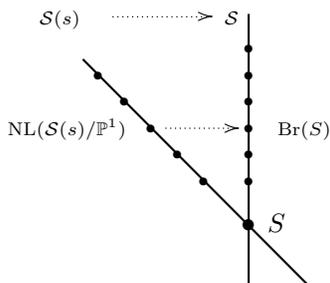
The precise meaning of the notions involved in this statement will be explained in Sections 2.4 and 3.1. Viewing the Brauer (or Tate–Šafarevič) family as a degeneration of twistor lines is not new and has been discussed already by E. Markman [10, Rem. 4.6], see also work of M. Verbitsky [15]. Our discussion adds the comparison of Noether–Lefschetz loci and Brauer groups to the picture. The proof can be adapted to show analogously that the Dwork family with its Noether–Lefschetz locus specializes to the Brauer family and the Brauer group.

0.4. The three types of brilliant families, twistor, Brauer, and Dwork, can be put in a two-dimensional family. Those parameterizing algebraic K3 surfaces form a countable union of curves, each of which intersects the three brilliant deformation types in their Noether–Lefschetz locus. This allows one to view the transcendental twistor construction as a combination of the Brauer family and a family of algebraic K3 surfaces.

THEOREM 0.4. — *Assume S is an elliptic K3 surface with an ample line bundle L. For every point $t \in \mathrm{NL}(\mathcal{S}/\mathbb{P}^1)$ in the Noether–Lefschetz locus of the associated twistor space $\mathcal{S} \longrightarrow \mathbb{P}^1$, there exists a Brauer class $\alpha \in \mathrm{Br}(S)$ such that the associated K3 surface S_α and $\mathcal{S}_t$ are naturally fibres of a holomorphic family of algebraic K3 surfaces.*

The interest in this viewpoint stems from the fact that we do understand the propagation of algebraic classes in Theorem 0.2 for brilliant families of Brauer type, see Section 1.2, and that the propagation along families of algebraic K3 surfaces should be more accessible to geometric arguments than

for the purely transcendental twistor families.
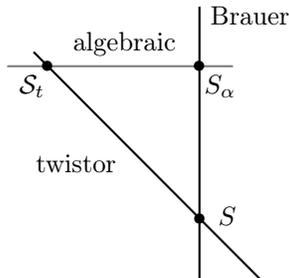
Once again, a similar result holds linking Dwork and Brauer families.

Acknowledgement

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1. Geometric families

We start by reviewing some of the geometric properties of the three families mentioned in the introduction.

1.1. The Dwork (or Fermat) pencil of quartic surfaces $\mathcal{S}_t \subset \mathbb{P}^3$ defined by $\sum_{i=0}^3 x_i^4 - 4t \prod_{i=0}^3 x_i$ parameterizes K3 surfaces for $t^4 \neq 1$. The four singular surfaces $\mathcal{S}_t$, $t \in \{\pm 1, \pm i\}$, have 16 ordinary double points and their minimal resolutions are K3 surfaces. In fact, by passing to a double cover $C \longrightarrow \mathbb{A}^1$ ramified at the four points and then taking a small resolution of the singular base change, the Dwork pencil can be turned into an algebraic family of quasi-polarized K3 surfaces.

The Fermat quartic $S = \mathcal{S}_0$ is known to have maximal Picard number $\rho(S) = 20$, in fact its Néron–Severi group $\mathrm{NS}(S) \simeq \mathbb{Z}^{\oplus 20}$ is generated by the lines contained in S, and its transcendental lattice $T(S)$ is of rank two with intersection matrix $\mathrm{diag}(8, 8)$, see [13]. Furthermore, there exists a primitive sublattice $\ell^\perp \subset \mathrm{NS}(S)$ of corank one that stays algebraic along the whole family and $\ell^\perp \simeq \mathrm{NS}(\mathcal{S}_t)$ for very general t. By Hodge index theorem, $(\ell.\ell) < 0$ and, more precisely, ℓ can be chosen of the form $L_1 + L_2 - L_3 - L_4$ for two

disjoint pairs of intersecting lines $L_i \subset S$ such that $(\ell.\ell) = -4$.[1] Thus, under parallel transport,

$$H^{2,0}(\mathcal{S}_t) \subset H^{2,0}(S) \oplus H^{0,2}(S) \oplus \mathbb{C} \cdot \ell = T(S)_{\mathbb{C}} \oplus \mathbb{C} \cdot \ell.$$

For another example, consider a non-trivial deformation E_t of a CM elliptic curve E and let $\mathcal{S}_t$ be the family of Kummer surfaces associated with $E_t \times E_t$. Again, $\rho(\mathcal{S}_t) = 19$ for the very general E_t and $\rho(\mathcal{S}_t) = 20$ whenever E_t has complex multiplication. Clearly, $H^{2,0}(\mathcal{S}_t) = H^{1,0}(E_t) \otimes H^{1,0}(E_t) \subset H^2(E_t \times E_t, \mathbb{C}) \subset H^2(\mathcal{S}_t, \mathbb{C})$ is contained in the three-dimensional space orthogonal to the classes $[E_t \times \{\mathrm{pt}\}], [\{\mathrm{pt}\} \times E_t], [\Delta_t] \in H^2(E_t \times E_t, \mathbb{Z})$ or, in other words,

$$H^{2,0}(\mathcal{S}_t) \subset (H^{1,0}(E) \otimes H^{1,0}(E)) \oplus (H^{0,1}(E) \otimes H^{0,1}(E)) \oplus \mathbb{C} \cdot \ell$$

where ℓ is an appropriate linear combination of the graph Γ of a complex multiplication of E and the aforementioned three algebraic classes.

More generally, any family of projective K3 surfaces $\mathcal{S}_t$ for which the very general fibre has Picard number $\rho(\mathcal{S}_t) = 19$ is of the type covered by our considerations. These families yield one-dimensional Shimura special cycles in the moduli space of polarized K3 surfaces. It is possible to construct holomorphic families of projective K3 surfaces $\mathcal{S}_t$ over a one-dimensional base that also fit our theory and have $\rho(\mathcal{S}_t) < 19$ for the very general fibre $\mathcal{S}_t$, but then the family does not correspond to an algebraic curve in the moduli space, cf. Remark 2.7.

1.2. We next come to the Brauer family. Consider a K3 surface S together with a fibration $\pi \colon S \longrightarrow \mathbb{P}^1$ by curves of genus one. We shall assume that the fibration has a section, so that all smooth fibres are elliptic curves, in which case π is called an elliptic fibration. The Fermat quartic is an example of an elliptic K3 surface and it is so in more than one way.

The Tate–Šafarevič group of S parameterizes all K3 surfaces with a genus one fibration and an isomorphism of its relative Jacobian fibration with S. It comes in two flavors: The algebraic and the analytic Tate–Šafarevič group. More precisely, $\mathrm{III}^{\mathrm{an}}(S)$ is the set of all isomorphism classes of pairs (S', ψ) consisting of a complex K3 surface S', typically not projective, with a fixed fibration $S' \longrightarrow \mathbb{P}^1$ by curves of genus one and an isomorphism $\psi \colon J(S'/\mathbb{P}^1) \xrightarrow{\sim} S$ over $\mathbb{P}^1$ between the relative Jacobian $J(S'/\mathbb{P}^1)$ and S, see [3, Ch. 1.5] or [6, Ch. 11.5]. The algebraic version is $\mathrm{III}(S)$, which is defined similarly but now requiring S' to be projective. Clearly, then

$$\mathrm{III}(S) \subset \mathrm{III}^{\mathrm{an}}(S).$$

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[1] Thanks to E. Sertöz for this information.

Both sets are indeed groups and as such they are naturally isomorphic to the algebraic resp. analytic Brauer groups

$$
\begin{array}{ccc}
\Sha(S) & \subset & \Sha^{\mathrm{an}}(S) \\
\wr\| & & \wr\| \\
\mathrm{Br}(S) & \subset & \mathrm{Br}^{\mathrm{an}}(S).
\end{array}
$$

Here, the algebraic Brauer group $\mathrm{Br}(S)$ can be described cohomologically as the torsion subgroup of $H^2(S,\mathcal{O}_S^*)$ or, more directly, as $H^2_{\mathrm{\acute et}}(S,\mathbb{G}_m)$, while the analytic Brauer group is all of $H^2(S,\mathcal{O}_S^*)$. From the exponential sequence one obtains an exact sequence

$$
0 \longrightarrow \mathrm{Pic}(S) \longrightarrow H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathcal{O}_S) \longrightarrow H^2(S,\mathcal{O}_S^*) \longrightarrow 0,
$$

which leads to the description of the analytic Brauer group as

$$
\mathrm{Br}^{\mathrm{an}}(S) \simeq H^2(S,\mathcal{O}_S)/\operatorname{coker}\left(\mathrm{NS}(S)\hookrightarrow H^2(S,\mathbb{Z})\right) \simeq \mathbb{C}/\mathbb{Z}^{\oplus 22-\rho(S)}.
$$

Note that the obvious inclusion $T(S) \subset H^2(S,\mathbb{Z})/\mathrm{NS}(S)$ is of finite index and, therefore, the natural surjection

$$
H^{0,2}(S)/T(S) \simeq H^2(S,\mathcal{O}_S)/T(S) \longrightarrow \mathrm{Br}^{\mathrm{an}}(S)
$$

has a finite, typically non-trivial, kernel. From a Hodge theoretic perspective $H^2(S,\mathcal{O}_S)/T(S)$ is more natural, but the difference between the two groups will be of no importance to us. Note that unless $\rho = 20$, the groups

$$
\Sha^{\mathrm{an}}(S) \simeq \mathrm{Br}^{\mathrm{an}}(S) \simeq H^{0,2}(S)/\mathbb{Z}^{\oplus 22-\rho(S)}
$$

have no reasonable geometric structure and cannot serve as a basis for a family of all K3 surfaces S' parameterized by $\Sha^{\mathrm{an}}(S)$. However, there exists a family, the *Brauer family*

$$
\mathcal{S} \longrightarrow H^{0,2}(S) \simeq \mathbb{C}, \tag{1.1}
$$

for which the fibre $\mathcal{S}_t$ over $t \in H^{0,2}(S)$ is isomorphic to the K3 surface S' corresponding to the image (S',ψ) of t under $H^{0,2}(S) \longrightarrow \mathrm{Br}(S) \simeq \Sha^{\mathrm{an}}(S)$, see [3, Ch. 1.5] for a detailed discussion. Note that $\mathcal{S} \longrightarrow \mathbb{C}$ is actually a family of K3 surfaces together with a genus one fibration which is the one that is given by considering (S',ψ) as an element in $\Sha^{\mathrm{an}}(S)$.

By construction, we again have

$$
H^{2,0}(\mathcal{S}_t) \subset H^{2,0}(S) \oplus \mathbb{C}\cdot\ell \subset H^{2,0}(S) \oplus H^{0,2}(S) \oplus \mathbb{C}\cdot\ell,
$$

where $\ell = f \in H^2(S,\mathbb{Z})$ denotes the class of a fibre of the elliptic fibration $S \longrightarrow \mathbb{P}^1$. The construction does not provide a topological or $\mathcal{C}^\infty$-trivialization, but see Remark 1.7.

Remark 1.1. — Instead of studying the Brauer family $\mathcal{S} \longrightarrow \mathbb{C} \simeq H^{0,2}(S)$ one can consider the constant family $S \times \mathbb{C} \longrightarrow \mathbb{C}$ and endow it with a universal Brauer class $\boldsymbol{\alpha} \in H^2(S \times \mathbb{C}, \mathcal{O}^*)$. Explicitly, using $H^2(S \times \mathbb{C}, \mathcal{O}) \simeq H^0(\mathbb{C}, \mathcal{O}) \otimes H^{0,2}(S)$, which under the identification $\mathbb{C} = H^{0,2}(S)$ contains $H^{0,2}(S)^* \otimes H^{0,2}(S)$, the universal Brauer class $\boldsymbol{\alpha}$ is realized as the image of $\mathrm{id}_{H^{0,2}}$ under the exponential map $H^2(S \times \mathbb{C}, \mathcal{O}) \longrightarrow H^2(S \times \mathbb{C}, \mathcal{O}^*)$. By construction, if $\alpha \in H^2(S, \mathcal{O}_S^*)$ is the image of $t \in \mathbb{C} = H^{0,2}(S)$, then $\boldsymbol{\alpha}|_{S \times \{t\}} = \alpha$. The situation was discussed for supersingular K3 surfaces in positive characteristic in [2].

Remark 1.2. — The analogy between the Brauer groups of supersingular K3 surfaces and twistor spaces promoted in [2] is sometimes met with the following objection. The Hochschild cohomology $HH^2(S) \simeq H^2(S, \mathcal{O}_S) \oplus H^1(S, \mathcal{T}_S) \oplus H^0(S, \bigwedge^2 \mathcal{T}_S)$ parameterizes all infinitesimal deformations of S, classical commutative as well as non-commutative ones. First order classical deformations correspond to classes in $H^1(S, \mathcal{T}_S)$, while non-commutative ones associated to deformations of the standard bounded derived category $\mathrm{D}^\mathrm{b}(S)$ of coherent sheaves to the bounded derived category $\mathrm{D}^\mathrm{b}(S, \alpha)$ of α-twisted coherent sheaves correspond to classes in $H^2(S, \mathcal{O}_S)$. How could then possibly the tangent space of $\mathrm{Br}^\mathrm{an}(S)$, which is naturally identified with $H^2(S, \mathcal{O}_S)$, end up in the direct summand $H^1(S, \mathcal{T}_S)$?

The answer to this is that it is actually not S that is deformed in the Brauer family above but its Fourier–Mukai partner, which just happens to be isomorphic to S. More precisely, the automorphism of $HH^2(S)$ induced by the relative Poincaré sheaf $\mathcal{P}$ on $S \times_{\mathbb{P}^1} S$ is compatible with the action of $\mathcal{P}$ on Hochschild homology $HH_*(S) \simeq \bigoplus_{q-p=*} H^{p,q}(S)$ considered as a module over $HH^*(S)$. Identifying $HH_*(S)$ with (ungraded) de Rham cohomology $H^*(S, \mathbb{C})$ and using that the action of $\mathcal{P}$ on $H^*(S, \mathbb{Z})$ identifies $(H^0 \oplus H^4)(S, \mathbb{Z})$ with the hyperbolic plane in $\mathrm{NS}(S) \subset H^2(S, \mathbb{Z})$ spanned by the fibre class ℓ and the class of the section, one finds that $\mathcal{P}$ indeed sends $H^2(S, \mathcal{O}_S) \subset HH^2(S)$ into $H^1(S, \mathcal{T}_S) \subset HH^2(S)$.

A similar point of view is exploited in Section 3.2, where non-commutative deformations of S corresponding to Brauer classes are interpreted in terms of classical commutative deformations of a certain moduli space of sheaves on S.

For $\alpha \in \mathrm{Br}(S)$, let $(S_\alpha, \psi_\alpha) \in \mathrm{III}(S)$ be the corresponding algebraic K3 surface $\pi_\alpha \colon S_\alpha \longrightarrow \mathbb{P}^1$ together with an isomorphism $\psi_\alpha \colon J(S_\alpha/\mathbb{P}^1) \xrightarrow{\sim} S$. The relative Jacobian $J(S_\alpha/\mathbb{P}^1)$ can be interpreted as a moduli space of stable sheaves concentrated on the fibres of π_α, but it is not a fine moduli space. In fact, the obstruction to the existence of a universal family is exactly $\alpha \in \mathrm{Br}(S)$. In other words, a universal sheaf $\mathcal{P}_\alpha$ on $S_\alpha \times_{\mathbb{P}^1} S$ exists but only

as a twisted sheaf, where the twist is with respect to $1 \boxtimes \alpha \in \mathrm{Br}(S_\alpha \times S)$. The usual Fourier–Mukai formalism then yields an exact equivalence

$$\mathrm{D}^b(S_\alpha) \simeq \mathrm{D}^b(S, \alpha)$$

between the bounded derived category $\mathrm{D}^b(S_\alpha)$ of coherent sheaves on S_α and the bounded derived category $\mathrm{D}^b(S, \alpha)$ of α-twisted coherent sheaves on S, for more information and references see [6, §10.2.2 & Rem. 11.5.9]. The Mukai vector of the twisted universal sheaf yields a Hodge isometry $\widetilde{H}(S_\alpha, \mathbb{Z}) \simeq \widetilde{H}(S, \alpha, \mathbb{Z})$, which restricts to a Hodge isometry between their transcendental lattices $T(S_\alpha) \simeq T(S, \alpha)$, see [5, 9] for details. Since the transcendental lattice of the twisted K3 surface (S, α) is a finite index sub-Hodge structure of $T(S)$, this yields a Hodge isometry

$$T(S_\alpha)_{\mathbb{Q}} \simeq T(S)_{\mathbb{Q}}. \tag{1.2}$$

Moreover, (1.2) is algebraic, i.e. it is defined by an algebraic class in $(T(S_\alpha) \otimes T(S))^{2,2}_{\mathbb{Q}} \subset H^{2,2}(S_\alpha \times S, \mathbb{Q})$. Here, we use that the $(1 \boxtimes \alpha)$-twisted sheaf $\mathcal{P}$ becomes untwisted by passing to its derived tensor power $\mathcal{P}^{\otimes r}$ for $r = \mathrm{ord}(\alpha)$. The Chern classes of the untwisted sheaves $\mathcal{P}^{\otimes r}$ exist in the traditional sense, which eventually leads to a description of (1.2) in terms of algebraic classes associated to untwisted sheaves.

Remark 1.3. — There is a slightly confusing point about the relation between the transcendental lattices $T(S_\alpha)$, $T(S)$, and $T(S, \alpha)$. By definition, $T(S_\alpha)_{\mathbb{Q}} \subset H^2(S_\alpha, \mathbb{Q})$ and $T(S)_{\mathbb{Q}} \subset H^2(S, \mathbb{Q})$, while $T(S, \alpha)_{\mathbb{Q}} \subset \widetilde{H}(S, \alpha, \mathbb{Q})$ is more naturally viewed as $T(S, \alpha)_{\mathbb{Q}} = \exp(B) \cdot T(S)_{\mathbb{Q}} \subset (H^2 \oplus H^4)(S, \mathbb{Q})$, where $B \in H^2(S, \mathbb{Q})$ lifts α. The cohomological Fourier–Mukai formalism above then yields $T(S_\alpha)_{\mathbb{Q}} \simeq T(S, \alpha)_{\mathbb{Q}} \subset (H^2 \oplus H^4)(S, \mathbb{Q})$. On the other hand, parallel transport along the Brauer family leads to an inclusion $T(S_\alpha)_{\mathbb{Q}} \subset T(S)_{\mathbb{Q}} \oplus \mathbb{Q} \cdot f \subset H^2(S, \mathbb{Q})$, where f is the class of the fibre of the elliptic fibration $S \longrightarrow \mathbb{P}^1$. Furthermore, the image of $T(S_\alpha) \simeq T(S, \alpha) \subset (H^2 \oplus H^4)(S, \mathbb{Z})$ under the projection onto $H^2(S, \mathbb{Z})$ identifies $T(S_\alpha)$ with the kernel of $T(S) \longrightarrow \mathbb{Q}/\mathbb{Z}$, $\gamma \longmapsto (B.\gamma)$, which eventually leads to the Hodge isometry $T(S_\alpha)_{\mathbb{Q}} \simeq T(S)_{\mathbb{Q}}$. In particular, the one-dimensional $H^{2,0}(S_\alpha) \subset H^2(S_\alpha, \mathbb{C})$ is via (1.2) mapped into $H^{2,0}(S) \oplus H^4(S, \mathbb{C})$ while parallel transport maps it into $H^{2,0}(S) \oplus \mathbb{C} \cdot f$.

Remark 1.4. — The preceding discussion can be made to work with suitable modifications in the case when α is not torsion. Again S can be viewed as a coarse moduli space of torsion sheaves on S_α for which the universal sheaf is twisted with respect to the non-torsion class α on S. For moduli spaces of stable sheaves on non-projective K3 surfaces see [12].[2] There is

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[2] Note however that the case of torsion sheaves needs extra care, but as it is not essential for our purpose, we ignore this point.

still a Hodge isometry $T(S_\alpha) \simeq T(S, \alpha)$, only that now $T(S, \alpha)$ cannot be viewed just as the B-field shift of $T(S)$ (up to finite index). By definition, it is the minimal primitive sub-Hodge structure of $H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ with its $(2, 0)$-part spanned by $\sigma + \sigma \wedge t$, where $t \in \mathbb{C} \simeq H^{0,2}(S)$ lifts α under $H^{0,2}(S) \longrightarrow \mathrm{Br}^{\mathrm{an}}(S)$. In fact, when α is non-torsion, the rank of the transcendental lattice $T(S_\alpha)$ exceeds the rank of $T(S)$ and, moreover, the intersection form is degenerate on it.

Clearly, since S_α and S have Hodge isometric rational transcendental lattices for $\alpha \in \mathrm{Br}(S)$, their fields of Hodge endomorphisms are isomorphic:

$$\mathrm{End}_{\mathrm{Hdg}}(T(S_\alpha) \otimes \mathbb{Q}) \simeq \mathrm{End}_{\mathrm{Hdg}}(T(S) \otimes \mathbb{Q}).$$

Furthermore, viewing S as a moduli space of sheaves on S_α or, conversely, S_α as a moduli space of α-twisted sheaves on S, provides us with a geometric understanding of this isomorphism. The Mukai vector of (some tensor power of) $\mathcal{P} \boxtimes \mathcal{P}$ on $(S_\alpha \times S)^2$ maps any Hodge endomorphism $\varphi \in \mathrm{End}_{\mathrm{Hdg}}(T(S) \otimes \mathbb{Q})$ or, equivalently, any Hodge class $\varphi \in (T(S) \otimes T(S))_{\mathbb{Q}}^{2,2} \subset H^{2,2}(S \times S, \mathbb{Q})$ to a Hodge class on $S_\alpha \times S_\alpha$. Under this map, algebraic classes are mapped to algebraic classes. Note, however, that the Hodge conjecture for squares of K3 surfaces is only known for K3 surfaces with CM endomorphism field.

We summarize the above discussion as follows.

PROPOSITION 1.5. — *Consider the Brauer family* (1.1) $\mathcal{S} \longrightarrow \mathbb{C}$ *associated with an elliptic K3 surface $S \longrightarrow \mathbb{P}^1$. Then for any algebraic fibre $\mathcal{S}_t$ there exists an algebraic Hodge isometry*

$$T(S) \otimes \mathbb{Q} \simeq T(\mathcal{S}_t) \otimes \mathbb{Q}.$$

Furthermore, the induced isomorphism of their endomorphism fields

$$\mathrm{End}_{\mathrm{Hdg}}(T(S) \otimes \mathbb{Q}) \simeq \mathrm{End}_{\mathrm{Hdg}}(T(\mathcal{S}_t) \otimes \mathbb{Q})$$

maps algebraic classes to algebraic classes. □

In Section 3.2 we explain that the same principle applies to projective K3 surfaces S which are not elliptic by working with higher-dimensional moduli spaces. In the case of elliptic K3 surfaces one can state more algebraically that the rational Chow motives of all algebraic fibres of the Brauer family are isomorphic: $\mathfrak{h}(\mathcal{S}_t) \simeq \mathfrak{h}(S)$, cf. [7].

Remark 1.6. —

(i) The first assertion in the above proposition does not hold for the other two types of families considered in this article. A weaker version of the second assertion does hold, see Proposition 2.11, but a clear geometric reason for the recurrence of the same Hodge classes in all algebraic fibres is lacking.

(ii) Furthermore, for the Brauer family, we also know that if S is defined over $\overline{\mathbb{Q}}$, also all other algebraic fibres S_α are defined over $\overline{\mathbb{Q}}$. This is unknown for the algebraic fibres of the twistor family associated to a K3 surface without complex multiplication.

1.3. We conclude by a short review of twistor families which have been discussed already in the prequel [8]. To any K3 surface S together with a Kähler class $\omega \in H^{1,1}(S, \mathbb{R})$ there is naturally associated a complex structure on $S \times \mathbb{P}^1$, with the corresponding complex threefold denoted $\mathcal{S}$, such that the projection defines a holomorphic map $\mathcal{S} \longrightarrow \mathbb{P}^1$. The fibres $\mathcal{S}_t$ are K3 surfaces with

$$H^{2,0}(\mathcal{S}_t) \subset H^{2,0}(S) \oplus H^{0,2}(S) \oplus \mathbb{C} \cdot \omega \subset T(S)_{\mathbb{C}} \oplus \mathbb{C} \cdot \omega.$$

The construction of $\mathcal{S} \longrightarrow \mathbb{P}^1$, the twistor family associated with (S, ω), depends on the existence of the unique Ricci flat Kähler structure with Kähler class ω. In this sense, it is a transcendental construction. For more information and references see [8, §1].

For a projective K3 surface S, the construction can be applied to any ample line bundle L on S whose first Chern class $\ell := c_1(L)$ is of course a Kähler class. However, even in this case, only countably many of the fibres $\mathcal{S}_t$ are projective. According to [8, Prop. 3.2], any projective fibre $\mathcal{S}_t$ for which $H^{2,0}(\mathcal{S}_t) \oplus H^{0,2}(\mathcal{S}_t)$ does not contain ℓ satisfies $\rho(\mathcal{S}_t) = \rho(S)$. The set of all fibres for which ℓ is contained in $H^{2,0}(\mathcal{S}_t) \oplus H^{0,2}(\mathcal{S}_t)$ can be pictured as the equator of the sphere $\mathbb{P}^1 \simeq S^2$ with the original K3 surface S corresponding to the north pole.

Remark 1.7. — Inspired by the twistor construction one might try to find a $\mathcal{C}^\infty$-trivialization of the Brauer family. This is indeed possible but not in a canonical way. Concretely, if $\pi \colon S \longrightarrow \mathbb{P}^1$ is an elliptic K3 surface and S is viewed as a differentiable manifold M endowed with a complex structure I, then a generator σ of $H^{2,0}(S)$ is a closed complex two-form on M which uniquely determines the complex structure I, for $T^{0,1}(M, I) \subset T_{\mathbb{C}} M$ is the kernel of $\sigma \colon T_{\mathbb{C}} M \longrightarrow T_{\mathbb{C}}^* M$. Now, pick any non-exact closed $(1,1)$-form ω on $\mathbb{P}^1$, say with $\int \omega = 1$. Then $\sigma_t := \sigma + t\pi^*\omega$ is a closed complex two-form with the same properties as σ, namely $\sigma_t \wedge \sigma_t \equiv 0$ and $\sigma_t \wedge \bar{\sigma}_t > 0$ (pointwise), and therefore defines a complex structure I_t on M. Note that with respect to I_t the projection π is still holomorphic and, hence, $S_t := (M, I_t)$ comes with a natural genus one fibration. However, the section of $\pi \colon S \longrightarrow \mathbb{P}^1$ will not be holomorphic anymore with respect to I_t. Altogether, this yields a family of complex structures I_t on M that is isomorphic to the Brauer family. We emphasize that the construction of the family (M, I_t) depends on the choice of ω and in this sense the Brauer family does not come with a natural

trivialization. Also note that proving that the relative Jacobian of S_t is isomorphic to S for all t is not immediate.

2. Brilliant families of Hodge structures

This section deals with all purely Hodge theoretic aspects. The results can be applied to any geometric situation that involves Hodge structures of weight two with a one-dimensional $(2,0)$-part. Later we will focus exclusively on K3 surfaces, but the Hodge theory developed here equally well applies to hyperkähler manifolds.

2.1. We start by considering a real vector space $T_{\mathbb{R}}$ with a non-degenerate symmetric bilinear form $(\ .\ )$ of signature $(2, r - 2)$. A Hodge structure of K3 type on $T_{\mathbb{R}}$ is then given by a generator $\sigma_0 \in T_{\mathbb{R}} \otimes \mathbb{C}$ of its $(2,0)$-part, unique up to scaling, that satisfies the two conditions $(\sigma_0.\sigma_0) = 0$ and $(\sigma_0.\bar{\sigma}_0) > 0$. Alternatively, the Hodge structure can be thought of in terms of the oriented, positive plane $P_{\sigma_0} := \langle \mathrm{Re}(\sigma_0), \mathrm{Im}(\sigma_0) \rangle \subset T_{\mathbb{R}}$.

Next, we extend $T_{\mathbb{R}}$ to the orthogonal sum

$$T_\ell := T_{\mathbb{R}} \oplus \mathbb{R} \cdot \ell, \tag{2.1}$$

where $d := (\ell.\ell) \in \mathbb{R}$ can be arbitrary. The discussions in the three cases, $d > 0$, $d = 0$, and $d < 0$, will be similar, but there are interesting differences and special phenomena that we wish to explore.

For now we fix a Hodge structure on $T_{\mathbb{R}}$ with $\sigma_0 \in T_{\mathbb{R}} \otimes \mathbb{C}$ as above and consider it as a Hodge structure on T_ℓ by declaring ℓ to be of type $(1,1)$. We are interested in Hodge structures of K3 type on T_ℓ, i.e. for which the $(2,0)$-part is spanned by a class $\sigma_t \in T_\ell \otimes \mathbb{C}$ with $(\sigma_t.\sigma_t) = 0$ and $(\sigma_t.\bar{\sigma}_t) > 0$. Once again, even for $d = 0$, such a Hodge structure is uniquely determined by σ_t, for its $(1,1)$-part is the orthogonal complement of the positive plane $P_{\sigma_t} := \langle \mathrm{Re}(\sigma_t), \mathrm{Im}(\sigma_t) \rangle \subset T_{\mathbb{R}} \oplus \mathbb{R} \cdot \ell$.

However, we shall only consider Hodge structures for which

$$\sigma_t \in \mathbb{C} \cdot \sigma_0 \oplus \mathbb{C} \cdot \bar{\sigma}_0 \oplus \mathbb{C} \cdot \ell$$

or, equivalently, such that $P_{\sigma_t} \subset P_{\sigma_0} \oplus \mathbb{R} \cdot \ell$. They are parameterized by an open subset $D_\ell \subset Q_\ell$ of the conic $Q_\ell \subset \mathbb{P}(\mathbb{C} \cdot \sigma_0 \oplus \mathbb{C} \cdot \bar{\sigma}_0 \oplus \mathbb{C} \cdot \ell)$ defined by $(\sigma.\sigma) = 0$:

$$D_\ell := \{\, \sigma \in Q_\ell \mid (\sigma.\bar{\sigma}) > 0 \,\} \subset Q_\ell \subset \mathbb{P}(\mathbb{C} \cdot \sigma_0 \oplus \mathbb{C} \cdot \bar{\sigma}_0 \oplus \mathbb{C} \cdot \ell) \simeq \mathbb{P}^2. \tag{2.2}$$

For $d \neq 0$ the conic Q_ℓ is smooth, while for $d = 0$ it consists of the two conjugate lines $\mathbb{P}(\mathbb{C} \cdot \sigma_0 \oplus \mathbb{C} \cdot \ell)$ and $\mathbb{P}(\mathbb{C} \cdot \bar{\sigma}_0 \oplus \mathbb{C} \cdot \ell)$ with $[\ell]$ as their point of intersection. In the latter case, we shall write $L_\ell := \mathbb{P}(\mathbb{C} \cdot \sigma_0 \oplus \mathbb{C} \cdot \ell) \setminus \{[\ell]\}$. We

emphasize that, although not reflected by the notation, D_ℓ and Q_ℓ depend not only on $T_\mathbb{R}$ and ℓ, but also on σ_0.

LEMMA 2.1. — *Let $T_\mathbb{R} \subset T_\mathbb{R} \oplus \mathbb{R} \cdot \ell$ be as above. Then*

$$D_\ell = \begin{cases} Q_\ell \simeq \mathbb{P}^1 & \text{if } d > 0 \\ L_\ell \sqcup \bar{L}_\ell \simeq \mathbb{C} \sqcup \mathbb{C} & \text{if } d = 0 \\ D'_\ell \sqcup \bar{D}'_\ell \simeq \mathbb{H} \sqcup \bar{\mathbb{H}} & \text{if } d < 0, \end{cases}$$

where the connected component D'_ℓ is chosen to contain σ_0.

Proof. — The cases $d > 0$ and $d = 0$ are covered by the above discussion. For $d < 0$ observe that $\langle \mathrm{Re}(\sigma_0), \mathrm{Im}(\sigma_0), \ell \rangle_\mathbb{R}$ is isometric to $\mathbb{R}^3$ with e_1, e_2 isotropic, $(e_1.e_2) = 1$, and $(e_3.e_3) = 1$. Then $z \longmapsto [1 : -(z.z) : \sqrt{2}z]$ describes a biholomorphic identification of $\mathbb{H}$ with one of the two connected components of the period domain D_ℓ. □

To restore the symmetry between the three cases, we shall shrink the period domain for $d > 0$ to the complement of the equator $S^1_\ell \subset D_\ell \simeq \mathbb{P}^1$. Recall that the equator is described by the condition $\ell \in P_{\sigma_t}$. Hence,

$$D_\ell \setminus S^1_\ell \simeq \mathbb{H} \sqcup \bar{\mathbb{H}}$$

as in the case $d < 0$ and we usually follow the convention that $\sigma_0 \in \mathbb{H}$.

Remark 2.2. — Note that for $d \leqslant 0$ every $\sigma_t \in D_\ell$ satisfies $\ell \notin P_{\sigma_t}$. In fact, if one lets $d > 0$ approach $d = 0$, then the equator S^1_ℓ flows into the point of intersection $[\ell]$. Thus, removing the equator allows one to view the two connected components in each of the three cases as part of a family depending on the parameter $d = (\ell.\ell)$.

| $(\ell.\ell) > 0$ | $(\ell.\ell) = 0$ | $(\ell.\ell) < 0$ |
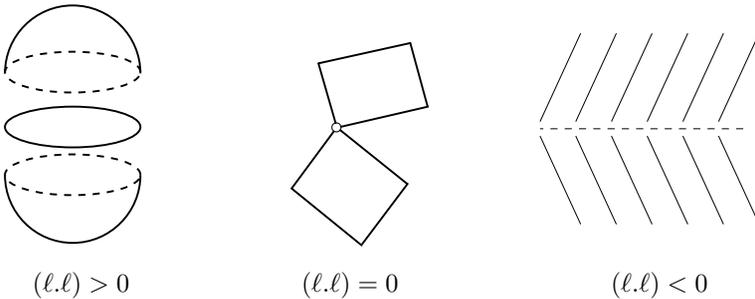
Those are the deformations of σ_0 that interest us here.

DEFINITION 2.3. — *For a fixed Hodge structure of K3 type on $T_\mathbb{R}$ given by $\sigma_0 \in T_\mathbb{R} \otimes \mathbb{C}$ the family of Hodge structures σ_t on $T_\mathbb{R} \oplus \mathbb{R} \cdot \ell$ satisfying*

$$P_{\sigma_t} \subset P_{\sigma_0} \oplus \mathbb{R} \cdot \ell \text{ and } \ell \notin P_{\sigma_t}$$

is called a brilliant family *of deformations of* σ_0. *We usually restrict to the connected component containing* σ_0, *the other one being obtained by complex conjugation.*

Occasionally, we will refer to the three types of brilliant deformations corresponding to $d < 0$, $d = 0$, and $d > 0$ as deformations of *Dwork*, *Brauer*, resp. *twistor type*.

2.2. In a second step, we start with a $\mathbb{Q}$-vector space T, associate to it the real vector space $T_{\mathbb{R}} := T \otimes \mathbb{R}$, and assume that the Hodge structure σ_0 is irreducible. In particular, orthogonal projection yields $\mathbb{Q}$-linear injections $T \hookrightarrow P_{\sigma_0}$ and $T \hookrightarrow \mathbb{C} \cdot \sigma_0$, cf. [8, Lem. 2.3]. Furthermore, the class ℓ is now considered to be rational and of type $(1,1)$, so that σ_0 can be viewed as a Hodge structure on the $\mathbb{Q}$-vector space

$$T \oplus \mathbb{Q} \cdot \ell. \tag{2.3}$$

For a brilliant deformation σ_t of σ_0 on $T \oplus \mathbb{Q} \cdot \ell$ we denote by

$$T_t \subset T \oplus \mathbb{Q} \cdot \ell$$

its transcendental lattice, i.e. the minimal sub-Hodge structure with non-trivial $(2,0)$-part. For example, since the original Hodge structure is assumed to be irreducible, we have $T_0 = T$. The Picard number of σ_t is then defined as

$$\rho(\sigma_t) := \dim(T \oplus \mathbb{Q} \cdot \ell) - \dim T_t.$$

We are used to think of the Picard number as the dimension of the subspace of all classes in $T \oplus \mathbb{Q} \cdot \ell$ that are $(1,1)$ with respect to σ_t. However, due to the fact that the quadratic form is possibly non-degenerate, this number can be bigger, cf. the discussion for $d = 0$ in Section 2.3 below.

With this definition, $\rho(\sigma_0) = 1$ and $\rho(\sigma_t) = 0$ for the very general σ_t. In fact, $\rho(\sigma_t) = 1$ is the maximal value that can be attained due to the assumption on σ_t to be brilliant. Slightly stronger, we have $\dim(T_t^{\perp}) \leqslant 1$ for all t. To see this adapt the proof of [8, Prop. 3.2] to cover also the case $d < 0$ and $d = 0$.

The set of brilliant deformations σ_t of σ_0 with $\rho(\sigma_t) = 1$ is countable and dense. Of particular interest to us are those for which the transcendental lattice $T_t \subset T \oplus \mathbb{Q} \cdot \ell$ has the same signature $(2, r-2)$ as $T_0 = T$. Then clearly $\rho(\sigma_t) = 1$ and the converse holds for $d \leqslant 0$. We define the Noether–Lefschetz locus as the following countable dense subset of brilliant deformations of σ_0:

$$\mathrm{NL}_\ell := \{\, \sigma_t \mid \mathrm{sign}(T_t) = \mathrm{sign}(T_0) = (2, r-2) \,\}.$$

LEMMA 2.4. — *For $\sigma_t \in \mathrm{NL}_\ell$ orthogonal projection $T_t \hookrightarrow T \oplus \mathbb{Q} \cdot \ell \twoheadrightarrow T_0 = T$ defines a bijection*

$$T_t \xrightarrow{\sim} T$$

of $\mathbb{Q}$-vector spaces, which for $d = 0$ is an isometry of Hodge structures up to conjugation.

Proof. — First observe that for $\rho(\sigma_t) = 1$ the obvious inclusion $P_{\sigma_t} \subset T_t \cap (P_{\sigma_0} \oplus \mathbb{R} \cdot \ell)$ is an equality. Next, since T_t and T_0 are $\mathbb{Q}$-vector spaces of the same dimension, it suffices to prove the injectivity of $T_t \longrightarrow T$, which follows from $\ell \notin T_t$ or, equivalently, $\ell \notin P_{\sigma_t} = T_t \cap (P_{\sigma_0} \oplus \mathbb{R} \cdot \ell)$. Indeed, as P_{σ_t} is positive definite, we clearly have $\ell \notin P_{\sigma_t}$ for $d \leqslant 0$. For $d > 0$ it is implied by the assumption that σ_t is brilliant deformations.

For the second assertion in the case $d = 0$ use that $\sigma_t = \sigma_0 + t\ell \in L_\ell$ or $\sigma_t = \bar{\sigma}_0 + t\ell \in \bar{L}_\ell$ for some $t \in \mathbb{C}$. Hence, the projection $T_t \longrightarrow T$ maps σ_t to σ_0 or $\bar{\sigma}_0$. As for $d = 0$ the set of $(1, 1)$-classes is the same for all $\sigma_0 + t\ell$, see Section 2.3 below, also $(1, 1)$-classes are preserved under the projection. □

2.3. Let us now study the case of an isotropic class ℓ, i.e. $d = 0$. We will restrict to the connected component L_ℓ, i.e. we only study brilliant deformations of the form $\sigma_t = \sigma_0 + t\ell$ for some $t \in \mathbb{C}$. Those parameterized by $\bar{L}_\ell$ are obtained by complex conjugation.

The first thing to observe is that, unlike the case $d \neq 0$, the class ℓ is always of type $(1, 1)$, for $(\sigma_t.\ell) = 0$. However, for the very general $\sigma_t \in L_\ell$, one has $T_t = T \oplus \mathbb{Q} \cdot \ell$, cf. Lemma 2.5, and, therefore, in spite of being of type $(1, 1)$ for σ_t, the class ℓ is contained in the transcendental lattice T_t. More generally, the space of $(1, 1)$-classes is the same for all $\sigma_t \in L_f$.

Next, any $\sigma_t \in L_\ell$ can be written as

$$\sigma_t = \sigma_0 + (\sigma_0.B)\ell$$

for some $B \in T \otimes \mathbb{C}$. Here, the class B is not uniquely defined, only its $(0, 2)$-part $B^{0,2} \in T^{0,2}$ is. This yields an identification

$$L_\ell \xrightarrow{\sim} T^{0,2}, \tag{2.4}$$

which is independent of the choice of σ_0. The following description of the image of the Noether–Lefschetz locus under (2.4) views the irreducible Hodge structure T as a $\mathbb{Q}$-linear subspace of $T^{0,2}$ via orthogonal projection.

LEMMA 2.5. — *A point $t \in L_\ell$ is contained in the Noether–Lefschetz locus NL_ℓ if and only if there exists a class $B \in T$ (in the $\mathbb{Q}$-vector space!) with $\sigma_t = \sigma_0 + (\sigma_0.B)\ell$. In other words, under the identification (2.4),*

$$\mathrm{NL}_\ell \simeq T.$$

Proof. — For any $B \in T$, the map $f_B : T \longrightarrow T \oplus \mathbb{Q} \cdot \ell$, $\alpha \longmapsto \alpha + (\alpha.B)\ell$ defines a $\mathbb{Q}$-linear isometric embedding with $\sigma_t = \sigma_0 + (\sigma_0.B)\ell \in f_B(T) \otimes \mathbb{C}$ and, moreover, its image $f_B(T) \subset T \oplus \mathbb{Q} \cdot \ell$ is a sub-Hodge structure with

respect to σ_t. Thus, $T_t \subset f_B(T)$ by minimality of T_t, which combined with $\rho(\sigma_t) \leqslant 1$ shows that $T_t = f_B(T)$ is of signature $(2, r-2)$, i.e. $t \in \mathrm{NL}_\ell$.

Conversely, if t is contained in the Noether–Lefschetz locus, then by virtue of Lemma 2.4, orthogonal projection yields an isomorphism $T_t \xrightarrow{\sim} T$. Its inverse is a map $T \longrightarrow T \oplus \mathbb{Q} \cdot \ell$, $\alpha \longmapsto \alpha + g(\alpha)\ell$ for some linear map $g \colon T \longrightarrow \mathbb{Q}$. As $(\,.\,)$ is non-degenerate on T, there exists a (non-unique) $B \in T$ with $g(\alpha) = (\alpha.B)$ for all α. □

If T is associated with an integral Hodge structure, i.e. an integral lattice $T_{\mathbb{Z}} \subset T$ with $T_{\mathbb{Z}} \otimes \mathbb{Q} \simeq T$ is fixed, then its Brauer group can be defined and it comes in two flavors. In the following definition we again use the injectivity of the orthogonal projection $T \longrightarrow T^{0,2}$ which allows us to view both, the $\mathbb{Z}$-module $T_{\mathbb{Z}}$ and the bigger $\mathbb{Q}$-vector space T, as subgroups of $T^{0,2}$.

Definition 2.6. — *The* analytic Brauer group *of an irreducible integral Hodge structure $T_{\mathbb{Z}}$ of K3 type as above is the quotient*

$$\mathrm{Br}^{\mathrm{an}}(T_{\mathbb{Z}}) := T^{0,2}/T_{\mathbb{Z}}.$$

The algebraic Brauer group $\mathrm{Br}(T_{\mathbb{Z}}) \subset \mathrm{Br}^{\mathrm{an}}(T_{\mathbb{Z}})$ *is defined as the subgroup of all torsion elements or, equivalently,*

$$\mathrm{Br}(T_{\mathbb{Z}}) := T/T_{\mathbb{Z}} \subset T^{0,2}/T_{\mathbb{Z}} = \mathrm{Br}^{\mathrm{an}}(T_{\mathbb{Z}}).$$

Then Lemma 2.5 expresses a relation between the Noether–Lefschetz locus and the Brauer group:

$$
\begin{array}{ccc}
\mathrm{NL}_\ell & \subset L_\ell \simeq T^{0,2} \\
\downarrow & \quad \downarrow \\
\mathrm{Br}(T_{\mathbb{Z}}) & \subset \ \mathrm{Br}^{\mathrm{an}}(T_{\mathbb{Z}})
\end{array}
$$

or, in other words,

$$\mathrm{NL}_\ell / T_{\mathbb{Z}} \simeq \mathrm{Br}(T_{\mathbb{Z}}),$$

where NL_ℓ is considered as an affine space over $T_{\mathbb{Z}}$.

2.4. There are various similarities between the three cases, $d < 0$, $d = 0$, and $d > 0$ and also a procedure to combine them.

Remark 2.7. — The most notable difference between the three cases is the signature of $T \oplus \mathbb{Q} \cdot \ell$. To simplify things, assume T is two-dimensional. Then $T \oplus \mathbb{Q} \cdot \ell$ is positive definite for $d > 0$, of signature $(2,1)$ for $d < 0$, and degenerate with one isotropic and two positive directions for $d = 0$. This not only has consequences for the description of the corresponding period domains as in Lemma 2.1, but also for the possible existence of algebraic quotients.

To make this more transparent, assume that we start with an integral Hodge structure $T_{\mathbb{Z}}$, let $T = T_{\mathbb{Z}} \otimes \mathbb{Q}$, and assume $d = (\ell.\ell) \in \mathbb{Z}$. Then the orthogonal group $\Gamma := \mathrm{O}(T_{\mathbb{Z}} \oplus \mathbb{Z} \cdot \ell)$ of the lattice $T_{\mathbb{Z}} \oplus \mathbb{Z} \cdot \ell$ is a finite group for $d > 0$ and in many cases essentially trivial. On the other hand, for brilliant families of Dwork type, so $d < 0$, the group Γ is infinite and D_{ℓ}/Γ will be the quotient of $\mathbb{H}$ by a modular group. Often, as in the case of the classical Dwork family, a family given over D_{ℓ} will descend to an algebraic family of the quotient of D_{ℓ} by some subgroup of Γ of finite index.

Something special happens in the case $d = 0$. Here, Γ is again an infinite group. Indeed, Γ contains the B-field shifts $\alpha + n\ell \longmapsto \alpha + ((\alpha.B) + n)\ell$ for any $B \in T_{\mathbb{Z}}$. Note that this also works when $\dim T > 2$ and that under the identification $D_{\ell} = L_{\ell} \simeq T^{0,2} \simeq \mathbb{C}$ the action corresponds to translation by elements in $T_{\mathbb{Z}} \subset T^{0,2}$. However, unlike the case $d < 0$, the quotient D_{ℓ}/Γ usually does not have any reasonable geometric structure.

The three types of brilliant deformations can be combined as follows. We extend the Hodge structure on T given by σ_0 to
$$T_{\ell_1,\ell_2} := T \oplus \mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$$
by declaring both classes ℓ_1 and ℓ_2 to be of type $(1,1)$. We assume $d := (\ell_1.\ell_1) > 0$, $(\ell_1.\ell_2) = 0$, and $(\ell_2.\ell_2) = -d$. In particular, $f := \ell_1 + \ell_2$ is an isotropic class. Analogously to $D_{\ell} \subset Q_{\ell}$, one defines
$$D_{\ell_1,\ell_2} \subset Q_{\ell_1,\ell_2} \subset \mathbb{P}(\mathbb{C} \cdot \sigma_0 \oplus \mathbb{C} \cdot \bar{\sigma}_0 \oplus \mathbb{C} \cdot \ell_1 \oplus \mathbb{C} \cdot \ell_2)$$
by the conditions $(\sigma.\sigma) = 0$ and $(\sigma.\bar{\sigma}) > 0$. Clearly, the three types of period domains D_{ℓ} with $(\ell.\ell) > 0, = 0$, and < 0 considered previously reappear as curves
$$D_{\ell_1}, D_{\ell_2}, D_f \subset D_{\ell_1,\ell_2}.$$
More generally, for any rational (or real) linear combination $\ell = c_1\ell_1 + c_2\ell_2$ the associated period domain D_{ℓ} describes a curve in D_{ℓ_1,ℓ_2}. Note that if an arbitrary $\sigma \in D_{\ell_1,\ell_2}$ is written as $\sigma = a\sigma + b\bar{\sigma} + (c_1\ell_1 + c_2\ell_2)$, then σ can be viewed as a point in the one-dimensional family of deformations D_{ℓ} with $\ell = c_1\ell_1 + c_2\ell_2$ as in Section 2.1, but typically $\ell = c_1\ell_1 + c_2\ell_2$ is not even real. In this context one would again say that σ is a brilliant deformation of σ_0 whenever P_{σ} intersects $\mathbb{R} \cdot \ell_1 \oplus \mathbb{R} \cdot \ell_2$ trivially. In Remark 2.2 we have noted already that the equators $S^1_{\ell_s} \subset D_{\ell_s}$ for $\ell_s = \ell_1 + s\ell_2$ with $s \in [0,1)$ specialize to the point $[f] \in Q_f$ for s approaching 1.

Putting the brilliant deformations associated with varying classes $\ell = c_1\ell_1 + c_2\ell_2$ into one family shows the discrete Noether–Lefschetz loci NL_{ℓ} as part of another continuous family. To make this precise, consider $\ell' \in T \oplus \mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$ not contained in $\mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$ such that
$$(\ell'.\ell') > 0 \quad \text{and} \quad (\ell'.f) > 0. \tag{2.5}$$

Then $D_{\ell_1,\ell_2} \cap \ell'^{\perp} \subset D_{\ell_1,\ell_2}$ is a one-dimensional family of deformations σ of σ_0 for which ℓ' is of type $(1,1)$.

PROPOSITION 2.8. — *For any ℓ' satisfying (2.5) and any rational $\ell = c_1\ell_1 + c_1\ell_2$, the curve $D_{\ell_1,\ell_2} \cap \ell'^{\perp}$ intersects the family of brilliant deformations parameterized by D_ℓ in the Noether–Lefschetz locus. Conversely, if $\sigma_t \neq \sigma_0$ is a brilliant deformation contained in $\mathrm{NL}_\ell \subset D_\ell$, then $\sigma_t \in \ell'^{\perp}$ for an appropriate ℓ' as above.*

Proof. — Let $\sigma_t \in D_\ell$ be a brilliant deformation of σ_0 orthogonal to ℓ'. If $(\ell.\ell) > 0$, then σ_t is contained in the orthogonal complement of the plane $\langle \ell', \ell'' \rangle_{\mathbb{Q}}$, where ℓ'' is a generator of $\ell^{\perp} \subset \mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$. As $\mathrm{sign} \langle \ell', \ell'' \rangle_{\mathbb{Q}} = (1,1)$ and σ_t is brilliant, T_t is of signature $(2, r-2)$ and, therefore, $\sigma \in \mathrm{NL}_\ell$. For $(\ell.\ell) = 0$, i.e. ℓ a multiple of $f = \ell_1 + \ell_2$ or $\ell_1 - \ell_2$, we apply Lemma 2.5. For example, if $\sigma_t = \sigma_0 + tf$, then it suffices to show that $t = (\sigma_0.B)$ for some $B \in T$. However, $(\sigma_t.\ell') = 0$ and $(\ell'.f) \neq 0$ allows us to write $t = -(\sigma_0.\ell')/(\ell'.f)$. Then let $B := -(1/(\ell'.f))\bar{\ell}'$, where $\bar{\ell}'$ is the image of ℓ' under the projection onto T.

Eventually, for $(\ell.\ell) < 0$ the transcendental lattice T_t is contained in $T \oplus \mathbb{Q} \cdot \ell$, which is of signature $(2, r-1)$. Since by assumption ℓ' is not contained in $\mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$, the transcendental lattice T_t is of dimension r and using $P_{\sigma_t} \subset T_t \otimes \mathbb{R}$ its signature must be $(2, r-2)$. Hence, also in this case, $\sigma_t \in \mathrm{NL}_\ell$.

To prove the converse, recall that σ_t is contained in the Noether–Lefschetz locus if its transcendental lattice $T_t \subset T \oplus \mathbb{Q} \cdot \ell$ has signature $(2, r-2)$. For $(\ell.\ell) > 0$, its orthogonal complement $T_t^{\perp}$ is spanned by some $\ell' \in T \oplus \mathbb{Q} \cdot \ell$ with $(\ell'.\ell') > 0$, which, after changing its sign if necessary, also satisfies $(\ell'.f) > 0$. For $(\ell.\ell) < 0$, one changes a generator of the orthogonal complement $T_t^{\perp} \subset T \oplus \mathbb{Q} \cdot \ell$ by a large multiple of a class in $\ell^{\perp} \subset \mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$ to ensure $(\ell'.\ell') > 0$. The condition $(\ell'.f) > 0$ can then be achieved by a sign change if necessary. If $(\ell.\ell) = 0$, then ℓ is a multiple of $\ell_1 \pm \ell_2$. We shall assume $\ell = f = \ell_1 + \ell_2$, the case $\ell = \ell_1 - \ell_2$ is similar. Then the transcendental lattice is the image of f_B for some $B \in T$, see Lemma 2.5. In other words, T_t is orthogonal to any element of the form $\ell' := -B + (1/d)\ell_1 + kf$ with $(\ell'.f) > 0$ and for $k \gg 0$ one has $(\ell'.\ell') > 0$. In all three cases, the assumption $\sigma_t \neq \sigma_0$ is needed to ensure that ℓ' is not contained in $\mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$. $\square$

Rephrasing the above discussion yields the Hodge theory underlying Theorem 0.4.

COROLLARY 2.9. — *Consider a brilliant twistor deformation $\sigma_t \in D_{\ell_1}$ of σ_0 that is contained in NL_{ℓ_1}. Then there exists a class $\ell' \in T \oplus \mathbb{Q} \cdot \ell_1 \oplus \mathbb{Q} \cdot \ell_2$ satisfying (2.5) and such that the curve $D_{\ell_1,\ell_2} \cap \ell'^{\perp}$ intersects the Brauer*

family $L_f \subset D_f$ in a point in the Noether–Lefschetz locus, i.e. in a point of the form $\sigma_0 + (\sigma_0.B)f$ for some $B \in T$.

Proof. — By virtue of the proposition, $\sigma_t \in \ell'^\perp$ for some ℓ' satisfying (2.5). Then $\sigma := \sigma_0 + (\sigma_0.B)f$, where B is the projection of $-(1/(f.\ell'))\ell'$ in T, is a brilliant deformation of σ_0 in the Brauer family. Moreover, σ is contained in the Noether–Lefschetz locus NL_f and, by construction, it is orthogonal to ℓ', which proves the assertion. $\qquad\square$

2.5. As before, we consider an irreducible rational Hodge structure of K3 type T and fix a generator σ_0 of $T^{2,0}$. Then the ring

$$K(\sigma_0) := \mathrm{End}_{\mathrm{Hdg}}(T)$$

of $\mathbb{Q}$-linear endomorphisms $\varphi \colon T \longrightarrow T$ of the Hodge structure T is a field. The map $\varphi \longmapsto \lambda$, with λ determined by $\varphi(\sigma_0) = \lambda \sigma_0$, describes an embedding $K(\sigma_0) \hookrightarrow \mathbb{C}$ and according to [17], $K(\sigma_0)$ is either a RM or a CM field, i.e. $K(\sigma_0)$ is either a totally real field or a purely imaginary quadratic extension $K^0(\sigma_0) \subset K(\sigma_0)$ of a totally real field $K^0(\sigma_0)$.

The Hodge structure T is said to have CM if $K(\sigma_0)$ is a CM field and, additionally, T is of dimension one as a vector space over $K(\sigma_0)$.

Remark 2.10. — For a brilliant Hodge structure σ_t contained in the Noether–Lefschetz locus, projection yields a $\mathbb{Q}$-linear isomorphism $\varpi_t \colon T_t \xrightarrow{\sim} T$, see Lemma 2.4. In particular, any Hodge endomorphism $\varphi \in K(\sigma_0)$ naturally describes a $\mathbb{Q}$-linear endomorphism φ_t of T_t. Observe that after linear extension, the projection ϖ_t maps $\sigma_t = a\sigma_0 + b\bar\sigma_0 + c\ell$ to $a\sigma_0 + b\bar\sigma_0$. Thus, if $\varphi(\sigma_0) = \lambda\sigma_0$, then $\varphi(\varpi_t(\sigma_t)) = \varphi(a\sigma_0 + b\bar\sigma_0) = a\lambda\sigma_0 + b\bar\lambda\bar\sigma_0$ and, therefore, $\varphi_t(\sigma_t) = \lambda\sigma_t$ for any φ with $\lambda \in \mathbb{R}$. Hence, in this case the $\mathbb{Q}$-linear map $\varphi_t \colon T_t \longrightarrow T_t$ respects the $(2,0)$-part. However, for $d \neq 0$ the map φ_t is not an endomorphism of the Hodge structure T_t, as it does not respect the space of $(1,1)$-forms. For $d = 0$, φ_t is a Hodge endomorphism, which also follows from Lemma 2.4.

In spite of there not being an obvious way of associating to a Hodge endomorphism φ of T a Hodge endomorphism φ_t of T_t, this is possible whenever T has CM and φ is contained in $K^0(\sigma_0)$. The proof of [8, Prop. 3.8], dealing with the case $d = (\ell.\ell) > 0$, carries over literally to the case $d < 0$ and yields the following.

PROPOSITION 2.11. — *Assume T is an irreducible Hodge structure of K3 type with complex multiplication. Then, for any brilliant deformation σ_t of σ_0 as a Hodge structure on $T \oplus \mathbb{Q}\cdot\ell$ that is contained in the Noether–Lefschetz locus NL_ℓ, there exists a natural embedding*

$$K^0(\sigma_0) \hookrightarrow K(\sigma_t) = \mathrm{End}_{\mathrm{Hdg}}(T_t).$$

Proof. — Only the case $d < 0$ needs to be checked. For a brilliant deformation σ_t in NL_ℓ there exists a class $\ell' \in T \oplus \mathbb{Q} \cdot \ell$ with $m := (\ell.\ell') \neq 0$ and $T_t = \ell'^\perp$, which is of signature $(2, r-2)$. Note that unlike the case considered in [8, §3], we have $(\ell'.\ell') < 0$, but this sign difference is of no consequence in the argument there, only $m \neq 0$ matters. □

The result immediately implies Theorem 0.2, see Section 3.1 for the translation.

COROLLARY 2.12. — *Under the assumption of the previous proposition, if σ_0 has complex multiplication, then also any brilliant deformation σ_t of σ_0 contained in the Noether–Lefschetz locus has complex multiplication and the maximal totally real subfields of σ_0 and σ_t coincide.*

Proof. — For dimension reasons, T_t as a vector space over $K^0(\sigma_0)$ is of dimension two. Hence, according to [14, Lem. 3.2], T_t has complex multiplication and $K^0(\sigma_0)$ is its maximally totally real subfield. □

Remark 2.13. — Note that the CM fields $K(\sigma_0)$ and $K(\sigma_t)$ will typically be different. In [8, Cor. 3.10] an equation of degree two $X^2 + \gamma X + \delta = 0$ is given that describes the extension $K^0(\sigma_0) \simeq K^0(\sigma_t) \subset K(\sigma_t)$. Unfortunately, there is a factor 2 missing in the computation of δ, which really should be $\delta = m^2 - (1/2)d(\sigma_0.\bar{\sigma}_0)^{-1}(\alpha^2 + \alpha^{-2} - 2)$.[3] The same computation then holds for any brilliant deformation in the Noether–Lefschetz locus as long as $d \neq 0$. Note that for $d = 0$ we have $K(\sigma_0) \simeq K(\sigma_t)$ for all t in the Noether–Lefschetz locus.

3. Geometric brilliant families

In this section we translate the Hodge theory of the previous section into geometric statements. We also indicate how to think of the Hodge theoretic Brauer family for K3 surfaces that are not elliptic.

3.1. We shall first explain how to reduce the geometric situation considered in the introduction and in Section 1 to the purely Hodge theoretic context in Section 2. The arguments are straightforward and we will be brief.

Assume S is a complex projective K3 surface with transcendental lattice $T(S) := \mathrm{NS}(S)^\perp \subset H^2(S, \mathbb{Z})$. Then the real Hodge structure $T_\mathbb{R} := T(S) \otimes \mathbb{R}$ and the rational Hodge structure $T := T(S) \otimes \mathbb{Q}$ satisfy the assumptions in Sections 2.1 and 2.2. In particular, $T_\mathbb{Q}$ is an irreducible Hodge structure. The generator $\sigma_0 \in T^{2,0}$ is nothing but a non-zero holomorphic $(2,0)$-form on S. For the additional class ℓ in (2.1) and (2.3) one picks a class $\ell \in H^{1,1}(S, \mathbb{Z})$.

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[3] Thanks to F. Viganò [16] for pointing this out to me.

The three cases $(\ell.\ell) > 0$, $= 0$, and < 0 are geometrically realized by the constructions in Sections 1.1, 1.2, and 1.3.

The surjectivity of the period map, see [6, Ch. 7] for the statement and references, immediately yields the following.

PROPOSITION 3.1. — *Consider the period domain D_ℓ in (2.2). Then there exists a smooth family of K3 surfaces $\mathcal{S} \longrightarrow D_\ell$ such that the fibre over σ_0 is S and parallel transport identifies $H^{2,0}(\mathcal{S}_t)$ with the Hodge structure on $T \oplus \mathbb{Q} \cdot \ell$ corresponding to σ_t.* $\qquad\square$

For an ample class ℓ the family can alternatively be viewed as the twistor family in Section 1.3. If ℓ is the class of the fibre of an elliptic fibration of S, the family can be constructed geometrically and without using the surjectivity of the period map, see Section 1.2. However, for $(\ell.\ell) < 0$ there typically is no alternative geometric construction. Moreover, in this case, unless $\rho(S) = 20$, the family $\mathcal{S} \longrightarrow D_\ell$ will not descend to an algebraic family despite all its fibres being polarized by any ample class in $\ell^\perp \cap H^{1,1}(S, \mathbb{Z})$, see Remark 2.7 and the discussion at the end of Section 1.1.

Remark 3.2. — For $(\ell.\ell) \leqslant 0$, the family $\mathcal{S} \longrightarrow D_\ell$ is a brilliant deformation of S, i.e. $H^{2,0}(\mathcal{S}_t) \subset (H^{2,0} \oplus H^{0,2})(S) \oplus \mathbb{C} \cdot \ell$ with $\ell \notin (H^{2,0} \oplus H^{0,2})(\mathcal{S}_t)$. For $(\ell.\ell) > 0$, the last condition forces us to restrict to the union of the two hemispheres $D_\ell \setminus S_\ell^1$, which we shall henceforth do.

Note that by definition $t \in \mathrm{NL}_\ell$ if and only of $\mathcal{S}_t$ is algebraic with $\rho(\mathcal{S}_t) = \rho(S)$, which justifies to call NL_ℓ the Noether–Lefschetz locus of the brilliant family.

Proof of Theorem 0.2. — Apply Proposition 2.11 and Corollary 2.12. $\quad\square$

Let us now focus on the case $(\ell.\ell) = 0$. As remarked already in Section 1.2, the two Brauer groups $\mathrm{Br}(S)$ and $\mathrm{Br}(T(S))$ differ by a finite group. More precisely, there is a surjection $\mathrm{Br}^{\mathrm{an}}(T(S)) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Br}^{\mathrm{an}}(S)$ with a finite kernel and inducing a surjection between the algebraic Brauer groups. However, as we rather work with the brilliant family over $H^{2,0}(S) \simeq T^{2,0} \simeq \mathbb{C}$, the difference is of no importance to us. If ℓ is the class f of an elliptic fibration $S \longrightarrow \mathbb{P}^1$, then the Brauer family $\mathcal{S} \longrightarrow \mathbb{C}$, see (1.1), is the geometric realization of the Hodge theoretic Brauer family. This yields

COROLLARY 3.3. — *Let S be a complex projective elliptic K3 surface. Then the Brauer family $\mathcal{S} \longrightarrow \mathbb{C} \simeq H^{2,0}(S)$ is a brilliant deformation of S such that its fibre $\mathcal{S}_t$ is isomorphic to the K3 surface S_α, where α is the image of t under $H^{2,0}(S) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Br}^{\mathrm{an}}(S) \simeq \mathrm{III}^{\mathrm{an}}(S)$. Furthermore, $\mathcal{S}_t$ is algebraic if and only if $t \in T(S) \otimes \mathbb{Q}$ or, equivalently, if $\alpha \in \mathrm{Br}(S)$.* $\qquad\square$

In other words, the fibre $\mathcal{S}_t$ is algebraic if and only if after parallel transport $H^{2,0}(\mathcal{S}_t)$ is generated by $\sigma_0 + (\sigma_0.B)f$ for some rational cohomology class $B \in H^2(S, \mathbb{Q})$.

Proof of Theorem 0.3. — In order to apply the Hodge theory in Section 2.4, we pick an ample line bundle L on S and let $\ell_1 := m \cdot c_1(L)$ with $m := (f.c_1(L))/(c_1(L).c_1(L))$. Then $\ell_2 := f - \ell_1$ satisfies the assumption $(\ell_2.\ell_2) = -(\ell_1.\ell_1)$.

Then the first assertion in Theorem 0.3 is the observation in Remark 2.2, while the second follows from Proposition 2.8 applied to $\ell = \ell_1$. $\qquad\square$

Proof of Theorem 0.4. — The result is a direct consequence of Corollary 2.9. The class ℓ' there satisfies $(\ell'.\ell') > 0$. Thus, it is a $(1,1)$-class in the positive cone of all fibres of $\mathcal{S} \dashrightarrow D_{\ell_1,\ell_2} \cap \ell'^\perp$, which therefore are all algebraic $\qquad\square$

3.2. In order to realize the Hodge theoretic description of the Brauer family in Section 2.3 geometrically, we need to give the additional isotropic class ℓ a geometric meaning. In Sections 1.2 and 3.1 this was achieved for elliptic K3 surfaces $S \longrightarrow \mathbb{P}^1$ by letting ℓ be the fibre class f. Of course, a general K3 surface is not elliptic, i.e. there is no isotropic $(1,1)$-class in $H^2(S, \mathbb{Z})$. However, there is an easy way out by passing to higher-dimensional hyperkähler manifolds endowed with a Lagrangian fibration. To this end, we first note the following standard fact.

PROPOSITION 3.4. — *Let S be a projective K3 surface. Then there exists a projective hyperkähler manifold X together with a Lagrangian fibration $X \longrightarrow \mathbb{P}^n$ with a section and an algebraic Hodge isometry $T(S) \simeq T(X)$.*

Proof. — Consider a primitive ample line bundle L on S and a Mukai vector $v \in H^*(S, \mathbb{Z})$ of the form $v = (0, c_1(L), \chi)$. Then the moduli space X of sheaves E with $v(E) = v$ that are Gieseker stable with respect to a v-generic polarization on S is a projective hyperkähler manifold, see [6, Prop. 10.2.5] for the statement and references. Furthermore, there exists a Hodge isometry $H^2(X, \mathbb{Z}) \simeq v^\perp \subset \widetilde{H}(S, \mathbb{Z})$, where the left hand side is endowed with the Beauville–Bogomolov form and the right hand side is the orthogonal complement of v in the Mukai lattice, see [11]. Moreover, the isomorphism is algebraic, i.e. it is induced by an algebraic class on the product $S \times X$, and, thus, yields an algebraic Hodge isometry between the transcendental lattices $T(S) \simeq T(X)$. Finally, mapping a sheaf to its Fitting support yields a Lagrangian fibration $X \longrightarrow \mathbb{P}^n \simeq |L|$, which for $\chi = (L.L)/2$ comes with a section given by the sheaves $L|_C$ for $C \in |L|$.

Alternatively, one could work with any primitive v with a square zero $(1,1)$-class in $v^\perp$ and use [1, Thm. 1.5] to produce a birational Lagrangian fibration. $\qquad\square$

Clearly, there is no canonical choice for X, in fact not even for its dimension. But note that this happens already for the case of elliptic K3 surfaces considered before, as frequently K3 surfaces admit more than one elliptic fibration.

The analogue of Corollary 3.3 for elliptic K3 surfaces is then the following.

COROLLARY 3.5. — *Let S be a projective K3 surface with*

$$(H^{2,0} \oplus H^{0,2})(S) \cap H^2(S, \mathbb{Q}) = 0.$$

  (i) *There exists a compact Lagrangian hyperkähler manifold $\pi\colon X \longrightarrow \mathbb{P}^n$ together with an algebraic Hodge isometry $T(S) \simeq T(X)$ and a brilliant deformation $\mathcal{X} \longrightarrow \mathbb{C}$ of X of Brauer type that is endowed with a family of Lagrangian fibrations $\mathcal{X}_t \longrightarrow \mathbb{P}^n$.*
 (ii) *If $\alpha_t \in \mathrm{Br}(X)$ is the image of a point $t \in \mathbb{C}$ under the natural projection $H^{0,2}(S) \simeq \mathbb{C} \longrightarrow \mathrm{Br}^{\mathrm{an}}(S) \simeq \mathrm{Br}^{\mathrm{an}}(X)$, then the Lagrangian fibration $\pi_t\colon \mathcal{X}_t \longrightarrow \mathbb{P}^n$ corresponds to the Tate–Šafarevič twist of $X \longrightarrow \mathbb{P}^n$ associated to α_t.*
(iii) *The hyperkähler manifold $\mathcal{X}_t$ is projective if and only if t is contained in the Noether–Lefschetz locus or, equivalently, if its image α_t is contained in the algebraic Brauer group $\mathrm{Br}(X) \subset \mathrm{Br}^{\mathrm{an}}(X)$.*

To spell this out a bit more, observe that $T(S) \simeq T(X)$ yields a natural isomorphism (up to a finite kernel) between the algebraic and the analytic Brauer groups of S and X. The statement that $\mathcal{X} \longrightarrow \mathbb{C}$ is a brilliant deformation of Brauer type means that there exists an isotropic class $f \in H^{1,1}(X, \mathbb{Z})$, namely $\pi^* c_1(\mathcal{O}(1))$, such that $H^{2,0}(\mathcal{X}_t) \subset H^{2,0}(X) \oplus H^{0,2}(X) \oplus \mathbb{C} \cdot f$. This allows one to use the identification $L_f \simeq H^{0,2}(X) \simeq H^{0,2}(S)$ in (2.4) and the surjection $\mathrm{NL}_f \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Br}(X) \simeq \mathrm{Br}(S)$.

Proof. — The idea of the proof is similar to the one for elliptic K3 surfaces. The existence of $X \longrightarrow \mathbb{P}^n$ is guaranteed by the previous proposition. For the construction of the family $\mathcal{X} \longrightarrow \mathbb{C}$ we refer to [10, §§7.1-7.2], it is analogous to the one for elliptic K3 surfaces [3, Ch. 1.5]. To prove (iii), one needs to relate projectivity of $\mathcal{X}_t$ to the existence of $(1,1)$-classes of positive Beauville–Bogomolov square, for which we use the criterion [4, Thm. 3.11]. $\qquad\square$

Ideally one would like to phrase (ii) in terms of an isomorphism $\mathrm{Br}^{\mathrm{an}}(X) \simeq \mathrm{III}^{\mathrm{an}}(X)$ (and similar for the algebraic variants), but the situation is more complicated in higher dimensions. Also note that the assumption on $(H^{2,0} \oplus$

$H^{0,2})(S)$ holds for most K3 surfaces and is probably not needed. It is used to ensure that all fibres in the Brauer family obtained by twisting the Lagrangian fibration $X \longrightarrow \mathbb{P}^n$ are actually hyperkähler.

Remark 3.6. — The main result of [10] roughly says that any Lagrangian fibration of a hyperkähler manifold that is deformation equivalent to a Hilbert scheme of a K3 surface is obtained as a fibre $\mathcal{X}_t$ of a family of the above type.

Remark 3.7. — The situation in positive characteristic shows certain features that are definitely not mirrored by Hodge theory. For example, the construction of "twistor spaces" in [2] starts with a supersingular K3 surface S together with a fixed isotropic Mukai vector v and interpretes the collection of moduli spaces of stable sheaves that are twisted with respect to the varying Brauer class α as a family of untwisted K3 surfaces. In characteristic zero, this does not work unless the Mukai vector $v = (r, \ell, s)$ satisfies $r = 0$. Indeed, if an α-twisted sheaf exists at all, then the order of α must divide r, which is impossible to ensure for varying Brauer classes.

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